Topics in Abstract Order Geometry

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Contents

Introduction

Ordered geometry, also called order geometry, starts with the study of a set X with a ternary relation $\langle\cdot,\, \cdot,\, \cdot\rangle$ on X satisfying some axioms that support the interpretation that $\langle x,\, y,\, z\rangle$ iff y is between x and z . It also includes the interaction of such a ternary relation with other structures such as a topology or a metric. According to the preface of [**10**],

"The choice of a system of axioms is inherently arbirtrary, since there will be many equivalent systems. However, the purpose of an axiom system is not only to provide a basis for rigorous proof, but also to reveal the structure of a subject. [...] The development should seem natural, almost inevitable."

According to [**35**, section 1], where a reference to [**34**] is made,

"Unlike most concepts of elementary geometry, whose origin is shrouded in the ineluctable fog of all cultural beginnings, the history of the notion of betweenness can be traced back to one person, one book, and one year: Moritz Pasch, Vorlesungen über neuere Geometrie, and 1882."

On the other hand, according to [**47**, chapter I, §1], referring to [**34**], [**36**] and others, order geometry extends into incidence geometry and beyond:

"The propositions brought forward as axioms in this paper are stated in terms of a class of elements called "points" and a relation among points called "order"; they thus follow the trend of development inaugurated by Pasch,† and continued by Peano‡ rather than that of Hilbert§ or Pieri. ‖ All other geometrical concepts, such as line, plane, space, motion, are defined in terms of point and order. In particular, the congruence relations are made the subject of definitions** rather than of axioms. This is accomplished by the aid of projective geometry according to the method first given analytically by Cayley and Klein."

[**11**, 1.4] provides an example of the applicability of order geometry outside mathematics:

"We need to throw most of the specialized apparatus of Euclidean geometry overboard. Once we've stripped our geometry to a bare minimum, then we can go back and build up a different set of equipment that will be better suited to relativity. The stripped-down geometry we want is called ordered geometry, and was developed by Moritz Pasch around 1882."

The following examples of branches of order geometry, given in chronological order of their supposed foundation, shows that order geometry spans a broad range of topics in mathematics, with a lot of concepts and results available, some of them integrating continuous and discrete structures:

- classical order geometry: Here, the focus is on the order geometry of affine spaces, including normed real vector spaces, for instance the Hahn-Banach theorem: [**34**], [**36**], [**47**], [**13**], [**38**], [**48**, chapter I, §7 and chapter IV, §1], [**10**], [**35**] and others. It also includes the order geometry of spherical spaces and hyperbolic spaces.

- general order geometry: Here, the focus is on the order geometry of general interval spaces and geometric interval spaces: [**37**], [**6**], [**20**], [**49**], [**48**] and others. This is the "ultimately stripped-down order geometry" and can serve as the basis of all other branches, far remote from the above-cited "ineluctable fog of all cultural beginnings". General order geometry is embedded in the theory of algebraic closure spaces: In [**48**] and [**10**, chapter I], some topics, for instance the theorems of Helly, Caratheodory and Radon, are developed in this more general context.
- arboric interval spaces: [**43**], [**14**] [**9**] and others. Here, some of the axioms of classical order geometry are replaced by other axioms that are suited to the study of tree-like structures of various kinds.
- topological order geometry: [**44**], [**24**], [**23**], [**48**, chapter III], [**8**], [**28**], [**46**] and others. Here the betweenness relation interacts with a general topology. This includes generalizations of von Neumann's minimax theorem in game theory.
- modular and median interval spaces: [**3**], [**49**], [**48**], [**7**] and others. It is intermediate between general order geometry and arboric interval spaces.

In this introduction, for some of these branches of order geometry the main new results are introduced and some known results are listed. The focus is on characterizations by fundamental combinatorial concepts, then the concepts of a modular (median) interval space and an arboric interval space and, between these concepts, the new concept of a quadrimodular (quadrimedian) interval space. The new and the less familiar concepts used to formulate them are defined. Some examples and references are given. For proofs and more definitions, examples and references see the main text, including the chapter on preliminaries.

General interval spaces. Let

$$R = \langle \cdot,\, \cdot,\, \cdot \rangle$$

be a ternary relation on a set X , i.e. $R \subseteq X \times X \times X$ and

$$\langle x,\, y,\, z\rangle \text{ iff } (x,\, y,\, z) \in R\,.$$

The *sections* of $\langle \cdot,\, \cdot,\, \cdot \rangle$ are:

- for $a \in X$, the *binary sections*, defined as the following binary relations on X : the $(1,\, 2)$-section $\langle \cdot,\, \cdot,\, a\rangle$, the $(1,\, 3)$-section $\langle \cdot,\, a,\, \cdot\rangle$ and the $(2,\, 3)$-section $\langle a,\, \cdot,\, \cdot\rangle$ defined by

$$\begin{aligned}(u,\, v) \in \langle \cdot,\, \cdot,\, a\rangle &:\Leftrightarrow \langle u,\, v,\, a\rangle \ ,\\ (u,\, v) \in \langle \cdot,\, a,\, \cdot\rangle &:\Leftrightarrow \langle u,\, a,\, v\rangle \ ,\\ (u,\, v) \in \langle a,\, \cdot,\, \cdot\rangle &:\Leftrightarrow \langle a,\, u,\, v\rangle \ ,\end{aligned}$$

- for $a,\, b \in X$, the *unary sections*, defined as the following subsets of X : the 1-section $\langle \cdot,\, a,\, b\rangle$, the 2-section $\langle a,\, \cdot,\, b\rangle$ and the 3-section $\langle a,\, b,\, \cdot\rangle$ defined by

$$\begin{aligned}u \in \langle \cdot,\, a,\, b\rangle &:\Leftrightarrow \langle u,\, a,\, b\rangle \ .\\ u \in \langle a,\, \cdot,\, b\rangle &:\Leftrightarrow \langle a,\, u,\, b\rangle \ ,\\ u \in \langle a,\, b,\, \cdot\rangle &:\Leftrightarrow \langle a,\, b,\, u\rangle \ .\end{aligned}$$

The associated *strict, left-strict and right-strict relations* are the ternary relations $\langle \cdot \neq \cdot \neq \cdot \rangle$, $\langle \cdot \neq \cdot, \cdot \rangle$, $\langle \cdot, \cdot \neq \cdot \rangle$, on X defined by

$$\begin{aligned} \langle x \neq y \neq z\rangle &\text{iff } \langle x,\, y,\, z\rangle \text{ and } x \neq y \text{ and } y \neq z\,, \\ \langle x \neq y,\, z\rangle &\text{iff } \langle x,\, y,\, z\rangle \text{ and } x \neq y\,, \\ \langle x,\, y \neq z\rangle &\text{iff } \langle x,\, y,\, z\rangle \text{ and } y \neq z\,. \end{aligned}$$

Let X be a set. A *pseudointerval relation* on X is a ternary relation $\langle \cdot, \cdot, \cdot \rangle$ on X such that the following conditions are satisfied:

- For $a,\, x \in X$, $\langle x,\, x,\, a\rangle$ and $\langle a,\, x,\, x\rangle$.
- For $a,\, x,\, y \in X$, if $\langle x,\, a,\, y\rangle$, then $\langle y,\, a,\, x\rangle$.

A *pseudointerval space* is a pair consisting of a set X and a pseudointerval relation on X .

An *interval relation* on X is a pseudointerval relation $\langle \cdot, \cdot, \cdot \rangle$ on X such that the following additional condition is satisfied:

- For $x,\, y \in X$, $\langle x,\, y,\, x\rangle$ implies $y = x$.

An *interval space* is a pair consisting of a set X and an interval relation on X . For $a,\, b \in X$,

$$[a,\, b] := \{x \in X |\, \langle a,\, x,\, b\rangle\}\ .$$

The set $[a,\, b]$ is called the *interval* from a to b . An interval space X is called *geometric* iff the following conditions are satisfied:

- For $a \in X$, the binary relation $\langle a, \cdot, \cdot \rangle$ is transitive. This condition is the interval relation version of the strict interval relation condition [**34**, §1, IV. Grundsatz].
- For $a,\, b,\, x,\, y \in X$, $x,\, y \in [a,\, b]$ and $\langle a,\, x,\, y\rangle$ imply $\langle x,\, y,\, b\rangle$.

Let X be a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$. The ternary relation $\langle \cdot, \cdot, \cdot \rangle$ on X defined by

$$\langle x,\, y,\, z\rangle :\Leftrightarrow \text{There is a } \lambda \in K \text{ such that } 0 \leq \lambda \leq 1 \text{ and } y = x + \lambda\,(z - x)\ ,$$

is a geometric interval relation on X . It is called the *vector interval relation* of X . $(X,\, \langle \cdot, \cdot, \cdot \rangle)$ is called the *vector interval space* associated with X . For $a,\, b \in X$, $[a,\, b]$ is the straight-line segment from a to b . This example has been taken from [**48**, chapter I, 4.2].

Unless otherwise stated, interval space concepts, applied to a vector space, refer to its vector interval relation.

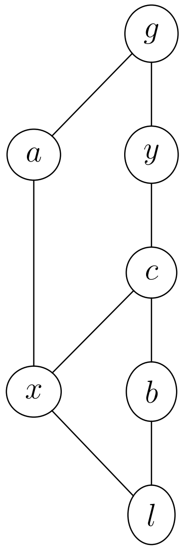
Let X be a lattice, for example $\mathbb{R}^2$ with the componentwise partial order. The ternary relation $\langle \cdot, \cdot, \cdot \rangle$ on X defined by

$$\langle x,\, y,\, z\rangle :\Leftrightarrow x \wedge z \leq y \leq x \vee z\,,$$

is an interval relation on X . It is called the *lattice interval relation* of X . $(X,\, \langle \cdot, \cdot, \cdot \rangle)$ is called the *lattice interval space* associated with X . This example has also been taken from [**48**, chapter I, 4.2].

Unless otherwise stated, interval space concepts, applied to a lattice, refer to its lattice interval relation.

For example, in $\mathbb{R}^2$ with the componentwise partial order, intervals are rectangles parallel to the coordinate axes. When X is a chain, for example $(\mathbb{R}, \leq)$, then it is geometric. The lattice with the following Hasse diagram is not geometric: $[a,\, b] = [a \wedge b,\, a \vee b] = [l,\, g]$; thus, $x,\, y \in [a,\, b]$; and $x = a \wedge y$, therefore, $\langle a,\, x,\, y \rangle$. But $y \not\leq c = x \vee b$, consequently not $\langle x,\, y,\, b \rangle$.

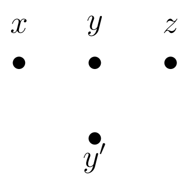


Let $(X,\, d)$ be a metric space. By 1.4.4 (interval relation of a metric space) and 1.4.15 (geometricity of metric spaces), the ternary relation $\langle \cdot,\, \cdot,\, \cdot \rangle = \langle \cdot,\, \cdot,\, \cdot \rangle_d$ on X defined by

$$\langle x,\, y,\, z \rangle_d :\Leftrightarrow d_{xz} = d_{xy} + d_{yz} \,,$$

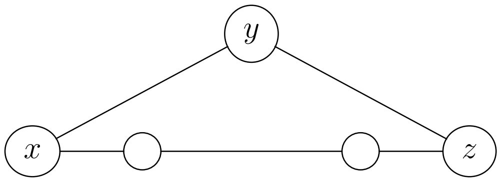
is a geometric interval relation on X . It is called the *geodesic interval relation* of X . The interval space $(X,\, \langle \cdot,\, \cdot,\, \cdot \rangle_d)$ is called the *geodesic interval space* associated with $(X,\, d)$. In the interval space $(X,\, \langle \cdot,\, \cdot,\, \cdot \rangle_d)$, for $a,\, b \in X$, $[a,\, b]$ is called the *geodesic interval* from a to b . For example:

- In $(\mathbb{R}^2,\, \|\cdot - \cdot\|_1)$, balls are squares parallel to the coordinate diagonals, and geodesic intervals are rectangles parallel to the coordinate axes.
- For $n \in \mathbb{Z}_{\geq 1}$, in $(\mathbb{R}^n,\, \|\cdot - \cdot\|_2)$, the geodesic interval relation coincides with the vector interval relation. In the following example in $(\mathbb{R}^2,\, \|\cdot - \cdot\|_2)$, $\langle x,\, y,\, z \rangle$, but not $\langle x,\, y',\, z \rangle$.

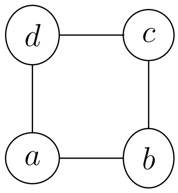


- In $(\mathbb{R}^2,\, \|\cdot - \cdot\|_\infty)$, balls are squares parallel to the coordinate axes, and geodesic intervals are rectangles parallel to the coordinate diagonals.

- In a connected graph $(N,\, E)$, by 1.4.2 (distance function of a connected graph), the distance function d is a metric on N . $\langle x,\, y,\, z\rangle$ iff y is on a geodesic from x to z . In the following graph, $[x,\, z] = \{x,\, y,\, z\}$.



In the following graph, $[a,\, c] = \{a,\, b,\, c,\, d\}$.



Graphs will mainly be used as counterexamples to illustrate the limits of concepts and results.

Unless otherwise stated, interval space concepts, applied to a metric space, refer to its geodesic interval relation.

When X is a geometric interval space, then for $p \in X$, according to 1.4.16 (Hedlíková's criterion for geometric interval spaces), the $(2,\, 3)$-section $\langle p,\, \cdot,\, \cdot\rangle$ is a partial order on X .

The concept of a pseudointerval space coincides with the concept of an *interval convexity* in [**6**, after corollary 2.1]. There it has been defined in terms of intervals. Here, relational notation is used, as has been done before in several variants, for example in [**37**], [**13**], [**20**], [**9**] and [**35**]. The relational notation $'\,\langle x,\, y,\, z\rangle'$ directly visualizes a geometric situation, immediately generalizes to more than three terms and makes explicit that order geometry starts as a first-order theory. Here, the set operational expression $'\,[x,\, z]'$ is used only when it is more convenient. In [**48**, chapter I, 4.1], a pseudointerval space is called an interval space. The concept of an interval space as defined in [**44**, section 2] involves a topology on the set X . An interval space as defined there for which the topology is indiscrete is the same as a pseudointerval space with the indiscrete topology added.

The concept of an interval space has been taken from [**49**, chapter I, 3.1]. There it has been defined in terms of intervals. In [**48**, chapter I, 4.10], an interval space is called an *idempotent interval space*. The terminology adopted here is parallel to metric space terminology: interval spaces correspond to metric spaces, see 1.4.4 (interval relation of a metric space), while pseudointerval spaces correspond to pseudometric spaces.

By 1.4.16 (Hedlíková's criterion for geometric interval spaces), the concept of a geometric interval space coincides with the concept of a *ternary space* defined by the conditions (T_1) to (T_4) in [**20**, section 1]. The terminology used here has been taken from [**49**, chapter I, 3.1]. The definition used here is equivalent to the definition in [**48**, chapter I, 4.1]. The conditions T_1 , T_2 in [**20**, section 1] coincide with the conditions $\alpha,\, \beta$ in [**37**, part I, section 1]. They define

an intermediate concept between the concepts of an interval space and of a geometric interval space.
Note that interval spaces are first-order structures. In particular:

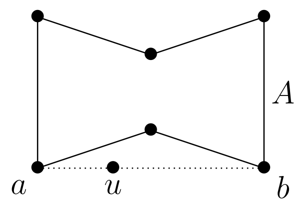
(1) A map $f : X \to Y$ of interval spaces is
  - a homomorphism iff for all $a,\, b,\, c \in X$, $\langle a,\, b,\, c\rangle$ implies $\langle f\,(a)\,,\, f\,(b)\,,\, f\,(c)\rangle$,
  - an embedding iff it is injective and for all $a,\, b,\, c \in X$,
$$\langle a,\, b,\, c\rangle \text{ iff } \langle f\,(a)\,,\, f\,(b)\,,\, f\,(c)\rangle\ ,$$
  - an isomorphism of X onto Y iff it is an embedding of X onto Y .

(2) A substructure of an interval space X is a pair consisting of a subset Y of X and the relation $\langle \cdot,\, \cdot,\, \cdot\rangle \cap (Y \times Y \times Y)$. It is an interval space. In [**48**, chapter I, 4.3], it is called a *subspace* of X . There it has been defined in terms of intervals.

The concepts of a convex set, of a convex closure and of a half-space in a vector space over a totally ordered field have natural generalizations to an interval space:

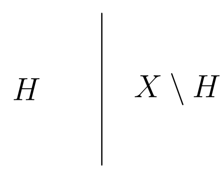
Let X be an interval space.

A subset C of X is called *convex* iff for all $x,\, z \in C$, $[x,\, z] \subseteq C$. For example, for $n \in \mathbb{Z}_{\geq 1}$, in $(\mathbb{R}^n,\, \|\cdot - \cdot\|_2)$, open and closed balls are convex. In the real vector space $\mathbb{R}^2$, the following set A is not convex: $a,\, b \in A$, but $[a,\, b] \nsubseteq A$ because $u \in [a,\, b]$ and $u \notin A$.



For $A \subseteq X$, the *convex closure* of A is the intersection of all convex sets in X containg A . The following notation has been taken from [**10**, chapter II, section 2]: For $A \subseteq X$,
$$[A] := \text{the convex closure of } A\,.$$
When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then the convex closure of A is the set of all $\sum_{j=1}^{k} \lambda_j x_j$ such that $k \in \mathbb{Z}_{\geq 1}$, $\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_k \in K_{\geq 0}$ and $\sum_{j=1}^{k} \lambda_j = 1$. In [**6**, section 1], the concepts of a convex set and a convex closure have been defined in a more general context of set systems, and a convex closure has been called a *convex hull*.

A subset H of X is called a *half-space* iff H and $X \setminus H$ are convex.

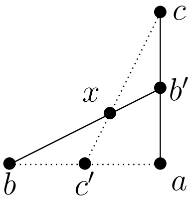


When X is a vector space over a totally ordered field K and f is a non-zero linear map from X to K , then for $\lambda \in K$, $f^{-1}\,(\downarrow \lambda)$ is a half-space. In [**48**, chapter I, 3.1], the concept of a half-space has been defined in the more general context of an algebraic closure space.

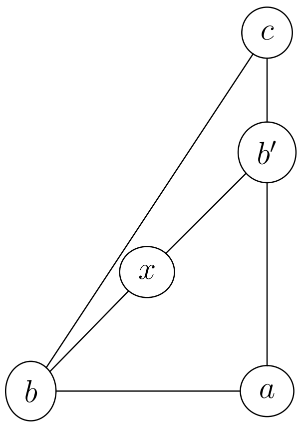
Characterizations by fundamental combinatorial concepts. Let X be an interval space. X is called *triangle-convex* iff one and therefore each of the following equivalent conditions is satisfied:

- For all $a,\, b,\, c,\, x \in X$, if there is a $b' \in [a,\, c]$ such that $\langle b,\, x,\, b' \rangle$, then there is a $c' \in [a,\, b]$ such that $\langle c,\, x,\, c' \rangle$.
- For all $a,\, b,\, c, \in X$, $\bigcup_{b' \in [a,\, c]} [b,\, b'] \subseteq \bigcup_{c' \in [a,\, b]} [c,\, c']$.
- For all $a,\, b,\, c, \in X$, $\bigcup_{b' \in [a,\, c]} [b,\, b']$ is convex.
- For all $a,\, b,\, c, \in X$, $[\{a,\, b,\, c\}] = \bigcup_{b' \in [a,\, c]} [b,\, b']$.

The first of these conditions is the interval relation version of the strict interval relation condition [**36**, §10, Assioma XIII]. In [**48**, chapter I, 4.9] it is called the *Peano Property*. When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X is triangle-convex.

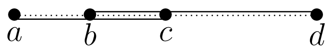


For $n \in \mathbb{Z}_{\geq 1}$, the metric spaces $(\mathbb{R}^n,\, \|\cdot - \cdot\|_1)$ and $(\mathbb{R}^n,\, \|\cdot - \cdot\|_2)$ are triangle-convex. Each convex subspace of a triangle-convex interval space is triangle-convex. 1.6.6 (medianity criterion for a geometric interval space) provides further examples of triangle-convex interval spaces. The following complete-bipartite graph is not triangle-convex: $b' \in [a,\, c]$ and $\langle b,\, x,\, b' \rangle$, but there is no $c' \in [a,\, b]$ such that $\langle c,\, x,\, c' \rangle$:

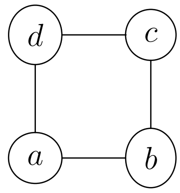


PROPOSITION. [**10**, chapter II, proposition 3] *Let X be a triangle-convex interval space. For C a non-empty convex set and $a \in X$, $[C \cup \{a\}] = \bigcup_{c \in C} [c,\, a]$.*

Let X be an interval space. X is called *one-way* iff for all $a,\, b,\, c,\, d \in X$, $\langle a,\, b,\, c \rangle$, $b \neq c$ and $\langle b,\, c,\, d \rangle$ imply $\langle a,\, b,\, d \rangle$. This condition is the interval relation version of the strict interval relation condition [**34**, §1, VIII. Grundsatz]. When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X is one-way.



Each tree is one-way. The following graph is not one-way: $\langle a,\, b,\, c\rangle$, $b \neq c$, $\langle b,\, c,\, d\rangle$, but not $\langle a,\, b,\, d\rangle$.



For $n \in \mathbb{Z}_{\geq 2}$, the metric space $(\mathbb{R}^n,\, \|\cdot - \cdot\|_1)$ is not one-way.

Let X be a set. A *closure system* on X is a set C of subsets of X such that $X \in C$ and for each non-empty $D \subseteq C$, $\bigcap D \in C$.

A *closure space* is a pair consisting of a set X and a closure system C on X . A set $A \subseteq X$ is called *closed* iff $A \in C$. When $(X,\, O)$ is a topological space, then the pair consisting of X and the set of closed sets in $(X,\, O)$ is a closure space. When $(X,\, \langle\cdot,\, \cdot,\, \cdot\rangle)$ is an interval space, then the pair consisting of X and the set of convex sets is a closure space. The concept of a closure space as defined here is slighly more general than in [**48**, chapter I, 1.2], where it is required that $\emptyset \in C$ and a closure system is called a protopology.

A closure space $(X,\, C)$ is also simply denoted by X when it is clear from the context whether the closure space or only the set is meant.

The concepts of a closure in a topological space and of a convex closure in an interval space have a natural generalization to a closure space.

Let $(X,\, C)$ be a closure space.

For $A \subseteq X$, the *closure* of A is the intersection of all closed sets $B \supseteq A$. It is the smallest closed superset of A . When X is an interval space and C is the system of convex sets in X , then for $A \subseteq X$, the closure of A is the convex closure of A .

For $A \subseteq X$, the *entailment relation* of C relative to A or A-entailment relation is the binary relation $\vdash_A$ on X defined by

$$x \vdash_A y :\Leftrightarrow y \in \text{ the closure of } A \cup \{x\} \ .$$

X is called an *antiexchange space* iff for each closed $A \subseteq X$, one and therefore both of the following conditions hold, which are equivalent by 3.1.1 (2) (relative entailment relation):

- The restriction $\vdash_A \mid (X \setminus A)$ antisymmetric.
- The restriction $\vdash_A \mid (X \setminus A)$ is a partial order on $X \setminus A$.

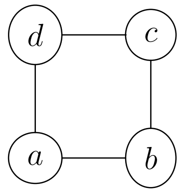
Let X be a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$. The pair consisting of X and its vector interval relation is a triangle-convex and one-way interval space. By 3.2.1 (triangle-convex one-way interval spaces), the pair consisting of X and the set of convex sets is an antiexchange space. In [**48**, chapter I, 2.24], an exchange space that is an algebraic closure space with $\emptyset$ closed is called an *anti-matroid* or *convex geometry*.

The first main new result characterizes a geometric property of an interval relation in terms of a fundamental property of a family of derived binary relations:

THEOREM. *3.2.4 (antiexchange criterion for triangle-convex geometric interval spaces) Let $(X, \langle\cdot, \cdot, \cdot\rangle)$ be a triangle-convex geometric interval space. The pair consisting of X and the set of convex sets is an antiexchange space iff X is one-way.*

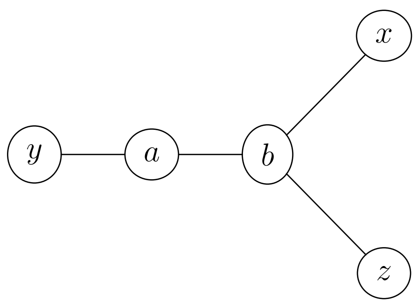
Let X be a geometric interval space.

X is called *interval-linear* iff for all $a, b \in X$, $([a, b], \langle a, \cdot, \cdot\rangle)$ is a chain. This condition is contained in [**43**, (5.2)]. When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then it is interval-linear. For $n \in \mathbb{Z}_{\geq 1}$, the metric space $(\mathbb{R}^n, \|\cdot - \cdot\|_2)$ is interval-linear. Each chain, for example $(\mathbb{R}, \leq)$, with its lattice interval relation is interval-linear. Each subspace of an interval-linear interval space is interval-linear. The following graph is not interval-linear: $b, d \in [a, c]$, but not $\langle a, b, d\rangle$ and not $\langle a, d, b\rangle$.



For $n \in \mathbb{Z}_{\geq 2}$, the metric spaces $(\mathbb{R}^n, \|\cdot - \cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot - \cdot\|_\infty)$ are not interval-linear.

X is called *ray-linear* iff for all $a, b \in X$, $a \neq b$ implies that $(\langle a, b, \cdot\rangle, \langle a, \cdot, \cdot\rangle)$ is a chain. This condition is the interval relation version of the strict interval relation condition [**34**, §1, VII. Grundsatz]. When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X is ray-linear. Each subspace of a ray-linear interval space is ray-linear. A tree that has a point of degree ≥ 3 is not ray-linear. For example, the following tree is not ray-linear: $y \neq b$, $x, z \in \langle y, b, \cdot\rangle$, but not $\langle y, x, z\rangle$ and not $\langle y, z, x\rangle$.



For $n \in \mathbb{Z}_{\geq 2}$, the metric space $(\mathbb{R}^n, \|\cdot - \cdot\|_1)$ is not ray-linear.

The second main new result, like the first one, 3.2.4 (antiexchange criterion for triangle-convex geometric interval spaces), characterizes a geometric property of an interval relation in terms of a fundamental property of a family of derived binary relations:

THEOREM. *3.3.2 (perspectivity relation) Let X be a one-way geometric interval space. Define the binary relation $\sim_p$ on $X \setminus \{p\}$ by*

$$a \sim_p b :\Leftrightarrow \langle p \neq a,\, b\rangle \text{ or } \langle a \neq p \neq b\rangle \text{ or } \langle a,\, b \neq p\rangle \ .$$

The following conditions are equivalent:

(1) *For each $p \in X$, $\sim_p$ is transitive.*
(2) *For each $p \in X$, $\sim_p$ is an equivalence relation on $X \setminus \{p\}$.*
(3) *X is interval-linear and ray-linear.*

Let X be an interval space. Define the binary relation $\sim_p$ on $X \setminus \{p\}$ by

$$a \sim_p b :\Leftrightarrow \langle p \neq a,\, b\rangle \text{ or } \langle a \neq p \neq b\rangle \text{ or } \langle a,\, b \neq p\rangle \ .$$

X is called *perspective* iff it is geometric and one-way and satisfies one and therefore each of the following conditions, which are equivalent by 3.3.2 (perspectivity relation):

- For each $p \in X$, $\sim_p$ is transitive.
- For each $p \in X$, $\sim_p$ is an equivalence relation on $X \setminus \{p\}$.
- X is interval-linear and ray-linear.

Let X be a perspective interval space. Three points $p,\, a,\, b$ in X are called *collinear* iff $\langle p,\, a,\, b\rangle$ or $\langle a,\, p,\, b\rangle$ or $\langle a,\, b,\, p\rangle$. For $p \in X$:

- For $a,\, b \in X$, a is called *perspective* to b from p iff $a \sim_p b$, i.e. $\langle p \neq a,\, b\rangle$ or $\langle a \neq p \neq b\rangle$ or $\langle a,\, b \neq p\rangle$.
- For $n \in \mathbb{Z}_{\geq 2}$ and $a,\, b \in X^n$, a is called *perspective* to b from p iff the following conditions are satisfied:
  - For $\nu \in [n]$, a_ν is persepective to b_ν from p .
  - For $\mu,\, \nu \in [n]$ satisfying $\mu \neq \nu$, a_μ is not perspective to b_ν from p .

The concept of perspectivity of elements of X^n generalizes the concept of triangles in perspective in [**27**, chapter VIII, 160.].

Let X be an interval space.

X is called *desarguesian* iff it is perspective and for all $p,\, q_{12},\, q_{13},\, q_{23} \in X$ and $a,\, b \in X^3$, if $a \sim_p b$ and for all $(\mu,\, \nu) \in \{(1,\, 2)\, ,\, (1,\, 3)\, ,\, (2,\, 3)\}$, $(a_\mu,\, a_\nu) \sim_{q_{\mu\nu}} (b_\mu,\, b_\nu)$, then $q_{12,} q_{13}\, q_{23}$ are collinear.

X is called *unending* iff for all $a,\, b \in X$, $a \neq b$ implies $\langle a,\, b \neq \cdot\rangle \neq \emptyset$. This condition is the interval relation version of the strict interval relation condition [**34**, §1, VI. Grundsatz].

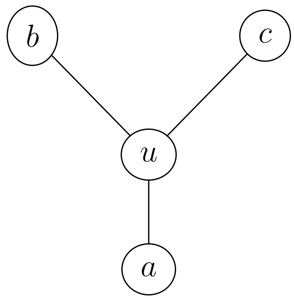
X is called *complete* iff for all $a,\, b \in X$ and each convex set C , $a \in C \subseteq [a,\, b]$ implies that there is a $c \in [a,\, b]$ such that $C = \langle a,\, \cdot,\, c\rangle$, which is $[a,\, c]$, or $C = \langle a,\, \cdot \neq c\rangle$. This condition is axiom (S) from [**10**, chapter VIII, section 1]. It generalizes the defintion of completeness of a totally ordered field by Dedekind cuts.

Examples and indeed a complete classification of triangle-convex desarguesian unending complete interval spaces with three non-collinear points are provided by the following coordinatization theorem. The direction ($\Rightarrow$) follows immediately from [**13**, section 5] in connection with

3.3.1(3) (interval-linear geometric interval spaces) and has been called the *fundamental theorem of ordered geometry* in [**10**, chapter VIII, section 3].

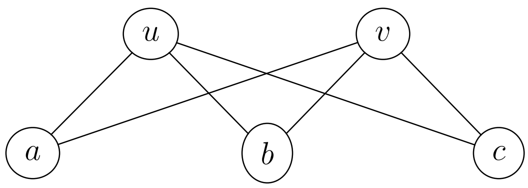
THEOREM. *An interval space that has three non-collinear points is triangle-convex, desarguesian, unending and complete iff it is isomorphic to an unending convex subset of a real vector space.*

Modular and median spaces. For $a,\, b,\, c,\, u \in X$, u is called a *median* of $a,\, b,\, c$ iff $\langle a,\, u,\, b\rangle$ and $\langle b,\, u,\, c\rangle$ and $\langle c,\, u,\, a\rangle$.

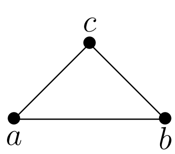


A *median triangle* in X is a partial matrix $\begin{bmatrix} & c & \\ a & u & b \end{bmatrix}$ in X such that u is a median of $a,\, b,\, c$. When X is a lattice, then $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ is a median of $a,\, b,\, c$, i.e. $\begin{bmatrix} & c & \\ a & (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) & b \end{bmatrix}$ is a median triangle. Medians were introduced in [**5**] for the particular case that X is a distributive lattice.

An interval space X is called *modular (median)* iff for all $a,\, b,\, c \in X$, there is at least one (exactly one) median of $a,\, b,\, c$. For example, the following bipartite graph is modular, but not median: $a,\, b,\, c$ have the two different medians $u,\, v$.



For $n \in \mathbb{Z}_{\geq 1}$, the metric space $(\mathbb{R}^n,\, \|\cdot - \cdot\|_1)$ is median. For $n \in \mathbb{Z}_{\geq 2}$, the metric space $(\mathbb{R}^n,\, \|\cdot - \cdot\|_2)$ is not modular. In the following example in $(\mathbb{R}^2,\, \|\cdot - \cdot\|_2)$, the points $a,\, b,\, c \in \mathbb{R}^2$ have no median.



In [**3**, 1.4] the concept of modularity of an interval space has been defined under the assumption that the interval space is geometric.

A *topological interval space* is a triple $(X, \langle\cdot, \cdot, \cdot\rangle, O)$ such that:

- $(X, \langle\cdot, \cdot, \cdot\rangle)$ is an interval space
- (X, O) is a topological space
- $\langle\cdot, \cdot, \cdot\rangle$ is a closed subset of the product space $X \times X \times X$.

Each real topological vector space is a topological interval space. By 2.4.2 (topological interval space property of a metric space), each metric space with the geodesic interval relation and the topology determined by its metric is a topological interval space. By 2.4.3 (discrete topological interval spaces), each interval space with the discrete topology is a topological interval space. The concept of a topological interval space is analogous to the concept of a topological poset, which is implicit in the results on topological spaces equipped with a closed order in [**33**, chapter 1, §1 and §3]. It is related to the concept of a topological convex structure as defined in [**48**, chapter III, 1.1.1]. The concept of an interval space as defined in [**44**, section 2] also involves a topology, but there the interval space structure and the topology are connected by a different condition.

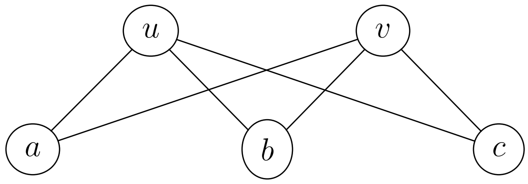
Let X be a geometric interval space. For $A \subseteq X$ and $x \in X$, if the poset $(A, \langle x, \cdot, \cdot\rangle)$ has a least element, then this least element is called the *gate* of x into A . A set $G \subseteq X$ is called *gated* iff each element of X has a gate into G . For a gated set G in X , the map from X to G mapping x to the gate of x into G is called the *gate map* of G . It is the unique map $g : X \to G$ such that for $x, a \in X$, if $a \in G$, then $\langle x, g(x), a\rangle$. Examples for gated sets are provided by 4.3.1 (modular geometric topological interval spaces) and 5.1.5(2e) (arboric interval spaces). The concept of a gated set has been taken from [**16**]. There it has been defined for the particular case of a metric space and further examples have been provided. When X is a metric space, then each gated set G is a *Chebyshev set*, i.e. for each $x \in X$, there is exactly one $a \in G$ such that $d(x, G) = d(x, a)$. a is the gate of x into G .

The third main new result characterizes compact gated sets in a modular geometric topological interval space:

THEOREM. *4.3.1 (modular geometric topological interval spaces) Let X be a modular geometric topological interval space. For C a non-empty compact set, C is gated iff it is convex.*

Let X be an interval space.

X is said to have *point-interval separation* iff for all $x, y, z \in X$, if $x \notin [y, z]$, then there is a half-space H such that $x \in H$ and $[y, z] \subseteq X \setminus H$. If X has point-interval separation, then each subspace of X has point-interval separation. When X is a real vector space, for example $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X has point-interval separation. 4.7.1 (separation in a median geometric interval space) provides further examples of interval spaces with point-interval separation. The following complete-bipartite graph X does not have point-interval separation: The only half-spaces are X and $\emptyset$.



X is called *metrizable* iff it has an isomorphism onto a metric space. The next theorem provides sufficient criteria for metrizability of an interval space.

X is called *submedian-metrizable* iff it has an embedding into a median metric space. For example, by part (1) of the next theorem a finite subset of a real vector space is submedian-metrizable. The complete-bipartite graph $K_{2,3}$ is not submedian-metrizable.

The fourth main new result consists of sufficient criteria for submedian metrizability and for metrizability of a finite interval space.

THEOREM. *4.7.2 (metrizability criterion) Let X be a finite geometric interval space.*

(1) *If X has point-interval separation, then it is submedian-metrizable.*
(2) *If X is median, then it is metrizable.*

Here are some results from [**3**] and [**49**]:

THEOREM. [**3**, 2.3] *Let X be a modular geometric interval space. For $u \in X$, the following conditions are equivalent:*

(1) *For all $x, y \in X$, x, y, u have exactly one median.*
(2) *The poset $(X, \langle u, \cdot, \cdot \rangle)$ is a meet semilattice such that every principal down-set is a modular lattice and any three elements have a common upper bound whenever any two of them do.*

THEOREM. [**3**, 2.5] *Let X be a modular geometric interval space. For $u \in X$, if for all $x, y \in X$, x, y, u have exactly one median, then X has an embedding into a bounded modular lattice with least element u.*

The second part of the following theorem shows that the interval relation of a modular geometric interval space cannot be refined.

THEOREM. [**3**, 3.1] *Let X be a set with geometric interval relations $\langle \cdot, \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot, \cdot \rangle_2$.*

- *If for all $a, b, c \in X$, a, b, c have a common median with respect to both interval relations, then $\langle \cdot, \cdot, \cdot \rangle_1 = \langle \cdot, \cdot, \cdot \rangle_2$.*
- *If $(X, \langle \cdot, \cdot, \cdot \rangle_1)$ is modular and $\langle \cdot, \cdot, \cdot \rangle_1 \subseteq \langle \cdot, \cdot, \cdot \rangle_2$, then $\langle \cdot, \cdot, \cdot \rangle_1 = \langle \cdot, \cdot, \cdot \rangle_2$.*

The following theorem is analogous to [**25**, 4.50] (Jordan-Hölder theorem) for groups.

THEOREM. [**3**, 4.3] *Let X be a modular geometric interval space. For $a, b \in X$, if the poset $([a, b], \langle a, \cdot, \cdot \rangle)$ has a maximal finite chain with least element a and greatest element b, then all such chains have the same size.*

THEOREM. [**3**, 4.5] *Let X be a modular geometric interval space. Suppose that for each $u \in X$, in the poset $(X, \langle u, \cdot, \cdot \rangle)$ each bounded chain is finite. Then X is isomorphic to a connected graph.*

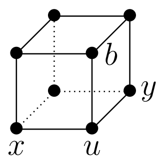
THEOREM. [**3**, 4.7] *Let X be a modular geometric interval space. The following conditions are equivalent:*

(1) *For all $a, b \in X$, the poset $([a, b], \langle a, \cdot, \cdot \rangle)$ is a modular lattice.*
(2) *The complete-bipartite graph $K_{3,3}$ minus an edge has no embedding into X.*

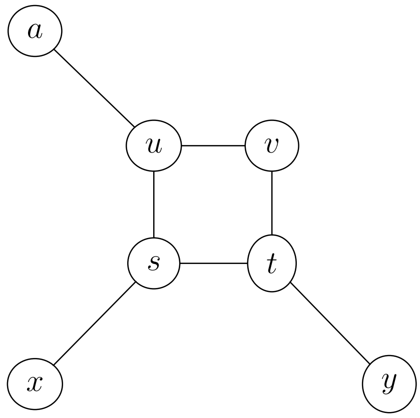
Let X be a median interval space. For $a, b, c \in X$,

$$m(a, b, c) := \text{the median of } a, b, c.$$

For example, the edge graph of a cube is median with $m(x, y, b) = u$:



For $M \subseteq X$, M is called *median* in X iff for all $a, b, c \in M$, $m(a, b, c) \in M$. For example, for $n \in \mathbb{Z}_{\geq 0}$, in the median metric space $(\mathbb{R}^n, \|\cdot - \cdot\|_1)$, $\mathbb{Z}^n$ and $\mathbb{Q}^n$ are median. For $A \subseteq X$, the *median closure* of A in X is the intersection of all median sets in X containing A. For example, in the following graph, the median closure of $\{x, y, a\}$ equals $\{x, y, a, s\}$.



LEMMA. [**48**, chapter I, 6.20] *Let X be a median geometric interval space. For A a finite subset of X and $p \in X$:*

(1) *p belongs to the median closure of A in X iff for all half-spaces H_1 and H_2 , $p \in H_1 \cap H_2$ implies $H_1 \cap H_2 \cap A \neq \emptyset$.*
(2) *The median closure of A in X is finite.*

The concept of a lattice can be defined in terms of axioms for an order relation or in terms of axioms for algebraic operations, namely the join and meet operations. Analogously, the concept of a median interval space, which has been defined in terms of axioms for an interval relation, can also be defined in terms of axioms for an algebraic operation, namely the median operation. A *median algebra* is a set X with a ternary operation m on X such that the following conditions are satisfied:

- For $a \in X$, the binary operation $m(a, \cdot, \cdot)$ is idempotent.
- For $a_1, a_2, a_3 \in X$ and $\sigma \in S_3$, $m(a_1, a_2, a_3) = m\left(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}\right)$.
- For $a \in X$, the binary operation $m(a, \cdot, \cdot)$ is associative.

These conditions are equivalent to conditions (2), (3) in section 1 and (12) in section 2 of [**26**]. The equivalence of conditions (a) and (c) in [**26**, section 2, theorem 2] entails that they are also equivalent to conditions (T1), (T2) and (T3) defining the concept of a *ternary distributive semilattice* in [**2**]. Here are some other synonyms for the concept of a median algebra that have appeared in the literature: *median semilattice*, *normal graphic algebra*, *simple graphic algebra*, *simple ternary algebra*, *symmetric medium* and *distributive median algebra*. The following proposition shows that results about median algebras are at the same time results about median geometric interval spaces and vice versa. In this sense, it provides a link of the theory of median geometric interval spaces to the extensive literature on median algebras.

PROPOSITION. [**48**, chapter I, 6.11] *If $(X, \langle\cdot, \cdot, \cdot\rangle)$ is a median geometric interval space, then (X, m) is a median algebra. If (X, m) is a median algebra, then with*

$$\langle x, y, z\rangle :\Leftrightarrow m(x, y, z) = y ,$$

$(X, \langle\cdot, \cdot, \cdot\rangle)$ is a median geometric interval space. These two transformations between median geometric interval spaces and median algebras are inverse to each other.

Arboric spaces. Let X be a poset.

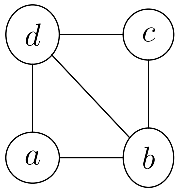
X is called *arboric* iff X is a meet semilattice and for each $a \in X$, the poset $(\downarrow a, \leq)$ is a chain. Each chain, for example $(\mathbb{R}, \leq)$, is an arboric poset. Let (N, E) be a tree. For $u \in N$, define the binary relation $\leq_u$ on N by $x \leq_u y$ iff x is on the path from u to y . Then $(N, \leq_u)$ is an arboric poset. The concept of an arboric poset has been taken from [**48**, chapter I, 5.3]. There an arboric poset has been called an *order tree*. Since an arboric poset is a poset, and not a tree in the usual sense of graph theory, here the substantive 'poset' together with the adjective 'arboric', which means 'tree-like', is preferred.

Let X be an interval space.

X is called *interval-concatenable* iff for all $a, b, c \in X$, $[a, b] \cap [b, c] = \{b\}$ implies $\langle a, b, c\rangle$. For the particular case of a connected graph, this condition has been used in [**31**, 3.1.7]. For example, for $n \in \mathbb{Z}_{\geq 1}$, the metric space $(\mathbb{R}^n, \|\cdot - \cdot\|_1)$ is interval-concatenable.

For $n \in \mathbb{Z}_{\geq 2}$, the metric space $(\mathbb{R}^n,\ \|\cdot - \cdot\|_2)$ is not interval-concatenable. The cycle C_5 is not interval-concatenable.

X is called *arboric* iff X is geometric and interval-concatenable and for each $a \in X$, the poset $(X,\ \langle a,\ \cdot,\ \cdot\rangle)$ is arboric. For example, the metric space $(\mathbb{R},\ |\cdot - \cdot|)$ is arboric. Each tree is arboric. The following graph is not arboric: $\langle [a,\ c]\ ,\ \langle a,\ \cdot,\ \cdot\rangle\rangle$ it not a chain because $b,\ d \in [a,\ c]$, but not $\langle a,\ b,\ d\rangle$ and not $\langle a,\ d,\ b\rangle$.



For $n \in \mathbb{Z}_{\geq 2}$, the metric space $(\mathbb{R}^n,\ \|\cdot - \cdot\|_2)$ is not arboric.

The concept of an arboric interval space is implicit in [**43**, (1.2), (1.3), (1.1), (1.4), (1.5), (4.7), (2.1), (S), (2.1) in connection with the last part of (4.8)]. There an arboric interval space is called a tree. Since an arboric interval space is an interval space, and not what is called a tree in graph theory, here also the substantive 'interval space' together with the adjective 'arboric', which means 'tree-like', is preferred.

The implication ($\Leftarrow$) of the following proposition is implicit in [**43**, (1.2), (1.3), (1.1), (1.4), (1.5), (4.7), (2.1), (S), (2.1) in connection with the last part of (4.8)]. Parts (1) and (2) are axioms (S) and (T) from section 1, and part (3) is axiom (U_1) from section 2 of [**43**].

PROPOSITION. *1.7.1 (Sholander's criterion for arboric interval spaces) Let* X *be a set and* $\langle\cdot,\ \cdot,\ \cdot\rangle$ *a ternary relation on* X *. For* $a,\ b \in X$ *, define* $[a,\ b] := \{x \in X |\ \langle a,\ x,\ b\rangle\}$ *.* $(X,\ \langle\cdot,\ \cdot,\ \cdot\rangle)$ *is an arboric interval space iff for all* $u,\ a,\ b \in X$ *the following conditions are satisfied:*

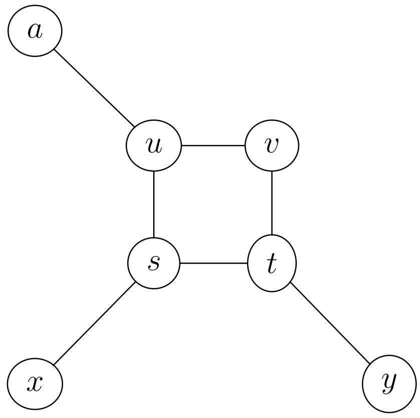
(1) *There is a* $c \in X$ *such that* $[u,\ a] \cap [u,\ b] = [u,\ c]$.
(2) *If* $[u,\ a] \subseteq [u,\ b]$ *, then* $[u,\ a] \cap [a,\ b] = \{a\}$.
(3) *If* $[u,\ a] \cap [a,\ b] = \{a\}$ *, then* $[u,\ a] \cup [a,\ b] = [u,\ b]$.

PROPOSITION. [**43**, (4.6)] *Each arboric interval space is one-way.*

The concept of an extremal point of a set in a vector space over a totally ordered field has a natural generalization to an interval space: Let X be an interval space. For $p \in X$, p is called *extremal* iff for all $a,\ b \in X$, the following implication holds: If $\langle a,\ p,\ b\rangle$, then $p \in \{a,\ b\}$. For example, when X is a triangle in the Euclidean plane $(\mathbb{R}^2,\ \|\cdot - \cdot\|_2)$, i.e. the convex closure of three affinely independent points $x,\ y,\ z$, then the set of its extremal points equals $\{x,\ y,\ z\}$. In [**48**, chapter I, 1.23], the concept of an extremal point has been defined in the more general context of an algebraic closure space. The new concept of a median-extremal point generalizes the concept of an extremal point: For $p \in X$, p is called *median-extremal* iff for all $a,\ b,\ c \in X$, the following implication holds: If $\begin{bmatrix} & c & \\ a & p & b \end{bmatrix}$ is a median triangle, then $p \in \{a,\ b,\ c\}$. Each extremal point is median-extremal. The *median boundary* of X is the set

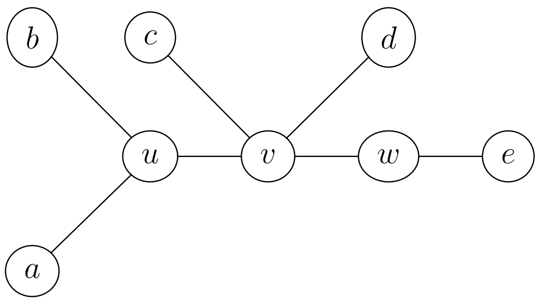
$$\partial_M (X) := \{x \in X | x \text{ is median-extremal.}\} \ .$$

For example, in the following graph, the median-extremal points are $x,\ y,\ a,\ v$, i.e. $\partial_M (X) = \{x,\ y,\ a,\ v\}$. $x,\ y,\ a$ are also extremal, while v is not extremal.



