New Combinatorial Proofs for Enumeration Problems and Random Anchored Structures

On the *Selberg Integral Formula*, *Domino Towers*,
Rook Paths and *Random Anchored Structures*

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Abstract

This thesis is divided into four parts. In the first part we present a combinatorial proof of Selberg's integral formula. This solves a problem posed by Stanley in 2009 in his collection of bijective proof problems. In the second part we analyse S-omino towers – a generalisation of domino towers that was proposed by Brown in 2017. We find a closed formula for the number of such towers using the Lagrange inversion formula, show a connection to generalised Dyck paths and describe a bijective proof. Finally, we consider the set of row-convex k-omino towers, introduced by Brown in 2017, and calculate their ordinary generating function. In the third part we answer another question from Stanleys collection of problems. We present a bijective proof for the enumeration of walks of length k a chess rook can move along on an $m \times n$ board starting and ending on the same square. Finally, in the last part we study a new probabilistic version of a combinatorial problem posed by Freedman in 1969. Let $\mathcal{P}$ be a set of $n \to \infty$ points chosen uniformly from the unit square. We maximize the total area covered by non-overlapping structures anchored at $\mathcal{P}$ and prove lower and upper bounds for the expected optimum.

Diese Dissertation besteht aus vier Teilen. Im ersten Teil finden wir einen kombinatorischen Beweis der Selbergschen Integralformel mit Hilfe von Sijektionen. Dies beantwortet eine Frage, die Stanley 2009 in seiner Sammlung von bijektiven Beweisproblemen gestellt hat. Im zweiten Teil analysieren wir S-omino-Türme – eine Verallgemeinerung von Domino-Türmen, welche 2017 von Brown vorgeschlagen wurde. Wir finden einen geschlossenen Ausdruck für die Anzahl solcher Türme über die Lagrangesche Inversionsformel und beschreiben einen bijektiven Beweis. Außerdem betrachten wir reihenkonvexe k-omino-Türme, 2017 von Brown eingeführt, und berechnen deren gewöhnliche erzeugene Funktion. Im dritten Teil beantorten wir eine weitere Frage von Stanley's Sammlung. Wir beschreiben einen bijektiven Beweis für die Aufzählung von Rundwegen auf einem $m \times n$ Schachbrett, die ein Turm mit genau k Zügen ablaufen kann. Abschließend beschäftigen wir uns mit einer neuen probabilistischen Version eines kombinatorischen Problems, welches 1969 von Freedman gestellt wurde. Sei eine Menge von $n \to \infty$ zufälligen Punkten im Einheitsquadrat gegeben. Wir maximieren den Gesamtflächeninhalt von sich nicht überlappenden, verankerten Strukturen und beweisen untere und obere Schranken für das erwartete Optimum.

Zusammenfassung

Die abzählende Kombinatorik beschäftigt sich mit der Bestimmung der Größe von endlichen Mengen, die durch kombinatorische Eigenschaften definiert sind. Meistens betrachten wir eine Mengenfamilie $S_1, S_2, S_3, \ldots$ und interessieren uns für die Anzahl der Elemente in S_n für jedes beliebige n. Eine Antwort beschreibt zum Beispiel eine Formel abhängig von n, eine erzeugene Funktion, die Information über alle n beinhaltet, oder eine asympotische Formel, die nur eine Aussage über das Verhalten für große n trifft.

Es wurden viele Methoden entwickelt um Mengen abzuzählen. Dazu zählen unter anderem algebraische Methoden, die die Eigenschaften von erzeugenden Funktionen ausnutzen oder auch von vorhandenen Werkzeugen der Linearen Algebra Gebrauch machen. Eine andere Beweismethode liefern Bijektionen. Die Grundidee ist hier einen strukturerhaltenen Zusammenhang zwischen zwei verschiedenen Mengen zu finden. Die Erhaltung der Struktur führt dazu, dass beide Mengen gleich groß sind. Einerseits liefert diese Idee eine Methode zur Abzählung der einen Menge, falls die Größe der anderen Menge bereits bekannt war. Andererseits fungiert eine Bijektion als Brücke zwischen verschiedenen mathematischen Fragestellungen und ist an sich von Interesse. Sijektionen (von signed bijection) übernehmen die Rolle von Bijektionen für vorzeichenbehaftete Mengen, die Fischer und Konvalinka einführten. Vorzeichenbehaftete Mengen sind lediglich Mengen, in denen jedes Element zusätzlich mit einem Vorzeichen versehen ist. Wir verallgemeinern diese Idee etwas, indem wir jedem Element außerdem ein Gewicht zuweisen. Diese Dissertation besteht aus folgenden vier Teilen.

Im ersten Teil beweisen wir die Selbergsche Integralformel kombinatorisch mit Hilfe von verallgemeinerten Vandermonde Determinanten und gewichteten Sijektionen. Damit beantworten wir eine offene Frage von Stanley. Außerdem liefert ein kombinatorischer Beweis eines Zwischenergebnisses eine Antwort auf ein offenes Problem von Kim und Oh, welches im Zusammenhang zur Selbergschen Integralformel gestellt wurde.

Im zweiten Teil betrachten wir eine Verallgemeinerung von Dominotürmen, die wir S-omino Türme nennen. Wir zeigen einen geschlossenen Ausdruck für die Anzahl dieser Türme mit Hilfe der Lagranschen Inversionsformel und lösen damit ein offenes Problem gestellt von Brown. Wir beschreiben einen bijektiven Beweis mit Hilfe von Raney's Lemma und zeigen einen Zusammenhang zu Dyck-Pfaden. Dann betrachten wir reihenkonvexe k-omino Türme und finden deren erzeugende Funktion, indem wir einen Beweis von Privman und Svrakic adaptieren.

Im dritten Teil betrachten wir Rundwege eines Schachturms auf einem Schachbrett. Mit Hilfe einer Bijektion zählen wir die Anzahl Rundwege auf einem $m \times n$ Schachbrett, die ein Turm in genau k Zügen ablaufen kann. Damit lösen wir ein weiteres Problem von Stanley. Der Beweis benutzt ungewichtete Sijketionen.

Im letzten Teil der Dissertation analysieren wir eine neue probabilistische Version eines kombinatorischen Problems, welches 1969 von Freedman gestellt wurde. Sei $\mathcal{P}$ eine Menge von $n \to \infty$ zufälligen Punkten im Einheitsquadrat. Wir maximieren den Gesamtflächeninhalt von sich nicht überlappenden, an $\mathcal{P}$ verankerten Strukturen und beweisen obere und untere Schranken für das erwartete Optimum oder auch von dem Flächeninhalt, den Greedy Algorithmen überdecken. Wir betrachten verschiedene Szenarien, wie zum Beispiel unten-links verankerte Rechecke, wie im ursprünglichen Problem.

Contents

1 Introduction **1**

1.1 Enumerative Combinatorics 1

1.2 Main Results 3

1.2.1 Combinatorial Proof of Selberg's Integral Formula 3

1.2.2 Generalised Domino Towers 7

1.2.3 Rook Paths 10

1.2.4 Random anchored structures 10

1.3 Tools and Notation 11

1.3.1 Definitions 11

1.3.2 Signed Sets and Sijections 12

1.3.3 Garsia and Milne's involution principle 14

1.3.4 Ordinary Generating Functions 15

1.3.5 Topological Orderings of Directed Acyclic Graphs 16

2 Combinatorial Proof of Selberg's Integral Formula **17**

2.1 Introduction 17

2.2 Proof of Selberg's integral formula 19

2.3 Bijective proof of Theorem 2.8 28

2.3.1 Binomial coefficients. 29

2.3.2 Topological orderings. 30

2.3.3 Weighted sijections 31

2.3.4 Matrices. 33

2.3.5 Putting it all together. 36

2.4 Comments and further research 39

3 Enumeration of Generalised Domino Towers **40**

3.1 Introduction 40

3.2 Representation of towers as sequences 42

3.3 Proof using the Lagrange inversion formula 46

3.4 Bijective proof 52

3.5 Row-convex k-omino towers . . . 56
3.6 Comments and further research . . . 60

4 Bijective Proof for the Enumeration of Rook Walks **61**
4.1 Introduction . . . 61
4.2 Proof of Theorem 4.3 . . . 62
4.3 Comments and further research . . . 65

5 Random anchored structures **66**
5.1 Introduction . . . 66
5.2 Tools . . . 67
5.3 Lower-left-anchored rectangles . . . 74
5.4 Lower-left-anchored squares . . . 76
5.5 Center-anchored discs . . . 84
5.6 Center-anchored squares . . . 85
5.7 Proofs . . . 87
5.8 Comments and open questions . . . 92

Acknowledgements **93**

Bibliography **94**

1 Introduction

1.1 Enumerative Combinatorics

In how many ways can you arrange a deck of 52 cards? In how many ways can we deal 13 cards to each of four players? In how many ways can we cover a chess board with 2×1 domino pieces? Questions like this are answered using techniques from the field of enumerative combinatorics. To convey an idea of the nature of the problems, we examine some concrete well-known examples in more detail:

Example 1.1 (Staircase walks)**.** *A* staircase walk *on a rectangular grid is a walk starting at* $(0,0)$*, where each step is in the rightward or upward direction, see Figure 1.1. For given integers* $m, n \geq 0$*, how many staircase walks end at position* (m,n)*? The answer is* $\frac{(n+m)!}{m!n!}$.

Figure 1.1: All $\frac{4!}{2!2!} = 6$ staircase walks on a 2×2 grid.

Example 1.2 (Dyck paths)**.** *We define a* Dyck path *of size* n *to be a staircase walk ending on* (n,n) *with the additional property, that for every point* (x_i, y_i) *on the path we have* $x_i \geq y_i$*, see Figure 1.2. What is the number of Dyck paths of size* n*? The answer turns out to be* $\frac{(2n)!}{(n+1)!n!}$*, the* n^{th} *Catalan number. The first five Catalan numbers are* $1, 2, 5, 14, 42$.

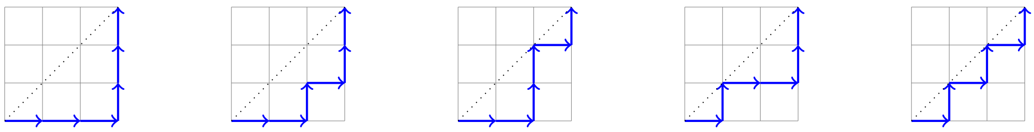


Figure 1.2: All Dyck paths of size 3.

Example 1.3 (Balanced Parentheses)**.** *Consider strings built from the two characters '(' and ')'. We call such a string* balanced *if every open parenthesis has a matching closed* *parenthesis to its right. How many strings with exactly n pairs of parantheses are balanced? For example the strings () and (()()) are balanced, while (() and ())(() are not. The answer is again* $\frac{(2n)!}{(n+1)!n!}$. *See Figure 1.3 for all* 5 *balanced strings for* $n = 3$.

((()))   (()())   (())()   ()(())   ()()()

Figure 1.3: All strings with 3 pairs of balanced parentheses.

Example 1.4 (Triangulations of an $(n+2)$-gon)**.** *In how many ways can a convex* $(n+2)$-*gon be cut it into triangles by cutting along non-intersecting diagonals? See Figure 1.4 for all* 5 *triangulations of a convex pentagon. The answer is again* $\frac{(2n)!}{(n+1)!n!}$ *[13]. Why is it the same again?*

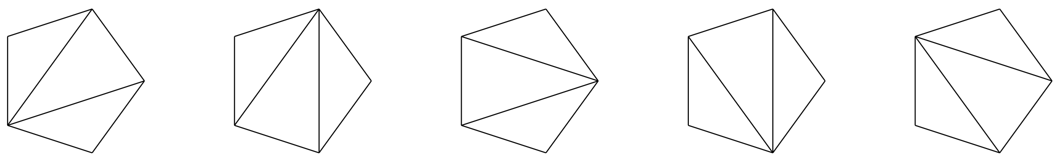


Figure 1.4: Illustration of Example 1.4

As you can see, all examples have the same answer, and it is not a coincidence. Let us first show the connection between Examples 1.2 and 1.3. It is straightforward to see that we can construct balanced parentheses from a Dyck path as follows: Starting with an empty string, for every rightward step we append an open parenthesis to the string, while for every upward step we append a closed parenthesis to it. Then the string being balanced corresponds exactly to the condition $x_i \geq y_i$. This is an example of a bijection: We have taken an object from one set, the set of Dyck paths, and turned it into an object from another set, in this case the set of balanced parentheses. It is important that we have done so in a 1-to-1 manner, as in, every balanced string of parentheses corresponds to exactly one Dyck path and vice versa. This proves that both sets have the same size.

The connection to triangulations is not quite as apparent. One bijection can be described as follows: Fix one side of the $(n+2)$-gon, drawn blue in Figure 1.5. Then draw a path which connects the inner sides of the polygon starting from the marked side in anti-clockwise order. Note that the path has three segments in each triangle, because the path passes through each of its corners exactly once. Now, for every triangle in the triangulation mark the first segment of the path red and the second green. Finally, create a Dyck path by following the path inside the polygon as follows: Starting from $(0, 0)$, for every red edge we walk rightward, while for every green edge we walk upward on the lattice. It can be shown that this procedure is also a bijection. We will see another occurrence of Catalan numbers in Chapter 3.

Sequences of numbers can be described in several different ways. In the previous discussion we have seen the *closed formula*

$$c_n = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n}$$

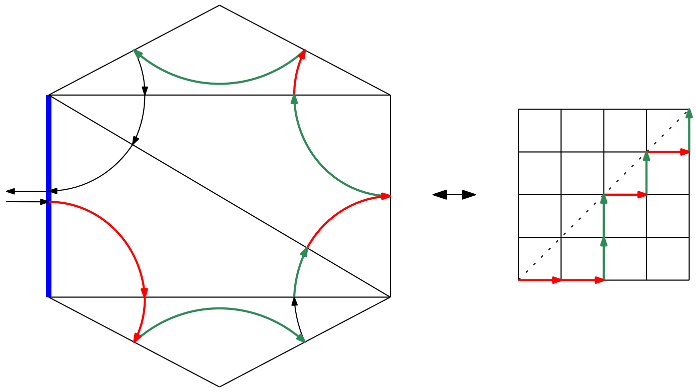


Figure 1.5: Illustration of a bijection between triangulations and Dyck paths

correspondoing to the sequence $(c_n)_{n\geq 0} = 1, 1, 2, 5, 14, 42, \ldots$. In this thesis we also use *ordinary generating functions*, which, in the case of Catalan numbers, is

$$C(x) = \sum_{n\geq 0} c_n x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + \ldots = \frac{1-\sqrt{1-4x}}{2x}.$$

A formal definition of ordinary generating functions follows in Subsection 1.3.4. The exceptional frequency with which sequences like the Catalan numbers, or similarly Fibonacci numbers, appear can be explained by their generating function. For example $C(x)$ satisfies the simple equation:

$$C(x) = 1 + xC(x)^2. \tag{1.1}$$

Bijective proofs uncover hidden structure between different sets. Often this gives powerful insights, such as the preservation of certain statistics. Furthermore, a bijective proof is constructive and algorithmic. Unfortunately using sijections or the Garsia and Milne's involution principle, see Subsections 1.3.2 and 1.3.3, produces indirect and somewhat complicated bijections. These may be unsatisfactory as compared to a direct bijection. However, most of above advantages are not lost.

1.2 Main Results

The present thesis covers four main problems. The first three problems are in the field of enumerative combinatorics, while the last one is a new probabilistic version of an old combinatorial problem. In this section, we summarise the main results of the present thesis. If a theorem or definition appears in a later chapter we take over its number from there.

1.2.1 Combinatorial Proof of Selberg's Integral Formula

Stanley stated the following result in his collection of bijective proof problems.

Theorem 2.2 (Problem 27 [36]). *Let $n \geq 2$ and $c \geq 0$. Let $f(n,c)$ be the number of sequences of length $n + 2c\binom{n}{2}$ with n symbols named x and $2c$ symbols named r_{ij} where*

$1 \leq i < j \leq n$, such that each r_{ij} occurs between the i^{th} and the j^{th} copy of x in the sequence. Then we have

$$f(n,c) = \frac{(n+2c\binom{n}{2})!}{n!(2c)!^{\binom{n}{2}}} \prod_{j=1}^{n} \frac{((j-1)c)!^2(jc)!}{c!(1+(n+j-2)c)!}.$$

For example, for $n = 3$ and $c = 1$ we have 9 symbols in total. The symbols, apart from the two copies of r_{13}, must appear in the following order:

$$x, r_{12}, r_{12}, x, r_{23}, r_{23}, x.$$

Additionally, we insert two copies of r_{13} between the first and third copy of x. There are $\binom{7}{2} = 21$ such sequences. Using Theorem 2.2 we can check $\frac{(3+6)!}{3!2^3} \cdot \frac{0!^2 1!}{1!3!} \cdot \frac{1!^2 2!}{1!4!} \cdot \frac{2!^2 3!}{1!5!} = 21$. The former approach does not generalise to larger values of n. The striking fact about this result is that the only known proof of this is non-combinatorial, in that it uses an analytical result known as Selberg's integral formula. A combinatorial proof would be very interesting, as the problem definition and solution are both of combinatorial nature. We now give a very brief sketch of the analytical proof.

Sketch of the proof using Selberg's integral formula. For each of the $n + 2c\binom{n}{2}$ symbols we choose a point independently and uniformly at random from the interval $[0, 1]$. The order of the points in $[0, 1]$ gives a sequence of symbols, where every permutation is equally likely. We can deduce that the number of sequences that follow the rules above equals:

$$\frac{(n+2c\binom{n}{2})!}{n!(2c)!^{\binom{n}{2}}} \int_0^1 \cdots \int_0^1 \prod_{j>i} |x_j - x_i|^{2c} dx_1 \ldots dx_n.$$

Now for the last step we evaluate the integral using the Selberg's integral formula, which immediately gives the result. □

Selberg's integral formula was first proved by Selberg in 1944 and then was rediscovered and generalised a number of times. It can be thought of as a generalisation of the beta function.

Theorem 2.1 (Selberg's integral formula [16])**.** *Define*

$$S_n(a,b,c) := \int_0^1 \cdots \int_0^1 \prod_i x_i^{a-1}(1-x_i)^{b-1} \prod_{i<j} |x_i - x_j|^{2c} dx_1 \ldots dx_n.$$

Then for all natural numbers $n \in \mathbb{N}$ and complex parameters $a, b, c \in \mathbb{C}$ with $\mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, \mathrm{Re}(c) > -\min\{1/n, \mathrm{Re}(a)/(n-1), \mathrm{Re}(b)/(n-1)\}$ *we have*

$$S_n(a,b,c) = \prod_{j=0}^{n-1} \frac{\Gamma(a+cj)\Gamma(b+cj)\Gamma(1+(j+1)c)}{\Gamma(a+b+(j+n-1)c)\Gamma(1+c)}.$$

Stanley states a more general version of Theorem 2.2 for any integers $a, b \geq 1$ and $c \geq 0$ and proved it using Theorem 2.1 [37]. In Chapter 2 we prove this generalised version combinatorially using the concept of signed sets and sijections, which we introduce in Subsection 1.3.2. Therefore we give an answer to Stanley's problem [36, Problem 27].

Standard Young tableaux and the hook-length formula

Consider the following problem. Fix a set $S \subset \{(i,j) : i,j \in \mathbb{Z}, i,j \geq 1\}$ with $N := |S|$. For now suppose that the set S, from now on referred to as the *shape*, satisfies

$$\begin{aligned} \forall i > 1, (i,j) \in S &\implies (i-1,j) \in S \text{ and} \\ \forall j > 1, (i,j) \in S &\implies (i,j-1) \in S, \end{aligned}$$

but we will consider other shapes later. The enumeration problem is to calculate the number of bijections $f : S \to \{1,2,\ldots,N\}$ such that:

- $\forall (i,j_1),(i,j_2) \in S$ with $j_1 < j_2$ we have $f((i,j_1)) < f((i,j_2))$ and
- $\forall (i_1,j),(i_2,j) \in S$ with $i_1 < i_2$ we have $f((i_1,j)) < f((i_2,j))$.

In other words, the integers are increasing from left to right in the rows and from top to bottom in the columns. This is called a standard Young tableau. The enumeration of standard Young tableaux is a well-studied problem. The following celebrated formula is the so-called hook-length formula, which was discovered by Frame, Robinson and Thrall in 1954 [18]:

$$\frac{N!}{\prod_{(i,j)\in S} h(i,j)},$$

where $h(i,j) = 1 + |\{(i,j') \in S : j' > j\}| + |\{(i',j) \in S : i' > i\}|$. See Figure 1.6 for a standard Young tableau and the hook of $(2,3)$.

1	2	3	6	12
4	5	10	13	14
7	11	16	18	
8	15	17		
9				

Figure 1.6: Standard Young tableau with a hook of length $h(2,3) = 5$.

Example 1.5 ([1, Prop. 14.3.3])**.** *Consider the shape $S = [2] \times [n]$. Then the number of standard Young tableaux with shape S equals*

$$\frac{(2n)!}{1!2!\cdots n!2!3!\cdots(n+1)!} = C_n$$

by the product formula. See Figure 1.7 for the correspondence between standard Young tableaux of this shape and Dyck paths.

A bijective proof of the hook-length formula was, for example, given by Franzblau and Zeilberger in 1981 [17]. The formula is similar to Theorem 2.2 in that both formulas are products of relatively small integers or their reciprocals. Such product formulas have been proven for various shapes, e.g., shifted shapes [38] or some skew shapes. However, there are skew shapes for which the number of standard Young tableau contains a large prime factor and hence such a product formula does not exist.

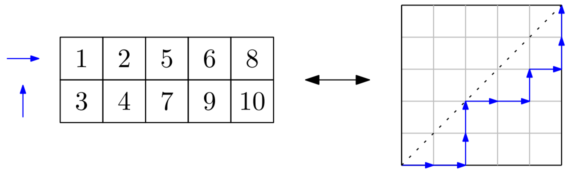


Figure 1.7: Relation to Dyck paths.

Shifted staircases, Young pages and Selberg pages

Let us now consider the shape $S = \{(i,j) : 1 \leq i \leq j \leq n\}$. This is called a *shifted standard Young tableau with staircase shape*, or also a *Young page of size n*. A *Selberg page of size n* is defined as a bijection $f : S \to \{1, 2, \dots, N\}$ such that for all $(i,j) \in S$ we have $f((i,i)) < f((i,j)) < f((j,j))$. See Figures 1.8 and 1.9 for examples. Note that every Young page is a Selberg page.

1	2	4	5
	3	6	8
		7	9
			10

Figure 1.8: One of twelve Young pages of size 4.

1	2	5	6
	3	4	9
		7	8
			10

Figure 1.9: One of one hundred forty-four Selberg pages of size 4.

We can represent the conditions on f using directed acyclic graphs. Then every bijection f corresponds to a topological ordering of the directed graph. Topological orderings are defined in Subsection 1.3.5. See Figures 1.10 and 1.11 for the graphs corresponding to Young and Selberg pages respectively.

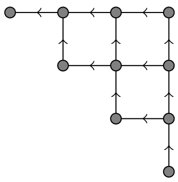


Figure 1.10: Directed graph corresponding to a Young page.

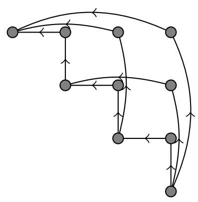


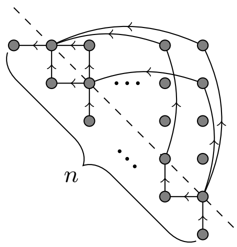
Figure 1.11: Directed graph corresponding to a Selberg page.

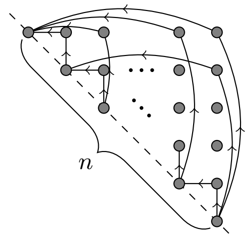
The connection between Young pages, Selberg pages and Selberg's integral formula was studied by Kim and Oh [26]. One result we want to highlight here is the following: The number of Selberg pages divided by the number of Young pages is exactly

$$1!2!3! \cdots (n-1)!.$$

An intermediate step of our proof of Theorem 2.2 is Theorem 2.8. It is essentially a generalisation of above result. For now, we only state a simplified version.

Theorem 2.8 (Simplified version)**.** *Fix any $n \geq 1$ and let A_n and B_n be the following directed graphs:*

$A_n :=$ 

and $B_n :=$ 

where we have omitted some edges for clarity. Note that B_n is the directed graph corresponding to a Selberg page, while A_n is B_{n-1} with n additional vertices below the diagonal.

Then B_n has $(n-1)!$ times as many topological orderings as A_n. Moreover, the ratio between topological orders of A_n and B_n remains $1 : (n-1)!$ even if we fix the labels of the copies of the n vertices on the main diagonal.

For $n = 4$,Theorem 2.8 shows that the graph in Figure 1.11 has 3! times as many topological orderings as the graph in Figure 1.12. Using Theorem 2.8 again with $n = 3$ shows that the graph in Figure 1.12 has 2! times as many topological orderings as the graph in Figure 1.10. This explains the factor $1!2! \cdots (n-1)!$ in above result. By proving Theorem 2.8 combinatorially we provide an answer to an open problem posed by Kim and Oh [27, Problem 6.1].

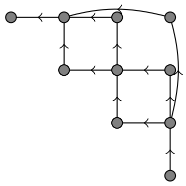


Figure 1.12: Illustration of an intermediate graph between the graphs in Figures 1.11 and 1.10.

The results in Chapter 2 are based on [23].

1.2.2 Generalised Domino Towers

In Chapter 3 we enumerate a generalisation of so-called *domino towers*. Domino towers are two-dimensional structures made out of dominoes, i.e., rectangular blocks of width 2 and height 1, arranged in a brickwork pattern, such that:

(i) The dominoes on the bottom level are contiguous, i.e., the row is convex;
(ii) Every domino above the bottom row is (half) supported on at least one domino in the row below it;
(iii) No domino lies directly on top of another domino, such as in a brickwork pattern.

See Figure 1.13 for an example. In 1985 Viennot proved that there are exactly 3^{n-1} domino towers of size n, i.e., made out of n dominoes [40]. This striking result has since attracted much attention and many variations of the problem have been studied.

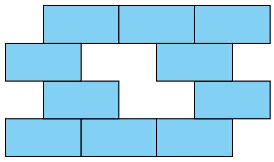


Figure 1.13: A domino tower with 10 dominoes.

One simple variation is allowing exact alignment of domino pieces on top of another, i.e., we remove rule (iii) from above. We call these towers *unrestricted*. Zeilberger and Brown independently proved that there are 4^{n-1} unrestricted towers [8, 11]. We give an alternative proof of this fact in Corollary 3.8 using symbolic methods and a construction called substitution.

Then we can generalise the problem by using rectangles of width k instead of 2. Brown called these structures k-omino towers and proved that the number of towers with exactly b blocks in the bottom row is $\binom{kn-1}{n-b}$ [8]. It should be noted here that changing the heights of the rectangles does not change the number of towers. We explain the reason for this in Section 3.2. In 2016, Brown proposed the generalisation of allowing rectangles of mixed widths, e.g., horizontal pieces of lengths from a set $S \subset \mathbb{N}$ [7]. In 2012, Zeilberger had already considered this setting and provided a Maple package that uses generating functions to calculate the number of towers computationally [11]. We find the following closed formula:

Theorem 3.1. *Fix a list of allowed widths $S = (s_1, \dots, s_m)$. The number of S-omino towers that have exactly n_i blocks of width s_i and b blocks in the bottom row, which has to be convex, equals*

$$\binom{n}{n_1, \dots, n_m}\binom{-1 + \sum n_i s_i}{n - b}.$$

Summing over all $b \in [n]$ we get

$$\binom{n}{n_1, \dots, n_m}\binom{-1 + \sum_{i=1}^m s_i n_i}{n-1} \cdot {}_2F_1\left(1, 1-n; 1 + \sum_{i=1}^m (s_i - 1) n_i; -1\right),$$

where ${}_2F_1$ is the Gaussian hypergeometric function.

We prove Theorem 3.1 using generating functions and the Lagrange inversion formula. We then turn the proof into an explicit bijection using Raney's lemma. The bijection is a 1-to-1 correspondence between S-omino towers and a set of sequences, whose cardinality can easily be calculated.

Directed lattice animals

The problem which the domino towers arose from is the so-called directed lattice animals-enumeration problem. A *directed lattice animal* of size n is a subset of n points $P \subset \mathbb{Z}^2$ such that every point in P can be reached from the *root* $\in P$ via a path contained in P using only rightward and upward steps. We are interested in the enumeration of animals up to a translation. See Figure 1.14 for all 13 directed animals of size 4. The ordinary

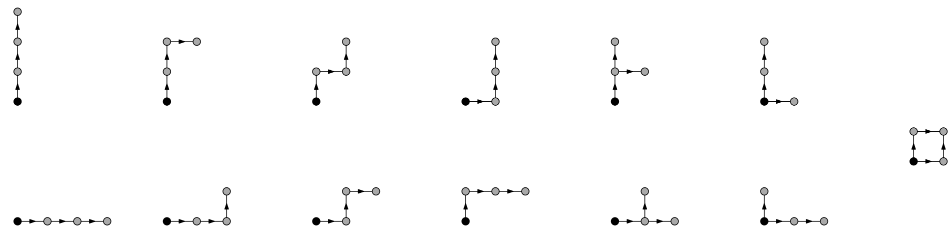


Figure 1.14: All 13 directed animals of size 4.

generating function is

$$\sum_{n\geq 1} a_n x^n = \frac{1}{2}\left(\sqrt{\frac{1+x}{1-3x}} - 1\right) = x + 2x^2 + 5x^3 + 13x^4 + 35x^5 + \dots,$$

where a_n is the number of such animals of size n [9]. A directed lattice animal is said to be *compact* if $\forall z, \forall x_1 < x_2 < x_3$ we have

$$(x_1, z - x_1), (x_3, z - x_3) \in P \implies (x_2, z - x_2) \in P.$$

In other words, the points in P with fixed $x + y$ form at most one diagonal chain. In 1988 Vladimir Privman and Nenad Svrakic enumerated compact directed lattice animals by calculating the corresponding ordinary generating function [32].

One can generalise the problem by allowing multiple roots: Fix the *roots* to be a set of points in P on the line $y = -x$. Then a set $P \subset \mathbb{Z}^2$ is called a lattice animal if every point is reachable by a rightward/upward path from at least one of the roots. In 1988 D. Gouyou-Beauchamps and G. Viennot introduced *compact-rooted* directed lattice animals, where the roots have to form a line of consecutive points. They proved that the number of compact-rooted directed lattice animals of size n equals 3^{n-1} using a bijection to one-dimensional paths. Zeilberger gave another bijective proof of this formula [41]. Restricted domino towers are related to these structures by a simple bijection, see Figure 1.15.

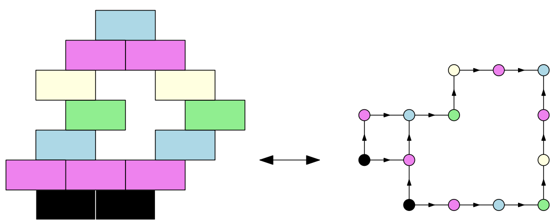


Figure 1.15: Illustration of the bijection between *restricted* domino towers and directed lattice animals.

Row-convex domino towers

In 2016 Brown studied *convex* k-omino towers, which are defined as follows:

Definition 3.2. A tower is called *column-convex* or *row-convex* if all its columns or respectively rows are convex. Further, a tower is called *convex* if it is both column- and row-convex.

In 2016, Brown calculated the generating function for convex domino towers and posed the question whether it was also possible to enumerate row-convex domino towers [7, p. 17].

Note that a restricted domino tower is row-convex if and only if the corresponding directed lattice animal is compact. Therefore enumerating restricted row-convex domino towers follows from a result by Privman et al. [32]. In Section 3.5 we calculate the generating function for unrestricted domino towers using a similar method. Moreover, we generalise it to row-convex k-omino towers, thus giving a solution to Brown's question.

The results in Chapter 3 are based on [24].

1.2.3 Rook Paths

Suppose we are given a rectangular chess board of width m and height n. The board is empty except for one single rook in the bottom left corner. Let $S_{m,n,k}$ be the set of walks of length k this rook can move along, such that its final position is its starting square. In his book "Enumerative Combinatorics" Richard P. Stanley proved a k-dimensional generalisation of the following result [37, Exer. 79 (b), pp. 626, 652] using the transfer-matrix method:

Theorem 4.3. *The number of rook paths is*

$$|S_{m,n,k}| = \frac{(m+n-2)^k + (n-1)(m-2)^k + (m-1)(n-2)^k + (m-1)(n-1)(-2)^k}{mn}.$$

Stanley states that a bijective proof of this result is unknown [36, Problem 240]. In Chapter 4, we prove Theorem 4.3 bijectively by making use of the concept of signed sets and sijections, introduced in Subsection 1.3.2.

In our proof we first find a bijection for the one-dimensional analogue of the problem. We then use that bijection to construct a bijection for the original two-dimensional problem.

The results in Chapter 4 are based on [22].

1.2.4 Random anchored structures

Consider the following combinatorial problem: Fix a finite set of points in the unit square $\mathcal{P} = \{P_1, \ldots, P_n\} \subset [0,1]^2$, where $P_1 = (0,0)$. We define a *rectangle packing* anchored at $\mathcal{P}$ to be a set of non-overlapping rectangles $\{R_1, \ldots, R_n\}$ in $[0,1]^2$ such that P_i is at the lower-left corner of R_i.

In 1969, Allen Freedman conjectured [39, p. 345] that for any such set $\mathcal{P}$, there exists a rectangle packing with total area exceeding $\frac{1}{2}$. This is the best possible constant lower bound, because for $\mathcal{P} := \{(\frac{i}{n}, \frac{i}{n}) : i \in \{0, \ldots, n\}\}$ every rectangle packing has total area $\leq \frac{1}{2} + \frac{1}{2n}$, see Figure 1.16.

In 2011, Dumitrescu and Tóth proved a lower bound of 0.09 [10], which is currently the best known lower bound. Many other variants of this problem have been studied. For example we can consider squares instead of rectangles. Note that for the $\mathcal{P}$ given above, every square packing has total area $\leq \frac{1}{n}$, i.e., there exists no constant lower bound.

Balas et al. studied the variant where rectangles can be anchored at any of the four corners [6]. Additionally they studied the same variant, but with squares instead of general rectangles.

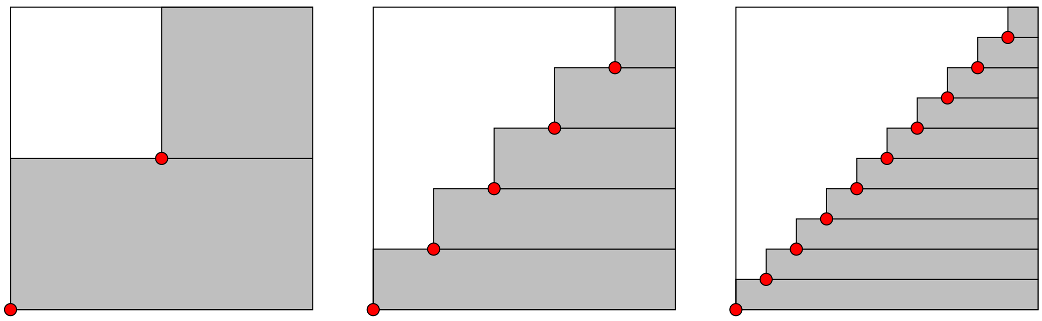


Figure 1.16: The maximum area covered by a rectangle packing is approaching $\frac{1}{2}$ as $n \to \infty$.

Other research has been done in the development of polynomial-time approximation schemes and NP-hardness of the problem [4, 6]. For instance, it is unknown whether the decision variant of the original lower-left-anchored rectangle problem is NP-hard. On the other hand, Antoniadis et al. showed that the center-anchored rectangle packing problem is NP-hard.

In Chapter 5 of the present thesis we study new probabilistic variants of the problem. We consider points chosen uniformly at random. Additionally we examine the limiting behaviour as $n \to \infty$. We analyse a greedy algorithm for lower-left-anchored rectangles which achieves an expected area of exactly 0.6. For the variant where all rectangles have to be square, we prove that the expected maximum total area we can cover is between 0.56 and 0.861.

In the second part of the chapter we consider anchored structures, for which the anchor is in the center of the shape. We first consider center-anchored disc packings and then center-anchored squares packings. In both cases we prove that the expected maximum total area is between 0.25 and 0.81.

The results in Chapter 5 are based on joint work with Albrechtsen, Lamothe, and Taraz [2, 31].

1.3 Tools and Notation

Finally, we define notation and introduce tools used throughout this thesis. This includes the definitions of weighted signed sets, sijections, ordinary generating functions and topological orders of directed acyclic graphs.

We start by defining common mathematical symbols. Then we define signed sets and sijections. We proceed by defining ordinary generating functions and introducing the symbolic method. Finally, we define topological orderings of directed acyclic graphs.

1.3.1 Definitions

For a set S we write $i < j \in S$ instead of $i, j \in S$ with $i < j$. For a positive integer $n \in \mathbb{N}$ we define $[n] := \{1, \ldots, n\}$. Further, we define $[0] := \varnothing$ to be the empty set. We define the *multinomial coefficient* as follows: For integers $n_1, \ldots, n_m \geq 0$ with sum $n = \sum_i n_i$ we define

$$\binom{n}{n_1, \ldots, n_m} := \frac{n!}{n_1! \cdots n_m!}.$$

Most often it appears with $m = 2$, then it is called the *binomial coefficient* and defined as follows:

$$\binom{n}{k} := \binom{n}{k, n-k} = \frac{n!}{k!(n-k)!}.$$

Let $\mathbb{Z}[[x_1, x_2, \ldots, x_n]]$ denote the ring of formal power series in n formal variables. Informally, its elements are infinite sums, where each summand is an integer multiplied by finitely many formal variables x_i, where repetitions are allowed.