Let X be an arboric interval space. For $a,\ b \in X$, a is called a u-*neighbor* of b iff u is not between a and b . For $u \in X$, the u-neighbor relation is an equivalence relation on $X \setminus \{u\}$. A u-*branch* is an equivalence class of the u-neighbor relation. For example, in the arboric metric space $(\mathbb{R},\ |\cdot - \cdot|)$, the 0-branches are the sets $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. In the following tree, $a,\ b,\ c,\ d,\ e$ are extremal. With respect to each of them, there is only one branch. Furthermore:

- The u-branches are $\{a\}$, $\{b\}$, $\{v,\ c,\ d,\ w,\ e\}$.
- The v-branches are $\{u,\ a,\ b\}$, $\{c\}$, $\{d\}$, $\{w,\ e\}$.
- The w-branches are $\{u,\ a,\ b,\ v,\ c,\ d\}$, $\{e\}$.



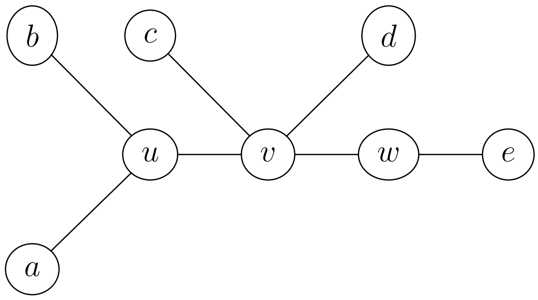
The concept of a branch in an arboric interval space is similar to the concept of a branch in an arboric poset as defined in [**48**, chapter I, 5.3].

Let X be an arboric interval space and $u \in X$. By 5.1.3 (extremal points in an arboric interval space), u is extremal iff there is at most one u-branch. u is called *pre-extremal* iff u is non-extremal and there is at most one u-branch of size ≥ 2 . For pre-extremal $u \in X$, the *extremal neighborhood* of u is the set

$$\begin{aligned}\mathrm{EN}(u) &:= \{u\} \cup \{x \in X \mid \{x\} \text{ is a } u\text{-branch.}\} \\ &= \{u\} \cup \{x \in X \mid x \text{ is extremal and adjacent to } u\,.\}\ ,\end{aligned}$$

where equality holds by 5.1.5(3b) (arboric interval spaces). For example, in the following tree,

- The u-branches are $\{a\}$, $\{b\}$, $\{v, c, d, w, e\}$, so u is pre-extremal and $\mathrm{EN}(u) = \{u, a, b\}$.
- The v-branches are $\{u, a, b\}$, $\{c\}$, $\{d\}$, $\{w, e\}$, so v is not pre-extremal although it is adjacent to the two different extremal points c, d .
- The w-branches are $\{u, a, b, v, c, d\}$, $\{e\}$, so w is pre-extremal and $\mathrm{EN}(w) = \{w, e\}$.



Let X be a metric space.

For $x, y, a \in X$, the *modular distance* of the point x from the pair (y, a) or *Gromov product* of y and a with respect to x is the number

$$d_{x,ya} := \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right)\ .$$

In [**17**, 1.1], this has been denoted by $(y, a)_x$. In [**50**, 2.7], it has been denoted by $(y|a)_x$. The notation $(y \cdot a)_x$ is also in use. The reason why here the term 'modular distance' and the notation $d_{x,ya}$ have been added is explained in the main text.

For finite $Y \subseteq X$ and $u \in X$, the *distance sum* of u along Y is the number

$$\lambda_u^Y := \sum_{x \in Y} d_{xu}\,.$$

For finite $Y \subseteq X$ and $a, b \in X$, the *augmented modular distance sum* of the pair (a, b) along Y is the number

$$\lambda_{ab}^Y := d_{ab} + \sum_{x \in Y} d_{x,ab}\,.$$

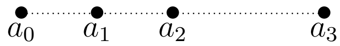
In [**45**], the expression λ_u^X is written as R_u . In [**30**, 7.3.2], it is written as d_{u+} . The concept of an augmented modular distance sum coincides with the concept of a *centrality index* in [**30**, 7.3.2], which has been defined there under addtional assumptions. There. λ_{ab}^X is written as $c(a, b)$.

The fifth main new result places the neighbor-joining method from [**41**] for reconstructing a weighted tree from the distances between its leaves in the conceptual framework of arboric metric spaces:

THEOREM. *5.3.3 (finite arboric metric spaces) Let X be a finite arboric metric space. For $Y \subseteq X$, if $\partial_M(X) \subseteq Y$, then:*

(1) *For $u \in X$, if u is non-extremal with greatest λ_u^Y , then u is pre-extremal.*
(2) *For $a,\ b \in X$, if $a,\ b \in Y$ and $a \neq b$ with greatest λ_{ab}^Y , then:*
   (a) *For $u \in X$, if u is non-extremal and $\langle a,\ u,\ b\rangle$, then u is non-extremal with greatest λ_u^Y .*
   (b) *If $|X| \geq 3$, then there is exactly one pre-extremal $u \in X$ such that $a,\ b \in EN(u)$.*
   (c) *If $|X| \geq 3$, then for the unique pre-extremal $u \in X$ such that $a,\ b \in EN(u)$, $EN(u) = \{u,\ a\} \cup \left\{y \in Y | \lambda_{ay}^Y = \lambda_{ab}^Y\right\}$.*

Quadrimodular and quadrimedian spaces. Let X be an interval space. For $k \in \mathbb{Z}_{\geq 0}$, a finite sequence $(a_0,\ a_1,\ ...,\ a_k)$ in X is called *aligned* iff it satisfies the following condition: For all $p,\ q,\ r \in \{0,\ 1,\ ...,\ k\}$, $p < q < r$ implies $\langle a_p,\ a_q,\ a_r\rangle$.



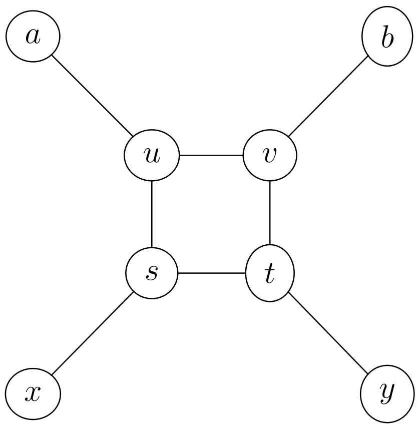
When X is a metric space, then for $k \in \mathbb{Z}_{\geq 2}$ and $a = (a_0,\ a_1,\ ...,\ a_k)$ a sequence in X , a is aligned iff $d_{a_0 a_k} = \sum_{\kappa=1}^{k} d_{a_{\kappa-1} a_\kappa}$. The concept of an aligned sequence has been taken from [**43**, section 4]. There an aligned sequence is called a chain, and the concept has been defined in a similar context.

The new concept of a median quadrangle is analogous to the concept of a median triangle: A *median quadrangle* in X is a partial matrix

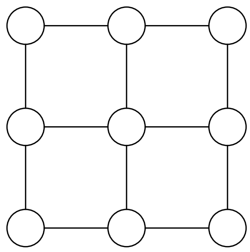
$$Q = \begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$$

in X such that the four-term sequences $(x,\ s,\ t,\ y)$, $(y,\ t,\ v,\ b)$, $(b,\ v,\ u,\ a)$, $(a,\ u,\ s,\ x)$ and the five-term sequences $(x,\ s,\ u,\ v,\ b)$, $(y,\ t,\ v,\ u,\ a)$, $(y,\ t,\ s,\ u,\ a)$, $(x,\ s,\ t,\ v,\ b)$ are aligned.

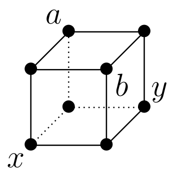
For example, in the following graph, $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle.



X is called *quadrimodular* iff for all $x, y, a, b \in X$, there are $s, t, u, v \in X$ such that $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ or $\begin{bmatrix} b & & & a \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ or $\begin{bmatrix} b & & & y \\ & u & v & \\ & s & t & \\ x & & & a \end{bmatrix}$. An interval space X is called *quadrimedian* iff it is median and quadrimodular. The following graph is quadrimedian.



The edge graph of a cube is median, but not quadrimodular: There are no $s, t, u, v \in X$ such that $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ or $\begin{bmatrix} b & & & a \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ or $\begin{bmatrix} b & & & y \\ & u & v & \\ & s & t & \\ x & & & a \end{bmatrix}$.



The sixth main new result characterizes quadrimodular metric spaces:

THEOREM. *6.2.3 (quadrimodularity criterion for metric spaces) Let X be a metric space. X is quadrimodular iff for $x,\, y,\, a,\, b \in X$, $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$ implies that there are $s,\, t,\, u,\, v \in X$ such that* $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$.

Part (3) of the seventh main new result is for a compact quadrimedian geometric topological interval space what [**40**, 3.24] (Krein-Milman theorem) is for a compact convex set in a locally convex real topological vector space:

THEOREM. *6.4.1 (convex closure of the median boundary) Let X be a compact quadrimedian geometric topological interval space.*

(1) *For $a,\, p \in X$:*
  (a) *The poset $(\langle a,\, p,\, \cdot\rangle\,,\, \langle a,\, \cdot,\, \cdot\rangle)$ has a maximal element.*
  (b) *There is a $b \in \partial_M(X)$ such that $\langle a,\, p,\, b\rangle$.*
(2) *For $p \in X$, there are $a,\, b \in \partial_M(X)$ such that $\langle a,\, p,\, b\rangle$.*
(3) *$[\partial_M(X)] = X$.*

Here, 'quadrimedian' cannot be replaced by 'median'.

Let X be metric space.

A *geodesic median closure* of X is a pair $(Y,\, i)$ such that Y is a median metric space, i is an isometric map from X into Y and the median closure of $i(X)$ in Y equals Y . In particular, when Y is a median metric space, X a subspace of Y with median closure Y in Y and i is the inclusion map of X into Y , then $(Y,\, i)$ is a geodesic median closure of X .

X is called *subquadrimedian* iff there is an isometric map from X into a quadrimedian metric space. Each subspace of a quadrimedian metric space is subquadrimedian. A *geodesic quadrimedian closure* of X is a pair $(Y,\, i)$ such that Y is a quadrimedian metric space, i is an isometric map from X into Y and the median closure of $i(X)$ in Y equals Y . In particular, when Y is a quadrimedian metric space, X a subspace of Y with median closure Y in Y and i is the inclusion map of X into Y , then $(Y,\, i)$ is a geodesic quadrimedian closure of X .

The eighth main new result is for the geodesic quadrimedian closure of a metric space what [**25**, 9.22] (theorem of Steinitz) is for the algebraic closure of a field:

THEOREM. *6.5.3 (existence and structural uniqueness of geodesic quadrimedian closure) Let X be a subquadrimedian metric space.*

(1) *X has a geodesic quadrimedian closure.*
(2) *Let $(Y,\, i_Y)$, $(Z,\, i_Z)$ be geodesic quadrimedian closures of X . Then Z is an isometric copy of Y .*

Here, 'geodesic quadrimedian closure' cannot be replaced by 'geodesic median closure'.

Thus, the two structure theorems 6.4.1 (3) (convex closure of the median boundary) and 6.5.3 (existence and structural uniqueness of geodesic quadrimedian closure), which are valid for quadrimedian spaces, but unvalid for median spaces, are analogous to two central structure theoremes of analysis and algebra, the Krein-Milman theorem for a compact convex set in a locally convex real topological vector space and the theorem of Steinitz on the algebraic closure of a field. Therefore, the concept of a quadrimedian interval space seems to be a natural sharpening of the concept of a median interval space.

The nineth main new result answers the question when in a compact arboric topological interval space the median closure of a set equals the whole space:

THEOREM. *6.6.1 (median closure of the median boundary) Let X be a compact arboric topological interval space.*

(1) *For $p \in X$, there are $a,\, b,\, c \in \partial_M(X)$ such that $p = m(a,\, b,\, c)$.*
(2) *The median closure of $\partial_M(X)$ in X equals X .*
(3) *For $Y \subseteq X$, the median closure of Y in X equals X iff $\partial_M(X) \subseteq Y$.*

Finally, 6.5.3 (existence and structural uniqueness of geodesic quadrimedian closure) and 6.6.1 (median closure of the median boundary) are used together to prove the tenth main new result:

THEOREM. *6.6.2 (compact arboric determination by the median boundary) Let X and Y be compact arboric metric spaces. If $\partial_M(Y)$ is an isometric copy of $\partial_M(X)$, then Y is an isometric copy of X .*

CHAPTER 1

Preliminaries

In this chapter, preliminaries on sets, algebraic structures, graphs, order structures, topological structures, metric spaces and interval spaces are collected for reference.

1.1. Sets, Functions, Relations

The following standard concepts have been taken from [**32**]: *map*, as a synomym for *function*, *domain* of a map, *restriction* of a map, $f|A_0$, *composite* of two maps, $g \circ f$, *injective* map, map *onto* a set, *image* of a set under a map, $f(A_0)$, and *preimage* of a set under a map, $f^{-1}(B_0)$ from §2, *equivalence relation* on a set and *equivalence class* of an element with respect to an equivalence relation from §3.

For $m \in \mathbb{Z}_{\geq 0}$,

$$[m] := \{1, 2, ..., m\} \ ,$$
$$S_m := \text{the set of permutations of } [m] \ .$$

Let X be a set.

- For $m \in \mathbb{Z}_{\geq 0}$,

$$\binom{X}{m} := \text{the set of subsets of } X \text{ of size } m \, .$$

- For $m, n \in \mathbb{Z}_{\geq 1}$,

$$X^{m \times n} := \text{the set of } m \times n\text{-matrices in } X \, .$$

Let f be a function.

$$\operatorname{dom} f := \text{the domain of } f \, .$$

1.2. First-Order Structures

General first-order structures. The following standard concepts have been taken from [**21**], where also examples have been provided: *structure*, as a synonym for *first-order structure*, from section 1.1, *homomorphism*, *embedding*, *isomorphism* of first-order structures, *isomorphic* first-order structures and *substructure* of a first-order structure from section 1.2, *product* of a family of first-order structures from section 8.5.

Algebraic structures. The following standard concepts have been taken from [**25**, chapter 4], where also examples have been provided: *vector space* from chapter II, section 1, *linear map of vector spaces* from chapter II, section 3, *group* and *abelian group* from chapter IV, section 1, *ring*, *commutative ring* and *field* from chapter IV, section 4, *algebraic closure* of a field from chapter IX, section 4.

Let X be a set. As usual, a binary operation $\star$ on X is called.

- *idempotent* iff for all $a \in X$, $a \star a = a$.
- *associative* iff for all $a,\ b,\ c \in X$, $(a \star b) \star c = a \star (b \star c)$.

Graphs. Graphs will mainly be used as counterexamples to illustrate the limits of concepts and results. The following standard concepts have been taken from [**19**], where also examples have been provided: *graph*, *point*, as a synonym for *vertex* and *node*, *line*, as a synonym for *edge*, *adjacent points*, as a synonym for *adjacent vertices*, *isomorphic graphs*, *walk*, *path*, *cycle*, *connected graph*, *length* of a walk, *geodesic*, *distance*, as a synonym for the *distance function of a connected graph*, *bipartite graph* and *complete bigraph*, as a synonym for *complete-bipartite graph* , from chapter 2, *tree* from chapter 4.

A graph is a first order structure in the sense that it is determined by the adjacency relation between its vertices.

Let $G = (N,\ E)$ be a graph. For $a,\ b \in N$ and w an a-b-walk, an a-b-*subwalk* of w is a subsequence of w that also is an a-b-walk, and an a-b-*subpath* of w is an an a-b-subwalk that is a path.

The following notations are used:

$$\begin{aligned} C_n &:= \text{cycle with } n \text{ vertices } (n \in \mathbb{Z}_{\geq 3}), \\ K_{m,n} &:= \text{complete-bipartite graph for a partition of } m \text{ and } n \text{ vertices}, \\ l\,(w) &:= \text{length of the walk } w\,. \end{aligned}$$

PROPOSITION 1.2.1. *(connected graphs) Let $(N,\ E)$ be a connected graph.*

(1) *For $b \in N$, $p := (b)$ is a b-b-geodesic with $l\,(p) = 0$.*
(2) *For $p = (a_0,\ a_1,\ ...,\ a_k)$ a geodesic, $p' := (a_k,\ a_{k-1},\ ...,\ a_0)$ is a geodesic with $l\,(p') = l\,(p)$.*
(3) *For $a,\ b \in N$ and w an a-b-walk,*
   (a) *w has an a-b-subpath.*
   (b) *$l\,(w) \geq d\,(a,\ b)$. If $l\,(w) = d\,(a,\ b)$, then w is a geodesic.*
(4) *For $p = (a_0,\ a_1,\ ...,\ a_k)$ a geodesic and $\kappa,\ \lambda \in \{0,\ 1,\ ...,\ k\}$, if $\kappa \leq \lambda$, then the path $q := (a_\kappa,\ a_{\kappa+1},\ ...,\ a_\lambda)$ is a geodesic.*

PROOF.

(1) p is a path, and $l\,(p) = 0$. Consequently, p is a b-b-path of least length, i.e. a b-b-geodesic.
(2) For $q = (b_0,\ b_1,\ ...,\ b_l)$ an a_k-a_0-path it is to be proved that $l \geq k$. Define $q' := (b_l,\ b_{l-1},\ ...,\ b_0)$. From the assumption that q is an a_k-a_0-path it follows:

$$q' \text{ is an } a_0\text{-}a_k\text{-path of length } l\,. \tag{1.2.1}$$

From (1.2.1) and the assumption that p is a geodesic it follows that $l \geq k$.

(3)

(a) The walk w has an a-b-subwalk $p = (b_0, b_1, ..., b_l)$ of least length. It suffices to prove that p is a path. Seeking a contradiction, suppose that p is not a path, i.e. there are $\kappa, \lambda \in \{0, 1, ..., l\}$ such that $\kappa < \lambda$ and $b_\kappa = b_\lambda$. The sequence $(b_0, b_1, ..., b_\kappa, b_{\lambda+1}, b_{\lambda+2}, ..., b_k)$ is an a-b-subwalk of p . Therefore, it is an a-b-subwalk of w of length $k - (\lambda - \kappa) < k$, contradicting minimality of the length of p .

(b) By (3a), there is

$$p \text{ , an } a\text{-}b\text{-subpath of } w \text{ .} \tag{1.2.2}$$

In particular,

$$l(w) \geq l(p) \text{ ,} \tag{1.2.3}$$

$$l(p) \geq d(a, b) \text{ .} \tag{1.2.4}$$

Step 1. $l(w) \geq d(a, b)$ follows from (1.2.3) and (1.2.4).
Step 2. Proof that $l(w) = d(a, b)$ implies that w is a geodesic. (1.2.3), (1.2.4) and the assumption $d(a, b) = l(w)$ imply:

$$l(w) = l(p) \text{ .} \tag{1.2.5}$$

(1.2.5) and (1.2.2) imply $w = p$, and consequently w is a path.

(4) Seeking a contradiction, assume that q is not a geodesic, i.e. there is an a_κ-a_λ-path q' of length less than $\lambda - \kappa$. Then $\kappa < \lambda$, and the concatenation w of $(a_0, a_1, ..., a_\kappa)$, q' and $(a_\lambda, a_{\lambda+1}, ..., a_k)$ is an a_0-a_k-walk satisfying $l(w) < k = d(a_0, a_k)$, contradicting (3b).

□

The following theorem has been cited from [**19**, theorem 4.1].

THEOREM 1.2.2. *(tree criterion) Let $G = (N, E)$ be a graph. G is a tree iff for all $a, b \in N$ there is exactly one a-b-path.*

PROOF. [**19**, theorem 4.1] □

Let $T = (N, E)$ be a tree. For $a, b \in N$, according to 1.2.2 (tree criterion), define

$$\begin{aligned} p_{ab} &:= p^T_{ab} \\ &:= \text{the } a\text{-}b\text{-path,} \\ E(a, b) &:= \text{the set of edges of } p_{ab} \text{ .} \end{aligned}$$

Posets. The following standard concepts have been taken from [**12**], where also examples have been provided: relation *reflexive* on a set, *antisymmetric* relation, *transitive* relation, *order* as a synonym for *partial order* on a set and *ordered set* as a synonym for *poset* from 1.2, *chain* as a synonym for *totally ordered set* from 1.3, *Hasse diagram* from 1.15, *dual of an ordered set* from 1.19, *maximal element*, *minimal element*, *greatest element* and *least element* from 1.23, *down-set* and *up-set* from 1.27, *order-preserving map* from 1.34, *upper bound* and *lower bound* from 2.1, *lattice* from 2.4, *bounded lattice* from 2.12, *distributive lattice* and *modular lattice* from 4.4, *directed set* from 7.7, *directed union* from 7.9, familiy of sets *closed under directed unions* from 7.10.

A poset $(X, \leq)$ is also simply denoted by X when it is clear from the context whether the poset or only the set is meant.

Let X be a poset. For $a \in X$:

- The *principal down-set* of a is the set

$$\begin{aligned}\downarrow a &:= X_{\leq a}\\ &:= \{x \in X | x \leq a\} \ .\end{aligned}$$

- The *principal up-set* of y is the set

$$\begin{aligned}\uparrow a &:= X_{\geq a}\\ &:= \{x \in X | x \geq a\} \ .\end{aligned}$$

These concepts have been taken from [**12**, 1.27].

PROPOSITION 1.2.3. *(posets) Let X be a poset. For $a \in X$ and U an up-set, if a is a maximal element of U , then a is a maximal element of X .*

PROOF. For $x \in X$ it is to be proved that from $a \leq x$ it follows that $a = x$. From the assumptions that U is an up-set, $a \in U$ and $a \leq x$ it follows:

$$x \in U \ . \tag{1.2.6}$$

The assumption that a is a maximal element of U , (1.2.6) and the assumption $a \leq x$ imply $a = x$. □

PROPOSITION 1.2.4. *(directed posets) Let X be a directed poset.*

(1) *Each non-empty finite subset of X has an upper bound in X .*
(2) *Each up-set in X is directed.*

PROOF.

(1) Let $X_0 = \{x_1, x_2, ..., x_k\}$ be a non-empty finite subset of X . Define $u_1 := x_1$. For $\kappa \in \{2, 3, ..., k\}$, once $u_{\kappa-1}$ has been defined, then from the assumption that X is directed it follows that $\{u_{\kappa-1}, x_\kappa\}$ has an upper bound u_κ . By induction, for $\kappa \in [k]$, u_κ is an upper bound of $\{x_1, x_2, ..., x_\kappa\}$. In particular, u_k is an upper bound of X_0 .
(2) Let Y be an up-set in X . For $a, b \in Y$ it is to be proved that there is a $u \in Y$ such that $a \leq u$ and $b \leq u$. From the assumption that X is directed it follows that there is a $u \in X$ such that

$$\begin{aligned}a &\leq u \,, \\ b &\leq u \,.\end{aligned} \tag{1.2.7}$$

It suffices to prove $u \in Y$. This claim follows from the assumptions that $a \in Y$ and Y is an up-set and (1.2.7).

□

The following theorem has been cited from [**12**, 10.3]. Note that the existence of an upper bound of the empty chain in X entails $X \neq \emptyset$.

THEOREM 1.2.5. *(Zorn's Lemma) Let X be a poset. If each chain in X has an upper bound, then X has a maximal element.*

PROOF. [**12**, 10.3] □

Let X be a poset.

X is called a *meet semilattice* iff for all $x,\, y \in X$, $\{x,\, y\}$ has a greatest lower bound $x \wedge y$. In this case, a *meet subsemilattice* of X is a set $Y \subseteq X$ that is closed under passing from $x,\, y$ to $x \wedge y$, i.e. for all $x,\, y \in X$, if $x,\, y \in Y$, then $x \wedge y \in Y$. Each lattice is a meet semilattice. Further examples of meet semilattices are provided by 4.5.1(2) (median geometric interval spaces) below. The concept of a meet semilattice has been taken from [**48**, chapter I, 5.3].

X is called *arboric* iff X is a meet semilattice and for each $a \in X$, the poset $(\downarrow a,\, \leq)$ is a chain. Each chain, for example $(\mathbb{R},\, \leq)$, is an arboric poset. Let $(N,\, E)$ be a tree. For $u \in N$, define the binary relation $\leq_u$ on N by $x \leq_u y$ iff x is on p_{uy} . Then $(N,\, \leq_u)$ is an arboric poset. The concept of an arboric poset has been taken from [**48**, chapter I, 5.3]. There an arboric poset has been called an *order tree*. Since an arboric poset is a poset, and not a tree in the usual sense of graph theory, here the substantive 'poset' together with the adjective 'arboric' or 'tree-like' is preferred.

The following concept has been taken from [**48**, chapter I, 1.5.1]: *totally ordered field*. An example is the field of real numbers with its usual order relation.

1.3. Topological Structures

Topological spaces. The following standard concepts have been taken from [**32**], where also examples have been provided: *topology*, *topological space*, *open set*, *discrete topology* and *indiscrete topology* from §12, *order topology* of a chain from §14, *subspace topology* and *subspace* of a topological space from §16, *closed set* and *neighborhood* of a point from §17, *continuous function*, as a synonym for *continuous map*, from §18, *product topology* and *product space*, as a synonym for *product* of topological spaces, from §19, *compact topological space* from §26. Each finite topological space is compact.

A topological space $(X,\, O)$ is also simply denoted by X when it is clear from the context whether the topological space or only the set is meant.

Part (1) of the following theorem has been cited from [**32**, theorem 18.1], part (2) from [**32**, theorem 18.2(a)] and part (3) from [**32**, theorem 18.2(b)].

THEOREM 1.3.1. *(continuous maps) Let X, Y be topological spaces and $f : X \to Y$.*

(1) *f is continuous iff for each closed set A in Y , $f^{-1}(A)$ is closed in X .*
(2) *If f is constant, then f is continuous.*
(3) *If $X \subseteq Y$ and f is the inclusion map, then f is continuous.*

PROOF.

(1) [**32**, theorem 18.1]
(2) [**32**, theorem 18.2(a)]
(3) [**32**, theorem 18.2(b)]

□

The following theorem has been cited from [**32**, theorem 18.2(c)].

THEOREM 1.3.2. *(composite of continuous maps) Let X, Y, Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

PROOF. [**32**, theorem 18.2(c)] □

The following theorem has been cited from [**32**, theorem 21.5].

THEOREM 1.3.3. *(sum and difference of continuous functions) Let X be a topological space. For f, $g : X \to \mathbb{R}$, if f and g are continuous, then $f + g$ and $f - g$ are continuous.*

PROOF. [**32**, theorem 21.5] □

Part (1) of the following proposition has been cited from [**32**, theorem 19.5]. Part (2) has been cited from the proof of [**32**, theorem 19.6]. Part (3) has been cited from [**32**, theorem 19.6].

PROPOSITION 1.3.4. *(product of topological spaces) $(X_\alpha)_{\alpha\in J}$ be a family of topological spaces.*

(1) *For $(A_\alpha)_{\alpha\in J}$ a family such that for $\alpha \in J$, $A_\alpha \subseteq X_\alpha$, $\overline{\prod_{\alpha\in J} A_\alpha} = \prod_{\alpha\in J} \overline{A_\alpha}$.*
(2) *For $\alpha_0 \in J$, the map from $\prod_{\alpha\in J} X_\alpha$ to X_{α_0} mapping x to x_{α_0} is continuous.*
(3) *For A a topological space and $(f_\alpha)_{\alpha\in J}$ a familiy such that $f_\alpha : A \to X_\alpha$, the map $f : A \to \prod_{\alpha\in J} X_\alpha$ mapping x to $(f_\alpha(x))_{\alpha\in J}$ is continuous iff for each $\alpha \in J$, f_α is continuous.*

PROOF.

(1) [**32**, theorem 19.5]
(2) [**32**, theorem 19.6]
(3) [**32**, theorem 19.6]

□

PROPOSITION 1.3.5. *(discrete topological spaces) The product of a finite family of discrete topological spaces is discrete.*

PROOF. Let J be a finite set and $(X_\alpha)_{\alpha \in J}$ a family of discrete topological spaces with product X . It suffices to prove that for $a = (a_\alpha)_{\alpha \in J} \in X$, $\{a\}$ is open. For $\alpha \in J$, let $\pi_\alpha : X \to X_\alpha$ be the projection map. Then

$$\{a\} = \bigcap_{\alpha \in J} \pi_\alpha^{-1}(\{a_\alpha\}) \ . \tag{1.3.1}$$

The assumption that $(X_\alpha)_{\alpha \in J}$ is a family of discrete topological spaces entails:

$$\text{For } \alpha \in J \text{ , } \{a_\alpha\} \text{ is open.} \tag{1.3.2}$$

From (1.3.2) and continuity of π_α it follows:

$$\text{For } \alpha \in J \text{ , } \pi_\alpha^{-1}(\{a_\alpha\}) \text{ is open.} \tag{1.3.3}$$

(1.3.1), (1.3.3) and the assumption that J is finite imply that $\{a\}$ is open. □

The following theorem has been cited from [**32**, theorem 26.9].

THEOREM 1.3.6. *(compactness criterion) Let X be a topological space. X is compact iff the following condition is satisfied: For each C a non-empty set of closed sets, if for all non-empty finite $C_0 \subseteq C$, $\bigcap C_0 \neq \emptyset$, then $\bigcap C \neq \emptyset$.*

PROOF. [**32**, theorem 26.9] □

The following theorem has been cited from [**32**, theorem 26.2].

THEOREM 1.3.7. *(compact topological spaces) Let X be a compact topological space. Then each closed subspace of X is compact.*

PROOF. [**32**, theorem 26.2] □

Topological vector spaces. The following standard concepts have been taken from [**40**]: *topological vector space* over $\mathbb{R}$, as a synonym for *real topological vector space* from section 1.6, *local base* and *locally convex topological vector space* from section 1.8.

Topological posets. A *topological poset* is a triple $(X, \leq, O)$ such that:

- $(X, \leq)$ is a poset.
- (X, O) is a topological space.
- $\leq$ is a closed subset of the product space $X \times X$.

When $(X, \leq)$ is a chain with order topology O , then $(X, \leq, O)$ is a topological poset. Each poset with the discrete topology is a topological poset. Further examples of topological posets are provided by 2.5.1 (geometric topological interval spaces) below. The concept of a topological poset is implicit in the results on topological spaces equipped with a closed order in [**33**, chapter 1, §1 and §3]. Sometimes, for instance in [**48**, chapter III, 1.2.3], a topological poset is called a *pospace*.

Part (1) of the following proposition has been cited from [**48**, chapter III, 1.2.3].

PROPOSITION 1.3.8. *(topological posets) Let X be a topological poset.*

(1) *For $a \in X$, $\uparrow a$ is closed.*
(2) *For a chain C in X , $\bar{C}$ is also a chain.*

PROOF.

(1) The assumption that X is a topological poset entails:
$$\leq \text{ is a closed subset of the product space } X \times X\,. \tag{1.3.4}$$
By 1.3.1(3) (continuous maps),
$$\text{The map from } X \text{ to } X \text{ mapping } x \text{ to } x \text{ is continuous.} \tag{1.3.5}$$
By 1.3.1(2) (continuous maps),
$$\text{The map from } X \text{ to } X \text{ mapping } x \text{ to } a \text{ is continuous.} \tag{1.3.6}$$
From (1.3.5) and (1.3.6) it follows by 1.3.4(3) (product of topological spaces):
$$\text{The map } i_a : X \to X \times X \text{ mapping } x \text{ to } (a,\, x) \text{ is continuous.} \tag{1.3.7}$$
$\uparrow a = i_a^{-1}(\leq)$, (1.3.4) and (1.3.7) imply by 1.3.1(1) (continuous maps) that $\uparrow a$ is closed.
(2) It is to be proved that for $a,\, b \in \bar{C}$, $a \leq b$ or $a \geq b$, i.e. $\bar{C} \times \bar{C} \subseteq \leq \cup \geq$. The assumption that C is a chain in X says that for $a,\, b \in C$, $a \leq b$ or $a \geq b$, i.e.
$$C \times C \subseteq \leq \cup \geq\ . \tag{1.3.8}$$
The assumption that X is a topological poset entails:
$$\leq \text{ is closed in } X \times X\,. \tag{1.3.9}$$
The projection maps from $X \times X$ to X mapping $(x,\, y)$ to x and y respectively are continuous. By 1.3.4(3) (product of topological spaces),
$$\text{The map } f \text{ from } X \times X \text{ to } X \times X \text{ mapping } (x,\, y) \text{ to } (y,\, x) \text{ is continuous.} \tag{1.3.10}$$
From (1.3.10) and $\geq = f^{-1}(\leq)$ it follows by 1.3.1(1) (continuous maps):
$$\geq \text{ is closed in } X \times X\,. \tag{1.3.11}$$
(1.3.9) and (1.3.11) imply:
$$\leq \cup \geq \text{ is closed in } X \times X\,. \tag{1.3.12}$$
From (1.3.8) and (1.3.12) it follows:
$$\overline{C \times C} \subseteq \leq \cup \geq\ . \tag{1.3.13}$$
By 1.3.4(1) (product of topological spaces),
$$\overline{C \times C} = \bar{C} \times \bar{C}\,. \tag{1.3.14}$$
Substituting (1.3.14) into (1.3.13), $\bar{C} \times \bar{C} \subseteq \leq \cup \geq$.

□

PROPOSITION 1.3.9. *(subspaces of a topological poset) Let $(X, \leq, O)$ be a topological poset. For $Y \subseteq X$, the triple consisting of Y, $\leq |Y$ and the subspace topology on Y is a topological poset.*

PROOF. Immediate from the definitions. □

Let $(X, \leq, O)$ be a topological poset. For $Y \subseteq X$, unless otherwise stated, the partial order and the topology as specified in 1.3.9 (subspaces of a topological poset) are used.

1.4. General Metric Spaces and Interval Spaces

Metric spaces and normed vector spaces. The following standard concepts have been taken from [**42**], where also examples have been provided: *metric* and *metric space* from definition 1.1.1, *metric subspace* as a synonym for *subspace* of a metric space and *metric superspace* as a synonym for *superspace* of a metric space from definition 1.3.1, *isometric map* and *isometric copy* from defintion 1.4.1, *norm* on a vector space and *normed linear space* over $\mathbb{R}$, as a synonym for *real normed vector space*, from definition 1.7.1, *metric determined by a norm* from definition 1.7.3, *distance from a point to a set* from definition 2.2.1, *distance from a set to a set* from definition 2.7.1, *topology determined by a metric* from definition 4.3.1, *open ball* and *closed ball* from defintion 5.1.1. The following notation is used:

$$O_d := \text{the topology determined by the metric } d\,.$$

Examples of real normed vector spaces and metric spaces that serve as running examples for a number of concepts are provided by the next theorem and the next proposition.

A metric space (X, d) is also simply denoted by X when it is clear from the context whether the metric space or only the set is meant.

For $n \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ define the function

$$\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$$

by

$$\|x\|_p := \begin{cases} \left(\sum_{\nu=1}^{n} |x_\nu|^p\right)^{\frac{1}{p}} & \text{if } p \in \mathbb{R}_{\geq 1} \\ \max\{|x_1|, |x_2|, \ldots, |x_n|\} & \text{if } p = \infty \end{cases}.$$

The following theorem has been cited from [**42**, theorem 12.11.3] and is a particular case of Minkowski's theorem on L^p-spaces over a measure space.

THEOREM 1.4.1. *(Minkowski's Theorem) For $n \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$, $\left(\mathbb{R}^n, \|\cdot\|_p\right)$ is a real normed vector space. In particular, $\left(\mathbb{R}^n, \|\cdot - \cdot\|_p\right)$ is a metric space.*

PROOF. [**42**, theorem 12.11.3] □

The following proposition has been cited from [**19**, chapter 2].

PROPOSITION 1.4.2. *(distance function of a connected graph) Let* $(N,\,E)$ *be a connected graph with distance function* d *. Then* d *is a metric on* N *.*

PROOF. Step 1. For $x \in N$, $d\,(x,\,x) = 0$ by 1.2.1(1) (connected graphs).

Step 2. Proof that for $x,\,y \in N$, if $x \neq y$, then $d\,(x,\,y) > 0$. There is an x-y-geodesic $p = (a_0,\,a_1,\,...,\,a_k)$. From the assumption $x \neq y$ it follows that $k > 0$, i.e. $l\,(p) > 0$, i.e. $d\,(x,\,y) > 0$.

Step 3. For $x,\,y \in N$, $d\,(x,\,y) = d\,(y,\,x)$ by 1.2.1(2) (connected graphs).

Step 4. Proof that for or $x,\,y,\,z \in N$, $d\,(x,\,z) \leq d\,(x,\,y) + d\,(y,\,z)$. There are an x-y-geodesic and a y-z-geodesic. Their concatenation is an x-z-walk of length $d\,(x,\,y) + d\,(y,\,z)$. By 1.2.1(3b) (connected graphs), $d\,(x,\,y) + d\,(y,\,z) \geq d\,(x,\,z)$. ☐

1.4.2 (distance function of a connected graph) may be used implicitly by applying results on metric spaces to connected graphs. Unless otherwise stated, metric space concepts and interval space concepts, applied to a connected graph, refer to the metric in 1.4.2 (distance function of a connected graph).

Let X be a metric space. For convenience, the following simplified notations are used:

- For $a,\,b \in X$,

$$d_{ab} := d\,(a,\,b)\ .$$

- For $a \in X$ and non-empty $B \subseteq X$,

$$d_{aB} := d\,(a,\,B)$$

- For non-empty $A,\,B \subseteq X$,

$$d_{AB} := d\,(A,\,B)$$

The same letter d may be employed for the metrics of several metric spaces at the same time. This simultaneous use is in accordance with the usual employment of the same operation and relation symbols in several structures of the same type. In such cases, the meaning of d is clear from the context.

Part (1) of the following proposition has been cited from [**39**, chapter III, §1].

PROPOSITION 1.4.3. *(metric spaces) Let* X *be a metric space. For* $k \in \mathbb{Z}_{\geq 0}$ *and* $a_0,\,a_1,\,...,\,a_k \in X$:

(1) $d_{a_0 a_k} \leq \sum_{\kappa=1}^{k} d_{a_{\kappa-1} a_\kappa}$.

(2) *If* $d_{a_0 a_k} = \sum_{\kappa=1}^{k} d_{a_{\kappa-1} a_\kappa}$ *, then for* $p,\,q \in \{0,\,1,\,...,\,k\}$: *if* $p < q$ *, then* $d_{a_p a_q} = \sum_{\kappa=p+1}^{q} d_{a_{\kappa-1} a_\kappa}$.

PROOF.

(1) [**39**, III.1]

(2) Seeking a contradiction, assume $d_{a_p a_q} \neq \sum_{\kappa=p+1}^{q} d_{a_{\kappa-1} a_\kappa}$. By (1),

$$d_{a_0 a_p} \leq \sum_{\kappa=1}^{p} d_{a_{\kappa-1} a_\kappa} \ ,$$
$$d_{a_p a_q} < \sum_{\kappa=p+1}^{q} d_{a_{\kappa-1} a_\kappa} \ ,$$
$$d_{a_q a_k} \leq \sum_{\kappa=q+1}^{k} d_{a_{\kappa-1} a_\kappa} \ .$$

Therefore by (1):

$$\begin{aligned} d_{a_0 a_k} &\leq d_{a_0 a_p} + d_{a_p a_q} + d_{a_q a_k} \\ &< \sum_{\kappa=1}^{p} d_{a_{\kappa-1} a_\kappa} + \sum_{\kappa=p+1}^{q} d_{a_{\kappa-1} a_\kappa} + \sum_{\kappa=q+1}^{k} d_{a_{\kappa-1} a_\kappa} \\ &= \sum_{\kappa=1}^{k} d_{a_{\kappa-1} a_\kappa} \ . \end{aligned}$$

Thus, $d_{a_0 a_k} < \sum_{\kappa=1}^{k} d_{a_{\kappa-1} a_\kappa}$, contradicting the assumption

$$d_{a_0 a_k} = \sum_{\kappa=1}^{k} d_{a_{\kappa-1} a_\kappa} \ .$$

□

Interval spaces. Let

$$R = \langle \cdot,\, \cdot,\, \cdot \rangle$$

be a ternary relation on a set X , i.e. $R \subseteq X \times X \times X$ and

$$\langle x,\, y,\, z \rangle \text{ iff } (x,\, y,\, z) \in R \, .$$

The *sections* of $\langle \cdot,\, \cdot,\, \cdot \rangle$ are:

- for $a \in X$, the *binary sections*, defined as the following binary relations on X : the $(1,\, 2)$-section $\langle \cdot,\, \cdot,\, a \rangle$, the $(1,\, 3)$-section $\langle \cdot,\, a,\, \cdot \rangle$ and the $(2,\, 3)$-section $\langle a,\, \cdot,\, \cdot \rangle$ defined by

$$\begin{aligned} (u,\, v) \in \langle \cdot,\, \cdot,\, a \rangle &:\Leftrightarrow \langle u,\, v,\, a \rangle \ , \\ (u,\, v) \in \langle \cdot,\, a,\, \cdot \rangle &:\Leftrightarrow \langle u,\, a,\, v \rangle \ , \\ (u,\, v) \in \langle a,\, \cdot,\, \cdot \rangle &:\Leftrightarrow \langle a,\, u,\, v \rangle \ , \end{aligned}$$

- for $a,\, b \in X$, the *unary sections*, defined as the following subsets of X : the 1-section $\langle \cdot,\, a,\, b \rangle$, the 2-section $\langle a,\, \cdot,\, b \rangle$ and the 3-section $\langle a,\, b,\, \cdot \rangle$ defined by

$$\begin{aligned} u \in \langle \cdot,\, a,\, b \rangle &:\Leftrightarrow \langle u,\, a,\, b \rangle \ . \\ u \in \langle a,\, \cdot,\, b \rangle &:\Leftrightarrow \langle a,\, u,\, b \rangle \ , \\ u \in \langle a,\, b,\, \cdot \rangle &:\Leftrightarrow \langle a,\, b,\, u \rangle \ . \end{aligned}$$

The associated *strict*, *left-strict* and *right-strict* ternary relations are the ternary relations $\langle \cdot \neq \cdot \neq \cdot \rangle$, $\langle \cdot \neq \cdot,\, \cdot \rangle$, $\langle \cdot,\, \cdot \neq \cdot \rangle$, on X defined by

$$\begin{aligned} \langle x \neq y \neq z \rangle &\text{iff } \langle x,\, y,\, z \rangle \text{ and } x \neq y \text{ and } y \neq z \, , \\ \langle x \neq y,\, z \rangle &\text{iff } \langle x,\, y,\, z \rangle \text{ and } x \neq y \, , \\ \langle x,\, y \neq z \rangle &\text{iff } \langle x,\, y,\, z \rangle \text{ and } y \neq z \, . \end{aligned}$$

Examples of ternary relations are the pseudointerval relations, which are interpreted in terms of some of their sections: A *pseudointerval relation* on X is a ternary relation $\langle \cdot,\, \cdot,\, \cdot \rangle$ on X such that the following conditions are satisfied:

- For $a \in X$, the binary relations $\langle \cdot,\, \cdot,\, a \rangle$ and $\langle a,\, \cdot,\, \cdot \rangle$ are reflexive on X ,
- for $a \in X$, the binary relation $\langle \cdot,\, a,\, \cdot \rangle$ is symmetric,

i.e. with the map $[\cdot,\, \cdot] :\ X \times X \to P\,(X)$ defined by

$$\begin{aligned} [a,\, b] &:= \langle a,\, \cdot,\, b \rangle \\ &= \{ x \in X \mid \langle a,\, x,\, b \rangle \} \ , \end{aligned}$$

for $a,\, b \in X$, $a,\, b \in [a,\, b]$, and for $a,\, b \in X$, $[a,\, b] = [b,\, a]$. Symmetry of $\langle \cdot,\, x,\, \cdot \rangle$ implies that reflexivity of $\langle a,\, \cdot,\, \cdot \rangle$ can be omitted from the definition of a pseudointerval relation. The set $[a,\, b]$ is called the *interval* between a and b . A *pseudointerval space* is a pair consisting of a set X and a pseudointerval relation on X . A pseudointerval relation can be interpreted in terms of some of its sections as follows: For $a,\, b \in X$,

- $u \in \langle a,\, \cdot,\, b \rangle$ says: u is between a and b .

For $a \in X$,

- $(u,\, v) \in \langle a,\, \cdot,\, \cdot \rangle$ says: u is in front of v when viewed from a .
- $(u,\, v) \in \langle \cdot,\, a,\, \cdot \rangle$ says: u is separated from v by a .

An *interval relation* on X is a pseudointerval relation $\langle \cdot,\, \cdot,\, \cdot \rangle$ on X such that one and therefore each of the following equivalent conditions is satisfied:

- For $x,\, y \in X$, $\langle x,\, y,\, x \rangle$ implies $y = x$,
- For $a,\, b,\, c \in X$, $\langle a \neq b,\, c \rangle$ implies $a \neq c$,
- For $x \in X$, $[x,\, x] = \{x\}$.

An *interval space* is a pair consisting of a set X and an interval relation on X .

Let X be a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$. The ternary relation $\langle \cdot,\, \cdot,\, \cdot \rangle$ on X defined by

$$\langle x,\, y,\, z \rangle :\Leftrightarrow \text{There is a } \lambda \in K \text{ such that } 0 \leq \lambda \leq 1 \text{ and } y = x + \lambda\,(z - x) \ ,$$

is an interval relation on X . It is called the *vector interval relation* of X . $(X, \langle\cdot, \cdot, \cdot\rangle)$ is called the *vector interval space* associated with X . For $a, b \in X$, $[a, b]$ is the straight-line segment from a to b . This example has been taken from [**48**, chapter I, 4.2].

Unless otherwise stated, interval space concepts, applied to a vector space, refer to its vector interval relation.

Let X be a lattice, for example $\mathbb{R}^2$ with the componentwise partial order. The ternary relation $\langle\cdot, \cdot, \cdot\rangle$ on X defined by

$$\langle x, y, z\rangle :\Leftrightarrow x \wedge z \leq y \leq x \vee z ,$$

is an interval relation on X . It is called the *lattice interval relation* of X . $(X, \langle\cdot, \cdot, \cdot\rangle)$ is called the *lattice interval space* associated with X . This example has also been taken from [**48**, chapter I, 4.2]. For example, in $\mathbb{R}^2$ with the componentwise partial order, intervals are rectangles parallel to the coordinate axes.

Unless otherwise stated, interval space concepts, applied to a lattice, refer to its lattice interval relation.

Further examples of interval relations are provided by and presented after the next two propositions.

The concept of a pseudointerval space coincides with the concept of an *interval convexity* in [**6**, after corollary 2.1]. There it has been defined in terms of intervals. Here, relational notation is used, as has been done before in several variants, for example in [**37**], [**13**], [**20**], [**9**] and [**35**]. The relational notation $'\langle x, y, z\rangle'$ directly visualizes a geometric situation, immediately generalizes to more than three terms and makes explicit that order geometry starts as a first-order theory. Here, the set operational expression $'[x, z]'$ is used only when it is more convenient. In [**48**, chapter I, 4.1], a pseudointerval space is called an interval space. The concept of an interval space as defined in [**44**, section 2] involves a topology on the set X . An interval space as defined there for which the topology is indiscrete is the same as a pseudointerval space with the indiscrete topology added.

The concept of an interval space has been taken from [**49**, chapter I, 3.1]. There it has been defined in terms of intervals. In [**48**, chapter I, 4.10], an interval space is called an *idempotent interval space*. The terminology adopted here is parallel to metric space terminology: Interval spaces correspond to metric spaces, see the next proposition, while pseudointerval spaces correspond to pseudometric spaces.

Sometimes, the condition $\langle x, y, z\rangle$ is simply written by juxtaposition as xyz . Here, the more elaborate notation $\langle x, y, z\rangle$ has been chosen because juxtapostion already has several other uses in mathematics, and also in order to make the notation for the binary and unary sections, $\langle\cdot, \cdot, z\rangle$, $\langle\cdot, y, \cdot\rangle$, $\langle x, \cdot, \cdot\rangle$, $\langle\cdot, y, z\rangle$, $\langle x, \cdot, z\rangle$ and $\langle x, y, \cdot\rangle$, more distinct than the analogous notation $\cdot\cdot z$, $\cdot y\cdot$, $x\cdot\cdot$, $\cdot yz$, $x\cdot z$, $xy\cdot$ might be. The notation $[a, b]$ for $\langle a, \cdot, b\rangle$ has been taken from [**10**, chapter II, section 2]. In [**49**, chapter I, 3.1], $\langle a, \cdot, b\rangle$ is written as $I\,(a, b)$, and in [**48**, chapter I, 4.1], it is also written as ab , and $\langle a, b, \cdot\rangle$ is written as b/a .

An interval space $(X, \langle\cdot, \cdot, \cdot\rangle)$ is also simply denoted by X when it is clear from the context whether the interval space or only the set is meant.

The following proposition has been cited from [**48**, chapter I, 4.2].

PROPOSITION 1.4.4. *(interval relation of a metric space) Let X be a metric space. The ternary relation $\langle\cdot,\,\cdot,\,\cdot\rangle = \langle\cdot,\,\cdot,\,\cdot\rangle_d$ on X be defined by*

$$\langle x,\, y,\, z\rangle_d :\Leftrightarrow d_{xz} = d_{xy} + d_{yz}$$

is an interval relation on X .

PROOF. Step 1. Proof that for $x \in X$, the binary relation $\langle x,\,\cdot,\,\cdot\rangle$ is reflexive on X , i.e. for $y \in X$, $\langle x,\, y,\, y\rangle$, i.e. $d_{xy} = d_{xy} + d_{yy}$. This claim follows from $d_{yy} = 0$.

Step 2. Proof that for $y \in X$, the binary relation $\langle\cdot,\, y,\,\cdot\rangle$ is symmetric, i.e. for $x,\, z \in X$, $\langle x,\, y,\, z\rangle$ implies $\langle z,\, y,\, x\rangle$, i.e. $d_{xz} = d_{xy} + d_{yz}$ implies $d_{zx} = d_{zy} + d_{yx}$. This claim follows from the symmetry of d .

Step 3. Proof that for $x,\, y \in X$, $\langle x,\, y,\, x\rangle$ implies $x = y$. The assumption $\langle x,\, y,\, x\rangle$ says $d_{xx} = d_{xy} + d_{yx}$. Therefore, $0 = d_{xy} + d_{xy}$. Thus, $d_{xy} = 0$. Consequently, $x = y$. ☐

Let $(X,\, d)$ be a metric space. By 1.4.4 (interval relation of a metric space), the ternary relation $\langle\cdot,\,\cdot,\,\cdot\rangle = \langle\cdot,\,\cdot,\,\cdot\rangle_d$ on X defined by

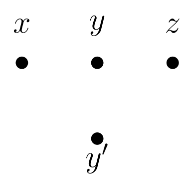
$$\langle x,\, y,\, z\rangle_d :\Leftrightarrow d_{xz} = d_{xy} + d_{yz}\,,$$

is an interval relation on X . It is called the *geodesic interval relation* of X . The interval space $(X,\, \langle\cdot,\,\cdot,\,\cdot\rangle_d)$ is called the *geodesic interval space* associated with $(X,\, d)$. In the interval space $(X,\, \langle\cdot,\,\cdot,\,\cdot\rangle_d)$, for $a,\, b \in X$,

$$\begin{aligned}[a,\, b]_d &:= [a,\, b]\\ &= \{x \in X | d_{ab} = d_{ax} + d_{xb}\}\end{aligned}$$

is called the *geodesic interval* from a to b . For example:

- In $(\mathbb{R}^2,\, \|\cdot - \cdot\|_1)$, where balls are squares parallel to the coordinate diagonals, geodesic intervals are rectangles parallel to the coordinate axes.
- For $n \in \mathbb{Z}_{\geq 1}$, in $(\mathbb{R}^n,\, \|\cdot - \cdot\|_2)$, the geodesic interval relation coincides with the vector interval relation. In the following example in $(\mathbb{R}^2,\, \|\cdot - \cdot\|_2)$, $\langle x,\, y,\, z\rangle$, but not $\langle x,\, y',\, z\rangle$.



- In $(\mathbb{R}^2,\, \|\cdot - \cdot\|_\infty)$, where balls are squares parallel to the coordinate axes, geodesic intervals are rectangles parallel to the coordinate diagonals.

1.4.4 (interval relation of a metric space) may be used implicitly by applying results on interval spaces to metric spaces. Unless otherwise stated, interval space concepts, applied to a metric space, refer to its geodesic interval relation.

The geodesic interval relation of a metric space has been defined in [**29**, Erste Untersuchung, section 2]. The term 'geodesic interval' has been taken from [**48**, chapter I, 4.2].

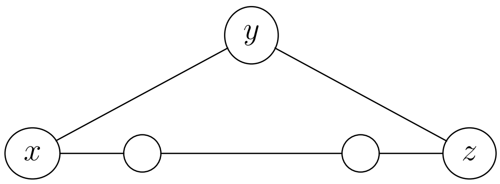
PROPOSITION 1.4.5. *(interval relation of a connected graph) Let* $(N,\ E)$ *be a connected graph with distance function* d *. For* $x,\ y,\ z\ \in\ N$ *,* $\langle x,\ y,\ z\rangle_d$ *iff* y *is on an* x*-*z*-geodesic, i.e.* $[x,\ z]_d$ *is the union of the* x*-*z*-geodesics.*

PROOF. Step 1. ($\Rightarrow$) From the assumption $\langle x,\ y,\ z\rangle_d$, i.e. $d\,(x,\ z)\ =\ d\,(x,\ y) + d\,(y,\ z)$, it is to be proved that y is on an x-z-geodesic. There are an x-y-geodesic of length $d\,(x,\ y)$ and a y-z-geodesic of length $d\,(y,\ z)$. y is on their concatenation w . w is an x-z-walk of length $d\,(x,\ y) + d\,(y,\ z) = d\,(x,\ z)$. By 1.2.1(3b) (connected graphs), w is an x-z-geodesic.

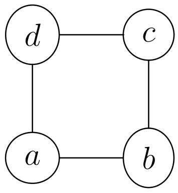
Step 2. ($\Leftarrow$) Suppose that y is on an x-z-geodesic p , say $p\ =\ (a_0,\ a_1,\ ...,\ a_k)$ and $\kappa \in \{0,\ 1,\ ...,\ k\}$ is such that $y\ =\ a_\kappa$. 1.2.1(4) (connected graphs), $(a_0,\ a_1,\ ...,\ a_\kappa)$ is an x-y-geodesic, and $(a_\kappa,\ a_{\kappa+1},\ ...,\ a_k)$ is a y-z-geodesic. Therefore, $d\,(x,\ y) = \kappa$ and $d\,(y,\ z) = k-\kappa$. Consequently, $d\,(x,\ y) + d\,(y,\ z) = k$, i.e. $d\,(x,\ y) + d\,(y,\ z) = d\,(x,\ z)$. □

For example, according to 1.4.5 (interval relation of a connected graph),

- In the following graph, $[x,\ z] = \{x,\ y,\ z\}$.



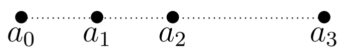
- In the following graph, $[a,\ c] = \{a,\ b,\ c,\ d\}$.



- In a tree $(N,\ E)$ with distance function d , for $x,\ y,\ z \in N$, $\langle x,\ y,\ z\rangle_d$ iff y is on p_{xz} .

Let X be an interval space. For $k \in \mathbb{Z}_{\geq 0}$, a finite sequence $(a_0,\ a_1,\ ...,\ a_k)$ in X is called *aligned* iff it satisfies one and therefore each of the following equivalent conditions:

- For all $p,\ q,\ r \in \{0,\ 1,\ ...,\ k\}$, $p < q < r$ implies $\langle a_p,\ a_q,\ a_r\rangle$.
- For all $p,\ q,\ r \in \{0,\ 1,\ ...,\ k\}$, $p \leq q \leq r$ implies $\langle a_p,\ a_q,\ a_r\rangle$.



Each one-term sequence (a_0) and each two-term sequence $(a_0,\ a_1)$ are aligned. A three-term sequence $(a_0,\, a_1,\, a_2)$ is aligned iff $\langle a_0,\, a_1,\, a_2\rangle$. Therefore the notation '$\langle a_0,\, a_1,\, a_2\rangle$' can be generalized as follows:

$$\langle a_0,\, a_1,\, ...,\, a_k\rangle :\Leftrightarrow (a_0,\, a_1,\, ...,\, a_k) \text{ is aligned.}$$

Examples of aligned sequences with more than 3 terms are provided by the next proposition. The concept of an aligned sequence has been taken from [**43**, section 4]. There an aligned sequence is called a chain, the concept has been defined in a similar context, and the condition $\langle a_0,\, a_1,\, ...,\, a_k\rangle$ is simply written by juxtaposition as $a_0 a_1 ... a_k$. Here, the more elaborate notation $\langle a_0,\, a_1,\, ...,\, a_k\rangle$ has been chosen for general k for the same reason as for the case $k = 3$.

PROPOSITION 1.4.6. *(aligned sequences in a metric space) Let X be a metric space. For $k \in \mathbb{Z}_{\geq 2}$ and $a = (a_0,\, a_1,\, ...,\, a_k)$ a sequence in X , a is aligned iff $d_{a_0 a_k} = \sum_{\kappa=1}^{k} d_{a_{\kappa-1} a_\kappa}$.*

PROOF. Step 1. ($\Rightarrow$) Suppose that $(a_0,\, a_1,\, ...,\, a_k)$ is aligned. It suffices to prove by induction that for $l \in \{2,\, 3,\, ...,\, k\}$, $d_{a_0 a_l} = \sum_{\kappa=1}^{l} d_{a_{\kappa-1} a_\kappa}$.

Step 1.1. $l = 2$. The assumption $\langle a_0,\, a_1,\, ...,\, a_k\rangle$ entails $\langle a_0,\, a_1,\, a_2\rangle$, i.e. $d_{a_0 a_2} = d_{a_0 a_1} + d_{a_1 a_2}$.

Step 1.2. $l - 1 \to l$. Suppose $l \in \{3,\, 4,\, ...,\, k\}$ and

$$d_{a_0 a_{l-1}} = \sum_{\kappa=1}^{l-1} d_{a_{\kappa-1} a_\kappa} \,. \tag{1.4.1}$$

The assumption $\langle a_0,\, a_1,\, ...,\, a_k\rangle$ entails $\langle a_0,\, a_{l-1},\, a_l\rangle$, i.e.

$$d_{a_0 a_l} = d_{a_0 a_{l-1}} + d_{a_{l-1} a_l} \,. \tag{1.4.2}$$

Substituting (1.4.1) into (1.4.2),

$$\begin{aligned} d_{a_0 a_l} &= \left(\sum_{\kappa=1}^{l-1} d_{a_{\kappa-1} a_\kappa}\right) + d_{a_{l-1} a_l} \\ &= \sum_{\kappa=1}^{l} d_{a_{\kappa-1} a_\kappa} \,. \end{aligned}$$

Step 2. ($\Leftarrow$) From the assumption $d_{a_0 a_k} = \sum_{\kappa=1}^{k} d_{a_{\kappa-1} a_\kappa}$ it is to be proved that for $p,\, q,\, r \in \{0,\, 1,\, ...,\, k\}$, $p < q < r$ implies $d_{a_p a_r} = d_{a_p a_q} + d_{a_q a_r}$. By 1.4.3(2) (metric spaces),

$$\begin{aligned} d_{a_p a_r} &= \sum_{\kappa=p+1}^{r} d_{a_{\kappa-1} a_\kappa} \,, \\ d_{a_p a_q} &= \sum_{\kappa=p+1}^{q} d_{a_{\kappa-1} a_\kappa} \,, \\ d_{a_q a_r} &= \sum_{\kappa=q+1}^{r} d_{a_{\kappa-1} a_\kappa} \,. \end{aligned}$$

Consequently,

$$\begin{aligned} d_{a_p a_r} &= \sum_{\kappa=p+1}^{r} d_{a_{\kappa-1} a_\kappa} \\ &= \left(\sum_{\kappa=p+1}^{q} d_{a_{\kappa-1} a_\kappa} \right) + \left(\sum_{\kappa=q+1}^{r} d_{a_{\kappa-1} a_\kappa} \right) \\ &= d_{a_p a_q} + d_{a_q a_r} \,. \end{aligned}$$

□

In [**7**, definition 2.2] an aligned sequence in a metric space is called a geodesic sequence.

Let $a = (a_0, a_1, ..., a_k)$ be a finite sequence. A *finite pseudosubsequence* of a is a composition $a \circ i = \left(a_{i(0)}, a_{i(1)}, ..., a_{i(l)}\right)$ for an order-preserving function $i : \{0, 1, ..., l\} \to \{0, 1, ..., k\}$ for an $l \in \mathbb{Z}_{\geq 0}$. A finite pseudosubsequence of a may be longer than a . For example, (a_1, a_1, u, a_2) is a finite pseudosubsequence of (a_1, u, a_2) .

Let X be an interval space. The new concept of an interval-spanning set is defined as follows: A subset A of X is called *interval-spanning* iff for all $x, y \in X$, there are $w, z \in A$ such that $\langle w, x, y, z\rangle$. Examples of interval-spanning sets are provided by part (1d) of the following proposition, 1.4.9(2b) (median triangles) below and 2.2.2(3) (median quadrangles) below.

PROPOSITION 1.4.7. *(aligned sequences) Let X be an interval space.*

(1) *For $k \in \mathbb{Z}_{\geq 1}$ and $S = (x_0, x_1, ..., x_k)$ a sequence in X :*
   (a) *If S is aligned, then each finite pseudosubsequence of S is aligned.*
   (b) *For S, T finite sequences, if S and T are pseudosubsequences of each other, then: S is aligned iff T is aligned.*
   (c) *$(x_0, x_1, ..., x_k)$ is aligned iff $(x_k, x_{k-1}, ..., x_0)$ is aligned.*
   (d) *If S is aligned, then in $(\{x_0, x_1, ..., x_k\}, \langle\cdot, \cdot, \cdot\rangle)$, $\{x_0, x_k\}$ is an interval-spanning set.*
(2) *For $k \in \mathbb{Z}_{\geq 1}$ and $T = (x_0, x_1, ..., x_{k-1}, x_k, x_0)$ a sequence in X , if T is aligned, then T is constant.*
(3) *Define the binary relation $\sim_p$ on $X \setminus \{p\}$ by*

$$a \sim_p b \text{ iff } \langle p \neq a, b\rangle \text{ or } \langle a \neq p \neq b\rangle \text{ or } \langle a, b \neq p\rangle \,.$$

   (a) *For $p \in X$, $\sim_p$ is reflexive on $X \setminus \{p\}$.*
   (b) *For $p \in X$, $\sim_p$ is symmetric.*

PROOF.

(1)
   (a) Let T be a finite pseudosubsequence of S . Each three-term pseudosubsequence of T is also a three-term pseudosubsequence of S and therefore aligned. Consequently, T is aligned.
   (b) follows from (1a).

(c) The assumption $\langle x_0,\, x_1,\, ...,\, x_k\rangle$ and the claim $\langle x_k,\, x_{k-1},\, ...,\, x_0\rangle$ both are equivalent to the condition that for all $p,\, q,\, r\, \in\, \{0,\, 1,\, ...,\, k\}$, $p\, <\, q\, <\, r$ implies $\langle a_p,\, a_q,\, a_r\rangle$.

(d) For $i,\, j \in \{0,\, 1,\, ...,\, k\}$ it is to be proved that there are $w,\, z \in \{x_0,\, x_k\}$ such that $\langle w,\, x_i,\, x_j,\, z\rangle$.
Case 1. $i\, \leq\, j$. From the assumption that S is aligned it follows by (1a) that $\langle x_0,\, x_i,\, x_j,\, x_k\rangle$.
Case 2. $j\, <\, i$. From the assumption that S is aligned it follows by (1a) that $\langle x_0,\, x_j,\, x_i,\, x_k\rangle$. By (1c), $\langle x_k,\, x_i,\, x_j,\, x_0\rangle$.

(2) The claim is proved by induction on k .
Step 1. $k = 1$. From the assumption $\langle x_0,\, x_1,\, x_0\rangle$ it follows that $x_0 = x_1$.
Step 2. $k \to k + 1$. Assume that for $k \in \mathbb{Z}_{\geq 1}$, for

$$T = (x_0,\, x_1,\, ...,\, x_{k-1},\, x_k,\, x_0)$$

a sequence in X , if T is aligned, then T is constant. For

$$T' = (x_0,\, x_1,\, ...,\, x_{k-1},\, x_k,\, x_{k+1},\, x_0)$$

an aligned sequence in X it is to be proved that T' is constant. From the assumption that T' is aligned it follows by (1a):

$$T := (x_0,\, x_1,\, ...,\, x_{k-1},\, x_k,\, x_0) \text{ is aligned.} \tag{1.4.3}$$

(1.4.3) and the induction hypothesis imply:

$$T \text{ is constant.} \tag{1.4.4}$$

In particular,

$$x_k = x_0\, . \tag{1.4.5}$$

The assumption that T' is aligned entails:

$$\langle x_k,\, x_{k+1},\, x_0\rangle\ . \tag{1.4.6}$$

Substituting (1.4.5) into (1.4.6), $\langle x_0,\, x_{k+1},\, x_0\rangle$. Therefore,

$$x_{k+1} = x_0\, . \tag{1.4.7}$$

From (1.4.4) and (1.4.7) it follows that T' is constant.

(3)

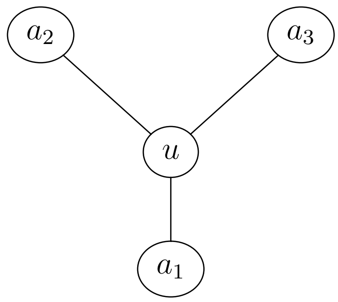
(a) For $a \in X \setminus \{p\}$, $a \sim_p a$ follows from $\langle p \neq a,\, a\rangle$.

(b) For $a,\, b \in X \setminus \{p\}$ it is to be proved that $a \sim_p b$ implies $b \sim_p a$. The assumption $a\, \sim_p\, b$ says $\langle p \neq a,\, b\rangle$ or $\langle a \neq p \neq b\rangle$ or $\langle a,\, b \neq p\rangle$. Therefore, $\langle b,\, a \neq p\rangle$ or $\langle b \neq p \neq a\rangle$ or $\langle p \neq b,\, a\rangle$, i.e. $b \sim_p a$.

□

Let X be an interval space. For $a_1,\, a_2,\, a_3,\, u \in X$, u is called a *median* of $a_1,\, a_2,\, a_3$ iff one and therefore each of the following equivalent conditions is satisfied:

- ○ $\langle a_1,\, u,\, a_2\rangle$ and $\langle a_2,\, u,\, a_3\rangle$ and $\langle a_3,\, u,\, a_1\rangle$.
- ○ for $j,\, k \in [3]$, if $j \neq k$, then $\langle a_j,\, u,\, a_k\rangle$.



Here is another way to express the condition that u is a median of $a_1,\ a_2,\ a_3$: A *median triangle* in X is a partial matrix

$$T = \begin{bmatrix} & a_3 & \\ a_1 & u & a_2 \end{bmatrix}$$

in X such that u is a median of $a_1,\ a_2,\ a_3$. The points $a_1,\ a_2,\ a_3$ are called the *vertices* of T . The aligned sequences $(a_1,\ u,\ a_2)$, $(a_2,\ u,\ a_3)$ and $(a_3,\ u,\ a_1)$ are called the *sides* of T . When X is a lattice, then $(a_1 \wedge a_2) \vee (a_2 \wedge a_3) \vee (a_3 \wedge a_1)$ is a median of $a_1,\ a_2,\ a_3$, i.e. $\begin{bmatrix} & a_3 & \\ a_1 & (a_1 \wedge a_2) \vee (a_2 \wedge a_3) \vee (a_3 \wedge a_1) & a_2 \end{bmatrix}$ is a median triangle. This expression defining the median of three points has been given in [**5**] for the particular case that X is a distributive lattice. Further examples of median triangles are provided by the following proposition.

PROPOSITION 1.4.8. *(median triangles in the plane) For* $w_{xu},\ w_{yu},\ w_{au} \in \mathbb{R}_{\geq 0}$, *define* $u := (0,\ 0)$, $x := (-w_{xu},\ 0)$, $y := (w_{yu},\ 0)$, $a := (0,\ w_{au})$.

(1) $\|x - u\|_1 = w_{xu}$, $\|y - u\|_1 = w_{yu}$, $\|a - u\|_1 = w_{au}$.

(2) *In* $(\mathbb{R}^2,\ \|\cdot - \cdot\|_1)$, $T = \begin{bmatrix} & a & \\ x & u & y \end{bmatrix}$ *is a median triangle.*

PROOF.

(1) Step 1. Proof of the equation $\|x - u\|_1 = w_{xu}$. From the assumption $w_{xu} \in \mathbb{R}_{\geq 0}$ it follows:

$$\begin{aligned} \|x - u\|_1 &= \|(-w_{xu},\ 0) - (0,\ 0)\|_1 \\ &= \|(-w_{xu},\ 0)\|_1 \\ &= |-w_{xu}| + |0| \\ &= w_{xu} + 0 \\ &= w_{xu} . \end{aligned}$$

Step 2. The proofs of the other two equations are analogous.

(2) Step 1. Proof of the alignment $\langle a,\, u,\, x\rangle$. It is to be proved that $\|a-x\|_1 = \|a-u\|_1 + \|u-x\|_1$. The assumption $w_{au},\, w_{xu} \in \mathbb{R}_{\geq 0}$ implies by (1):

$$\begin{aligned} \|a-x\|_1 &= \|(0,\, w_{au}) - (-w_{xu},\, 0)\|_1 \\ &= \|(w_{xu},\, w_{au})\|_1 \\ &= |w_{xu}| + |w_{au}| \\ &= w_{au} + w_{xu} \\ &= \|a-u\|_1 + \|x-u\|_1 \\ &= \|a-u\|_1 + \|u-x\|_1 \ . \end{aligned}$$

Step 2. The proofs of the other two alignments defining the condition that T is a median triangle are analogous.

□

For $a_1,\, a_2,\, a_3 \in X$,

$$\begin{aligned} M\left(a_1,\, a_2,\, a_3\right) &:= \{u \in X | u \text{ is a median of } a_1,\, a_2,\, a_3\} \\ &= \left\{ u \in X | \begin{bmatrix} & a_3 & \\ a_1 & u & a_2 \end{bmatrix} \text{ is a median triangle} \right\} \\ &= [a_1,\, a_2] \cap [a_2,\, a_3] \cap [a_3,\, a_1] \ . \end{aligned}$$

PROPOSITION 1.4.9. *(median triangles) Let X be an interval space.*

(1) *For $x,\, y,\, z \in X$, $\begin{bmatrix} & a & \\ x & a & y \end{bmatrix}$ is a median triangle iff $\langle x,\, a,\, y\rangle$.*

(2) *For a median triangle $\begin{bmatrix} & a_3 & \\ a_1 & u & a_2 \end{bmatrix}$ in X :*
   (a) *For each permutation $\sigma \in S_3$, $\begin{bmatrix} & a_{\sigma(3)} & \\ a_{\sigma(1)} & u & a_{\sigma(2)} \end{bmatrix}$ is a median triangle.*
   (b) *Setting $A := \{a_1,\, a_2,\, a_3\}$ and $T := \{a_1,\, a_2,\, a_3,\, u\}$, A is an interval-spanning set in $(T,\, \langle\cdot,\, \cdot,\, \cdot\rangle)$.*

(3) *For $x,\, y \in X$, $M\left(x,\, y,\, y\right) = \{y\}$.*

PROOF.

(1) The following conditions are equivalent:
   ◦ $\begin{bmatrix} & a & \\ x & a & y \end{bmatrix}$ is a median triangle.
   ◦ $\langle x,\, a,\, y\rangle$ and $\langle y,\, a,\, a\rangle$ and $\langle a,\, a,\, x\rangle$.
   ◦ $\langle x,\, a,\, y\rangle$.

(2)
   (a) For $j,\, k \in [3]$ it is to be proved that if $j \neq k$, then $\left\langle a_{\sigma(j)},\, u,\, a_{\sigma(k)}\right\rangle$. From the assumptions $\sigma \in S_3$ and $j \neq k$ it follows:

$$\sigma\left(j\right) \neq \sigma\left(k\right) \ . \tag{1.4.8}$$

the assumption that $\begin{bmatrix} & a_3 & \\ a_1 & u & a_2 \end{bmatrix}$ is a median triangle and (1.4.8) imply $\left\langle a_{\sigma(j)},\, u,\, a_{\sigma(k)} \right\rangle$.

(b) For $x,\, y \in T$ it is to be proved that there are $w,\, z \in A$ such that $\langle w,\, x,\, y,\, z\rangle$.
Case 1. $x,\, y \in A$. By 1.4.7(1a) (aligned sequences), $\langle x,\, x,\, y,\, y\rangle$.
Case 2. $x \in A$, $y = u$. Let without loss of generality $x = a_1$. It is to be proved there are $w,\, z \in A$ such that $\langle w,\, a_1,\, u,\, z\rangle$. The assumption that $\begin{bmatrix} & a_3 & \\ a_1 & u & a_2 \end{bmatrix}$ is a median triangle entails $\langle a_1,\, u,\, a_2\rangle$. By 1.4.7(1a) (aligned sequences), $\langle a_1,\, a_1,\, u,\, a_2\rangle$.
Case 3. $x = u$, $y \in A$. This case is analogous to case 2.
Case 4. $x = u$, $y = u$. It is to be proved that there are $w,\, z \in A$ such that $\langle w,\, u,\, u,\, z\rangle$. The assumption that $\begin{bmatrix} & a_3 & \\ a_1 & u & a_2 \end{bmatrix}$ is a median triangle entails $\langle a_1,\, u,\, a_2\rangle$. By 1.4.7(1a) (aligned sequences), $\langle a_1,\, u,\, u,\, a_2\rangle$.

(3) From $y \in [x,\, y]$ and $y \in [y,\, x]$ it follows:

$$\begin{aligned} M\,(x,\, y,\, y) &= [x,\, y] \cap [y,\, y] \cap [y,\, x] \\ &= [x,\, y] \cap \{y\} \cap [y,\, x] \\ &= \{y\}\ . \end{aligned}$$

□

Let X be an interval space. A partial matrix $T = \begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ of points in X is called *modular* iff one and therefore each of the following equivalent conditions is satisfied:

- There is an $s \in X$ such that $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ is a median triangle.
- $M\,(x,\, y,\, a) \neq \emptyset$.

Notation:

$$\left\langle \begin{matrix} & a & \\ x & & y \end{matrix} \right\rangle :\Leftrightarrow \begin{bmatrix} & a & \\ x & & y \end{bmatrix} \text{ is modular.}$$

PROPOSITION 1.4.10. *(modular matrices) Let X be an interval space. For $a_1,\, a_2,\, a_3 \in X$ and $\sigma \in S_3$, if $\begin{bmatrix} & a_3 & \\ a_1 & & a_2 \end{bmatrix}$ is modular, then so is $\begin{bmatrix} & a_{\sigma(3)} & \\ a_{\sigma(1)} & & a_{\sigma(2)} \end{bmatrix}$.*

PROOF. The claim follows by 1.4.9(2a) (median triangles). □

Note that interval spaces are first-order structures. In particular:

(1) A map $f : X \to Y$ of interval spaces is
   - a homomorphism iff for all $a, b, c \in X$, $\langle a, b, c\rangle$ implies $\langle f(a), f(b), f(c)\rangle$,
   - an embedding iff it is injective and for all $a, b, c \in X$,
   $$\langle a, b, c\rangle \text{ iff } \langle f(a), f(b), f(c)\rangle ,$$
   - an isomorphism of X onto Y iff it is an embedding of X onto Y.

(2) A substructure of an interval space X is a pair consisting of a subset Y of X and the relation $\langle\cdot, \cdot, \cdot\rangle \cap (Y \times Y \times Y)$. It is an interval space. In [**48**, chapter I, 4.3], it is called a *subspace* of X. There it has been defined in terms of intervals.

(3) The product of a family of interval spaces $\left(\left(X_q, \langle\cdot, \cdot, \cdot\rangle_q\right)\right)_{q\in Q}$ is the first-order structure
$$\prod_{q\in Q}\left(X_q, \langle\cdot, \cdot, \cdot\rangle_q\right) := (X, \langle\cdot, \cdot, \cdot\rangle) ,$$
where
$$X := \prod_{q\in Q} X_q$$
and for $f, g, h \in X$,
$$\langle f, g, h\rangle \text{ iff for all } q \in Q, \langle f(q), g(q), h(q)\rangle .$$
By 1.4.14 (product of interval spaces) below, it is an interval space.

The following proposition shows that for interval spaces the condition of injectivity is redundant in the definition of an embedding of first-order structures.

PROPOSITION 1.4.11. *(embedding of interval spaces) Let $f : X \to Y$ be a map of interval spaces such that for all $a, b, c \in X$, $\langle a, b, c\rangle$ iff $\langle f(a), f(b), f(c)\rangle$. Then f is an embedding.*

PROOF. It is to be proved that f is injective. For $x, y \in X$ it is to be proved that $f(x) = f(y)$ implies $x = y$. From the assumption $f(x) = f(y)$ it follows:
$$\langle f(x), f(y), f(x)\rangle . \tag{1.4.9}$$
(1.4.9) and the assumption that for $a, b, c \in X$, $\langle a, b, c\rangle \Leftrightarrow \langle f(a), f(b), f(c)\rangle$ imply $\langle x, y, x\rangle$. Consequently, $x = y$. □

PROPOSITION 1.4.12. *(isometric maps) Let $f : X \to Y$ be an isometric map of metric spaces. Then f is an embedding of $(X, \langle\cdot, \cdot, \cdot\rangle_d)$ into $(Y, \langle\cdot, \cdot, \cdot\rangle_d)$.*

PROOF. By 1.4.11 (embedding of interval spaces) it suffices to prove that for $a, b, c \in X$, $\langle a, b, c\rangle$ iff $\langle f(a), f(b), f(c)\rangle$. From the asumption that f is isometric it follows that the following equivalences hold:

$$\begin{aligned}\langle a,\, b,\, c\rangle &\Leftrightarrow d_{ac} = d_{ab} + d_{bc}\\ &\Leftrightarrow d_{f(a)f(c)} = d_{f(a)f(b)} + d_{f(b)f(c)}\\ &\Leftrightarrow \langle f(a),\, f(b),\, f(c)\rangle\ .\end{aligned}$$

□

PROPOSITION 1.4.13. *(homomorphisms of interval spaces) Let $f : X \to Y$ be a homomorphism of interval spaces.*

(1) *For $k \in \mathbb{Z}_{\geq 0}$ and $S = (a_0,\, a_1,\, ...,\, a_k)$ an aligned sequence in X, $f \circ S = (f(a_0),\, f(a_1),\, ...,\, f(a_k))$ is aligned in Y.*
(2) *For $T = \begin{bmatrix} & a & \\ x & s & y\end{bmatrix}$ a median triangle in X, $f \circ T = \begin{bmatrix} & f(a) & \\ f(x) & f(s) & f(y)\end{bmatrix}$ is a median triangle in Y.*
(3) *For $x,\, y,\, a \in X$, if $\begin{bmatrix} & a & \\ x & & y\end{bmatrix}$ is modular, then so is $\begin{bmatrix} & f(a) & \\ f(x) & & f(y)\end{bmatrix}$.*

PROOF.

(1) It is to be proved that for $p,\, q,\, r \in \{0,\, 1,\, ...,\, k\}$, $p < q < r$ implies $\langle f(a_p),\, f(a_q),\, f(a_r)\rangle$. From the assumption that S is aligned it follows:
$$\langle a_p,\, a_q,\, a_r\rangle\ . \tag{1.4.10}$$
(1.4.10) and the assumption that f is a homomorphism imply:
$$\langle f(a_p),\, f(a_q),\, f(a_r)\rangle\ .$$
(2) It is to be proved that each side S' of $f \circ T$ is aligned. There is a side S of T such that $S' = f \circ S$. It remains to be proved that $f \circ S$ is aligned. The assumption that T is a median triangle entails:
$$\text{The three-term sequence } S \text{ is aligned.} \tag{1.4.11}$$
From (1.4.11) and the assumption that f is a homomorphism it follows that $f \circ S$ is aligned.
(3) follows by (2).

□

Part (1) of the following proposition has been cited from [**48**, chapter I, 4.3].

PROPOSITION 1.4.14. *(product of interval spaces) Let $(X_q)_{q\in I}$ be a family of interval spaces with product X.*

(1) *X is an interval space.*
(2) *For $k \in \mathbb{Z}_{\geq 1}$ and $A = (f_0,\, f_1,\, ...,\, f_k)$ a sequence in X, A is aligned iff for each $q \in I$, $(f_0(q),\, f_1(q),\, ...,\, f_k(q))$ is aligned.*

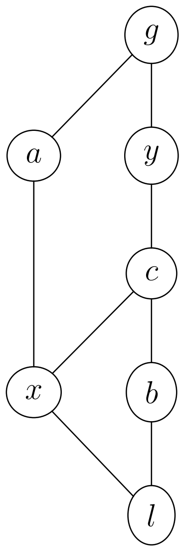
PROOF. The claims follow from the componentwise definition of the interval relation on X. □

Geometric interval spaces. An interval space X and its interval relation $\langle\cdot,\,\cdot,\,\cdot\rangle$ are called *geometric* iff the following conditions are satisfied:

- For $a \in X$, the binary relation $\langle a,\,\cdot,\,\cdot\rangle$ is transitive. This condition is the interval relation version of the strict interval relation condition [**34**, §1, IV. Grundsatz].
- For $a,\, b,\, x,\, y \in X$, $x,\, y \in [a,\, b]$ and $\langle a,\, x,\, y\rangle$ imply $\langle x,\, y,\, b\rangle$.

The condition that the binary relation $\langle a,\,\cdot,\,\cdot\rangle$ is transitive is equivalent to the condition that for all $y,\, z$, if $\langle a,\, y,\, z\rangle$, then $[a,\, y] \subseteq [a,\, z]$, see [**43**, (1.5)].

Each vector space over a totally ordered field, for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, is geometric. Each chain, for example $(\mathbb{R},\, \leq)$, with its lattice interval relation is geometric. Further examples of geometric interval spaces are provided by and presented after the next proposition. The lattice with the following Hasse diagram is not geometric: $[a,\, b] = [a \wedge b,\, a \vee b] = [l,\, g]$; thus, $x,\, y \in [a,\, b]$; and $x = a \wedge y$, therefore, $\langle a,\, x,\, y\rangle$. But $y \nleq c = x \vee b$, consequently not $\langle x,\, y,\, b\rangle$.



By 1.4.16 (Hedlíková's criterion for geometric interval spaces) below, the concept of a geometric interval space coincides with the concept of a *ternary space* defined by the conditions (T_1) to (T_4) in [**20**, section 1]. The terminology used here has been taken from [**49**, chapter I, 3.1]. The definition used here is equivalent to the definition in [**48**, chapter I, 4.1]. The conditions T_1, T_2 in [**20**, section 1] coincide with the conditions $\alpha,\, \beta$ in [**37**, part I, section 1]. They define an intermediate concept between the concepts of an interval space and of a geometric interval space.

The following proposition has been cited from [**48**, chapter I, 4.6.1]. By 1.4.5 (interval relation of a connected graph), it generalizes [**31**, proposition 1.1.2]. There the concept of a geometric interval space is implicit.

PROPOSITION 1.4.15. *(geometricity of metric spaces) Let X be a metric space. Then the interval space $(X,\, \langle\cdot,\,\cdot,\,\cdot\rangle_d)$ is geometric.*

PROOF. [**48**, chapter I, 4.6.1] □

1.4.15 (geometricity of metric spaces) may be used implicitly by applying results on geometric interval spaces to metric spaces.

By 1.4.15 (geometricity of metric spaces), for example the interval spaces associated with the following metric spaces are geometric: For $\left(\mathbb{R}^n,\ \|\cdot - \cdot\|_p\right)$ for each $n \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{R}_{\geq 1}$, $(N,\ d)$ for each graph $(N,\ E)$ with distance function d .

The following proposition has been cited from [**48**, chapter I, 4.22.1]. It characterizes a bundle of properties of a ternary relation in terms of fundamental properties of two families of binary relations, plus a property of the ternary relation. The axioms (T_1), (T_2, T_4) and (T_3) are from [**20**, section 1], where a geometric interval space has been called a ternary space. (T_2, T_4) has been proved for a geometric interval space also in [**49**, chapter I, 3.3(1)].

PROPOSITION 1.4.16. *(Hedlíková's criterion for geometric interval spaces) Let X be a set and $\langle\cdot,\ \cdot,\ \cdot\rangle$ a ternary relation on X . $(X,\ \langle\cdot,\ \cdot,\ \cdot\rangle)$ is a geometric interval space iff the following conditions are satisfied:*

- *(T_1) For $a \in X$, the binary relation $\langle\cdot,\ a,\ \cdot\rangle$ is symmetric.*
- *(T_2, T_4) For $a \in X$, the binary relation $\langle a,\ \cdot,\ \cdot\rangle$ on X is a partial order on X .*
- *(T_3) For $a,\ b,\ c,\ d \in X$, if $\langle a,\ b,\ c\rangle$ and $\langle a,\ c,\ d\rangle$, then $\langle b,\ c,\ d\rangle$.*

PROOF. Step 1. ($\Rightarrow$) From the assumption that $(X,\ \langle\cdot,\ \cdot,\ \cdot\rangle)$ is a geometric interval space it remains to be proved that the conditions (T_3) and (T_2, T_4) hold.

Step 1.1. Proof of (T_3). The assumptions that $\langle a,\ b,\ c\rangle$, $\langle a,\ c,\ d\rangle$ and X is geometric that $\langle a,\ b,\ d\rangle$, i.e.

$$b \in [a,\ d] \ . \tag{1.4.12}$$

The assumption $\langle a,\ c,\ d\rangle$ says:

$$c \in [a,\ d] \ . \tag{1.4.13}$$

From (1.4.12), (1.4.13) and the assumptions that $\langle a,\ b,\ c\rangle$ and X is geometric it follows that $\langle b,\ c,\ d\rangle$.

Step 1.2. Proof of (T_2, T_4). It remains to be proved that the binary relation $\langle a,\ \cdot,\ \cdot\rangle$ is antisymmetric, i.e. for $x,\ y \in X$, $\langle a,\ x,\ y\rangle$ and $\langle a,\ y,\ x\rangle$ imply $x = y$. From the assumptions $\langle a,\ x,\ y\rangle$, $\langle a,\ y,\ x\rangle$ it follows by step 1.1 that $\langle x,\ y,\ x\rangle$. Consequently, $x = y$.

Step 2. ($\Leftarrow$) From the assumptions (T_1), (T_2, T_4) and (T_3) it remains to be proved that for $a,\ b \in X$, $\langle a,\ b,\ b\rangle$, for $x,\ y \in X$, $\langle x,\ y,\ x\rangle$ implies $x = y$ and for $a,\ b,\ x,\ y \in X$, $x,\ y \in [a,\ b]$ and $\langle a,\ x,\ y\rangle$ imply $\langle x,\ y,\ b\rangle$.

Step 2.1. (T_2, T_4) entails that for $a,\ b \in X$, $\langle a,\ b,\ b\rangle$.

Step 2.2. Proof that for $x,\ y \in X$, $\langle x,\ y,\ x\rangle$ implies $x = y$. This implication follows from $\langle x,\ x,\ y\rangle$ and the assumption (T_2, T_4).

Step 2.3. Proof that for $a,\ b,\ x,\ y \in X$, $x,\ y \in [a,\ b]$ and $\langle a,\ x,\ y\rangle$ imply $\langle x,\ y,\ b\rangle$. The assumption $y \in [a,\ b]$ says:

$$\langle a,\ y,\ b\rangle \ . \tag{1.4.14}$$

From the assumption $\langle a,\ x,\ y\rangle$, (1.4.14) and the assumption (T_3) it follows that $\langle x,\ y,\ b\rangle$. □

Let X be a geometric interval space. 1.4.16 (Hedlíková's criterion for geometric interval spaces) entails that for $a \in X$, the binary relation $(X,\ \langle a,\ \cdot,\ \cdot\rangle)$ is a poset with dual $(X,\ \langle\cdot,\ \cdot,\ a\rangle)$ and may be applied by making use of both posets.

Part (1a) of the following proposition has been cited from [**20**, 1.1(1)], part (1b) from [**20**, 1.1(2)] and part (2a) from [**48**, chapter I, 4.5].

PROPOSITION 1.4.17. *(aligned sequences in a geometric interval space) Let X be a geometric interval space.*

(1) *For $k \in \mathbb{Z}_{\geq 1}$, $S = (x_0,\ x_1,\ \ldots,\ x_k)$ an aligned sequence and $x \in X$:*
   (a) *If $\langle x_0,\ x_k,\ x\rangle$, then $\langle x_0,\ x_1,\ \ldots,\ x_k,\ x\rangle$.*
   (b) *For $\kappa \in [k]$, if $\langle x_{\kappa-1},\ x,\ x_\kappa\rangle$, then $\langle x_0,\ x_1,\ \ldots,\ x_{\kappa-1},\ x,\ x_\kappa,\ x_{\kappa+1},\ \ldots,\ x_k\rangle$.*

(2) *For $a,\ b,\ c \in X$, if $\langle a,\ b,\ c\rangle$, then:*
   (a) $[a,\ b] \cap [b,\ c] = \{b\}$.
   (b) $[a,\ b] \cup [b,\ c] \subseteq [a,\ c]$.

PROOF.

(1)
   (a) [**20**, 1.1(1)]
   (b) [**20**, 1.1(2)]

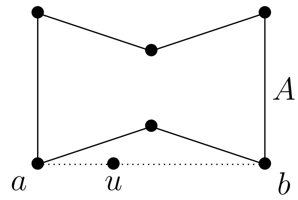
(2)
   (a) Step 1. ($\supseteq$) $\langle a,\ b,\ b\rangle$ and $\langle b,\ b,\ c\rangle$, i.e. $b \in [a,\ b]$ and $b \in [b,\ c]$, i.e. $[a,\ b] \cap [b,\ c] \supseteq \{b\}$.
   Step 2. ($\subseteq$) For $x \in [a,\ b] \cap [b,\ c]$, i.e. $\langle a,\ x,\ b\rangle$ and $\langle b,\ x,\ c\rangle$, it is to be proved that $x = b$. From the assumptions $\langle a,\ b,\ c\rangle$, $\langle a,\ x,\ b\rangle$ and $\langle b,\ x,\ c\rangle$ it follows by two applications of (1b) that $\langle a,\ x,\ b,\ x,\ c\rangle$. In particular, $\langle x,\ b,\ x\rangle$. Therefore, $x = b$.
   (b) It is to be proved that $[a,\ b] \subseteq [a,\ c]$ and $[b,\ c] \subseteq [a,\ c]$.
   Step 1. Proof that $[a,\ b] \subseteq [a,\ c]$. For $x \in X$ it is to be proved that $\langle a,\ x,\ b\rangle$ implies $\langle a,\ x,\ c\rangle$. From the assumptions $\langle a,\ x,\ b\rangle$ and $\langle a,\ b,\ c\rangle$ it follows that $\langle a,\ x,\ c\rangle$.
   Step 2. Proof that $[b,\ c] \subseteq [a,\ c]$, i.e. $[c,\ b] \subseteq [c,\ a]$. The assumption $\langle a,\ b,\ c\rangle$ implies $\langle c,\ b,\ a\rangle$. By step 1, $[c,\ b] \subseteq [c,\ a]$.

□

The concepts of a convex set, of a convex closure and of a half-space in a vector space over a totally ordered field have natural generalizations to an interval space:

Let X be an interval space.

A subset C of X is called *convex* in X iff it is closed under passing from $x,\ z$ to y when $\langle x,\ y,\ z\rangle$, i.e. for all $x,\ y,\ z \in X$, if $\langle x,\ y,\ z\rangle$ and $x,\ z \in C$, then $y \in C$, i.e. for all $x,\ z \in C$, $[x,\ z] \subseteq C$. For example, for $n \in \mathbb{Z}_{\geq 1}$, in $(\mathbb{R}^n,\ \|\cdot - \cdot\|_2)$, open and closed balls are convex. In the real vector space $\mathbb{R}^2$, the following set A is not convex: $a,\ b \in A$, but $[a,\ b] \nsubseteq A$ because $u \in [a,\ b]$ and $u \notin A$.



The set of convex sets in X is a closure system on X , i.e. X is convex in X , and the intersection of a non-empty set of convex sets in X is convex in X . For $A \subseteq X$, the *convex closure* of A in X is the intersection of all convex sets in X containg A . It is the smallest convex set in X containg A . The following notation has been taken from [**10**, chapter II, section 2]: For $A \subseteq X$,

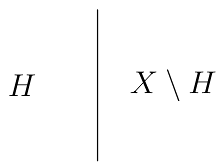
$$[A] := \text{the convex closure of } A\,.$$

$\emptyset$ is convex. Thus,

$$[\emptyset] = \emptyset\,.$$

When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then the convex closure of A is the set of all $\sum_{j=1}^{k} \lambda_j x_j$ such that $k \in \mathbb{Z}_{\geq 1}$, $\lambda_1,\, \lambda_2,\, ...,\, \lambda_k \in K_{\geq 0}$ and $\sum_{j=1}^{k} \lambda_j = 1$. In [**6**, section 1], the concepts of a convex set and of a convex closure have been defined in a more general context of set systems, and a convex closure has been called a *convex hull*.

A subset H of X is called a *half-space* iff H and $X \setminus H$ are convex.



When X is a vector space over a totally ordered field K and f is a non-zero linear map from X to K , then for $\lambda \in K$, $f^{-1}(\downarrow \lambda)$ is a half-space. Further examples of half-spaces are provided by 5.1.5(2f) (arboric interval spaces) below. In [**48**, chapter I, 3.1], the concept of a half-space has been defined in the more general context of an algebraic closure space.

Now let X be a geometric interval space.

For $A \subseteq X$ and $x \in X$, if the poset $(A,\, \langle x,\, \cdot,\, \cdot\rangle)$ has a least element, then this least element is called the *gate* of x into A . A set $G \subseteq X$ is called *gated* iff each element of X has a gate into G . For a gated set G in X , the map from X to G mapping x to the gate of x into G is called the *gate map* of G . It is the unique map $g\,:\, X \to G$ such that for $x,\, a \in X$, if $a \in G$, then $\langle x,\, g(x)\,,\, a\rangle$. Examples for gated sets are provided by 4.3.1 (modular geometric topological interval spaces) below and 5.1.5(2e) (arboric interval spaces) below. The concept of a gated set has been taken from [**16**]. There it has been defined for the particular case of a metric space and further examples have been provided. When X is a metric space, then each gated set G is a *Chebyshev set*, i.e. for each $x \in X$, there is exactly one $a \in G$ such that $d(x,\, G) = d(x,\, a)$. a is the gate of x into G .