By $A_1 \sqcup A_2$ we denote the disjoint union of two sets A_1 and A_2:

$$A_1 \sqcup A_2 := \{(a,1) : a \in A_1\} \cup \{(a,2) : a \in A_2\},$$

where 1 and 2 are two distinct auxiliary indices that indicate which A_i the element x came from.

1.3.2 Signed Sets and Sijections

In a recent paper by Fischer and Konvalinka the concept of signed sets and sijections was introduced [14]. Sijections are the analogue of bijections for signed sets. We introduce a weighted version of this concept.

A *signed set* is a finite set S equipped with a weight function $w : S \to R$. For example we can choose R to be the commutative ring $R = \mathbb{Z}[[x_1, x_2, \ldots, x_n]]$, where $x_1, x_2, \ldots, x_n$ are formal variables. Define the weight of the whole set as $w(S) := \sum_{s \in S} w(s)$. We use the subscript notation w_S if the signed set S is not clear from the context.

A *sijection* f between signed sets S and T is an involution on the set $S \sqcup T$, such that for $(x,i) \in S \sqcup T$ with $f((x,i)) = (y,j)$ we have

$$w((y,j)) = \begin{cases} -w((x,i)), & \text{if } i = j; \\ w((x,i)), & \text{otherwise.} \end{cases}$$

The motivation behind this definition is the following: If we have a sijection between A and B then $w(A) = w(B)$. This is analogous to the fact that if we have a bijection between A and B, then $|A| = |B|$. Fischer and Konvalinka used sijections with $w : S \to \{-1, 1\}$, in which case we call the signed set *unweighted*.

If A is a signed set, we define $-A$ as a copy of A, except we have $w_{-A}(a) := -w_A(a)$. If A and B are signed sets, define $A + B$ as the set $A \sqcup B$ together with the weight function

$$w_{A+B}((x,1)) := w_A(x)$$
$$w_{A+B}((x,2)) := w_B(x)$$

Similarly we define $A - B$ as the set $A + (-B)$. Finally, for signed sets A and B we define $A \times B$ as the signed set $\{(a,b) : a \in A, b \in B\}$ with weight function

$$w_{A \times B}((a,b)) = w_A(a) \cdot w_B(b).$$

Example 1.6. *Let A and B be the following weighted signed sets. Here the symbols are arbitrary, but for illustrative purposes their shape is chosen in accordance to their weight. We use the formal variables x and y for weights.*

$$A := \{\bullet, \blacksquare\} \times \{\circ, \square\}$$

$$
\begin{aligned}
&= \{(\bullet,\circ),(\blacksquare,\circ),(\bullet,\square),(\blacksquare,\square)\}\\
B &:= \{\circledcirc,\boxdot\}
\end{aligned}
$$

with weights:

$$
\begin{aligned}
w(\bullet) &= x, & w(\blacksquare) &= y, & w(\circ) &= x,\\
w(\square) &= -y, & w(\circledcirc) &= x^2, & w(\boxdot) &= -y^2
\end{aligned}
$$

and implicit weights:

$$
\begin{aligned}
w((\bullet,\circ)) &= x^2, & w((\blacksquare,\circ)) &= xy,\\
w((\bullet,\square)) &= -xy, & w((\blacksquare,\square)) &= -y^2.
\end{aligned}
$$

Then the following involution is a sijection:

$$
(\bullet,\circ)\leftrightarrow\circledcirc,\qquad (\blacksquare,\square)\leftrightarrow\boxdot,\qquad (\blacksquare,\circ)\leftrightarrow(\bullet,\square).
$$

Often we will not use different symbols for every single object, as seen above, but instead write the weights immediately:

$$
\begin{aligned}
A &= \{x,y\}\times\{x,-y\}\\
B &= \{x^2,-y^2\}
\end{aligned}
$$

From this the underlying reason for the existence of a sijection is clear:

$$
w(A) = (x+y)(x-y) = x^2 - y^2 = w(B).
$$

Figure 1.17 shows a useful illustration of the sijection above. Note that when following a line, the sign of the weight flips if and only if the line changes direction.

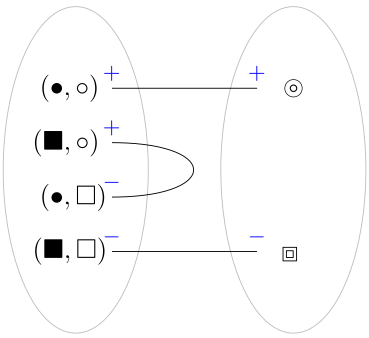


Figure 1.17: Illustration of Example 1.6

Remark 1.7. Now note that sijections can be composed much like regular functions. If we have a sijection $f : A \leftrightarrow B$ and a sijection $g : B \leftrightarrow C$ then we also have a sijection $g\circ f : A \leftrightarrow C$. See Figure 1.18 for an example. Since the sign of the weight flips if and only if the line changes direction, we know that every chain of lines from the left set A to the right set C must change its sign an even number of times, i.e., the weight is unchanged. Similarly, the sign flips for a chain of lines from A to itself.

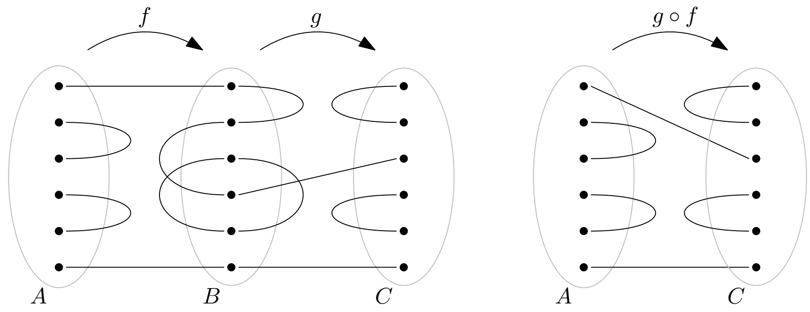


Figure 1.18: Illustration of sijection composition

Remark 1.8. If we have a sijection $A \leftrightarrow B + C$ then we have also a sijection $A - C \leftrightarrow B$. In fact it is the same involution on the set $A \sqcup B \sqcup C$ apart from having different auxiliary indices. See Figure 1.19 for an example.

Similarly we can show that, given a sijection $A + C \leftrightarrow B + C$, we can construct a sijection $A \leftrightarrow B$.

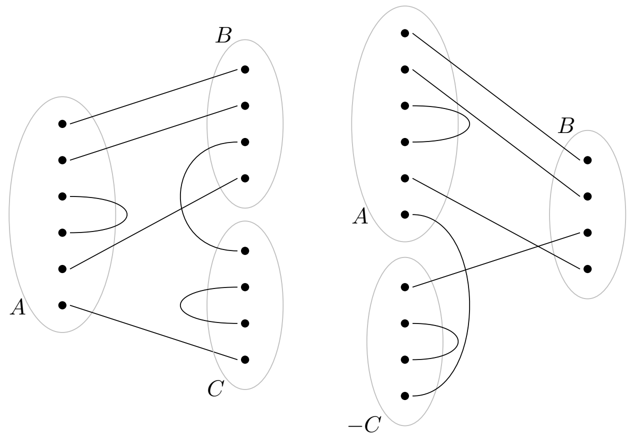


Figure 1.19: Illustration of subtraction of signed sets.

1.3.3 Garsia and Milne's involution principle

Finally, we would like to point out the connection between sijections and the well known Garsia-Milne involution principle.

Theorem 1.9 ([19])**.** *Suppose we have four sets A, B, X and Y such that $A \subset X$ and $B \subset Y$. Let $\phi : Y \to X \setminus A$ and $\psi : X \setminus B \to Y$ be bijections. Then we have a bijection $h : A \to B$.*

The Garsia-Milne involution principle can also be stated in terms of sijections. See Figure 1.20 for an illustration, where the arrow from $-X \setminus B$ to $-X \setminus A$ corresponds to the bijection $\phi^{-1} \circ \psi$. We have sijections $A \leftrightarrow X + (-X \setminus A) \leftrightarrow X + (-X \setminus B) \leftrightarrow B$ and as all elements in A and B have positive weights, we have a bijection between A and B.

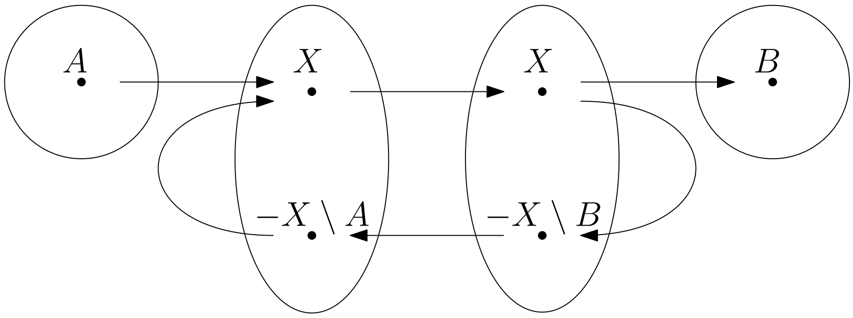


Figure 1.20: Illustration of the Garsia-Milne involution principle.

1.3.4 Ordinary Generating Functions

In this subsection, we introduce the well-known concept of ordinary generating functions. The ordinary generating function of a sequence $(a_n)_{n\geq 0}$ is defined to be the formal power series

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

Here x is a formal variable. We write $[x^n]A(x)$ for a_n, the coefficient of x^n in A. As an example of a formal power series, consider the sequence $(c_n)_{n\geq 0}$, where c_n is the number of Dyck paths of size n. Then using the result from Example 1.2, the corresponding ordinary generating function is

$$C(x) := \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n.$$

Another alternative, but equivalent, way to define generating functions is as follows: Consider the set of Dyck paths of any size. We now give each path a weight. The weight we assign to a Dyck path is x^n, where n is its size. The ordinary generating function is now simply the total of all weights. The advantage of this definition is that we can easily keep track of multiple statistics using additional formal variables. For example, we could define the weight to be $x_1^n x_2^k$ where n is its size, as before, and k is the maximum difference between x and y coordinates along the path.

The usefulness of ordinary generating functions comes from the fact that combinatorial set operations have corresponding algebraic operations on the generating functions. This approach is called the symbolic method and is introduced in the book Analytic Combinatorics by Flajolet and Sedgewick [15].

As it is relevant to Chapter 3, we would like to explain why $C(x)$ satisfies the equation

$$C(x) = 1 + xC(x)^2.$$

The symbolic method can answer this question with ease. Let $\mathcal{D}_n$ be the set of Dyck paths of length $n \geq 0$. For $n = 0$ there is only one empty Dyck path starting and ending at $(0, 0)$. This path has weight $x^0 = 1$. For $n \geq 1$ the first step must be rightward to $(1, 0)$. Then consider the point at which the path touches the main diagonal for the first time, say at (k, k). Then, by definition of k, the path from $(1, 0)$ to $(k, k-1)$ is a Dyck path of size $k-1$, shifted by $(1, 0)$. Also, the path from (k, k) to (n, n) is a Dyck path of size $n-k$, shifted by (k, k). See Figure 1.21 for an example. This is a bijection between $\mathcal{D}_n$

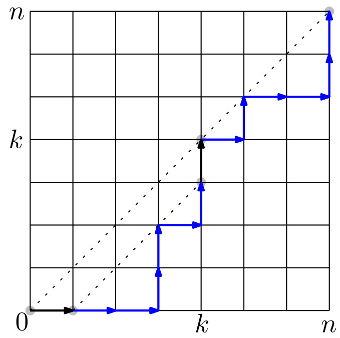


Figure 1.21: A Dyck path of size 7 decomposed into two Dyck paths of size 3.

and $\bigcup_{k=1}^{n} \mathcal{D}_{k-1} \times \mathcal{D}_{n-k}$. This explains the term $xC(x)^2$ in the equation. The extra factor x comes from the fact that elements in the former set have weight x^n while elements in the latter set have weight $x^{k-1}x^{n-k} = x^{n-1}$.

Given this equation, we can easily solve for $C(x)$ and calculate the coefficients c_n using Newton's generalised binomial theorem.

1.3.5 Topological Orderings of Directed Acyclic Graphs

In Chapter 2 we use topological orderings of directed acyclic graphs, which we define now. For a more detailed introduction of topological orderings, see [28].

A *directed graph* G is a pair (V, A) of sets V and A with $A \subset \{(u, v) : u, v \in V\}$. An element in V is called a *vertex* and an element in A is called an *directed edge*. We also write $u \to v$ instead of (u, v).

Let G be a directed graph with $n := |V|$. Then $(v_1, \ldots, v_k)$ is a *directed cycle* in G if $v_i \in V$ and $\forall i \in [k-1]$ $(v_i, v_{i+1}) \in A$ and $(v_k, v_1) \in A$. We define a directed graph to be *acyclic* if G does not contain a directed cycle. We define a topological ordering of G as a bijection $f : V \to [n]$ such that for each directed edge $i \to j$ we have $f(i) > f(j)$. A directed graph G has a topological ordering if and only if G is acyclic, see Figure 1.22 for an example. We denote by $\mathrm{TOP}(G)$ the set of topological orderings of G and define $\mathrm{top}(G) := |\mathrm{TOP}(G)|$.

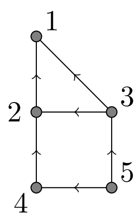


Figure 1.22: One of two topological orderings

2 Combinatorial Proof of Selberg's Integral Formula

2.1 Introduction

The following result known as Selberg's integral formula was first proved by Selberg in 1944.

Theorem 2.1 (Selberg's integral formula [16]). *Define*

$$S_n(a,b,c) := \int_0^1 \cdots \int_0^1 \prod_i x_i^{a-1}(1-x_i)^{b-1} \prod_{i<j} |x_i - x_j|^{2c} dx_1 \dots dx_n.$$

Then for all natural numbers $n \in \mathbb{N}$ *and complex parameters* $a, b, c \in \mathbb{C}$ *with* $\mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, \mathrm{Re}(c) > -\min\{1/n, \mathrm{Re}(a)/(n-1), \mathrm{Re}(b)/(n-1)\}$ *we have*

$$S_n(a,b,c) = \prod_{j=0}^{n-1} \frac{\Gamma(a+cj)\Gamma(b+cj)\Gamma(1+(j+1)c)}{\Gamma(a+b+(j+n-1)c)\Gamma(1+c)}.$$

The integral can be thought of as a generalisation of the beta function, because

$$S_1(a,b,0) = B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt.$$

The formula was rediscovered and generalised several times. It is also used in mathematical physics. For example, in the statistical theory of energy levels of complex systems it can be applied to evaluate Mehta's integral. It is related to Macdonald's conjectures and has applications in random matrix theory. For details on the history and further applications of the formula, we refer the reader to the survey by Forrester and Warnaar [16].

Stanley has stated the following interesting application of Selberg's integral formula in their collection of open bijective problems:

Theorem 2.2 (Problem 27 [36]). *Let* $n \geq 2$ *and* $c \geq 0$. *Let* $f(n,c)$ *be the number of sequences of length* $n + 2c\binom{n}{2}$ *with* n *symbols named* x *and* $2c$ *symbols named* r_{ij} *where* $1 \leq i < j \leq n$, *such that each* r_{ij} *occurs between the* i^{th} *and the* j^{th} *copy of* x *in the sequence. Then we have*

$$f(n,c) = \frac{(n+2c\binom{n}{2})!}{n!(2c)!^{\binom{n}{2}}} \prod_{j=1}^{n} \frac{((j-1)c)!^2(jc)!}{c!(1+(n+j-2)c)!}.$$

The striking fact about this result is that the only known proof is non-combinatorial in that it uses Selberg's integral formula. They stress that a combinatorial proof would be very interesting. As the problem definition and solution are both of combinatorial nature, a combinatorial proof might exist. We now give a very brief sketch of the analytical proof:

Sketch of the proof using Selberg's integral formula. For each of the $n + 2c\binom{n}{2}$ symbols we choose a point independently and uniformly at random from the interval $[0, 1]$. The order of the points in $[0, 1]$ gives a sequence of symbols, where every permutation is equally likely. We can deduce that the number of sequences that follow the rules above equals:

$$\frac{\left(n + 2c\binom{n}{2}\right)!}{n!(2c)!^{\binom{n}{2}}} \int_0^1 \cdots \int_0^1 \prod_{j>i} |x_j - x_i|^{2c} \, dx_1 \, \ldots \, dx_n.$$

Now for the last step we evaluate the integral using the Selberg's integral formula, which immediately gives the result. □

In the sketch of proof, we have set $a = b = 1$. A more general version of Theorem 2.2 for any positive integers a and b is the following theorem. It can be proven similarly to Theorem 2.2.

Theorem 2.3 (Problem 11 [37])**.** *Let $n \geq 2$ and $a, b \geq 1, c \geq 0$. Let $f(n, a, b, c)$ be the number of sequences of length $(a + b - 1)n + 2c\binom{n}{2}$ with*

- *n symbols named x;*
- *$a - 1$ symbols named p_i with $i \in [n]$, such that each p_i occurs before the i^{th} copy of x;*
- *$b - 1$ symbols named q_i with $i \in [n]$, such that each q_i occurs after the i^{th} copy of x.*
- *$2c$ symbols named r_{ij} with $1 \leq i < j \leq n$, such that each r_{ij} occurs between the i^{th} and the j^{th} copy of x;*

Then we have

$$f(n, a, b, c) = \frac{\left(2c\binom{n}{2} + (a + b - 1)n\right)!}{n!((2c)!)^{\binom{n}{2}}((a - 1)!(b - 1)!)^n} \prod_{j=0}^{n-1} \frac{\Gamma(jc + c + 1)\Gamma(a + jc)\Gamma(b + jc)}{\Gamma(c + 1)\Gamma(a + b + (j + n - 1)c)}.$$

With the aim of finding a combinatorial proof of Theorem 2.3, Kim and Oh defined Selberg pages and Young pages [27, 26] and proposed that one could find a combinatorial proof via these constructs. However, they showed that not for all Young pages there exist "nice" product formulas. This means that arguments via induction on Young pages will most likely fail.

In this chapter we prove Selberg's integral formula combinatorially for integer-valued parameters $a, b \geq 1, c \geq 0$ using the concept of signed sets and sijections.

Remark 2.4. Knowing that Selberg's integral formula holds for all integer-valued $a, b \geq 1, c \geq 0$ one can use Carlson's theorem to show that it holds for all $(a, b, c) \in D$, where

$$\begin{aligned} D := \{(a, b, c) : \ & a, b, c \in \mathbb{C}, \mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, \\ & \mathrm{Re}(c) > -\min\{1/n, \mathrm{Re}(a)/(n - 1), \mathrm{Re}(b)/(n - 1)\}\}. \end{aligned}$$

We define $f(a, b, c)$ as the difference of the right-hand side and the left-hand side of Theorem 2.1. Then f is analytic on D. Carlson's theorem says that if a function $g(z)$ is analytic for $\mathrm{Re}(z) \geq 0$ satisfying the bound $|g(z)| = \mathcal{O}(e^{k|z|})$ for $k < \pi$, and also $g(z) = 0$ on the non-negative integers, then g is identically 0. Now, as f is bounded by a constant for $a, b, c \geq 1$ we can apply Carlson's theorem three times on $f(a+1, b+1, c+1)$. Then Selberg's formula holds on all of D by the identity theorem.

Structure of the chapter In Section 2.2 we introduce topological orderings of directed acyclic graphs. Then we state Theorem 2.6, which is a reformulation of Theorem 2.1 in terms of topological orderings for integer-valued parameters a, b, c. We then motivate and state Theorem 2.8. We finish Section 2.2 by proving Theorem 2.6 by induction using Theorem 2.8. For better understanding we also give an example, which shows how Theorem 2.8 is applied. In Section 2.3 we prove Theorem 2.8 by constructing a sijection. We conclude the chapter with remarks and open questions.

2.2 Proof of Selberg's integral formula

In Theorem 2.2 Stanley interpreted Selberg's integral formula in terms of sequences. In this chapter we use graphs instead of sequences, as they are easier to visualise. However, both views are essentially equivalent, as rules for these sequences can be translated into directed graphs, and vice versa. For example the rule that a symbol named r_{ij} must be between x_i and x_j can be expressed with two edges $x_j \to r_{ij}$ and $r_{ij} \to x_i$, which we sometimes combine to $x_j \to r_{ij} \to x_i$. Then counting sequences corresponds to counting topological orderings of directed graphs, which are defined in Subsection 1.3.5. The following special graph captures the rules for our sequences.

Definition 2.5. For integers $n, a, b \geq 1, c \geq 0$ let $G_S(n, a, b, c) = (V_S, E_S)$ be the graph with vertex set:

$$\begin{aligned} V_S = &\{x_1, x_2, \dots, x_n\} \\ &\cup \ \{p_{i,k} : \ i \in [n], k \in [a-1]\} \\ &\cup \ \{q_{i,k} : \ i \in [n], k \in [b-1]\} \\ &\cup \ \{r_{i,j,k} : \ i < j \in [n], k \in [2c]\} \end{aligned}$$

and edge set

$$\begin{aligned} E_S = &\{x_{i+1} \to x_i : i \in [n-1]\} \\ &\cup \ \{x_i \to p_{i,k} : i \in [n], k \in [a-1]\} \\ &\cup \ \{q_{i,k} \to x_i : i \in [n], k \in [b-1]\} \\ &\cup \ \{x_j \to r_{i,j,k} \to x_i : i < j \in [n], k \in [2c]\}. \end{aligned}$$

See Figure 2.1 for a drawing of such a graph, where we have grouped some vertices and edges together. Theorem 2.1 for integers $n, a, b \geq 1, c \geq 0$ can be stated combinatorially in terms of topological orderings as follows:

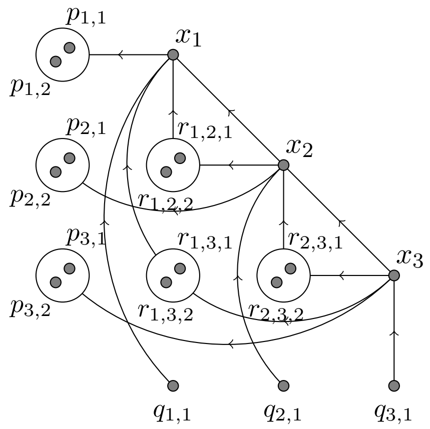


Figure 2.1: The directed graph $G_S(3,3,2,1)$

Theorem 2.6. *For integers $n, a, b \geq 1, c \geq 0$, we have:*

$$\begin{aligned}&\operatorname{top}(G_S(n,a,b,c))\\ &= \frac{(n(a+b-1)+\binom{n}{2}\cdot 2c)!}{n!}\prod_{j=0}^{n-1}\frac{(a+cj-1)!(b+cj-1)!(c+cj)!}{c!(a+b+(j+n-1)c-1)!}.\end{aligned}$$

Note that for $a = b = 1$ this implies Theorem 2.2, and in general implies the more general version as seen in [37][Chapter 1, Problem 11].

The proof of Theorem 2.6 is essentially a combinatorial analogue of Anderson's proof of Theorem 2.1 [3]. Most of the steps in their proof have combinatorial interpretations, and so the problem basically reduces to finding a combinatorial proof of Theorem 2.8, which we state after the following definition:

Definition 2.7. For fixed $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \geq 0$, let $G_X(\alpha) = (V_X, E_X)$ be the directed graph with vertex set:

$$\begin{aligned}V_X = \{&u_1, u_2, \ldots, u_n\}\\ &\cup\ \{w_{i,j,k} : i < j \in [n], k \in [\alpha_i \cdot \alpha_j]\}\end{aligned}$$

and edge set

$$\begin{aligned}E_X = \{&u_{i+1} \to u_i : i \in [n-1]\}\\ &\cup\ \{u_j \to w_{i,j,k} \to u_i : i < j \in [n], k \in [\alpha_i \cdot \alpha_j]\}.\end{aligned}$$

Figure 2.2 shows an example of such a graph for $\alpha = (2, 1, 3)$.

Theorem 2.8. *Fix $\alpha_1, \ldots, \alpha_{2n-1} \geq 0$ with $\alpha_i = 1$ for i even and define the two graphs $G_1 := G_X(\alpha_1, \alpha_2, \ldots, \alpha_{2n-1})$ and $G_2 := G_X(\alpha_1 + 1, \alpha_3 + 1, \ldots, \alpha_{2n-1} + 1)$. Note that both graphs have*

$$N := n + \sum_{1 \leq i < j \leq n} (\alpha_{2i-1} + 1)(\alpha_{2j-1} + 1)$$

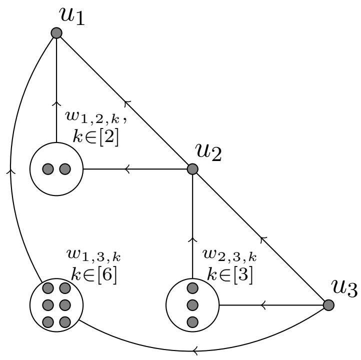


Figure 2.2: The directed graph $G_X(2,1,3)$

vertices. Fix any distinct $p_1, p_3, \ldots, p_{2n-1} \in [N]$ and let

$$A := \{f \in \mathrm{TOP}(G_1) : f(u_i) = p_i \ \forall i \text{ odd}\}$$

$$B := \{f \in \mathrm{TOP}(G_2) : f(u_i) = p_{2i-1} \ \forall i\}.$$

In other words, in B we have fixed labels of the diagonal. In A we have fixed every second label of the diagonal. We have a bijection

$$\left[\left(\sum \alpha_i\right)!\right] \times A \leftrightarrow \prod[\alpha_i!] \times B.$$

Theorem 2.8 is essentially a combinatorial interpretation of the following result.

Theorem 2.9. *Fix $\alpha_1, \ldots, \alpha_{2n-1} \geq 0$ with $\alpha_i = 1$ for i even. We have*

$$\begin{aligned}&\left(\sum_{i=1}^{2n-1} \alpha_i\right)! \int_{x_1}^{x_3} \cdots \int_{x_{2n-3}}^{x_{2n-1}} \prod_{i<j} (x_j - x_i)^{\alpha_i \alpha_j} dx_{2n-2} \ldots dx_2 \\ &= \prod_{i=1}^{2n-1} (\alpha_i!) \prod_{\substack{i<j \\ \text{both odd}}} (x_j - x_i)^{(\alpha_i+1)(\alpha_j+1)}.\end{aligned}$$

Here we can divide through the common factor

$$\prod_{\substack{i<j \\ \text{both odd}}} (x_j - x_i)^{\alpha_i \alpha_j},$$

which results in Corollary 2.10. It is used in Anderson's proof of Theorem 2.1 and proved there using a cunning change of variables. We now show a new idea, which does not rely on change of variables and which can then be turned into a bijection.

Corollary 2.10. *Fix $\alpha_1, \ldots, \alpha_{2n-1} \geq 0$ with $\alpha_i = 1$ for i even. We have*

$$\begin{aligned}&\int_{x_1}^{x_3} \cdots \int_{x_{2n-3}}^{x_{2n-1}} \prod_{\substack{i<j \\ \text{not both odd}}} (x_j - x_i)^{\alpha_i \alpha_j} dx_{2n-2} \ldots dx_2 \\ &= \frac{\prod \alpha_i!}{(\sum \alpha_i)!} \prod_{\substack{i<j \\ \text{both odd}}} (x_j - x_i)^{\alpha_i + \alpha_j + 1}.\end{aligned}$$

Example 2.11. *For $\alpha = (2,1,3)$ we have*

$$\int_{x_1}^{x_3} (x_2 - x_1)^2 (x_3 - x_2)^3 (x_3 - x_1)^6 \, dx_2 = \frac{1}{60}(x_3 - x_1)^{12};$$

and after dividing through the common factor:

$$\int_{x_1}^{x_3} (x_2 - x_1)^2 (x_3 - x_2)^3 \, dx_2 = \frac{1}{60}(x_3 - x_1)^{6}.$$

We want to prove Example 2.11 without a bijection first, to show the idea of the proof. In Section 2.3 we write the proof as a bijection.

Proof of Example 2.11. We write the expression as a determinant of a matrix, in which x_2 appears in one row only. This allows us to integrate each entry in that row separately.

$$\int_{x_1}^{x_3} (x_2 - x_1)^2 (x_3 - x_2)^3 \, dx_2$$

$$= \int_{x_1}^{x_3} (x_3 - x_1)^{-6} \cdot \det \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \\ 0 & 1 & 2x_3 & 3x_3^2 & 4x_3^3 & 5x_3^4 \\ 0 & 0 & 1 & 3x_3 & 6x_3^2 & 10x_3^3 \end{pmatrix} dx_2$$

$$= (x_3 - x_1)^{-6} \cdot \det \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ x_3 - x_1 & \frac{1}{2}(x_3^2 - x_1^2) & \frac{1}{3}(x_3^3 - x_1^3) & \frac{1}{4}(x_3^4 - x_1^4) & \frac{1}{5}(x_3^5 - x_1^5) & \frac{1}{6}(x_3^6 - x_1^6) \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \\ 0 & 1 & 2x_3 & 3x_3^2 & 4x_3^3 & 5x_3^4 \\ 0 & 0 & 1 & 3x_3 & 6x_3^2 & 10x_3^3 \end{pmatrix}$$

$$= (x_3 - x_1)^{-6} \cdot \frac{1}{6!} \cdot \det \begin{pmatrix} 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 & 6x_1^5 \\ 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 & 30x_1^4 \\ x_3 - x_1 & x_3^2 - x_1^2 & x_3^3 - x_1^3 & x_3^4 - x_1^4 & x_3^5 - x_1^5 & x_3^6 - x_1^6 \\ 1 & 2x_3 & 3x_3^2 & 4x_3^3 & 5x_3^4 & 6x_3^5 \\ 0 & 2 & 6x_3 & 12x_3^2 & 20x_3^3 & 30x_3^4 \\ 0 & 0 & 3 & 12x_3 & 30x_3^2 & 60x_3^3 \end{pmatrix}$$

$$= (x_3 - x_1)^{-6} \cdot \frac{2!3!}{6!} \cdot \det \begin{pmatrix} 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 & 6x_1^5 \\ 0 & 1 & 3x_1 & 6x_1^2 & 10x_1^3 & 15x_1^4 \\ x_3 - x_1 & x_3^2 - x_1^2 & x_3^3 - x_1^3 & x_3^4 - x_1^4 & x_3^5 - x_1^5 & x_3^6 - x_1^6 \\ 1 & 2x_3 & 3x_3^2 & 4x_3^3 & 5x_3^4 & 6x_3^5 \\ 0 & 1 & 3x_3 & 6x_3^2 & 10x_3^3 & 15x_3^4 \\ 0 & 0 & 1 & 4x_3 & 10x_3^2 & 20x_3^3 \end{pmatrix}$$

$$= (x_3 - x_1)^{-6} \cdot \frac{2!3!}{6!} \cdot \det \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 & x_1^6 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 & 6x_1^5 \\ 0 & 0 & 1 & 3x_1 & 6x_1^2 & 10x_1^3 & 15x_1^4 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 & x_3^3 - x_1^3 & x_3^4 - x_1^4 & x_3^5 - x_1^5 & x_3^6 - x_1^6 \\ 0 & 1 & 2x_3 & 3x_3^2 & 4x_3^3 & 5x_3^4 & 6x_3^5 \\ 0 & 0 & 1 & 3x_3 & 6x_3^2 & 10x_3^3 & 15x_3^4 \\ 0 & 0 & 0 & 1 & 4x_3 & 10x_3^2 & 20x_3^3 \end{pmatrix}$$

$$= (x_3 - x_1)^{-6} \cdot \frac{2!3!}{6!} \cdot \det \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 & x_1^6 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 & 6x_1^5 \\ 0 & 0 & 1 & 3x_1 & 6x_1^2 & 10x_1^3 & 15x_1^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 & x_3^6 \\ 0 & 1 & 2x_3 & 3x_3^2 & 4x_3^3 & 5x_3^4 & 6x_3^5 \\ 0 & 0 & 1 & 3x_3 & 6x_3^2 & 10x_3^3 & 15x_3^4 \\ 0 & 0 & 0 & 1 & 4x_3 & 10x_3^2 & 20x_3^3 \end{pmatrix}$$

$$= \frac{1}{60} \cdot (x_3 - x_1)^6.$$

The first and last matrix in above proof can be evaluated using the generalised Vandermonde determinant, as found for example in [25]. The second equality follows by integrating each entry in the third row. The other steps were: Multiply columns by factors, divide rows by factors, enlarge matrix without changing the determinant and, finally, add row 1 to row 4. ☐

Theorem 2.12 (Generalised Vandermonde determinant [25] [29, Thm. 20])**.** *Fix $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \geq 1$. Let $m = \sum \alpha_i$ and $M(a, m, x)$ be the following $a \times m$ matrix:*

$$M(a,m,x) := \begin{pmatrix} 1 & x & x^2 & \ldots & \binom{a-1}{0}x^{a-1} & \ldots & \binom{m-1}{0}x^{m-1} \\ 0 & 1 & 2x & \ldots & \binom{a-1}{1}x^{a-2} & \ldots & \binom{m-1}{1}x^{m-2} \\ 0 & 0 & 1 & \ldots & \binom{a-1}{2}x^{a-3} & \ldots & \binom{m-1}{2}x^{m-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ 0 & 0 & 0 & \ldots & 1 & \ldots & \binom{m-1}{a-1}x^{m-a} \end{pmatrix}$$
$$= \left(\binom{j-1}{i-1} x^{j-i} \right)_{1 \leq i \leq a, 1 \leq j \leq m.}$$

Then we have:

$$\det \begin{pmatrix} M(\alpha_1, m, x_1) \\ \vdots \\ M(\alpha_n, m, x_n) \end{pmatrix} = \prod_{i<j} (x_j - x_i)^{\alpha_i \alpha_j}.$$

Note that if we have $\alpha_i = 1$ for all i this reduces to the usual Vandermonde determinant. In Example 2.11 we have divided both sides by the common factor $(x_3 - x_1)^6$. We can make sense of this combinatorially by removing the corresponding vertices from the graphs G_1, G_2 in Theorem 2.8. The resulting graphs are G_A and G_B which we define now:

Definition 2.13. For fixed $\alpha_1, \ldots, \alpha_{2n-1} \geq 0$ with $\alpha_i = 1$ for even i, we define the graph $G_A(\alpha_1, \ldots, \alpha_{2n-1}) = (V_A, E_A)$ to have vertex set

$$\begin{aligned} V_A = \{&u_1, u_2, \ldots, u_{2n-1}\} \\ &\cup \ \{w_{i,j,k} : i < j \in [2n-1] \text{ not both odd}, k \in [\alpha_i \cdot \alpha_j]\} \end{aligned}$$

and edge set

$$\begin{aligned} E_A = \{&u_{i+1} \to u_i : i \in [2n-2]\} \\ &\cup \ \{u_j \to w_{i,j,k} \to u_i : i < j \in [2n-1] \text{ not both odd}, k \in [\alpha_i \cdot \alpha_j]\}. \end{aligned}$$

Definition 2.14. For fixed $\alpha_1, \ldots, \alpha_{2n-1} \geq 0$ with $\alpha_i = 1$ for even i, we define the graph $G_B(\alpha_1, \ldots, \alpha_{2n-1}) = (V_B, E_B)$ to have vertex set

$$\begin{aligned} V_B = \{&u_1, u_3, \ldots, u_{2n-1}\} \\ &\cup \ \{w_{i,j,k} : i < j \in \{1, 3, \ldots, 2n-1\}, k \in [\alpha_i + \alpha_j + 1]\} \end{aligned}$$

and edge set

$$\begin{aligned} E_B = \{&u_{i+2} \to u_i : i \in [2n-3]\} \\ &\cup \ \{u_j \to w_{i,j,k} \to u_i : i < j \in \{1, 3, \ldots, 2n-1\}, k \in [\alpha_i + \alpha_j + 1]\}. \end{aligned}$$

Corollary 2.15. *Fix* $\alpha = (\alpha_1, \dots, \alpha_{2n-1})$ *with* $\alpha_i \geq 0$ *for* i *odd and* $\alpha_i = 1$ *for even* i *and let* $G_A = G_A(\alpha)$ *and* $G_B = G_B(\alpha)$ *be as in Definitions 2.13 and 2.14. Fix any integers* $p_1 < p_3 < \cdots < p_{2n-1}$. *Then we have*

$$\begin{aligned}&\Big(\sum \alpha_i\Big)! \cdot |\{f \in \mathrm{TOP}(G_A)\colon f(u_i) = p_i \ \forall i \ \textit{odd}\}|\\&= \prod(\alpha_i!) \cdot |\{f \in \mathrm{TOP}(G_B)\colon f(u_i) = p_i \ \forall i \ \textit{odd}\}|.\end{aligned}$$

Proof. Define $N := |V(G_A)| = |V(G_B)|$ and $\mathcal{P} := \{(p_1, p_3, \dots, p_{2n-1}) : p_1 < p_3 < \cdots < p_{2n-1} \in [N]\}$. For $P \in \mathcal{P}$ define $\mathrm{idx}(P)$ as the index of P in $\mathcal{P}$ in lexicographical order. For example, we have $\mathrm{idx}((1, 2, \dots, n)) = 1$ and $\mathrm{idx}((N - n + 1, \dots, N - 1, N)) = |\mathcal{P}| = \binom{N}{n}$. Finally define the two graphs $G_1 := G_X(\alpha)$ and $G_2 := G_X(\alpha_1 + 1, \alpha_3 + 1, \dots, \alpha_{2n-1} + 1)$. We now prove the result for all $P \in \mathcal{P}$ by induction on $\mathrm{idx}(P)$.