Part (1) of the following proposition has been cited from [**48**, chapter I, 5.12(1)].

PROPOSITION 1.4.18. *(gated sets) Let X be a geometric interval space. For $G \subseteq X$:*

(1) *If G is gated, then G is convex.*
(2) *For each $x \in G$, x is a gate of x into G .*
(3) *G is gated iff each element of $X \setminus G$ has a gate into G .*

PROOF.

(1) [**48**, chapter I, 5.12(1)]
(2) For each $y \in G$, $\langle x,\, x,\, y \rangle$.
(3) follows from (2).

$\square$

The following theorem has been cited from [**48**, chapter I, 4.8(2)].

THEOREM 1.4.19. *(product of geometric interval spaces) Let $(X_q)_{q \in Q}$ be a family of geometric interval spaces with product X . Then X is geometric.*

PROOF. The claim follows from the componentwise definition of the interval relation on X . $\square$

Metric spaces continued.

PROPOSITION 1.4.20. *(median triangles in a metric space) Let X be a metric space. For $\begin{bmatrix} & a & \\ x & u & y \end{bmatrix}$ a median triangle in X , $d_{xu} = \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right)$.*

PROOF. The assumption that $\begin{bmatrix} & a & \\ x & u & y \end{bmatrix}$ is a median triangle says:

$$d_{xu} + d_{uy} = d_{xy}\,, \tag{1.4.15}$$
$$d_{yu} + d_{ua} = d_{ya}\,, \tag{1.4.16}$$
$$d_{au} + d_{ux} = d_{ax}\,. \tag{1.4.17}$$

Addition of (1.4.15) and (1.4.17), subtraction of (1.4.16) and the symmetry of d yield $2d_{xu} = d_{xy} + d_{xa} - d_{ya}$. Upon division by 2 , $d_{xu} = \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right)$. $\square$

PROPOSITION 1.4.21. *(point-interval distance) Let X be a metric space. For $x,\, y,\, a \in X$,*

(1) $d_{x,[y,\,a]} \geq \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right)$.
(2) *If $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular, then $d_{x,[y,\,a]} = \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right)$.*

PROOF.

(1) For $z \in [y,\, a]$ it is to be proved that $d_{xz} \geq \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right)$. The assumption $z \in [y,\, a]$ says $d_{ya} = d_{yz} + d_{za}$. Therefore, the following inequalities, of which the first one

is to be proved, are equivalent:

$$\begin{aligned} d_{xz} &\geq \frac{1}{2}\left(d_{xy}+d_{xa}-d_{ya}\right) ,\\ 2d_{xz} &\geq d_{xy}+d_{xa}-d_{ya} ,\\ 2d_{xz} &\geq d_{xy}+d_{xa}-\left(d_{yz}+d_{za}\right) ,\\ \left(d_{xz}+d_{za}\right)+\left(d_{xz}+d_{zy}\right) &\geq d_{xa}+d_{xy} . \end{aligned}$$

The last inequality follows by adding two instances of the triangle inequality: $d_{xz} + d_{za} \geq d_{xa}$ and $d_{xz}+d_{zy} \geq d_{xy}$.

(2) It suffices to prove $d_{x,[y,\,a]} \geq \frac{1}{2}\left(d_{xy}+d_{xa}-d_{ya}\right)$ and $d_{x,[y,\,a]} \leq \frac{1}{2}\left(d_{xy}+d_{xa}-d_{ya}\right)$.
Step 1. $d_{x,[y,\,a]} \geq \frac{1}{2}\left(d_{xy}+d_{xa}-d_{ya}\right)$ by (1).
Step 2. Proof that $d_{x,[y,\,a]} \leq \frac{1}{2}\left(d_{xy}+d_{xa}-d_{ya}\right)$. The assumption that $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular says that there is an $s \in X$ such that

$$\begin{bmatrix} & a & \\ x & s & y \end{bmatrix} \text{ is a median triangle.} \tag{1.4.18}$$

In particular, $s \in [y,\,a]$. Therefore,

$$d_{x,[y,\,a]} \leq d_{xs} . \tag{1.4.19}$$

From (1.4.18) it follows by 1.4.20 (median triangles in a metric space):

$$d_{xs} = \frac{1}{2}\left(d_{xy}+d_{xa}-d_{ya}\right) . \tag{1.4.20}$$

Substituting (1.4.20) into (1.4.19), $d_{x,[y,\,a]} \leq \frac{1}{2}\left(d_{xy}+d_{xa}-d_{ya}\right)$.

□

PROPOSITION 1.4.22. *(modular matrix representation) Let A be a metric space with $A = \{x,\,y,\,a\}$. There is an isometric map i from A into $(\mathbb{R}^2,\,\|\cdot-\cdot\|_1)$ such that* $\begin{bmatrix} & i(a) & \\ i(x) & & i(y) \end{bmatrix}$ *is modular.*

PROOF. From the three instances of the triangle inequality $d_{ya} \leq d_{xy}+d_{xa}$, $d_{xa} \leq d_{yx}+d_{ya}$, $d_{xy} \leq d_{ax}+d_{ay}$ it follows that $\frac{1}{2}\left(d_{xy}+d_{xa}-d_{ya}\right) \in \mathbb{R}_{\geq 0}$, $\frac{1}{2}\left(d_{yx}+d_{ya}-d_{xa}\right) \in \mathbb{R}_{\geq 0}$ and $\frac{1}{2}\left(d_{ax}+d_{ay}-d_{xy}\right) \in \mathbb{R}_{\geq 0}$. By 1.4.8 (median triangles in the plane), there is

$$\begin{bmatrix} & a' & \\ x' & u' & y' \end{bmatrix} \text{, a median triangle in } \left(\mathbb{R}^2,\,\|\cdot-\cdot\|_1\right) , \tag{1.4.21}$$

such that

$$\|x' - u'\|_1 = \frac{1}{2}(d_{xy} + d_{xa} - d_{ya}) \ , \tag{1.4.22}$$

$$\|y' - u'\|_1 = \frac{1}{2}(d_{yx} + d_{ya} - d_{xa}) \ , \tag{1.4.23}$$

$$\|a' - u'\|_1 = \frac{1}{2}(d_{ax} + d_{ay} - d_{xy}) \ . \tag{1.4.24}$$

(1.4.21) entails that

$$\begin{bmatrix} & a' & \\ x' & & y' \end{bmatrix} \text{ is modular} \tag{1.4.25}$$

and $\langle x',\, u',\, y'\rangle$, i.e.

$$\|x' - y'\|_1 = \|x' - u'\|_1 + \|u' - y'\|_1 \ , \tag{1.4.26}$$

From (1.4.26), (1.4.22) and (1.4.23) it follows:

$$\begin{aligned} \|x' - y'\|_1 &= \|x' - u'\|_1 + \|u' - y'\|_1 \\ &= \|x' - u'\|_1 + \|y' - u'\|_1 \\ &= \frac{1}{2}(d_{xy} + d_{xa} - d_{ya}) + \frac{1}{2}(d_{yx} + d_{ya} - d_{xa}) \\ &= d_{xy} \ . \end{aligned} \tag{1.4.27}$$

Analogously,

$$\|y' - a'\|_1 = d_{ya} \ , \tag{1.4.28}$$

$$\|a' - x'\|_1 = d_{ax} \ . \tag{1.4.29}$$

From (1.4.27), (1.4.28) and (1.4.29) it follows that an isometric map i from A into $(\mathbb{R}^2,\ \|\cdot - \cdot\|_1)$ is well-defined by setting $i\,(x) = x'$, $i\,(y) = y'$ and $i\,(a) = a'$. (1.4.25) says that $\begin{bmatrix} & i\,(a) & \\ i\,(x) & & i\,(y) \end{bmatrix}$ is modular. □

Part (1) of the following proposition has been cited from [**42**, theorem 1.6.1]. Part (2) has been cited from [**48**, chapter I, 4.3.2]. Part (3) is a particular case of [**42**, theorem 4.5.1].

PROPOSITION 1.4.23. *(sum metric) Let $((X_q,\, d_q))_{q\in Q}$ be a finite family of metric spaces and $X = \prod_{q\in Q} X_q$. For the map $d:\ X \times X \to \mathbb{R}_{\geq 0}$ defined by $d\,(x,\, y) = \sum_{q\in Q} d_q\,(x_q,\, y_q)$:*

1. *d is a metric on X .*
2. *The interval space $(X,\, \langle\cdot,\, \cdot,\, \cdot\rangle_d)$ is the product of the family of interval spaces $\left(\left(X_q,\, \langle\cdot,\, \cdot,\, \cdot\rangle_{d_q}\right)\right)_{q\in Q}$.*
3. *The topological space $(X,\, O_d)$ is the product of the family of topological spaces $\left((X_q,\, O_{d_q})\right)_{q\in Q}$.*

PROOF.

1. [**42**, theorem 1.6.1]

(2) For $f,\, g,\, h \in X$ it is to be proved: If for all $q \in Q$, $d_q\,(f\,(q)\,,\, h\,(q)) = d_q\,(f\,(q)\,,\, g\,(q)) + d_q\,(g\,(q)\,,\, h\,(q))$, then $d\,(f,\, h) = d\,(f,\, g) + d\,(g,\, h)$, and otherwise $d\,(f,\, h) \neq d\,(f,\, g) + d\,(g,\, h)$.
Step 1. If for all $q \in Q$, $d_q\,(f\,(q)\,,\, h\,(q)) = d_q\,(f\,(q)\,,\, g\,(q)) + d_q\,(g\,(q)\,,\, h\,(q))$, then addition of those equations yields $d\,(f,\, h) = d\,(f,\, g) + d\,(g,\, h)$.
Step 2. If, on the other hand, there is a $q_0 \in Q$ such that $d_{q_0}\,(f\,(q_0)\,,\, h\,(q_0)) \neq d_{q_0}\,(f\,(q_0)\,,\, g\,(q_0)) + d_{q_0}\,(g\,(q_0)\,,\, h\,(q_0))$, then

$$d_{q_0}\,(f\,(q_0)\,,\, h\,(q_0)) < d_{q_0}\,(f\,(q_0)\,,\, g\,(q_0)) + d_{q_0}\,(g\,(q_0)\,,\, h\,(q_0)) \ .$$

Addition of that inequality and the triangle inequality

$$d_q\,(f\,(q)\,,\, h\,(q)) \leq d_q\,(f\,(q)\,,\, g\,(q)) + d_q\,(g\,(q)\,,\, h\,(q))$$

for all $q \in Q \setminus \{q_0\}$ yields $d\,(f,\, h) < d\,(f,\, g) + d\,(g,\, h)$.
(3) is a particular case of [**42**, theorem 4.5.1].

□

Let $((X_q,\, d_q))_{q\in Q}$ be a finite family of metric spaces and $X = \prod_{q\in Q} X_q$. By 1.4.23(1) (sum metric) the map $d:\ X\times X \to \mathbb{R}_{\geq 0}$ defined by $d\,(x,\, y) = \sum_{q\in Q} d_q\,(x_q,\, y_q)$ is a metric on X . d is called the *sum metric* and the metric space $(X,\, d)$ is called the *sum* of the family $((X_q,\, d_q))_{q\in Q}$.

Let X and Y be metric spaces. A *nonexpansive map* from X to Y is a map f from X to Y such that for all $a,\, b \in X$, $d_{f(a)f(b)} \leq d_{ab}$. In [**22**, section 1] a nonexpansive map between metric spaces is just called a mapping between metric spaces. For each finite family of metric spaces, the projection maps of its sum are nonexpansive. Each isometric map of metric spaces is nonexpansive. Further examples of nonexpansive maps are provided by the following proposition.

PROPOSITION 1.4.24. *(nonexpansiveness of metrics) Let X be a metric space. With respect to the sum metric on $X \times X$, the map from $X \times X$ to $\mathbb{R}$ mapping $(x,\, y)$ to d_{xy} is nonexpansive.*

PROOF. For $(a,\, b)\,,\, (x,\, y) \in X \times X$ it is to be proved that $|d_{xy} - d_{ab}| \leq d_{ax} + d_{by}$. It suffices to prove $d_{xy} - d_{ab} \leq d_{ax} + d_{by}$ and $d_{ab} - d_{xy} \leq d_{ax} + d_{by}$. By 1.4.3(1) (metric spaces), $d_{xy} \leq d_{xa} + d_{ab} + d_{by}$. Therefore, $d_{xy} - d_{ab} \leq d_{ax} + d_{by}$. Interchanging x with a and y with b, $d_{ab} - d_{xy} \leq d_{ax} + d_{by}$. □

Part (1) of the following proposition is a particular case of [**42**, theorem 9.4.2(i)].

PROPOSITION 1.4.25. *(nonexpansive maps) Let X, Y be metric spaces and f a nonexpansive map from X to Y .*

(1) *f is continuous.*
(2) *For A an interval-spanning set in X , if $f|A$ is isometric, then f is isometric.*

PROOF.

(1) is a particular case of [**42**, theorem 9.4.2(i)].
(2) For $x,\, y \in X$ it is to be proved that $d_{f(x)f(y)} = d_{xy}$. From the assumption that f is nonexpansive it follows that $d_{f(x)f(y)} \leq d_{xy}$. Therefore, it suffices to prove

$d_{f(x)f(y)} \geq d_{xy}$. Seeking a contradiction, assume

$$d_{f(x)f(y)} < d_{xy} \,. \tag{1.4.30}$$

The assumption that A is interval-spanning implies that there are

$$w,\, z \in A$$

such that $\langle w,\, x,\, y,\, z\rangle$. By 1.4.6 (aligned sequences in a metric space),

$$d_{wz} = d_{wx} + d_{xy} + d_{yz} \,. \tag{1.4.31}$$

From the assumption that $f|A$ is isometric, $w,\, z \in A$, the assumption that f is nonexpansive, (1.4.30) and (1.4.31) it follows by 1.4.3(1) (metric spaces):

$$\begin{aligned} d_{wz} &= d_{f(w)f(z)} \\ &\leq d_{f(w)f(x)} + d_{f(x)f(y)} + d_{f(y)f(z)} \\ &< d_{wx} + d_{xy} + d_{yz} \\ &= d_{wz} \,, \end{aligned}$$

Therefore, $d_{wz} < d_{wz}$, a contradiction.

□

Let Y be a metric space.

Let f a nonexpansive map from a metric space X to Y . f is called *universally extendable* iff it has an extension to a nonexpansive map from each superspace of X to Y .

Y is called *injective* iff each nonexpansive map from a metric space to Y is universally extendable. For example, for $n \in \mathbb{Z}_{\geq 1}$, the metric space $(\mathbb{R}^n,\, \|\cdot - \cdot\|_\infty)$ is injective. The metric space $(\mathbb{R}^2,\, \|\cdot - \cdot\|_1)$, being an isometric copy of $(\mathbb{R}^2,\, \|\cdot - \cdot\|_\infty)$, is injective. But for $n \in \mathbb{Z}_{\geq 3}$, $(\mathbb{R}^n,\, \|\cdot - \cdot\|_1)$ is not injective. The concept of an injective metric space has been taken from [**22**, introduction]. It is equivalent to the concept of a *hyperconvex metric space* as defined in [**1**, section 2, definition 1]. Therefore, the two terms 'injective metric space' and 'hyperconvex metric space' may be used synonymously.

The following theorem has been cited from [**22**, 2.1].

THEOREM 1.4.26. *(existence of injective closure) For each metric space X , there is an injective metric space Y and an isometric map $i:\ X \to Y$ such that Y has no injective proper subspace containing $i\,(X)$.*

PROOF. [**22**, 2.1] □

PROPOSITION 1.4.27. *(isometric maps into an injective metric space) Let X be a metric space and Y an injective metric space. For A an interval-spanning set in X , each isometric map from A into Y has an extension to an isometric map from X into Y .*

PROOF. Let f be an isometric map from A into Y . In particular,

$$f \text{ is a nonexpansive map from } A \text{ to } Y \,. \tag{1.4.32}$$

From (1.4.32) and the assumption that Y is injective it follows that there is

$$g\,,\ \text{a nonexpansive map from } X \text{ to } Y\,, \tag{1.4.33}$$

such that

$$g|A = f\,. \tag{1.4.34}$$

(1.4.34) and the assumption that f is isometric imply:

$$g|A \text{ is an isometric map from } A \text{ to } Y\,. \tag{1.4.35}$$

From the assumption that A is interval-spanning, (1.4.33) and (1.4.35) it follows by 1.4.25(2) (nonexpansive maps) that g is an isometric map from X into Y . □

1.5. Interval-Convex and Triangle-Convex Spaces

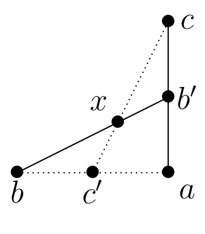
Let X be an interval space.

X is called *interval-convex* iff for all $a,\, b \in X$, $[a,\, b]$ is convex. When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X is interval-convex. Each subspace of an interval-convex interval space is interval-convex. Further examples of interval-convex interval spaces are provided by the next proposition. The complete-bipartite graph $K_{2,3}$ is not interval-convex. The concept of an interval-convex interval space generalizes the concept of an interval monotone graph in [**31**, 1.1.6]. According to [**31**, 1.1], each tree is interval-convex. This fact is a particular case of 3.3.1(1) (interval-linear geometric interval spaces) below.

X is called *triangle-convex* iff one and therefore each of the following equivalent conditions is satisfied:

- For all $a,\, b,\, c,\, x \in X$, if there is a $b' \in [a,\, c]$ such that $\langle b,\, x,\, b' \rangle$, then there is a $c' \in [a,\, b]$ such that $\langle c,\, x,\, c' \rangle$.
- For all $a,\, b,\, c, \in X$, $\bigcup_{b' \in [a,\, c]} [b,\, b'] \subseteq \bigcup_{c' \in [a,\, b]} [c,\, c']$.
- For all $a,\, b,\, c, \in X$, $\bigcup_{b' \in [a,\, c]} [b,\, b']$ is convex.
- For all $a,\, b,\, c, \in X$, $[\{a,\, b,\, c\}] = \bigcup_{b' \in [a,\, c]} [b,\, b']$.

The first of these conditions is the interval relation version of the strict interval relation condition [**36**, §10, Assioma XIII]. In [**48**, chapter I, 4.9] it is called the *Peano Property*. When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X is triangle-convex.



For $n \in \mathbb{Z}_{\geq 1}$, the metric spaces $(\mathbb{R}^n,\, \|\cdot - \cdot\|_1)$ and $(\mathbb{R}^n,\, \|\cdot - \cdot\|_2)$ are triangle-convex. Each convex subspace of a triangle-convex interval space is triangle-convex. 1.6.6 (medianity criterion for a geometric interval space) below provides further examples of triangle-convex interval spaces. The complete-bipartite graph $K_{2,3}$ is not triangle-convex.

The following proposition has been cited from [**48**, chapter I, 4.10].

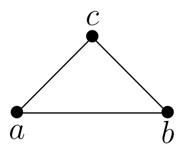
PROPOSITION 1.5.1. *(triangle-convex interval spaces) Let X be a triangle-convex interval space.*

- *X is interval-convex.*
- *For $a \in X$, the binary relation $\langle a, \cdot, \cdot \rangle$ is transitive.*

PROOF. [**48**, chapter I, 4.10] $\square$

1.6. Modular and Median Spaces

Modular interval spaces. Let X be an interval space. X is called *modular* iff for all $a, b, c \in X$, there is at least one median of a, b, c, i.e. a $u \in X$ such that $\begin{bmatrix} & c & \\ a & u & b \end{bmatrix}$ is a median triangle, i.e. for all $a, b, c \in X$, $\begin{bmatrix} & c & \\ a & & b \end{bmatrix}$ is modular. For example, for $n \in \mathbb{Z}_{\geq 1}$, the metric spaces $(\mathbb{R}^n, \|\cdot - \cdot\|_\infty)$ and $(\mathbb{R}^n, \|\cdot - \cdot\|_1)$ are modular. Further examples of modular interval spaces are provided by 6.1.4 (modularity of injective metric spaces) below. For $n \in \mathbb{Z}_{\geq 2}$, the metric space $(\mathbb{R}^n, \|\cdot - \cdot\|_2)$ is not modular. In the following example in $(\mathbb{R}^2, \|\cdot - \cdot\|_2)$, the points $a, b, c \in \mathbb{R}^2$ have no median.



In [**3**, 1.4] the concept of modularity of an interval space has been defined under the assumption that the interval space is geometric.

PROPOSITION 1.6.1. *(metrics for a modular interval space) Let $(X, \langle \cdot, \cdot, \cdot \rangle)$ be a modular interval space and $d : X \times X \to \mathbb{R}_{\geq 0}$. d is a metric on X such that $\langle \cdot, \cdot, \cdot \rangle_d$ coincides with $\langle \cdot, \cdot, \cdot \rangle$ iff the following conditions are satisfied:*

- *For $x, y \in X$, $d(x, y) = d(y, x)$.*
- *$d(x, y) = 0$ iff $x = y$.*
- *For $x, y, z \in X$, $\langle x, y, z \rangle$ implies $d(x, z) = d(x, y) + d(y, z)$.*

PROOF. It suffices to prove the direction ($\Leftarrow$). For this, it is sufficient to prove that for $x, y, z \in X$, not $\langle x, y, z \rangle$ implies $d(x, z) < d(x, y) + d(y, z)$. From the assumption that X is modular it follows that x, y, z have a median u. Therefore, $\langle x, u, y \rangle$,

$$\langle x, u, z \rangle$$

and $\langle y,\, u,\, z\rangle$. Thus,

$$d(x,\, y) = d(x,\, u) + d(u,\, y)\ , \tag{1.6.1}$$
$$d(x,\, z) = d(x,\, u) + d(u,\, z)\ , \tag{1.6.2}$$
$$d(y,\, z) = d(y,\, u) + d(u,\, z)\ . \tag{1.6.3}$$

$\langle x,\, u,\, z\rangle$ and the assumption that not $\langle x,\, y,\, z\rangle$ imply:

$$u \neq y\,. \tag{1.6.4}$$

From (1.6.2), (1.6.1) and (1.6.3) it follows that the following inequalities, of which the first one is to be proved, are equivalent:

$$\begin{aligned} d(x,\, z) &< d(x,\, y) + d(y,\, z)\ , \\ d(x,\, u) + d(u,\, z) &< d(x,\, u) + d(u,\, y) + d(y,\, u) + d(u,\, z)\ , \\ 0 &< 2d(u,\, y)\ , \\ 0 &< d(u,\, y)\ . \end{aligned}$$

The last inequality follows from (1.6.4). □

Modular metric spaces. The following proposition has been cited from [**3**, abstract].

PROPOSITION 1.6.2. *(modularity of injective metric spaces) Each injective metric space is modular.*

PROOF. Let $(Y,\, d)$ be an injective metric space. For $x,\, y,\, a \in Y$ it is to be proved that $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular. Setting

$$A := \{x,\, y,\, a\}\ ,$$

by 1.4.22 (modular matrix representation) there is

$$i\,,\ \text{an isometric map from } A \text{ into } \left(\mathbb{R}^2,\, \|\cdot - \cdot\|_1\right)\ , \tag{1.6.5}$$

such that

$$\begin{bmatrix} & i(a) & \\ i(x) & & i(y) \end{bmatrix} \text{ is modular,} \tag{1.6.6}$$

i.e. there is an $s' \in \mathbb{R}^2$ such that

$$\begin{bmatrix} & i(a) & \\ i(x) & s' & i(y) \end{bmatrix} \text{ is a median triangle in } \left(\mathbb{R}^2,\, \|\cdot - \cdot\|_1\right)\ . \tag{1.6.7}$$

Furthermore,

$$i(A) = \{i(x)\,,\, i(y)\,,\, i(a)\}\ . \tag{1.6.8}$$

Setting $T' := \{i(x)\,,\, i(y)\,,\, i(a)\,,\, s'\}$, from (1.6.7) and (1.6.8) it follows by 1.4.9(2b) (median triangles):

$$i(A) \text{ is an interval-spanning set in } \left(T',\, \|\cdot - \cdot\|_1\right)\ . \tag{1.6.9}$$

(1.6.5) implies:

$$i^{-1} \text{ is an isometric map from } \left(i\left(A\right),\, \left\|\cdot - \cdot\right\|_1\right) \text{ into } \left(Y,\, d\right) . \tag{1.6.10}$$

The assumption that Y is injective, (1.6.9) and (1.6.10) imply by 1.4.27 (isometric maps into an injective metric space) that there is a

$$g\,, \text{ an isometric map from } \left(T',\, \left\|\cdot - \cdot\right\|_1\right) \text{ into } \left(Y,\, d\right) \tag{1.6.11}$$

such that g extends i^{-1} , i.e.:

$$\text{For each } z \in A\,,\; g\left(i\left(z\right)\right) = z\,. \tag{1.6.12}$$

From (1.6.11) it follows by 1.4.12 (isometric maps) that g is an embedding from $\left(T',\, \langle\cdot,\,\cdot,\,\cdot\rangle_{\|\cdot-\cdot\|_1}\right)$ into $\left(Y,\, \langle\cdot,\,\cdot,\,\cdot\rangle_d\right)$. In particular,

$$g \text{ is a homomorphism from } \left(T',\, \langle\cdot,\,\cdot,\,\cdot\rangle_{\|\cdot-\cdot\|_1}\right) \text{ into } \left(Y,\, \langle\cdot,\,\cdot,\,\cdot\rangle_d\right) . \tag{1.6.13}$$

(1.6.13) and (1.6.6) imply by 1.4.13(3) (homomorphisms of interval spaces):

$$\begin{bmatrix} & g\left(i\left(a\right)\right) & \\ g\left(i\left(x\right)\right) & & g\left(i\left(y\right)\right) \end{bmatrix} \text{ is modular.} \tag{1.6.14}$$

Substituting (1.6.12) into (1.6.14), $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular. □

PROPOSITION 1.6.3. *(point-interval distance in a modular metric space) Let Y be a modular metric space. For $x,\, y,\, a \in Y$,*

(1) $d_{x[y,\,a]} = \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right)$.
(2) $d_{xy} = d_{x[y,\,a]} + d_{y[x,\,a]}$.

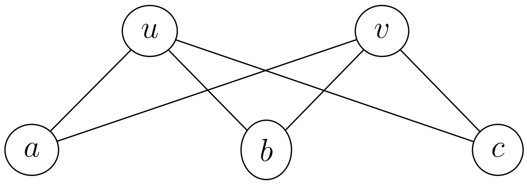
PROOF.

(1) From the assumption that Y is modular it follows that $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular. By 1.4.21(2) (point-interval distance), $d_{x[y,\,a]} = \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right)$.
(2) The assumption that Y is modular implies by (1):

$$\begin{aligned} d_{x[y,\,a]} + d_{y[x,\,a]} &= \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right) + \frac{1}{2}\left(d_{yx} + d_{ya} - d_{xa}\right) \\ &= d_{xy}\,. \end{aligned}$$

□

Median interval spaces. Let X be an interval space. X is called *median* iff for all $a,\, b,\, c \in X$, there is exactly one median of $a,\, b,\, c$, i.e. exactly one $u \in X$ such that $\begin{bmatrix} & c & \\ a & u & b \end{bmatrix}$ is a median triangle. For example, for $n \in \mathbb{Z}_{\geq 1}$, the metric space $\left(\mathbb{R}^n,\, \left\|\cdot - \cdot\right\|_1\right)$ is median. The metric space $\left(\mathbb{R}^2,\, \left\|\cdot - \cdot\right\|_\infty\right)$, being an isometric copy of $\left(\mathbb{R}^2,\, \left\|\cdot - \cdot\right\|_1\right)$, is median. For $n \in \mathbb{Z}_{\geq 2}$, the metric space $\left(\mathbb{R}^n,\, \left\|\cdot - \cdot\right\|_2\right)$ is not median. Each median interval space is modular. The following bipartite graph is modular, but not median: $a,\, b,\, c$ have the two different medians $u,\ v$.

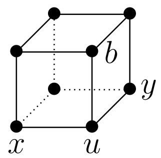


1.6.6 (medianity criterion for a geometric interval space) below, cited from [**3**, theorem 4.6], is a necessary and sufficient criterion for medianity of a modular geometric interval space.

Let X be a median interval space. For $a,\ b,\ c \in X$,

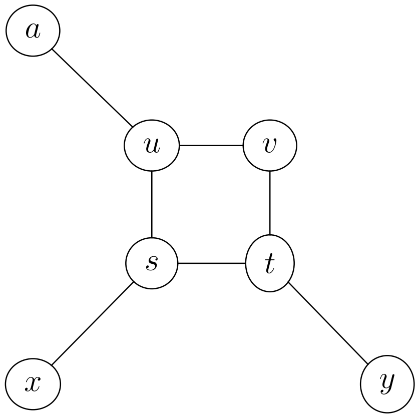
$$m\,(a,\ b,\ c) := \text{the median of } a,\ b,\ c\,.$$

For example, the edge graph of a cube is median with $m\,(x,\ y,\ b) = u$:



For $M \subseteq X$, M is called *median* in X iff it is closed under passing from $a,\ b,\ c$ to u when $m\,(a,\ b,\ c) = u$, i.e. for all $a,\ b,\ c,\ u \in X$, if $m\,(a,\ b,\ c) = u$ and $a,\ b,\ c \in M$, then $u \in M$, i.e. for all $a,\ b,\ c \in M$, $m\,(a,\ b,\ c) \in M$. For example, for $n \in \mathbb{Z}_{\geq 1}$, in the median metric space $(\mathbb{R}^n,\ \|\cdot - \cdot\|_1)$, $\mathbb{Z}^n$ and $\mathbb{Q}^n$ are median. The set of median sets in X is a closure system on X , i.e. X is median in X , and the intersection of a non-empty set of median sets in X is median in X . For $A \subseteq X$, the *median closure* of A in X is the intersection of all median sets in X containg A . It is the smallest median set in X containg A .

For example, in the following graph, the median closure of $\{x,\ y,\ a\}$ equals $\{x,\ y,\ a,\ s\}$.



Further examples of median closures are provided by 6.6.1(2), (3) (median closure of the median boundary) below.

In [**49**, chapter I, 2.18] a median set is called *median stable*, and the median closure is called the *median stabilization*. In [**48**, chapter I, section 6], these terms have been used with a more general meaning in the context of modular geometric interval spaces. Here the terms 'median' and 'median closure' have been chosen because for $M \subseteq X$, M is median iff the subspace $(M, \langle\cdot, \cdot, \cdot\rangle)$ is median.

PROPOSITION 1.6.4. *(small interval spaces) Let X be an interval space. If $|X| \leq 2$, then X is median.*

PROOF. From the assumption $|X| \leq 2$ it follows that any three elements of X can be denoted by a, b, b . By 1.4.9(3) (median triangles), $M(a, b, b) = \{b\}$. Consequently, X is median. □

For the particular case of median geometric interval spaces, the following proposition has been stated in [**3**, 1.4].

PROPOSITION 1.6.5. *(product of median interval spaces) Let $(X_q)_{q \in Q}$ be a family of median interval spaces with product X . Then the interval space X is median.*

PROOF. The claim follows from the componentwise definition of the interval relation on X . □

Median geometric interval spaces. The following theorem has been cited from [**3**, theorem 4.6].

THEOREM 1.6.6. *(medianity criterion for a geometric interval space) Let X be a modular geometric interval space. Then the following conditions are equivalent:*

(1) *For all $a, b \in X$, the poset $([a, b], \langle a, \cdot, \cdot\rangle)$ is a distributive lattice.*
(2) *For all $a, b \in X$, in the poset $([a, b], \langle a, \cdot, \cdot\rangle)$, for all $x, y \in [a, b]$:*
    - *Each lower bound of $\{x, y\}$ is less than or equal to a maximal one.*
    - *Each upper bound of $\{x, y\}$ is greater than or equal to a minimal one.*
    - *For $x, y, y' \in [a, b]$, if $\{x, y\}$ and $\{x, y'\}$ have a maximal lower bound and a minimal upper bound in common, then $y = y'$.*
(3) *X is triangle-convex.*
(4) *There is no embedding of the complete-bipartite graph $K_{2,3}$ into X .*
(5) *X is interval-convex.*
(6) *X is median.*

PROOF. [**3**, theorem 4.6] □

Median metric spaces. Let Q be a finite set. The metric space $X := \left(\{0, 1\}^Q, \|\cdot - \cdot\|_1\right)$ is called the *binary Hamming space* over Q . For $f, g \in X$, $\|f - g\|_1 = |\{q \in Q | f(q) \neq g(q)\}|$. This metric space has been used in [**18**]. There it has been defined in part II, section 5.

PROPOSITION 1.6.7. *(binary Hamming spaces) Let Q be a finite set. The metric space* $X := \left(\{0,\, 1\}^{Q},\, \|\cdot - \cdot\|_1\right)$ *with*

$$\|f\|_1 := \sum_{q \in Q} |f(q)| \ .$$

has the following properties:

(1) *X is an interval space product of equal factors* $(\{0,\, 1\},\, |\cdot - \cdot|)$.
(2) *X is median.*

PROOF.

(1) is a particular case of 1.4.23(2) (sum metric).
(2) By 1.6.4 (small interval spaces), the metric space $(\{0,\, 1\},\, |\cdot - \cdot|)$ is median. By (1), X is an interval space product of median metric spaces. By 1.6.5 (product of median interval spaces), X is median.

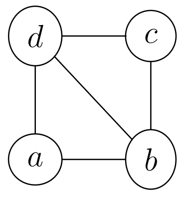
□

1.7. Arboric Spaces

Let X be an interval space.

X is called *interval-concatenable* iff for all $a,\, b,\, c \in X$, $[a,\, b] \cap [b,\, c] = \{b\}$ implies $\langle a,\, b,\, c\rangle$. For the particular case of a connected graph, this condition has been used in [**31**, 3.1.7]. For example, for $n \in \mathbb{Z}_{\geq 1}$, the metric space $(\mathbb{R}^n,\, \|\cdot - \cdot\|_1)$ is interval-concatenable. For $n \in \mathbb{Z}_{\geq 2}$, the metric space $(\mathbb{R}^n,\, \|\cdot - \cdot\|_2)$ is not interval-concatenable. The cycle C_5 is not interval-concatenable. Further examples of interval-concatenable interval spaces are provided by 4.1.1(3) (modular interval spaces) below.

X is called *arboric* iff X is geometric and interval-concatenable and for each $a \in X$, the poset $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$ is arboric. For example, by proposition 1.7.3(2) (arboricity of the real line) below, the metric space $(\mathbb{R},\, |\cdot - \cdot|)$ is arboric. By 1.7.4 (tree representation of finite arboric interval spaces) below, each tree is arboric. The following graph is not arboric: $\langle [a,\, c]\,,\, \langle a,\, \cdot,\, \cdot\rangle\rangle$ it not a chain because $b,\, d \in [a,\, c]$, but not $\langle a,\, b,\, d\rangle$ and not $\langle a,\, d,\, b\rangle$.



For $n \in \mathbb{Z}_{\geq 2}$, the metric space $(\mathbb{R}^n,\, \|\cdot - \cdot\|_2)$ is not arboric.

The concept of an arboric interval space is implicit in [**43**, (1.2), (1.3), (1.1), (1.4), (1.5), (4.7), (2.1), (S), (2.1) in connection with the last part of (4.8)]. There an arboric interval space is called a tree. Since an arboric interval space is an interval space, and not what is called a tree in graph theory, here the substantive 'interval space' together with the adjective 'arboric', which means 'tree-like', is preferred.

The implication ($\Leftarrow$) of the following proposition is implicit in [**43**, (1.2), (1.3), (1.1), (1.4), (1.5), (4.7), (2.1), (S), (2.1) in connection with the last part of (4.8)]. Parts (1) and (2) are axioms (S) and (T) from section 1, respectively, and part (3) is axiom (U_1) from section 2 of [**43**].

PROPOSITION 1.7.1. *(Sholander's criterion for arboric interval spaces) Let* X *be a set and* $\langle\cdot,\,\cdot,\,\cdot\rangle$ *a ternary relation on* X *. For* $a,\,b \in X$ *, define* $[a,\,b] := \{x \in X |\, \langle a,\,x,\,b\rangle\}$ *.* $(X,\,\langle\cdot,\,\cdot,\,\cdot\rangle)$ *is an arboric interval space iff for all* $u,\,a,\,b \in X$ *the following conditions are satisfied:*

(1) *There is a* $c \in X$ *such that* $[u,\,a] \cap [u,\,b] = [u,\,c]$ *.*
(2) *If* $[u,\,a] \subseteq [u,\,b]$ *, then* $[u,\,a] \cap [a,\,b] = \{a\}$ *.*
(3) *If* $[u,\,a] \cap [a,\,b] = \{a\}$ *, then* $[u,\,a] \cup [a,\,b] = [u,\,b]$ *.*

PROOF. Step 1. ($\Rightarrow$) The assumption that X is arboric entails that

$$X \text{ is geometric,} \tag{1.7.1}$$

$$X \text{ is interval-concatenable} \tag{1.7.2}$$

and the poset $(X,\,\langle u,\,\cdot,\,\cdot\rangle)$ is arboric. In particular:

$$(X,\,\langle u,\,\cdot,\,\cdot\rangle) \text{ is a meet semilattice.} \tag{1.7.3}$$

$$([u,\,b]\,,\,\langle u,\,\cdot,\,\cdot\rangle) \text{ is a chain.} \tag{1.7.4}$$

(1) (1.7.3) entails that in $(X,\,\langle u,\,\cdot,\,\cdot\rangle)$, $\{a,\,b\}$ has a greatest lower bound c. Therefore, $(\downarrow a) \cap (\downarrow b) = \downarrow c$, i.e. $[u,\,a] \cap [u,\,b] = [u,\,c]$.
(2) From $a \in [u,\,a]$ and the assumption $[u,\,a] \subseteq [u,\,b]$ it follows that $a \in [u,\,b]$, i.e. $\langle u,\,a,\,b\rangle$. By 1.4.17(2a) (aligned sequences in a geometric interval space), $[u,\,a] \cap [a,\,b] = \{a\}$.
(3) From the assumption $[u,\,a] \cap [a,\,b] = \{a\}$ and (1.7.2) it follows:

$$\langle u,\,a,\,b\rangle\ . \tag{1.7.5}$$

It is to be proved that $[u,\,a] \cup [a,\,b] \subseteq [u,\,b]$ and $[u,\,a] \cup [a,\,b] \supseteq [u,\,b]$.
Step 1. ($\subseteq$) This follows from (1.7.1) and (1.7.5) by 1.4.17(2b) (aligned sequences in a geometric interval space).
Step 2. ($\supseteq$) For $x \in X$ it is to be proved that $\langle u,\,x,\,b\rangle$ implies $\langle u,\,x,\,a\rangle$ or $\langle a,\,x,\,b\rangle$. From (1.7.4), (1.7.5) and the assumption $\langle u,\,x,\,b\rangle$ it follows that $\langle u,\,x,\,a\rangle$ or $\langle u,\,a,\,x\rangle$.
Case 1. $\langle u,\,x,\,a\rangle$. Nothing remains to be proved.
Case 2. $\langle u,\,a,\,x\rangle$. This assumption and the assumption $\langle u,\,x,\,b\rangle$ imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $\langle a,\,x,\,b\rangle$.

Step 2. ($\Leftarrow$) [**43**, (1.2), (1.3), (1.1), (1.4), (1.5), (4.7), (2.1), (S), (2.1) in connection with the last part of (4.8)] □

PROPOSITION 1.7.2. *(arboricity of chains) Each chain with its lattice interval relation is an arboric interval space.*

PROOF. Let X be a chain. By 1.7.1 (Sholander's criterion for arboric interval spaces), it suffices to prove for $u,\, a,\, b \in X$ that there is a $c \in X$ such that $[u,\, a] \cap [u,\, b] = [u,\, c]$, $[u,\, a] \subseteq [u,\, b]$ implies $[u,\, a] \cap [a,\, b] = \{a\}$, and $[u,\, a] \cap [a,\, b] = \{a\}$ implies $[u,\, a] \cup [a,\, b] = [u,\, b]$.

Step 1. Proof that there is a $c \in X$ such that $[u,\, a] \cap [u,\, b] = [u,\, c]$.

Case 1.1. $a \geq u$ and $b \geq u$. Then $[u,\, a] = \{x \in X | u \leq x \leq a\}$, $[u,\, b] = \{x \in X | u \leq x \leq b\}$ and with $c := a \wedge b$, $[u,\, c] = \{x \in X | u \leq x \leq c\}$. Consequently, $[u,\, a] \cap [u,\, b] = [u,\, c]$.

Case 1.2. $a \geq u$ and $b < u$. Then $[u,\, a] = \{x \in X | u \leq x \leq a\}$, $[u,\, b] = \{x \in X | b \leq x \leq u\}$ and with $c := u$, $[u,\, c] = \{u\}$. Consequently, $[u,\, a] \cap [u,\, b] = [u,\, c]$.

Case 1.3. $a < u$ and $b < u$. This case is analogous to case 1.1.

Step 2. Proof that $[u,\, a] \subseteq [u,\, b]$ implies $[u,\, a] \cap [a,\, b] = \{a\}$. From $a \in [u,\, a]$ and the assumption $[u,\, a] \subseteq [u,\, b]$ it follows:

$$a \in [u,\, b] \ . \tag{1.7.6}$$

Case 2.1. $u \leq b$. Then (1.7.6) says $u \leq a \leq b$. Therefore,

$$[u,\, a] = \{x \in X | u \leq x \leq a\} \ , \tag{1.7.7}$$

$$[a,\, b] = \{x \in X | a \leq x \leq b\} \ . \tag{1.7.8}$$

(1.7.7) and (1.7.8) imply $[u,\, a] \cap [a,\, b] = \{a\}$.

Case 2.2. $u > b$. This case is analogous to case 2.1.

Step 3. Proof that $[u,\, a] \cap [a,\, b] = \{a\}$ implies $[u,\, a] \cup [a,\, b] = [u,\, b]$.

Step 3.1. Proof that $b \leq a \leq u$ or $u \leq a \leq b$. Seeking a contradiction, assume ($u > a$ and $b > a$) or ($a < u$ and $b < a$) , without loss of generality $u > a$ and $b > a$. From this assumption it follows:

$$\begin{aligned} [u,\, a] \cap [a,\, b] &= [a,\, u] \cap [a,\, b] \\ &= [a,\, u \wedge b] \ . \end{aligned}$$

In particular,

$$u \wedge b \in [u,\, a] \cap [a,\, b] \ . \tag{1.7.9}$$

(1.7.9) and the assumption $[u,\, a] \cap [a,\, b] = \{a\}$ imply $u \wedge b = a$. From this and the assumption that X is a chain it follows that $u = a$ or $b = a$, contradicting the assumptions $u > a$ and $b > a$.

Step 3.2. Proof that $[u,\, a] \cup [a,\, b] = [u,\, b]$. By step 3.1., $b \leq a \leq u$ or $u \leq a \leq b$. Suppose without loss of generality that $u \leq a \leq b$. Then $[u,\, a] = \{x \in X | u \leq x \leq a\}$, $[a,\, b] = \{x \in X | a \leq x \leq b\}$ and $[u,\, b] = \{x \in X | u \leq x \leq b\}$. Consequently, $[u,\, a] \cup [a,\, b] = [u,\, b]$.

□

PROPOSITION 1.7.3. *(arboricity of the real line)*

(1) *The geodesic interval relation of the metric space* $(\mathbb{R},\, |\cdot - \cdot|)$ *coincides with the lattice interval relation of* $(\mathbb{R},\, \leq)$.
(2) $(\mathbb{R},\, |\cdot - \cdot|)$ *is arboric.*

PROOF.

(1) For $a,\, b,\, x \in \mathbb{R}$, $|a-x| + |x-b| = |a-b|$ iff $a \leq x \leq b$ or $b \leq x \leq a$.
(2) By 1.7.2 (arboricity of chains), $\mathbb{R}$ with the lattice interval relation of $(\mathbb{R},\, \leq)$ is arboric. By (1), $(\mathbb{R},\, |\cdot - \cdot|)$ is arboric.

□

The equivalence (2) ⇔ (3) of the following proposition has been cited from [**9**, theorem 1]. The implication (2) ⇒ (3) has already been stated in [**43**, section 2] as follows: "Trees in our sense which are finite are trees in König's sense."

PROPOSITION 1.7.4. *(tree representation of finite arboric interval spaces) Let* X *be a finite set and* $\langle \cdot,\, \cdot,\, \cdot \rangle$ *a ternary relation on* X *. The following conditions are equivalent:*

(1) $(X,\, \langle \cdot,\, \cdot,\, \cdot \rangle)$ *is an arboric interval space.*
(2) *For* $u,\, a,\, b \in X$:
   (a) *There is a* $c \in X$ *such that* $[u,\, a] \cap [u,\, b] = [u,\, c]$.
   (b) *If* $[u,\, a] \subseteq [u,\, b]$ *, then* $[u,\, a] \cap [a,\, b] = \{a\}$.
   (c) *If* $[u,\, a] \cap [a,\, b] = \{a\}$ *, then* $[u,\, a] \cup [a,\, b] = [u,\, b]$.
(3) *There is an* $E \in \binom{X}{2}$ *such that* $T = (X,\, E)$ *is a tree and its distance function* d *satisfies* $\langle \cdot,\, \cdot,\, \cdot \rangle = \langle \cdot,\, \cdot,\, \cdot \rangle_d$.

PROOF.

Step 1. (1) ⇔ (2) is 1.7.1 (Sholander's criterion for arboric interval spaces).
Step 2. (2) ⇔ (3) is [**9**, theorem 1]. □

The following proposition is cited from [**43**, (2.5)].

PROPOSITION 1.7.5. *(medianity of arboric interval spaces) Each arboric interval space is median.*

PROOF. [**43**, (2.5)] □

1.7.5 (medianity of arboric interval spaces) may be used implicitly by applying results on modular and median interval spaces to arboric interval spaces.

A *weighted tree* is a triple $(N,\, E,\, w)$ such that $(N,\, E)$ is a tree and w is a function from E to $\mathbb{R}_{>0}$. This concept has been taken from [**4**]. Let $T = (N,\, E,\, w)$ be a weighted tree. The statement that d_w is a metric on N in the following proposition has been cited from [**4**].

PROPOSITION 1.7.6. *(weighted trees) Let* $T = (N,\, E,\, w)$ *be a weighted tree. Let* d *be the distance function of the graph* $(N,\, E)$ *. The mapping* $d_w :\, N \times N \to \mathbb{R}_{\geq 0}$ *defined by* $d_w\,(a,\, b) = \sum_{e \in E(a,\, b)} w\,(e)$ *is a metric on* N *, and* $\langle \cdot,\, \cdot,\, \cdot \rangle_{d_w}$ *coincides with* $\langle \cdot,\, \cdot,\, \cdot \rangle_d$ *. In particular,* $\langle \cdot,\, \cdot,\, \cdot \rangle_{d_w}$ *is determined by* $(N,\, E)$ *, i.e. independent of the choice of* w .

PROOF. By 1.7.4 (tree representation of finite arboric interval spaces), $(N, \langle\cdot, \cdot, \cdot\rangle_d)$ is an arboric interval space. By 1.7.5 (medianity of arboric interval spaces), it is median. In particular, it is modular. By 1.6.1 (metrics for a modular interval space) it suffices to prove that for $x, y \in N$, $d_w(x, y) = d_w(y, x)$ and $d_w(x, y) = 0$ iff $x = y$, and for $x, y, z \in N$, $\langle x, y, z\rangle_d$ implies $d_w(x, z) = d_w(x, y) + d_w(y, z)$.

Step 1. Proof that for $x, y \in N$, $d_w(x, y) = d_w(y, x)$. $E(x, y) = E(y, x)$. Thus, $\sum_{e \in E(x, y)} w(e) = \sum_{e \in E(y, x)} w(e)$, i.e. $d_w(x, y) = d_w(y, x)$.

Step 2. $d_w(x, y) = 0$ iff $E(x, y) = \emptyset$ iff $x = y$. The first equivalence follows from $w(e) > 0$ for each $e \in E$.

Step 3. Proof that for $x, y, z \in N$, $\langle x, y, z\rangle_d$ implies $d_w(x, z) = d_w(x, y) + d_w(y, z)$. From the assumption $\langle x, y, z\rangle_d$ it follows by 1.4.5 (interval relation of a connected graph) that y is on p_{xz}. Thus, $E(x, z)$ is the disjoint union of $E(x, y)$ and $E(y, z)$. Therefore, $\sum_{e \in E(x, z)} w(e) = \sum_{e \in E(x, y)} w(e) + \sum_{e \in E(y, z)} w(e)$, i.e. $d_w(x, z) = d_w(x, y) + d_w(y, z)$. $\square$

Let $T = (N, E, w)$ be a weighted tree. By 1.7.6 (weighted trees), the function $d_w : N \times N \to \mathbb{R}_{\geq 0}$ defined by

$$d_w(a, b) := \sum_{e \in E(a, b)} w(e)$$

is a metric on N. It is called the *metric induced by* T. This concept has been taken from [**4**].

1.8. Metric Spaces in their Modular Surroundings

PROPOSITION 1.8.1. *(existence of modular extension) For each metric space X, there is an isometric map from X into a modular metric space.*

PROOF. By 1.4.26 (existence of injective closure), there is an isometric map i from X into an injective metric space, which is modular by 1.6.2 (modularity of injective metric spaces). $\square$

Let X be a metric space.

By 1.8.1 (existence of modular extension), there is an isometric map i from X into a modular metric space Y. By 1.6.3(1) (point-interval distance in a modular metric space), the second equality of the following definition holds. In particular, the definition is independent of the choice of the modular metric space Y and the isometric map i. For $x, y, a \in X$, the *modular distance* of the point x from the pair (y, a) or *Gromov product* of y and a with respect to x is the number

$$\begin{aligned} d_{x,ya} &:= d_{i(x)[i(y), i(a)]} \\ &= \frac{1}{2}\left(d_{i(x)i(y)} + d_{i(x)i(a)} - d_{i(y)i(a)}\right) \\ &= \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right) \end{aligned}$$

In [**17**, 1.1], the last expression has been denoted by $(y, a)_x$. In [**50**, 2.7], it has been denoted by $(y|a)_x$. The notation $(y \cdot a)_x$ is also in use. Here, the term 'point-pair modular distance' and the notation $d_{x,ya}$ have been added as a reminder of the indicated geometric meaning of that number. If X is itself modular, then, taking for i the identity map of X , $d_{x,ab} = d_{x[a,\, b]}$. Another geometric interpretation of the Gromov product when X is a Euclidean plane is given in [**50**, 2.7]. Expressions in terms of point-pair modular distances are preferred to expressions in terms of point-point distances whenever they have a more direct geometric interpretation.

For finite $Y \subseteq X$ and $u \in X$, the *distance sum* of u along Y is the number

$$\lambda_u^Y := \sum_{x \in Y} d_{xu} .$$

For finite $Y \subseteq X$ and $a,\, b \in X$, the *augmented modular distance sum* of the pair $(a,\, b)$ along Y is the number

$$\lambda_{ab}^Y := d_{ab} + \sum_{x \in Y} d_{x,ab} .$$

In [**45**], the expression λ_u^X is written as R_u . In [**30**, 7.3.2], it is written as d_{u+} . The concept of an augmented modular distance sum coincides with the concept of a *centrality index* in [**30**, 7.3.2], which has been defined there under addtional assumptions. There, λ_{ab}^X is written as $c\,(a,\, b)$.

Parts (2) and (4) of the following proposition have been cited from [**50**, 2.8 (1)]. Part (5) has been cited from [**50**, 2.8 (2)]. In [**30**, 7.3.2], part (11) has been stated under additional assumptions.

PROPOSITION 1.8.2. *(point-pair modular distance) Let X be a metric space. For $a,\, b,\, x \in X$:*

(1) $d_{x,ab} \geq 0$.
(2) $d_{x,ab} = d_{x,ba}$.
(3) $x \in [a,\, b]$ *iff* $d_{x,ab} = 0$.
(4) *If* $x \in \{a,\, b\}$ *, then* $d_{x,ab} = 0$.
(5) $d_{ab} = d_{a,bx} + d_{b,ax}$.
(6) *For* $u \in X$ *, if* $\langle a,\, u,\, b\rangle$ *, then:*
    (a) $d_{x,ab} \leq d_{xu}$.
    (b) *If not* $\langle x,\, u,\, b\rangle$ *, then* $d_{x,ab} < d_{xu}$.
(7) $d_{[x,\, a][y,\, b]} \geq d_{x,yb} - d_{x,ya}$.
(8) *For* $y \in X$ *, if* $[x,\, b] \cap [y,\, a] \neq \emptyset$ *, then* $d_{x,yb} \geq d_{x,ya}$.
(9) *For* $y \in X$ *, if* $[x,\, b] \cap [y,\, a] \neq \emptyset$ *, then* $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$.
(10) *For finite* $Y \subseteq X$ *and* $u \in X$ *, if* $a,\, b \in Y$ *and* $\langle a,\, u,\, b\rangle$ *, then:*
    (a) $\lambda_{ab}^Y \leq \lambda_u^Y$.
    (b) *If there is a* $y \in Y \setminus \{a,\, b\}$ *such that* $d_{y,ab} < d_{yu}$ *, then* $\lambda_{ab}^Y < \lambda_u^Y$.
    (c) *If there is a* $y \in Y \setminus \{a,\, b\}$ *such that not* $\langle y,\, u,\, b\rangle$ *, then* $\lambda_{ab}^Y < \lambda_u^Y$.
(11) *For finite* $Y \subseteq X$ *and* $u \in X$ *, if* $a,\, b \in Y$ *and* $a \neq b$ *, then*

$$\lambda_{ab}^Y = \frac{1}{2}\left(\lambda_a^Y + \lambda_b^Y - (|Y| - 2)\, d_{ab}\right) .$$

PROOF. By 1.8.1 (existence of modular extension), there is

$$i\,,\ \text{an isometric map from } X \text{ into a modular metric space } Y\,. \tag{1.8.1}$$

(1)

$$\begin{aligned} d_{x,ab} &= d_{i(x)[i(a),\,i(b)]} \\ &\geq 0\,. \end{aligned}$$

(2) $[i\,(a)\,,\,i\,(b)] = [i\,(b)\,,\,i\,(a)]$. Therefore,

$$\begin{aligned} d_{x,ab} &= d_{i(x)[i(a),\,i(b)]} \\ &= d_{i(x)[i(b),\,i(a)]} \\ &= d_{x,ba}\,. \end{aligned}$$

(3) The following conditions are equivalent:

$$\begin{aligned} x &\in [a,\,b]\ , \\ d_{ab} &= d_{ax} + d_{xb}\,, \\ d_{xa} + d_{xb} - d_{ab} &= 0\,, \\ \frac{1}{2}\left(d_{xa} + d_{xb} - d_{ab}\right) &= 0\,, \\ d_{x,ab} &= 0\,. \end{aligned}$$

(4) From the assumption $x \in \{a,\,b\}$ and $a,\,b \in [a,\,b]$ it follows that $x \in [a,\,b]$. By (3), $d_{x,ab} = 0$.

(5) By 1.6.3(2) (point-interval distance in a modular metric space),

$$\begin{aligned} d_{a,bx} + d_{b,ax} &= d_{i(a)[i(b),\,i(x)]} + d_{i(b)[i(a),\,i(x)]} \\ &= d_{i(a)i(b)} \\ &= d_{ab}\,. \end{aligned}$$

(6)

(a) From (1.8.1) it follows by 1.4.12 (isometric maps):

$$i \text{ is an embedding of } \left(X,\,\langle\cdot,\,\cdot,\,\cdot\rangle_d\right) \text{ into } \left(Y,\,\langle\cdot,\,\cdot,\,\cdot\rangle_d\right)\,. \tag{1.8.2}$$

The assumption $\langle a,\,u,\,b\rangle$ and (1.8.2) imply $\langle i\,(a)\,,\,i\,(u)\,,\,i\,(b)\rangle$, i.e. $i\,(u) \in [i\,(a)\,,\,i\,(b)]$. Consequently,

$$\begin{aligned} d_{x,ab} &= d_{i(x)[i(a),\,i(b)]} \\ &\leq d_{i(x)i(u)} \\ &= d_{xu}\,. \end{aligned}$$

(b) From the assumption $\langle a,\, u,\, b\rangle$, i.e. $d_{ab} = d_{au} + d_{ub}$, it follows that the following backward implications hold:

$$\begin{aligned}
&d_{x,ab} < d_{xu}\\
\Leftarrow&\frac{1}{2}\left(d_{xa} + d_{xb} - d_{ab}\right) < d_{xu}\\
\Leftarrow&d_{xa} + d_{xb} - d_{ab} < 2d_{xu}\\
\Leftarrow&d_{xa} + d_{xb} < d_{ab} + 2d_{xu}\\
\Leftarrow&d_{xa} + d_{xb} < d_{ua} + d_{ub} + d_{xu} + d_{xu}\\
\Leftarrow&d_{xa} + d_{xb} < \left(d_{xu} + d_{ua}\right) + \left(d_{xu} + d_{ub}\right) \ .
\end{aligned}$$

The last inequality follows by addition of the triangle inequality $d_{xa} \le d_{xu} + d_{ua}$ and the inequality $d_{xb} < d_{xu} + d_{ub}$, which is the assumption that not $\langle x,\, u,\, b\rangle$.

(7) For $u \in [x,\, a]$ and $v \in [y,\, b]$, i.e. $d_{xa} = d_{xu} + d_{ua}$ and $d_{yb} = d_{yv} + d_{vb}$, it is to be proved that $d_{uv} \ge d_{x,yb} - d_{x,ya}$. The following inequalities, of which the first one is to be proved, are equivalent:

$$\begin{aligned}
d_{uv} &\ge d_{x,yb} - d_{x,ya} \ ,\\
d_{uv} &\ge \frac{1}{2}\left(d_{xy} + d_{xb} - d_{yb}\right) - \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right) \ ,\\
2d_{uv} &\ge d_{xb} - d_{yb} - d_{xa} + d_{ya} \ ,\\
2d_{uv} &\ge d_{xb} - \left(d_{yv} + d_{vb}\right) - \left(d_{xu} + d_{ua}\right) + d_{ya} \ ,\\
\left(d_{xu} + d_{uv} + d_{vb}\right) + \left(d_{yv} + d_{vu} + d_{ua}\right) &\ge d_{xb} + d_{ya} \ .
\end{aligned}$$

The last inequality follows by addition of the two instances of 1.4.3(1) (metric spaces), $d_{xu} + d_{uv} + d_{vb} \ge d_{xb}$ and $d_{yv} + d_{vu} + d_{ua} \ge d_{ya}$.

(8) From the assumption $[x,\, b] \cap [y,\, a] \neq \emptyset$ it follows by (7):

$$\begin{aligned}
0 &= d_{[x,\, b][y,\, a]}\\
&\ge d_{x,ya} - d_{x,yb} \ .
\end{aligned}$$

Consequently, $d_{x,yb} \ge d_{x,ya}$.

(9) It is to be proved that $d_{x,ab} \ge d_{x,ya}$ and $d_{x,yb} \ge d_{x,ya}$.
Step 1. Proof that $d_{x,ab} \ge d_{x,ya}$. From the assumption $[x,\, b] \cap [y,\, a] \neq \emptyset$ and $[y,\, a] = [a,\, y]$ it follows that $[x,\, b] \cap [a,\, y] \neq \emptyset$. By (8), $d_{x,ab} \ge d_{x,ay}$. By (2), $d_{x,ab} \ge d_{x,ya}$.
Step 2. $d_{x,yb} \ge d_{x,ya}$ follows from the assumption $[x,\, b] \cap [y,\, a] \neq \emptyset$ by (8).

(10)

(a) From the assumption $\langle a,\, u,\, b\rangle$ it follows by (6a):

$$\text{For all } x \in X \ , \ d_{x,ab} \le d_{xu} \ . \tag{1.8.3}$$

(1.8.3) and the assumption $a,\, b \in Y$ imply by (4):

$$\begin{aligned}\lambda^Y_{ab} &= d_{ab} + \sum_{x\in Y} d_{x,ab} \\ &= d_{ab} + d_{a,ab} + d_{b,ab} + \sum_{x\in Y\setminus\{a,\,b\}} d_{x,ab} \\ &= d_{ab} + 0 + 0 + \sum_{x\in Y\setminus\{a,\,b\}} d_{x,ab} \\ &\le d_{au} + d_{bu} + \sum_{x\in Y\setminus\{a,\,b\}} d_{xu} \\ &= \sum_{x\in Y} d_{xu} \\ &= \lambda^Y_u\,.\end{aligned}$$

(b) In the fourth step of the proof of (10a), '$\le$' can be replaced by '$<$' if there is a $x \in Y \setminus \{a,\, b\}$ such that $d_{x,ab} < d_{yu}$.

(c) From the assumption that not $\langle y,\, u,\, b\rangle$ it follows by (6b), $d_{y,ab} < d_{yu}$. By (10b), $\lambda^Y_{ab} < \lambda^Y_u$.

(11) From the assumptions $a,\, b \in Y$ and $a \ne b$ it follows by (4):

$$\begin{aligned}\lambda^Y_{ab} &= d_{ab} + \sum_{x\in Y} d_{x,ab} \\ &= d_{ab} + \left(\sum_{x\in Y\setminus\{a,\,b\}} d_{x,ab}\right) + d_{a,ab} + d_{b,ab} \\ &= d_{ab} + \left(\sum_{x\in Y\setminus\{a,\,b\}} d_{x,ab}\right) + 0 + 0 \\ &= \frac{1}{2}\left(d_{ab} + d_{ab}\right) + \sum_{x\in Y\setminus\{a,\,b\}} \frac{1}{2}\left(d_{ax} + d_{bx} - d_{ab}\right) \\ &= \frac{1}{2}\left(d_{ab} + d_{ab}\right) + \left(\sum_{x\in Y\setminus\{a,\,b\}} \frac{1}{2}\left(d_{ax} + d_{bx}\right) - \sum_{x\in Y\setminus\{a,\,b\}} \frac{1}{2} d_{ab}\right) \\ &= \frac{1}{2}\left(\left(d_{ab} + d_{ba}\right) + \sum_{x\in Y\setminus\{a,\,b\}} \left(d_{ax} + d_{bx}\right) - \sum_{x\in Y\setminus\{a,\,b\}} d_{ab}\right) \\ &= \frac{1}{2}\left(\left(d_{aa} + d_{ba}\right) + \left(d_{ab} + d_{bb}\right) + \sum_{x\in Y\setminus\{a,\,b\}} \left(d_{ax} + d_{bx}\right) - \left|Y \setminus \{a,\, b\}\right| d_{ab}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left([d_{ax} + d_{bx}]_{x=a} + [d_{ax} + d_{bx}]_{x=b} + \sum_{x \in Y \setminus \{a, b\}} (d_{ax} + d_{bx}) - (|Y| - 2)\, d_{ab} \right) \\
&= \frac{1}{2}\left(\sum_{x \in Y} (d_{ax} + d_{bx}) - (|Y| - 2)\, d_{ab} \right) \\
&= \frac{1}{2}\left(\sum_{x \in Y} d_{ax} + \sum_{x \in Y} d_{bx} - (|Y| - 2)\, d_{ab} \right) \\
&= \frac{1}{2}\left(\lambda_a^Y + \lambda_b^Y - (|Y| - 2)\, d_{ab} \right) .
\end{aligned}
$$

$\square$

CHAPTER 2

General Interval Spaces and Metric Spaces

In this and subsequent chapters, the following known results on general interval spaces and metric spaces are used:

- ◦ 1.4.16 (Hedlíková's criterion for geometric interval spaces)
- ◦ 1.4.17 (aligned sequences in a geometric interval space)
- ◦ 1.4.18(1) (gated sets)
- ◦ 1.4.19 (product of geometric interval spaces)
- ◦ 1.4.26 (existence of injective closure)
- ◦ 1.8.2(2), (4), (5) (point-pair modular distance)

In this chapter, the following new concepts are introduced:

- ◦ median quadrangle
- ◦ quadrimodular matrix of points
- ◦ median-extremal point
- ◦ median boundary
- ◦ topological interval space

2.1. Topological Posets

PROPOSITION 2.1.1. *(compact directed topological posets) Let X be a non-empty compact directed topological poset. Then X has a greatest element.*

PROOF. The assumption that X is non-empty implies by 1.3.8(1) (topological posets):

$$\text{The set } C := \{\uparrow a | a \in X\} \text{ is a non-empty set of closed sets in } X\,. \tag{2.1.1}$$

Step 1. Proof that for non-empty finite $C_0 \subseteq C$, $\bigcap C_0 \neq \emptyset$. There is a non-empty finite $X_0 \subseteq X$ such that $C_0 = \{\uparrow a | a \in X_0\}$. From the assumption that X is directed it follows by 1.2.4(1) (directed posets) that X_0 has an upper bound u_0, i.e. for $a \in X_0$, $u_0 \in\uparrow a$, i.e. $u_0 \in \bigcap C_0$. In particular, $\bigcap C_0 \neq \emptyset$.

Step 2. Proof that X has a greatest element. The assumption that X is compact, (2.1.1) and step 1 imply by 1.3.6 (compactness criterion) that $\bigcap C \neq \emptyset$, i.e. there is a $u \in X$ such that for each $a \in X$, $u \in\uparrow a$, i.e. u is a greatest element of X. □

PROPOSITION 2.1.2. *(compact topological posets) Each non-empty compact topological poset has a maximal element.*

PROOF. Let X be a non-empty compact topological poset. From the assumption that X is non-empty it follows by 1.2.5 (Zorn's Lemma) that it suffices to prove that each non-empty chain C

in X has an upper bound. By 1.3.8(2) (topological posets), $\bar{C}$ is a non-empty chain in X. In particular,

$$\bar{C} \text{ is non-empty and directed.} \tag{2.1.2}$$

The assumption that X is compact implies by 1.3.7 (compact topological spaces):

$$\bar{C} \text{ is compact.} \tag{2.1.3}$$

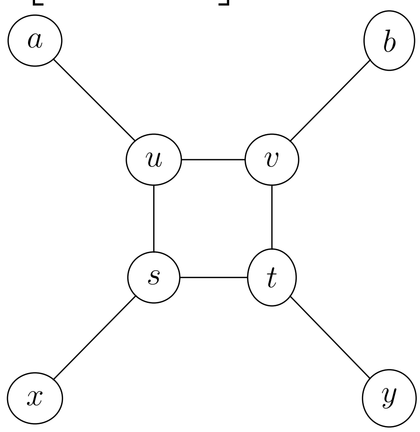
From (2.1.2) and (2.1.3) it follows by 2.1.1 (compact directed topological posets) that $\bar{C}$ has a greatest element s. In particular, s is an upper bound of C. □

2.2. Interval Spaces

The new concept of a median quadrangle is analogous to the concept of a median triangle: Let X be an interval space. A *median quadrangle* in X is a partial matrix

$$Q = \begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$$

in X such that the four-term sequences (x, s, t, y), (y, t, v, b), (b, v, u, a), (a, u, s, x) and the five-term sequences (x, s, u, v, b), (y, t, v, u, a), (y, t, s, u, a), (x, s, t, v, b) are aligned. Of these sequences, the four-term sequences are called the *sides* and the five-term sequences are called the *diagonals* of Q. The points x, y, a, b are called the *vertices* of Q. For example, in the following graph, $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle.



Further examples of median quadrangles are provided by the following proposition.

PROPOSITION 2.2.1. *(median quadrangles in the plane) For*

$$w_{st},\, w_{su},\, w_{xs},\, w_{yt},\, w_{au},\, w_{bv} \in \mathbb{R}_{\geq 0}\,,$$

define $s := (0,\, 0)$, $t := (w_{st},\, 0)$, $u := (0,\, w_{su})$, $v := (w_{st},\, w_{su})$, $x := (0,\, -w_{xs})$, $y := (w_{st} + w_{yt},\, 0)$, $a := (-w_{au},\, w_{su})$, $b := (w_{st},\, w_{su} + w_{bv})$.

(1) $\|s-t\|_1 = w_{st}$, $\|u-v\|_1 = w_{st}$, $\|s-u\|_1 = w_{su}$, $\|t-v\|_1 = w_{su}$, $\|x-s\|_1 = w_{xs}$, $\|y-t\|_1 = w_{yt}$, $\|a-u\|_1 = w_{au}$, $\|b-v\|_1 = w_{bv}$.

(2) *In* $(\mathbb{R}^2,\, \|\cdot - \cdot\|_1)$, $Q := \begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ *is a median quadrangle.*

PROOF.

(1) Step 1. Proof of the equation $\|u-v\|_1 = w_{st}$. From the assumption $w_{st} \in \mathbb{R}_{\geq 0}$ it follows:

$$\begin{aligned} \|u-v\|_1 &= \|(0,\, w_{su}) - (w_{st},\, w_{su})\|_1 \\ &= \|(-w_{st},\, 0)\|_1 \\ &= |-w_{st}| + |0| \\ &= w_{st} + 0 \\ &= w_{st}\,. \end{aligned}$$

Step 2. The proofs of the other seven equations are analogous to the proof of the equation $\|u-v\|_1 = w_{st}$.

(2) Step 1. Proof of the alignment $\langle x,\, s,\, t,\, y\rangle$. By 1.4.6 (aligned sequences in a metric space) it suffices to prove $\|x-y\|_1 = \|x-s\|_1 + \|s-t\|_1 + \|t-y\|_1$. The assumption $w_{st},\, w_{yt}\,, w_{xs} \in \mathbb{R}_{\geq 0}$ implies by (1):

$$\begin{aligned} \|x-y\|_1 &= \|(0,\, -w_{xs}) - (w_{st} + w_{yt},\, 0)\|_1 \\ &= \|(-w_{st} - w_{yt},\, -w_{xs})\|_1 \\ &= |-w_{st} - w_{yt}| + |-w_{xs}| \\ &= (w_{st} + w_{yt}) + w_{xs} \\ &= w_{xs} + w_{st} + w_{yt} \\ &= \|x-s\|_1 + \|s-t\|_1 + \|t-y\|_1\,. \end{aligned}$$

Step 2. The proofs of the other seven alignments defining the condition that Q is a median quadrangle are analogous to the proof of the alignment $\langle x,\, s,\, t,\, y\rangle$.

□

PROPOSITION 2.2.2. *(median quadrangles) Let X be an interval space. For* $Q = \begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ *a median quadrangle in* X :

(1) *For* $x'' \in \{x,\, s\}$, $y'' \in \{y,\, t\}$, $a'' \in \{a,\, u\}$, $b'' \in \{b,\, v\}$:
  (a) $\begin{bmatrix} & a'' & \\ x'' & s & y'' \end{bmatrix}$ *is a median triangle.*
  (b) $\begin{bmatrix} & b'' & \\ x'' & t & y'' \end{bmatrix}$ *is a median triangle.*
  (c) $\begin{bmatrix} & b'' & \\ x'' & u & a'' \end{bmatrix}$ *is a median triangle.*
  (d) $\begin{bmatrix} & b'' & \\ y'' & v & a'' \end{bmatrix}$ *is a median triangle.*
  (e) $\langle x,\, x'',\, b'',\, b\rangle$.
  (f) $\langle y,\, y'',\, a'',\, a\rangle$.

(2) $s \in M(x,\, y,\, a) \cap [x,\, b]$.

(3) *Setting* $A := \{x,\, y,\, a,\, b\}$ *and* $T := \{x,\, y,\, a,\, b,\, s,\, t,\, u,\, v\}$, A *is an interval-spanning set in* $(T,\, \langle\cdot,\, \cdot,\, \cdot\rangle)$.

(4) *The rotated partial matrices* $\begin{bmatrix} b & & & y \\ & v & t & \\ & u & s & \\ a & & & x \end{bmatrix}$, $\begin{bmatrix} y & & & x \\ & t & s & \\ & v & u & \\ b & & & a \end{bmatrix}$, $\begin{bmatrix} x & & & a \\ & s & u & \\ & t & v & \\ y & & & b \end{bmatrix}$ *and the reflected partial matrices* $\begin{bmatrix} x & & & y \\ & s & t & \\ & u & v & \\ a & & & b \end{bmatrix}$, $\begin{bmatrix} y & & & b \\ & t & v & \\ & s & u & \\ x & & & a \end{bmatrix}$, $\begin{bmatrix} b & & & a \\ & v & u & \\ & t & s & \\ y & & & x \end{bmatrix}$, $\begin{bmatrix} a & & & x \\ & u & s & \\ & v & t & \\ b & & & y \end{bmatrix}$ *are median quadrangles.*

PROOF.

(1)
  (a) The assumption that Q is a median quadrangle entails $\langle x,\, s,\, t,\, y\rangle$, $\langle y,\, t,\, s,\, u,\, a\rangle$ and $\langle a,\, u,\, s,\, x\rangle$. In particular, $\langle x'',\, s,\, y''\rangle$, $\langle y'',\, s,\, a''\rangle$ and $\langle a'',\, s,\, x''\rangle$, i.e. $\begin{bmatrix} & a'' & \\ x'' & s & y'' \end{bmatrix}$ is a median triangle.
  (b) is analogous to (1a).
  (c) is analogous to (1a).
  (d) is analogous to (1a).

(e) The assumption that R is a median quadrangle entails $\langle x,\, s,\, t,\, v,\, b\rangle$. In particular, $\langle x,\, x'',\, b'',\, b\rangle$.

(f) is analogous to (1e).

(2) The assumption that Q is a median quadrangle entails

$$\langle x,\, s,\, t,\, v,\, b\rangle \tag{2.2.1}$$

and implies by (1a) that $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ is a median triangle, i.e.

$$s \in M\,(x,\, y,\, a)\ . \tag{2.2.2}$$

(2.2.1) entails that $\langle x,\, s,\, b\rangle$, i.e.

$$s \in [x,\, b]\ . \tag{2.2.3}$$

(2.2.2) and (2.2.3) together say that $s \in M\,(x,\, y,\, a) \cap [x,\, b]$.

(3) For $x',\, y' \in T$ it is to be proved that there are $w,\, z \in A$ such that $\langle w,\, x',\, y',\, z\rangle$. Let without loss of generality $x' \in \{x,\, s\}$.

Case 1. $y' \in \{x,\, s,\, t,\, y\}$. Then

$$x',\, y' \in \{x,\, s,\, t,\, y\}\ . \tag{2.2.4}$$

The assumption that Q is a median quadrangle entails:

$$\langle x,\, s,\, t,\, y\rangle\ . \tag{2.2.5}$$

From (2.2.5) and (2.2.4) it follows by 1.4.7(1d) (aligned sequences) that there are $w,\, z \in \{x,\, y\}$ such that $\langle w,\, x',\, y',\, z\rangle$.

Case 2. $y' \in \{u,\, a\}$. Then

$$x',\, y' \in \{a,\, u,\, s,\, x\}\ . \tag{2.2.6}$$

The assumption that Q is a median quadrangle entails:

$$\langle a,\, u,\, s,\, x\rangle\ . \tag{2.2.7}$$

(2.2.7) and (2.2.6) imply by 1.4.7(1d) (aligned sequences) that there are $w,\, z \in \{a,\, x\}$ such that $\langle w,\, x',\, y',\, z\rangle$.

Case 3. $y' \in \{v,\, b\}$. Then

$$x',\, y' \in \{x,\, s,\, u,\, v,\, b\}\ . \tag{2.2.8}$$

The assumption that Q is a median quadrangle entails:

$$\langle x,\, s,\, u,\, v,\, b\rangle\ . \tag{2.2.9}$$

From (2.2.9) and (2.2.8) it follows by 1.4.7(1d) (aligned sequences) that there are $w,\, z \in \{x,\, b\}$ such that $\langle w,\, x',\, y',\, z\rangle$.

(4) The sides and diagonals of the rotated and reflected partial matrices are the sides and diagonals of Q and the inverse sequences of the sides and diagonals of Q . The assumption that Q is a median quadrangle implies by 1.4.7(1c) (aligned sequences) that they are aligned.

□

PROPOSITION 2.2.3. *(degenerate median quadrangles) Let X be an interval space.*

(1) *For* $Q = \begin{bmatrix} a & & & b \\ & u & u & \\ & s & s & \\ x & & & y \end{bmatrix}$ *a partial matrix in* X ,

(a) *Q is a median quadrangle iff the sequences* $(x,\, s,\, y)$, $(y,\, s,\, u,\, b)$, $(b,\, u,\, a)$, $(a,\, u,\, s,\, x)$, $(x,\, s,\, u,\, b)$, $(y,\, s,\, u,\, a)$ *are aligned.*

(b) *If Q is a median quadrangle, then* $Q' = \begin{bmatrix} b & & & a \\ & u & u & \\ & s & s & \\ x & & & y \end{bmatrix}$ *is also a median quadrangle.*

(2) *For* $Q = \begin{bmatrix} a & & & a \\ & u & u & \\ & s & s & \\ x & & & y \end{bmatrix}$ *a partial matrix in* X , *Q is a median quadrangle iff* $u = a$ *and* $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ *is a median triangle.*

(3) *For* $Q = \begin{bmatrix} a & & & b \\ & s & t & \\ & s & t & \\ x & & & y \end{bmatrix}$ *a partial matrix in* X ,

(a) *Q is a median quadrangle iff the sequences* $(x,\, s,\, t,\, y)$, $(y,\, t,\, b)$, $(b,\, t,\, s,\, a)$, $(a,\, s,\, x)$, $(x,\, s,\, t,\, b)$, $(y,\, t,\, s,\, a)$ *are aligned.*

(b) *If Q is a median quadrangle, then* $Q' = \begin{bmatrix} a & & & y \\ & s & t & \\ & s & t & \\ x & & & b \end{bmatrix}$ *is also a median quadrangle.*

(4) *For* $Q = \begin{bmatrix} x_3 & & & x_4 \\ & s & s & \\ & s & s & \\ x_1 & & & x_2 \end{bmatrix}$ *a partial matrix in* X ,

(a) *Q is a median quadrangle iff for* $j,\, k \in [4]$ *satisfying* $j < k$, $\langle x_j,\, s,\, x_k \rangle$.

(b) *Q is a median quadrangle iff for* $j,\, k \in [4]$ *satisfying* $j \neq k$, $\langle x_j,\, s,\, x_k \rangle$.

(c) *If Q is a median quadrangle, then for each permutation* $\pi \in S_4$, $Q' = \begin{bmatrix} x_{\pi(3)} & & & x_{\pi(4)} \\ & s & s & \\ & s & s & \\ x_{\pi(1)} & & & x_{\pi(2)} \end{bmatrix}$ *is also a median quadrangle.*

(5) *For* $x,\ y,\ a \in X$, *if* $\langle x,\ a,\ y\rangle$, *then* $\begin{bmatrix} a & & & a \\ & a & a & \\ & a & a & \\ x & & & y \end{bmatrix}$ *is a median quadrangle.*

PROOF.

(1)
  (a) Q is a median quadrangle iff the four-term sequences $(x,\ s,\ s,\ y)$, $(y,\ s,\ u,\ b)$, $(b,\ u,\ u,\ a)$, $(a,\ u,\ s,\ x)$ and the five-term sequences $(x,\ s,\ u,\ u,\ b)$, $(y,\ s,\ u,\ u,\ a)$, $(y,\ s,\ s,\ u,\ a)$, $(x,\ s,\ s,\ u,\ b)$ are aligned. By 1.4.7(1b) (aligned sequences), alignment of these sequences is equivalent to alignment of the sequences $(x,\ s,\ y)$, $(y,\ s,\ u,\ b)$, $(b,\ u,\ a)$, $(a,\ u,\ s,\ x)$ and the sequences $(x,\ s,\ u,\ b)$, $(y,\ s,\ u,\ a)$, $(y,\ s,\ u,\ a)$, $(x,\ s,\ u,\ b)$. Two of these alignment conditions are redundant.
  (b) By (1a), the condition that Q is a median quadrangle is equivalent to the alignments

$$\langle x,\ s,\ y\rangle\ , \tag{2.2.10}$$
$$\langle y,\ s,\ u,\ b\rangle\ , \tag{2.2.11}$$
$$\langle b,\ u,\ a\rangle\ , \tag{2.2.12}$$
$$\langle a,\ u,\ s,\ x\rangle\ , \tag{2.2.13}$$
$$\langle x,\ s,\ u,\ b\rangle\ , \tag{2.2.14}$$
$$\langle y,\ s,\ u,\ a\rangle\ . \tag{2.2.15}$$

  Interchanging a and b, the condition that Q' is a median quadrangle is equivalent to the alignments

$$\langle x,\ s,\ y\rangle\ , \tag{2.2.16}$$
$$\langle y,\ s,\ u,\ a\rangle\ , \tag{2.2.17}$$
$$\langle a,\ u,\ b\rangle\ , \tag{2.2.18}$$
$$\langle b,\ u,\ s,\ x\rangle\ , \tag{2.2.19}$$
$$\langle x,\ s,\ u,\ a\rangle\ , \tag{2.2.20}$$
$$\langle y,\ s,\ u,\ b\rangle\ . \tag{2.2.21}$$

  By 1.4.7(1c) (aligned sequences), the following equivalences hold: (2.2.10) $\Leftrightarrow$ (2.2.16), (2.2.11) $\Leftrightarrow$ (2.2.21), (2.2.12) $\Leftrightarrow$ (2.2.18), (2.2.13) $\Leftrightarrow$ (2.2.20), (2.2.14) $\Leftrightarrow$ (2.2.19), (2.2.15) $\Leftrightarrow$ (2.2.17). Consequently, if Q is a median quadrangle, then Q' is also a median quadrangle.

(2) Step 1. (⇒) Suppose that Q is a median quadrangle. By (1a):

$$\langle x,\, s,\, y\rangle \ , \tag{2.2.22}$$

$$\langle y,\, s,\, u,\, a\rangle \ , \tag{2.2.23}$$

$$\langle a,\, u,\, a\rangle \ , \tag{2.2.24}$$

$$\langle a,\, u,\, s,\, x\rangle \ . \tag{2.2.25}$$

(2.2.23), (2.2.25) entail:

$$\langle y,\, s,\, a\rangle \ , \tag{2.2.26}$$

$$\langle a,\, s,\, x\rangle \ . \tag{2.2.27}$$

(2.2.22), (2.2.26) and (2.2.27) together say that $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ is a median triangle. (2.2.24) implies $u = a$.

Step 2. (⇐) The assumption $u = a$ implies by (1a) that it suffices to prove that the sequences $(x,\, s,\, y)$, $(y,\, s,\, a,\, a)$, $(a,\, a,\, a)$, $(a,\, a,\, s,\, x)$, $(x,\, s,\, a,\, a)$, $(y,\, s,\, a,\, a)$ are aligned. By 1.4.7(1a) (aligned sequences) ot suffices to prove that $(x,\, s,\, y)$, $(y,\, s,\, a)$, $(a,\, a,\, a)$, $(a,\, s,\, x)$, $(x,\, s,\, a)$, $(y,\, s,\, a)$ are aligned. These alignments are entailed by the assumption that $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ is a median triangle.

(3) is analogous to (1).

(4)
 (a) By (1a) and 1.4.7(1b) (aligned sequences), the following conditions are equivalent:
  - Q is a median quadrangle.
  - $\langle x_1,\, s,\, x_2\rangle$, $\langle x_2,\, s,\, s,\, x_4\rangle$, $\langle x_4,\, s,\, x_3\rangle$, $\langle x_3,\, s,\, s,\, x_1\rangle$, $\langle x_1,\, s,\, s,\, x_4\rangle$, $\langle x_2,\, s,\, s,\, x_3\rangle$.
  - $\langle x_1,\, s,\, x_2\rangle$, $\langle x_2,\, s,\, x_4\rangle$, $\langle x_4,\, s,\, x_3\rangle$, $\langle x_3,\, s,\, x_1\rangle$, $\langle x_1,\, s,\, x_4\rangle$, $\langle x_2,\, s,\, x_3\rangle$.
  - $\langle x_1,\, s,\, x_2\rangle$, $\langle x_1,\, s,\, x_3\rangle$,$\langle x_1,\, s,\, x_4\rangle$, $\langle x_2,\, s,\, x_3\rangle$. $\langle x_2,\, s,\, x_4\rangle$, $\langle x_3,\, s,\, x_4\rangle$.
  - For all $j,\, k \in [4]$ satisfying $j < k$, $\langle x_j,\, s,\, x_k\rangle$.

 (b) follows from (4a).

 (c) From the assumption that Q is a median quadrangle if follows by (4b) that for $j,\, k \in [4]$ satisfying $j \neq k$, $\langle x_j,\, s,\, x_k\rangle$. Therefore, for $j,\, k \in [4]$ satisfying $j \neq k$, $\pi(j) \neq \pi(k)$ and thus $\left\langle x_{\pi(j)},\, s,\, x_{\pi(k)}\right\rangle$. By (4b), Q' is a median quadrangle.

(5) By (2) it suffices to prove that $\begin{bmatrix} & a & \\ x & a & y \end{bmatrix}$ is a median triangle. This claim follows from the assumption $\langle x,\, a,\, y\rangle$ by 1.4.9(1) (median triangles).

□

Let X be an interval space. A matrix $Q = \begin{bmatrix} a & b \\ x & y \end{bmatrix}$ of points in X is called

- *quadrimodular*, in symbols $\left\langle \begin{matrix} a & b \\ x & y \end{matrix} \right\rangle$, iff there are $s, t, u, v \in Y$ such that $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle.
- *star-quadrimodular*, in symbols $\left\langle \begin{matrix} a & & b \\ & \cdot & \\ x & & y \end{matrix} \right\rangle$, iff there is an $s \in X$ such that $\begin{bmatrix} a & & & b \\ & s & s & \\ & s & s & \\ x & & & y \end{bmatrix}$ is a median quadrangle.
- *vertical-quadrimodular*, in symbols $\left\langle \begin{matrix} a & | & b \\ x & & y \end{matrix} \right\rangle$, iff there are $s, u \in X$ such that $\begin{bmatrix} a & & & b \\ & u & u & \\ & s & s & \\ x & & & y \end{bmatrix}$ is a median quadrangle.
- *properly vertical-quadrimodular*, in symbols $\left\langle \begin{matrix} a & : & b \\ x & & y \end{matrix} \right\rangle$, iff $\left\langle \begin{matrix} a & | & b \\ x & & y \end{matrix} \right\rangle$ and not $\left\langle \begin{matrix} a & & b \\ & \cdot & \\ x & & y \end{matrix} \right\rangle$.
- *horizontal-quadrimodular*, in symbols $\left\langle \begin{matrix} a & & b \\ & - & \\ x & & y \end{matrix} \right\rangle$, iff there are $s, u \in X$ such that $\begin{bmatrix} a & & & b \\ & s & t & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle.
- *properly horizontal-quadrimodular*, in symbols $\left\langle \begin{matrix} a & & b \\ & \cdot\cdot & \\ x & & y \end{matrix} \right\rangle$, iff $\left\langle \begin{matrix} a & & b \\ & - & \\ x & & y \end{matrix} \right\rangle$ and not $\left\langle \begin{matrix} a & & b \\ & \cdot & \\ x & & y \end{matrix} \right\rangle$.
- *properly quadrimodular*, in symbols $\left\langle \begin{matrix} a & & b \\ & :: & \\ x & & y \end{matrix} \right\rangle$, iff $\left\langle \begin{matrix} a & b \\ x & y \end{matrix} \right\rangle$ and not $\left\langle \begin{matrix} a & | & b \\ x & & y \end{matrix} \right\rangle$ and not $\left\langle \begin{matrix} a & & b \\ & - & \\ x & & y \end{matrix} \right\rangle$.

In 2.2.1 (median quadrangles in the plane), $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular. If $w_{st} = 0$ and $w_{su} = 0$, then $s = t = u = v$, and $\left\langle \begin{matrix} a & & b \\ x & \cdot & y \end{matrix} \right\rangle$. If $w_{st} = 0$ and $w_{su} > 0$, then $s = t$, $u = v$, $s \neq u$ and $\left\langle \begin{matrix} a & & b \\ x & : & y \end{matrix} \right\rangle$. If $w_{st} > 0$ and $w_{su} = 0$, then $s = u$, $t = v$, $s \neq t$ and $\left\langle \begin{matrix} a & & b \\ x & \cdot\cdot & y \end{matrix} \right\rangle$. If $w_{st} > 0$ and $w_{su} > 0$, then $s \neq t$, $u \neq v$, $s \neq u$, $t \neq v$ and $\left\langle \begin{matrix} a & & b \\ x & :: & y \end{matrix} \right\rangle$.

PROPOSITION 2.2.4. *(symmetries of quadrimodularity properties) Let X be an interval space.*

(1) *For $x, y, a, b \in X$, the condition $\left\langle \begin{matrix} a & | & b \\ x & & y \end{matrix} \right\rangle$ is equivalent to each of the three rotated conditons $\left\langle \begin{matrix} b & & y \\ a & - & x \end{matrix} \right\rangle$, $\left\langle \begin{matrix} y & | & x \\ b & & a \end{matrix} \right\rangle$, $\left\langle \begin{matrix} x & & a \\ y & - & b \end{matrix} \right\rangle$ and to each of the four reflected conditions $\left\langle \begin{matrix} x & | & y \\ a & & b \end{matrix} \right\rangle$, $\left\langle \begin{matrix} y & & b \\ x & - & a \end{matrix} \right\rangle$, $\left\langle \begin{matrix} b & | & a \\ y & & x \end{matrix} \right\rangle$, $\left\langle \begin{matrix} a & & x \\ b & - & y \end{matrix} \right\rangle$.*

(2) *For $x, y, a, b \in X$, the conditions $\left\langle \begin{matrix} a & & b \\ x & - & y \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} y & | & b \\ x & & a \end{matrix} \right\rangle$ are invariant under the transpositions (yb) and (xa).*

(3) *For $x, y, a, b \in X$, the condition $\left\langle \begin{matrix} a & & b \\ x & \cdot & y \end{matrix} \right\rangle$ is invariant under each permutation of a, b, x, y.*

(4) *For $x, y, a, b \in X$, the condition $\left\langle \begin{matrix} a & & b \\ x & : & y \end{matrix} \right\rangle$ is equivalent to each of the three rotated conditions $\left\langle \begin{matrix} b & & y \\ a & \cdot\cdot & x \end{matrix} \right\rangle$, $\left\langle \begin{matrix} y & & x \\ b & : & a \end{matrix} \right\rangle$, $\left\langle \begin{matrix} x & & a \\ y & \cdot\cdot & b \end{matrix} \right\rangle$ and to each of the four reflected conditions $\left\langle \begin{matrix} x & & y \\ a & : & b \end{matrix} \right\rangle$, $\left\langle \begin{matrix} y & & b \\ x & \cdot\cdot & a \end{matrix} \right\rangle$, $\left\langle \begin{matrix} b & & a \\ y & : & x \end{matrix} \right\rangle$, $\left\langle \begin{matrix} a & & x \\ b & \cdot\cdot & y \end{matrix} \right\rangle$.*

(5) *For $x, y, a, b \in X$, the conditions $\left\langle \begin{matrix} a & & b \\ x & \cdot\cdot & y \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} y & & b \\ x & : & a \end{matrix} \right\rangle$ are invariant under the transpositions (yb) and (xa).*

PROOF.

(1) holds by 2.2.2(4) (median quadrangles).
(2) holds by 2.2.3(3b) and (1b) (degenerate median quadrangles).
(3) holds by 2.2.3(4c) (degenerate median quadrangles).
(4) holds by (1) and (3).
(5) holds by (2) and (3).

$\square$

PROPOSITION 2.2.5. *(quadrimodularity properties) Let X be an interval space.*

(1) *For $x,\, y,\, a,\, b\, \in\, X$, $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular iff $\left\langle \begin{matrix} a & :: & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & : & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & \cdot\cdot & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & \cdot & b \\ x & & y \end{matrix} \right\rangle$.*

(2) *For $x,\, y,\, a,\, b \in X$, if $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular, then the partial matrices $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$, $\begin{bmatrix} & b & \\ x & & y \end{bmatrix}$, $\begin{bmatrix} & b & \\ x & & a \end{bmatrix}$, $\begin{bmatrix} & b & \\ y & & a \end{bmatrix}$ are modular.*

(3) *For $x,\, y,\, a, \in X$, $\left\langle \begin{matrix} a & | & a \\ x & & y \end{matrix} \right\rangle$ iff $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular.*

(4) *For $x,\, y,\, a, \in X$, $\left\langle \begin{matrix} a & & a \\ x & \cdot & y \end{matrix} \right\rangle$ iff $\langle x,\, a,\, y\rangle$.*

(5) *For $x,\, y,\, a,\, b \in X$, if $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular, then $M\,(x,\, y,\, a) \cap [x,\, b] \neq \emptyset$.*

PROOF.

(1) Nothing remains to be proved.
(2) follows by 2.2.2(1a) to (1d) (median quadrangles).
(3) Step 1. ($\Rightarrow$) The assumption $\left\langle \begin{matrix} a & | & a \\ x & & y \end{matrix} \right\rangle$ says that there are $s,\, u\, \in\, X$ such that $\begin{bmatrix} a & & & a \\ & u & u & \\ & s & s & \\ x & & & y \end{bmatrix}$ is a median quadrangle. By 2.2.3(2) (degenerate median quadrangles), $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ is a median triangle. Consequently, $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular.

Step 2. ($\Leftarrow$) The assumption that $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular says that there is an $s \in X$ such that $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ is a median triangle. By 2.2.3(2) (degenerate median quadrangles), $\begin{bmatrix} a & & & a \\ & a & a & \\ & s & s & \\ x & & & y \end{bmatrix}$ is a median quadrangle. Consequently, $\left\langle \begin{matrix} a & | & a \\ x & & y \end{matrix} \right\rangle$.

(4) Step 1. ($\Rightarrow$) The assumption $\left\langle \begin{matrix} a & \cdot & a \\ x & & y \end{matrix} \right\rangle$ says that there is an $s \in X$ such that $\begin{bmatrix} a & & & a \\ & s & s & \\ & s & s & \\ x & & & y \end{bmatrix}$ is a median quadrangle. By 2.2.3(4b) (degenerate median quadrangles),

$$\langle x,\, s,\, y\rangle \ , \tag{2.2.28}$$

$$\langle a,\, s,\, a\rangle \ . \tag{2.2.29}$$

From (2.2.29) it follows:

$$s = a\,. \tag{2.2.30}$$

Substituting (2.2.30) into (2.2.28), $\langle x,\, a,\, y\rangle$.
Step 2. ($\Leftarrow$) follows by 2.2.3(5) (degenerate median quadrangles).

(5) The assumption that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular says that there are $s,\, t,\, u,\, v \in Y$ such that $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle. By 2.2.2(2) (median quadrangles), $s \in M\,(x,\, y,\, a) \cap [x,\, b]$. In particular, $M\,(x,\, y,\, a) \cap [x,\, b] \neq \emptyset$.

$\square$

PROPOSITION 2.2.6. *(homomorphic image of a median quadrangle) Let $f : X \to Y$ be a homomorphism of interval spaces.*

(1) *For* $Q = \begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ *a median quadrangle in* X ,

$$f \circ Q = \begin{bmatrix} f\,(a) & & & f\,(b) \\ & f\,(u) & f\,(v) & \\ & f\,(s) & f\,(t) & \\ (x) & & & (y) \end{bmatrix}$$

is a median quadrangle in Y .

(2) *For* $x,\, y,\, a,\, b \in X$, *if* $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ *is quadrimodular, then so is* $\begin{bmatrix} f\,(a) & f\,(b) \\ f\,(x) & f\,(y) \end{bmatrix}$.

PROOF.