Given $P = (p_1, p_3, \dots, p_{2n-1})$ define $p'_1, p'_3, \dots, p'_{2n-1}$ as follows:

$$p'_\ell := p_\ell + \sum_{\substack{1 \leq i < j \leq \ell \\ i,j \text{ odd}}} \alpha_i \alpha_j.$$

We can enumerate $\{f \in \mathrm{TOP}(G_1)\colon f(u_i) = p'_i, \forall i \text{ odd}\}$ by first considering all possible labels of the vertices $w_{i,j,k}$ for i and j odd, and then the rest, which is a topological order of G_A, for which the labels of $u_1, u_3, \dots, u_{2n-1}$ has been fixed to be, say, $P'' = (p''_1, p''_3, \dots, p''_{2n-1})$. We have

$$p''_\ell = p'_\ell - |\{w_{i,j,k} : i, j \in [2n-1] \text{ odd } k \in [\alpha_i \alpha_j] \text{ and } f(w_{i,j,k}) < p'_\ell\}|.$$

The crucial point here is that although P'' depends on the choices of $f(w_{i,j,k})$, we always have $\mathrm{idx}(P'') \leq \mathrm{idx}(P)$. Hence we get

$$\begin{aligned}&= |\{f \in \mathrm{TOP}(G_1)\colon f(u_i) = p'_i \ \forall i \text{ odd}\}|\\&= \sum_{P''\colon \mathrm{idx}(P'') \leq \mathrm{idx}(P)} c(P'') \cdot |\{f \in \mathrm{TOP}(G_A)\colon f(u_i) = p''_i \ \forall i \text{ odd}\}|\end{aligned}$$

for some constants $c(P'')$, where in particular we have $c(P) \geq 1$. We can do the same for G_2 with B_2 and get the same constants c:

$$\begin{aligned}&= |\{f \in \mathrm{TOP}(G_2)\colon f(u_i) = p'_{2i-1} \ \forall i\}|\\&= \sum_{P''\colon \mathrm{idx}(P'') \leq \mathrm{idx}(P)} c(P'') \cdot |\{f \in \mathrm{TOP}(G_B)\colon f(u_i) = p''_i \ \forall i \text{ odd}\}|.\end{aligned}$$

By Theorem 2.8, we have

$$\begin{aligned}&= \Big(\sum \alpha_i\Big)! \cdot |\{f \in \mathrm{TOP}(G_1)\colon f(u_i) = p'_i \ \forall i \text{ odd}\}|\\&= \prod(\alpha_i!) \cdot |\{f \in \mathrm{TOP}(G_2)\colon f(u_i) = p'_{2i-1} \ \forall i\}|\end{aligned}$$

and by induction we have for all P'' with $\mathrm{idx}(P'') < \mathrm{idx}(P)$:

$$\begin{aligned}&\Big(\sum \alpha_i\Big)! \cdot |\{f \in \mathrm{TOP}(G_A)\colon f(u_i) = p''_i \ \forall i \text{ odd}\}|\\&= \prod(\alpha_i!) \cdot |\{f \in \mathrm{TOP}(G_B)\colon f(u_i) = p''_i \ \forall i \text{ odd}\}|.\end{aligned}$$

As $c(P) \neq 0$ the result follows for P. □

Remark 2.16. One could have stated Corollary 2.15 as a bijection. But above proof would result in a bijection

$$\text{LHS} \times X \leftrightarrow \text{RHS} \times X,$$

for some large set X, which we cannot simply divide through.

Finally, we define the last graph G_K which we relate to G_S in two different ways. This part of the proof is analogous to Anderson's proof [3], except that rather than defining a graph G_K, they instead define an integral K.

Definition 2.17. For fixed integers $n, a, b, c \geq 1$ let $G_K(n, a, b, c) = (V_K, E_K)$ be the graph with vertex set:

$$\begin{aligned} V_K = \{&x_1, x_2, \dots, x_n\} \\ &\cup \ \{y_1, y_2, \dots, y_{n-1}\} \\ &\cup \ \{v_{i,j} : i < j \in [n]\} \\ &\cup \ \{w_{i,j} : i < j \in [n-1]\} \\ &\cup \ \{p_{i,k} : i \in [n], k \in [a-1]\} \\ &\cup \ \{q_{i,k} : i \in [n], k \in [b-1]\} \\ &\cup \ \{r_{i,j,k} : i \in [n], j \in [n-1], k \in [c-1]\} \end{aligned}$$

and edge set

$$\begin{aligned} E_K = \{&x_{i+1} \to y_i \to x_i : i \in [n-1]\} \\ &\cup \ \{x_j \to v_{i,j} \to x_i : i < j \in [n]\} \\ &\cup \ \{y_j \to w_{i,j} \to y_i : i < j \in [n-1]\} \\ &\cup \ \{x_i \to p_{i,k} : i \in [n], k \in [a-1]\} \\ &\cup \ \{q_{i,k} \to x_i : i \in [n], k \in [b-1]\} \\ &\cup \ \{y_j \to r_{i,j,k} \to x_i : j \in [n-1], i \in [j], k \in [c-1]\} \\ &\cup \ \{y_j \leftarrow r_{i,j,k} \leftarrow x_i : i \in [n], j \in [i-1], k \in [c-1]\}. \end{aligned}$$

Lemma 2.18. *For fixed integers $n, a, b, c \geq 1$ we enumerate the topological orderings of $G_K = G_K(n, a, b, c)$ and $G_S = G_S(n, a, b, c)$. We have*

$$\text{top}(G_K) = \frac{(c-1)!^n}{(nc-1)!} \cdot \text{top}(G_S).$$

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_{2n-1})$ with $\alpha_i = c - 1$ for i odd and $\alpha_i = 1$ for i even. Note that the induced subgraph H of G_K on the vertices $x_i, y_i, w_{i,j}, r_{i,j,k}$ is isomorphic to the graph $G_A(\alpha)$ (c.f. Definition 2.13). Replacing this copy of G_A within G_K by the graph $G_B(\alpha)$ (keeping the old vertices x_i and $v_{i,j}$) yields the graph G_S (c.f. Definition 2.5). Note that the vertices $v_{i,j}$ are not connected to the subgraph we replaced and hence we can consider orderings of the two parts separately.

Now we compare the number of topological orderings of G_K and G_S. Fix the labels of all vertices x_i to be $g(x_i)$ and the labels of all vertices $v_{i,j}$ to be $g(v_{i,j})$. Now define

$p_{2k-1} := g(x_k) - |\{v_{i,j} : i < j \in [n], g(v_{i,j}) < \ell_i\}|$. By Corollary 2.15 with parameters α and $p_1, p_3, \ldots, p_{2n-1}$ we have

$$\begin{aligned}&|\{f \in \mathrm{TOP}(G_K) : f(x_i) = g(x_i)\ \forall i \in [n],\\ &\qquad\qquad f(v_{i,j}) = g(v_{i,j})\ \forall i < j \in [n]\}|\\ &= \frac{(c-1)!^n}{(nc-1)!} \cdot |\{f \in \mathrm{TOP}(G_S) : f(x_i) = g(x_i)\ \forall i \in [n],\\ &\qquad\qquad f(v_{i,j}) = g(v_{i,j})\ \forall i < j \in [n]\}|.\end{aligned}$$

Since this holds for any fixed labels of x_i and $v_{i,j}$, the result follows. □

Lemma 2.19. *For fixed integers $n, a, b, c \geq 1$ we enumerate the topological orderings of $G_K = G_K(n, a, b, c)$ and $G_S = G_S(n-1, a+c, b+c, c)$. We have*

$$\mathrm{top}(G_K) = \frac{(n(a+b+c(n-1)-1))!}{((n-1)(a+b+cn-1))!} \frac{(a-1)!(b-1)!(c-1)!^{n-1}}{(a+b+(n-1)c-1)!} \cdot \mathrm{top}(G_S).$$

Proof. First construct from G_K the graph G'_K by adding vertices y_0 and y_n to the vertex set and edges

$$\begin{aligned}&\{x_1 \to y_0, y_n \to x_n\} \cup \{p_{i,k} \to y_0 : i \in [n], k \in [a-1]\}\\ &\qquad \cup \{y_n \to q_{i,k} : i \in [n], k \in [b-1]\}\end{aligned}$$

to the edge set. It is easy to see that $\mathrm{top}(G_K) = \mathrm{top}(G'_K)$, because vertices y_0 (resp. y_n) need to be assigned the smallest (resp. largest) label. Now let $\alpha = (\alpha_1, \ldots, \alpha_{2n-1}) = (a-1, 1, c-1, 1, \ldots, 1, c-1, 1, b-1)$, i.e., $\alpha_i = 1$ for i even and $\alpha_i = c-1$ for $3 \leq i \leq 2n-3$ odd. Then note that the induced subgraph H of G'_K on the vertices $x_i, y_i, v_{i,j}, p_{i,k}, q_{i,k}, r_{i,j,k}$ is isomorphic to the graph $G_A(\alpha)$ (c.f. Definition 2.13). Replacing this copy of G_A within G'_K by the graph $G_B(\alpha)$ (keeping the old vertices $y_0, \ldots, y_n$) and removing y_0 and y_n yields the graph G_S (c.f. Definition 2.5) together with $a+b-1$ isolated vertices. Now, there are $\frac{(n(a+b+c(n-1)-1))!}{((n-1)(a+b+cn-1))!}$ ways of choosing labels for these $a+b-1$ vertices. The other factor follows immediately by Corollary 2.15. □

Now Theorem 2.6 for $c \geq 1$ follows by chaining these lemmas together and using induction on n with base case $\mathrm{top}(G_S(0, a, b, c)) = 1$:

$$\begin{aligned}&\mathrm{top}(G_S(n, a, b, c))\\ &= \frac{(n(a+b+c(n-1)-1))!}{((n-1)(a+b+cn-1))!} \frac{(nc-1)!(a-1)!(b-1)!}{(c-1)!(a+b+(n-1)c-1)!}\\ &\qquad \times \mathrm{top}(G_S(n-1, a+c, b+c, c)).\end{aligned}$$

For the special case $c = 0$ we do not use Lemmas 2.18 and 2.19. Instead note that in a topological ordering of $G_S(1, a, b, 0)$ the label of x_1 is fixed to be a and we have $\mathrm{top}(G_S(1, a, b, 0)) = (a-1)!(b-1)!$. Let H be the graph composed of n disjoint copies of $G_S(1, a, b, 0)$. The number of topological orderings of H equals

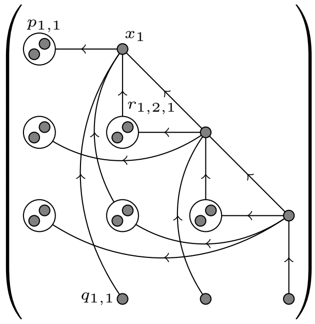
$$\binom{n(a+b-1)}{a+b-1, \ldots, a+b-1} ((a-1)!(b-1)!)^n.$$

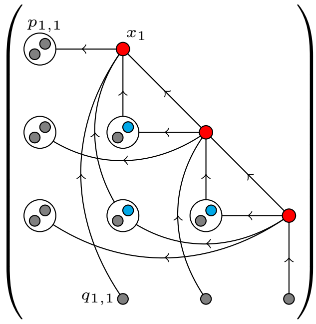
The graph H is isomorphic to $G_S(n,a,b,0)$ without the edges $x_n \to x_{n-1} \to \cdots \to x_2 \to x_1$. Exactly $\frac{1}{n!}$ of the topological orderings of H correspond to topological orderings of $G_S(n,a,b,0)$ as the vertices $x_1,\ldots,x_n$ need to have increasing labels. Therefore the result follows:

$$\begin{aligned}
&\operatorname{top}(G_S(n,a,b,0)) \\
&= \frac{1}{n!}\binom{n(a+b-1)}{a+b-1,\ldots,a+b-1}((a-1)!(b-1)!)^n \\
&= \frac{(n(a+b-1))!}{n!}\prod_{j=0}^{n-1}\frac{(a-1)!(b-1)!}{(a+b-1)!}.
\end{aligned}$$

We illustrate an induction step using an example:

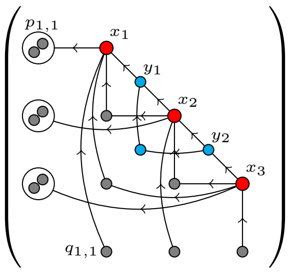
Example 2.20. *Let* $n=3, a=3, b=2, c=1$. *In this example we want to evaluate* $\operatorname{top}(G_S(3,3,2,1))$. *Vertices are marked cyan if they are part of the subgraph replaced in Lemma 2.18 or Lemma 2.19. We start by summing over all possible labels of the red vertices on the diagonal as follows:*

top 

$= \sum$ *labels of red vertices* top 

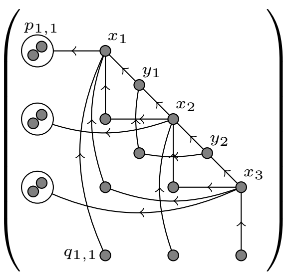
.

Having fixed the labels of the red vertices, we can apply Corollary 2.15 with $\alpha = (0,1,0,1,0)$ *to each summand.*

$= \sum$ *labels of red vertices* $2 \cdot$ top 

.

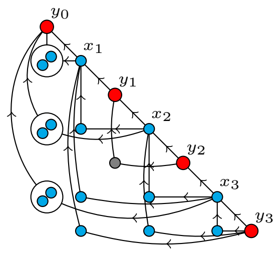
Undoing the first step then yields:

$= 2 \cdot$ top 

.

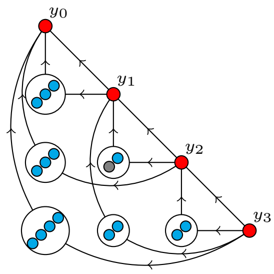
We introduce two new vertices which have to be labelled with the minimum and maximum label. Again we sum over all possible labels of the red (now different) vertices along the diagonal:

$$= 2 \cdot \sum_{\substack{\textit{labels of red}\\ \textit{vertices}}} \operatorname{top} \left(\quad \right).$$



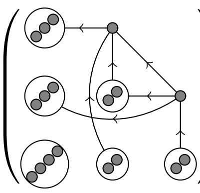
Applying Corollary 2.15 with $\alpha = (2,1,0,1,0,1,1)$ *yields:*

$$= \frac{1}{180} \cdot \sum_{\substack{\textit{labels of red}\\ \textit{vertices}}} \operatorname{top} \left(\quad \right).$$



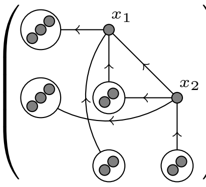
Next we remove again the new vertices:

$$= \frac{1}{180} \cdot \operatorname{top} \left(\quad \right)$$



There are $18 \cdot 17 \cdot 16 \cdot 15$ *possibilities for the four independent vertices, and hence we get:*

$$= 408 \cdot \operatorname{top} \left(\quad \right) = 408 \cdot \operatorname{top}(G_S(2,4,3,1))$$



2.3 Bijective proof of Theorem 2.8

Before we start with the bijective proof of Theorem 2.8, we want to highlight the following result: Gessel found a bijective proof of the Vandermonde determinant [21], by comparing terms in the expansion. For example

$$(x_2 - x_1)(x_3 - x_1)(x_3 - x_2) = \det \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix},$$

where the terms $x_1x_2x_3$ and $-x_1x_2x_3$ on the left hand side cancel. We will reformulate this using signed sets and sijections and then generalise it to the generalised Vandermonde determinant above.

2.3.1 Binomial coefficients.

Definition 2.21. Let $x_1, x_2, \ldots, x_n$ be formal variables and k be any integer. Define the following weighted signed set

$$B((x_1, x_2, \ldots, x_n), k) := \{(a_1, \ldots, a_n) \colon a_i \geq 0, \sum a_i = k\},$$

where an element $a = (a_1, \ldots, a_n)$ is given weight $w(a) = \prod x_i^{a_i}$. For $k < 0$ the set is empty.

Example 2.22. *The weighted signed set $B((x_1, x_1, x_2), 2)$ has $\binom{n+k-1}{k} = \binom{4}{2} = 6$ elements, for example by the stars and bars method, and their weights are: $x_1^2, x_1^2, x_1^2, x_1x_2, x_1x_2, x_2^2$.*

Lemma 2.23. *Fix any integers $1 \leq q \leq j$. We have a weight-preserving bijection*

$$[q] \times B((\underbrace{x, \ldots, x}_{q+1 \text{ times}})), j - q) \leftrightarrow [j] \times B((\underbrace{x, \ldots, x}_{q \text{ times}}), j - q).$$

Proof. All elements have weight x^{j-q}, so we can ignore the weights for our bijection. Given an element $(\ell, (a_1, \ldots, a_{q+1}))$ with $\ell \in [q]$ and $\sum_{i \in [q+1]} a_i = j - q$, we create a q-tuple by merging a_ℓ and $a_{\ell+1}$ as follows: $(a_1, \ldots, a_{\ell-1}, a_\ell + a_{\ell+1}, a_{\ell+2}, \ldots, a_{q+1})$. Of course the total sum is unchanged and still equals $j - q$. Additionally we record $\ell + \sum_{i \leq \ell} a_i \in [j]$, as to be able to undo the merging step. This completes our bijection f:

$$f((\ell, (a_1, \ldots, a_{q+1}))) = (\ell + \sum_{i \leq \ell} a_i, (a_1, \ldots, a_{\ell-1}, a_\ell + a_{\ell+1}, a_{\ell+2}, \ldots, a_{q+1})).$$

The inverse f^{-1} can be described as follows: Given an element $(m, (b_1, \ldots, b_q))$ with $m \in [j]$ and $\sum_{i \in [q]} b_i = j - q$, find the minimal ℓ for which $b_1 + \ldots + b_\ell + \ell \geq m$. Then let $x = b_1 + \ldots + b_\ell + \ell - m$ and

$$f^{-1}((m, (b_1, \ldots, b_q))) = (\ell, (b_1, \ldots, b_{\ell-1}, b_\ell - x, x, b_{\ell+1}, \ldots, b_q)).$$ □

Example 2.24. *For $j = 4, q = 2$, Lemma 2.23 gives the following bijection:*

$$\begin{array}{lll}
(1,(2,0,0)) \leftrightarrow (3,(2,0)), & (2,(2,0,0)) \leftrightarrow (4,(2,0)), & (1,(1,1,0)) \leftrightarrow (2,(2,0)) \\
(2,(1,1,0)) \leftrightarrow (4,(1,1)), & (1,(1,0,1)) \leftrightarrow (2,(1,1)), & (2,(1,0,1)) \leftrightarrow (3,(1,1)) \\
(1,(0,2,0)) \leftrightarrow (1,(2,0)), & (2,(0,2,0)) \leftrightarrow (4,(0,2)), & (1,(0,1,1)) \leftrightarrow (1,(1,1)) \\
(2,(0,1,1)) \leftrightarrow (3,(0,2)), & (1,(0,0,2)) \leftrightarrow (1,(0,2)), & (2,(0,0,2)) \leftrightarrow (2,(0,2))
\end{array}$$

We need one more lemma. By setting $k = 2, m = 0$, one can see that it is a generalisation of Example 1.6.

Lemma 2.25. *Fix integers $k \geq 1, m \geq 0$ and formal variables $x_i, x_j, y_1, \ldots, y_m$. Then we have a weighted sijection*

$$\begin{aligned}
&B((x_j, y_1, \ldots, y_m), k) - B((x_i, y_1, \ldots, y_m), k) \\
&\leftrightarrow \{x_j, -x_i\} \times B((x_i, x_j, y_1, \ldots, y_m), k - 1).
\end{aligned}$$

Proof. Let $A := B((x_i, x_j, y_1, \ldots, y_m), k)$. We clearly have bijections

$$\{a \in A : a_1 = 0\} + \{a \in A : a_1 \geq 1\} \leftrightarrow A \leftrightarrow \{a \in A : a_2 = 0\} + \{a \in A : a_2 \geq 1\}.$$

Rearranging gives the sijection

$$\{a \in A : a_1 = 0\} - \{a \in A : a_2 = 0\} \leftrightarrow \{a \in A : a_2 \geq 1\} - \{a \in A : a_1 \geq 1\}.$$

We also have a bijection $\{a \in A : a_1 = 0\} \leftrightarrow B((x_j, y_1, \ldots, y_m), k)$ by simply discarding the first entry in the tuple, which is always 0. Similarly we have $\{a \in A : a_2 = 0\} \leftrightarrow B((x_i, y_1, \ldots, y_m), k)$.

On the other hand we have a bijection

$$\{a \in A : a_2 \geq 1\} \leftrightarrow \{x_j\} \times B((x_i, x_j, y_1, \ldots, y_m), k - 1)$$

by decreasing a_2 by 1, but instead multiply by a singleton set with an element with weight x_j to keep the bijection weight-preserving. Similarly we have a bijection

$$\{a \in A : a_1 \geq 1\} \leftrightarrow \{x_i\} \times B((x_i, x_j, y_1, \ldots, y_m), k - 1).$$

This completes the proof, as $\{x_j\} - \{x_i\} = \{x_j, -x_i\}$. □

2.3.2 Topological orderings.

We start this subsection by showing a sijection between unweighted signed sets of topological orderings.

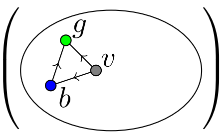
Lemma 2.26. *Let G be any directed graph with three vertices b, g, v with $v \to g$, $v \to b$ and $b \to g$. Let G_1 be a copy of G without the edge $b \to g$ and G_2 a copy of G without $v \to b$ and $b \to g$ but instead $g \to b$.*

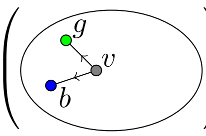
We have a sijection

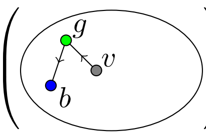
$$\mathrm{TOP}(G) \leftrightarrow \mathrm{TOP}(G_1) - \mathrm{TOP}(G_2).$$

Proof. Any topological order f of G_1 either has $f(b) > f(g)$ or $f(b) < f(g)$. In the first case, f is also a topological order of G, while in the second case it is instead a topological order of G_2. On the other hand, any topological order of G or G_2 is also a topological order of G_1. □

Comparing the sizes of the signed sets on both sides yields the equation $\mathrm{top}(G) = \mathrm{top}(G_1) - \mathrm{top}(G_2)$. Or graphically:

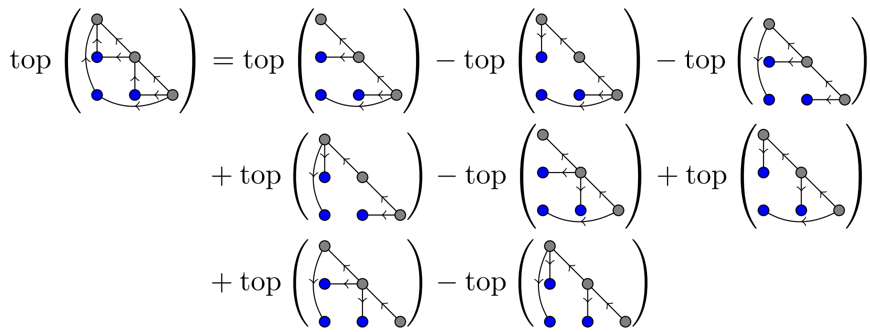
top (


) = top (


) − top (


). (2.1)

Example 2.27. *We now apply this expansion on all three vertices below the diagonal of $G_X(1,1,1)$ as follows:*



Note that the sign equals -1 to the power of the number of vertical arrows.

2.3.3 Weighted sijections

Definition 2.28. Fix integers $n \leq N \in \mathbb{N}$ and let $X = (x_1, \ldots, x_n)$ be a tuple of formal symbols. Let S be a weighted signed set, in which all elements have weights of the form $\pm \prod_{i \in [n]} x_i^{c_i}$ with $c_i \geq 0\ \forall i$ and $\sum c_i = N - n$ constant, and let $P = (p_1, \ldots, p_n)$ with all $p_i \in [N]$ distinct.

We define an unweighted signed set:

$$\begin{aligned}\phi(S, X, P) := \{(s, L) \colon s \in S, w_S(s) = \pm \prod_{i \in [n]} x_i^{c_i}, L = (L_1, \ldots, L_n),\\ L_i = (\ell_{i,1}, \ldots, \ell_{i,c_i}), \text{all } \ell_{i,j} \text{ and } p_i \text{ distinct}\\ \text{integers in } [N], \ell_{i,j} < p_i\}\end{aligned}$$

with weight $w((s, L)) = \operatorname{sign}(w_S(s))$.

Remark 2.29. Note that every integer in $[N]$ appears either in P or in some L_i. Also note that we allow $c_i = 0$, in which case the corresponding tuple L_i is an empty tuple, which we write as $()$. Furthermore for all signed sets A, B with suitable weights as in Definition 2.28 we have

$$\phi(A, X, P) + \phi(B, X, P) \leftrightarrow \phi(A + B, X, P)$$

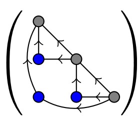
and for an unweighted signed set C there is a bijection

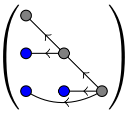
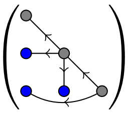
$$C \times \phi(A, X, P) \leftrightarrow \phi(C \times A, X, P).$$

Finally, if we have a weighted sijection ψ between A and B, then there is an unweighted sijection between $\phi(A, X, P)$ and $\phi(B, X, P)$. This holds as weight-preservation corresponds to the lists L_i having the same lengths.

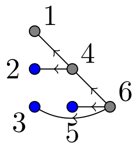
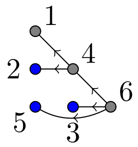
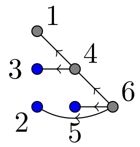
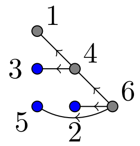
Now, we can formalise Example 2.27 as follows:

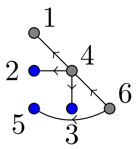
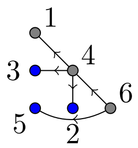
Example 2.30. *We again let $\alpha = (1,1,1)$. And here we set $p_1 = 1, p_2 = 4, p_3 = 6$. Then*

$\{f \in \mathrm{TOP}$  $: f(u_i) = p_i\}$

$\leftrightarrow \{f \in \mathrm{TOP}$ () $: f(u_i) = p_i\} - \{f \in \mathrm{TOP}$ () $: f(u_i) = p_i\},$

where we have omitted the sets corresponding to vertical arrows down from u_1, because these sets are empty. We continue above calculation as follows:

$= \Bigg\{$  ,  ,  ,  $\Bigg\}$

$- \Bigg\{$  ,  $\Bigg\}$

$$
\begin{aligned}
&\leftrightarrow \big\{(x_2x_3^2, ((),(2),(3,5))), (x_2x_3^2, ((),(2),(5,3))),\\
&\qquad (x_2x_3^2, ((),(3),(2,5))), (x_2x_3^2, ((),(3),(5,2)))\big\}\\
&\quad - \big\{(x_2^2x_3, ((),(2,3),(5))), (x_2^2x_3, ((),(3,2),(5)))\big\}\\
&= \phi(\{x_2x_3^2\}, (x_1,x_2,x_3), (1,4,6)) - \phi(\{x_2^2x_3\}, (x_1,x_2,x_3), (1,4,6))\\
&= \phi(\{x_2x_3^2\}, (x_1,x_2,x_3), (1,4,6)) - \phi(\{x_2^2x_3\}, (x_1,x_2,x_3), (1,4,6))\\
&\quad - \phi(\{x_1x_2x_3\}, (x_1,x_2,x_3), (1,4,6)) + \phi(\{x_1x_2^2\}, (x_1,x_2,x_3), (1,4,6))\\
&\quad - \phi(\{x_1x_3^2\}, (x_1,x_2,x_3), (1,4,6)) + \phi(\{x_1x_2x_3\}, (x_1,x_2,x_3), (1,4,6))\\
&\quad + \phi(\{x_1^2x_3\}, (x_1,x_2,x_3), (1,4,6)) - \phi(\{x_1^2x_2\}, (x_1,x_2,x_3), (1,4,6))\\
&= \phi(\{x_2,-x_1\} \times \{x_3,-x_1\} \times \{x_3,-x_2\}, (x_1,x_2,x_3), (1,4,6)),
\end{aligned}
$$

where in the second last line we have reintroduced the empty sets corresponding to vertical arrows down from u_1.

More generally we have:

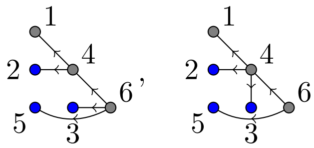
Lemma 2.31. *Fix $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \geq 1$ and let $X = (x_1, \dots, x_n)$ a tuple of formal variables. Then for any tuple $P = (p_1, \dots, p_n)$ with $p_1 < \dots < p_n$ and $p_i \in [n + \sum_{1 \leq i < j \leq n} \alpha_i\alpha_j]$ we have*

$$\{f \in \mathrm{TOP}(G_X(\alpha)) \colon f(u_i) = p_i\} \leftrightarrow \phi(\prod_{1 \leq i < j \leq n} \{x_j, -x_i\}^{\alpha_i\alpha_j}, (x_1, \dots, x_n), P).$$

Proof. We apply Lemma 2.26 on all vertices $w_{i,j,k}$, as in Example 2.30. For each vertex $w_{i,j,k}$ we can choose between a horizontal arrow from u_j or a vertical arrow from u_i. The sign equals -1 to the power of times we chose the vertical arrow.

Now, for each term in this expansion, we construct the tuples L_p as follows: We iterate over all vertices $w_{i,j,k}$ sorted by their indices in lexicographical order. If $u_p \to w_{i,j,k}$, we add $f(w_{i,j,k})$ to the tuple L_p. This way the lists are guaranteed to have the correct length, and we can undo the process by iterating over all vertices $w_{i,j,k}$ in the same order. □

Remark 2.32. In the proof of Lemma 2.31 we compose multiple sijections from Lemma 2.26. We use one such sijection for each $w_{i,j,k}$. Note that if we have $f(w_{i,j,k}) < f(u_i)$ and also $f(w_{i,j,k}) < f(u_j)$ in above proof, then the sijection constructed in Lemma 2.26 maps the topological ordering of the directed graph with edge $u_i \to w_{i,j,k}$ to the ordering of the graph with edge $u_j \to w_{i,j,k}$ with opposite sign. In Example 2.30 one such pair is:



corresponding to the elements

$$(x_2x_3^2, ((), (2), (5,3))) \in \phi(\{x_2x_3^2\}, (x_1,x_2,x_3), (1,4,6)), \text{ and}$$
$$(x_2^2x_3, ((), (2,3), (5))) \in \phi(\{x_2^2x_3\}, (x_1,x_2,x_3), (1,4,6)).$$

Lemma 2.33. *Fix $1 \le j \ne k < n$ and $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n$ and $X = (x_1, \dots, x_n)$.*

Let S be a signed set for which all weights are of the form $\pm \prod_{i\in[n]} x_i^{c_i}$ with $c_j =: \ell$. We have a sijection

$$[\ell+1] \times \sum_{p_j<p_k} \phi(S, X, (p_1, p_2, \dots, p_n))$$
$$\leftrightarrow \phi(S', (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), (p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n)),$$

where S' is a copy of S, for which x_j^ℓ is replaced by $x_k^{\ell+1}$ in all weights.

Proof. Consider an element $(x, (s, L)) \in [\ell+1] \times \phi(S, X, (p_1, p_2, \dots, p_n))$ for any p_j. Note that the tuple L_j contains exactly ℓ elements. Insert p_j into this tuple at position x and append the resulting tuple to L_k. Since p_j is the larger than all elements in L_j, this process is reversible. □

Lemma 2.34. *Fix $p_1 < \dots < p_{j-1} < p_{j+1} < \dots < p_n$. We have a sijection*

$$\sum_{p_{j-1}<p_j<p_{j+1}} \phi(S, X, (p_1, p_2, \dots, p_n))$$
$$\leftrightarrow \sum_{p_j<p_{j+1}} \phi(S, X, (p_1, p_2, \dots, p_n)) - \sum_{p_j<p_{j-1}} \phi(S, X, (p_1, p_2, \dots, p_n)).$$

Proof. Every choice of $p_j < p_{j+1}$ either satisfies $p_{j-1} < p_j < p_{j+1}$ or $p_j < p_{j-1}$. The result follows. □

2.3.4 Matrices.

We start this subsection with a definition of a signed set D:

Definition 2.35. Let M be an $n \times n$ matrix, where each $M_{i,j}$ is any signed set with weight function $w_{M_{i,j}}$. Define a signed set

$$D(M) := \{(\sigma, (m_1, \dots, m_n)) : \sigma \in S_n, m_i \in M_{i,\sigma(i)}\}$$

together with the weight function

$$w_{D(M)}(\sigma, (m_1, \dots, m_n)) := \operatorname{sgn}(\sigma) \cdot \prod_i w_{M_{i,\sigma(i)}}(m_i).$$

Here S_n is the set of all permutations of $[n]$ and $\operatorname{sgn}(\sigma) \in \{-1, 1\}$ denotes the parity of σ.

Note that $w(D(M)) = \det(M')$, where M' is the matrix with entries $M'_{i,j} = w(M_{i,j})$. Using Definition 2.35, one could rewrite Gessel's proof of the Vandermonde Determinant as a weighted sijection between the two signed sets

$$\prod_{1\le i<j\le n} \{x_j, -x_i\} \text{ and } D\left(\begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ \vdots & & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix}\right).$$

For the rest of this subsection we extend this to a weighted sijection for the generalised Vandermonde determinant. For usual matrices, we know that we can add or subtract a row from another without changing the determinant. We can express this fact for $D(M)$ using a sijection as follows:

Lemma 2.36. *Fix any $i \neq j \in [n]$. Let A be an $n \times n$ matrix and A' be the result after adding (resp. subtracting) the i^{th} row of A to (resp. from) the j^{th} row. In other words we have*

$$(A')_{s,t} = \begin{cases} A_{s,t}, & \text{if } s \neq j \\ A_{j,t} \pm A_{i,t}, & \text{if } s = j \end{cases}$$

where addition and subtraction of signed sets is as defined above. We have a sijection

$$D(A) \leftrightarrow D(A').$$

Proof. We have $D(A) \subset D(A')$, as $A_{j,t} \subset A_{j,t} \pm A_{i,t}$, so we only need to find a sign-reversing involution ϕ on $D(A') \setminus D(A)$. Take any $x = (\sigma, (m_1, \dots, m_n)) \in D(A') \setminus D(A)$, so we have $m_i \in A_{i,\sigma(i)}$ and $m_j \in A_{i,\sigma(j)}$. Now define $\phi(x) := (\sigma', (m'_1, \dots, m'_n))$ with $\sigma' := \sigma(i\ j)$ and

$$m'_k := \begin{cases} m_j, & \text{if } k = i \\ m_i, & \text{if } k = j \\ m_k, & \text{otherwise.} \end{cases}$$

We have $m'_i = m_j \in A_{i,\sigma(j)} = A_{i,\sigma'(i)}$ and $m'_j = m_i \in A_{i,\sigma(i)} = A_{i,\sigma'(j)}$, so $\phi(x) \in D(A') \setminus D(A)$. Additionally we have $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$, i.e. ϕ is sign-reversing. Also, ϕ is clearly an involution. □

Definition 2.37. Let $a, m \geq 0$ and $y_1, \dots, y_k, x$ be formal variables. Define the matrix $M((y_1, \dots, y_k), x, a, m)$ as the $a \times m$ matrix with entries:

$$M((y_1, \dots, y_k), x, a, m)_{i,j} = B((y_1, \dots, y_k, \underbrace{x, \dots, x}_{i \text{ times}}), j - i).$$

Definition 2.38. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \geq 0$, $m = \sum \alpha_i$ and $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_k)$ tuples of formal variables.

Define the following $m \times m$ block matrix:

$$H(Y, X, \alpha) = \begin{pmatrix} M(Y, x_1, \alpha_1, m) \\ \vdots \\ M(Y, x_n, \alpha_n, m) \end{pmatrix}.$$

Before stating the next theorem in its general form, we would like to show how to use Lemmas 2.36 and 2.25 with an example.

Example 2.39. *Let $n = 2, \alpha_1 = 2, \alpha_2 = 2$. Consider the corresponding signed set $A = D(H((), (x_1, x_2), (2, 2)))$. Then*

$$A = D\left(\begin{pmatrix} B((x_1),0) & B((x_1),1) & B((x_1),2) & B((x_1),3) \\ \varnothing & B((x_1,x_1),0) & B((x_1,x_1),1) & B((x_1,x_1),2) \\ B((x_2),0) & B((x_2),1) & B((x_2),2) & B((x_2),3) \\ \varnothing & B((x_2,x_2),0) & B((x_2,x_2),1) & B((x_2,x_2),2) \end{pmatrix}\right).$$

Now we apply Lemma 2.36 by subtracting row 1 from row 3. Using Lemma 2.25 the entry at position $(3, j)$ then becomes

$$B((x_2), j-1) - B((x_1), j-1) \leftrightarrow \{x_2, -x_1\} \times B((x_1, x_2), j-2).$$

We can pull out the common factor $\{x_2, -x_1\}$ out of row 3 and we get:

$$A \leftrightarrow \{x_2, -x_1\} \times D\left(\begin{pmatrix} B((x_1),0) & B((x_1),1) & B((x_1),2) & B((x_1),3) \\ \varnothing & B((x_1,x_1),0) & B((x_1,x_1),1) & B((x_1,x_1),2) \\ \varnothing & B((x_1,x_2),0) & B((x_1,x_2),1) & B((x_1,x_2),2) \\ \varnothing & B((x_2,x_2),0) & B((x_2,x_2),1) & B((x_2,x_2),2) \end{pmatrix}\right).$$

Now we apply Lemma 2.36 by subtracting row 3 from row 4. The entry at position $(4, j)$ then becomes

$$B((x_2, x_2), j-2) - B((x_1, x_2), j-2) \leftrightarrow \{x_2, -x_1\} \times B((x_1, x_2, x_2), j-3).$$

We can pull out the common factor $\{x_2, -x_1\}$ out of row 4 and we get:

$$A \leftrightarrow \{x_2, -x_1\}^2 \times D\left(\begin{pmatrix} B((x_1),0) & B((x_1),1) & B((x_1),2) & B((x_1),3) \\ \varnothing & B((x_1,x_1),0) & B((x_1,x_1),1) & B((x_1,x_1),2) \\ \varnothing & B((x_1,x_2),0) & B((x_1,x_2),1) & B((x_1,x_2),2) \\ \varnothing & \varnothing & B((x_1,x_2,x_2),0) & B((x_1,x_2,x_2),1) \end{pmatrix}\right).$$

Now all entries in the left-most column are empty sets, except the top-left corner, which contains a single element with weight 1. Hence we have a sijection

$$A \leftrightarrow \{x_2, -x_1\}^2 \times D\left(\begin{pmatrix} B((x_1,x_1),0) & B((x_1,x_1),1) & B((x_1,x_1),2) \\ B((x_1,x_2),0) & B((x_1,x_2),1) & B((x_1,x_2),2) \\ \varnothing & B((x_1,x_2,x_2),0) & B((x_1,x_2,x_2),1) \end{pmatrix}\right).$$

We have shown:

$$D(H((), (x_1, x_2), (2, 2))) \leftrightarrow \{x_2, -x_1\}^2 \times D\left(H((x_1), (x_1, x_2), (1, 2))\right).$$

Lemma 2.40. *Fix $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \geq 0$, $m = \sum \alpha_i$ and $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_k)$ tuples of formal variables.*

a) For $\alpha_1 \geq 1$ there exists a sijection

$$\begin{aligned} &D(H(Y, X, \alpha)) \\ &\leftrightarrow \prod_{2 \leq j \leq n} \{x_j, -x_1\}^{\alpha_j} \times D(H((y_1, \ldots, y_k, x_1), X, (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n))). \end{aligned}$$

b) For $\alpha_1 = 0$ we have

$$D(H(Y, X, \alpha)) = D(H(Y, (x_2, \dots, x_n), (\alpha_2, \dots, \alpha_n))).$$

Proof. For part a) we do the same as in Example 2.39. We perform row operations on the matrix $D(H(Y, X, \alpha))$ with $\alpha_1 \geq 1$. We apply Lemmas 2.36 and 2.25 multiple times as follows:

We subtract row 1 from row $\alpha_1 + \ldots + \alpha_{j-1} + 1$ for all $j \in \{2, \dots, n\}$ with $\alpha_j \geq 1$. Then for all $q \in \{2, \dots, \alpha_j\}$ we subtract row $\alpha_1 + \ldots + \alpha_{j-1} + q - 1$ from row $\alpha_1 + \ldots + \alpha_{j-1} + q$. Each time we pull out the factor $\{x_j, -x_1\}$.

In the resulting matrix the left-most column only has empty sets, except for the top-left corner, which contains a single element with weight 1. We can therefore delete the first row and column and are left with the matrix $D(H((y_1, \dots, y_k, x_1), X, (\alpha_1 - 1, \alpha_2, \dots, \alpha_n)))$.

Part b) follows directly from the definition of H: The top-most block simply has height 0. □

Theorem 2.41. *Fix $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \geq 0$, $m = \sum \alpha_i$ and $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_k)$ tuples of formal variables.*

Then there exists a sijection

$$D(H(Y, X, \alpha)) \leftrightarrow \prod_{1 \leq i < j \leq n} \{x_j, -x_i\}^{\alpha_i \alpha_j}.$$

Proof. Proof by induction on n. For $n = 0$, $D(H(Y, (), ())) = \{1\}$, the statement is true. Now suppose $n \geq 1$. We apply Lemma 2.40 part a) until α_1 is decreased down to 0. In total we have pulled out a factor $\prod_{2 \leq j \leq n} \{x_j, -x_1\}^{\alpha_1 \alpha_j}$. Then we remove α_1 and x_1 by applying Lemma 2.40 b) and by the induction hypothesis, the result follows. □

For $Y = ()$ this theorem is a sijective version of Theorem 2.12.

2.3.5 Putting it all together.

We need to define some more matrices

Definition 2.42. Let $a, m \geq 0$ and x and y be formal variables. Define $M_1(x, a, m)$ as the $a \times m$ matrix with entries:

$$M_1(x, a, m)_{i,j} = B((\underbrace{x, \dots, x}_{i+1 \text{ times}}), j - i).$$

Define $M_2(x, m)$ as the $1 \times m$ matrix with entries:

$$M_2(x, m)_{1,j} = [j] \times B((x), j - 1).$$

Define $M_3(x, y, m)$ as the $1 \times m$ matrix with entries:

$$M_3(x, y, m)_{1,j} = B((y), j) - B((x), j).$$

Definition 2.43. Let $\alpha = (\alpha_1, \alpha_3, \ldots, \alpha_{2n-1})$ with $\alpha_i \geq 0$, $m = \sum \alpha_i$. Define the following $m \times m$ block matrices:

$$H_1((x_1, \ldots, x_{2n-1}), \alpha) := \begin{pmatrix} M_1(x_1, \alpha_1, m) \\ M_2(x_2, m) \\ M_1(x_3, \alpha_3, m) \\ \vdots \\ M_1(x_{2n-1}, \alpha_{2n-1}, m) \end{pmatrix} \text{ and}$$

$$H_2((x_1, x_3, \ldots, x_{2n-1}), \alpha) := \begin{pmatrix} M_1(x_1, \alpha_1, m) \\ M_3(x_1, x_3, m) \\ M_1(x_3, \alpha_3, m) \\ \vdots \\ M_1(x_{2n-1}, \alpha_{2n-1}, m) \end{pmatrix}.$$

Lemma 2.44. *Fix $X = (x_1, \ldots, x_n)$, $P = (p_1, \ldots, p_n)$ with $p_1 < \ldots < p_n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. We have a sijection*

$$\{f \in \mathrm{TOP}(G_X(\alpha)) : f(u_i) = p_i \ \forall i\} \leftrightarrow \phi(D(H((), X, \alpha)), X, P).$$

Proof. Follows directly from Lemmas 2.31 and 2.41. □

Lemma 2.45. *Fix $\alpha = (\alpha_1, \ldots, \alpha_{2n-1})$ with $\alpha_i \geq 0$ for i odd and $\alpha_i = 1$ for i even. Also fix $X = (x_1, \ldots, x_{2n-1})$ a tuple of formal variables. Then we have a sijection*

$$[(\sum \alpha_i)!] \times D(H((), X, \alpha)) \leftrightarrow \prod_{i \text{ odd}} [\alpha_i!] \times D(H_1(X, (\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}))).$$

Proof. By Lemma 2.23 we have for any $a \geq 0$:

$$\begin{aligned} [j] \times M((), x, a, m)_{i,j} &= [j] \times B((\underbrace{x, \ldots, x}_{i \text{ times}}), j - i) \\ &\leftrightarrow [i] \times B((\underbrace{x, \ldots, x}_{i+1 \text{ times}}), j - i) = [i] \times M_1(x, a, m)_{i,j}. \end{aligned}$$

We also have $[j] \times M((), x, 1, m)_{1,j} = [j] \times B((x), j - 1) = M_2(x, m)_{1,j}$. From these two facts it follows that multiplying the j^{th} column of $D(H((), \alpha, X))$ through by $[j]$ for all j yields the same signed set as if one multiplies the j^{th} row of the block $M_1(x_{2i-1}, \alpha_{2i-1}, m)$ through by $[j]$, for all j and i. □

Example 2.46. *Let $\alpha = (1, 1, 2), X = (x_1, x_2, x_3)$. Then*

$$\begin{aligned} &[4!] \times D(H((), X, \alpha)) \\ &\leftrightarrow [4!] \times D\left(\begin{pmatrix} B((x_1),0) & B((x_1),1) & B((x_1),2) & B((x_1),3) \\ B((x_2),0) & B((x_2),1) & B((x_2),2) & B((x_2),3) \\ B((x_3),0) & B((x_3),1) & B((x_3),2) & B((x_3),3) \\ \varnothing & B((x_3,x_3),0) & B((x_3,x_3),1) & B((x_3,x_3),2) \end{pmatrix}\right) \\ &\leftrightarrow D\left(\begin{pmatrix} [1]\times B((x_1),0) & [2]\times B((x_1),1) & [3]\times B((x_1),2) & [4]\times B((x_1),3) \\ [1]\times B((x_2),0) & [2]\times B((x_2),1) & [3]\times B((x_2),2) & [4]\times B((x_2),3) \\ [1]\times B((x_3),0) & [2]\times B((x_3),1) & [3]\times B((x_3),2) & [4]\times B((x_3),3) \\ \varnothing & [2]\times B((x_3,x_3),0) & [3]\times B((x_3,x_3),1) & [4]\times B((x_3,x_3),2) \end{pmatrix}\right) \end{aligned}$$

$$\leftrightarrow D\left(\begin{pmatrix} [1]\times B((x_1,x_1),0) & [1]\times B((x_1,x_1),1) & [1]\times B((x_1,x_1),2) & [1]\times B((x_1,x_1),3) \\ [1]\times B((x_2),0) & [2]\times B((x_2),1) & [3]\times B((x_2),2) & [4]\times B((x_2),3) \\ [1]\times B((x_3,x_3),0) & [1]\times B((x_3,x_3),1) & [1]\times B((x_3,x_3),2) & [1]\times B((x_3,x_3),3) \\ \varnothing & [2]\times B((x_3,x_3,x_3),0) & [2]\times B((x_3,x_3,x_3),1) & [2]\times B((x_3,x_3,x_3),2) \end{pmatrix}\right)$$
$$\leftrightarrow [1!2!]\times D\left(\begin{pmatrix} B((x_1,x_1),0) & B((x_1,x_1),1) & B((x_1,x_1),2) & B((x_1,x_1),3) \\ [1]\times B((x_2),0) & [2]\times B((x_2),1) & [3]\times B((x_2),2) & [4]\times B((x_2),3) \\ B((x_3,x_3),0) & B((x_3,x_3),1) & B((x_3,x_3),2) & B((x_3,x_3),3) \\ \varnothing & B((x_3,x_3,x_3),0) & B((x_3,x_3,x_3),1) & B((x_3,x_3,x_3),2) \end{pmatrix}\right)$$
$$\leftrightarrow [1!2!]\times D(H_1(X,(1,2))).$$

Lemma 2.47. *Fix* $\alpha = (\alpha_1, \alpha_3, \dots, \alpha_{2n-1})$ *with* $\alpha_i \geq 0$ *and a tuple of formal variables* $X = (x_1, \dots, x_{2n-1})$*. Also fix* $P = (p_1, p_3, \dots, p_{2n-1})$ *with* $p_1 < p_3 < \dots < p_{2n-1}$*. Then we have a sijection*

$$\sum_{p_1<p_2<\dots<p_{2n-1}} \phi(D(H_1(X,\alpha)), X, (p_1,p_2,\dots,p_{2n-1}))$$
$$\leftrightarrow \phi(D(H_2((x_1,x_3,\dots,x_{2n-1}),\alpha)),(x_1,x_3,\dots,x_{2n-1}),P).$$

Proof. By Lemma 2.33 and Lemma 2.34 we have for fixed $j \in [m]$:

$$\sum_{p_1<p_2<p_3} \phi(M_2(x_2,m)_{1,j},(x_1,x_2,x_3),(p_1,p_2,p_3))$$
$$\leftrightarrow \phi(M_3(x_1,x_3,m)_{1,j},(x_1,x_3),(p_1,p_3)).$$

For $i = \alpha_1 + 1$ and $\sigma \in S_m$ we have $H_1(X,\alpha)_{i,\sigma(i)} = M_2(x_2,m)_{1,\sigma(i)}$ and thus we can use the sijection above for each entry in the $(\alpha_1+1)^{\text{st}}$ row of H_1 as follows:

$$\sum_{p_1<p_2<\dots<p_{2n-1}} \phi(D(H_1(X,\alpha)),X,(p_1,p_2,\dots,p_{2n-1}))$$
$$\leftrightarrow \sum_{p_1<p_2<\dots<p_{2n-1}} \phi(D(\begin{pmatrix} M_1(x_1,\alpha_1,m) \\ M_2(x_2,m) \\ M_1(x_3,\alpha_3,m) \\ M_2(x_4,m) \\ \vdots \\ M_1(x_{2n-1},\alpha_{2n-1},m) \end{pmatrix}),X,(p_1,p_2,\dots,p_{2n-1}))$$
$$\leftrightarrow \sum_{p_3<p_4<\dots<p_{2n-1}} \phi(D(\begin{pmatrix} M_1(x_1,\alpha_1,m) \\ M_3(x_1,x_3,m) \\ M_1(x_3,\alpha_3,m) \\ M_2(x_4,m) \\ \vdots \\ M_1(x_{2n-1},\alpha_{2n-1},m) \end{pmatrix}),(x_1,x_3,x_4,\dots),(p_1,p_3,p_4,\dots)).$$

In the calculation above we sum over all values of $p_2, p_4, ..., p_{2n-2}$ and the values for $p_1, p_3, \dots, p_{2n-1}$ are fixed. We repeat this argument for every p_i for i even and the result follows:

$$\sum_{p_1<p_2<\dots<p_{2n-1}} \phi(D(H_1(X,\alpha)),X,(p_1,p_2,\dots,p_{2n-1}))$$
$$\leftrightarrow \phi(D(H_2((x_1,x_3,\dots,x_{2n-1}),\alpha)),(x_1,x_3,\dots,x_{2n-1}),P). \qquad \square$$

Lemma 2.48. *Fix* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ *with* $\alpha_i \geq 0$*. Also fix* $X = (x_1, \dots, x_n)$*,* $P = (p_1, \dots, p_n)$ *with* $p_1 < \dots < p_n$*. Then we have a sijection*

$$\phi(D(H((),X,(\alpha_1+1,\dots,\alpha_n+1))),X,P) \leftrightarrow \phi(D(H_2(X,\alpha),X,P).$$

Proof. We start with the matrix $H((), X, (\alpha_1 + 1, \ldots, \alpha_n + 1))$ and do the following row operations: We subtract row $\alpha_1 + \ldots + \alpha_{i-1} + 1$ from row $\alpha_1 + \ldots + \alpha_i + 1$ for every $i = n-1, n-2, \ldots, 2, 1$ in this order.

In the resulting matrix the entry in row $\alpha_1 + \ldots + \alpha_i + 1$ and column j is $B((x_i), j-1) - B((x_{i-1}), j-1)$. For $j = 1$, we have a sijection from $B((x_i), 0) - B((x_{i-1}), 0) \leftrightarrow \varnothing$. Therefore the first column contains only empty sets, except for the entry in row 1, which contains a single positive element of weight 1. Hence we can delete the first row and first column.

Also note that for $j \geq 2$, we have $B((x_i), j-1) - B((x_{i-1}), j-1) = M_3(x_{i-1}, x_i, m)_{1,j-1}$. □

Proof of Theorem 2.8. In the definition of A, the values of f for all odd i are fixed. Therefore we can sum over all possible values of the even i:

$$[(\sum \alpha_i)!] \times A \leftrightarrow \sum_{p_1 < p_2 < \ldots < p_{2n-1}} [(\sum \alpha_i)!] \times \{f \in \mathrm{TOP}(G_1) : f(u_i) = p_i \ \forall i\}$$

where the sum is over all p_i with even i. Now we apply Lemma 2.44 to move from topological orderings to determinants.

$$\leftrightarrow \sum_{p_1 < p_2 < \ldots < p_{2n-1}} [(\sum \alpha_i)!] \times \phi(D(H((), (\alpha_1, \ldots, \alpha_{2n-1}), X)), X, (p_1, \ldots, p_{2n-1})).$$

Now apply Lemma 2.45, which does some row operations on the matrix.

$$\leftrightarrow \prod_{i \text{ odd}} [\alpha_i!] \times \sum_{p_1 < p_2 < \ldots < p_{2n-1}} \phi(D(H_1(X, (\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}))), X, (p_1, \ldots, p_{2n-1})).$$

Apply Lemma 2.47, which corresponds to the integrals in Theorem 2.9.

$$\leftrightarrow \prod_{i \text{ odd}} [\alpha_i!] \times \phi(D(H_2(X', (\alpha_1, \ldots, \alpha_{2n-1}))), X', (p_1, p_3, \ldots, p_{2n-1})),$$

where $X' := (x_1, x_3, \ldots, x_{2n-1})$. Apply Lemma 2.48, which does some more row operations on the matrix.

$$\leftrightarrow \prod_{i \text{ odd}} [\alpha_i!] \times \phi(D(H((), X', (\alpha_1 + 1, \ldots, \alpha_{2n-1} + 1))), X', (p_1, p_3, \ldots, p_{2n-1})).$$

Finally apply Lemma 2.44 again, which relates the determinant back to topological orderings:

$$\leftrightarrow \prod_{i \text{ odd}} [\alpha_i!] \times B.$$ □

2.4 Comments and further research

We have presented a bijective proof of Theorem 2.8 or equivalently Corollary 2.15. Unfortunately, because of the *division* going to Corollary 2.15, we don't have a bijection from topological orderings of $G_S(n, a, b, c)$, but only a bijection from $G_S(n, a, b, c) \times X$ for some set X. A direct bijective proof for Corollary 2.15 would be very interesting.

Aomoto's integral formula is a slight generalisation of Selberg's integral formula. It seems plausible that the approach presented in this chapter can by generalised to Aomoto's integral.

3 Enumeration of Generalised Domino Towers

3.1 Introduction

In this chapter, we enumerate a generalisation of so-called *domino towers*. Domino towers are two-dimensional structures made out of n dominoes, i.e., rectangular blocks of width 2 and height 1, with the following properties:

1. The dominoes on the bottom level are contiguous, i.e., the row is convex;
2. Every domino above the bottom row is (half) supported on at least one domino in the row below it;
3. No domino lies directly on top of another domino, such as in a brickwork pattern.

See Figure 3.1 for the domino towers with $n \in \{1, 2, 3\}$. The problem of counting domino towers was first mentioned by Viennot [40, Cor. 4]. Surprisingly, the number of domino towers made up of exactly n blocks is simply 3^{n-1}. Zeilberger showcased the result together with a proof using ordinary generating functions and a bijective proof [41]. The problem also appears as an example in the *Handbook of Enumerative Combinatorics* [5, p. 25].

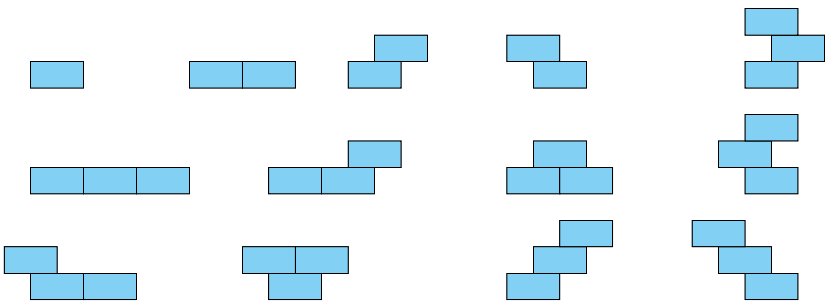
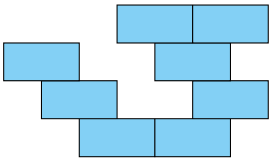
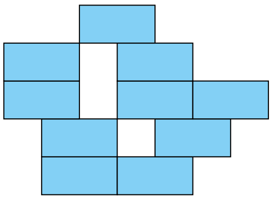


Figure 3.1: Small restricted domino towers for $n \in \{1, 2, 3\}$.

In this chapter, we drop the restriction that blocks cannot be placed directly on top of another and call the structures *unrestricted* towers. Brown proved that there are 4^{n-1} unrestricted towers [8, Cor. 2.3]. We give an alternative proof of this fact in Corollary 3.8 using symbolic methods and a construction called substitution. See Figures 3.2a and 3.2b for examples of restricted and unrestricted domino towers.



(a) A restricted domino tower with 8 blocks.



(b) An unrestricted domino tower with 10 blocks.

Figure 3.2: Examples of domino towers.

In 2016, Brown generalised the problem to unrestricted towers made up of rectangles of width k, which they called k-omino towers [8]. They also introduced a variable $b \geq 1$ for the number of blocks in the bottom row. The number of k-omino towers is $\binom{kn-1}{n-b}$.

Brown also suggests that enumerating towers using rectangles of mixed widths could be interesting for other applications [7, p. 17]. In this chapter, we study this generalisation by allowing rectangles with any width in a fixed finite list $S = (s_1, \ldots, s_m)$ of positive integers. We call this set of towers S-*omino* towers. We additionally fix a list $(n_1, \ldots, n_m)$, where n_i denotes the number of blocks of width s_i, and $b \geq 1$ the number of blocks in the bottom row. Furthermore, let $n := n_1 + \cdots + n_m \geq 1$ be the total number of blocks as before. We now state the first result of this chapter. Note that for $S = (k)$ we, of course, recover the same formula as found by Brown.

Theorem 3.1. *Fix a list of allowed widths $S = (s_1, \ldots, s_m)$. The number of S-omino towers that have exactly n_i blocks of width s_i and b blocks in the bottom row, which has to be convex, equals*

$$\binom{n}{n_1, \ldots, n_m}\binom{-1 + \sum n_i s_i}{n-b}.$$

Summing over all $b \in [n]$ we get

$$\binom{n}{n_1, \ldots, n_m}\binom{-1 + \sum_{i=1}^m s_i n_i}{n-1} \cdot {}_2F_1\left(1, 1-n; 1 + \sum_{i=1}^m (s_i - 1)n_i; -1\right),$$

where ${}_2F_1$ is the Gaussian hypergeometric function.

Note that the heights of the blocks do not change the result, as we will explain in the next section. In particular, setting $S = (1, k)$ for $k \geq 2$ corresponds to stacking k-ominoes horizontally or vertically. In this chapter, we assume that S does not contain duplicate entries for ease of notation. However, the methods would work and yield the same formula. Duplicate entries could be interpreted as having multiple distinguishable versions of dominoes with equal width.

At the end of the chapter we turn our attention to convex k-omino towers, which are defined as follows:

Definition 3.2. A tower is called *column-convex* or *row-convex* if all its columns or respectively rows are convex. Further, a tower is called *convex* if it is both column- and row-convex.

In 2016, Brown calculated the generating function for convex towers and asked whether row-convex towers can be enumerated as well [7, p. 17].

Definition 3.3. Let $g(n)$ be the number of row-convex k-omino towers made up of n k-ominoes. We also define $f_\ell(n)$ to be the number of row-convex k-omino towers made up of n k-ominoes resting on a platform of width ℓk. In other words, the blocks on the bottom row need to rest on this platform, but the platform does not count towards the number of blocks.

By adapting a method that Privman and Švrakić used in 1988 to calculate so-called fully directed compact lattice animals [32], we calculate the ordinary generating functions $G(z) = \sum_{n=0}^{\infty} g(n)z^n$ and $F_\ell(z) = \sum_{n=0}^{\infty} f_\ell(n)z^n$.

Theorem 3.4. *We have*

$$G(z) = \sum_{\ell=1}^{\infty} z^\ell F_\ell(z),$$

where

$$\begin{aligned} F_\ell(z) &= \Big((1+kz)T_{1,\ell} + (kz^2-1)T_{2,\ell} + (k-1)z^3T_{3,\ell}\Big) \Big/ \Big((k-1)^2z^5T_{2,3} \\ &\qquad + (1-(2k-1)(1+z)z + k^2z^3)T_{1,2} + (k-1)((2k-1)z-1)z^3T_{1,3}\Big), \\ T_{s,t}(z) &= A_sB_t - A_tB_s, \\ A_\ell(z) &= \sum_{j=0}^{\infty} \frac{z^{\ell j}h_j}{(z;z)_j^2}, \\ B_\ell(z) &= \sum_{j=0}^{\infty} \frac{z^{\ell j}h_j}{(z;z)_j^2}\left(\ell + \sum_{m=1}^{j}\left(1 + \frac{2}{1-z^m} - \frac{1}{1+(k-1)z^m}\right)\right), \quad \textit{and} \\ h_j(z) &:= z^{j(j+1)}\left((1-k)z;z\right)_j. \end{aligned}$$

The structure of the chapter is as follows: In Section 3.2, we will introduce a few ideas and notation, which we need in later sections. In Section 3.3, we prove Theorem 3.1 using ordinary generating functions and the Lagrange inversion formula. In Section 3.4, we will turn the proof into an explicit bijection and show the connection to generalised Dyck paths [34]. Finally, in Section 3.5, we prove Theorem 3.4.

3.2 Representation of towers as sequences

We can think of domino towers as being built by dropping single dominoes one by one straight down from an infinite height. However, there may be multiple ways of building the same tower. However, there is a unique order as described in Lemma 3.5.

Lemma 3.5. *For any domino tower there is a unique order $b_1, \ldots, b_n$ of its dominoes with the following properties:*

i) The tower can be built by dropping the dominoes straight down from an infinite height in the order $b_1, \dots, b_n$;

ii) For all $i \in [n-1]$, the left border of b_{i+1} is strictly to the left of the right border of b_i.

Proof. Consider the set of blocks B, which have no other blocks above them, i.e., could be the last block according to condition i). Let b' be the left-most block in B and $b^\star \in B$ be the block, which is dropped last. Suppose, for a contradiction, that $b' \neq b^\star$. Then all blocks after b' must be strictly to the left of b'. In particular, $b^\star$ is to the left of b'. This contradicts the definition of b', and therefore b' is dropped last and we are done by induction on n. □

Hence, we can use Lemma 3.5 as a bijection between towers and valid sequences of x-coordinates of the left borders of blocks $b_1, \dots, b_n$. We fix $x_1 = 0$ to keep this sequence unique. For example, $\langle 0, 1, 0, 1, -2, -1, -3, -4 \rangle$ is the sequence for the tower in Figure 3.2a. We now define the set $\mathcal{W}_b$ and then prove that it contains exactly those sequences of x-coordinates corresponding to domino towers with b blocks in the bottom row.

Definition 3.6. For $b \geq 1$ we define $\mathcal{W}_b$ to be the set of sequences $\langle x_1, \dots, x_n \rangle$ with the following properties:

i) We have $x_1 = 0$;

ii) For all $i \in [n-1]$ we have $x_{i+1} < x_i + 2$;

iii) There exists a set $D \subset [n]$ containing b numbers $1 = d_1 < d_2 < \cdots < d_b$ such that

  a) for all $j \in [b-1]$ we have $x_{d_{j+1}} + 2 = x_{d_j}$,

  b) for all $i \notin D$ with $i < d_{j+1}$ we have $x_i \geq x_{d_j}$,

  c) for all $i \notin D$ with $d_b < i$ we have $x_i + 2 > \min_{j<i} x_j$.

Proposition 3.7. *The set $\mathcal{W}_b$ contains exactly the sequences corresponding to domino towers with b blocks in the bottom row. In other words, Lemma 3.5 always produces a sequence in $\mathcal{W}_b$ and any sequence in $\mathcal{W}_b$ corresponds to a valid domino tower.*

Proof. Given a tower with b blocks in the bottom row, let $b_1, \dots, b_n$ be the order produced by Lemma 3.5. Let $\langle x_1, \dots, x_n \rangle$ be the x-coordinates of the blocks $b_1, \dots, b_n$ with $x_1 = 0$. We now show that $\langle x_1, \dots, x_n \rangle \in \mathcal{W}_b$ by checking all properties:

i) We have $x_1 = 0$ by definition.

ii) From property ii) in Lemma 3.5 follows $x_{i+1} < x_i + 2$.

iii) Let D be the set of indices of the blocks in the bottom row of the tower. As the tower has b blocks in the bottom row, we have $|D| = b$. Order $D = \{d_1, \dots, d_b\}$ as $d_1 < \cdots < d_b$.

a) It follows immediately from the properties in Lemma 3.5 that the rightmost block in the bottom row is always dropped first. Hence we have $d_1 = 1$. Similarly, we have that the bottom row must be dropped from right to left. Hence block $b_{d_{j+1}}$ is dropped two units to the left of block b_{d_j}, because the bottom row is convex by definition. Hence we have $x_{d_{j+1}} + 2 = x_{d_j}$. See Figure 3.3 for an illustration.

b) If a block b_i is not in the bottom row, dropped before $b_{d_{j+1}}$ and $x_i < x_{d_j}$, then we cannot anymore drop $b_{d_{j+1}}$ from an infinite height such that $b_{d_{j+1}}$ reaches the bottom row.

c) If we have $x_i + 2 \leq \min_{j<i} x_j$, then block b_i falls down to the bottom row. This is not possible if $i > d_b$, as at this point the bottom row is complete.

For the other direction, fix a sequence $\langle x_1, \ldots, x_n \rangle \in \mathcal{W}_b$ and consider the structure built by dropping blocks at positions x_1 to x_n. We now show that the structure is a valid domino tower, such that the set D from property iii) contains the indices of blocks which land in the bottom row. Clearly, the first block lands in the bottom row. Properties iii. a) and iii. b) guarantee that the block with index d_{j+1} lands in the bottom row and keeps the row convex. In Figure 3.3 all blocks with indices strictly between d_j and d_{j+1} are dropped to the right of the vertical line. Finally, properties ii) and iii. c) guarantee that all other blocks land on top of a previous block. □

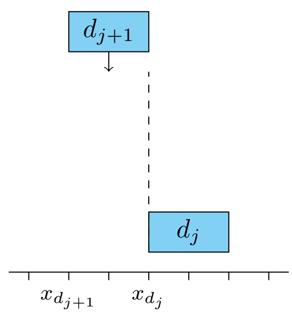


Figure 3.3: Illustration of property iii. a) in Definition 3.6.

Note that by thinking about sequences of x-coordinates instead of towers, it is now clear that the heights of blocks do not change the number of towers. Only the x-coordinates and order matter. Also note that a tower is restricted if and only if the corresponding sequence does not have repeated consecutive entries. Using this insight we can now explain the relation between restricted and unrestricted towers:

Corollary 3.8 ([8, Cor. 2.3])**.** *The number of unrestricted domino towers made out of n dominoes is equal to 4^{n-1}.*

Proof. Consider the sequences in $\mathcal{W} := \bigcup_b \mathcal{W}_b$ that have no repeated consecutive entries. We know from the introduction that there are 3^{n-1} such sequences of length n. The corresponding ordinary generating function is therefore $f(x) = x + 3x^2 + 9x^3 + \cdots = \frac{x}{1-3x}$. Now, if we take such a sequence and replace every entry x_i by a sequence $x_i, \ldots, x_i$ of arbitrary positive length, we get a sequence in $\mathcal{W}$, where repeated consecutive entries are allowed, i.e., a sequence corresponding to an unrestricted tower. This process is reversible: To recover the original sequence, we simply delete repeated consecutive entries. See Figure 3.4 for an illustration. This is an example of a *substitution* as defined by Flajolet [15, Def. I.14]. In terms of the generating functions this procedure therefore corresponds to replacing x with $x + x^2 + x^3 + \cdots = \frac{x}{1-x}$. We can now deduce the generating function for the unrestricted domino towers:

$$\frac{x}{1-3x} \rightsquigarrow \frac{\frac{x}{1-x}}{1-3\cdot\frac{x}{1-x}} = \frac{x}{1-4x},$$

from which we can read off the number: 4^{n-1}. □

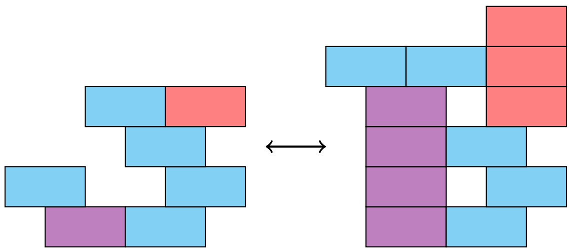


Figure 3.4: Illustration of the substitution of x with $\frac{x}{1-x}$. The corresponding sequences are $\langle 0, 1, 0, 1, -2, -1, -3\rangle$ and $\langle 0, 1, 0, 1, 1, 1, -2, -2, -2, -2, -1, -3\rangle$ respectively.

We can generalise sequences $\mathcal{W}_b$ for S-omino towers by also keeping track of the widths ℓ_i. For that we redefine $\mathcal{W}_b$ as follows:

Definition 3.9. We define $\mathcal{W}_b$ to be the set of sequences of pairs $\langle (x_1, \ell_1), \ldots, (x_n, \ell_n)\rangle$ with $n \geq b$ with the following properties:

i) We have $x_1 = 0$;

ii) For all $i \in [n-1]$ we have $x_{i+1} < x_i + \ell_i$;

iii) There exists a set $D \subset [n]$ containing b numbers $1 = d_1 < d_2 < \cdots < d_b$ such that

   a) for $j \in [b-1]$ we have $x_{d_{j+1}} + \ell_{d_{j+1}} = x_{d_j}$,
   b) for all $i \notin D$ with $i < d_{j+1}$ we have $x_i \geq x_{d_j}$,
   c) for all $i \notin D$ with $d_b < i$ we have $x_i + \ell_i > \min_{j<i} x_j$.

Proposition 3.7 still holds analogously, as 2 was merely replaced by the length of the appropriate block. The purposes of the properties remain exactly the same.

As we want to keep track of how many blocks of each length we have used, we define the weight of a sequence $t = \langle (x_1, \ell_1), \ldots, (x_n, \ell_n)\rangle \in \mathcal{W}$ as $w(t) := z^n y_{\ell_1} y_{\ell_2} \cdots y_{\ell_n}$. Hence, the exponent of y_ℓ is the number of pairs in t with second entry equal to ℓ and the exponent of z is the total number of pairs. We define the multivariate ordinary generating function of $\mathcal{W}_b$ with formal variables $z, y_1, y_2, y_3, \ldots$ as $W = \sum_{t\in\mathcal{W}} w(t)$. Lemma 3.5 now immediately generalises to:

Lemma 3.10. *Fix s_i, n_i and b as before. Then there is a bijection between such S-omino towers and elements in $\mathcal{W}_b$ of weight $z^n \prod_i y_{s_i}^{n_i}$. Furthermore, a tower is restricted if and only if there are no repeated consecutive elements in the corresponding sequence in $\mathcal{W}_b$.*

Proof. Similarly to Lemma 3.5, the last element of the sequence must correspond to the left-most block of the tower, among the blocks that do not have any other blocks vertically above it. The statement follows from induction on n. □

We will often need to offset sequences of pairs of the form (x, ℓ) horizontally, so we define

$$\langle (x_1, \ell_1), \ldots, (x_n, \ell_n) \rangle + \alpha := \langle (x_1 + \alpha, \ell_1), \ldots, (x_n + \alpha, \ell_n) \rangle.$$

Also, we define *concatenation* of two sequences as follows:

$$\langle a_1, \ldots, a_n \rangle \parallel \langle b_1, \ldots, b_m \rangle := \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle.$$

3.3 Proof using the Lagrange inversion formula

The main tool we use in this section is the following version of the Lagrange inversion formula [33, Section 2.6]. Here, $[x^n]G(x)$ denotes the coefficient of x^n in the formal power series $G(x)$.

Proposition 3.11 (The Lagrange inversion formula [33])**.** *Let $Y(x) = x\Phi(Y)$, where $\Phi(Y)$ is a power series such that $\Phi(0) \neq 0$. Then for any power series $g(Y)$ and $n \geq 1$ we have*

$$[x^n]g(Y) = \frac{1}{n}[y^{n-1}]g'(y)(\Phi(y))^n.$$

We prove an immediate corollary:

Corollary 3.12. *Let $Y(x) = x\Phi(Y)$, where $\Phi(Y)$ is a power series such that $\Phi(0) \neq 0$. Then for any power series $h(Y)$ and $n \geq 1$ we have*

$$[x^n]x\frac{dY}{dx}h(Y) = [y^{n-1}]h(y)(\Phi(y))^n.$$

Proof. Let $g(Y) = \int h(Y)dY$ and apply Proposition 3.11 on g.

$$\begin{aligned}
&[x^n]x\frac{dY}{dx}h(Y) \\
&= n \cdot [x^n] \int \frac{dY}{dx}h(Y)dx \\
&= n \cdot [x^n] \int h(Y)dY \\
&= [y^{n-1}]h(y)(\Phi(y))^n.
\end{aligned}$$

□

The idea of the proof is to relate $\mathcal{W}_b$ to other sets of sequences.

Definition 3.13. We define $\mathcal{U}_\star$ to be the set of sequences of pairs $\langle (x_1, \ell_1), \ldots, (x_n, \ell_n) \rangle$ with $n \geq 1$ with the following properties:

i) We have $x_1 = 0$;

ii) For all $i > 1$ we have $x_i \geq 1$ and $x_i < x_{i-1} + \ell_{i-1}$.