(1) It is to be proved that each side or diagonal S' of $f \circ Q$ is aligned. There is a side or diagonal S of Q such that $S' = f \circ S$. It remains to be proved that $f \circ S$ is aligned. The

assumption that Q is a median quadrangle entails:

$$S \text{ is aligned.} \tag{2.2.31}$$

From (2.2.31) and the assumption that f is a homomorphism it follows by 1.4.13(1) (homomorphisms of interval spaces) that $f \circ S$ is aligned.

(2) follows by (1).

□

PROPOSITION 2.2.7. *(median quadrangles in a product of interval spaces) Let $(X_q)_{q \in I}$ be a family of interval spaces with product X.*

(1) *For $Q = \begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ a partial matrix in X, Q is a median quadrangle iff for each $q \in I$, $\begin{bmatrix} a(q) & & & b(q) \\ & u(q) & v(q) & \\ & s(q) & t(q) & \\ x(q) & & & y(q) \end{bmatrix}$ is a median quadrangle.*

(2) *For $x,\, y,\, a,\, b \in X$, $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular iff for each $q \in I$, $\begin{bmatrix} a(q) & b(q) \\ x(q) & y(q) \end{bmatrix}$ is quadrimodular.*

PROOF.

(1) follows by 1.4.14(2) (product of interval spaces).

(2) follows by (1) and, if I is infinite, the axiom of choice for the direction ($\Leftarrow$).

□

In the following proposition, note how the positions of corresponding symbols in the three partial matrices are different.

PROPOSITION 2.2.8. *(product of two interval spaces) Let $X_1,\, X_2$ be interval spaces.*

(1) *For $Q_1 = \begin{bmatrix} a_1 & & & b_1 \\ & s_1 & t_1 & \\ & s_1 & t_1 & \\ x_1 & & & y_1 \end{bmatrix}$ a median quadrangle in X_1 and $Q_2 = \begin{bmatrix} b_2 & & & a_2 \\ & s_2 & t_2 & \\ & s_2 & t_2 & \\ x_2 & & & y_2 \end{bmatrix}$ a median quadrangle in X_2, $Q := \begin{bmatrix} (b_1,\, b_2) & & & (y_1,\, y_2) \\ & (t_1,\, s_2) & (t_1,\, t_2) & \\ & (s_1,\, s_2) & (s_1,\, t_2) & \\ (x_1,\, x_2) & & & (a_1,\, a_2) \end{bmatrix}$ is a median quadrangle in $X_1 \times X_2$.*

(2) *For* $x,\, y,\, a,\, b \in X_1 \times X_2$ *, if* $\left\langle \begin{matrix} a_1 & - & b_1 \\ x_1 & & y_1 \end{matrix} \right\rangle$ *and* $\left\langle \begin{matrix} b_2 & - & a_2 \\ x_2 & & y_2 \end{matrix} \right\rangle$ *, then* $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ *is quadrimodular.*

PROOF.

(1) By 2.2.7(1) (median quadrangles in a product of interval spaces), it suffices to prove that $Q_1' = \begin{bmatrix} b_1 & & & y_1 \\ & t_1 & t_1 & \\ & s_1 & s_1 & \\ x_1 & & & a_1 \end{bmatrix}$ is a median quadrangle in X_1 and $Q_2' = \begin{bmatrix} b_2 & & & y_2 \\ & s_2 & t_2 & \\ & s_2 & t_2 & \\ x_2 & & & a_2 \end{bmatrix}$ is a median quadrangle in X_2 .
Step 1. Proof that Q_1' is a median quadrangle. From the assumption that Q_1 is a median quadrangle it follows by 2.2.2(4) (median quadrangles) that $\begin{bmatrix} y_1 & & & b_1 \\ & t_1 & t_1 & \\ & s_1 & s_1 & \\ x_1 & & & a_1 \end{bmatrix}$ is a median quadrangle. By 2.2.3(1b) (degenerate median quadrangles), Q_1' is a median quadrangle.
Step 2. Proof that Q_2' is a median quadrangle. The assumption that Q_2 is a median quadrangle implies by 2.2.3(3b) (degenerate median quadrangles) that Q_2' is a median quadrangle.

(2) follows from (1).

□

The concept of an extremal point of a set in a vector space over a totally ordered field also has a natural generalization to an interval space:

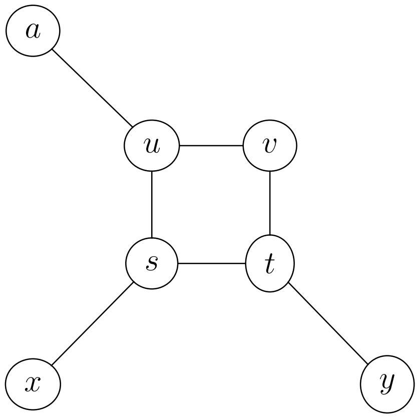
Let X be an interval space.

For $p \in X$, p is called *extremal* iff for all $a,\, b \in X$, the following implication holds: If $\langle a,\, p,\, b\rangle$, then $p \in \{a,\, b\}$. For example, when X is a triangle in the Euclidean plane $(\mathbb{R}^2,\, \|\cdot - \cdot\|_2)$, i.e. the convex closure of three affinely independent points $x,\, y,\, z$, then the set of its extremal points equals the set of the vertices $x,\, y,\, z$. Further examples of extremal points are provided by 5.1.5(1a) and (2b) (arboric interval spaces) below. In [**48**, chapter I, 1.23], the concept of an extremal point has been defined in the more general context of an algebraic closure space, and an extremal point has been called an *extreme point*.

The new concept of a median-extremal point generalizes the concept of an extremal point: For $p \in X$, p is called *median-extremal* iff for all $a,\, b,\, c \in X$, the following implication holds: If $\begin{bmatrix} & c & \\ a & p & b \end{bmatrix}$ is a median triangle, then $p \in \{a,\, b,\, c\}$. Each extremal point is median-extremal. The *median boundary* of X is the set

$$\partial_M (X) := \{x \in X | x \text{ is median-extremal.}\} \ .$$

For example, in the following graph, the median-extremal points are $x,\, y,\, a,\, v$, i.e. $\partial_M (X) = \{x,\, y,\, a,\, v\}$. $x,\, y,\, a$ are also extremal, while v is not extremal.



Further examples of median-extremal points are provided by propositions 6.3.1 (quadrimedian geometric interval spaces) below and 5.1.4 (median boundary of an arboric interval space) below.

2.3. Geometric Interval Spaces

PROPOSITION 2.3.1. *(geometric interval spaces) Let X be a geometric interval space.*

(1) *For $k \in \mathbb{Z}_{\geq 1}$, $S = (x_0,\, x_1,\, ...,\, x_k)$ an aligned sequence, $\kappa \in [k]$, $l \in \mathbb{Z}_{\geq 2}$ and $(x_{\kappa-1} = y_0,\, y_1,\, ...,\, y_l = x_\kappa)$ an aligned sequence,*
$(x_0,\, x_1,\, ...,\, x_{\kappa-1},\, y_1,\, y_2,\, ...,\, y_{l-1},\, x_\kappa,\, x_{\kappa+1},\, ...,\, x_k)$ is also aligned.
(2) *For $a,\, b \in X$, if $([a,\, b]\, ,\, \langle a,\, \cdot,\, \cdot\rangle)$ is a chain, then $[a,\, b]$ is convex.*
(3) *For $a,\, b \in X$, on $\langle a,\, b,\, \cdot\rangle$ the partial order $\langle a,\, \cdot,\, \cdot\rangle$ coincides with the partial order $\langle b,\, \cdot,\, \cdot\rangle$.*
(4) *For $a,\, v,\, x,\, y \in X$, if $\langle a,\, v,\, x\rangle$ and $\langle a,\, v,\, y\rangle$, then each median of $v,\, x,\, y$ is a median of $a,\, x,\, y$.*

PROOF.

(1) The claim is proved by induction on l .
Step 1. $l = 2$. This is 1.4.17(1b) (aligned sequences in a geometric interval space).
Step 2. $l \to l+1$. Suppose that $l \in \mathbb{Z}_{\geq 2}$ and for each
aligned sequence $(x_{\kappa-1} = y_0,\, y_1,\, ...,\, y_l = x_\kappa)$,
$\langle x_0,\, x_1,\, ...,\, x_{\kappa-1},\, y_1,\, y_2,\, ...,\, y_{l-1},\, x_\kappa,\, x_{\kappa+1},\, ...,\, x_k\rangle$. For
$(x_{\kappa-1} = y_0,\, y_1,\, ...,\, y_{l+1} = x_\kappa)$ an aligned sequence, it is to be proved that
$\langle x_0,\, x_1,\, ...,\, x_{\kappa-1},\, y_1,\, y_2,\, ...,\, y_l,\, x_\kappa,\, x_{\kappa+1},\, ...,\, x_k\rangle$. From the assumption that
$(x_{\kappa-1} = y_0,\, y_1,\, ...,\, y_{l+1} = x_\kappa)$ is aligned it follows by 1.4.7(1a) (aligned sequences):

$$\langle x_{\kappa-1} = y_0,\, y_1,\, ...,\, y_{l-1},\, y_{l+1} = x_\kappa\rangle \ . \tag{2.3.1}$$

(2.3.1) and the induction hypothesis imply:

$$\langle x_0,\, x_1,\, ...,\, x_{\kappa-1},\, y_1,\, y_2,\, ...,\, y_{l-1},\, y_{l+1},\, x_\kappa,\, x_{\kappa+1},\, ...,\, x_k\rangle \ . \tag{2.3.2}$$

The assumption that $(x_{\kappa-1} = y_0, y_1, ..., y_{l+1} = x_\kappa)$ is aligned entails:

$$\langle y_{l-1}, y_l, y_{l+1}\rangle . \tag{2.3.3}$$

From (2.3.2) and (2.3.3) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle x_0, x_1, ..., x_{\kappa-1}, y_1, y_2, ..., y_l, x_\kappa, x_{\kappa+1}, ..., x_k\rangle .$$

(2) For $x, y, z \in X$ it is to be proved that $x, z \in [a, b]$ and $\langle x, y, z\rangle$ imply $y \in [a, b]$, i.e. $\langle a, y, b\rangle$. From the assumptions that $([a, b], \langle a, \cdot, \cdot\rangle)$ is a chain and $x, z \in [a, b]$ it follows that $\langle a, x, z\rangle$ or $\langle a, z, x\rangle$. Assume without loss of generality that $\langle a, x, z\rangle$. The assumption $z \in [a, b]$ says:

$$\langle a, z, b\rangle . \tag{2.3.4}$$

(2.3.4) and the assumption $\langle a, x, z\rangle$ imply by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle a, x, z, b\rangle . \tag{2.3.5}$$

From (2.3.5) and the assumption $\langle x, y, z\rangle$ it follows by 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle a, x, y, z, b\rangle$. In particular, $\langle a, y, b\rangle$.

(3) For $x, y \in X$ satisfying $\langle a, b, x\rangle$ and $\langle a, b, y\rangle$ it is to be proved that $\langle a, x, y\rangle$ iff $\langle b, x, y\rangle$.
Step 1. ($\Rightarrow$) From the assumption $\langle a, x, y\rangle$ it is to be proved that $\langle b, x, y\rangle$. The assumptions $\langle a, b, x\rangle$ and $\langle a, x, y\rangle$ imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $\langle b, x, y\rangle$.
Step 2. ($\Leftarrow$) From the assumption $\langle b, x, y\rangle$ it is to be proved that $\langle a, x, y\rangle$. The assumptions $\langle a, b, y\rangle$ and $\langle b, x, y\rangle$ imply by 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle a, b, x, y\rangle$. In particular, $\langle a, x, y\rangle$.

(4) For $u \in X$ it is to be proved: If u is a median of v, x, y , then u is a median of a, x, y , i.e. $\langle a, u, x\rangle$, $\langle a, u, y\rangle$ and $\langle x, u, y\rangle$. The assumption that u is a median of v, x, y entails:

$$\langle v, u, x\rangle \text{ and } \langle v, u, y\rangle \tag{2.3.6}$$

and $\langle x, u, y\rangle$. It remains to be proved that $\langle a, u, x\rangle$, $\langle a, u, y\rangle$. From the assumptions $\langle a, v, x\rangle$ and $\langle a, v, y\rangle$ and (2.3.6) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle a, v, u, x\rangle$, $\langle a, v, u, y\rangle$. In particular, $\langle a, u, x\rangle$ and $\langle a, u, y\rangle$.

□

PROPOSITION 2.3.2. *(median quadrangles in a geometric interval space) Let X be a geometric interval space.*

(1) *A partial matrix* $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ *in X is a median quadrangle iff* $s \in M(x, y, a) \cap [x, b]$, $t \in M(s, y, b)$, $v \in M(t, b, a)$ *and* $u \in M(v, a, s)$.

(2) *For* $Q = \begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ *a median quadrangle:*

(a) *If* $s = t$, *then:* $u = v$. *In particular,* $\left\langle \begin{matrix} a \\ x \end{matrix} \mid \begin{matrix} b \\ y \end{matrix} \right\rangle$.
(b) *If* $s = u$, *then:* $t = v$. *In particular,* $\left\langle \begin{matrix} a \\ x \end{matrix} - \begin{matrix} b \\ y \end{matrix} \right\rangle$.
(c) *If* $\langle x, t, u \rangle$, *then:* $s = t$ *and* $u = v$. *In particular,* $\left\langle \begin{matrix} a \\ x \end{matrix} \mid \begin{matrix} b \\ y \end{matrix} \right\rangle$.
(d) *If* $\langle x, u, t \rangle$, *then:* $s = u$ *and* $t = v$. *In particular,* $\left\langle \begin{matrix} a \\ x \end{matrix} - \begin{matrix} b \\ y \end{matrix} \right\rangle$.
(e) *If* $\langle x, a, y \rangle$, *then* $a = s$.
(f) *If* $\langle x, b, y \rangle$, *then* $b = t$.

PROOF.

(1) Step 1. ($\Rightarrow$) Suppose that $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle. By 2.2.2(2) (median quadrangles), $s \in M(x, y, a) \cap [x, b]$. From the assumption that $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle it follows by 2.2.2(1b), (1d) and (1c) (median quadrangles) that $\begin{bmatrix} & b & \\ s & t & y \end{bmatrix}$, $\begin{bmatrix} & a & \\ t & v & b \end{bmatrix}$ and $\begin{bmatrix} & s & \\ v & u & a \end{bmatrix}$ are median triangles, i.e. $t \in M(s, y, b)$, $v \in M(t, b, a)$ and $u \in M(v, a, s)$.
Step 2. ($\Leftarrow$) Suppose $s \in M(x, y, a) \cap [x, b]$, $t \in M(s, y, b)$, $v \in M(t, b, a)$ and $u \in M(v, a, s)$. It is to be proved that the four-term sequences (x, s, t, y) , (y, t, v, b) , (b, v, u, a) , (a, u, s, x) and the five-term sequences (x, s, u, v, b) , (y, t, v, u, a) , (y, t, s, u, a) , (x, s, t, v, b) are aligned.

The assumption $s \in M(x, y, a)$ says:

$$\langle x, s, y\rangle , \tag{2.3.7}$$

$$\langle y, s, a\rangle , \tag{2.3.8}$$

$$\langle x, s, a\rangle . \tag{2.3.9}$$

The assumption $t \in M(s, y, b)$ says:

$$\langle s, t, y\rangle , \tag{2.3.10}$$

$$\langle y, t, b\rangle , \tag{2.3.11}$$

$$\langle s, t, b\rangle . \tag{2.3.12}$$

From (2.3.7) and (2.3.10) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle x, s, t, y\rangle .$$

The assumption $v \in M(t, b, a)$ says:

$$\langle t, v, b\rangle , \tag{2.3.13}$$

$$\langle b, v, a\rangle , \tag{2.3.14}$$

$$\langle t, v, a\rangle . \tag{2.3.15}$$

(2.3.11) and (2.3.13) imply by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle y, t, v, b\rangle .$$

The assumption $u \in M(v, a, s)$ says:

$$\langle v, u, a\rangle , \tag{2.3.16}$$

$$\langle s, u, a\rangle , \tag{2.3.17}$$

$$\langle s, u, v\rangle . \tag{2.3.18}$$

From (2.3.14) and (2.3.16) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle b, v, u, a\rangle .$$

(2.3.9) and (2.3.17) imply by 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle x, s, u, a\rangle$. By 1.4.7(1c),

$$\langle a, u, s, x\rangle .$$

From the assumption $\langle x, s, b\rangle$ and (2.3.12) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle x, s, t, b\rangle . \tag{2.3.19}$$

(2.3.19) and (2.3.13) imply by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle x,\, s,\, t,\, v,\, b\rangle \ .$$

By 1.4.7(1a) (aligned sequences),

$$\langle x,\, s,\, v,\, b\rangle \ . \tag{2.3.20}$$

From (2.3.20) and (2.3.18) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle x,\, s,\, u,\, v,\, b\rangle \ .$$

(2.3.10) implies:

$$\langle y,\, t,\, s\rangle \ . \tag{2.3.21}$$

From (2.3.8) and (2.3.21) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle y,\, t,\, s,\, a\rangle \ . \tag{2.3.22}$$

In particular,

$$\langle y,\, t,\, a\rangle \ . \tag{2.3.23}$$

(2.3.22) and (2.3.17) imply by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle y,\, t,\, s,\, u,\, a\rangle \ .$$

From (2.3.23) and (2.3.15) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle y,\, t,\, v,\, a\rangle \ . \tag{2.3.24}$$

(2.3.24) and (2.3.16) imply by 1.4.17(1b) (aligned sequences in a geometric interval space):

$$\langle y,\, t,\, v,\, u,\, a\rangle \ .$$

(2) From the assumption that Q is a median quadrangle it follows by 2.2.2(4) (median quadrangles):

$$\begin{bmatrix} y & & & b \\ & t & v & \\ & s & u & \\ x & & & a \end{bmatrix} \text{ is a median quadrangle.} \tag{2.3.25}$$

(a) The assumption that Q is a median quadrangle entails $\langle x,\, s,\, u,\, v,\, b\rangle$ and $\langle y,\, t,\, v,\, u,\, a\rangle$. In particular,

$$\langle s,\, u,\, v\rangle \ , \tag{2.3.26}$$

$$\langle t,\, v,\, u\rangle \ . \tag{2.3.27}$$

Substituting the assumption $s = t$ into (2.3.27),

$$\langle s,\, v,\, u\rangle \ . \tag{2.3.28}$$

(2.3.26) and (2.3.28) imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces):

$$u = v\,. \tag{2.3.29}$$

Substituting the assumption $s = t$ and (2.3.29) into the assumption that Q is a median quadrangle, $\begin{bmatrix} a & & & b \\ & u & u & \\ & s & s & \\ x & & & y \end{bmatrix}$ is a median quadrangle. In particular, $\left\langle \begin{array}{c|c} a & b \\ x & y \end{array} \right\rangle$.

(b) (2.3.25) and the assumption $s = u$ imply by (2a):

$$t = v\,. \tag{2.3.30}$$

Substituting the assumption $s = u$ and (2.3.30) into the assumption that Q is a median quadrangle, $Q = \begin{bmatrix} a & & & b \\ & s & t & \\ & s & t & \\ x & & & y \end{bmatrix}$ a median quadrangle. In particular, $\left\langle \begin{array}{ccc} a & & b \\ & - & \\ x & & y \end{array} \right\rangle$.

(c) By (2a) it suffices to prove $s = t$. The assumption that Q is a median quadrangle entails $\langle y,\, t,\, s,\, u,\, a\rangle$. In particular, $\langle t,\, s,\, u\rangle$. Therefore,

$$\langle u,\, s,\, t\rangle \ . \tag{2.3.31}$$

The assumption that Q is a median quadrangle also entails $\langle x,\, s,\, t,\, y\rangle$. In particular,

$$\langle x,\, s,\, t\rangle \ . \tag{2.3.32}$$

From (2.3.32) and the assumption $\langle x,\, t,\, u\rangle$ it follows by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $\langle s,\, t,\, u\rangle$. Thus,

$$\langle u,\, t,\, s\rangle \ . \tag{2.3.33}$$

(2.3.31) and (2.3.33) imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $s = t$.

(d) By (2b) it suffices to prove $s = u$. From (2.3.25) and the assumption $\langle x,\, u,\, t\rangle$ it follows by (2c) that $s = u$.

(e) The assumption that Q is a median quadrangle entails:

$$\langle x,\, s,\, u,\, a\rangle \ , \tag{2.3.34}$$

$$\langle a,\, u,\, s,\, t,\, y\rangle \ . \tag{2.3.35}$$

From the assumption $\langle x,\, a,\, y\rangle$, (2.3.34) and (2.3.35) it follows by two applications of 2.3.1(1) (geometric interval spaces) that $\langle x,\, s,\, u,\, a,\, u,\, s,\, t,\, y\rangle$. By 1.4.7(1a) (aligned sequences), $\langle s,\, u,\, a,\, u,\, s\rangle$. By 1.4.7(2) (aligned sequences), $a = s$.

(f) By 2.2.2(4) (median quadrangles)

$$\begin{bmatrix} b & & & a \\ & v & u & \\ & t & s & \\ y & & & x \end{bmatrix} \text{ is a median quadrangle.} \tag{2.3.36}$$

From the assumption $\langle x,\, b,\, y\rangle$ it follows:

$$\langle y,\, b,\, x\rangle\ . \tag{2.3.37}$$

(2.3.36) and (2.3.37) imply by (2e) that $b = t$.

□

PROPOSITION 2.3.3. *(quadrimodular matrices in a geometric interval space) Let X be a geometric interval space. For $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ a quadrimodular matrix in X , if $a,\, b \in [x,\, y]$, then $\langle x,\, a,\, b\rangle$.*

PROOF. The assumption that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular says that there are $s,\, t,\, u,\, v \in X$ such that

$$\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix} \text{ is a median quadrangle.} \tag{2.3.38}$$

In particular, $\langle x,\, s,\, t,\, y\rangle$. Thus,

$$\langle x,\, s,\, t\rangle\ . \tag{2.3.39}$$

From (2.3.38) and the assumption $a,\, b \in [x,\, y]$ it follows by 2.3.2(2e) and (2f) (median quadrangles in a geometric interval space):

$$a = s\,, \tag{2.3.40}$$

$$b = t\,. \tag{2.3.41}$$

Substituting (2.3.40) and (2.3.41) into (2.3.39), $\langle x,\, a,\, b\rangle$. □

PROPOSITION 2.3.4. *(interval-concatenable geometric interval spaces) Let X be an interval-concatenable geometric interval space. For $a,\, b,\, c \in X$, each maximal element of the poset $([a,\, b] \cap [a,\, c]\,,\, \langle a,\, \cdot,\, \cdot\rangle)$ is a median of $a,\, b,\, c$.*

PROOF. For $u \in X$ it is to be proved: If u is a maximal element of $([a,\, b] \cap [a,\, c]\,,\, \langle a,\, \cdot,\, \cdot\rangle)$, then u is a median of $a,\, b,\, c$. The assumption that u is an element of $[a,\, b] \cap [a,\, c]$ says:

$$\langle a,\, u,\, b\rangle \;, \tag{2.3.42}$$

$$\langle a,\, u,\, c\rangle \;. \tag{2.3.43}$$

Therefore, it suffices to prove $\langle b,\, u,\, c\rangle$. From the assumption that X is interval-concatenable it follows that it suffices to prove $[b,\, u] \cap [u,\, c] = \{u\}$. For $v \in X$ it is to be proved that $v \in [u,\, b] \cap [u,\, c]$ implies $v = u$. The assumption $v \in [u,\, b] \cap [u,\, c]$ says:

$$\langle u,\, v,\, b\rangle \;, \tag{2.3.44}$$

$$\langle u,\, v,\, c\rangle \;. \tag{2.3.45}$$

From (2.3.42) and (2.3.44) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle a,\, u,\, v,\, b\rangle$. In particular,

$$\langle a,\, u,\, v\rangle \;, \tag{2.3.46}$$

$$\langle a,\, v,\, b\rangle \;. \tag{2.3.47}$$

(2.3.43) and (2.3.45) imply by 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle a,\, u,\, v,\, c\rangle$. In particular,

$$\langle a,\, v,\, c\rangle \;. \tag{2.3.48}$$

(2.3.47) and (2.3.48) together say:

$$v \in [a,\, b] \cap [a,\, c] \;. \tag{2.3.49}$$

From the assumption that u is a maximal element of $([a,\, b] \cap [a,\, c]\,,\, \langle a,\, \cdot,\, \cdot\rangle)$, (2.3.49) and (2.3.46) it follows that $v = u$. ☐

2.4. Topological Interval Spaces

Part (1) of the following proposition has been cited from [**32**, §20, exercise 3(a)].

PROPOSITION 2.4.1. *(continuity of metrics) Let X be a metric space.*

(1) *The metric $d:\; X \times X \to \mathbb{R}$ is continuous with respect to the product topology.*
(2) *The function $f:\; X \times X \times X \to \mathbb{R}$ mapping $(x,\, y,\, z)$ to $d_{xy} + d_{yz} - d_{xz}$ is continuous with respect to the product topology.*

PROOF.

(1) By 1.4.23(3) (sum metric), it suffices to prove that f is continuous with respect to the sum metric. By 1.4.24 (nonexpansiveness of metrics), f is nonexpansive. By 1.4.25(1) (nonexpansive maps), f is continuous.
(2) By 1.3.4(2) and (3) (product of topological spaces),

$$\text{The map } p:\; X \times X \times X \to X \times X \text{ mapping } (x,\, y,\, z) \text{ to } (x,\, y) \text{ is continuous.} \tag{2.4.1}$$

By (1),

$$\text{The metric } d:\; X \times X \to \mathbb{R} \text{ is continuous.} \tag{2.4.2}$$

From (2.4.1) and (2.4.2) it follows by 1.3.2 (composite of continuous maps):

The map from $X \times X \times X$ to $\mathbb{R}$ mapping $(x,\, y,\, z)$ to d_{xy} is continuous. (2.4.3)

Analogously:

The map from $X \times X \times X$ to $\mathbb{R}$ mapping $(x,\, y,\, z)$ to d_{yz} is continuous. (2.4.4)

The map from $X \times X \times X$ to $\mathbb{R}$ mapping $(x,\, y,\, z)$ to d_{xz} is continuous. (2.4.5)

(2.4.3), (2.4.4) and (2.4.5) imply by two applications of 1.3.3 (sum and difference of continuous functions) that the map $f:\, X \times X \times X \to \mathbb{R}$ mapping $(x,\, y,\, z)$ to $d_{xy} + d_{yz} - d_{xz}$ is continuous.

□

A *topological interval space* is a triple $(X,\, \langle\cdot,\,\cdot,\,\cdot\rangle\,,\, O)$ such that:

- ∘ $(X,\, \langle\cdot,\,\cdot,\,\cdot\rangle)$ is an interval space
- ∘ $(X,\, O)$ is a topological space
- ∘ $\langle\cdot,\,\cdot,\,\cdot\rangle$ is a closed subset of the product space $X \times X \times X$.

Each real topological vector space is a topological interval space. Further examples of topological interval spaces are provided by the next two propositions. The concept of a topological interval space is analogous to the concept of a topological poset, which is implicit in the results on topological spaces equipped with a closed order in [**33**, chapter 1, §1 and §3]. It is related to the concept of a topological convex structure as defined in [**48**, chapter III, 1.1.1]. The concept of an interval space as defined in [**44**, section 2] also involves a topology, but there the interval space structure and the topology are connected by a different condition.

PROPOSITION 2.4.2. *(topological interval space property of a metric space) Each metric space with the geodesic interval relation and the topology determined by its metric is a topological interval space.*

PROOF. Let X be a metric space. By 1.4.4 (interval relation of a metric space), it remains to be proved that $\langle\cdot,\,\cdot,\,\cdot\rangle$ is closed in $X \times X \times X$.

$$\begin{aligned}\langle\cdot,\,\cdot,\,\cdot\rangle &= \{(x,\, y,\, z) \in X \times X \times X | d_{xz} = d_{xy} + d_{yz}\} \\ &= \{(x,\, y,\, z) \in X \times X \times X | d_{xy} + d_{yz} - d_{xz} = 0\}\end{aligned}$$

Therefore, with the map $f:\, X \times X \times X \to \mathbb{R}$ mapping $(x,\, y,\, z)$ to $d_{xy} + d_{yz} - d_{xz}$,

$$\langle\cdot,\,\cdot,\,\cdot\rangle = f^{-1}\left(\{0\}\right)\,. \tag{2.4.6}$$

By 2.4.1(2) (continuity of metrics),

$$f \text{ is continuous.} \tag{2.4.7}$$

From (2.4.6), (2.4.7) and closedness of $\{0\}$ it follows by 1.3.1(1) (continuous maps) that $\langle\cdot,\,\cdot,\,\cdot\rangle$ is closed. □

2.4.2 (topological interval space property of a metric space) may be used implicitly by applying results on topological interval spaces to metric spaces.

PROPOSITION 2.4.3. *(discrete topological interval spaces) Each interval space with the discrete topology is a topological interval space. In particular, each finite interval space with the discrete topology is a compact topological interval space.*

PROOF. Let X be an interval space with the discrete topology. By 1.3.5 (discrete topological spaces), $X \times X \times X$ is discrete. In particular, each subset of $X \times X \times X$ has an open complement, i.e. is closed. In particular, $\langle \cdot,\, \cdot,\, \cdot \rangle$ is closed. Consequently, X is a topological interval space. $\square$

PROPOSITION 2.4.4. *(topological interval spaces) Let X be a topological interval space. For $a,\, b \in X$, the unary sections $\langle a,\, b,\, \cdot \rangle$, $\langle a,\, \cdot,\, b \rangle = [a,\, b]$ and $\langle \cdot,\, a,\, b \rangle$ are non-empty and closed in X . For $a \in X$, the binary sections $\langle a,\, \cdot,\, \cdot \rangle$, $\langle \cdot,\, a,\, \cdot \rangle$ and $\langle \cdot,\, \cdot,\, a \rangle$ are non-empty and closed in $X \times X$.*

PROOF. The assumption that X is a topological interval space entails:

$$\langle \cdot,\, \cdot,\, \cdot \rangle \text{ is a closed subset of the product space } X \times X \times X\,. \tag{2.4.8}$$

Step 1. Proof that $\langle a,\, b,\, \cdot \rangle$ is non-empty and closed in X .

Step 1.1. From $\langle a,\, b,\, b \rangle$ it follows that $\langle a,\, b,\, \cdot \rangle$ is non-empty.

Step 1.2. Proof that $\langle a,\, b,\, \cdot \rangle$ is closed in X . By 1.3.1(2) (continuous maps),

$$\text{The map from } X \text{ to } X \text{ mapping } x \text{ to } a \text{ is continuous.} \tag{2.4.9}$$

$$\text{The map from } X \text{ to } X \text{ mapping } x \text{ to } b \text{ is continuous.} \tag{2.4.10}$$

By 1.3.1(3) (continuous maps),

$$\text{The map from } X \text{ to } X \text{ mapping } x \text{ to } x \text{ is continuous.} \tag{2.4.11}$$

(2.4.9), (2.4.10) and 2.4.11) imply by 1.3.4(3) (product of topological spaces):

$$\text{The map } i_{ab} : X \to X \times X \times X \text{ mapping } x \text{ to } (a,\, b,\, x) \text{ is continuous.} \tag{2.4.12}$$

From $\langle a,\, b,\, \cdot \rangle = i_{ab}^{-1}\left(\langle \cdot,\, \cdot,\, \cdot \rangle\right)$, (2.4.8) and (2.4.12) it follows by 1.3.1(1) (continuous maps) that $\langle a,\, b,\, \cdot \rangle$ is closed in X .

Step 2. The proofs that $[a,\, b]$ and $\langle \cdot,\, a,\, b \rangle$ are non-empty and closed in X are analogous to Step 1.

Step 3. Proof that $\langle a,\, \cdot,\, \cdot \rangle$ is non-empty and closed in $X \times X$.

Step 3.1. From $\langle a,\, a,\, a \rangle$ it follows that $\langle a,\, \cdot,\, \cdot \rangle$ is non-empty.

Step 3.2. Proof that $\langle a,\, \cdot,\, \cdot \rangle$ is closed in $X \times X$. By 1.3.1(2) (continuous maps),

$$\text{The map from } X \times X \text{ to } X \text{ mapping } (x,\, y) \text{ to } a \text{ is continuous.} \tag{2.4.13}$$

By 1.3.4(2) (product of topological spaces),

$$\text{The map from } X \times X \text{ to } X \text{ mapping } (x,\, y) \text{ to } x \text{ is continuous.} \tag{2.4.14}$$

$$\text{The map from } X \times X \text{ to } X \text{ mapping } (x,\, y) \text{ to } y \text{ is continuous.} \tag{2.4.15}$$

(2.4.13), (2.4.14) and (2.4.15) imply by 1.3.4(3) (product of topological spaces):

$$\text{The map } i_a : X \times X \to X \times X \times X \text{ mapping } (x,\, y) \text{ to } (a,\, x,\, y) \text{ is continuous.} \tag{2.4.16}$$

From $\langle a,\, \cdot,\, \cdot \rangle = i_a^{-1}\left(\langle \cdot,\, \cdot,\, \cdot \rangle\right)$, (2.4.8) and (2.4.16) it follows by 1.3.1(1) (continuous maps) that $\langle a,\, \cdot,\, \cdot \rangle$ is closed in $X \times X$.

Step 4. The proofs that $\langle\cdot,\ a,\ \cdot\rangle$ and $\langle\cdot,\ \cdot,\ a\rangle$ are non-empty and closed in $X\times X$ are analogous to step 3. $\square$

2.5. Geometric Topological Interval Spaces

PROPOSITION 2.5.1. *(geometric topological interval spaces) Let $(X,\ \langle\cdot,\ \cdot,\ \cdot\rangle\ ,\ O)$ be a geometric topological interval space. For $p\in X$, $(X,\ \langle p,\ \cdot,\ \cdot\rangle\ ,\ O)$ is a topological poset.*

PROOF. By 1.4.16 (Hedlíková's criterion for geometric interval spaces) $(X,\ \langle p,\ \cdot,\ \cdot\rangle)$ is a poset, and by 2.4.4 (topological interval spaces), $\langle p,\ \cdot,\ \cdot\rangle$ is closed with respect to the product topology. Consequently, $(X,\ \langle p,\ \cdot,\ \cdot\rangle\ ,\ O)$ is a topological poset. $\square$

2.5.1 (geometric topological interval spaces) may be used implicitly by applying results on topological posets to geometric topological interval spaces.

2.6. Metric Spaces

PROPOSITION 2.6.1. *(median quadrangles in a metric space) Let X be a metric space. For $Q=\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ a median quadrangle in X :*

(1) $d_{sb}=d_{xb}-d_{x,ya}$.
(2) $d_{st}=d_{x,yb}-d_{x,ya}$.
(3) $d_{su}=d_{x,ab}-d_{x,ya}$.
(4)
  (a) $d_{uv}=d_{st}$.
  (b) $d_{tv}=d_{su}$.
  (c) $d_{sv}=d_{st}+d_{su}$.
  (d) $d_{tu}=d_{st}+d_{su}$.

PROOF.

(1) From the assumption that Q is a median quadrangle it follows by 2.2.2(1a) (median quadrangles) that $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ is a median triangle. By 1.4.20 (median triangles in a metric space), $d_{xs}=\frac{1}{2}\left(d_{xy}+d_{xa}-d_{ya}\right)$, i.e.

$$d_{xs}=d_{x,ya}\,. \tag{2.6.1}$$

The assumption that Q is a median quadrangle implies by 2.2.2(2) (median quadrangles) that $s\in[x,\ b]$, i.e.

$$d_{xb}=d_{xs}+d_{sb}\,. \tag{2.6.2}$$

From (2.6.2) and (2.6.1) it follows:

$$\begin{aligned} d_{sb}&=d_{xb}-d_{xs}\\ &=d_{xb}-d_{x,ya}\,. \end{aligned}$$

(2) The assumption that Q is a median quadrangle entails

$$\langle x,\, s,\, t,\, y\rangle \tag{2.6.3}$$

and implies by 2.2.2(1a) and (1b) (median quadrangles):

$$\begin{bmatrix} & a & \\ x & s & y \end{bmatrix} \text{ is a median triangle,} \tag{2.6.4}$$

$$\begin{bmatrix} & b & \\ x & t & y \end{bmatrix} \text{ is a median triangle.} \tag{2.6.5}$$

(2.6.3) implies by 1.4.6 (aligned sequences in a metric space) that $d_{xy} = d_{xs} + d_{st} + d_{ty}$. Therefore,

$$d_{st} = d_{xy} - d_{xs} - d_{ty}\,. \tag{2.6.6}$$

From (2.6.5) it follows by 1.4.9(2a) (median triangles):

$$\begin{bmatrix} & b & \\ y & t & x \end{bmatrix} \text{ is a median triangle.} \tag{2.6.7}$$

(2.6.4) and (2.6.7) imply by 1.4.20 (median triangles in a metric space): $d_{xs} = \frac{1}{2}\left(d_{xy} + d_{xa} - d_{ya}\right)$ and $d_{ty} = \frac{1}{2}\left(d_{yx} + d_{yb} - d_{xb}\right)$, i.e.

$$d_{xs} = d_{x,ya}\,, \tag{2.6.8}$$

$$d_{ty} = d_{y,xb}\,. \tag{2.6.9}$$

Substituting (2.6.8) and (2.6.9) into (2.6.6),

$$\begin{aligned} d_{st} &= d_{xy} - d_{x,ya} - d_{y,xb} \\ &= (d_{xy} - d_{y,xb}) - d_{x,ya}\,. \end{aligned} \tag{2.6.10}$$

By 1.8.2(5) (point-pair modular distance), $d_{xy} = d_{x,yb} + d_{y,xb}$. Therefore,

$$d_{xy} - d_{y,xb} = d_{x,yb}\,. \tag{2.6.11}$$

Substituting (2.6.11) into (2.6.10), $d_{st} = d_{x,yb} - d_{x,ya}$.

(3) The assumption that Q is a median quadrangle implies by 2.2.2(4) (median quadrangles) that $\begin{bmatrix} y & & & b \\ & t & v & \\ & s & u & \\ x & & & a \end{bmatrix}$ is a median quadrangle. By (2), $d_{su} = d_{x,ab} - d_{x,ay}$. By 1.8.2(2) (point-pair modular distance), $d_{su} = d_{x,ab} - d_{x,ya}$.

(4) The assumption that Q is a median quadrangle entails:

$$\langle x,\, s,\, t,\, v,\, b\rangle\ ,$$
$$\langle x,\, s,\, u,\, v,\, b\rangle\ ,$$
$$\langle y,\, t,\, s,\, u,\, a\rangle\ ,$$
$$\langle y,\, t,\, v,\, u,\, a\rangle\ .$$

In particular,

$$\begin{aligned} &\langle s,\, t,\, v\rangle\ ,\\ &\langle s,\, u,\, v\rangle\ ,\\ &\langle t,\, s,\, u\rangle\ ,\\ &\langle t,\, v,\, u\rangle\ , \end{aligned}$$

i.e.

$$\begin{aligned} d_{sv} &= d_{st} + d_{tv}\ ,\\ d_{sv} &= d_{su} + d_{uv}\ ,\\ d_{tu} &= d_{st} + d_{su}\ ,\\ d_{tu} &= d_{tv} + d_{uv}\ . \end{aligned}$$

Solving this system of linear equations in terms of d_{st} and d_{su} yields

$$\begin{aligned} d_{uv} &= d_{st}\ ,\\ d_{tv} &= d_{su}\ ,\\ d_{sv} &= d_{st} + d_{su}\ ,\\ d_{tu} &= d_{st} + d_{su}\ . \end{aligned}$$

□

PROPOSITION 2.6.2. *(quadrimodular matrix representation) Let A be a metric space with $A = \{x,\, y,\, a,\, b\}$. If $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$, then there is an isometric map i from A into $(\mathbb{R}^2,\, \|\cdot - \cdot\|_1)$ such that $\begin{bmatrix} i\,(a) & i\,(b)\\ i\,(x) & i\,(y)\end{bmatrix}$ is quadrimodular.*

PROOF. $d_{x,ya},\, d_{y,xb},\, d_{a,xb},\, d_{b,ya}\ \in\ \mathbb{R}_{\geq 0}$, and from the assumption $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$ it follows that $d_{x,yb} - d_{x,ya},\, d_{x,ab} - d_{x,ay}\ \in\ \mathbb{R}_{\geq 0}$. By 2.2.1 (median quadrangles in the plane), there is

$$\begin{bmatrix} a' & & & b'\\ & u' & v' & \\ & s' & t' & \\ x' & & & y' \end{bmatrix}\ ,\ \text{a median quadrangle in}\ \left(\mathbb{R}^2,\, \|\cdot - \cdot\|_1\right)\ , \tag{2.6.12}$$

such that

$$\|s' - t'\|_1 = d_{x,yb} - d_{x,ya} \,, \tag{2.6.13}$$
$$\|u' - v'\|_1 = d_{x,yb} - d_{x,ya} \,, \tag{2.6.14}$$
$$\|s' - u'\|_1 = d_{x,ab} - d_{x,ay} \,, \tag{2.6.15}$$
$$\|t' - v'\|_1 = d_{x,ab} - d_{x,ay} \,, \tag{2.6.16}$$
$$\|x' - s'\|_1 = d_{x,ya} \,, \tag{2.6.17}$$
$$\|y' - t'\|_1 = d_{y,xb} \,, \tag{2.6.18}$$
$$\|a' - u'\|_1 = d_{a,xb} \,, \tag{2.6.19}$$
$$\|b' - v'\|_1 = d_{b,ya} \,. \tag{2.6.20}$$

(2.6.12) entails that

$$\begin{bmatrix} a' & b' \\ x' & y' \end{bmatrix} \text{ is quadrimodular} \tag{2.6.21}$$

and $\langle x', s', t', y' \rangle$, $\langle x', s', u', a' \rangle$, $\langle x', s', t', v', b' \rangle$, $\langle y', t', v', u', a' \rangle$, $\langle y', t', v', b' \rangle$, $\langle a', u', v', b' \rangle$. By 1.4.6 (aligned sequences in a metric space),

$$\|x' - y'\|_1 = \|x' - s'\|_1 + \|s' - t'\|_1 + \|t' - y'\|_1 \,, \tag{2.6.22}$$
$$\|x' - a'\|_1 = \|x' - s'\|_1 + \|s' - u'\|_1 + \|u' - a'\|_1 \,, \tag{2.6.23}$$
$$\|x' - b'\|_1 = \|x' - s'\|_1 + \|s' - t'\|_1 + \|t' - v'\|_1 + \|v' - b'\|_1 \,, \tag{2.6.24}$$
$$\|y' - a'\|_1 = \|y' - t'\|_1 + \|t' - v'\|_1 + \|v' - u'\|_1 + \|u' - a'\|_1 \,, \tag{2.6.25}$$
$$\|y' - b'\|_1 = \|y' - t'\|_1 + \|t' - v'\|_1 + \|v' - b'\|_1 \,, \tag{2.6.26}$$
$$\|a' - b'\|_1 = \|a' - u'\|_1 + \|u' - v'\|_1 + \|v' - b'\|_1 \,. \tag{2.6.27}$$

Substituting (2.6.17), (2.6.13) and (2.6.18) into (2.6.22), it follows by 1.8.2(5) (point-pair modular distance):

$$\begin{aligned} \|x' - y'\|_1 &= \|x' - s'\|_1 + \|s' - t'\|_1 + \|t' - y'\|_1 \\ &= \|x' - s'\|_1 + \|s' - t'\|_1 + \|y' - t'\|_1 \\ &= d_{x,ya} + (d_{x,yb} - d_{x,ya}) + d_{y,xb} \\ &= d_{x,yb} + d_{y,xb} \\ &= d_{xy} \,. \end{aligned} \tag{2.6.28}$$

Substituting (2.6.17), (2.6.15) and (2.6.19) into (2.6.23), it follows by 1.8.2 (5) and (2) (point-pair modular distance):

$$\begin{aligned}
\|x'-a'\|_1 &= \|x'-s'\|_1 + \|s'-u'\|_1 + \|u'-a'\|_1 \\
&= \|x'-s'\|_1 + \|s'-u'\|_1 + \|a'-u'\|_1 \\
&= d_{x,ya} + (d_{x,ab} - d_{x,ay}) + d_{a,xb} \\
&= d_{x,ya} + (d_{x,ab} - d_{x,ya}) + d_{a,xb} \\
&= d_{x,ab} + d_{a,xb} \\
&= d_{xa}\,. \qquad (2.6.29)
\end{aligned}$$

Substituting (2.6.17), (2.6.13), (2.6.16) and (2.6.20) into (2.6.24), it follows by 1.8.2 (2) (point-pair modular distance):

$$\begin{aligned}
\|x'-b'\|_1 &= \|x'-s'\|_1 + \|s'-t'\|_1 + \|t'-v'\|_1 + \|v'-b'\|_1 \\
&= \|x'-s'\|_1 + \|s'-t'\|_1 + \|t'-v'\|_1 + \|b'-v'\|_1 \\
&= d_{x,ya} + (d_{x,yb} - d_{x,ya}) + (d_{x,ab} - d_{x,ay}) + d_{b,ya} \\
&= d_{x,ya} + (d_{x,yb} - d_{x,ya}) + (d_{x,ab} - d_{x,ya}) + d_{b,ya} \\
&= d_{x,yb} + (d_{x,ab} - d_{x,ya}) + d_{b,ya} \\
&= \frac{1}{2}(d_{xy} + d_{xb} - d_{yb}) + \frac{1}{2}(d_{xa} + d_{xb} - d_{ab}) \\
&\quad - \frac{1}{2}(d_{xy} + d_{xa} - d_{ya}) + \frac{1}{2}(d_{by} + d_{ba} - d_{ya}) \\
&= d_{xb}\,. \qquad (2.6.30)
\end{aligned}$$

Substituting (2.6.18), (2.6.16), (2.6.14) and (2.6.19) into (2.6.25), it follows by 1.8.2(2) and (5) (point-pair modular distance):

$$\begin{aligned}
\|y'-a'\|_1 &= \|y'-t'\|_1 + \|t'-v'\|_1 + \|v'-u'\|_1 + \|u'-a'\|_1 \\
&= \|y'-t'\|_1 + \|t'-v'\|_1 + \|u'-v'\|_1 + \|a'-u'\|_1 \\
&= d_{y,xb} + (d_{x,ab} - d_{x,ay}) + (d_{x,yb} - d_{x,ya}) + d_{a,xb} \\
&= d_{y,xb} + (d_{x,ab} - d_{x,ya}) + (d_{x,yb} - d_{x,ya}) + d_{a,xb} \\
&= (d_{y,xb} + d_{x,yb}) + (d_{x,ab} + d_{a,xb}) - 2d_{x,ya} \\
&= d_{yx} + d_{xa} - 2\cdot\frac{1}{2}(d_{xy} + d_{xa} - d_{ya}) \\
&= d_{yx} + d_{xa} - (d_{xy} + d_{xa} - d_{ya}) \\
&= d_{ya}\,. \qquad (2.6.31)
\end{aligned}$$

Substituting (2.6.18), (2.6.16) and (2.6.20) into (2.6.26),

$$\begin{aligned}\|y'-b'\|_1 &= \|y'-t'\|_1+\|t'-v'\|_1+\|v'-b'\|_1\\ &= \|y'-t'\|_1+\|t'-v'\|_1+\|b'-v'\|_1\\ &= d_{y,xb}+(d_{x,ab}-d_{x,ay})+d_{b,ya}\\ &= \frac{1}{2}(d_{yx}+d_{yb}-d_{xb})+\frac{1}{2}(d_{xa}+d_{xb}-d_{ab})\\ &\quad -\frac{1}{2}(d_{xa}+d_{xy}-d_{ay})+\frac{1}{2}(d_{by}+d_{ba}-d_{ya})\\ &= d_{yb}\,. \end{aligned} \tag{2.6.32}$$

Substituting (2.6.19), (2.6.14) and (2.6.20) into (2.6.27), it follows by 2.2.1(1) (median quadrangles in the plane) and 1.8.2(2) (point-pair modular distance):

$$\begin{aligned}\|a'-b'\|_1 &= \|a'-u'\|_1+\|u'-v'\|_1+\|v'-b'\|_1\\ &= \|a'-u'\|_1+\|u'-v'\|_1+\|b'-v'\|_1\\ &= d_{a,xb}+(d_{x,yb}-d_{x,ya})+d_{b,ya}\\ &= \frac{1}{2}(d_{ax}+d_{ab}-d_{xb})+\frac{1}{2}(d_{xy}+d_{xb}-d_{yb})\\ &\quad -\frac{1}{2}(d_{xy}+d_{xa}-d_{ya})+\frac{1}{2}(d_{by}+d_{ba}-d_{ya})\\ &= d_{ab}\,. \end{aligned} \tag{2.6.33}$$

From (2.6.28), (2.6.29), (2.6.30), (2.6.31), (2.6.32) and (2.6.33) it follows that an isometric map i from A into $(\mathbb{R}^2,\, \|\cdot-\cdot\|_1)$ is well-defined by setting $i(x)=x'$, $i(y)=y'$, $i(a)=a'$ and $i(b)=b'$. (2.6.21) says that $\begin{bmatrix} i(a) & i(b)\\ i(x) & i(y)\end{bmatrix}$ is quadrimodular. □

PROPOSITION 2.6.3. *(quadrimodular matrices in a metric space) Let X be a metric space. For $x,\ y,\ a,\ b\in X$:*

(1) *If* $\left\langle \begin{matrix} a & & b\\ & \cdot & \\ x & & y\end{matrix}\right\rangle$ *, then* $d_{x,ya}=d_{x,yb}$ *and* $d_{x,ya}=d_{x,ab}$.