We also define $\mathcal{U}_1 := \mathrm{Seq}_{\geq 1}(\mathcal{U}_\star)$, the set of sequences that are a concatenation of at least one sequence in $\mathcal{U}_\star$. For convenience we similarly define $\mathcal{U} := \mathrm{Seq}_{\geq 0}(\mathcal{U}_\star) = \{\langle\rangle\} \cup \mathcal{U}_1$, which also contains the empty sequence.

Lemma 3.14. *Let $\langle (x_1, \ell_1), \ldots, (x_n, \ell_n) \rangle \in \mathcal{U}_\star$. Then there is exactly one choice of indices $d_1, \ldots, d_{\ell_1}$ such that*

- *we have $2 = d_1 \leq d_2 \leq \cdots \leq d_{\ell_1} = n + 1$ and*
- *for all $j \in [\ell_1 - 1]$ the subsequence $\langle (x_{d_j}, \ell_{d_j}), \ldots, (x_{d_{j+1}-1}, \ell_{d_{j+1}-1}) \rangle - \alpha_j \in \mathcal{U}$, where $\alpha_j := \ell_1 - j$. In particular, if $d_j < d_{j+1}$, then $x_{d_j} = \alpha_j$.*

Proof. Define the indices $d_j^\star := \min\{i \geq 2 : x_i \leq \alpha_j\}$, where $d_j^\star = n + 1$ if such an i does not exist. We now prove by induction on j that $d_{j+1} = d_{j+1}^\star$ is the unique choice for indices that satisfy the conditions in the lemma. Clearly, for $j = 1$ we are done, because $d_1 = 2 = d_1^\star$. Similarly, we have $d_{\ell_1} = n + 1 = d_{\ell_1}^\star$ for $j = \ell_1 - 1$. Now for $j \in [\ell_1 - 2]$ we can assume the induction hypothesis for $j - 1$, i.e., that $d_j = d_j^\star$. For a contradiction, we consider the cases $d_{j+1} > d_{j+1}^\star$ and $d_{j+1} < d_{j+1}^\star$ separately:

1. If $d_{j+1} > d_{j+1}^\star \geq d_j^\star = d_j$, then by definition of $d_{j+1}^\star$ we have $x_{d_{j+1}^\star} \leq \alpha_{j+1}$. However, by definition of $\mathcal{U}_\star$ and the fact that $\langle (x_{d_j}, \ell_{d_j}), \ldots, (x_{d_{j+1}-1}, \ell_{d_{j+1}-1}) \rangle - \alpha_j \in \mathcal{U}$ we have $x_{d_{j+1}^\star} \geq \alpha_j = \alpha_{j+1} + 1$, a contradiction.
2. If $d_{j+1} < d_{j+1}^\star$, then by definition of $d_{j+1}^\star$ we have $x_{d_{j+1}} > \alpha_{j+1}$. However, for $k := \max\{p : d_p = d_{j+1}\}$ we have $d_k < d_{k+1}$ and therefore $x_{d_{j+1}} = x_{d_k} = \alpha_k \leq \alpha_{j+1}$, a contradiction.

Hence $d_{j+1} = d_{j+1}^\star$ and we are done by induction on j. □

Example 3.15. *The sequence $\langle (0,4), (3,2), (3,2), (1,2), (2,2) \rangle \in \mathcal{U}_\star$ corresponds to the tower in Figure 3.5. The indices d_i in this example are: $d_1 = 2$, $d_2 = 4$, $d_3 = 4$ and $d_4 = 6$. We have $\langle (3,2), (3,2) \rangle - 3 \in \mathcal{U}$ drawn in violet, $\langle\rangle - 2 \in \mathcal{U}$, and finally $\langle (1,2), (2,2) \rangle - 1 \in \mathcal{U}$ drawn in red.*

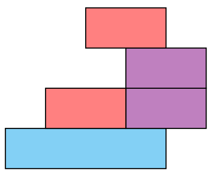


Figure 3.5: Illustration of Example 3.15.

We have just shown that every sequence in $\mathcal{U}_\star$ can be built by concatenating $\langle (0, \ell) \rangle$ with $\ell - 1$ sequences in $\mathcal{U}$, offset by $\ell - 1, \ell - 2, \ldots, 1$ respectively and that this construction is unique. Similarly, every sequence in $\mathcal{U}_1$ can be constructed uniquely using ℓ sequences in $\mathcal{U}$. This motivates the following proposition.

Lemma 3.16. *Let $u \in \mathcal{U}$ be non-empty. Define ℓ and n such that the first element of u is $(0, \ell)$ and $n = |u|$. Then u satisfies the following two properties:*

i) There exists a unique pair (x, y) with $x \in \mathcal{U}_\star$ and $y \in \mathcal{U}$ such that $u = x \parallel y$.

ii) There is exactly one choice of indices $d_1, \dots, d_{\ell+1}$ such that

- *we have $2 = d_1 \leq \dots \leq d_{\ell+1} = n + 1$ and*
- *we have $u = \langle (0, \ell) \rangle \parallel (c_{\ell-1} + \ell - 1) \parallel (c_{\ell-2} + \ell - 2) \parallel \dots \parallel (c_0 + 0)$, where $c_0, \dots, c_{\ell-1}$ in $\mathcal{U}$ with $|c_{\ell-i}| = d_{i+1} - d_i$.*

Proof. For part i) suppose that $u = \langle (x_1, \ell_1), \dots, (x_n, \ell_n) \rangle$. There are two cases:

1. Suppose that $\forall i > 1$ $x_i \geq 1$. In this case $u \in \mathcal{U}_\star$. Note that x must be non-empty by definition and if y were non-empty its first element would be $(0, \ell')$ for some ℓ'. Hence $(u, \langle\rangle)$ is the unique pair (x, y) such that $u = x \parallel y$.
2. Suppose that there exists $i > 1$ with $x_i = 0$ and let $i^\star > 1$ be the minimal such i. In this case $u \notin \mathcal{U}_\star$. Hence y cannot be empty and needs to start with $(0, \ell')$ for some ℓ' by definition. By definition of $\mathcal{U}_\star$ we must have $|x| < i^\star$. Hence the pair $(\langle (x_1, \ell_1), \dots, (x_{i^\star-1}, \ell_{i^\star-1}) \rangle, \langle (x_{i^\star}, \ell_{i^\star}), \dots, (x_n, \ell_n) \rangle)$ is the unique pair (x, y) such that $u = x \parallel y$.

This completes the proof of part i). We now prove part ii). By part i) there exists a unique pair (x, y) with $x \in \mathcal{U}_\star, y \in \mathcal{U}$ and $u = x \parallel y$. By Lemma 3.14 there is exactly one choice of indices $d_1, \dots, d_\ell$ such that $2 = d_1 \leq \dots \leq d_\ell = |x| + 1$ and for all $j \in [\ell - 1]$ we have $c_{\ell-j} := \langle (x_{d_j}, \ell_{d_j}), \dots, (x_{d_{j+1}-1}, \ell_{d_{j+1}-1}) \rangle - \ell + j \in \mathcal{U}$. Also define $c_0 := y$ and $d_{\ell+1} := n+1$. Then we have

$$x = \langle (0, \ell) \rangle \parallel (c_{\ell-1} + \ell - 1) \parallel (c_{\ell-2} + \ell - 2) \parallel \dots \parallel (c_1 + 1)$$

with $|c_{\ell-j}| = d_{j+1} - d_j$ and $|c_0| = |y| = n - |x| = d_{\ell+1} - d_\ell$. As $u = x \parallel y$, the result follows. ☐

We also define two other sets of sequences $\mathcal{X}$ and $\mathcal{V}$.

Definition 3.17. We define $\mathcal{X}_\ell$ to be the set of sequences

$$\mathcal{X}_\ell := \{u - \alpha : 0 \leq \alpha < \ell \text{ and } u \in \mathcal{U}_\star \text{ and } u \text{ starts with } (0, \ell)\}.$$

Let $\mathcal{X} := \bigcup_{\ell \in \mathbb{N}} \mathcal{X}_\ell$ be the union over all possible lengths ℓ. Further, we define $\mathcal{V}$ to be the minimal set with the following properties:

i) We have $\langle\rangle \in \mathcal{V}$;

ii) The set $\mathcal{V}$ is closed under the following procedure:

  a) Pick any $\ell \in \mathbb{N}$;
  b) Pick any elements $v \in \mathcal{V}$ and $x \in \mathcal{X}_\ell$ starting with, say, $(-\alpha, \ell)$;
  c) Then $x \parallel (v - \alpha) \in \mathcal{V}$.

Remark 3.18. To aid readability of the following arguments, we now describe the towers corresponding to the sets of sequences $\mathcal{W}_b, \mathcal{U}_\star, \mathcal{U}_1, \mathcal{U}, \mathcal{X}_\ell, \mathcal{X}$ and $\mathcal{V}$ informally.

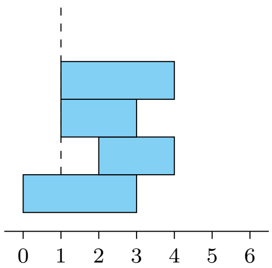
The sequences in $\mathcal{W}_b$ correspond to the towers we want to enumerate: Towers with b blocks in the bottom row as introduced in Section 3.1.

The sequences in $\mathcal{U}_\star$ correspond to towers with a single block in the bottom row with left edge at $x = 0$, and no other block crossing the vertical line $x = 1$. See Figure 3.6a for an example.

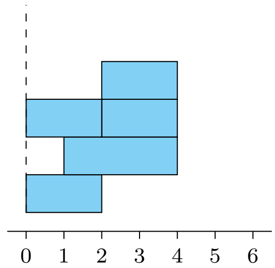
The sequences in $\mathcal{U}_1$ correspond to towers with a single block in the bottom row with left edge at $x = 0$, and no other block crossing the vertical line $x = 0$. See Figure 3.6b for an example. The set of sequences $\mathcal{U}$ equals $\mathcal{U}_1$ but including an empty tower with 0 blocks.

You can think of sequences in $\mathcal{X}_\ell$ as towers in $\mathcal{U}_\star$ with a base of length ℓ lying on top of a unit-length platform. The platform is fixed between 0 and 1 and its position relative to the base block corresponds to α in Definition 3.17. See Figure 3.6c for an example. The set $\mathcal{X}$ is simply the union over all possible lengths of the base.

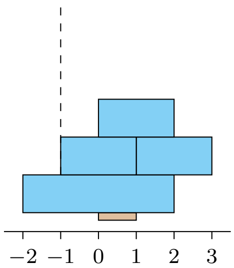
Similarly to $\mathcal{X}$, you can think of sequences in $\mathcal{V}$ as towers in $\mathcal{W}_1$ with their base lying on top of a unit-length platform. See Figure 3.6d for an example.



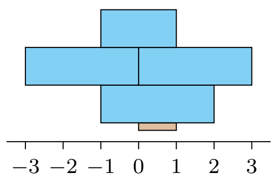
(a) The tower corresponding to $\langle (0,3),(2,2),(1,2),(1,3)\rangle \in \mathcal{U}_\star$.



(b) The tower corresponding to $\langle (0,2),(1,3),(2,2),(2,2),(0,2)\rangle \in \mathcal{U}_1$.



(c) The tower corresponding to $\langle (-2,4),(1,2),(-1,2),(0,2)\rangle \in \mathcal{X}_4$.



(d) The tower corresponding to $\langle (-1,3),(0,3),(-3,3),(-1,2)\rangle \in \mathcal{V}$.

Figure 3.6: Towers corresponding to sequences in $\mathcal{U}_\star, \mathcal{U}_1, \mathcal{X}_4$ and $\mathcal{V}$.

Lemma 3.19. *The set $\mathcal{W}_b$ is related to $\mathcal{X}$ and $\mathcal{V}$ as follows:*

i) For all $b \geq 2$ we have a weight-preserving bijection $\mathcal{W}_b \leftrightarrow \mathcal{U}_1 \times \mathcal{W}_{b-1}$;

ii) We have a weight-preserving bijection $\mathcal{W}_1 \leftrightarrow \mathcal{U}_\star \times \mathcal{V}$.

Proof. Consider any element $\langle (x_1,\ell_1),\ldots,(x_n,\ell_n)\rangle \in \mathcal{W}_b$. For $b \geq 2$ we know from Definition 3.9 that there exists an index $1 < d_2 = \min\{i \geq 2 : x_i < 0\}$ such that we have

$$\langle (x_1,\ell_1),\ldots,(x_{d_2-1},\ell_{d_2-1})\rangle \in \mathcal{U}_1 \text{ and } \langle (x_{d_2},\ell_{d_2}),\ldots,(x_n,\ell_n)\rangle + \ell_{d_2} \in \mathcal{W}_{b-1}.$$

For $b = 1$ we let $d = \min\{i \geq 2 : x_i \leq 0\}$ and $d = n + 1$ if such an i does not exist. Then

$$\langle (x_1, \ell_1), \ldots, (x_{d-1}, \ell_{d-1}) \rangle \in \mathcal{U}_\star \text{ and } \langle (x_d, \ell_d), \ldots, (x_n, \ell_n) \rangle \in \mathcal{V}.$$

In both cases the function has an inverse: Concatenate both parts back together. □

Example 3.20. *The sequence* $\langle (0, 2), (1, 2), (2, 2), (-2, 2), (-1, 2), (-3, 2), (-4, 2) \rangle \in \mathcal{W}_2$ *corresponds to the tower in Figure 3.7. We have* $\langle (0, 2), (1, 2), (2, 2) \rangle \in \mathcal{U}_1$ *drawn in blue,* $\langle (-2, 2), (-1, 2) \rangle + 2 \in \mathcal{U}_\star$ *drawn in violet, and finally* $\langle (-3, 2), (-4, 2) \rangle + 2 \in \mathcal{V}$ *drawn in red.*

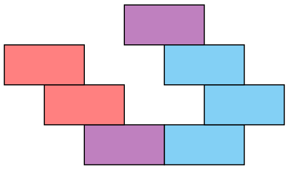


Figure 3.7: Illustration of Example 3.20.

Using the same weights as for W, let $U, U_1, U_\star, X_\ell, X$ and V be the multivariate ordinary generating functions of $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_\star, \mathcal{X}_\ell, \mathcal{X}$ and $\mathcal{V}$ respectively.

Theorem 3.21. *The ordinary generating functions satisfy the following equations:*

a) $U = 1 + U_1$ *and* $U_1 = U_\star \cdot U = \sum_{\ell \in \mathbb{N}} z y_\ell U^\ell$

b) $X_\ell = \ell z y_\ell U^{\ell-1}$, $X = \sum_{\ell \in \mathbb{N}} X_\ell$, *and* $V = 1 + X \cdot V$

c) $W_b = U_\star \cdot V \cdot U_1^{b-1}$

d) $z \frac{dU_1}{dz} = U_1 + X \cdot z \frac{dU_1}{dz}$

e) $z \frac{dU_1}{dz} = U_1 \cdot V$

f) $(1 + U_1) \cdot W_b = U_1^{b-1} z \frac{dU_1}{dz}$

Proof. Parts a), b) and c) follow from Lemma 3.16, Definition 3.17 and Lemma 3.19 respectively. For d) note that $z \frac{dU_1}{dz}$ is the ordinary generating function of $\Theta\mathcal{U}_1$, i.e., the set $\mathcal{U}_1$, where one element is marked. We can define $\Theta\mathcal{U}_1 := \{(u, k) : u \in \mathcal{U}_1, k \in [|u|]\}$. We now describe a bijection f between $\Theta\mathcal{U}_1$ and $\mathcal{U}_1 + X \times \Theta\mathcal{U}_1$. Let $(u, k) \in \Theta\mathcal{U}_1$. Now note that there are two cases:

1. For $k = 1$ we simply define $f((u, k)) := u \in \mathcal{U}_1$;

2. For $k \geq 2$ we know that by Lemma 3.16 there exists $\ell \in \mathbb{N}$ and ℓ sequences $c_0, \ldots, c_{\ell-1} \in \mathcal{U}$ and $2 = d_1 \leq \cdots \leq d_{\ell+1} = n + 1$ such that

$$u = \langle (0, \ell) \rangle \parallel (c_{\ell-1} + \ell - 1) \parallel (c_{\ell-2} + \ell - 2) \parallel \cdots \parallel (c_0 + 0),$$

where $|c_{\ell-i}| = d_{i+1} - d_i$. As $k \geq 2$, there exists exactly one $p \in [\ell]$ such that $d_p \leq k < d_{p+1}$. Hence $(c_{\ell-p}, k - d_p + 1) \in \Theta\mathcal{U}_1$. Let

$$x = \langle (-\ell + p, \ell) \rangle \parallel (c_{\ell-1} + p - 1) \parallel \cdots \parallel (c_{\ell-p+1} + 1)$$

$$\| (c_{\ell-p-1}) \| \cdots \| (c_0 - \ell + p + 1).$$

Then $x \in \mathcal{X}_\ell$. We define $f((u,k)) = (x, (c_{\ell-p}, k - d_p + 1)) \in \mathcal{X} \times \Theta\mathcal{U}_1$. This function is invertible, because x stores the variable p, which enables us to undo the shifts and reinsert $c_{\ell-p}$ into u at the correct position.

For e), note that from b) and d) follows $\Theta\mathcal{U}_1 = \mathcal{U}_1 \times \mathrm{Seq}_{\geq 0}(\mathcal{X}) = \mathcal{U}_1 \times \mathcal{V}$. Finally f) follows from a), c) and e). □

Proof of Theorem 3.1. We consider the multivariate power series W_b in $z, y_1, y_2, y_3, \ldots$ as a power series in z with coefficients in the ring of multivariate formal power series in $y_1, y_2, y_3, \ldots$ and use Corollary 3.12:

$$\begin{aligned}
[z^n]W_b &= [z^n] z \frac{dU_1}{dz} \frac{U_1^{b-1}}{1+U_1} \\
&= [u^{n-1}] \frac{u^{b-1}}{1+u} \Big(\sum_{i\in[m]} y_{s_i} (1+u)^{s_i} \Big)^n \\
&= [u^{n-b}] \frac{1}{1+u} \Big(\sum_{i\in[m]} y_{s_i} (1+u)^{s_i} \Big)^n .
\end{aligned}$$

Now we also fix the number of occurrences of length s_i to be n_i:

$$\begin{aligned}
[y_{s_1}^{n_1} \cdots y_{s_m}^{n_m}][z^n]W_b &= [y_{s_1}^{n_1} \cdots y_{s_m}^{n_m}][u^{n-b}] \frac{1}{1+u} \Big(\sum_{i\in[m]} y_{s_i} (1+u)^{s_i} \Big)^n \\
&= [u^{n-b}] \frac{1}{1+u} \binom{n}{n_1, \ldots, n_m} \prod_{i=1}^m (1+u)^{s_i n_i} \\
&= [u^{n-b}] \binom{n}{n_1, \ldots, n_m} (1+u)^{-1+\sum_{i=1}^m s_i n_i} \\
&= \binom{n}{n_1, \ldots, n_m} \binom{-1+\sum_{i=1}^m s_i n_i}{n-b}.
\end{aligned}$$

Now, summing over all $b \in [n]$ we can express the total number of S-omino towers for given $(n_1, \ldots, n_m)$ in terms of the Gaussian hypergeometric function ${}_2F_1$:

$$\begin{aligned}
&[z^n y_{s_1}^{n_1} \cdots y_{s_m}^{n_m}] \sum_{b=1}^n W_b \\
&= \sum_{b=1}^n \binom{n}{n_1, \ldots, n_m} \binom{-1+\sum_{i=1}^m s_i n_i}{n-b} \\
&= \binom{n}{n_1, \ldots, n_m} \binom{-1+\sum_{i=1}^m s_i n_i}{n-1} \sum_{b=0}^{n-1} \frac{(n-1)!(-n+\sum_{i=1}^m s_i n_i)!}{(n-1-b)!(b-n+\sum_{i=1}^m s_i n_i)!} \\
&= \binom{n}{n_1, \ldots, n_m} \binom{-1+\sum_{i=1}^m s_i n_i}{n-1} \sum_{b=0}^{n-1} \frac{(1)_b (1-n)_b}{(1-n+\sum_{i=1}^m s_i n_i)_b} \frac{(-1)^b}{b!}
\end{aligned}$$

$$= \binom{n}{n_1, \ldots, n_m} \binom{-1 + \sum_{i=1}^m s_i n_i}{n-1} \cdot {}_2F_1\left(1, 1-n; 1 + \sum_{i=1}^m (s_i - 1)n_i; -1\right).$$

This completes the proof of Theorem 3.1. □

Remark 3.22. We can find closed formulas for the other sets analogously. For example, for $s := \sum n_i s_i$ we have

$$[y_{s_1}^{n_1} \cdots y_{s_m}^{n_m}][z^n]V = \binom{s}{n_1, \ldots, n_m, s-n}$$

and

$$[y_{s_1}^{n_1} \cdots y_{s_m}^{n_m}][z^n]U = \frac{1}{s+1} \binom{s+1}{n_1, \ldots, n_m, s+1-n}.$$

Therefore, the number of sequences in $\mathcal{U}$ for $m = 1, n_1 = n, s_1 = 2$ is given by the Catalan numbers – sequence A000108 in the *On-Line Encyclopedia of Integer Sequences* [35].

3.4 Bijective proof

In this section, we give an explicit bijection between $\mathcal{U}$ and generalised Dyck paths and then extend it to a bijection between $\mathcal{W}_b$ and $\mathcal{D}_{W_b}$.

Definition 3.23. We define the weighted set of generalised Dyck paths $\mathcal{D}_U$ as the set of all integer sequences $\langle u_1, u_2, \ldots, u_N \rangle$ with the following properties:

i) We have $\forall i \in [N]\ u_i \geq -1$;

ii) We have $\forall j \in [N]\ \sum_{i<j} u_i \geq 0$;

iii) We have $\sum_{i \in [N]} u_i = -1$.

We define the weight function to be

$$w(\langle u_1, u_2, \ldots, u_N \rangle) := \prod_{i \in [N]} \begin{cases} z y_{u_i+1}, & \text{if } u \geq 0; \\ 1, & \text{if } u = -1. \end{cases}$$

From the properties above, it follows that any sequence in $\mathcal{D}_U$ satisfies $u_N = -1$ and $\sum_{i \in [N-1]} u_i = 0$. We include the fixed -1 at the end of each sequence for later convenience. Also note that Rukavicka gave a different definition of generalised Dyck paths [34]. They also consider paths with *flaws*, which allows the path to fall below 0. When disregarding the fixed -1 at the end of our sequence, our definition corresponds to Dyck paths with 0 flaws as defined by Rukavicka. See Figure 3.8 for the correspondence between the two definitions: Every -1 corresponds to a vertical step of length 1 and every $u_i \geq 1$ corresponds to a horizontal step of length u_i. Note that our definition allows us to have $u_i = 0$ which would correspond to a horizontal step of length 0.

Lemma 3.24. *We have a weight-preserving bijection f_U between $\mathcal{U}$ and $\mathcal{D}_U$.*

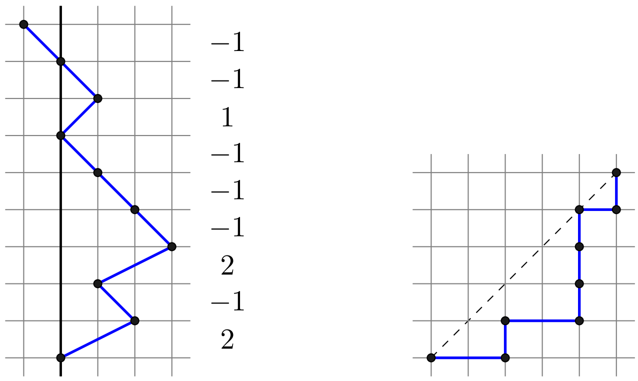


Figure 3.8: Correspondence between the sequence $\langle 2, -1, 2, -1, -1, -1, 1, -1, -1\rangle \in \mathcal{D}_U$ on the left and a Dyck path with 0 flaws as defined by Rukavicka on the right.

Proof. We define $f_U(\langle\rangle) := \langle -1\rangle$ and given a sequence $u = \langle (x_1, \ell_1), \dots, (x_n, \ell_n)\rangle \in \mathcal{U}_1$, we define

$$f_U(u) := \big(\, \|_{p=1}^{n-1} \langle \ell_p - 1, \underbrace{-1, -1, \dots, -1}_{x_p+\ell_p-1-x_{p+1} \text{ times}} \rangle\big) \,\|\, \langle \ell_n - 1, \underbrace{-1, -1, \dots, -1}_{x_n+\ell_n \text{ times}}\rangle.$$

Note that f_U is clearly weight-preserving, as for every pair $(x, \ell) \in u$ we have got exactly one copy of $\ell - 1$ in $f_U(u)$, both of which account for the same weight: zy_ℓ.

We now show $f_U(u) \in \mathcal{D}_U$ by checking all three properties of sequences in $\mathcal{D}_U$. For that let $\langle u_1, \dots, u_N\rangle = f_U(u)$.

i) By definition we have $\forall i \; u_i \geq -1$.

ii) Let $k_1 < k_2 < \dots < k_n \in [N]$ be all indices with $u_{k_m} \geq 0$. By construction of $f_U(u)$ it is enough to show property ii) for each $j = k_m$. We have

$$\begin{aligned}\sum_{i<k_m} u_i &= \sum_{p<m} ((\ell_p - 1) - 1 \cdot (x_p + \ell_p - 1 - x_{p+1})) \\ &= \sum_{p<m} (-1 \cdot (x_p - x_{p+1})) \\ &= x_m - x_1 = x_m \\ &\geq 0.\end{aligned}$$

iii) Finally, we have

$$\begin{aligned}\sum_{i\in[N]} u_i &= \sum_{p<n} \big((\ell_p - 1) - 1 \cdot (x_p + \ell_p - 1 - x_{p+1})\big) + (\ell_n - 1) - 1 \cdot (x_n + \ell_n) \\ &= \sum_{p<n} (-1 \cdot (x_p - x_{p+1})) - x_n - 1 \\ &= -1.\end{aligned}$$

Therefore $f_U(u) \in \mathcal{D}_U$. The inverse of above function can be described as follows: Fix any sequence $v = \langle u_1, \ldots, u_N \rangle \in \mathcal{D}_U$. Let $k_1 < k_2 < \cdots < k_n \in [N]$ be all indices with $u_{k_m} \geq 0$. Note that this determines the value of n. Define $f^{-1}(v) := \langle (x_1, \ell_1), \ldots, (x_n, \ell_n) \rangle$ with $x_m := \sum_{i<k_m} u_i$ and $\ell_m := u_{k_m} + 1$ for all $m \in [n]$. We now check both properties of sequences in $\mathcal{U}$:

i) We have $x_1 = \sum_{i<1} u_i = 0$.

ii) For all $m > 1$ we have $x_m = \sum_{i<k_m} u_i \geq 0$, by definition of $\mathcal{D}_U$. Also we have

$$\begin{aligned} x_m - x_{m-1} &= \sum_{k_{m-1} \leq i < k_m} u_i \\ &= u_{k_{m-1}} - 1 \cdot (k_m - k_{m-1} - 1) \\ &< u_{k_{m-1}} + 1 = \ell_{m-1}. \end{aligned}$$

Therefore $f^{-1}(v) \in \mathcal{U}$. □

As we have proven Theorem 3.21 bijectively, we already know how we can relate sequences in $\mathcal{V}$ and $\mathcal{W}$ with sequences in $\mathcal{U}$. We now reuse the ideas from the previous section and replace $\mathcal{U}$ with $\mathcal{D}_U$. Also, we use Raney's lemma [20] instead of the Lagrange inversion formula.

Lemma 3.25 (Version of Raney's lemma)**.** *For any sequence of integers $\langle a_1, \ldots, a_m \rangle$ with $a_i \geq -1$ and $\sum a_i = -1$, there exists exactly one $r \in [m]$ with the property that all proper partial sums, or in other words, the totals of all proper prefixes, of $\langle a_{r+1}, \ldots, a_m, a_1, \ldots, a_r \rangle$ are non-negative. Note that we must have $a_r = -1$, of course, as $a_{r+1} + \cdots + a_m + a_1 + \cdots + a_{r-1} \geq 0$, but $\sum a_i = -1$.*

We now generalise the idea from Dyck paths to the following sets $\mathcal{V}$ and $\mathcal{W}$:

Definition 3.26. Define

$$\mathcal{D}_V := \left\{ \langle u_1, u_2, \ldots, u_N \rangle : \forall i \; u_i \geq -1 \text{ and } \sum_{i \in [N]} u_i = -1 \text{ and } u_N = -1 \right\} \text{ and}$$

$$\mathcal{D}_{W_b} := \left\{ \langle u_1, u_2, \ldots, u_N \rangle : \forall i \in [b] \; u_i \geq 0 \text{ and } \sum_{i \in [N]} u_i = -b \text{ and } u_N = -1 \right\},$$

where the weight of a sequence is $w(\langle u_1, u_2, \ldots, u_N \rangle) := \prod_{i \in [N]} \begin{cases} z y_{u_i+1}, & \text{if } u \geq 0; \\ 1, & \text{if } u = -1. \end{cases}$

We can easily enumerate elements in V and W_b with given weight.

Lemma 3.27. *Fix a weight $w = z^n \prod_{i \in [m]} y_{s_i}^{n_i}$ with $n = \sum_{i \in [m]} n_i$ and let $s = \sum_{i \in [m]} n_i s_i$. Then the number of elements in $\mathcal{D}_V$ with weight w equals*

$$\binom{s}{n_1, n_2, \ldots, n_m, s-n}.$$

Also, the number of elements in $\mathcal{D}_{W_b}$ with weight w equals

$$\binom{n}{n_1, \ldots, n_m} \binom{s-1}{n-b}.$$

Proof. For the first part, note that every element in $\mathcal{D}_V$ with weight w has $s-n+1$ copies of -1 and total length $N = s+1$. The number of sequences follows from the fact that only u_N is fixed to be -1.

For the second part, note that every element in $\mathcal{D}_{W_b}$ with weight w has $s-n+b$ copies of -1 and total length $N = s+b$. We have $u_N = -1$ and $u_i \neq -1$ for $i \in [b]$. The number of elements in $\mathcal{D}_{W_b}$ follows by first considering the order and then the position of the non-negative integers. □

Corollary 3.28. *Define*

$$\mathcal{D}_Y := \{(v,k) : v = \langle u_1, u_2, \ldots, u_N\rangle \in \mathcal{D}_U \text{ and } k \in [N] \text{ and } u_k = -1\},$$

which can be thought of as the set of sequences in $\mathcal{D}_U$, where one copy of -1 is marked. Then there is a bijection f_{V_2} between $\mathcal{D}_Y$ and $\mathcal{D}_V$.

Proof. Let f_{V_2} be the following bijection: Take an element $(v,k) \in \mathcal{D}_Y$ and let $f_{V_2}((v,k)) := \langle u_{k+1}, \ldots, u_N, u_1, \ldots, u_k\rangle \in \mathcal{D}_V$. In other words, we have moved the marked element of an element in $\mathcal{D}_Y$ to the end by doing a cyclic rotation of the sequence. This can be undone using Raney's lemma. □

Lemma 3.29. *We have a weight-preserving bijection f_{V_1} between $\mathcal{V}$ and $\mathcal{D}_Y$. Hence, $f_V := f_{V_2} \circ f_{V_1}$ is a weight-preserving bijection between $\mathcal{V}$ and $\mathcal{D}_V$.*

Proof. We construct an explicit bijection $f_{V_1} : \mathcal{V} \to \mathcal{D}_Y$ as follows. First, let $f_{V_1}(\langle\rangle) := (\langle -1\rangle, 1)$ and now consider any non-empty $v \in \mathcal{V}$. Then by definition $\exists \ell \in \mathbb{N}, \alpha \in \{0, \ldots, \ell-1\}, x \in \mathcal{X}_\ell, v' \in \mathcal{V}$, where x starts with $(-\alpha, \ell)$, such that $v = x \parallel (v' - \alpha)$. By definition of $\mathcal{X}_\ell$ and Lemma 3.14 we can find $c_1, \ldots, c_{\ell-1} \in \mathcal{U}$ such that $x = \big(\langle(0,\ell)\rangle \parallel c_{\ell-1} + \ell - 1 \parallel c_{\ell-2} + \ell - 2 \parallel \cdots \parallel c_1 + 1\big) - \alpha$. Hence we have

$$v = \big(\langle(0,\ell)\rangle \parallel c_{\ell-1} + \ell - 1 \parallel c_{\ell-2} + \ell - 2 \parallel \cdots \parallel c_1 + 1 \parallel v'\big) - \alpha.$$

Let $(u,k) := f_{V_1}(v')$. We define $k' := k + 1 + |f_U(c_{\ell-1})| + \cdots + |f_U(c_{\alpha+1})|$ and

$$f_{V_1}(v) := \big(\langle \ell - 1\rangle \parallel f_U(c_{\ell-1}) \parallel \cdots \parallel f_U(c_{\alpha+1}) \parallel u \parallel f_U(c_\alpha) \parallel \cdots \parallel f_U(c_1), k'\big).$$

Note that k' is chosen such that the marked element in u remains marked. □

Lemma 3.30. *We have a weight-preserving bijection f_W between $\mathcal{W}_b$ and $\mathcal{D}_{W_b}$.*

Proof. From Lemma 3.19 we know that we can split an element $w \in \mathcal{W}_b$ into two parts. For $b = 1$ we have $w = u \parallel v$, where $u \in \mathcal{U}_\star$ and $v \in \mathcal{V}$. We define $f_W(w) := f_U(u) \parallel f_V(v)$. For $b \geq 2$ we have $w = u \parallel w' - \ell$, where $u \in \mathcal{U}_1, w' \in \mathcal{W}_{b-1}$ and u starts with $(0,\ell)$. Here we could choose to define $f_W(w) := f_U(u) \parallel f_W(w')$, but it would change the definition of $\mathcal{D}_W$ and make its enumeration more complicated. So we instead proceed as follows: Let $f_W(w') = \langle a_1, \ldots, a_n\rangle$. Then define $f_W(w) := \langle a_1, \ldots, a_{b-1}\rangle \parallel f_U(u) \parallel \langle a_b, \ldots, a_n\rangle$. We can undo the function in both cases, as we know that the sequence $f_U(u)$ sums to -1. The bijection f_W is weight-preserving, because f_U and f_V are. □

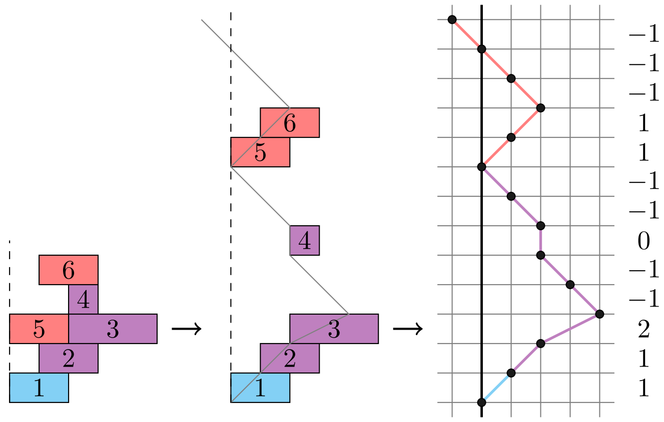
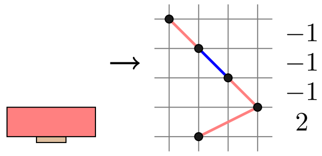


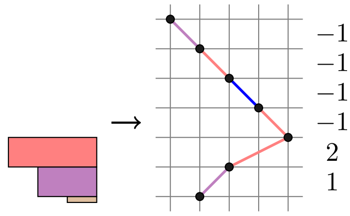
Figure 3.9: Illustration of the bijection f_U with the element $\langle 1, 1, 2, -1, -1, 0, -1, -1, 1, 1, -1, -1, -1 \rangle \in \mathcal{D}_U$.

We would like to conclude the section by providing examples of the bijection and show how the sequences are related to the towers described in Remark 3.18. Figure 3.9 shows the idea of the bijection for $\mathcal{D}_U$. The labels on the blocks correspond to the order of Lemma 3.5. The different colour correspond to the decomposition from Lemma 3.16.

As $\mathcal{U}$ corresponds to towers with no overhang to the left, the sequences in $\mathcal{D}_U$ are Dyck paths. To deal with overhang we use Lemma 3.29 to mark an element -1 in the sequences. The position of the marked element then determines the overhang to the left. See Figures 3.10 and 3.11 for an illustration, where the marked element is coloured dark blue. Figure 3.10 shows two examples of the bijection f_{V_1}, where in Figure 3.10b the result of Figure 3.10a is used.



(a) $(\langle 2, -1, -1, -1 \rangle, 3) \in \mathcal{D}_Y$.



(b) $(\langle 1, 2, -1, -1, -1, -1 \rangle, 4) \in \mathcal{D}_Y$.

Figure 3.10: Illustration of the bijection f_{V_1}.

Figure 3.11 shows a complete example of f_V. Note that the result from Figure 3.10b is used.

Finally, see Figure 3.12 for an example of the bijection between $\mathcal{W}_2$ and $\mathcal{D}_{W_2}$. The violet blocks are in $\mathcal{U}$, the blue blocks in $\mathcal{U}_\star$ and the red blocks in $\mathcal{V}$. Note that the result of f_{V_2} applied to Figure 3.10b is used.

3.5 Row-convex k-omino towers

In this section, we consider row-convex k-omino towers, as defined in Definition 3.2. By conditioning on the width of the bottom row, we see that f and g are related by the

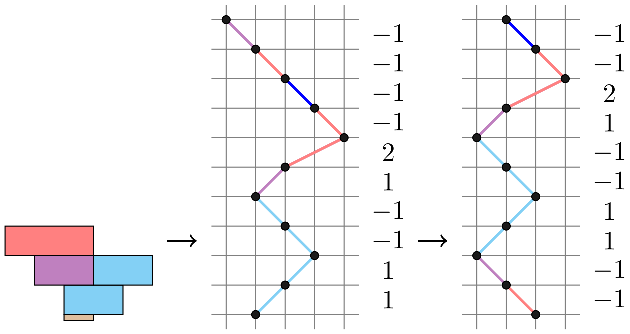


Figure 3.11: Illustration of the bijection $f_V := f_{V_2} \circ f_{V_1}$ with the element $\langle -1,-1,1,1,-1,-1,1,2,-1,-1\rangle \in \mathcal{D}_V$.

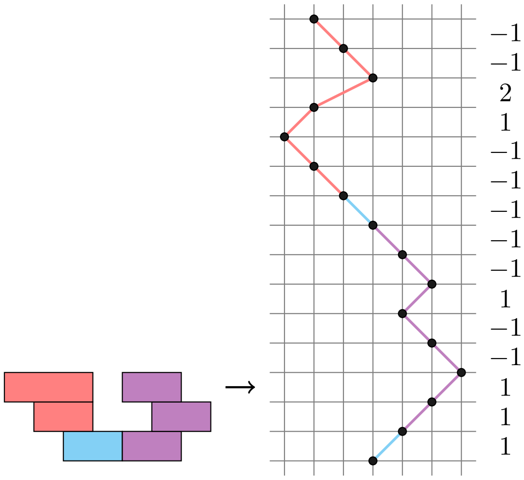


Figure 3.12: Illustration of the bijection f_W with the element $\langle 1,1,1,-1,-1,1,-1,-1,-1,-1,-1,1,2,-1,-1\rangle \in \mathcal{D}_{W_2}$.

equation

$$g(n) = \sum_{\ell=1}^{n} f_\ell(n-\ell), \text{ or equivalently } G(z) = \sum_{\ell=1}^{\infty} z^\ell F_\ell(z).$$

To improve readability, from now on we write $F_\ell(z)$ as F_ℓ and similarly for $G(z)$ and $h_n(z), \alpha(z)$ and $\beta(z)$ which are yet to be introduced. If the platform has width $k\ell$ and the bottom row consists of i blocks, then there are $(\ell+2-i)k-1$ positions the blocks in the row above could take such that the row is convex and they do not fall off the sides. This can be seen by an argument similar to the one given by Brown [7, Prop. 2.5]. We immediately find the recurrence:

$$f_\ell(n) = \sum_{i=1}^{\ell+1} \Big((\ell+2-i)k-1\Big) f_i(n-i), \text{ for } n \geq 1, \text{ where we define} \tag{3.1}$$
$$f_\ell(0) = 1 \text{ and } f_\ell(n) = 0, \text{ for } n < 0.$$

We now simplify this recurrence relation of f and rewrite it in terms of the generating functions F_ℓ.

Lemma 3.31. *The generating functions F_ℓ satisfy the following recurrence relation and boundary conditions:*

$$F_{\ell+2} - 2F_{\ell+1} + F_\ell = z^{\ell+2}F_{\ell+2} + (k-1)z^{\ell+3}F_{\ell+3}, \tag{3.2}$$

$$\begin{aligned} F_1 &= 1 + (2k-1)zF_1 + (k-1)z^2F_2, \\ F_2 &= 1 + (3k-1)zF_1 + (2k-1)z^2F_2 + (k-1)z^3F_3. \end{aligned} \tag{3.3}$$

Proof. First, we calculate

$$f_{\ell+1}(n) - f_\ell(n) = (k-1)f_{\ell+2}(n-\ell-2) + k\sum_{i=1}^{\ell+1} f_i(n-i)$$

and then use this result twice as follows:

$$\begin{aligned} &f_{\ell+2}(n) - 2f_{\ell+1}(n) + f_\ell(n) \\ &= \big(f_{\ell+2}(n) - f_{\ell+1}(n)\big) - \big(f_{\ell+1}(n) - f_\ell(n)\big) \\ &= (k-1)f_{\ell+3}(n-\ell-3) + f_{\ell+2}(n-\ell-2). \end{aligned}$$

The corresponding recurrence in terms of F_ℓ is

$$\begin{aligned} &F_{\ell+2} - 2F_{\ell+1} + F_\ell \\ &= \sum_{n=0}^{\infty}\Big(f_{\ell+2}(n) - 2f_{\ell+1}(n) + f_\ell(n)\Big)z^n \\ &= \sum_{n=0}^{\infty}\Big(f_{\ell+2}(n-\ell-2) + (k-1)f_{\ell+3}(n-\ell-3)\Big)z^n \\ &= z^{\ell+2}F_{\ell+2} + (k-1)z^{\ell+3}F_{\ell+3}. \end{aligned}$$

The boundary conditions are obtained by setting $\ell = 1$ and $\ell = 2$ in (3.1). □

To solve the recurrence (3.2), we first guess that there is a solution of the form

$$\sum_{j=0}^{\infty} \frac{z^{\ell j}h_j(z)}{(z;z)_j^2}$$

and then determine an $h_j(z)$ such that the recurrence relation holds. Here, $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$ denotes the q-Pochhammer symbol. That this method works is not surprising: In 1988, Privman and Švrakić successfully found an exact generating function for *fully directed compact lattice animals* using this approach [32]. The two problems are related, as there is a bijection between fully directed compact lattice animals and *restricted* row-convex domino towers. The number of dominoes in the bottom row maps to the number of compact sources of the directed animal. For an illustration of this bijection see Figure 3.13.

After adapting their method we end up with two solutions A_ℓ and B_ℓ, which we now check:

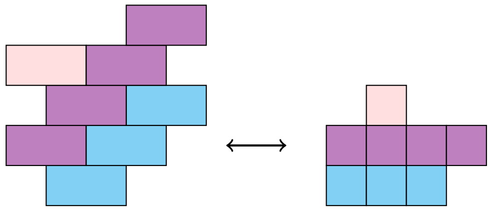


Figure 3.13: Illustration of the bijection between *restricted* row-convex domino towers and fully directed compact lattice animals

Lemma 3.32. *Two solutions of* (3.2) *are:*

$$A_\ell := \sum_{j=0}^{\infty} \frac{z^{\ell j} h_j}{(z;z)_j^2} \quad \textit{and}$$

$$B_\ell := \sum_{j=0}^{\infty} \frac{z^{\ell j} h_j}{(z;z)_j^2} \left(\ell + \sum_{m=1}^{j} \left(1 + \frac{2}{1-z^m} - \frac{1}{1+(k-1)z^m}\right)\right), \quad \textit{where}$$

$$h_j := z^{j(j+1)} \left((1-k)z;z\right)_j .$$

Proof. First, we calculate the ratio

$$\frac{h_j}{h_{j-1}} = \frac{z^{j(j+1)}}{z^{(j-1)j}} \frac{\prod_{i=1}^{j}(1-(1-k)z^i)}{\prod_{i=1}^{j-1}(1-(1-k)z^i)} = z^{2j}(1+(k-1)z^j).$$

Then we show that the recurrence holds

$$\begin{aligned}
&A_{\ell+2} - 2A_{\ell+1} + A_\ell \\
&= \sum_{j=0}^{\infty} \frac{h_j}{(z;z)_j^2} \left(z^{(\ell+2)j} - 2z^{(\ell+1)j} + z^{\ell j}\right) \\
&= \sum_{j=1}^{\infty} \frac{h_{j-1}\left(z^{2j} + (k-1)z^{3j}\right)}{(z;z)_j^2} z^{\ell j} \left(1 - z^j\right)^2 \\
&= \sum_{j=1}^{\infty} \frac{h_{j-1}\left(z^{(\ell+2)j} + (k-1)z^{(\ell+3)j}\right)}{(z;z)_{j-1}^2} \\
&= \sum_{j=0}^{\infty} \frac{h_j\left(z^{(\ell+2)(j+1)} + (k-1)z^{(\ell+3)(j+1)}\right)}{(z;z)_j^2} \\
&= z^{\ell+2}A_{\ell+2} + (k-1)z^{\ell+3}A_{\ell+3}.
\end{aligned}$$

Similarly, we can prove that B_ℓ is a solution. □

We have yet to find a solution that satisfies the boundary conditions. A suitable linear combination of A_ℓ and B_ℓ, however, does the trick. In general, setting $F_\ell = \alpha A_\ell + \beta B_\ell$ and solving the simultaneous equations $c_1F_1 + c_2F_2 + c_3F_3 = 1$ and $d_1F_1 + d_2F_2 + d_3F_3 = 1$ for α and β yields after some algebra:

$$\alpha = \big(-(c_1-d_1)B_1 - (c_2-d_2)B_2 - (c_3-d_3)B_3\big)/d,$$

$$\beta = \big((c_1 - d_1)A_1 + (c_2 - d_2)A_2 + (c_3 - d_3)A_3\big)/d, \text{ where}$$
$$d = (c_1A_1 + c_2A_2 + c_3A_3)(d_1B_1 + d_2B_2 + d_3B_3) - (c_1B_1 + c_2B_2 + c_3B_3)(d_1A_1 + d_2A_2 + d_3A_3).$$

Now setting

$$c_1 = 1 - (2k-1)z, \qquad c_2 = -(k-1)z^2, \qquad c_3 = 0,$$
$$d_1 = -(3k-1)z, \qquad d_2 = 1 - (2k-1)z^2, \qquad d_3 = -(k-1)z^3$$

as in (3.3) and plugging α and β into $F_\ell = \alpha A_\ell + \beta B_\ell$ yields the result of Theorem 3.4.

Remark 3.33. Note that for $k = 1$ the sequence $(g(n))_{n\geq 1} = 1, 2, 4, 8, 15, \dots$ is given by sequence A001523. It also counts the weakly unimodal compositions of n. Similarly, $f_\ell(n)$ counts the number of unimodal ℓ-tuples of positive integers summing to $n + \ell$. For example the sequence $(f_3(n))_{n\geq 0}$ is given by sequence A000212. For $k = 2$, the sequence $(g(n))_{n\geq 1} = 1, 4, 16, 61, 225, \dots$ is given by sequence A338531 in the *On-line Encyclopedia of Integer Sequences* [35].

3.6 Comments and further research

1. One might reconsider the *restricted* problem. Using a substitution similar to the one mentioned in Corollary 3.8, we can deduce the generating function for the restricted case, by replacing zy_i with $\frac{zy_i}{1+zy_i}$. However, is it possible to find a direct way of enumerating the generating function of restricted towers and a closed formula for its coefficients?

2. In this chapter, we have counted S-omino towers and row-convex towers. Is it possible to combine the two ideas and count row-convex S-omino towers?

4

Bijective Proof for the Enumeration of Rook Walks

4.1 Introduction

Definition 4.1. Let $S_{m,n,k}$ be the set of walks of length k a chess rook can move along on a rectangular board with width m and height n, starting and ending on the bottom left square.

Remark 4.2. First of all, note that it does not make a difference whether the board is plane or on a torus, by which we mean that opposite edges are identified. This is because the set of allowed moves a rook could take does not change. Secondly we note that by translational invariance of the torus, the starting square does not make a difference to the number of rook walks.

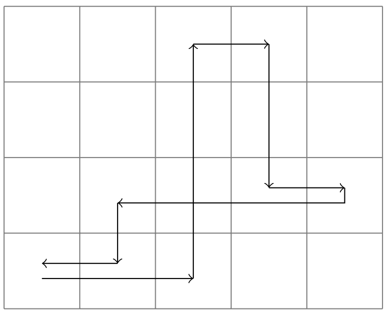


Figure 4.1: An example with $m = 5, n = 4, k = 8$.

In their collection of Bijection Proof Problems [36, Problem 240], Stanley gives the number of rook walks.

Theorem 4.3. *We have*

$$|S_{m,n,k}| = \frac{(m+n-2)^k + (n-1)(m-2)^k + (m-1)(n-2)^k + (m-1)(n-1)(-2)^k}{mn}.$$

Stanley states that the formula is known, but no combinatorial proof of it is. This problem also appears as an exercise in their book [37, p. 626, Problem 79b] as an example of the transfer-matrix method.

In this chapter, we provide a bijective proof of Theorem 4.3, based on the concept of sijections, which we defined in Subsection 1.3.2. The proof starts similarly to the one given by Stanley [37].

4.2 Proof of Theorem 4.3

We define a simple, but crucial signed set α_i now, which we use it to handle alternating signs. One example is the expression $(-2)^k$ in Theorem 4.3.

Definition 4.4. Define α_i to be the signed set containing a single element of weight $(-1)^i$.

We start the proof by considering the one-dimensional version of this problem, i.e., rook walks on grids of height 1. For that we define $S_{m,k} := S_{m,1,k}$. The two-dimensional problem is then related to the one-dimensional problem by the following lemma:

Lemma 4.5. *We have a bijection*

$$S_{m,n,k} \longleftrightarrow \sum_{i=0}^{k} \binom{[k]}{i} \times S_{m,i} \times S_{n,k-i}.$$

Proof. Let i be the number of horizontal steps of the whole two-dimensional walk. Projecting the two-dimensional walk on the horizontal and vertical axis, we get a pair of walks in $S_{m,i} \times S_{n,k-i}$. The information we lost in this way is exactly which i of the k steps belong to the horizontal walk, which is an element $I \in \binom{[k]}{i}$, an i-subset of $[k]$. The result follows. □

We represent elements in $S_{m,k}$ by sequences in $[m-1]^k$, whose total sum is $\equiv 0 \pmod{m}$. As noted in Remark 4.2, we can imagine the board to wrap around, such that we can interpret each number as the number of squares the rook moves rightwards. We then end on the starting square if and only if the sum of all steps is divisible by m. For example, in Figure 4.1 the walk moves horizontally at positions $\{1, 3, 5, 6, 8\}$ and vertically at positions $\{2, 4, 7\}$, so we have $i = 5$. The horizontal subwalk is $(2, 1, 1, 2, 4)$ while the vertical subwalk is $(3, 2, 3)$. Note that these two subwalks sum to $\equiv 0 \pmod 5$ and $\equiv 0 \pmod 4$ respectively.

Lemma 4.6. *We have a sijection*

$$[m] \times S_{m,i} \leftrightarrow [m-1]^i + \alpha_i \times [m-1],$$

where $S_{m,i}$ is now to be understood as signed set with weight function $w(x) := 1$.

Proof. We consider i even and i odd separately. So we want to show the following bijections:

i) For all m and even i: $[m] \times S_{m,i} \leftrightarrow [m-1] + [m-1]^i$.

ii) For all m and odd i: $[m] \times S_{m,i} + [m-1] \leftrightarrow [m-1]^i$.

We define the set of alternating sequences in $[m-1]^i$ as

$$[m-1]^i_a := \left\{ s \in [m-1]^i : s \text{ is of the form } s = (x, m-x, x, m-x, \ldots) \right\}$$

and the non-alternating sequences as

$$[m-1]^i_{na} := [m-1]^i \setminus [m-1]^i_a.$$

Note that we have a simple bijection $[m-1] \leftrightarrow [m-1]^i_a$, as an alternating sequence is determined by its first element. Now, for $i = 0$, note that there is only one walk of length 0, so we have

$$[m] \times S_{m,0} \leftrightarrow [m] \leftrightarrow [m-1] + \{m\} \leftrightarrow [m-1] + [m-1]^0.$$

Now suppose that $i \geq 1$, take an element in $[m] \times S_{m,i}$ and define a bijection f as follows:

a) For $x = (j, (a_1, \dots, a_i)) \in [m] \times S_{m,i}$ with $j \in [m-1]$, define $f(x) := (j, a_1, \dots, a_{i-1}) \in [m-1]^i$. The image of this function for this case is the set of sequences $(b_1, \dots, b_i) \in [m-1]^i$ with $\sum_{q=2}^i b_q \not\equiv 0 \pmod{m}$, because $a_i + \sum_{q=2}^i b_q \equiv \sum_{q=1}^i a_q \equiv 0$ and $a_i \not\equiv 0 \pmod{m}$.

b) For $x = (m, (a_1, \dots, a_i)) \in [m] \times S_{m,i}$ with $(a_1, \dots, a_i) \in [m-1]^i_a$, define $f(x) := a_1$. This case is only possible for i even and the image is $[m-1]$.

c) For $x = (m, (a_1, \dots, a_i)) \in [m] \times S_{m,i}$, with $(a_1, \dots, a_i) \in [m-1]^i_{na}$, we do the following: Let $0 \leq p < i$ be minimal such that $a_{i-p} + (-1)^{i-p} a_1 \not\equiv 0 \pmod{m}$, which exists as $(a_1, \dots, a_i)$ is not alternating. Now define $f(x) := (a_1, \dots, a_{i-p-1}, a_{i-p} + (-1)^{i-p} a_1, (-1)^{i-p+1} a_1, (-1)^{i-p+2} a_1, \dots, (-1)^i a_1)$, where we use modular arithmetic mod m. The image of this function for this case is the set of sequences $(b_1, \dots, b_i) \in [m-1]^i_{na}$ with $\sum_{q=2}^i b_q \equiv 0 \pmod{m}$.

For bijection i), we map to exactly $[m-1]$ from b) and $[m-1]^i$ from a) and c). For bijection ii), we map to exactly $[m-1]^i_{na}$ from a) and c). The result follows. □

Example 4.7. *Let $m = 3, i = 3$. The following describes the bijection from Lemma 4.6:*

$$\begin{aligned}&(1,(1,1,1)) \mapsto (1,1,1), \quad (2,(1,1,1)) \mapsto (2,1,1), \quad (3,(1,1,1)) \mapsto (1,1,2), \quad 1 \mapsto (1,2,1),\\ &(1,(2,2,2)) \mapsto (1,2,2), \quad (2,(2,2,2)) \mapsto (2,2,2), \quad (3,(2,2,2)) \mapsto (2,2,1), \quad 2 \mapsto (2,1,2)\end{aligned}$$

Example 4.8. *Let $m = 3, i = 4$. The following describes the bijection from Lemma 4.6:*

$$\begin{aligned}&(1,(1,1,2,2)) \mapsto (1,1,1,2), \quad (2,(1,1,2,2)) \mapsto (2,1,1,2), \quad (3,(1,1,2,2)) \mapsto (1,1,1,1),\\ &(1,(1,2,1,2)) \mapsto (1,1,2,1), \quad (2,(1,2,1,2)) \mapsto (2,1,2,1), \quad (3,(1,2,1,2)) \mapsto 1,\\ &(1,(1,2,2,1)) \mapsto (1,1,2,2), \quad (2,(1,2,2,1)) \mapsto (2,1,2,2), \quad (3,(1,2,2,1)) \mapsto (1,2,2,2),\\ &(1,(2,1,1,2)) \mapsto (1,2,1,1), \quad (2,(2,1,1,2)) \mapsto (2,2,1,1), \quad (3,(2,1,1,2)) \mapsto (2,1,1,1),\\ &(1,(2,1,2,1)) \mapsto (1,2,1,2), \quad (2,(2,1,2,1)) \mapsto (2,2,1,2), \quad (3,(2,1,2,1)) \mapsto 2,\\ &(1,(2,2,1,1)) \mapsto (1,2,2,1), \quad (2,(2,2,1,1)) \mapsto (2,2,2,1), \quad (3,(2,2,1,1)) \mapsto (2,2,2,2)\end{aligned}$$

Lemma 4.9. *We have a bijection*

$$[\ell-1]^k + \sum_{i=0:i \text{ odd}}^{k} \binom{[k]}{i} \times [\ell]^{k-i} \longleftrightarrow \sum_{i=0:i \text{ even}}^{k} \binom{[k]}{i} \times [\ell]^{k-i}.$$

Proof. First we define:

$$[\ell]_i^k := \left\{(S,x) \in \binom{[k]}{i} \times [\ell]^k : \forall j \in S\ x_j = \ell\right\},$$

for which we have a simple bijection

$$[\ell]_i^k \longleftrightarrow \binom{[k]}{i} \times [\ell]^{k-i},$$

that takes an element $(S,x) \in \binom{[k]}{i} \times [\ell]^k$ and removes from x the elements at the positions in S. As all of these were copies of ℓ, we can undo this operation. Now we can restate the lemma: We want to find a bijection

$$[\ell-1]^k + \sum_{i=0:i \text{ odd}}^{k} [\ell]_i^k \longleftrightarrow \sum_{i=0:i \text{ even}}^{k} [\ell]_i^k.$$

Using an idea from Garsia and Milne, as presented by Zeilberger [42], we define a bijection f as follows:

(a) For $x \in [\ell-1]^k$, let $f(x) = (\varnothing, x) \in [\ell]_0^k$.

(b) For $(S,x) \in [\ell]_i^k$ with i odd, let m be the smallest index m with $x_m = \ell$, which exists as $i \geq 1$. If $m \in S$ let $f((S,x)) = (S \setminus \{m\}, x) \in [\ell]_{i-1}^k$. If $m \notin S$ let $f((S,x)) = (S \cup \{m\}, x) \in [\ell]_{i+1}^k$.

To see why this is invertible, we state f^{-1}:

(a) For $(S,x) \in [\ell]_i^k$ with i even and x not containing a copy of ℓ, we define $f^{-1}((S,x)) := x$.

(b) For $(S,x) \in [\ell]_i^k$ with i even and x containing a copy of ℓ, let m be the smallest index m with $x_m = \ell$, which exists by assumption. If $m \in S$ let $f^{-1}((S,x)) = (S \setminus \{m\}, x) \in [\ell]_{i-1}^k$. If $m \notin S$ let $f^{-1}((S,x)) = (S \cup \{m\}, x) \in [\ell]_{i+1}^k$. □

Corollary 4.10. *We have a sijection*

$$\sum_{i=0}^{k} \binom{[k]}{i} \times \alpha_i \times [\ell]^{k-i} \longleftrightarrow [\ell-1]^k.$$

Proof. This follows from Lemma 4.9 directly by Remark 1.8 and Definition 4.4. □

Lemma 4.11. *We have a sijection*

$$\sum_{i=0}^{k} \binom{[k]}{i} \times \alpha_i \times \alpha_{k-i} \leftrightarrow [2]^k \times \alpha_k.$$

Proof. We have $\alpha_i \times \alpha_{k-i} \leftrightarrow \alpha_k$, as both sides are singleton sets with an element of equal weight whether k and i are odd or even. We also have a bijection

$$\sum_{i=0}^{k} \binom{[k]}{i} \leftrightarrow \mathcal{P}([k]) \leftrightarrow [2]^k.$$

The result follows. □

Theorem 4.12. *We have a bijection*

$$[m]\times[n]\times S_{m,n,k} \leftrightarrow [m+n-2]^k+[n-1]\times[m-2]^k+[m-1]\times[n-2]^k+[m-1]\times[n-1]\times\alpha_k\times[2]^k.$$

Proof. By Lemma 4.5, Lemma 4.6, Corollary 4.10 and Lemma 4.11 we have

$$\begin{aligned}
&[m] \times [n] \times S_{m,n,k} \\
&\leftrightarrow \sum_{i=0}^{k} \binom{[k]}{i} \times [m] \times S_{m,i} \times [n] \times S_{n,k-i} \\
&\leftrightarrow \sum_{i=0}^{k} \binom{[k]}{i} \times \big([m-1]^i + \alpha_i \times [m-1]\big) \times \Big([n-1]^{k-i} + \alpha_{k-i} \times [n-1]\Big) \\
&= \sum_{i=0}^{k} \binom{[k]}{i} \times [m-1]^i \times [n-1]^{k-i} + \sum_{i=0}^{k} \binom{[k]}{i} \times [m-1]^i \times \alpha_{k-i} \times [n-1] \\
&\qquad + \sum_{i=0}^{k} \binom{[k]}{i} \times \alpha_i \times [m-1] \times [n-1]^{k-i} + \sum_{i=0}^{k} \binom{[k]}{i} \times \alpha_i \times [m-1] \times \alpha_{k-i} \times [n-1] \\
&\leftrightarrow [m+n-2]^k + [n-1] \times [m-2]^k + [m-1] \times [n-2]^k + [m-1] \times [n-1] \times \alpha_k \times [2]^k.
\end{aligned}$$

□

Theorem 4.3 follows immediately from Theorem 4.12, by evaluating the weight of both sides.

4.3 Comments and further research

We have presented a combinatorial proof using sijections. Is it possible to rewrite the presented bijection as a direct bijection without signed sets – possibly one for even k and one for odd k?

The exercise in Stanley's book also generalises the problem to higher dimensions and the case when starting and ending square differ. It might be interesting to generalise the combinatorial proof presented in this chapter in this regard.

5 Random anchored structures

5.1 Introduction

In this chapter, we analyse a probabilistic variant of the following well-known combinatorial problem. Fix a finite set of points $\mathcal{P} = \{P_1, \dots, P_n\} = \{(x_1, y_1), \dots, (x_n, y_n)\} \subset [0,1]^2$ in the unit square, where $(x_1, y_1) := (0,0)$ is fixed. We define a *rectangle packing* to be a set of disjoint rectangles $\{R_1, \dots, R_n\}$ in $[0,1]^2$, such that P_i is at the lower-left corner of R_i, i.e., R_i is of the form $[x_i, \hat{x_i}] \times [y_i, \hat{y_i}]$ with $x_i \le \hat{x_i}$, $y_i \le \hat{y_i}$ and $(\hat{x_i}, \hat{y_i}) \in [0,1]^2$. We say that R_i is *lower-left-anchored* at P_i. Disjointness of rectangles can be expressed as follows: For all $i \neq j$, we have:

$$]x_i, \hat{x_i}[\times]y_i, \hat{y_i}[\quad \cap \quad]x_j, \hat{x_j}[\times]y_j, \hat{y_j}[\quad = \varnothing.$$

In other words, rectangles may touch, but not intersect each other. In 1969 Allen Freedman conjectured that for any $\mathcal{P}$ there exists a rectangle packing with total area exceeding $\frac{1}{2}$ [39, p. 345], i.e., $\mathrm{opt}(\mathcal{P}) > \frac{1}{2}$, where we define $\mathrm{opt}(\mathcal{P})$ to be the maximum area covered by a rectangle packing anchored at $\mathcal{P}$. More precisely, the conjecture is as follows:

Conjecture 5.1. *The set* $\mathcal{P}^\star := \{(\frac{i-1}{n}, \frac{i-1}{n}) : i \in [n]\}$ *minimises* $\mathrm{opt}(\mathcal{P})$ *over all sets* $\mathcal{P}$ *with* $|\mathcal{P}| = n$ *points. In other words, for all* $\mathcal{P}$ *we have* $\mathrm{opt}(\mathcal{P}) \ge \mathrm{opt}(\mathcal{P}^\star) = \frac{1}{2} + \frac{1}{2n}$.

See Figure 5.1 for an illustration of the point configuration $\mathcal{P}^\star$ for $n = 5$, whose maximum rectangle packing covers an area of about $\frac{1}{2}$. In 2011 Dumitrescu and Tóth proved a lower bound of 0.09 using a greedy algorithm [10].

Theorem 5.2 ([10, Thm. 8])**.** *Fix any set* $\mathcal{P}$ *and consider the following greedy algorithm: First sort the points by the sum of their two coordinates in decreasing order. Iterate over all points in this order and choose for a point* P *the maximum rectangle anchored at* P*, which is disjoint from all previously chosen rectangles. Then, the chosen rectangles are a rectangle packing of* $\mathcal{P}$ *with total area* ≥ 0.09.

In this chapter we study random sets of points instead of the worst possible configuration. More formally, let $P_1, P_2, \dots, P_n$ be independent and identically distributed random variables drawn uniformly at random from the unit square $[0,1]^2$. The resulting point configuration $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ is known as a *homogeneous binomial point process*. We

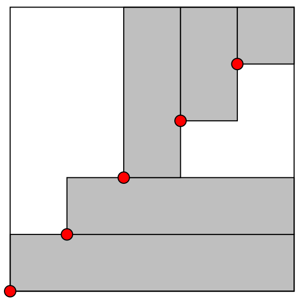


Figure 5.1: The maximum area covered by a rectangle packing anchored at the red points is $\frac{1}{2} + \frac{1}{10}$.

are interested in the *expected* maximum area, which can be covered. Furthermore, we are interested in the limit as $n \to \infty$, if the limit exists. We omit the fixed point at $(0, 0)$ as it becomes negligible in the limit.

This chapter is structured as follows. In Section 5.2 we prove results relating binomial point processes with Poisson processes. In Section 5.3 we study lower-left-anchored rectangle packings, as introduced above, in the random setting. In Section 5.4 we add the constraint that each rectangle must be a square, i.e., we study lower-left-anchored square packings. In Section 5.5 we consider discs, where the anchor is centered. Finally, in Section 5.6 we study center-anchored squares. To make the results more readable, we have put some calculations and lemmas into Section 5.7. We finish the chapter with comments and open questions in Section 5.8.

5.2 Tools

Independently choosing n points uniformly at random from $[0, 1]^2$ corresponds to a *homogeneous binomial point process*, as described in the introduction. If, instead of n points, we first randomise the number of points using a Poisson distribution with mean n, then the resulting point process is called a *homogeneous Poisson point process* with intensity n. One can think of binomial and Poisson point processes as random countable subsets of a space S. In this chapter most of the time we have $S = [0, 1]^2$ or $S = \mathbb{R}^2$.

Many calculations in this chapter become more tractable in the Poisson setting, because, unlike binomial point processes, Poisson point processes have the following remarkable property, known as the *independence property*: Given a homogeneous Poisson point process η on S with intensity n and two finite disjoint regions $A, B \subset S$, the number of points in A and in B is independent. Furthermore, from this follows the *Poisson property*, which states that for any finite region $A \subset S$ the number of points in A is a random variable with Poisson distribution and mean $\lambda = n \cdot \text{Area}(A)$ and we have

$$\mathbb{P}(|\eta \cap A| = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

For the questions asked in this chapter, both settings have the same limiting behaviour as $n \to \infty$. We will make this notion more precise in this section. Define the set of *locally* *finite point configurations* in S as

$$N_l(S) := \{A \subset S : |A \cap B| < \infty, \forall \text{ bounded } B \subseteq S\}.$$

We write N_l instead of $N_l(S)$ if the space S is clear from the context. Now let $k \geq 1$ be an integer and $q : ([0,1]^2)^k \times N_l \to [0,1]$ a function, where $q(x_1, \ldots, x_k; \nu)$ is defined for all $x_1, \ldots, x_k \in \nu$. We abuse the notation slightly and for $x_i \notin \nu$ write

$$q(x_1, \ldots, x_k; \nu) = q(x_1, \ldots, x_k; \nu \cup \{x_i\}). \tag{5.1}$$

Define the function $H_q : N_l \times N_l \to \mathbb{R}_+$ as follows:

$$H_q(\alpha, \beta) := \sum_{(x_1,\ldots,x_k) \in \alpha^k_{\neq}} q(x_1, \ldots, x_k; \beta).$$

Here $\alpha^k_{\neq}$ denotes the set of ordered k-tuples which consist of distinct elements from α. Define also the abbreviation $H_q(\nu) := H_q(\nu, \nu)$. We are interested in $\lim\limits_{n\to\infty} \mathbb{E}H_q(\nu)$ where ν is either a binomial or a Poisson point process on $[0,1]^2$ with mean number of points n. In our calculations we often need the following result, known as the multivariate Mecke equation for homogeneous Poisson point processes.

Lemma 5.3 (Special case of [30, Theorem 4.4])**.** *Let η_n be a homogeneous Poisson process on S with intensity n. Then, for every $k \in \mathbb{N}$ and for every function $\xi : S^k \times N_l \to [0,\infty]$ we have*

$$\mathbb{E}H_\xi(\eta_n) = n^k \int \mathbb{E}q(x_1, \ldots, x_k; \eta_n \cup \{x_1, \ldots, x_k\})\, d(x_1, \ldots, x_k).$$

Lemma 5.4. *Fix any $a, b > 0$ and let η_a, η_b be homogeneous Poisson point processes on S with intensities a and b respectively. Then, we have*

$$\mathbb{E}H_q(\eta_a, \eta_b) = \left(\frac{a}{b}\right)^k \mathbb{E}H_q(\eta_b).$$

Proof. Let η be a homogeneous Poisson point process on $S \times [0,\infty)$ with intensity 1. Then the random variables $\eta_t := \{x \in S : \exists z \leq t \text{ s.t. } (x,z) \in \eta\}$ are homogeneous Poisson point processes on S with intensity t. Apply Lemma 5.3 to η as follows:

$$\begin{aligned}
&\mathbb{E}H_q(\eta_a, \eta_b) \\
&= \int_{S^k} \int_{[0,a]^k} \mathbb{E}q(x_1, \ldots, x_k; (\eta \cup \{(x_i, t_i) : i \in [k]\})_b)\, d(t_1, \ldots, t_k)\, d(x_1, \ldots, x_k) \\
&= \int_{S^k} \int_{[0,a]^k} \mathbb{E}q\big(x_1, \ldots, x_k; \eta_b \cup \{x_i : i \in [k], t_i \leq b\}\big)\, d(t_1, \ldots, t_k)\, d(x_1, \ldots, x_k) \\
&= \int_{S^k} \int_{[0,a]^k} \mathbb{E}q\big(x_1, \ldots, x_k; \eta_b\big)\, d(t_1, \ldots, t_k)\, d(x_1, \ldots, x_k) \\
&= a^k \int_{S^k} \mathbb{E}q(x_1, \ldots, x_k; \eta_b)\, d(x_1, \ldots, x_k),
\end{aligned}$$

where the first equality follows from Lemma 5.3 and the third equality from (5.1). Analogously, we have

$$\mathbb{E}H_q(\eta_b, \eta_b) = b^k \int_{S^k} \mathbb{E}q(x_1, \ldots, x_k; \eta_b)\, d(x_1, \ldots, x_k).$$

Combining both parts yields the result. □

Definition 5.5. We say a function $q : ([0,1]^2)^k \times N_l \to [0,1]$ is *decreasing* if for all $x_1, \ldots, x_k \in [0,1]^2$ and $\nu_1 \subset \nu_2 \in N_l$ we have $q(x_1, \ldots, x_k; \nu_1) \geq q(x_1, \ldots, x_k; \nu_2)$. In other words, throwing more points into the square can only decrease a particular term of H_q disregarding, for a moment, that more terms are introduced.

Definition 5.6. We say a function $H_q : N_l \times N_l \to [0,1]$ is *bi-monotone* if for all $\alpha, \nu_1, \nu_2 \in N_l$ with $\nu_1 \subset \nu_2$ we have

a) $H_q(\nu_1, \alpha) \leq H_q(\nu_2, \alpha)$ and

b) $H_q(\alpha, \nu_1) \geq H_q(\alpha, \nu_2)$.

Lemma 5.7. *If q is a decreasing function, then H_q is bi-monotone.*

Proof. Property a) follows from the fact that $q \geq 0$. Property b) follows from the fact that q is decreasing. □

Lemma 5.8. *Let $S = [0,1]^2$, q be a function such that H_q is bi-monotone and β_n (resp. η_n) be a binomial (resp. Poisson) point process with n points (resp. intensity n). Then we have*

$$\limsup_{n\to\infty} \mathbb{E}H_q(\beta_n) = \limsup_{n\to\infty} \mathbb{E}H_q(\eta_n) \text{ and}$$
$$\liminf_{n\to\infty} \mathbb{E}H_q(\beta_n) = \liminf_{n\to\infty} \mathbb{E}H_q(\eta_n).$$

In particular, if the limits exist, then $\lim_{n\to\infty} \mathbb{E}H_q(\beta_n) = \lim_{n\to\infty} \mathbb{E}H_q(\eta_n)$.