(2) *If* $\left\langle \begin{matrix} a & & b\\ & : & \\ x & & y\end{matrix}\right\rangle$ *, then* $d_{x,ya}=d_{x,yb}$ *and* $d_{x,ya}<d_{x,ab}$.

(3) *If* $\left\langle \begin{matrix} a & & b\\ & \cdot\cdot & \\ x & & y\end{matrix}\right\rangle$ *, then* $d_{x,ya}<d_{x,yb}$ *and* $d_{x,ya}=d_{x,ab}$.

(4) *If* $\left\langle \begin{matrix} a & & b\\ & :: & \\ x & & y\end{matrix}\right\rangle$ *, then* $d_{x,ya}<d_{x,yb}$ *and* $d_{x,ya}<d_{x,ab}$.

(5) *If* $\begin{bmatrix} a & b\\ x & y\end{bmatrix}$ *is quadrimodular, then* $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\}=d_{x,ya}$.

PROOF.

(1) The assumption $\left\langle \begin{matrix} a & \cdot & b \\ x & & y \end{matrix} \right\rangle$ says that there is an $s \in X$ such that $\begin{bmatrix} a & & & b \\ & s & s & \\ & s & s & \\ x & & & y \end{bmatrix}$ is a median quadrangle. By 2.6.1(2) and (3) (median quadrangles in a metric space), $d_{ss} = d_{x,yb} - d_{x,ya}$ and $d_{ss} = d_{x,ab} - d_{x,ya}$. Therefore, $0 = d_{x,yb} - d_{x,ya}$ and $0 = d_{x,ab} - d_{x,ya}$. Consequently, $d_{x,ya} = d_{x,yb}$ and $d_{x,ya} = d_{x,ab}$.

(2) The assumption $\left\langle \begin{matrix} a & : & b \\ x & & y \end{matrix} \right\rangle$ says that

$$\text{not} \left\langle \begin{matrix} a & \cdot & b \\ x & & y \end{matrix} \right\rangle \tag{2.6.34}$$

and there are $s,\, u \in X$ such that

$$\begin{bmatrix} a & & & b \\ & u & u & \\ & s & s & \\ x & & & y \end{bmatrix} \text{ is a median quadrangle,} \tag{2.6.35}$$

From (2.6.35) and (2.6.34) it follows that $s \neq u$. Therefore,

$$d_{su} > 0\,. \tag{2.6.36}$$

(2.6.35) implies by 2.6.1(2) and (3) (median quadrangles in a metric space):

$$d_{ss} = d_{x,yb} - d_{x,ya}\,, \tag{2.6.37}$$

$$d_{su} = d_{x,ab} - d_{x,ya}\,. \tag{2.6.38}$$

From (2.6.37) it follows that $0 = d_{x,yb} - d_{x,ya}$. Consequently, $d_{x,ya} = d_{x,yb}$. Substituting (2.6.38) into (2.6.36), $d_{x,ab} - d_{x,ya} > 0$. Consequently, $d_{x,ya} < d_{x,ab}$.

(3) The assumption $\left\langle \begin{matrix} a & \cdot\cdot & b \\ x & & y \end{matrix} \right\rangle$ implies by 2.2.4(4) (symmetries of quadrimodularity properties) that $\left\langle \begin{matrix} y & : & b \\ x & & a \end{matrix} \right\rangle$. By (2), $d_{x,ay} < d_{x,yb}$ and $d_{x,ay} = d_{x,ab}$. By 1.8.2(2) (point-pair modular distance), $d_{x,ya} < d_{x,yb}$ and $d_{x,ya} = d_{x,ab}$.

(4) The assumption $\left\langle \begin{matrix} a & & b \\ & :: & \\ x & & y \end{matrix} \right\rangle$ says that there are $s,\ t,\ u,\ v \in X$ such that

$$\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix} \text{ is a median quadrangle,} \tag{2.6.39}$$

$$\text{not } \left\langle \begin{matrix} a & \mid & b \\ x & & y \end{matrix} \right\rangle , \tag{2.6.40}$$

$$\text{not } \left\langle \begin{matrix} a & & b \\ & - & \\ x & & y \end{matrix} \right\rangle . \tag{2.6.41}$$

From (2.6.39) and (2.6.40) it follows by 2.3.2(2a) (median quadrangles in a geometric interval space) that $s \neq t$. Therefore,

$$d_{st} > 0\,. \tag{2.6.42}$$

(2.6.39) and (2.6.41) imply by 2.3.2(2b) (median quadrangles in a geometric interval space) that $s \neq u$. Thus,

$$d_{su} > 0\,. \tag{2.6.43}$$

From (2.6.39) it follows by 2.6.1(2), (3) (median quadrangles in a metric space):

$$d_{st} = d_{x,yb} - d_{x,ya}\,, \tag{2.6.44}$$

$$d_{su} = d_{x,ab} - d_{x,ya}\,. \tag{2.6.45}$$

Substituting (2.6.44) into (2.6.42), $d_{x,yb} - d_{x,ya} > 0$. Consequently, $d_{x,ya} < d_{x,yb}$. Substituting (2.6.45) into (2.6.43), $d_{x,ab} - d_{x,ya} > 0$. Consequently, $d_{x,ya} < d_{x,ab}$.

(5) From the assumption that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular it follows by 2.2.5(1) (quadrimodularity properties) that $\left\langle \begin{matrix} a & & b \\ & :: & \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & & b \\ & : & \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & & b \\ & \cdot\cdot & \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & & b \\ & \cdot & \\ x & & y \end{matrix} \right\rangle$. By (4), (2), (3) and (1), $\min\{d_{x,ya},\ d_{x,yb},\ d_{x,ab}\} = d_{x,ya}$.

□

PROPOSITION 2.6.4. *(isometric invariance of point-pair modular distance) Let* $i : X \to Y$ *be an isometric map of metric spaces. For* $x,\ a,\ b \in X$, $d_{i(x),i(a)i(b)} = d_{x,ab}$.

PROOF. By 1.8.1 (existence of modular extension), there is an isometric map j from Y into a modular metric space Z , and $j \circ i$ is an isometric map from X into Z . Therefore,

$$\begin{aligned} d_{i(x),i(a)i(b)} &= d_{j(i(x))[j(i(a)),\,j(i(b))]} \\ &= d_{x,ab}\,. \end{aligned}$$

□

CHAPTER 3

Characterizations by Fundamental Combinatorial Concepts

In this chapter, the following known results on triangle-convex spaces are used:

- ◦ 1.5.1 (triangle-convex interval spaces)
- ◦ 3.2.1 (triangle-convex one-way interval spaces)

The following main new results are proved:

- ◦ 3.2.4 (antiexchange criterion for triangle-convex geometric interval spaces)
- ◦ 3.3.2 (perspectivity relation)

3.1. Closure Spaces

Let X be a set. A *closure system* on X is a set C of subsets of X such that $X \in C$ and for each non-empty $D \subseteq C$, $\bigcap D \in C$.

A *closure space* is a pair consisting of a set X and a closure system C on X. A set $A \subseteq X$ is called *closed* iff $A \in C$. When (X, O) is a topological space, then the pair consisting of X and the set of closed sets in (X, O) is a closure space. When $(X, \langle\cdot, \cdot, \cdot\rangle)$ is an interval space, then the pair consisting of X and the set of convex sets is a closure space. The concept of a closure space as defined here is slighly more general than in [**48**, chapter I, 1.2], where it is required that $\emptyset \in C$ and a closure system is called a protopology.

A closure space (X, C) is also simply denoted by X when it is clear from the context whether the closure space or only the set is meant.

The concepts of a closure in a topological space and of a convex closure in an interval space have a natural generalization to a closure space. Let (X, C) be a closure space. For $A \subseteq X$, the *closure* of A is the set

$$\mathrm{cl}\,(A) := \bigcap \{B \subseteq X | B \supseteq A \text{ and } B \in C\}$$

It is the smallest closed superset of A. When X is an interval space and C is the system of convex sets in X, then for $A \subseteq X$, the closure of A is the convex closure of A.

Let X be a closure space. For $A \subseteq X$, the *entailment relation* of C relative to A or A-entailment relation is the binary relation $\vdash_A$ on X defined by

$$x \vdash_A y :\Leftrightarrow y \in \mathrm{cl}\,(A \cup \{x\}) \ .$$

PROPOSITION 3.1.1. *(relative entailment relation) Let X be a closure space. For $A \subseteq X$,*

(1) *The relation $\vdash_A$ is reflexive on X and transitive.*
(2) *The restriction $\vdash_A | (X \setminus A)$ is a partial order on $X \setminus A$ iff it is antisymmetric, i.e. for all $x,\, y \in X \setminus A$,* if $y \in$ closure of $A \cup \{x\}$ and $x \in$ closure of $A \cup \{y\}$, then $x = y$.
(3) *The restriction $\vdash_A | (X \setminus A)$ is an equivalence relation on $X \setminus A$ iff it is symmetric, i.e. for all $x,\, y \in X \setminus A$,* if $y \in$ closure of $A \cup \{x\}$, then $x \in$ closure of $A \cup \{y\}$.

PROOF.

(1) Step 1. Reflexivity. For $x \in X$, $x \vdash_A x$, i.e. $x \in \mathrm{cl}\,(A \cup \{x\})$, follows from $x \in A \cup \{x\}$ and $A \cup \{x\} \subseteq \mathrm{cl}\,(A \cup \{x\})$.
Step 2. Transitivity. For $x,\, y,\, z \in X$, it is to be proved that $y \in \mathrm{cl}\,(A \cup \{x\})$ and $z \in \mathrm{cl}\,(A \cup \{y\})$ imply $z \in \mathrm{cl}\,(A \cup \{x\})$, i.e. from the assumption $y \in \mathrm{cl}\,(A \cup \{x\})$ it is to be proved that $\mathrm{cl}\,(A \cup \{x\}) \supseteq \mathrm{cl}\,(A \cup \{y\})$. $\mathrm{cl}\,(A \cup \{y\})$ is the smallest closed superset of $A \cup \{y\}$. Therefore, it suffices to prove that the closed set $\mathrm{cl}\,(A \cup \{x\})$ is a superset of $A \cup \{y\}$. From the assumption $y \in \mathrm{cl}\,(A \cup \{x\})$ it follows that it suffices to prove $A \subseteq \mathrm{cl}\,(A \cup \{x\})$. This claim follows from $A \subseteq A \cup \{x\}$ and $A \cup \{x\} \subseteq \mathrm{cl}\,(A \cup \{x\})$.
(2) follows from (1).
(3) follows from (1).

□

If $|A| \geq 2$, then the binary relation $\vdash_A$ on the whole of X is not antisymmetric.

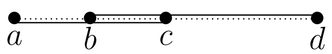
Let X be a closure space. X is called an *antiexchange space* iff for each closed $A \subseteq X$, one and therefore all of the following conditions hold, which are equivalent by 3.1.1 (2) (relative entailment relation):

- The restriction $\vdash_A | (X \setminus A)$ antisymmetric.
- The restriction $\vdash_A | (X \setminus A)$ is a partial order on $X \setminus A$.

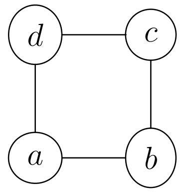
Examples of antiexchange spaces are provided by and presented after the first proposition in the next section. In [**48**, chapter I, 2.24], an exchange space that is an algebraic closure space with $\emptyset$ closed is called an *anti-matroid* or *convex geometry*.

3.2. One-Way Geometric Interval Spaces

Let X be an interval space. X is called *one-way* iff for all $a,\, b,\, c,\, d \in X$, $\langle a,\, b,\, c\rangle$, $b \neq c$ and $\langle b,\, c,\, d\rangle$ imply $\langle a,\, b,\, d\rangle$. This condition is the interval relation version of the strict interval relation condition [**34**, §1, VIII. Grundsatz]. When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X is one-way.



Each tree is one-way. The following graph is not one-way: $\langle a,\, b,\, c\rangle$, $b \neq c$, $\langle b,\, c,\, d\rangle$, but not $\langle a,\, b,\, d\rangle$.



The following proposition has been cited from [**10**, chapter II, proposition 10].

PROPOSITION 3.2.1. *(triangle-convex one-way interval spaces) Let X be a triangle-convex one-way interval space. Then the pair consisting of X and the set of convex sets is an antiexchange space.*

PROOF. [**10**, chapter II, proposition 10] □

Let X be a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$. The pair consisting of X and its vector interval relation is a triangle-convex one-way interval space. By 3.2.1 (triangle-convex one-way interval spaces), the pair consisting of X and the set of convex sets is an antiexchange space.

PROPOSITION 3.2.2. *(one-way criterion for a geometric interval space) Let X be a geometric interval space. The following conditions are equivalent:*

(1) *X is one-way.*
(2) *For all $a,\, b,\, c,\, d \in X$, $\langle a,\, b,\, c\rangle$, $b \neq c$ and $\langle b,\, c,\, d\rangle$ imply $\langle a,\, b,\, c,\, d\rangle$*
(3) *For all $a,\, b,\, c,\, d \in X$, $\langle a,\, b,\, c\rangle$, $b \neq c$ and $\langle b,\, c,\, d\rangle$ imply $\langle a,\, b,\, d\rangle$ or $\langle a,\, c,\, d\rangle$.*

PROOF. Step 1. (1) $\Rightarrow$ (2). From the assumption that X is one-way it is to be proved that for $a,\, b,\, c,\, d \in X$, $\langle a,\, b,\, c\rangle$, $b \neq c$ and $\langle b,\, c,\, d\rangle$ implies $\langle a,\, b,\, c,\, d\rangle$. From the assumptions that X is one-way, $\langle a,\, b,\, c\rangle$, $b \neq c$ and $\langle b,\, c,\, d\rangle$ it follows:

$$\langle a,\, b,\, d\rangle \ . \tag{3.2.1}$$

(3.2.1) and the assumption $\langle b,\, c,\, d\rangle$ imply by 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle a,\, b,\, c,\, d\rangle$.

Step 2. For (2) $\Rightarrow$ (3) nothing is to be proved.

Step 3. (3) $\Rightarrow$ (1). From the assumption that $\langle a,\, b,\, c\rangle$, $b \neq c$ and $\langle b,\, c,\, d\rangle$ imply $\langle a,\, b,\, d\rangle$ or $\langle a,\, c,\, d\rangle$, it is to be proved that $\langle a,\, b,\, d\rangle$.

Case 1. $\langle a,\, b,\, d\rangle$. Nothing remains to be proved.

Case 2. $\langle a,\, c,\, d\rangle$. From the assumptions $\langle a,\, b,\, c\rangle$ and $\langle a,\, c,\, d\rangle$ it follows that $\langle a,\, b,\, d\rangle$. □

PROPOSITION 3.2.3. *(antiexchange interval-convex geometric interval spaces) Let $(X,\, \langle\cdot,\, \cdot,\, \cdot\rangle)$ be an interval-convex geometric interval space such that the pair consisting of X and the set of convex sets is an antiexchange space. Then X is one-way.*

PROOF. By 3.2.2 (one-way criterion for a geometric interval space) it suffices to prove for $a,\, b,\, c,\, d \in X$ that $\langle a,\, b,\, c\rangle$, $b \neq c$ and $\langle b,\, c,\, d\rangle$ imply $\langle a,\, b,\, d\rangle$ or $\langle a,\, c,\, d\rangle$. Seeking a contradiction, suppose that not $\langle a,\, b,\, d\rangle$ and not $\langle a,\, c,\, d\rangle$, i.e.

$$b,\, c \notin [a,\, d] \ . \tag{3.2.2}$$

The assumption that X is interval-convex entails:

$$[a,\, d] \text{ is convex.} \tag{3.2.3}$$

The assumption $\langle a,\, b,\, c\rangle$ implies:

$$b \in [\{a,\, c\}] \tag{3.2.4}$$

From $a \in [a,\, d]$ and (3.2.4) it follows that $\{a,\, c\} \subseteq [a,\, d] \cup \{c\}$. Therefore,

$$[\{a,\, c\}] \subseteq [[a,\, d] \cup \{c\}] \ . \tag{3.2.5}$$

(3.2.4) and (3.2.5) imply:

$$b \in [[a,\, d] \cup \{c\}] \ , \tag{3.2.6}$$

Analogously,

$$c \in [[a,\, d] \cup \{b\}] \ . \tag{3.2.7}$$

From (3.2.3), (3.2.2), (3.2.6), (3.2.7) and the assumption that the pair consisting of X and the set of convex sets is an antiexchange space it follows that $b = c$, contradicting the assumption $b \neq c$. ☐

The following theorem characterizes a geometric property of an interval relation in terms of a fundamental property of a family of derived binary relations:

THEOREM 3.2.4. *(antiexchange criterion for triangle-convex geometric interval spaces) Let* $(X,\, \langle\cdot,\, \cdot,\, \cdot\rangle)$ *be a triangle-convex geometric interval space. The pair consisting of* X *and the set of convex sets is an antiexchange space iff* X *is one-way.*

PROOF. Step 1. ($\Rightarrow$) From the assumption that X is triangle-convex it follows by 1.5.1 (triangle-convex interval spaces):

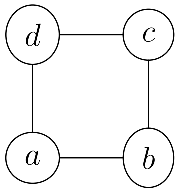
$$X \text{ is interval-convex.} \tag{3.2.8}$$

(3.2.8) and the assumptions that X is geometric and the pair consisting of X and the set of convex sets is an antiexchange space imply by 3.2.3 (antiexchange interval-convex geometric interval spaces) that X is one-way.

Step 2. ($\Leftarrow$) From the assumption that X is triangle-convex and one-way it follows by 3.2.1 (triangle-convex one-way interval spaces) that the pair consisting of X and the set of convex sets is an antiexchange space. ☐

3.3. Perspectivity Relations

Let X be a geometric interval space. X is called *interval-linear* iff for all $a,\, b \in X$, $[a,\, b]$ is a chain in the poset $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$. This condition is contained in [**43**, (5.2)]. When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then it is interval-linear. For $n \in \mathbb{Z}_{\geq 1}$, the metric space $(\mathbb{R}^n,\, \|\cdot - \cdot\|_2)$ is interval-linear. Each lattice interval space associated with a chain, for example $(\mathbb{R},\, \leq)$, is an interval-linear geometric interval space. Each subspace of an interval-linear interval space is interval-linear. The following graph is not interval-linear: $b,\, d \in [a,\, c]$, but not $\langle a,\, b,\, d\rangle$ and not $\langle a,\, d,\, b\rangle$.



For $n \in \mathbb{Z}_{\geq 2}$, the metric spaces $(\mathbb{R}^n,\, \|\cdot - \cdot\|_1)$ and $(\mathbb{R}^n,\, \|\cdot - \cdot\|_\infty)$ are not interval-linear.

In the following proposition, condition (3) is condition (L4) in [**10**, chapter II, section 3].

PROPOSITION 3.3.1. *(interval-linear geometric interval spaces) Let X be an interval-linear geometric interval space. Then:*

(1) *X is interval-convex.*
(2) *For $a,\, b,\, x,\, y \in X$, if $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular, then $\left\langle \begin{matrix} a & - & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & \mid & b \\ x & & y \end{matrix} \right\rangle$.*
(3) *For $a,\, b,\, c \in X$, if $\langle a,\, b,\, c\rangle$, then $[a,\, b] \cup [b,\, c] = [a,\, c]$.*

PROOF.

(1) For $a,\, b \in X$ it is to be proved that $[a,\, b]$ is convex. The assumption that the geometric interval space X is interval-linear entails that $([a,\, b]\, ,\, \langle a,\, \cdot,\, \cdot\rangle)$ is a chain. By 2.3.1(2) (geometric interval spaces), $[a,\, b]$ is convex.
(2) The assumption that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular says that there are $s,\, t,\, u,\, v \in X$ such that

$$Q := \begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$$

is a median quadrangle. In particular, $\langle x,\, s,\, t,\, v,\, b\rangle$ and $\langle x,\, s,\, u,\, v,\, b\rangle$. Thus, $\langle x,\, t,\, v\rangle$ and $\langle x,\, u,\, v\rangle$, i.e.

$$t,\, u \in [x,\, v]\ . \tag{3.3.1}$$

The assumption that X is interval-linear entails:

$$[x,\, v] \text{ is a chain in } (X,\, \langle x,\, \cdot,\, \cdot\rangle)\ . \tag{3.3.2}$$

From (3.3.1) and (3.3.2) it follows that $\langle x,\, t,\, u\rangle$ or $\langle x,\, u,\, t\rangle$.
Case 1.1. $\langle x,\, t,\, u\rangle$. The assumptions that Q is a median quadrangle and $\langle x,\, t,\, u\rangle$ imply by 2.3.2(2c) (median quadrangles in a geometric interval space) that $\left\langle \begin{matrix} a & | & b \\ x & & y \end{matrix} \right\rangle$.
Case 1.2. $\langle x,\, u,\, t\rangle$. The assumptions that Q is a median quadrangle and $\langle x,\, u,\, t\rangle$ imply by 2.3.2(2d) (median quadrangles in a geometric interval space) that $\left\langle \begin{matrix} a & - & b \\ x & & y \end{matrix} \right\rangle$.

(3) By 1.4.17(2b) (aligned sequences in a geometric interval space), it suffices to prove $[a,\, b] \cup [b,\, c] \supseteq [a,\, c]$, i.e. for $x \in X$, if $x \in [a,\, c]$, then $\langle a,\, x,\, b\rangle$ or $\langle b,\, x,\, c\rangle$. The assumption $\langle a,\, b,\, c\rangle$ says:

$$b \in [a,\, c] \ . \tag{3.3.3}$$

From (3.3.3) and the assumptions that $x \in [a,\, c]$ and X is interval-linear it follows that $\langle a,\, x,\, b\rangle$ or $\langle a,\, b,\, x\rangle$.
Case 1. $\langle a,\, x,\, b\rangle$. Nothing remains to be proved.
Case 2. $\langle a,\, b,\, x\rangle$. The assumption $x \in [a,\, c]$ says:

$$\langle a,\, x,\, c\rangle \ . \tag{3.3.4}$$

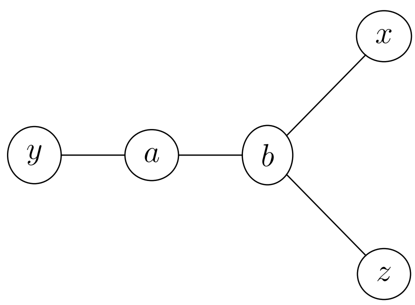
From the assumption $\langle a,\, b,\, x\rangle$ and (3.3.4) it follows by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $\langle b,\, x,\, c\rangle$.

□

Let X be an interval space. X is called *ray-linear* iff one and therefore each of the following conditions is satisfied, which are equivalent by 2.3.1(3) (geometric interval spaces):

- For all $a,\, b \in X$, $a \neq b$ implies that $(\langle a,\, b,\, \cdot\rangle\, ,\, \langle a,\, \cdot,\, \cdot\rangle)$ is a chain.
- For all $a,\, b \in X$, $a \neq b$ implies that $(\langle a,\, b,\, \cdot\rangle\, ,\, \langle b,\, \cdot,\, \cdot\rangle)$ is a chain.

The first condition is the interval relation version of the strict interval relation condition [**34**, §1, VII. Grundsatz]. When X is a vector space over a totally ordered field K , for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X is ray-linear. Each subspace of a ray-linear interval space is ray-linear. A tree that has a point of degree ≥ 3 is not ray-linear. For example, the following tree is not ray-linear: $y \neq b$, $x,\, z \in \langle y,\, b,\, \cdot\rangle$, but not $\langle y,\, x,\, z\rangle$ and not $\langle y,\, z,\, x\rangle$.



For $n \in \mathbb{Z}_{\geq 2}$, the metric space $(\mathbb{R}^n,\, \|\cdot - \cdot\|_1)$ is not ray-linear.

The following theorem, like 3.2.4 (antiexchange criterion for triangle-convex geometric interval spaces), characterizes a geometric property of an interval relation in terms of a fundamental property of a family of derived binary relations:

THEOREM 3.3.2. *(perspectivity relation) Let X be a one-way geometric interval space. Define the binary relation $\sim_p$ on $X \setminus \{p\}$ by*

$$a \sim_p b :\Leftrightarrow \langle p \neq a,\, b\rangle \text{ or } \langle a \neq p \neq b\rangle \text{ or } \langle a,\, b \neq p\rangle\ .$$

The following conditions are equivalent:

(1) *For each $p \in X$, $\sim_p$ is transitive.*
(2) *For each $p \in X$, $\sim_p$ is an equivalence relation on $X \setminus \{p\}$.*
(3) *X is interval-linear and ray-linear.*

PROOF. Step 1. (1) $\Rightarrow$(2) follows from 1.4.7(3a) and (3b) (aligned sequences).

Step 2. (2)$\Rightarrow$ (3). From the assumption that $\sim_p$ is an equivalence relation on $X \setminus \{p\}$ it is to be proved that X is interval-linear and ray-linear.

Step 2.1. Proof that X is interval-linear. For $a,\, b,\, x,\, y$ it is to be proved that $\langle a,\, x,\, b\rangle$ and $\langle a,\, y,\, b\rangle$ imply that $\langle a,\, x,\, y\rangle$ or $\langle a,\, y,\, x\rangle$, i.e. $\langle a,\, x,\, y\rangle$ or $\langle x,\, y,\, a\rangle$.

Case 1. $x = a$. Then $\langle a,\, a,\, y\rangle$ says $\langle a,\, x,\, y\rangle$.

Case 2. $y = a$. Then $\langle x,\, a,\, a\rangle$ says $\langle x,\, y,\, a\rangle$.

Case 3. $x \neq a$ and $y \neq a$. This assumption together with the assumptions $\langle a,\, x,\, b\rangle$ and $\langle a,\, y,\, b\rangle$ says $\langle a \neq x,\, b\rangle$ and $\langle a \neq y,\, b\rangle$. In particular,

$$x \sim_a b\,, \tag{3.3.5}$$
$$y \sim_a b\,. \tag{3.3.6}$$

(3.3.5), (3.3.6) and the assumption that $\sim_p$ is an equivalence relation imply $x \sim_a y$. Therefore, $\langle a,\, x,\, y\rangle$ or $\langle x,\, a,\, y\rangle$ or $\langle x,\, y,\, a\rangle$. Thus, it suffices to prove that the second condition $\langle x,\, a,\, y\rangle$ leads to a contradiction. The assumption that $\langle a,\, x,\, b\rangle$ implies:

$$\langle b,\, x,\, a\rangle\ . \tag{3.3.7}$$

From the assumptions that $\langle x,\, a,\, y\rangle$, $a \neq y$, $\langle a,\, y,\, b\rangle$ and X is one-way it follows by 3.2.2 (one-way criterion for a geometric interval space) that $\langle x,\, a,\, y,\, b\rangle$. In particular, $\langle x,\, a,\, b\rangle$. Therefore,

$$\langle b,\, a,\, x\rangle\ . \tag{3.3.8}$$

(3.3.7) and (3.3.8) imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $x = a$, contradicting the assumption $x \neq a$.

Step 2.2. Proof that X is ray-linear. For $a,\, b,\, x,\, y \in X$ it is to be proved that $a \neq b$, $\langle a,\, b,\, x\rangle$ and $\langle a,\, b,\, y\rangle$ imply that $\langle a,\, x,\, y\rangle$ or $\langle a,\, y,\, x\rangle$, i.e. $\langle a,\, x,\, y\rangle$ or $\langle x,\, y,\, a\rangle$. The assumptions $a \neq b$, $\langle a,\, b,\, x\rangle$ and $\langle a,\, b,\, y\rangle$ together say that $\langle a \neq b,\, x\rangle$ and $\langle a \neq b,\, y\rangle$. In particular,

$$x \sim_a b\,, \tag{3.3.9}$$
$$y \sim_a b\,. \tag{3.3.10}$$

(3.3.9), (3.3.10) and the assumption that $\sim_p$ is an equivalence relation imply $x \sim_a y$. Therefore, $\langle a,\, x,\, y\rangle$ or $\langle x,\, a,\, y\rangle$ or $\langle x,\, y,\, a\rangle$. Thus, it suffices to prove that the second condition $\langle x,\, a,\, y\rangle$ leads to a contradiction. The assumptions $\langle x,\, a,\, y\rangle$ and $\langle a,\, b,\, y\rangle$ imply by 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle x,\, a,\, b,\, y\rangle$. In particular,

$$\langle x,\, a,\, b\rangle \ . \tag{3.3.11}$$

From the assumption $\langle a,\, b,\, x\rangle$ it follows:

$$\langle x,\, b,\, a\rangle \ . \tag{3.3.12}$$

(3.3.11) and (3.3.12) imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $a = b$, contradicting the assumption $a \neq b$.

Step 3. (3) $\Rightarrow$ (1). From the assumption that X is interval-linear and ray-linear it is to be proved that for $p \in X$, $\sim_p$ is transitive., i.e. for $a,\, b,\, c \in X$, $a \sim_p b$ and $b \sim_p c$ imply $a \sim_p c$. The assumption $a \sim_p b$ says $\langle p \neq a,\, b\rangle$ or $\langle a \neq p \neq b\rangle$ or $\langle a,\, b \neq p\rangle$. Suppose without loss of generality that 1. $\langle p \neq a,\, b\rangle$ or 2. $\langle a \neq p \neq b\rangle$. The assumption $b \sim_p c$ says that a. $\langle p \neq b,\, c\rangle$ or b. $\langle b \neq p \neq c\rangle$ or c. $\langle b,\, c \neq p\rangle$.

Case 1a. $\langle p \neq a,\, b\rangle$ and $\langle p \neq b,\, c\rangle$. Therefore, $\langle p \neq a,\, c\rangle$. Consequently, $a \sim_p c$.
Case 1b. $\langle p \neq a,\, b\rangle$ and $\langle b \neq p \neq c\rangle$. Then

$$\langle b,\, a,\, p\rangle \ . \tag{3.3.13}$$

(3.3.13) and the assumption $\langle b \neq p \neq c\rangle$ imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces):

$$\langle a,\, p,\, c\rangle \ . \tag{3.3.14}$$

(3.3.14) and the assumptions $p \neq a$ and $p \neq c$ together say that $\langle a \neq p \neq c\rangle$. Consequently, $a \sim_p c$.
Case 1c. $\langle p \neq a,\, b\rangle$ and $\langle b,\, c \neq p\rangle$. Thus,

$$a,\, c \in [p,\, b] \ . \tag{3.3.15}$$

From (3.3.15) and the assumption that X is interval-linear it follows:

$$\langle p,\, a,\, c\rangle \ \text{ or } \ \langle p,\, c,\, a\rangle \ . \tag{3.3.16}$$

(3.3.16) and the assumptions $p \neq a$ and $c \neq p$ together say that $\langle p \neq a,\, c\rangle$ or $\langle p \neq c,\, a\rangle$. Consequently, $a \sim_p c$.

Case 2a. $\langle a \neq p \neq b\rangle$ and $\langle p \neq b,\, c\rangle$. This assumption and the assumption that X is one-way imply:

$$\langle a,\, p,\, c\rangle \ . \tag{3.3.17}$$

From the assumption $\langle p \neq b,\, c\rangle$ it follows:

$$p \neq c \tag{3.3.18}$$

because $\langle p,\, b,\, c\rangle$ and $p = c$ would imply $\langle p,\, b,\, p\rangle$ and therefore $p = b$. The assumption $a \neq p$, (3.3.17) and (3.3.18) together say that $\langle a \neq p \neq c\rangle$. Consequently, $a \sim_p c$.

Case 2b. $\langle a \neq p \neq b\rangle$ and $\langle b \neq p \neq c\rangle$. Thus,

$$\langle b \neq p,\, a\rangle \ . \tag{3.3.19}$$

From (3.3.19) and the assumptions that $\langle b \neq p,\, c\rangle$ and X is ray-linear it follows:

$$\langle p,\, a,\, c\rangle \ \text{or} \ \langle p,\, c,\, a\rangle \ . \tag{3.3.20}$$

(3.3.20) together with the assumptions $a \neq p$ and $p \neq c$ says $\langle p \neq a,\, c\rangle$ or $\langle p \neq c,\, a\rangle$. Consequently, $a \sim_p c$.

Case 2c. $\langle a \neq p \neq b\rangle$ and $\langle b,\, c \neq p\rangle$. Therefore,

$$\langle p,\, c,\, b\rangle \tag{3.3.21}$$

The assumption $\langle a \neq p \neq b\rangle$ and (3.3.21) imply by 1.4.17(1b) (aligned sequences in a geometric interval space), that $\langle a,\, p,\, c,\, b\rangle$. In particular,

$$\langle a,\, p,\, c\rangle \ . \tag{3.3.22}$$

(3.3.22) and the assumption $a \neq p$ together say that $\langle a \neq p,\, c\rangle$. Consequently, $a \sim_p c$. $\square$

CHAPTER 4

Modular and Median Spaces

In this and subsequent chapters, the following known results on median spaces are used:

- ◦ 1.6.5 (product of median interval spaces)
- ◦ 1.6.6 (medianity criterion for a geometric interval space)

The following new concept is introduced:

- ◦ submedian-metrizable interval space

The following main new results are proved:

- ◦ 4.3.1 (modular geometric topological interval spaces)
- ◦ 4.7.2 (metrizability criterion)

4.1. Modular Interval Spaces

Let X be an interval space. For $a,\, b \in X$, a is *adjacent* to b iff $a \neq b$ and $[a,\, b] = \{a,\, b\}$. This concept corresponds to the concept of an *underlying graph* that has been defined for particular cases in [**31**, 7.1.6] and in [**48**, chapter I, 5.5].

Part (3) of the following proposition generalizes one direction in [**31**, 3.1.7].

PROPOSITION 4.1.1. *(modular interval spaces) Let X be a modular interval space.*

(1) *For $a,\, b \in X$, if a is adjacent to b , then $X = \langle \cdot,\, a,\, b\rangle \cup \langle a,\, b,\, \cdot\rangle$.*
(2) *For $a,\, b \in X$, if $a \neq b$ and $|X| \geq 3$, then there is a non-extremal $u \in X$ such that $\langle a,\, u,\, b\rangle$.*
(3) *X is interval-concatenable.*
(4) *For $a,\, b,\, p \in X$, if $a \neq b$ and $a,\, b$ are adjacent to p , then $\langle a,\, p,\, b\rangle$.*
(5) *For $p \in X$, if p is extremal, then p is adjacent to at most one point.*

PROOF.

(1) For $x \in X$ it is to be proved that $\langle x,\, a,\, b\rangle$ or $\langle a,\, b,\, x\rangle$. From the assumption that X is modular it follows that $a,\, b,\, c$ have a median u , i.e.

$$\langle a,\, u,\, b\rangle \ , \tag{4.1.1}$$

$$\langle a,\, u,\, x\rangle \ , \tag{4.1.2}$$

$$\langle x,\, u,\, b\rangle \ . \tag{4.1.3}$$

From (4.1.1) and the assumption that a is adjacent to b it follows that $u = a$ or $u = b$.
Case 1. $u = a$. Substituting this assumption into (4.1.3), $\langle x,\, a,\, b\rangle$.
Case 2. $u = b$. Substituting this assumption into (4.1.2), $\langle a,\, b,\, x\rangle$.

(2) Case 1. a is adjacent to b . From $\langle a,\, a,\, b\rangle$ and $\langle a,\, b,\, b\rangle$ it follows that it suffices to prove that a is non-extremal or b is non-extremal. By (1),

$$X = \langle \cdot,\, a,\, b\rangle \cup \langle a,\, b,\, \cdot\rangle \ . \tag{4.1.4}$$

The assumption $|X| \geq 3$ entails that there is a $c \in X$ such that

$$c \neq a\,, \tag{4.1.5}$$

$$c \neq b\,. \tag{4.1.6}$$

(4.1.4) implies that $\langle c,\, a,\, b\rangle$ or $\langle a,\, b,\, c\rangle$.
Case 1.1. $\langle c,\, a,\, b\rangle$. From this assumption, (4.1.5) and the assumption $a \neq b$ it follows that a is non-extremal.
Case 1.2. $\langle a,\, b,\, c\rangle$. This assumption, (4.1.6) and the assumption $a \neq b$ imply that b is non-extremal.
Case 2. a is not adjacent to b , i.e. there is a $u \in X$ such that $\langle a,\, u,\, b\rangle$ and $u \notin \{a,\, b\}$. In particular, u is non-extremal.

(3) For $a,\, b,\, p \in X$ it is to be proved that $[p,\, a] \cap [p,\, b] = \{p\}$ implies $\langle a,\, p,\, b\rangle$ .The assumption that X is modular entails that $p,\, a,\, b$ have a median u , i.e.

$$u \in [p,\, a] \cap [p,\, b] \ , \tag{4.1.7}$$

$$\langle a,\, u,\, b\rangle \ . \tag{4.1.8}$$

From (4.1.7) and the assumption $[p,\, a] \cap [p,\, b] = \{p\}$ it follows:

$$u = p\,. \tag{4.1.9}$$

Substituting (4.1.9) into (4.1.8), $\langle a,\, p,\, b\rangle$.

(4) By (3), it suffices to prove $[p,\, a] \cap [p,\, b] = \{p\}$. From $\{p\} \subseteq [p,\, a] \cap [p,\, b]$ it follows that it is sufficient to prove $[p,\, a] \cap [p,\, b] \subseteq \{p\}$. The assumption that a , b are adjacent to p entails $[p,\, a] = \{p,\, a\}$ and $[p,\, b] = \{p,\, b\}$. Therefore, it suffices to prove $\{p,\, a\} \cap \{p,\, b\} \subseteq \{p\}$. For $x \in X$ it is to be proved that $x \in \{p,\, a\} \cap \{p,\, b\}$ implies $x = p$. Seeking a contradiction, assume $x \neq p$. From the assumptions $x \in \{p,\, a\} \cap \{p,\, b\}$ and $x \neq p$ it follows that $x = a$ and $x = b$. Therefore, $a = b$, contradicting the assumption $a \neq b$.

(5) For $a,\, b \in X$ it is to be proved: If $a,\, b$ are adjacent to p , then $a = b$. Seeking a contradiction, assume $a \neq b$. By (4),

$$\langle a,\, p,\, b\rangle \ . \tag{4.1.10}$$

From (4.1.10) and the assumption that p is extremal it follows that $p = a$ or $p = b$, contradicting the assumption that $a,\, b$ are adjacent to p .

□

4.2. Modular Geometric Interval Spaces

PROPOSITION 4.2.1. *(quadrimodular matrices in a modular geometric interval space) Let* X *be a modular geometric interval space. For* $x,\, y,\, a,\, b \in X$:

(1) $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ *is quadrimodular iff* $M\,(x,\, y,\, a) \cap [x,\, b] \neq \emptyset$.

(2) *For* $s,\, t \in X$, *if* $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ *and* $\begin{bmatrix} & b & \\ x & t & y \end{bmatrix}$ *are median triangles and* $\langle x,\, s,\, t\rangle$, *then* $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ *is quadrimodular.*

PROOF.

(1) Step 1. ($\Rightarrow$) From the assumption that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular it follows by 2.2.5(5) (quadrimodularity properties) that $M\,(x,\, y,\, a) \cap [x,\, b] \neq \emptyset$.
Step 2. ($\Leftarrow$) From the assumption $M\,(x,\, y,\, a) \cap [x,\, b] \neq \emptyset$ it is to be proved that there are $s,\, t,\, u,\, v \in X$ such that $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle. The assumption $M\,(x,\, y,\, a) \cap [x,\, b] \neq \emptyset$ says that there is an

$$s \in M\,(x,\, y,\, a) \cap [x,\, b]\ . \tag{4.2.1}$$

The assumption that X is modular implies that there are $t,\, v,\, u \in X$ such that

$$t \in M\,(s,\, y,\, b)\ , \tag{4.2.2}$$

$$v \in M\,(t,\, b,\, a)\ , \tag{4.2.3}$$

$$u \in M\,(v,\, a,\, s)\ . \tag{4.2.4}$$

From the assumption that X is geometric, (4.2.1), (4.2.2), (4.2.3) and (4.2.4) it follows by 2.3.2(1) (median quadrangles in a geometric interval space) that $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle.

(2) The assumption that X is modular and geometric implies by (1) that it suffices to prove $M\,(x,\, y,\, a) \cap [x,\, b] \neq \emptyset$. The assumption that $\begin{bmatrix} & b & \\ x & t & y \end{bmatrix}$ is a median triangle entails:

$$\langle x,\, t,\, b\rangle\ . \tag{4.2.5}$$

From the assumption $\langle x,\, s,\, t\rangle$ and (4.2.5) it follows that $\langle x,\, s,\, b\rangle$, i.e.

$$s \in [x,\, b]\ . \tag{4.2.6}$$

The assumption that $\begin{bmatrix} & a & \\ x & s & y \end{bmatrix}$ is a median triangle and (4.2.6) together say that $s \in M(x, y, a) \cap [x, b]$. Consequently, $M(x, y, a) \cap [x, b] \neq \emptyset$.

□

PROPOSITION 4.2.2. *(modular geometric interval spaces) Let X be a modular geometric interval space. For $p \in X$ and C a convex set, the poset $(C, \langle \cdot, \cdot, p \rangle)$ is directed.*

PROOF. Remember that $(C, \langle \cdot, \cdot, p \rangle)$ is the dual of the poset $(C, \langle p, \cdot, \cdot \rangle)$. For $x, y \in X$ it is to be proved: If $x, y \in C$, then there is a $u \in C$ such that $\langle x, u, p \rangle$ and $\langle y, u, p \rangle$. The assumption that X is modular entails that there is a $u \in X$ such that $\langle x, u, p \rangle$, $\langle y, u, p \rangle$ and

$$\langle x, u, y \rangle \ . \tag{4.2.7}$$

It sufffices to prove $u \in C$. This claim follows from (4.2.7) and the assumptions that $x, y \in C$ and C is convex. □

4.3. Modular Geometric Topological Interval Spaces

The following theorem characterizes compact gated sets in a modular geometric topological interval space. For a related result, see [**7**, lemma 2.13(3)]. There, the condion on the space X is stronger and the condition on the set C is weaker than here.

THEOREM 4.3.1. *(modular geometric topological interval spaces) Let X be a modular geometric topological interval space. For C a non-empty compact set, C is gated iff it is convex.*

PROOF. Step 1. ($\Rightarrow$) From the assumption that C is gated it follows by 1.4.18(1) (gated sets) that C is convex.
Step 2. ($\Leftarrow$) Suppose that C is convex. It is to be proved that for $p \in X$, the poset $(C, \langle p, \cdot, \cdot \rangle)$ has a least element. The assumption that C is non-empty says that there is a $c \in C$. By 2.4.4 (topological interval spaces), $[c, p]$ is closed. Therefore,

$$[c, p] \cap C \text{ is closed in } C \, . \tag{4.3.1}$$

From $c \in [c, p] \cap C$, the assumption that C is compact and (4.3.1) it follows by 1.3.7 (compact topological spaces):

$$[c, p] \cap C \text{ is non-empty and compact.} \tag{4.3.2}$$

The assumption that C is convex implies by 4.2.2 (modular geometric interval spaces):

$$\text{The poset } (C, \langle \cdot, \cdot, p \rangle) \text{ is directed.} \tag{4.3.3}$$

In $(C, \langle \cdot, \cdot, p \rangle)$, $[c, p] \cap C = \uparrow c$. In particular,

$$[c, p] \cap C \text{ is an up-set in } (C, \langle \cdot, \cdot, p \rangle) \ . \tag{4.3.4}$$

From (4.3.3) and (4.3.4) it follows by 1.2.4(2) (directed posets)

$$\text{The poset } ([c, p] \cap C, \langle \cdot, \cdot, p \rangle) \text{ is directed.} \tag{4.3.5}$$

(4.3.2) and (4.3.5) imply by 2.1.1 (compact directed topological posets) that $([c,\, p] \cap C,\, \langle\cdot,\, \cdot,\, p\rangle)$ has a greatest element s_0 .

$$s_0 \text{ is a least element of } ([p,\, c] \cap C,\, \langle p,\, \cdot,\, \cdot\rangle) \ . \tag{4.3.6}$$

In particular,

$$s_0 \in C\,, \tag{4.3.7}$$

and $s_0 \in [p,\, c]$, i.e.

$$\langle p,\, s_0,\, c\rangle \ . \tag{4.3.8}$$

It suffices to prove that s_0 is a gate of p into C , i.e. for each $s \in C$, $\langle p,\, s_0,\, s\rangle$. From the assumption that X is modular it follows that there is a $u \in X$ such that $\begin{bmatrix} & s & \\ p & u & s_0 \end{bmatrix}$ is a median triangle, i.e.

$$\langle p,\, u,\, s_0\rangle \ , \tag{4.3.9}$$

$$\langle s_0,\, u,\, s\rangle \ , \tag{4.3.10}$$

$$\langle p,\, u,\, s\rangle \ . \tag{4.3.11}$$

(4.3.11) implies that it suffices to prove $u = s_0$. (4.3.7), the assumption $s \in C$, (4.3.10) and the assumption that C is convex imply:

$$u \in C\,. \tag{4.3.12}$$

From (4.3.9) and (4.3.8) it follows that $\langle p,\, u,\, c\rangle$, i.e.

$$u \in [p,\, c] \ . \tag{4.3.13}$$

(4.3.13) and (4.3.12) together say:

$$u \in [p,\, c] \cap C\,. \tag{4.3.14}$$

(4.3.14) and (4.3.6) imply:

$$\langle p,\, s_0,\, u\rangle \ . \tag{4.3.15}$$

From (4.3.15) and (4.3.9) it follows by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $u = s_0$. $\square$

4.4. Median Interval Spaces

PROPOSITION 4.4.1. *(median boundary of a median interval space) Let X be a median interval space.*

(1) *For $x \in X$, $x \in \partial_M (X)$ iff $X \setminus \{x\}$ is median.*
(2) *For $A \subseteq X$, if the median closure of A in X equals X , then $\partial_M (X) \subseteq A$.*

PROOF.