Proof. Let η be a homogeneous Poisson point process on $S \times [0,\infty)$ with intensity 1. Define the random variables $\eta_t := \{x \in S : \exists z \leq t \text{ s.t. } (x,z) \in \eta\}$, $t_n := \min\{t : |\eta_t| = n\}$ and $\beta_n := \eta_{t_n}$. Note that η_n is a homogeneous Poisson point process with intensity n and β_n is a homogeneous binomial point process with n points. Note also that for all $0 < \varepsilon < 1$ we have $\eta_{(1-\varepsilon)n} \subset \eta_{(1+\varepsilon)n}$. Fix $\varepsilon > 0$ and define the event $A_{n,\varepsilon} := \{\eta_{(1-\varepsilon)n} \subset \beta_n \subset \eta_{(1+\varepsilon)n}\}$. We now show $\mathbb{P}(A^c_{n,\varepsilon}) = \mathcal{O}\left(e^{-n\left(1-e^{-\varepsilon}(1+\varepsilon)\right)}\right)$. For that we write $\mathbb{P}(A^c_{n,\varepsilon}) = \mathbb{P}(|\eta_{(1-\varepsilon)n}| > n) + \mathbb{P}(|\eta_{(1+\varepsilon)n}| < n)$ and use Chernoff's bound as follows:

$$\mathbb{P}(|\eta_{(1-\varepsilon)n}| > n) \leq \frac{\mathbb{E}\left(e^{\varepsilon|\eta_{(1-\varepsilon)n}|}\right)}{e^{\varepsilon n}} = e^{-n(1-e^{\varepsilon}(1-\varepsilon))}$$

and similarly

$$\mathbb{P}(|\eta_{(1+\varepsilon)n}| < n) \leq \frac{\mathbb{E}\left(e^{-\varepsilon|\eta_{(1+\varepsilon)n}|}\right)}{e^{-\varepsilon n}} = e^{-n\left(1-e^{-\varepsilon}(1+\varepsilon)\right)}.$$

Now, by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\Big(\mathbb{E}(1\{A_{n,\varepsilon}^c\} \cdot H_q(\eta_{(1+\varepsilon)n}))\Big)^2 \\
&\le \mathbb{E}(1\{A_{n,\varepsilon}^c\}^2) \cdot \mathbb{E}(H_q(\eta_{(1+\varepsilon)n})^2) \\
&\le \mathbb{E}(1\{A_{n,\varepsilon}^c\}) \cdot \mathbb{E}((|\eta_{(1+\varepsilon)n}|^k \cdot 1)^2) \\
&= \mathbb{P}\left(A_{n,\varepsilon}^c\right) \cdot \mathbb{E}(|\eta_{(1+\varepsilon)n}|^{2k}) \\
&= \mathcal{O}\Big(e^{-n\left(1-e^{-\varepsilon}(1+\varepsilon)\right)} \cdot n^{2k}\Big) \\
&\to 0,
\end{aligned}$$

where $1\{\cdot\}$ is the indicator function. For the binomial point process we have

$$\begin{aligned}
&\mathbb{E}\left(1\{A_{n,\varepsilon}^c\} \cdot H_q(\beta_n)\right) \\
&\le \mathbb{E}\left(1\{A_{n,\varepsilon}^c\} \cdot n^k\right) \\
&= \mathcal{O}\left(e^{-n\left(1-e^{-\varepsilon}(1+\varepsilon)\right)} \cdot n^k\right) \\
&\to 0.
\end{aligned}$$

One the one hand we have

$$\begin{aligned}
&\mathbb{E}H_q(\beta_n) \\
&\ge \mathbb{E}(1\{A_{n,\varepsilon}\} \cdot H_q(\beta_n)) \\
&\ge \mathbb{E}\left(1\{A_{n,\varepsilon}\} \cdot H_q(\eta_{(1-\varepsilon)n}, \eta_{(1+\varepsilon)n})\right) \\
&= \mathbb{E}\left(\left(1 - 1\{A_{n,\varepsilon}^c\}\right) \cdot H_q(\eta_{(1-\varepsilon)n}, \eta_{(1+\varepsilon)n})\right) \\
&\ge \mathbb{E}\left(H_q(\eta_{(1-\varepsilon)n}, \eta_{(1+\varepsilon)n})\right) - \mathbb{E}\left(1\{A_{n,\varepsilon}^c\} \cdot H_q(\eta_{(1+\varepsilon)n})\right) \\
&\ge \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^k \mathbb{E}(H_q(\eta_{(1+\varepsilon)n})) - \mathbb{E}\left(1\{A_{n,\varepsilon}^c\} \cdot H_q(\eta_{(1+\varepsilon)n})\right).
\end{aligned}$$

Letting $\varepsilon \to 0$ we can deduce

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{E}H_q(\beta_n) &\ge \limsup_{n\to\infty} \mathbb{E}H_q(\eta_n) \text{ and} \\
\liminf_{n\to\infty} \mathbb{E}H_q(\beta_n) &\ge \liminf_{n\to\infty} \mathbb{E}H_q(\eta_n).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\mathbb{E}H_q(\beta_n) \\
&= \mathbb{E}(1\{A_{n,\varepsilon}\} \cdot H_q(\beta_n)) + \mathbb{E}(1\{A_{n,\varepsilon}^c\} \cdot H_q(\beta_n)) \\
&\le \mathbb{E}\left(1\{A_{n,\varepsilon}\} \cdot H_q(\eta_{(1+\varepsilon)n}, \eta_{(1-\varepsilon)n})\right) + \mathbb{E}(1\{A_{n,\varepsilon}^c\} \cdot H_q(\beta_n)) \\
&\le \mathbb{E}\left(H_q(\eta_{(1+\varepsilon)n}, \eta_{(1-\varepsilon)n})\right) + \mathbb{E}(1\{A_{n,\varepsilon}^c\} \cdot H_q(\beta_n)) \\
&= \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^k \mathbb{E}(H_q(\eta_{(1-\varepsilon)n})) + \mathbb{E}(1\{A_{n,\varepsilon}^c\} \cdot H_q(\beta_n)).
\end{aligned}$$

Analogously to above we can deduce

$$\limsup_{n\to\infty} \mathbb{E}H_q(\beta_n) \le \limsup_{n\to\infty} \mathbb{E}H_q(\eta_n) \text{ and}$$

$$\liminf_{n\to\infty} \mathbb{E}H_q(\beta_n) \leq \liminf_{n\to\infty} \mathbb{E}H_q(\eta_n).$$

Together the result follows. □

Theorem 5.9. *Suppose we have functions*

$$q : ([0,1]^2)^k \times N_l([0,1]^2) \to [0,1] \text{ and}$$
$$q' : (\mathbb{R}^2)^k \times N_l(\mathbb{R}^2) \to [0,\infty].$$

Suppose also we are given functions

$$R : [0,1]^2 \times N_l([0,1]^2) \to [0,\infty] \text{ and}$$
$$R' : \mathbb{R}^2 \times N_l(\mathbb{R}^2) \to [0,\infty].$$

Define $\varepsilon(n) := n^{-1/2}\log(n)$, *let* $B_r(x)$ *be the ball of radius* r *around* x, *and let* $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$ *be the be the linear function defined by:* $\phi_n((x,y)) := ((x-1/2)\sqrt{n}, (y-1/2)\sqrt{n})$ *which maps the square* $[0,1]^2$ *to the square* $[-\sqrt{n}/2, \sqrt{n}/2]^2$ *of area* n. *Let* η_n *be a Poisson point process on* $[0,1]^2$ *with intensity* n. *Let* η *be a Poisson point process on* $\mathbb{R}^2$ *with intensity* 1. *We also make the following assumptions.*

i) A tuple $(x_1,\dots,x_k)$ *only gives a contribution if* $x_2,\dots,x_k$ *are close to* x_1*:*

$$q(x_1,\dots,x_k;\nu) \neq 0 \implies x_2,\dots,x_k \in B_{R(x_1,\nu)}(x_1).$$

ii) The values of R *and* q *depend only on the points within* $B_{R(x_1,\nu)}(x_1)$*: For all* $A \in N_l([0,1]^2 \setminus B_{R(x_1,\nu)}(x_1))$ *we have*

$$R(x_1,\nu) = R(x_1, (\nu \cap B_{R(x_1,\nu)}(x_1)) \cup A),$$

$$q(x_1,\dots,x_k;\nu) = q(x_1,\dots,x_k; (\nu \cap B_{R(x_1,\nu)}(x_1)) \cup A).$$

iii) If x_1 *is not affected by the boundary of* $[0,1]^2$, *then we can scale* q *and* R *using* ϕ_n*: For all* $x_1,\dots,x_k \in [0,1]^2, \nu \in N_l([0,1]^2)$ *with* $B_{R(x_1,\nu)}(x_1) \subset [0,1]^2$ *we have*

$$q(x_1,\dots,x_k;\nu)\cdot n = q'(\phi_n(x_1),\dots,\phi_n(x_k);\phi_n(\nu)),$$

$$R(x_1,\nu)\cdot\sqrt{n} = R'(\phi(x_1),\phi(\nu)).$$

iv) The function q' *is translation-invariant: For all* $x_i \in \mathbb{R}^2, \nu \in N_l(\mathbb{R}^2)$ *we have*

$$q'(x_1,\dots,x_k;\nu) = q'(0, x_2 - x_1, \dots, x_k - x_1; \nu - x_1).$$

v) Large radii are improbable: For all $x \in [0,1]^2$ *we have*

$$\mathbb{P}(R(x,\eta_n) \geq \varepsilon) \leq C_1 \exp(-nC_2\varepsilon^2)$$

for some constants $C_1, C_2 \geq 0$.

vi) If $R(x_1,\nu) \leq \varepsilon(n)$ *then* $q(x_1,\dots,x_k;\nu) < C_3\varepsilon(n)^2$.

Then we have

$$\lim_{n\to\infty} \mathbb{E}H_q(\eta_n) = \int_{x_2,\dots,x_k\in\mathbb{R}^2} \mathbb{E}q'(0,x_2,\dots,x_k;\eta)\,d(x_2,\dots,x_k).$$

Proof. We apply Theorem 5.3 and then use the assumptions in the statement as follows:

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}H_q(\eta_n)\\
&= \lim_{n\to\infty} n^k \int_{x_1,\dots,x_k\in[0,1]^2} \mathbb{E}q(x_1,\dots,x_k;\eta_n\cup\{x_1,\dots,x_k\})\,d(x_1,\dots,x_k)\\
&\geq \lim_{n\to\infty} n^k \int_{x_1,\dots,x_k\in[0,1]^2}\\
&\qquad \mathbb{E}(1\{R(x_1,\eta_n)\leq\varepsilon\}\cdot q(x_1,\dots,x_k;\eta_n\cup\{x_1,\dots,x_k\}))\,d(x_1,\dots,x_k)\\
&= \lim_{n\to\infty} n^k \int_{x_1\in[0,1]^2}\int_{x_2,\dots,x_k\in B_\varepsilon(x_1)}\\
&\qquad \mathbb{E}(1\{R(x_1,\eta_n)\leq\varepsilon\}\cdot q(x_1,\dots,x_k;\eta_n\cup\{x_1,\dots,x_k\}))\,d(x_2,\dots,x_k)\,dx_1\\
&\geq \lim_{n\to\infty} n^k \int_{x_1\in(\varepsilon,1-\varepsilon)^2}\int_{x_2,\dots,x_k\in B_\varepsilon(x_1)}\\
&\qquad \mathbb{E}(1\{R(x_1,\eta_n)\leq\varepsilon\}\cdot q(x_1,\dots,x_k;\eta_n\cup\{x_1,\dots,x_k\}))\,d(x_2,\dots,x_k)\,dx_1\\
&= \lim_{n\to\infty} \int_{x_1\in((-1/2+\varepsilon)\sqrt{n},(1/2-\varepsilon)\sqrt{n})^2}\int_{x_2,\dots,x_k\in B_{\varepsilon\sqrt{n}}(x_1)}\\
&\qquad \mathbb{E}(1\{R'(x_1,\eta)\leq\varepsilon\sqrt{n}\}\cdot \frac{1}{n}q'(x_1,\dots,x_k;\eta\cup\{x_1,\dots,x_k\}))\,d(x_2,\dots,x_k)\,dx_1\\
&= \lim_{n\to\infty} (1-2\varepsilon)^2 \int_{x_2,\dots,x_k\in B_{\varepsilon\sqrt{n}}(0)}\\
&\qquad \mathbb{E}(1\{R'(0,\eta)\leq\varepsilon\sqrt{n}\}\cdot q'(0,x_2,\dots,x_k;\eta\cup\{0,x_2,\dots,x_k\}))\,d(x_2,\dots,x_k)\\
&= \int_{x_2,\dots,x_k\in\mathbb{R}^2} \mathbb{E}q'(0,x_2,\dots,x_k;\eta\cup\{0,x_2,\dots,x_k\})\,d(x_2,\dots,x_k).
\end{aligned}$$

From the second to the third line, we have lost all terms with $R>\varepsilon$. However this case is very unlikely and the total tends to 0:

$$\begin{aligned}
&\leq n^k \int_{x_1\in[0,1]^2} \mathbb{P}(R(x_1,\eta_n)>\varepsilon)\,dx_1\\
&\leq n^k \int_{x_1\in[0,1]^2} C_1\exp(-nC_2\varepsilon^2)\,dx_1\\
&\to 0,\ \text{because } 1\ll n\varepsilon^2.
\end{aligned}$$

We have also lost all terms with x_1 too close to the boundary. However, since the boundary is small, the total area lost tends to 0:

$$\begin{aligned}
&= n^k \int_{x_1\in[0,1]^2\setminus(\varepsilon,1-\varepsilon)^2}\int_{x_2,\dots,x_k\in B_\varepsilon(x_1)}\\
&\qquad \mathbb{E}(1\{R(x_1,\eta_n)\leq\varepsilon\}\cdot q(x_1,\dots,x_k;\eta\cup\{x_1,\dots,x_k\}))\,d(x_2,\dots,x_k)\,dx_1
\end{aligned}$$

$$\leq n^k \int_{x_1\in[0,1]^2\setminus(\varepsilon,1-\varepsilon)^2} \int_{x_2,\dots,x_k\in B_\varepsilon(x_1)} C_3\varepsilon^2 \, d(x_2,\dots,x_k)\, dx_1$$
$$\leq n^k(1-(1-2\varepsilon)^2)(\pi\varepsilon^2)^{k-1}C_3\varepsilon^2$$
$$\to 0, \text{ because } n^k\varepsilon^{2k+1} \ll 1.$$

As all other steps were equalities, the result follows. □

Example 5.10. *Fix a point configuration $\nu \in N_l([0,1]^2)$. Suppose we draw for each point $x \in \nu$ the largest square contained in $[0,1]^2$ which is lower-left-anchored at x and does not contain any other point in ν nor intersects the boundary and let $q(x;\nu)$ be its area. See Figure 5.2 for two points with their squares. We are interested in evaluating*

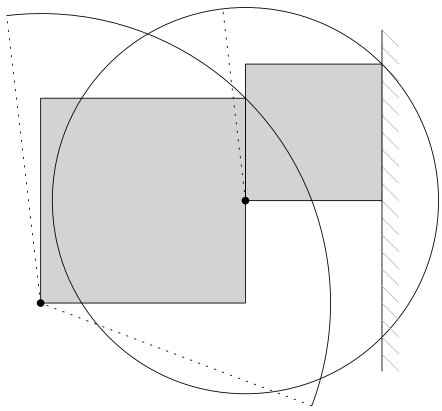


Figure 5.2: Illustration of Example 5.10.

the total area covered by the squares including overlap: $\lim_{n\to\infty} \mathbb{E}H_q(\beta_n)$. For that we want to use Theorem 5.9 to evaluate the limit and define $R(x,\nu)$ as the length of the diagonal of the square anchored at x. Property i) holds by default, as $k=1$. We have that the square anchored at x depends only on the point configuration within distance $R(x,\nu)$, see Figure 5.2. In other words property ii) holds. Let q' and R' be the same as q and R, but extended to $\mathbb{R}^2$ with no boundary around $[0,1]^2$. Then properties iii) and iv) hold. For property v) note that we have

$$\mathbb{P}(R(x,\eta_n) \geq \varepsilon\sqrt{2})$$
$$= \mathbb{P}(\textit{square anchored at } x \textit{ with side length } \varepsilon \textit{ is empty})$$
$$= \exp(-n\varepsilon^2),$$

where η_n is a Poisson point process on $[0,1]^2$ with intensity n. For property vi) note that we have

$$q(x;\nu) = \frac{1}{2}R(x,\nu)^2.$$

Now apply Lemma 5.8 and Theorem 5.9:

$$\lim_{n\to\infty} \mathbb{E}H_q(\beta_n)$$
$$= \lim_{n\to\infty} \mathbb{E}H_q(\eta_n)$$

$$
\begin{aligned}
&= \mathbb{E}q'(0;\eta) \\
&= \int_0^\infty 2s\exp(-s^2)\cdot s^2\,ds \\
&= 1,
\end{aligned}
$$

where η is a Poisson point process on $\mathbb{R}^2$ with intensity 1.

Definition 5.11. Let $V \subset R^2$ be a set of points. The *nearest neighbour* of $v \in V$ is a point $\mathrm{nn}(v) \in V \setminus \{v\}$ with minimum distance from v. We use either the Euclidean norm or the L^∞ norm depending on the context. Note that the nearest neighbour is unique if the points are in general position. The *nearest-neighbour graph* of V is the directed graph (V, E) with $E := \{(v, \mathrm{nn}(v)) : v \in V\}$. We call two points $x, y \in V$ *twins* if $(x, y), (y, x) \in E$. In other words twins are nearest neighbours of each other. Define $\mathrm{d_{nn}}(v) := |v - \mathrm{nn}(v)|$ as the distance from v to its nearest neighbour.

In 1997, Eppstein defined nearest-neighbour graphs and proved that the probability that a point is a twin converges to $6\pi/(3\sqrt{3} + 8\pi)$ [12].

5.3 Lower-left-anchored rectangles

When considering $n \to \infty$, the first question one might ask is whether there exists a limit for the optimal area covered by a rectangle packing anchored at β_n. Modelling the problem with $n = 300$ as an integer linear programming (ILP) problem and solving it with Gurobi 9.1 gives an average optimal area covered of ≈ 0.916. For large n our model becomes computationally untractable.

Question 5.12. *Is it true that* $\lim\limits_{n\to\infty} \mathbb{E}(\mathrm{opt}(\mathcal{P})) = 1$*?*

To answer this question positively, it would suffice to show that an algorithm produces an area tending towards 1, because every algorithm evidently gives a lower bound of the optimal area. We now give examples of potential algorithms and find their limits. Consider the following class of algorithms: Given the set of points $\mathcal{P}$, we draw from each point $(x_i, y_i) \in \mathcal{P}$ the line segment from (x_i, y_i) to $(1, y_i)$. We impose the additional constraint that no chosen rectangle is allowed to cross these line segments. Now it is impossible for two anchored rectangles to intersect - hence we can choose all rectangles independently. We can do this in several ways. For example, from all available rectangles for a given point, we could choose:

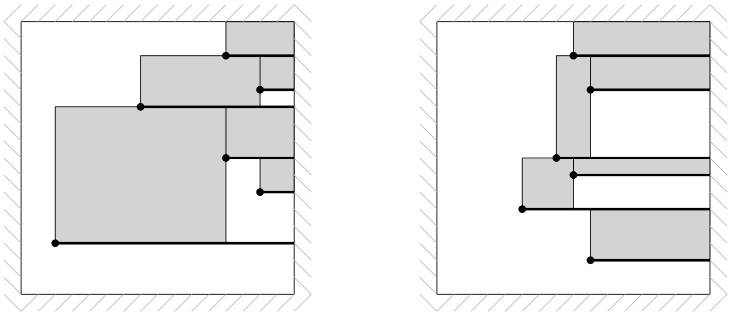
(1) the largest rectangle;

(2) the middle rectangle by index.

See Figure 5.3 for examples of which rectangles the algorithm with option (2) would choose. Especially in Figure 5.3b option (1) would perform better than option (2). We conjecture the following from computational evidence:

Conjecture 5.13. *The greedy algorithm with option (1) achieves expected area about*

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{gre}_1) \approx 0.77.$$



(a) An example where middle rectangles are large.

(b) An example where middle rectangles are small.

Figure 5.3: Illustration of the bijection f_{V_1}.

However, we can prove an exact limit for option (2).

Theorem 5.14. *The greedy algorithm with option (2) achieves expected area exactly*

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{gre}_2) = \frac{3}{5}.$$

Proof. By linearity of expectation and symmetry, the expected total area is the sum of the expected area for each point, multiplied by n. So we will proceed by calculating the expected area of a single point $P = (x, y)$. We condition over k, the number of rectangles we can choose for P. Let $P_1 = (x_0, y_1)$ and $P_2 = (x_i, y_i), \ldots, P_k = (x_k, y_k)$ be the points such that:

1. $0 < x_0 < x_1 < \ldots < x_k < 1$,
2. $0 < y_{k+1} < y_k < \ldots < y_1 < 1$,
3. there are no other points in $[0, x] \times [y_{k+1}, y_1]$,
4. there are no other points in $[x_1, x_{i+1}] \times [y_{i+1}, y_i]$ for $i \in [k]$,

where we define $x_1 := x, x_{k+1} := 1, y_{k+1} := y$ for convenience. See Figure 5.4 for an illustration.

Now suppose for all points with k options, we choose the rectangle with index m, where m ranges from $m = 1$ (high and narrow) to $m = k$ (flat and wide). The width of this rectangle is $x_{m+1} - x_1$ and the height is $y_m - y_{k+1}$. Therefore the expected area for all points that have k options is

$$\begin{aligned}
&= \lim_{n\to\infty} (n)_{k+1} \int_{\substack{0<x_0<x_1<\ldots<x_k<1 \\ 0<y_{k+1}<y_k<\ldots<y_1<1}} (x_{m+1} - x_1)(y_m - y_{k+1}) \cdot (1-A)^{n-k-1}\, d(x, y) \\
&= \frac{(3^m - 2^m)(3^{k+1-m} - 2^{k+1-m})}{6^k} := f(m),
\end{aligned}$$

where $A = \sum_{i=1}^{k} x_{i+1}(y_i - y_{i+1})$ is the area of the gray region in Figure 5.4 and $(n)_{k+1} = n(n-1)\cdots(n-k)$ is the falling factorial. We have used Theorem 5.36 to evaluate the integral. We are free to choose an m for every k independently. Note that

$$f(m+1) - f(m) = 6^{-k-m}\left(3^k 2^{2m} - 2^k 3^{2m}\right) > 0 \iff k > 2m.$$

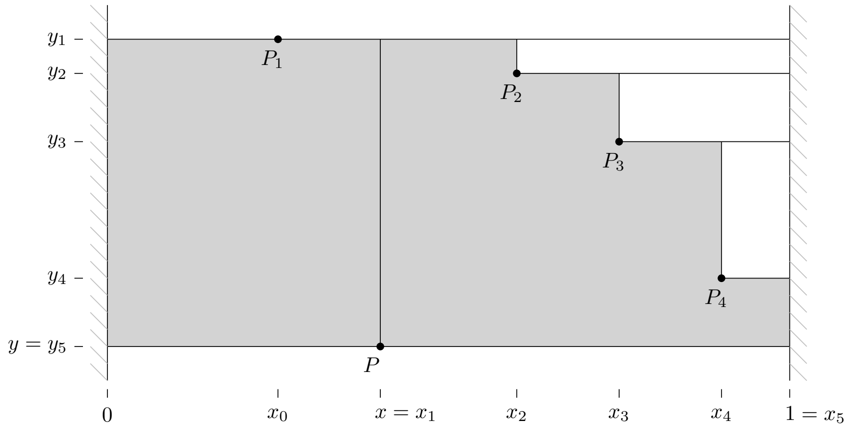


Figure 5.4: No other point may be inside the gray region. Here we have $k = 4$.

Hence the maximum

$$\max_{1 \le m \le k} \frac{(3^m - 2^m)(3^{k+1-m} - 2^{k+1-m})}{6^k}$$

is attained for $m = \lceil \frac{k}{2} \rceil$, i.e., for the middle rectangle. Summing over all k gives exactly $\frac{3}{5}$:

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{(3^{\lceil \frac{k}{2} \rceil} - 2^{\lceil \frac{k}{2} \rceil})(3^{k+1-\lceil \frac{k}{2} \rceil} - 2^{k+1-\lceil \frac{k}{2} \rceil})}{6^k} \\
&= \sum_{k=1}^{\infty} \frac{\left(2^k - 3^k\right)^2}{6^{2k-1}} + \sum_{k=1}^{\infty} \frac{\left(3^k - 2^k\right)\left(3^{k+1} - 2^{k+1}\right)}{6^{2k}} \\
&= \frac{1}{4} + \frac{7}{20} \\
&= \frac{3}{5}.
\end{aligned}$$

□

Remark 5.15. We can evaluate the integrals for option (1) for small k separately: Let $f(k)$ be the limit of the expected total area of all points, which have k options, when all of them choose the largest available rectangle.

Then we have $f(1) = \frac{1}{6}, f(2) = \frac{1}{180}(1 + 48\log(2))$, $f(3) \approx 0.152$, $f(4) \approx 0.103$ and $f(5) \approx 0.064$ numerically.

By the proof of Theorem 5.14 we have that $\sum_{k=6}^{\infty} f(k) \ge 0.05897$. Hence, the algorithm achieves an expected total area of at least 0.7347 if it chooses the maximum rectangle for points with at most 5 options, and the middle rectangle for points with at least 6 options.

5.4 Lower-left-anchored squares

We now consider packings of lower-left-anchored squares. The definition of such a packing is the same as in Section 5.3, except that we require for all i: $\hat{x_i} - x_i = \hat{y_i} - y_i$. For this setting we can prove that the optimum does not converge to 1. In particular, we can show the following upper bound.

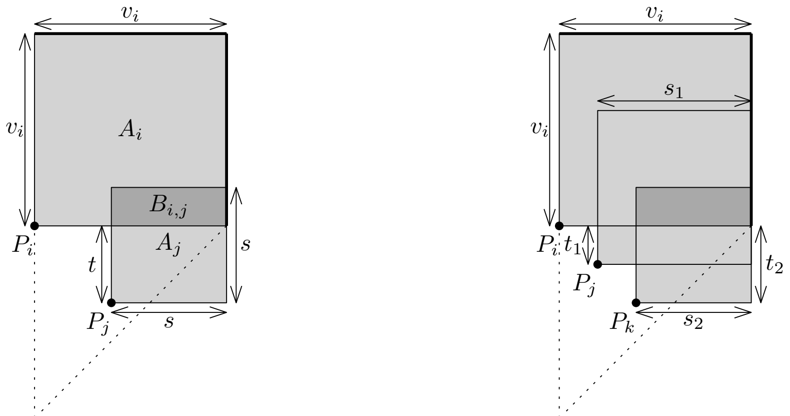
Theorem 5.16. *The expected area of the optimal lower-left-anchored square packing is bounded from above by* 0.861*:*

$$\lim_{n\to\infty} \mathbb{E}(\text{opt}) \leq \frac{5}{4} - \frac{\pi^2}{96} - \frac{1}{2}\log(2) + \frac{1}{8}\log^2(2) < 0.861.$$

Proof. The idea of this proof is the following: For each point $P_i = (x_i, y_i)$ let the random variable A_i be the maximum square anchored at P_i, which does not include any other point. Let v_i be the side length of this square, i.e., we have $A_i = [x_i, x_i + v_i] \times [y_i, y_i + v_i]$. Then an upper bound of the optimal packing is the expected area of the union $A_1 \cup \cdots \cup A_n$. Hence we wish to evaluate

$$\lim_{n\to\infty} \mathbb{E}(|A_1 \cup \cdots \cup A_n|).$$

Without loss of generality, we can assume that for $i < j$ we have $x_i < x_j$. For each $i < j$ we define areas $B_{i,j}$ with $B_{i,j} \subset A_i \cap A_j$ as follows: If $0 < y_i - y_j < x_i + v_i - x_j \leq v_j$ then let $B_{i,j} = [x_j, x_i + v_i] \times [y_i, y_j + x_i + v_i - x_j]$, otherwise $B_{i,j} = \varnothing$. See Figure 5.5a for an illustration. For convenience we also define $B_{i,i} := A_i$.



(a) $B_{i,j}$ is covered by both squares. (b) The intersection of $B_{i,k} \cap A_j$.

Figure 5.5: Illustrations of the rectangles A_i and $B_{i,j}$.

Now consider a point $P \in [0,1]^2$ and suppose P is contained in exactly $m \geq 1$ squares $(A_i)_{i\in I}$ for some index set I with $I = \{i_1, \ldots, i_m\}$ with $i_1 < \cdots < i_m$. Note that for all $a < b < c \in I$ we have $y_b > y_c$ and $x_b < x_c$ and hence $B_{a,c} \subset B_{a,b}$.

For all $j \in [m]$ define $f(j) := \max\{d : 0 \leq d \leq m - j \text{ and } P \in B_{i_j, i_{j+d}}\}$. Note that we have $d(m) = 0$. Hence we have

$$\begin{aligned}
&\sum_{i\in[n]} 1\{P \in A_i\} - \sum_{i<j\in[n]} 1\{P \in B_{i,j}\} + \sum_{i<j<k\in[n]} 1\{P \in B_{i,k} \cap A_j\} \\
&= \sum_{i\in I} 1\{P \in A_i\} - \sum_{i<j\in I} 1\{P \in B_{i,j}\} + \sum_{i<j<k\in I} 1\{P \in B_{i,k}\} \\
&= m + \sum_{i\in I}\left(-f(i) + \binom{f(i)}{2}\right) \\
&\geq m - (m-1) = 1 = 1\{P \in \bigcup_{i\in[n]} A_i\}.
\end{aligned}$$

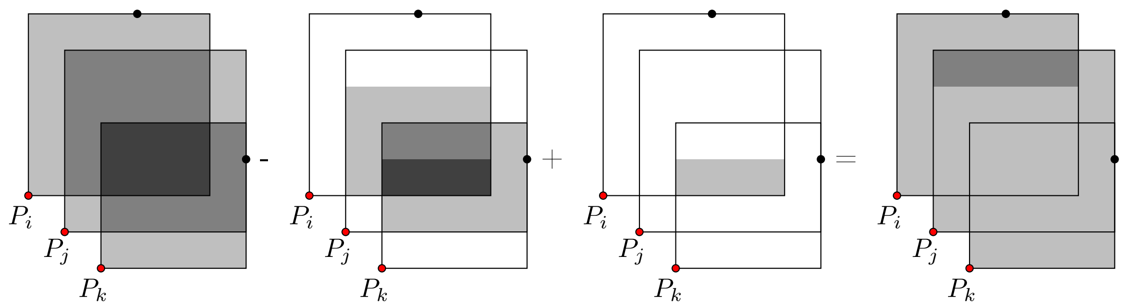


Figure 5.6: Illustration of the right-hand side of Equation 5.2.

Choosing $P \in [0,1]^2$ uniformly at random and taking the expectation on both sides yields:

$$|A_1 \cup \ldots \cup A_n| \leq \sum_{i \in [n]} |A_i| - \sum_{i<j \in [n]} |B_{i,j}| + \sum_{i<j<k \in [n]} |B_{i,k} \cap A_j|. \tag{5.2}$$

Figure 5.6 demonstrates the upper bound for the union $A_i \cup A_j \cup A_k$. We now apply Theorem 5.9 to each summand of the right-hand side of (5.2). We first calculate the expected total of all squares A_i. By Example 5.10 we have

$$\lim_{n\to\infty} \mathbb{E}\Big(\sum_i |A_i|\Big) = 1.$$

For the next term of (5.2) we again use Theorem 5.9. This time we let $k = 2$ and $q(P_i, P_j; \nu) = |B_{i,j}|$ with radius $R(P_i, \nu) = v_i\sqrt{2}$. Then the conditions of Theorem 5.9 hold and we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}(\sum_{i<j} |B_{i,j}|) \\
&= \int_0^\infty \int_0^v \int_0^s 2v \exp\left(-v^2 - st\right) \cdot s(s-t)\, dt\, ds\, dv \\
&= \frac{1-\log 2}{2}.
\end{aligned}$$

The meaning of s and t can be seen in Figure 5.5a. Finally, using another application of Theorem 5.9 we evaluate

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}(\sum_{i<j<k} |B_{i,k} \cap A_j|) \\
&= \int_0^\infty \int_0^v \int_0^{s_1} \int_0^{s_2} \int_0^{t_2} 2vs_2(s_2-t_2) \exp\left(-v^2 - s_1t_1 - s_2(t_2-t_1)\right)\, dt_1\, dt_2\, ds_2\, ds_1\, dv \\
&= \frac{3}{4} - \log(2) + \frac{\log^2(2)}{8} - \frac{\pi^2}{96}.
\end{aligned}$$

See Figure 5.5b for an illustration. The result now follows from (5.2). □

We have provided an upper bound for the area covered by lower-left-anchored squares. We now provide a lower bound using a greedy algorithm.

Algorithm 5.17. We define a greedy algorithm as follows:

1. Determine for every point the distance to the nearest-neighbour, $\mathrm{d_{nn}}$.
2. Sort points by $\mathrm{d_{nn}}$ in decreasing order.
3. In this order, each point chooses the maximum square which does not intersect with any previous squares.

Algorithm 5.18. We define another algorithm, which is easier to analyse but strictly worse than Algorithm 5.17:

1. Determine for every point the distance to the nearest-neighbour, $\mathrm{d_{nn}}$.
2. Every point does the following: It assumes that all other points with larger value of $\mathrm{d_{nn}}$ did indeed get a square of size $\mathrm{d_{nn}}$. Then it chooses the largest square that does not intersect any of them.

The result of Algorithm 5.18 is a valid covering, because every chosen square is at most as large as the corresponding square, which Algorithm 5.17 would have chosen for the corresponding point.

Theorem 5.19. *We have* $\lim_{n\to\infty} \mathbb{E}(\textit{Area covered by Algorithm 5.18}) > 0.56$.

Proof. Suppose that u is a point, which does not get its nearest-neighbour square. Then there is at least one other point v, which we call a *disturber*, whose nearest-neighbour square is larger and both squares intersect. The point v could be either below or to the left of u. Let

$$D_b(u) := \{v : \mathrm{d_{nn}}(v) > \mathrm{d_{nn}}(u) \text{ and } v_x \in [u_x, u_x + \mathrm{d_{nn}}(u)] \text{ and } v_y \in [u_y - \mathrm{d_{nn}}(v), u_y]\}$$

be the set of disturbers from the bottom and similarly

$$D_\ell(u) := \{v : \mathrm{d_{nn}}(v) > \mathrm{d_{nn}}(u) \text{ and } v_x \in [u_x - \mathrm{d_{nn}}(v), u_x] \text{ and } v_y \in [u_y, u_y + \mathrm{d_{nn}}(u)]\}$$

the set of disturbers from the left. For $v \in D_b(u)$ we define the *disturbance distance* $d(u,v) := v_x - u_x$. Similarly define $d(u,v) := v_y - u_y$ for $v \in D_\ell(u)$. Further, for $v \in D_b(u) \cup D_\ell(u)$ we define $S(u,v) := \mathrm{d_{nn}}(u)^2 - d(u,v)^2$ as the area that u does not get, because it is disturbed by v. Finally let $\mathrm{cd}(u)$ be the *closest disturber*, i.e., the point $v \in D_b(u) \cup D_\ell(u)$, which minimises $d(u,v)$. Note that $\mathrm{cd}(u)$ is the only disturber that actually matters for the point u and hence the total area achieved by this algorithm is exactly equal to

$$T := \sum_{u\in\mathcal{P}} \mathrm{d_{nn}}(u)^2 - \sum_{\substack{u\in\mathcal{P}:\\ u \text{ is disturbed}}} S(u, \mathrm{cd}(u)).$$

Now, we can bound this from below by

$$T \geq \sum_{u\in\mathcal{P}} \mathrm{d_{nn}}(u)^2 - \sum_{\substack{u\in\mathcal{P}:\\ u \text{ is disturbed}}} \Bigg(\sum_{v\in D_b(u)\cup D_\ell(u)} S(u,v) - \sum_{v\in D'_b(u)} S(u,v) - \sum_{v\in D'_\ell(u)} S(u,v) \Bigg),$$

where D'_b and D'_ℓ are any sets with $D'_b(u) \subset D_b(u) \setminus \{\mathrm{cd}(u)\}$ and $D'_\ell(u) \subset D_\ell(u) \setminus \{\mathrm{cd}(u)\}$. We define D'_b shortly. By symmetry, we find

$$T \geq \sum_{u \in \mathcal{P}} \mathrm{d_{nn}}(u)^2 - 2 \cdot \sum_{\substack{u \in \mathcal{P}: \\ u \text{ is disturbed}}} \Bigg(\sum_{v \in D_b(u)} S(u,v) - \sum_{v \in D'_b(u)} S(u,v) \Bigg).$$

We partition $D_b(u)$ into $A(u) \cup B(u)$ with $A(u) := \{v \in D_b(u) : v_y > u_y - \mathrm{d_{nn}}(u)\}$ and $B(u) := \{v \in D_b(u) : v_y < u_y - \mathrm{d_{nn}}(u)\}$. Also, we define

$$\begin{aligned}
D'_b(u) &:= A_1(u) \cup B_1(u) \cup A_2(u) \cup B_2(u), \text{where} \\
A_1(u) &:= \{v \in A(u) : \exists w \in D_b(u) \cap ([u_x, v_x] \times [v_y, u_y]) \text{ s.t. } \mathcal{P} \cap ((w_x, v_x] \times [v_y, u_y]) = \varnothing\} \\
B_1(u) &:= \{v \in B(u) : \exists w \in D_b(u) \cap ([u_x, v_x] \times [u_y - \mathrm{d_{nn}}(u), u_y]) \text{ s.t.} \\
&\qquad \mathcal{P} \cap ([w_x, v_x] \times [u_y - \mathrm{d_{nn}}(u), u_y]) = \varnothing\} \\
A_2(u) &:= \{v \in A(u) : \mathcal{P} \cap ([u_x, v_x] \times [v_y, u_y]) = \varnothing \\
&\quad \text{and } \exists w \in D_\ell(u) \cap ([u_x - \mathrm{d_{nn}}(u), u_x] \times [u_y, u_y + (v_x - u_x)]) \text{ s.t.} \\
&\qquad \mathcal{P} \cap ((w_x, u_x] \times [u_y, u_y + (v_x - u_x)]) = \varnothing\} \\
B_2(u) &:= \{v \in B(u) : \mathcal{P} \cap ([u_x, v_x] \times [u_y - \mathrm{d_{nn}}(u), u_y]) = \varnothing \\
&\quad \text{and } \exists w \in D_\ell(u) \cap ([u_x - \mathrm{d_{nn}}(u), u_x] \times [u_y, u_y + (v_x - u_x)]) \text{ s.t.} \\
&\qquad \mathcal{P} \cap ((w_x, u_x] \times [u_y, u_y + (v_x - u_x)]) = \varnothing\}.
\end{aligned}$$

Now we calculate each term separately. Each time we use Theorem 5.9. As in Example 5.10 we have

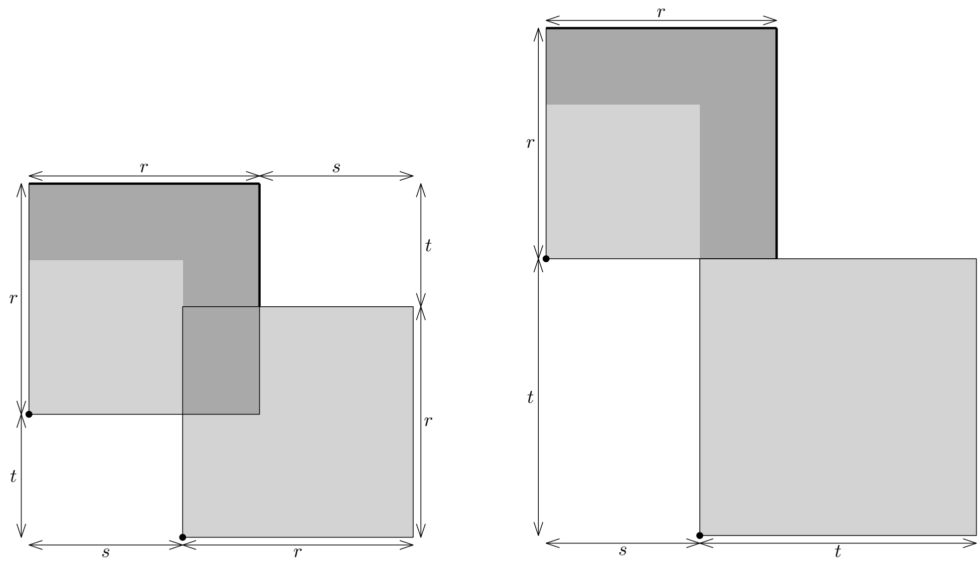
$$\lim_{n\to\infty} \mathbb{E}(\sum_{u \in \mathcal{P}} \mathrm{d_{nn}}(u)^2) = 1.$$

Now we consider the case $v \in A(u)$. See Figure 5.7a for an illustration.

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}(\sum_{\substack{u \in \mathcal{P}: \\ u \text{ is disturbed}}} \sum_{v \in A(u)} S(u,v)) \\
&= \int_0^\infty \int_{0 \le s,t \le r} \left(r^2 - s^2\right)(r+t) e^{-(2r^2 - (r-s)(r-t))} \, d(s,t)\, dr \\
&= \frac{1}{8}(\log(16) - 1) \\
&\approx 0.221574.
\end{aligned}$$

For the case $v \in B(u)$, as illustrated in Figure 5.7b, we evaluate:

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}(\sum_{\substack{u \in \mathcal{P}: \\ u \text{ is disturbed}}} \sum_{v \in B(u)} S(u,v)) \\
&= \int_0^\infty \int_{0 \le s \le r \le t} \left(r^2 - s^2\right)(r+r) e^{-(r^2+t^2)} \, d(r,s)\, dt \\
&= \frac{\pi}{8} - \frac{1}{3} \\
&\approx 0.0593657.
\end{aligned}$$



(a) The case $v \in A(u)$. (b) The case $v \in B(u)$.

Figure 5.7: Disturbers in $A(u)$ and $B(u)$.

For the case $v \in A_1(u)$, as illustrated in Figure 5.8, we evaluate:

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}(\sum_{\substack{u\in\mathcal{P}: \\ u \text{ is disturbed}}} \sum_{v\in A_1(u)} S(u,v)) \\
&= \int_0^\infty \int_{\substack{0\le a\le t\le r \\ 0\le b\le s\le r}} \left(r^2 - s^2\right)(r+a)e^{-(r(s+t+r)-ab)}\, d(a,b,s,t)\, dr \\
&\approx 0.0303546.
\end{aligned}$$

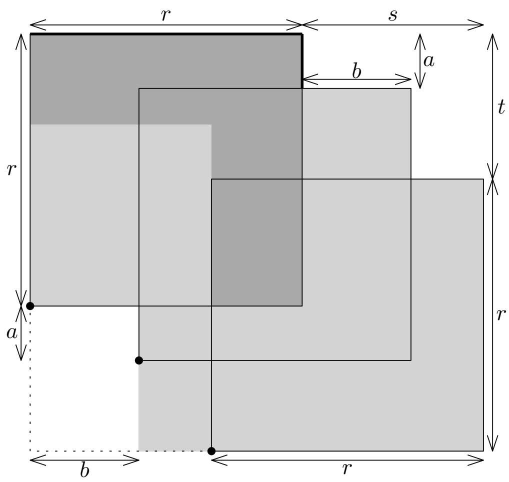


Figure 5.8: Disturbers in $A_1(u)$.

For the case $v \in A_2(u)$, as illustrated in Figure 5.9, we evaluate:

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\Big(\sum_{\substack{u\in\mathcal{P}:\\ u \text{ is disturbed}}} \sum_{v\in A_2(u)} S(u,v)\Big) \\
&= \int_0^\infty \int_{\substack{0\le a\le s\le r\\ 0\le b,t\le r}} \left(r^2 - s^2\right)(b+t)e^{-r(a+b+s+t+r)}\, d(a,b,s,t)\, dr \\
&\approx 0.0139693.
\end{aligned}$$

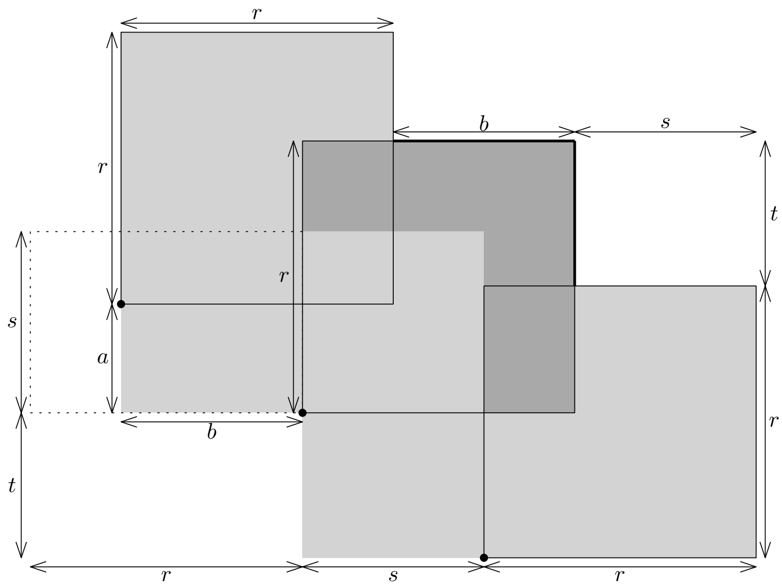


Figure 5.9: Disturbers in $A_2(u)$.

For the case $v \in B_1(u)$, as illustrated in Figure 5.10, we evaluate:

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\Big(\sum_{\substack{u\in\mathcal{P}:\\ u \text{ is disturbed}}} \sum_{v\in B_1(u)} S(u,v)\Big) \\
&= \int_0^\infty \int_{\substack{0\le b\le s\le r\le t\\ 0\le a\le r}} \left(r^2 - s^2\right)(r+a)e^{-(t^2+r(s+r)-ab)}\, d(a,b,r,s)\, dt \\
&\approx 0.0109992.
\end{aligned}$$

For the case $v \in B_2(u)$, as illustrated in Figure 5.11, we evaluate:

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\Big(\sum_{\substack{u\in\mathcal{P}:\\ u \text{ is disturbed}}} \sum_{v\in B_2(u)} S(u,v)\Big) \\
&= \int_0^\infty \int_{\substack{0\le a\le s\le r\le t\\ 0\le b\le r}} \left(r^2 - s^2\right)(b+r)e^{-(t^2+r(a+b+r+s))}\, d(a,b,r,s)\, dt \\
&\approx 0.00404983.
\end{aligned}$$

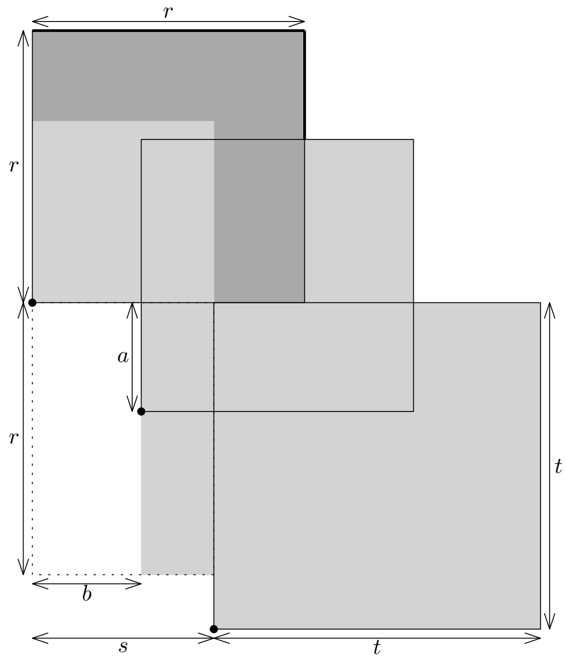


Figure 5.10: Disturbers in $B_1(u)$.

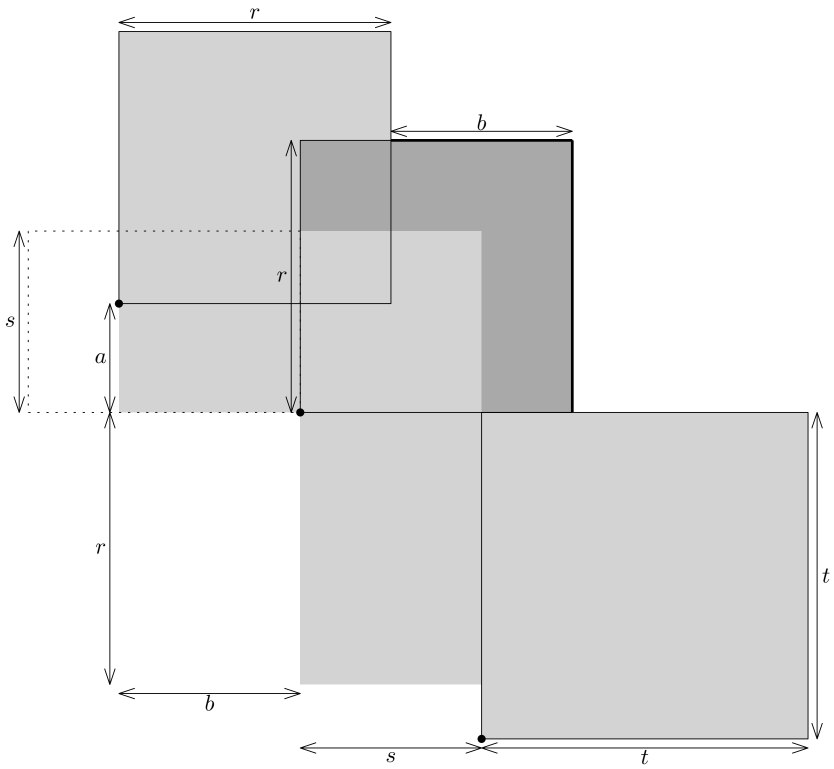


Figure 5.11: Disturbers in $B_2(u)$.

Now by substituting the results, we get:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}(\text{gre}) &\approx 1 - 2 \cdot \big(0.221574 + 0.0593657 - 0.0303546 \\ &\qquad - 0.0139693 - 0.0109992 - 0.0035581\big) \\ &\approx 0.563514.\end{aligned}$$

□

However, we believe that the performance of above greedy algorithm is better than what we proved and conjecture the following.

Conjecture 5.20. *Algorithm 5.18 covers an expected area of* 0.58. *Algorithm 5.17 covers an expected area of* 0.6.

5.5 Center-anchored discs

In this section we consider the following type of packings. Given a finite set of points in the unit square $\mathcal{P} = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ we consider partial coverings of $[0, 1]$ with n disjoint discs, each centered at a different point in $\mathcal{P}$. For a point $u \in \mathcal{P}$ we call the disc center-anchored at u with radius $\mathrm{d}_{\mathrm{nn}}(u)$ the *nearest neighbour disc at* u. Now we calculate the total area of all nearest neighbour discs, including overlapping areas.

Theorem 5.21. *We have*

$$\lim_{n\to\infty} \mathbb{E}(\textit{Total area of nearest neighbour discs}) = 1.$$

We can deduce a simple lower bound:

$$\lim_{n\to\infty} \mathbb{E}(\text{opt}) \geq \frac{1}{4}.$$

Proof. By a calculation similar to the one from Example 5.10 we have that the total area of all nearest neighbour discs is

$$\int_0^\infty 2\pi z \cdot \exp\left(-\pi z^2\right) \cdot \pi z^2 \, dz = 1.$$

Now, for the lower bound we simply half all radii: For every point u, we choose the disc with radius $\mathrm{d}_{\mathrm{nn}}(u)/2$. These discs clearly are disjoint and thus form a valid packing. Also, since every disc has area $\frac{1}{4}$ of the corresponding nearest neighbour disc, the total area is equal to $\frac{1}{4} \cdot 1 = \frac{1}{4}$. □

Next we prove an upper bound. The intuition of the argument is as follows. As mentioned after Definition 5.11, a significant proportion of points are twins. In the limit the proportion of twins tends towards $6\pi/(3\sqrt{3} + 8\pi)$. These twins contribute a significant area to the total of 1 from Theorem 5.21, although their circles are small. However, not all twins can get their whole nearest neighbour disc in a valid packing, and thus we cannot reach the total of 1.

Lemma 5.22. *We have* $\lim_{n\to\infty} \mathbb{E}(\textit{sum of nearest neighbour discs of twins}) = \theta^2$.

Proof. We again use Theorem 5.9. If u and v are twins with $\mathrm{d_{nn}}(u) = \mathrm{d_{nn}}(v) = z$, let $q(u, v; \nu) = \pi z^2$ and 0 otherwise. Let $R(u, \nu) = 2 \cdot \mathrm{d_{nn}}(u)$. Then the assumptions in Theorem 5.9 hold and we evaluate

$$\int_0^\infty 2\pi z \exp\left(-\pi z^2 - \frac{3\sqrt{3}+2\pi}{6} z^2\right) \cdot \pi z^2 \, dz = \left(\frac{6\pi}{3\sqrt{3}+8\pi}\right)^2. \qquad \square$$

See Figure 5.12 for an illustration.

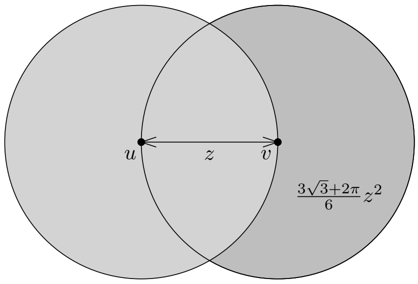


Figure 5.12: No other point may be inside the gray region.

We can now deduce an upper bound as follows:

Theorem 5.23. *We have* $\lim_{n\to\infty} \mathbb{E}(\mathrm{opt}) \le 1 - \frac{1}{2}\theta^2 < 0.81$.

Proof. We know from Theorem 5.21 that the total area of all nearest neighbour discs is 1. By Lemma 5.22 we know that the twins contribute θ^2 while the non-twins contribute $1 - \theta^2$ to this sum. As nearest-neighbour discs are as large as possible $1 - \theta^2$ is an upper bound of the total area of disjoint discs anchored at non-twins. We now argue that $\frac{1}{2}\theta^2$ is an upper bound of the total area of disjoint discs anchored at twins, giving a total upper bound of $1 - \frac{1}{2}\theta^2$.

If u and v are twins with $\mathrm{d_{nn}}(u) = \mathrm{d_{nn}}(v) = z$, the maximum sum of areas of their two discs is

$$\max\{\pi r^2 + \pi(z - r)^2 : 0 \le r \le z\} = \pi z^2.$$

In other words, one disc is as large as possible, while the other has radius 0, effectively halving the area contributed by twins. $\square$

We conclude this section with a conjecture.

Conjecture 5.24. *We define the following simple greedy algorithm. Repeat the following step n times: Choose the largest disc which does not intersect any previously chosen discs. Then we have*

$$\lim_{n\to\infty} \mathbb{E}(\textit{Area covered by this algorithm}) = 0.5.$$

5.6 Center-anchored squares

Most of the proofs in this section are analogous to the proofs in Section 5.5. Again, $\mathrm{d_{nn}}$ is the distance to the nearest neighbour – now with respect to the maximum norm. Nearest neighbour squares are the analogue of nearest neighbour discs.

Theorem 5.25. *We have*

$$\lim_{n\to\infty} \mathbb{E}(\textit{Total area of nearest neighbour sqaures}) = 1.$$

We can deduce a simple lower bound:

$$\lim_{n\to\infty} \mathbb{E}(\text{opt}) \geq \frac{1}{4}.$$

Proof. Analogously to Theorem 5.21 we calculate the expected total area of all nearest-neighbour squares is

$$\int_0^\infty 8z\ \exp\left(-4z^2\right) \cdot 4z^2\, dz = 1,$$

where z is half the side-length of the square. A lower bound is therefore

$$\int_0^\infty 8z \exp\left(-4z^2\right) \cdot z^2\, dz = \frac{1}{4}.$$

□

We now prove an upper bound for the optimum area covered.

Theorem 5.26. *We have* $\lim_{n\to\infty} \mathbb{E}(\textit{optimal covering}) \leq \frac{17}{21} < 0.81$.

Proof. The expected total area of all nearest-neighbour squares centered at twins is

$$\begin{aligned} &8 \cdot \int_0^\infty \int_0^z \exp\left(-4z^2 - 4z^2 + z(y+z)\right) \cdot 4z^2\, dy\, dz \\ &= 8 \cdot \int_0^\infty \frac{e^{-6z^2} - e^{-7z^2}}{z} \cdot 4z^2\, dz \\ &= \frac{8}{21}. \end{aligned}$$

See Figure 5.13 for an illustration. Therefore, the expected total area of non-twins is

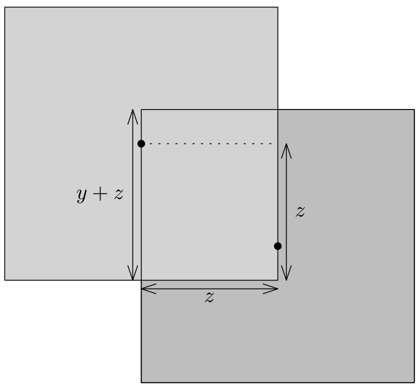


Figure 5.13: No other point may be inside the gray region.

$1 - \frac{8}{21} = \frac{13}{21}$. Analogous to the proof of Theorem 5.23, we therefore have an upper bound of $\frac{1}{2} \cdot \frac{8}{21} + \frac{13}{21} = \frac{17}{21}$. □

For center-anchored squares we also conjecture that the greedy algorithm covers an expected area of 0.5.

Conjecture 5.27. *We define the following simple greedy algorithm. Repeat the following step n times: Choose the largest square which does not intersect any previously chosen squares. Then we have*

$$\lim_{n\to\infty} \mathbb{E}(\textit{Area covered by this algorithm}) = 0.5.$$

5.7 Proofs

This section contains more technical proofs of results referred to in other sections.

Lemma 5.28 (Multivariate beta function)**.** *For any $m \geq 1$, $\ell_1, \dots, \ell_m \geq 0$ and fixed $0 \leq z_0 \leq z_m \leq 1$ we have*

$$\int_{z_0<z_1<\cdots<z_{m-1}<z_m} \prod_{i=1}^{m} \frac{(z_{i+1}-z_i)^{\ell_i}}{\ell_i!} d(z_1, \dots, dz_{m-1}) = \frac{(z_m - z_0)^{m-1+\sum_{i=1}^m \ell_i}}{(m-1+\sum_{i=1}^m \ell_i)!}.$$

Lemma 5.29. *For integers $0 \leq m < n$ we have*

$$\int_{0<a<b<1} \frac{(1-(b-a))^{n-1-m}}{(n-1-m)!} \frac{(b-a)^m}{m!} d(a,b) = \frac{n-m}{(n+1)!}.$$

Proof.

$$\begin{aligned}
&\int_{0<a<b<1} \frac{(1-(b-a))^{n-1-m}}{(n-1-m)!} \frac{(b-a)^m}{m!} d(a,b) && (5.3)\\
&= \int_{0<a<b<1} \sum_{p=0}^{n-1-m} \frac{(a-0)^p}{p!} \frac{(b-a)^m}{m!} \frac{(1-b)^{n-1-m-p}}{(n-1-m-p)!} d(a,b) && (5.4)\\
&= \sum_{p=0}^{n-1-m} \int_{0<a<b<1} \frac{(a-0)^p}{p!} \frac{(b-a)^m}{m!} \frac{(1-b)^{n-1-m-p}}{(n-1-m-p)!} d(a,b) && (5.5)\\
&= \sum_{p=0}^{n-1-m} \frac{1}{(n+1)!} && (5.6)\\
&= \frac{n-m}{(n+1)!},
\end{aligned}$$

where from (5.3) to (5.4) we have used the binomial theorem, from (5.4) to (5.5) we have used Tonelli's theorem. Finally (5.6) follows from Lemma 5.28. □

Lemma 5.30. *We have*

$$\begin{aligned}
&\lim_{n\to\infty} (n)_{k+\sum_{i=2}^k \ell_i} \int_{0<y_{k+1}<y_k<\cdots<y_1<1} (1-(y_1-y_{k+1}))^{n-(k+\sum_{i=2}^k \ell_i)-1}\\
&\quad \cdot (y_q - y_{q+1}) \prod_{i=2}^{k} \frac{(y_{i-1}-y_i)^{\ell_i}}{\ell_i!} d(y_1, \dots, y_{k+1}) && (5.7)\\
&= \ell_{q+1} + 1. && (5.8)
\end{aligned}$$

Proof. We calculate

$$\begin{aligned}
&\lim_{n\to\infty}(n)_{k+\sum_{i=2}^k \ell_i}\int_{0<y_{k+1}<y_k<\cdots<y_1<1}(1-(y_1-y_{k+1}))^{n-(k+\sum_{i=2}^k \ell_i)-1}\\
&\qquad\cdot(y_q-y_{q+1})\prod_{i=2}^{k}\frac{(y_{i-1}-y_i)^{\ell_i}}{\ell_i!}\,d(y_1,\dots,y_{k+1}) &(5.9)\\
&=\lim_{n\to\infty}(n)_{k+\sum_{i=2}^k \ell_i}\int_{0<y_{k+1}<y_1<1}(1-(y_1-y_{k+1}))^{n-(k+\sum_{i=2}^k \ell_i)-1}(\ell_{q+1}+1)\\
&\quad\cdot\int_{y_{k+1}<y_k<\cdots<y_1}\frac{(y_q-y_{q+1})^{\ell_{q+1}+1}}{(\ell_{q+1}+1)!}\prod_{i=2}^{q}\frac{(y_{i-1}-y_i)^{\ell_i}}{\ell_i!}\prod_{i=q+2}^{k}\frac{(y_{i-1}-y_i)^{\ell_i}}{\ell_i!}\,d(y_2,\dots,y_k)\\
&\qquad\qquad d(y_1,y_{k+1}) &(5.10)\\
&=\lim_{n\to\infty}(n)_{k+\sum_{i=2}^k \ell_i}\int_{0<y_{k+1}<y_1<1}(1-(y_1-y_{k+1}))^{n-(k+\sum_{i=2}^k \ell_i)-1}(\ell_{q+1}+1)\\
&\quad\frac{(y_1-y_{k+1})^{k+\sum_{i=2}^k \ell_i}}{(k+\sum_{i=2}^k \ell_i)!}\,d(y_1,y_{k+1}) &(5.11)\\
&=(\ell_{q+1}+1)\lim_{n\to\infty}(n)_{k+\sum_{i=2}^k \ell_i}\frac{(n-(k+\sum_{i=2}^k \ell_i))!}{(n+1)!} &(5.12)\\
&=(\ell_{q+1}+1)\lim_{n\to\infty}\frac{n-(k+\sum_{i=2}^k \ell_i)}{n+1} &(5.13)\\
&=\ell_{q+1}+1, &(5.14)
\end{aligned}$$

where from (5.10) to (5.11) we have used Lemma 5.28 and from (5.11) to (5.12) we have used Lemma 5.29. □

Lemma 5.31. *For $k\geq 1$ define*

$$A(k,x_k):=\int_{0<x_1<x_2<\ldots<x_k}\sum_{\ell_2,\dots,\ell_k\geq 0}x_1\prod_{j=2}^{k}(1-x_j)^{\ell_j}\,d(x_1,\dots,x_{k-1}).$$

For all $k\geq 1$ we have

$$A(k,x)=2^{1-k}x.$$

Proof. By induction on k. For $k=1$ we simply have $A(1,x)=x$. For $k\geq 2$ we can use the following recurrence:

$$\begin{aligned}
A(k,x_k)=&\int_0^{x_k}\left(\sum_{\ell_k=0}^{\infty}A(k-1,x_{k-1})(1-x_k)^{\ell_k}\right)dx_{k-1}\\
=&\frac{1}{x_k}\int_0^{x_k}A(k-1,x_{k-1})\,dx_{k-1}\\
=&\frac{1}{x_k}\int_0^{x_k}2^{2-k}x_{k-1}\,dx_{k-1}\\
=&2^{1-k}x_k.
\end{aligned}$$

□

Lemma 5.32. *For* $1 \le p \le k$ *define*

$$B(k, x_k, p) := \int_{0<x_1<x_2<\ldots<x_k} \sum_{\ell_2,\ldots,\ell_k \ge 0} x_1 \prod_{j=2}^{k} (1-x_j)^{\ell_j} x_p \, d(x_1, \ldots, x_{k-1}).$$

For all $k \ge 1, p \in [k]$ *we have*

$$B(k, x, p) = 2^{1-p} 3^{p-k} x^2.$$

Proof. Proof by induction on k. For $p = k$ we have

$$B(k, x, k) = A(k, x)x = 2^{1-k} x^2 = 2^{1-k} 3^{k-k} x^2.$$

Note that this includes the base case $k = 1$. For $p \le k-1$ we have

$$\begin{aligned} B(k, x, p) &= \int_0^x \left(\sum_{\ell_k=0}^{\infty} B(k-1, x_{k-1}, p)(1-x)^{\ell_k} \right) dx_{k-1} \\ &= \frac{1}{x} \int_0^x B(k-1, x_{k-1}, p) dx_{k-1} \\ &= \frac{1}{x} \int_0^x 2^{1-p} 3^{p-k+1} x_{k-1}^2 dx_{k-1} \\ &= 2^{1-p} 3^{p-k} x^2. \end{aligned}$$

□

Lemma 5.33. *For* $1 \le p \le q+1 \le k$ *and* $1 \le q$ *define*

$$C(k, x_k, p, q) := \int_{0<x_1<x_2<\ldots<x_k} \sum_{\ell_2,\ldots,\ell_k \ge 0} x_1 \prod_{j=2}^{k} (1-x_j)^{\ell_j} x_p (\ell_{q+1}+1) \, d(x_1, \ldots, x_{k-1}).$$

We have

$$C(k, x, p, q) = 2^{-k-p+q+2} 3^{-1+p-q} x.$$

Proof. Proof by induction on k. If $p = k, q = k-1$ then

$$\begin{aligned} C(k, x_k, k, k-1) &= \int_{0<x_1<x_2<\ldots<x_k} \sum_{\ell_2,\ldots,\ell_k \ge 0} x_1 \prod_{j=2}^{k} (1-x_j)^{\ell_j} x_k (\ell_k+1) \, d(x_1, \ldots, x_{k-1}) \\ &= \int_{0<x_{k-1}<x_k} \sum_{l_k=0}^{\infty} (1-x_k)^{\ell_k} x_k (\ell_k+1) A(k-1, x_{k-1}) \, dx_{k-1} \\ &= \frac{1}{x_k} \int_{0<x_{k-1}<x_k} 2^{2-k} x_{k-1} \, dx_{k-1} \\ &= 2^{1-k} x_k. \end{aligned}$$

If $p < k, q = k-1$ then

$$C(k, x_k, p, q) = \int_{0<x_1<x_2<\ldots<x_k} \sum_{\ell_2,\ldots,\ell_k \ge 0} x_1 \prod_{j=2}^{k} (1-x_j)^{\ell_j} x_p (\ell_{q+1}+1) \, d(x_1, \ldots, x_{k-1})$$

$$\begin{aligned}
&= \int_{0<x_{k-1}<x_k} \sum_{l_k=0}^{\infty} (1-x_k)^{\ell_k}(\ell_k+1)B(k-1,x_{k-1},p)\,dx_{k-1} \\
&= \frac{1}{x_k^2}\int_{0<x_{k-1}<x_k} 2^{1-p}3^{p-k+1}x_{k-1}^2\,dx_{k-1} \\
&= 2^{1-p}3^{p-k}x_k.
\end{aligned}$$

Note that the previous two cases include the induction base case $k=2$. If $q \le k-2$ then

$$\begin{aligned}
C(k,x_k,p,q) &= \int_{0<x_1<x_2<\ldots<x_k} \sum_{\ell_2,\ldots,\ell_k\ge 0} x_1 \prod_{j=2}^{k}(1-x_j)^{\ell_j} x_p(\ell_{q+1}+1)\,d(x_1,\ldots,x_{k-1}) \\
&= \int_{0<x_{k-1}<x_k} \sum_{l_k=0}^{\infty}(1-x_k)^{\ell_k} C(k-1,x_{k-1},p,q)\,dx_{k-1} \\
&= \frac{1}{x_k}\int_{0<x_{k-1}<x_k} 2^{-k-p+q+3}3^{-1+p-q}x_{k-1}\,dx_{k-1} \\
&= 2^{-k-p+q+2}3^{-1+p-q}x_k.
\end{aligned}$$

□

Lemma 5.34. *For $1 \le p \le q \le k$ we define:*

$$D(k,p,q) := \int_{0<x_1<x_2<\cdots<x_k<1} \sum_{\ell_2,\ldots,\ell_k\ge 0} x_1 \left(\prod_{i=2}^{k}(1-x_i)^{\ell_i}\right)(x_{p+1}-x_p)(\ell_{q+1}+1)\,d(x),$$

where $x_{k+1} := 1, \ell_{k+1} := 0$. We have

$$D(k,p,q) = 2^{q-p-k}3^{p-q-1}.$$

Proof. If $q=k, p=k$ then we have

$$\begin{aligned}
D(k,p,q) &= \int_0^1 (A(k,x_k)-B(k,x_k,k))\,dx_k \\
&= \int_0^1 (2^{1-k}x_k - 2^{1-k}x_k^2)\,dx_k \\
&= 2^{-k}3^{-1}.
\end{aligned}$$

If $q=k, p<k$ then

$$\begin{aligned}
D(k,p,q) &= \int_0^1 (B(k,x_k,p+1)-B(k,x_k,p))\,dx_k \\
&= \int_0^1 (2^{-p}3^{p-k+1} - 2^{1-p}3^{p-k})x_k^2\,dx_k \\
&= 2^{-p}3^{p-k-1}.
\end{aligned}$$

If $q<k$ we have

$$D(k,p,q) = \int_0^1 (C(k,x_k,p+1,q)-C(k,x_k,p,q))\,dx_k$$

$$= \int_0^1 (2^{-k-p+q+1}3^{p-q} - 2^{-k-p+q+2}3^{-1+p-q}) x_k \, dx_k$$
$$= 2^{-k-p+q}3^{p-q-1}. \qquad \square$$

Theorem 5.35. *For $1 \le p \le q \le k$ we have*

$$E(p,q) := \lim_{n\to\infty} (n)_{k+1} \int_{\substack{0<x_0<x_1<\ldots<x_k<1\\ 0<y_{k+1}<y_k<\ldots<y_1<1}} (x_{p+1}-x_p)(y_q-y_{q+1})(1-A)^{n-k-1}\, d(x,y)$$
$$= 2^{q-p-k}3^{p-q-1},$$

where $A = \sum_{i=1}^k x_{i+1}(y_i - y_{i+1})$ and $x_{k+1} := 1$.

Proof. We calculate

$$\lim_{n\to\infty} (n)_{k+1} \int_{\substack{0<x_0<x_1<\ldots<x_k<1\\ 0<y_{k+1}<y_k<\ldots<y_1<1}} (x_{p+1}-x_p)(y_q-y_{q+1})\,(1-A)^{n-k-1}\; d(x,y) \tag{5.15}$$

$$= \lim_{n\to\infty} \int_{\substack{0<x_1<\ldots<x_k<1\\ 0<y_{k+1}<y_k<\ldots<y_1<1}} x_1(x_{p+1}-x_p)(y_q-y_{q+1})(n)_{k+1}$$
$$\sum_{\ell_2,\ldots,\ell_k\ge 0} \binom{n-k-1}{l_2,\ldots,l_k,n-k-1-\sum_i l_i} (1-(y_1-y_{k+1}))^{n-k-1-\sum_i l_i}$$
$$\prod_{i=2}^k ((1-x_i)(y_{i-1}-y_i))^{l_i}\, d(x,y) \tag{5.16}$$

$$= \lim_{n\to\infty} \int_{0<x_1<x_2<\ldots<x_k<1} \sum_{\ell_2,\ldots,\ell_k\ge 0} x_1 \int_{0<y_{k+1}<y_k<\ldots<y_2<y_1<1} (x_{p+1}-x_p)(y_q-y_{q+1})(n)_{k+1}$$
$$\binom{n-k-1}{l_2,\ldots,l_k,n-k-1-\sum_i l_i} (1-(y_1-y_{k+1}))^{n-k-1-\sum_i l_i}$$
$$\prod_{i=2}^k ((1-x_i)(y_{i-1}-y_i))^{l_i}\, d(x,y) \tag{5.17}$$

$$= \lim_{n\to\infty} \int_{0<x_1<x_2<\ldots<x_k<1} \sum_{\ell_2,\ldots,\ell_k\ge 0} x_1 \left(\prod_{i=2}^k (1-x_i)^{\ell_i}\right)(x_{p+1}-x_p)$$
$$(n)_{k+\sum_i l_i} \int_{0<y_{k+1}<y_k<\ldots<y_2<y_1<1} (y_q-y_{q+1})(1-(y_1-y_{k+1}))^{n-k-1-\sum_i l_i}$$
$$\prod_{i=2}^k \frac{(y_{i-1}-y_i)^{l_i}}{l_i!}\, d(x,y) \tag{5.18}$$

$$= \int_{0<x_1<x_2<\ldots<x_k<1} \sum_{\ell_2,\ldots,\ell_k\ge 0} x_1 \left(\prod_{i=2}^k (1-x_i)^{\ell_i}\right)(x_{p+1}-x_p)$$
$$\lim_{n\to\infty} (n)_{k+\sum_i l_i} \int_{0<y_{k+1}<y_k<\ldots<y_2<y_1<1} (y_q-y_{q+1})(1-(y_1-y_{k+1}))^{n-k-1-\sum_i l_i}$$
$$\prod_{i=2}^k \frac{(y_{i-1}-y_i)^{l_i}}{l_i!}\, d(x,y) \tag{5.19}$$

$$= \int_{0<x_1<x_2<\ldots<x_k<1} \sum_{\ell_2,\ldots,\ell_k \geq 0} x_1 \left(\prod_{i=2}^{k} (1-x_i)^{\ell_i} \right) (x_{p+1} - x_p)(l_{q+1}+1) \tag{5.20}$$
$$= 2^{q-p-k} 3^{p-q-1}. \tag{5.21}$$

From (5.15) to (5.16) we have used the multinomial theorem. Interchanging sums and integrals from (5.16) to (5.17) is allowed by Tonelli's theorem. Interchanging limit and sum from (5.18) to (5.19) is allowed by the dominated convergence theorem (summand $\leq \ell_{q+1}+1$). From (5.20) to (5.21) we use Lemma 5.34.

□

Theorem 5.36. *We have*

$$\lim_{n\to\infty} (n)_{k+1} \int_{\substack{0<x_0<x_1<\ldots<x_k<1 \\ 0<y_{k+1}<y_k<\ldots<y_1<1}} (x_{m+1} - x_1)(y_m - y_{k+1}) \cdot (1-A)^{n-k-1}\, d(x,y)$$
$$= \frac{(3^m - 2^m)(3^{k+1-m} - 2^{k+1-m})}{6^k}.$$

Proof. We calculate

$$\lim_{n\to\infty} (n)_{k+1} \int_{\substack{0<x_0<x_1<\ldots<x_k<1 \\ 0<y_{k+1}<y_k<\ldots<y_1<1}} (x_{m+1} - x_1)(y_m - y_{k+1}) \cdot (1-A)^{n-k-1}\, d(x,y)$$
$$= \sum_{p=1}^{m} \sum_{q=m}^{k} E(p,q)$$
$$= \sum_{p=1}^{m} \sum_{q=m}^{k} 2^{q-p-k} 3^{p-q-1}$$
$$= \frac{(3^m - 2^m)(3^{k+1-m} - 2^{k+1-m})}{6^k}.$$

□

5.8 Comments and open questions

For lower-left-anchored rectangles we have proven lower bounds and conjecture that the greedy algorithm achieves an expected area of about 0.77. It remains an open problem whether the whole square can be covered in the limit.

For lower-left-anchored squares we have proven a lower bound of 0.56 and an upper bound of 0.861 for the expected optimal area covered. An improvement of either bound would be very interesting.

For center-anchored discs and center-anchored squares we have proven a lower bound of 0.25 and an upper bound of 0.81 for the expected optimal area covered. In both cases we additionally conjecture, that the greedy algorithms described in Conjectures 5.24 and 5.27 achieve an area of 0.5 in the limit.

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