(1) The following conditions are equivalent:
   - $x \in \partial_M (X)$.
   - For all $a,\, b,\, c \in X$, if $x = m(a,\, b,\, c)$, then $x \in \{a,\, b,\, c\}$.
   - For all $a,\, b,\, c \in X$, if $x \notin \{a,\, b,\, c\}$, then $x \neq m(a,\, b,\, c)$.
   - For all $a,\, b,\, c \in X$, if $a,\, b,\, c \in X \setminus \{x\}$, then $m(a,\, b,\, c) \in X \setminus \{x\}$.
   - $X \setminus \{x\}$ is median.

(2) Seeking a contradiction, assume $\partial_M (X) \nsubseteq A$, i.e. there is an

$$x \in \partial_M (X) \tag{4.4.1}$$

such that $x \notin A$, i.e.

$$A \subseteq X \setminus \{x\} \ . \tag{4.4.2}$$

From (4.4.1) it follows by (1):

$$X \setminus \{x\} \text{ is median.} \tag{4.4.3}$$

(4.4.2) and (4.4.3) contradict the assumption that the median closure of A in X equals X .

□

PROPOSITION 4.4.2. *(median quadrangles in a median interval space) Let X be a median interval space. For a median quadrangle $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ in X , $x'' \in \{x,\, s\}$, $y'' \in \{y,\, t\}$, $a'' \in \{a,\, u\}$ and $b'' \in \{b,\, v\}$:*

(1) $s = m(x'',\, y'',\, a'')$.
(2) $t = m(x'',\, y'',\, b'')$.
(3) $u = m(x'',\, a'',\, b'')$.
(4) $v = m(y'',\, a'',\, b'')$.

PROOF. The claim is a restatement of 2.2.2(1a) to (1d) (median quadrangles) for the case of a median interval space. □

4.5. Median Geometric Interval Spaces

PROPOSITION 4.5.1. *(median geometric interval spaces) Let X be a median geometric interval space. For $a \in X$:*

(1) *For $x,\, y \in X$, in the poset $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$, $m\,(a,\, x,\, y)$ is a greatest lower bound of $\{x,\, y\}$.*
(2) *$(X,\, \langle a,\, \cdot,\, \cdot\rangle)$ is a meet semilattice.*

PROOF.

(1) For $u := m\,(a,\, x,\, y)$ it is to be proved that in the poset $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$, u is a greatest lower bound of $\{x,\, y\}$, i.e. $\langle a,\, u,\, x\rangle$, $\langle a,\, u,\, y\rangle$ and for $v \in X$, $\langle a,\, v,\, x\rangle$ and $\langle a,\, v,\, y\rangle$ imply $\langle a,\, v,\, u\rangle$.
Step 1. $\langle a,\, u,\, x\rangle$ and $\langle a,\, u,\, y\rangle$ follow from the assumption $u = m\,(a,\, x,\, y)$.
Step 2. Proof that for $v \in X$, $\langle a,\, v,\, x\rangle$ and $\langle a,\, v,\, y\rangle$ imply $\langle a,\, v,\, u\rangle$. The assumptions $\langle a,\, v,\, x\rangle$ and $\langle a,\, v,\, y\rangle$ imply by 2.3.1(4) (geometric interval spaces) that $m\,(v,\, x,\, y)$ is a median of $a,\, x,\, y$, i.e. $m\,(v,\, x,\, y) = m\,(a,\, x,\, y)$, i.e. $m\,(v,\, x,\, y) = u$. In particular,
$$\langle v,\, u,\, x\rangle\ . \tag{4.5.1}$$
From the assumption $\langle a,\, v,\, x\rangle$ and (4.5.1) it follows by 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle a,\, v,\, u,\, x\rangle$. In particular, $\langle a,\, v,\, u\rangle$.
(2) follows by (1).

□

4.6. Median Geometric Topological Interval Spaces

The following proposition generalizes [**31**, 3.1.7].

PROPOSITION 4.6.1. *(medianity criterion for a compact geometric topological interval space) Let X be a compact geometric topological interval space. X is median iff it is interval-convex and interval-concatenable.*

PROOF. Step 1. ($\Rightarrow$) Assume that X is median.

Step 1.1. X is interval-convex by 1.6.6 (medianity criterion for a geometric interval space).

Step 1.2. X is interval-concatenable by 4.1.1(3) (modular interval spaces).

Step 2. ($\Leftarrow$) Assume that X is interval-convex and interval-concatenable. From the assumption that X is interval-convex it follows by 1.6.6 (medianity criterion for a geometric interval space) that it suffices to prove that X is modular, i.e. that all $a,\, b,\, c \in X$ have a median. By 2.4.4 (topological interval spaces), $[a,\, b]$ and $[a,\, c]$ are closed. Therefore,
$$[a,\, b] \cap [a,\, c] \ \text{ is closed.} \tag{4.6.1}$$
The assumption that X is compact and (4.6.1) imply by 1.3.7 (compact topological spaces):
$$[a,\, b] \cap [a,\, c] \ \text{ is compact.} \tag{4.6.2}$$
From $a \in [a,\, b] \cap [a,\, c]$ it follows:
$$[a,\, b] \cap [a,\, c] \neq \emptyset\,. \tag{4.6.3}$$

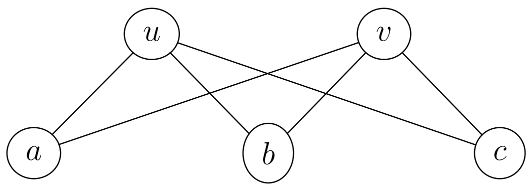
(4.6.2) and (4.6.3) imply by 2.1.2 (compact topological posets)

$$\text{The poset } ([a,\, b] \cap [a,\, c]\,,\, \langle a,\, \cdot,\, \cdot\rangle) \text{ has a maximal element } u\,. \tag{4.6.4}$$

From the assumption that X is interval-concatenable and (4.6.4) it follows by 2.3.4 (interval-concatenable geometric interval spaces) that u is a median of $a,\, b,\, c$. $\square$

4.7. Median Metric Spaces

Let X be an interval space. X is said to have *point-interval separation* iff for all $x,\, y,\, z \in X$, if $x \notin [y,\, z]$, then there is a half-space H such that $x \in H$ and $[y,\, z] \subseteq X \setminus H$. If X has point-interval separation, then each subspace of X has point-interval separation. When X is a real vector space, for example $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X has point-interval separation. The next proposition provides further examples of interval spaces with point-interval separation. The following complete-bipartite graph X does not have point-interval separation: The only half-spaces are X and $\emptyset$.



Part (1) of the following proposition has been cited from [**49**, chapter I, 4.19].

PROPOSITION 4.7.1. *(separation in a median geometric interval space) Let X be a median geometric interval space.*

(1) *For $C_1,\, C_2$ convex sets, if $C_1 \cap C_2 = \emptyset$, then there is a half-space H such that $C_1 \subseteq H$, and $C_2 \subseteq X \setminus H$.*
(2) *X has point-interval separation.*

PROOF.
(1) [**49**, chapter I, 4.19]
(2) By 1.6.6 (medianity criterion for a geometric interval space),

$$X \text{ is interval-convex.} \tag{4.7.1}$$

For $x,\, y,\, z \in X$ satisfying $x \notin [y,\, z]$, it is to be proved that there is a half-space H such that $\{x\} \subseteq H$ and $[y,\, z] \subseteq X \setminus H$.

$$\{x\} \text{ is convex.} \tag{4.7.2}$$

From (4.7.1) it follows:

$$[y,\, z] \text{ is convex.} \tag{4.7.3}$$

The assumption $x \notin [y,\, z]$ says:

$$\{x\} \cap [y,\, z] = \emptyset\,. \tag{4.7.4}$$

From (4.7.2), (4.7.3) and (4.7.4) it follows by (1) that there is a half-space H such that $\{x\} \subseteq H$ and $[y,\, z] \subseteq X \setminus H$.

□

Let X be an interval space.

X is called *metrizable* iff it has an isomorphism onto a metric space. The next theorem provides sufficient criteria for the metrizability of an interval space.

X is called *submedian-metrizable* iff it has an embedding into a median metric space. For example, by part (1) of the next theorem a finite subset of a real vector space is submedian-metrizable. The complete-bipartite graph $K_{2,3}$ is not submedian-metrizable.

The following theorem consists of sufficient criteria for submedian metrizability and for metrizability of a finite interval space.

THEOREM 4.7.2. *(metrizability criterion) Let X be a finite geometric interval space.*

(1) *If X has point-interval separation, then it is submedian-metrizable.*
(2) *If X is median, then it is metrizable.*

PROOF.

(1) Suppose that X has point-interval separation. It is to be proved that X has an embedding into a median metric space. Let Q denote the set of all half-spaces in X and $f : X \to \{0,\, 1\}^Q$ be defined by

$$(f(x))(H) = \begin{cases} 1 & \text{if } x \in H\,, \\ 0 & \text{if } x \notin H\,. \end{cases}$$

By 1.6.7(2) (binary Hamming spaces), the metric space $\left(\{0,\, 1\}^Q,\, \|\cdot - \cdot\|_1\right)$ is median. Therefore, it suffices to prove that f is an embedding of X into the metric space $\left(\{0,\, 1\}^Q,\, \|\cdot - \cdot\|_1\right)$. By 1.4.11 (embedding of interval spaces) it suffices to prove that for all $x,\, y,\, z \in X$, $\langle x,\, y,\, z\rangle$ iff $\langle f(x),\, f(y),\, f(z)\rangle$. The following equivalences hold, where H runs through all half-spaces of X :
$\langle f(x),\, f(y),\, f(z)\rangle$
$\Leftrightarrow \forall H\ \langle f(x)(H),\, f(y)(H),\, f(z)(H)\rangle$ (by 1.6.7(1) (binary Hamming spaces))
$\Leftrightarrow\ \forall H\ (f(x)(H) = f(z)(H) \Rightarrow f(y)(H) = f(x)(H))$ (alignment in a two-point interval space)
$\Leftrightarrow \forall H\, (f(x)(H) = f(z)(H) = 1 \Rightarrow f(y)(H) = 1)$
and $(f(x)(H) = f(z)(H) = 0 \Rightarrow f(y)(H) = 0)$
$\Leftrightarrow \forall H\, (x,\, z \in H \Rightarrow y \in H)$ and $(x,\, z \in X \setminus H \Rightarrow y \in X \setminus H)$
$\Leftrightarrow \forall H\ (x,\, z \in H \Rightarrow y \in H)$ and $\forall H\ (x,\, z \in X \setminus H \Rightarrow y \in X \setminus H)$
$\Leftrightarrow\ \forall H\ (x,\, z \in H \Rightarrow y \in H)$ (because when H runs through all half-spaces, then so does $X \setminus H$).
Therefore, it suffices to prove that the last condition, that for each half-space H, $x,\, z \in H \Rightarrow y \in H$, is equivalent to $\langle x,\, y,\, z\rangle$.
Step 1. ($\Rightarrow$) Suppose that for each half-space H, $x,\, z \in H \Rightarrow y \in H$. Seeking a

contradiction, assume that not $\langle x,\, y,\, z\rangle$, i.e. $y \notin [x,\, z]$. From this assumption and the assumption that X has point-interval separation it follows that there is

$$H_0\,,\ \text{a half-space.} \tag{4.7.5}$$

such that

$$y \in H_0\,, \tag{4.7.6}$$

$$[x,\, z] \subseteq X \setminus H_0\,. \tag{4.7.7}$$

From $x,\, z \in [x,\, z]$ and (4.7.7) it follows:

$$x,\, z \in X \setminus H_0\,. \tag{4.7.8}$$

(4.7.5) implies:

$$X \setminus H_0 \text{ is a half-space.} \tag{4.7.9}$$

(4.7.9), (4.7.8) and the assumption that for each half-space H , $x,\, z \in H \Rightarrow y \in H$ imply $y \in X \setminus H_0$, contradicting (4.7.6).
Step 2. ($\Leftarrow$) Suppose $\langle x,\, y,\, z\rangle$. The assumption that H is a half-space entails:

$$H \text{ is convex.} \tag{4.7.10}$$

From the assumption $\langle x,\, y,\, z\rangle$ and (4.7.10) it follows that $x,\, z \in H \Rightarrow y \in H$.

(2) By 4.7.1(2) (separation in a median geometric interval space), X has point-interval separation. By (1), X is submedian-metrizable, i.e. there is an embedding i of X into a median metric space Y . i is an isomorphism of X onto the metric subspace $i\,(X)$. Consequently, X is metrizable.

□

When X is a finite interval subspace of a real vector space, for example $\mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$, then X is a finite geometric interval space with point-interval separation. By 4.7.2(1) (metrizability criterion), X is submedian-metrizable.

PROPOSITION 4.7.3. *(distance from a median) Let X be a median metric space. For $x,\, y,\, a,\, b \in X$, if $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular, then $d_{m(x,\,y,\,a)b} = d_{xb} - d_{x,ya}$.*

PROOF. The assumption that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular says that there are $s,\, t,\, u,\, v \in Y$ such that

$$\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix} \text{ is a median quadrangle.} \tag{4.7.11}$$

By 4.4.2(1) (median quadrangles in a median interval space),

$$s = m\,(x,\, y,\, a)\ . \tag{4.7.12}$$

(4.7.11) implies by 2.6.1(1) (median quadrangles in a metric space):

$$d_{sb} = d_{xb} - d_{x,ya}\,. \tag{4.7.13}$$

Substituting (4.7.12) into (4.7.13), $d_{m(x,\,y,\,a)b} = d_{xb} - d_{x,ya}$. □

CHAPTER 5

Arboric Spaces

In this and subsequent chapters, the following known result on arboric spaces is used:

- ∘ 1.7.5 (medianity of arboric interval spaces)

The following new concepts are introduced:

- ∘ relative neighbor with respect to a point
- ∘ branch with respect to a point
- ∘ pre-extremal point
- ∘ extremal neighborhood of a pre-extremal point

The following main new result is proved:

- ∘ 5.3.3 (finite arboric metric spaces)

5.1. Arboric Interval Spaces

PROPOSITION 5.1.1. *(arboricity criterion) Let X be a geometric interval space. The following conditions are equivalent:*

(1) *X is arboric.*
(2) *X is interval-linear and median.*
(3) *X is interval-linear and modular.*
(4) *X is interval-linear and median, and for all $x,\, y,\, a,\, b \in X$, $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ or $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$ is quadrimodular.*
(5) *X is median, and for all $x,\, y,\, a,\, b \in X$, $\left\langle \begin{matrix} a & - & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & \mid & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} b & - & a \\ x & & y \end{matrix} \right\rangle$.*

PROOF. Step 1. (1) ⇒(2).

Step 1.1. Proof that X is interval-linear. For $a,\, b \in X$, the assumption that $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$ is arboric entails that in $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$, $\downarrow b$ is a chain, i.e. $[a,\, b]$ is a chain. Consequently, X is interval-linear.

Step 1.2. By 1.7.5 (medianity of arboric interval spaces), X is median.

Step 2. (2)⇒(3). Each median interval space is modular.

Step 3. (3)⇒(4).

Step 3.1. Proof that X is median. For $x,\, y,\, a,\, u_1,\, u_2 \in X$ it is to be proved that $u_1,\, u_2 \in M\,(x,\, y,\, a)$ implies $u_1 = u_2$. The assumption $u_1,\, u_2 \in M\,(x,\, y,\, a)$ entails:

$$u_1,\, u_2 \in [x,\, y] \tag{5.1.1}$$

and

$$\langle x,\, u_2,\, y\rangle \ , \tag{5.1.2}$$

$$\langle x,\, u_2,\, z\rangle \ , \tag{5.1.3}$$

$$\langle y,\, u_1,\, z\rangle \ . \tag{5.1.4}$$

The assumption that X is interval-linear entails that $[x,\, y]$ is a chain in $(X,\, \langle x,\, \cdot,\, \cdot\rangle)$. From this and (5.1.1) it follows that $\langle x,\, u_1,\, u_2\rangle$ or $\langle x,\, u_2,\, u_1\rangle$. Suppose without loss of generality that $\langle x,\, u_1,\, u_2\rangle$. This assumption and (5.1.2) imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $\langle u_1,\, u_2,\, y\rangle$. Therefore,

$$\langle y,\, u_2,\, u_1\rangle \ . \tag{5.1.5}$$

From the assumption $\langle x,\, u_1,\, u_2\rangle$ and (5.1.3) it follows by 1.4.16 (Hedlíková's criterion for geometric interval spaces):

$$\langle u_1,\, u_2,\, z\rangle \ . \tag{5.1.6}$$

(5.1.4), (5.1.5) and (5.1.6) imply by two applications of 1.4.17(1b) (aligned sequences in a geometric interval space) that $\langle y,\, u_2,\, u_1,\, u_2,\, z\rangle$. In particular, $\langle u_2,\, u_1,\, u_2\rangle$. Consequently, $u_1 = u_2$.

Step 3.2. Proof that for $x,\, y,\, a,\, b \in X$, $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ or $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$ is quadrimodular. The assumption that X is modular implies that there are $s,\, t \in X$ such that

$$\begin{bmatrix} & a & \\ x & s & y \end{bmatrix},\ \begin{bmatrix} & b & \\ x & t & y \end{bmatrix} \text{ are median triangles.} \tag{5.1.7}$$

In particular,

$$s,\, t \in [x,\, y] \ . \tag{5.1.8}$$

The assumption that X is interval-linear entails that $[x,\, y]$ is a chain in $(X,\, \langle x,\, \cdot,\, \cdot\rangle)$. From this and (5.1.8) it follows that $\langle x,\, s,\, t\rangle$ or $\langle x,\, t,\, s\rangle$. Let without loss of generality $\langle x,\, s,\, t\rangle$. From the assumption that X is modular and geometric, (5.1.7) and the assumption $\langle x,\, s,\, t\rangle$ it follows by 4.2.1(2) (quadrimodular matrices in a modular geometric interval space) that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular.

Step 4. (4)⇒(1). It is to be proved that X is interval-concatenable, and for each $a \in X$, the poset $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$ is arboric.

Step 4.1. Proof that X is interval-concatenable. The assumption that X is median entails that X is modular. By 4.1.1(3) (modular interval spaces) , X is interval-concatenable.

Step 4.2. Proof that for $a \in X$, the poset $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$ is arboric. It is to be proved that $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$ is a meet semilattice and for $b \in X$, $\downarrow b$ is a chain in $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$.

Step 4.2.1. From the assumption that X is median it follows by 4.5.1(2) (median geometric interval spaces) that for $a \in X$, the poset $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$ is a meet semilattice.

Step 4.2.2. Proof that for $b \in X$, $\downarrow b$ is a chain in $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$. The assumption that X is interval-linear entails that in $(X,\, \langle a,\, \cdot,\, \cdot\rangle)$, $[a,\, b]$ is a chain, i.e. $\downarrow b$ is a chain.

Step 5. (4)⇒(5). For $x,\, y,\, a,\, b \in X$ it is to be proved that $\left\langle \begin{matrix} a & - & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & \mid & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} b & - & a \\ x & & y \end{matrix} \right\rangle$.

Case 1. $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$. From this assumption and the assumption that X is interval-linear and geometric it follows by 3.3.1(2) (interval-linear geometric interval spaces) that $\left\langle \begin{matrix} a & - & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & \mid & b \\ x & & y \end{matrix} \right\rangle$.

Case 2. $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$. From this assumption and the assumption that X is interval-linear and geometric it follows by 3.3.1(2) (interval-linear geometric interval spaces) that $\left\langle \begin{matrix} b & - & a \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} b & \mid & a \\ x & & y \end{matrix} \right\rangle$. In the latter case, $\left\langle \begin{matrix} b & \mid & a \\ x & & y \end{matrix} \right\rangle$, by 2.2.4(2) (symmetries of quadrimodularity properties), $\left\langle \begin{matrix} a & \mid & b \\ x & & y \end{matrix} \right\rangle$.

Step 6. (5)⇒(4). It is to be proved that X is interval-linear, i.e. for $x,\, y \in X$, $[x,\, y]$ is a chain in $(X,\, \langle x,\, \cdot,\, \cdot\rangle)$. For $a,\, b \in X$ it is to be proved that $a,\, b \in [x,\, y]$ implies $\langle x,\, a,\, b\rangle$ or $\langle x,\, b,\, a\rangle$. The assumption that $\left\langle \begin{matrix} a & - & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} a & \mid & b \\ x & & y \end{matrix} \right\rangle$ or $\left\langle \begin{matrix} b & - & a \\ x & & y \end{matrix} \right\rangle$ entails that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ or $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$ is quadrimodular. Let without loss of generality $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ be quadrimodular. From this assumption and the assumption $a,\, b \in [x,\, y]$ it follows by 2.3.3 (quadrimodular matrices in a geometric interval space) that $\langle x,\, a,\, b\rangle$. □

Let X be an arboric interval space. For $u \in X$:

For $a,\, b \in X$, a is called a *relative neighbor* of b with respect to u or u-neighbor of b iff not $\langle a,\, u,\, b\rangle$. The u-neighbor relation on X is denoted by $\neg\langle \cdot,\, u,\, \cdot\rangle$, i.e. for $a,\, b \in X$,

$$(a,\, b) \in \neg\langle \cdot,\, u,\, \cdot\rangle \text{ iff not } \langle a,\, u,\, b\rangle \ .$$

In the metric space $(\mathbb{R},\, |\cdot - \cdot|)$, which is arboric by 1.7.3(2) (arboricity of the real line), for $a,\, b \in \mathbb{R}$, a is a 0-neighbor of b iff a and b are different from 0 and have the same sign. The concept of a relative neighbor with respect to a point can be immediately generalized to the concept of a relative neighbor with respect to a set.

PROPOSITION 5.1.2. *(neighbors relative to a point) Let X be an arboric interval space. For $u \in X$:*

(1) *For $x,\, y,\, z \in X$, if $\langle x,\, u,\, z\rangle$, then $\langle x,\, u,\, y\rangle$ or $\langle y,\, u,\, z\rangle$.*
(2) *The relative neighbor relation $\neg\,\langle \cdot,\, u,\, \cdot\rangle$ is an equivalence relation on $X \setminus \{u\}$.*

PROOF.

(1) The assumption that X is arboric entails that the poset $(X,\, \langle x,\, \cdot,\, \cdot\rangle)$ is arboric. In particular, $\downarrow z$ is a chain in $(X,\, \langle x,\, \cdot,\, \cdot\rangle)$, i.e.
$$[x,\, z] \text{ is a chain in } (X,\, \langle x,\, \cdot,\, \cdot\rangle) \ . \tag{5.1.9}$$
The assumption $\langle x,\, u,\, z\rangle$ says:
$$u \in [x,\, z] \ . \tag{5.1.10}$$
The assumption that X is arboric implies by 1.7.5 (medianity of arboric interval spaces) that X is median. Define $v := m\,(x,\, y,\, z)$. In particular,
$$v \in [x,\, z] \ , \tag{5.1.11}$$
$$\langle x,\, v,\, y\rangle \ . \tag{5.1.12}$$
From (5.1.9), (5.1.10) and (5.1.11) it follows that $\langle x,\, u,\, v\rangle$ or $\langle x,\, v,\, u\rangle$. Suppose without loss of generality that
$$\langle x,\, u,\, v\rangle \ . \tag{5.1.13}$$
(5.1.13) and (5.1.12) imply $\langle x,\, u,\, y\rangle$.

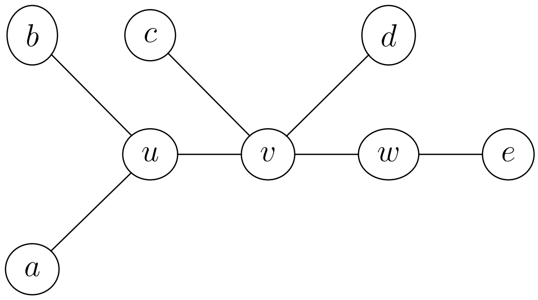
(2) Step 1. $\neg\,\langle \cdot,\, u,\, \cdot\rangle$ is a binary relation on $X \setminus \{u\}$ because $\langle u,\, u,\, u\rangle$.
Step 2. Reflexivity. For $y \in X \setminus \{u\}$, not $\langle y,\, u,\, y\rangle$ because $\langle y,\, u,\, y\rangle$ implies $y = u$.
Step 3. Symmetry. For $y,\, z \in X$, not $\langle y,\, u,\, z\rangle$ implies that not $\langle z,\, u,\, y\rangle$.
Step 4. Transitivity. For $x,\, y,\, z \in X$ it is to be proved that not $\langle x,\, u,\, y\rangle$ and not $\langle y,\, u,\, z\rangle$ imply that not $\langle x,\, u,\, z\rangle$. This claim is the contrapositive of (1).

□

Let X be an arboric interval space. For $u \in X$:

A *branch* with respect to u , or u-branch, is an equivalence class of the relative neighbor relation $\neg\,\langle \cdot,\, u,\, \cdot\rangle$, which is an equivalence relation on $X \setminus \{u\}$ by 5.1.2(2) (neighbors relative to a point). For $x \in X$,
$$[x]_u := \text{the } u\text{-branch of } x \,.$$
For example, in the metric space $(\mathbb{R},\, |\cdot - \cdot|)$, which is arboric by 1.7.3(2) (arboricity of the real line), the 0-branches are the sets $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. In the following tree, which is arboric by 1.7.4 (tree representation of finite arboric interval spaces), $a,\, b,\, c,\, d,\, e$ are extremal. With respect to each of them, there is only one branch. Furthermore:

- The u-branches are $\{a\}$, $\{b\}$, $\{v,\, c,\, d,\, w,\, e\}$.
- The v-branches are $\{u,\, a,\, b\}$, $\{c\}$, $\{d\}$, $\{w,\, e\}$.
- The w-branches are $\{u,\, a,\, b,\, v,\, c,\, d\}$, $\{e\}$.



The concept of a branch in an arboric interval space is similar to the concept of a branch in an arboric poset as defined in [**48**, chapter I, 5.3]. The degree of u is the cardinal number

$$\deg(u) := \text{the number of } u\text{-branches},$$

which may be finite or infinite. For a tree, this order-geometric concept of degree coincides with the graph-theoretic concept of degree.

PROPOSITION 5.1.3. *(extremal points in an arboric interval space) Let X be an arboric interval space. For $x \in X$, x is extremal iff* $\deg(x) \leq 1$.

PROOF. It suffices to prove the contrapositive of the claim, i.e. that x is not extremal iff $\deg(x) \geq 2$. The following conditions are equivalent:

- x is not extremal.
- There are $a,\, b \in X$ such that $\langle a,\, x,\, b\rangle$, but $x \notin \{a,\, b\}$.
- There are $a,\, b \in X \setminus \{x\}$ such that $\langle a,\, x,\, b\rangle$.
- There are $a,\, b \in X \setminus \{x\}$ such that $(a,\, b) \notin \neg\langle\cdot,\, x,\, \cdot\rangle$.
- The equivalence relation $\neg\langle\cdot,\, x,\, \cdot\rangle$ has at least two equivalence classes.
- $\deg(x) \geq 2$.

□

PROPOSITION 5.1.4. *(median boundary of an arboric interval space) Let X be an arboric interval space. For $x \in X$, $x \in \partial_M(X)$ iff* $\deg(x) \leq 2$.

PROOF. It suffices to prove the contrapositive of the claim, i.e. that $x \notin \partial_M(X)$ iff $\deg(x) \geq 3$. The following conditions are equivalent:

- $x \notin \partial_M(X)$.
- There are $a,\, b,\, c \in X$ such that $\begin{bmatrix} & c & \\ a & x & b \end{bmatrix}$ is a median triangle, but $x \notin \{a,\, b,\, c\}$.
- There are $a,\, b,\, c \in X \setminus \{x\}$ such that $\begin{bmatrix} & c & \\ a & x & b \end{bmatrix}$ is a median triangle.
- There are $a,\, b,\, c \in X \setminus \{x\}$ such that $\langle a,\, x,\, b\rangle$, $\langle b,\, x,\, c\rangle$, $\langle c,\, x,\, a\rangle$.
- There are $a,\, b,\, c \in X \setminus \{x\}$ such that $(a,\, b)$, $(b,\, c)$, $(c,\, a) \notin \neg\langle\cdot,\, x,\, \cdot\rangle$.
- The equivalence relation $\neg\langle\cdot,\, x,\, \cdot\rangle$ has at least three equivalence classes.
- $\deg(x) \geq 3$.

□

PROPOSITION 5.1.5. *(arboric interval spaces) Let X be an arboric interval space.*

(1) *For $u,\ a \in X$:*
   (a) *If a is maximal in the poset $(X,\ \langle u,\ \cdot,\ \cdot \rangle)$, then a is extremal.*
   (b) *For U an up-set in the poset $(X,\ \langle u,\ \cdot,\ \cdot \rangle)$, if a is a maximal element of U , then a is extremal.*

(2) *For $u \in X$ and B a u-branch:*
   (a) *In the poset $(X,\ \langle u,\ \cdot,\ \cdot \rangle)$, B is an up-set.*
   (b) *If s is maximal in the poset $(B,\ \langle u,\ \cdot,\ \cdot \rangle)$, then s is extremal.*
   (c) *B is a meet subsemilattice of the meet semilattice $(X,\ \langle u,\ \cdot,\ \cdot \rangle)$.*
   (d) *If B is finite, then $(B,\ \langle u,\ \cdot,\ \cdot \rangle)$ has a least element.*
   (e) *$X \setminus B$ is gated. For g the gate map of $X \setminus B$ and $s \in X$,*
   $$g(s) = \begin{cases} u & \text{if } s \in B \\ s & \text{if } s \in X \setminus B \end{cases} .$$
   (f) *B is a half-space.*
   (g) *For $v \in B$ and C a v-branch, $C \subseteq B$ iff $u \notin C$.*

(3) *For $u,\ x \in X$:*
   (a) *If x is adjacent to u , then $[x]_u = \langle u,\ x,\ \cdot \rangle$.*
   (b) *$\{x\}$ is a u-branch iff x is extremal and adjacent to u .*

PROOF.

(1)
   (a) For $x,\ z \in X$ it is to be proved that $\langle x,\ a,\ z \rangle$ implies $a \in \{x,\ z\}$. From the assumption $\langle x,\ a,\ z \rangle$ it follows by 5.1.2(1) (neighbors relative to a point) that $\langle x,\ a,\ u \rangle$ or $\langle u,\ a,\ z \rangle$.
   Case 1. $\langle x,\ a,\ u \rangle$, i.e. $\langle u,\ a,\ x \rangle$. From this assumption and the assumption that a is maximal in $(X,\ \langle u,\ \cdot,\ \cdot \rangle)$ it follows that $a = x$. In particular, $a \in \{x,\ z\}$.
   Case 2. $\langle u,\ a,\ z \rangle$. This assumption and the assumption that a is maximal in $(X,\ \langle u,\ \cdot,\ \cdot \rangle)$ imply $a = z$. In particular, $a \in \{x,\ z\}$.
   (b) From the assumptions that U is an up-set and a is a maximal element of U it follows by 1.2.3 (posets) that a is a maximal element of X . By (1a), a is extremal.

(2)
   (a) For $b,\ x \in X$ it is to be proved that $b \in B$ and $\langle u,\ b,\ x \rangle$ imply $x \in B$, i.e. not $\langle b,\ u,\ x \rangle$. Seeking a contradiction, assume $\langle b,\ u,\ x \rangle$. Therefore,
   $$\langle x,\ u,\ b \rangle\ . \tag{5.1.14}$$
   From the assumption that B is a u-branch it follows by 5.1.2(2) (neighbors relative to a point):
   $$u \notin B\,. \tag{5.1.15}$$
   From the assumption $\langle u,\ b,\ x \rangle$ it follows:
   $$\langle x,\ b,\ u \rangle\ . \tag{5.1.16}$$

(5.1.14) and (5.1.16) imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces):

$$b = u\,. \tag{5.1.17}$$

Substituting (5.1.17) into the assumption $b \in B$, $u \in B$, contradicting (5.1.15).

(b) By (2a), in the poset $(X,\ \langle u,\ \cdot,\ \cdot\rangle)$,

$$B \text{ is an up-set.} \tag{5.1.18}$$

From (5.1.18) and the assumption that s is maximal in the poset $(B,\ \langle u,\ \cdot,\ \cdot\rangle)$ it follows by (1b) that s is extremal.

(c) The assumption that X is arboric implies by 1.7.5 (medianity of arboric interval spaces) that X is median. By 4.5.1 (median geometric interval spaces), $(X,\ \langle u,\ \cdot,\ \cdot\rangle)$ is a meet semilattice and for $x,\ y \in X$, $x \wedge y = m\,(u,\ x,\ y)$. Therefore, it is to be proved that $x,\ y \in B$ implies $v := m\,(u,\ x,\ y) \in B$, i.e. not $\langle v,\ u,\ x\rangle$. Seeking a contradiction, assume $\langle v,\ u,\ x\rangle$. Thus,

$$\langle x,\ u,\ v\rangle\ . \tag{5.1.19}$$

The assumption that $v = m\,(u,\ x,\ y)$ entails:

$$\langle x,\ v,\ u\rangle\ . \tag{5.1.20}$$

From (5.1.19) and (5.1.20) it follows by 1.4.16 (Hedlíková's criterion for geometric interval spaces):

$$v = u\,. \tag{5.1.21}$$

Substituting (5.1.21) into the assumption $v = m\,(u,\ x,\ y)$, $u = m\,(u,\ x,\ y)$. In particular, $\langle x,\ u,\ y\rangle$, contradicting the assumption that $x,\ y$ belong to the same u-branch B .

(d) From the assumption that B is non-empty and finite it follows by (2c) that B has a greatest lower bound in $(B,\ \langle u,\ \cdot,\ \cdot\rangle)$, i.e. B has a least element with respect to $\langle u,\ \cdot,\ \cdot\rangle$.

(e) By 1.4.18(3) and (2) (gated sets) it suffices to prove that for each $s \in B$, u is a gate of s into $X \setminus B$, i.e. $(X \setminus B,\ \langle s,\ \cdot,\ \cdot\rangle)$ has least element u , i.e. for each $a \in X \setminus B$, $\langle s,\ u,\ a\rangle$.
Case 1. $a \neq u$. From this assumption and the assumption $a \in X \setminus B$ it follows that a belongs to a u-branch different from the u-branch B of s , i.e. a is not a u-neighbor of s , i.e. $\langle s,\ u,\ a\rangle$.
Case 2. $a = u$. $\langle s,\ u,\ u\rangle$ says $\langle s,\ u,\ a\rangle$.

(f) It is to be proved that B and $X \setminus B$ are convex.
Step 1. Proof that B is convex. For $a,\ x,\ b \in X$ it is to be proved that $\langle a,\ x,\ b\rangle$ and $a,\ b \in B$ imply $x \in B$, i.e. not $\langle a,\ u,\ x\rangle$. Seeking a contradiction, assume $\langle a,\ u,\ x\rangle$. From the assumptions $\langle a,\ u,\ x\rangle$ and $\langle a,\ x,\ b\rangle$ it follows that $\langle a,\ u,\ b\rangle$, contradicting the assumption that $a,\ b$ belong to the same u-branch B .
Step 2. Proof that $X \setminus B$ is convex. By (2e), $X \setminus B$ is gated. By 1.4.18(1) (gated sets), $X \setminus B$ is convex.

(g) Step 1. ($\Rightarrow$) Suppose $C \subseteq B$. Seeking a contradiction, assume $u \in C$. From this assumption and the assumption $C \subseteq B$ it follows that $u \in B$, contradicting 5.1.2(2) (neighbors relative to a point).
Step 2. ($\Leftarrow$) Suppose $u \notin C$. For $x \in X$ it is to be proved that $x \in C$ implies $x \in B$. Seeking a contradiction, assume $x \notin B$. This assumption and the assumptions that B is a u-branch and $v \in B$ imply:

$$\langle x,\, u,\, v\rangle \ . \tag{5.1.22}$$

From the assumptions that $x \in C$, C is a v-branch and $u \notin C$ it follows:

$$\langle x,\, v,\, u\rangle \ . \tag{5.1.23}$$

(5.1.22) and (5.1.23) imply by 1.4.16 (Hedlíková's criterion for geometric interval spaces):

$$v = u\,. \tag{5.1.24}$$

Substituting (5.1.24) into the assumption $v \in B$, $u \in B$, contradicting 5.1.2(2) (neighbors relative to a point).

(3)

(a) It suffices to prove $[x]_u \subseteq \langle u,\, x,\, \cdot\rangle$ and $[x]_u \supseteq \langle u,\, x,\, \cdot\rangle$.
Step 1. ($\subseteq$) For $y \in X$ it is to be proved that $y \in [x]_u$ implies $\langle u,\, x,\, y\rangle$. The assumption that X is arboric implies by 1.7.5 (medianity of arboric interval spaces) that X is median. In particular,

$$X \text{ is modular.} \tag{5.1.25}$$

From (5.1.25) and the assumption that x is adjacent to u it follows by 4.1.1(1) (modular interval spaces):

$$\langle y,\, u,\, x\rangle \ \text{or}\ \langle u,\, x,\, y\rangle \ . \tag{5.1.26}$$

The assumption $y \in [x]_u$ says:

$$\text{not}\ \langle y,\, u,\, x\rangle \ . \tag{5.1.27}$$

(5.1.26) and (5.1.27) imply $\langle u,\, x,\, y\rangle$.
Step 2. ($\supseteq$) follows by (2a).

(b) Step 1. ($\Rightarrow$) Assume that $\{x\}$ is a u-branch. In particular,

$$x \neq u\,. \tag{5.1.28}$$

Step 1.1. Proof that x is adjacent to u . From (5.1.28) it follows that it suffices to prove: For $y \in X$, if $\langle x,\, y,\, u\rangle$, then $y = x$ or $y = u$.
Case 1. $\langle x,\, u,\, y\rangle$. From this assumption and the assumption $\langle x,\, y,\, u\rangle$ it follows by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $y = u$.
Case 2. Not $\langle x,\, u,\, y\rangle$, i.e. y belongs to the same u-branch as x . This assumption and the assumption that $\{x\}$ is a u-branch imply $y \in \{x\}$, i.e. $y = x$.
Step 1.2. Proof that x is extremal. By 5.1.3 (extremal points in an arboric interval space) it is to be proved that $\deg(x) \leq 1$, i.e. there is at most one x-branch. It

suffices to prove that for $y \in X \setminus \{x\}$, $y \in [u]_x$, i.e. not $\langle y, x, u\rangle$. Seeking a contradiction, assume $\langle y, x, u\rangle$. The assumptions that $y \in X \setminus \{x\}$ and $\{x\}$ is a u-branch imply that y does not belong to the same u-branch as x , i.e.

$$\langle y, u, x\rangle \ . \tag{5.1.29}$$

From the assumption $\langle y, x, u\rangle$ and (5.1.29) it follows by 1.4.16 (Hedlíková's criterion for geometric interval spaces) that $x = u$, contradicting (5.1.28).

Step 2. ($\Leftarrow$) From the assumption that x is extremal and adjacent to u it is to be proved that $\{x\}$ is a u-branch, i.e. $[x]_u = \{x\}$, i.e. $[x]_u \subseteq \{x\}$. From the assumption that x is adjacent to u it follows by (3a) that $[x]_u = \langle u, x, \cdot\rangle$. Therefore, it suffices to prove $\langle u, x, \cdot\rangle \subseteq \{x\}$, i.e. for $y \in X$, if $\langle u, x, y\rangle$, then $x = y$. The assumptions that x is extremal and $\langle u, x, y\rangle$ imply:

$$x = u \text{ or } x = y \, . \tag{5.1.30}$$

The assumption that x is adjacent to u entails:

$$x \neq u \, . \tag{5.1.31}$$

From (5.1.30) and (5.1.31) it follows that $x = y$.

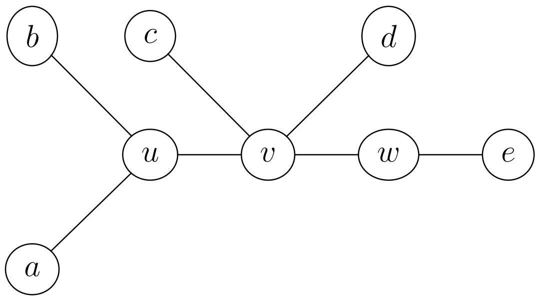
□

Let X be an arboric interval space and $u \in X$. By 5.1.3 (extremal points in an arboric interval space), u is extremal iff there is at most one u-branch. u is called *pre-extremal* iff u is non-extremal and there is at most one u-branch of size ≥ 2 . For pre-extremal $u \in X$, the *extremal neighborhood* of u is the set

$$\begin{aligned} \text{EN}(u) &:= \{u\} \cup \{x \in X | \, \{x\} \text{ is a } u\text{-branch.}\} \\ &= \{u\} \cup \{x \in X | x \text{ is extremal and adjacent to } u\, .\} \ , \end{aligned}$$

where equality holds by 5.1.5(3b) (arboric interval spaces). For example, in the following tree,

- ∘ The u-branches are $\{a\}$, $\{b\}$, $\{v, c, d, w, e\}$, so u is pre-extremal and $\text{EN}(u) = \{u, a, b\}$.
- ∘ The v-branches are $\{u, a, b\}$, $\{c\}$, $\{d\}$, $\{w, e\}$, so v is not pre-extremal although it is adjacent to the two extremal points c, d .
- ∘ The w-branches are $\{u, a, b, v, c, d\}$, $\{e\}$, so w is pre-extremal and $\text{EN}(w) = \{w, e\}$.



PROPOSITION 5.1.6. *(extremal neighborhoods) Let X be an arboric interval space.*

(1) *For $u \in X$, when u is pre-extremal, then:*
   (a) $|EN(u) \cap \partial_M(X)| \geq 2$.
   (b) *For $a,\ b \in X$, if $a \in EN(u)$ and $a \neq b$, then $\langle a,\ u,\ b\rangle$.*
   (c) *For $a,\ b,\ x \in X$, if $a,\ b \in EN(u)$, $a \neq b$ and $x \notin \{a,\ b\}$ then $u = m(x,\ a,\ b)$.*

(2) *For pre-extremal $u,\ v \in X$, if $u \neq v$, then $EN(u) \cap EN(v) = \emptyset$.*

PROOF.

(1)
   (a) The assumption that u is pre-extremal says:

$$u \text{ is non-extremal.} \tag{5.1.32}$$

$$\text{There is at most one } u\text{-branch of size } \geq 2\,. \tag{5.1.33}$$

(5.1.32) implies by 5.1.3 (extremal points in an arboric interval space) that $\deg(u) \geq 2$.
Case 1. $\deg(u) = 2$. By 5.1.4 (median boundary of an arboric interval space), $u \in \partial_M(X)$. Therefore,

$$u \in \text{EN}(u) \cap \partial_M(X)\ . \tag{5.1.34}$$

From the assumption $\deg(u) = 2$ and (5.1.33) it follows that there is at least one $x \in X$ such that

$$\{x\} \text{ is a } u\text{-branch.} \tag{5.1.35}$$

By 5.1.2(2) (neighbors relative to a point),

$$x \neq u\,. \tag{5.1.36}$$

(5.1.35) implies:

$$x \in \text{EN}(u) \tag{5.1.37}$$

From (5.1.35) it follows by 5.1.5(3b) (arboric interval spaces) that x is extremal. In particular,

$$x \in \partial_M(X)\ . \tag{5.1.38}$$

(5.1.37) and (5.1.38) together say:

$$x \in \text{EN}(u) \cap \partial_M(X)\ . \tag{5.1.39}$$

(5.1.34), (5.1.39) and (5.1.36) imply $|\text{EN}(u) \cap \partial_M(X)| \geq 2$.
Case 2. $\deg(u) > 2$. From this assumption and (5.1.33) it follows that there are at least two u-branches of size 1 , i.e. there are $x,\ y \in X$ such that

$$x \neq y\,, \tag{5.1.40}$$

$$\{x\}\,,\ \{y\} \text{ are } u\text{-branches.} \tag{5.1.41}$$

(5.1.41) implies:

$$x,\, y \in \mathrm{EN}\,(u)\ . \tag{5.1.42}$$

(5.1.41) implies by 5.1.5(3b) (arboric interval spaces) that $x,\, y$ are extremal. In particular,

$$x,\, y \in \partial_M\,(X)\ . \tag{5.1.43}$$

(5.1.42) and (5.1.43) together say:

$$x,\, y \in \mathrm{EN}\,(u) \cap \partial_M\,(X)\ . \tag{5.1.44}$$

From (5.1.44) and (5.1.40) it follows that $|\mathrm{EN}\,(u) \cap \partial_M\,(X)| \geq 2$.

(b) The assumption $a \in \mathrm{EN}\,(u)$ says that $a = u$ or $\{a\}$ is a u-branch.
Case 1. $\{a\}$ is a u-branch. From this assumption and the assumption $a \neq b$ it follows that b is not a u-neighbor of a , i.e. $\langle a,\, u,\, b\rangle$.
Case 2. $a = u$. $\langle u,\, u,\, b\rangle$ says $\langle a,\, u,\, b\rangle$.

(c) From the assumption $a \in \mathrm{EN}\,(u)$ and $a \neq b$ it follows by (1b):

$$\langle a,\, u,\, b\rangle\ . \tag{5.1.45}$$

The assumptions $a,\, b \in \mathrm{EN}\,(u)$ and $x \notin \{a,\, b\}$ imply by (1b):

$$\langle a,\, u,\, x\rangle\ , \tag{5.1.46}$$

$$\langle b,\, u,\, x\rangle\ . \tag{5.1.47}$$

(5.1.45), (5.1.46) and (5.1.47) together say that $u = m\,(x,\, a,\, b)$.

(2) $\mathrm{EN}\,(u) = \{u\} \cup \{x \in X | x \text{ is extremal and adjacent to } u\,.\}$, and $\mathrm{EN}\,(v) = \{v\} \cup \{x \in X | x \text{ is extremal and adjacent to } v\,.\}$. Therefore, setting

$$\begin{aligned}
A &:= \{u\} \cap \{v\}\ ,\\
B &:= \{u\} \cap \{x \in X | x \text{ is extremal and adjacent to } v\,.\}\ ,\\
C &:= \{x \in X | x \text{ is extremal and adjacent to } u\,.\} \cap \{v\}\ ,\\
D &:= \{x \in X | x \text{ is extremal and adjacent to } u\,.\}\\
&\quad\ \cap \{x \in X | x \text{ is extremal and adjacent to } v\,.\}\ ,
\end{aligned}$$

$\mathrm{EN}\,(u) \cap \mathrm{EN}\,(v) = A \cup B \cup C \cup D$. It is to be proved that $A,\, B,\, C,\, D$ are empty.
Step 1. From the assumption $u \neq v$ it follows that $A = \emptyset$.
Step 2. The assumption that u is pre-extremal entails that u is non-extremal. Consequently, $B = \emptyset$.
Step 3. The assumption that v is pre-extremal entails that v is non-extremal. Consequently, $C = \emptyset$.
Step 4. The assumption that X is arboric implies by 1.7.5 (medianity of arboric interval spaces) that X is median. In particular,

$$X \text{ is modular.} \tag{5.1.48}$$

From (5.1.48) and the assumption $u \neq v$ it follows by 4.1.1(5) (modular interval spaces) that $D = \emptyset$.

□

5.2. Arboric Topological Interval Spaces

The following proposition is for arboric interval spaces what 4.6.1 (medianity criterion for a compact geometric topological interval space) is for median interval spaces.

PROPOSITION 5.2.1. *(arboricity criterion for a compact geometric topological interval space) Let X be a compact geometric topological interval space. X is arboric iff it is interval-linear and interval-concatenable.*

PROOF. By 5.1.1 (arboricity criterion), it suffices to prove that X is interval-linear and median iff it is interval-linear and interval-concatenable, i.e. if X is interval-linear, then: X is median iff it is interval-concatenable.

Step 1. ($\Rightarrow$) Suppose that X is median. In particular, X is modular. By 4.1.1(3) (modular interval spaces), it is interval-concatenable.

Step 2. ($\Leftarrow$) Suppose that X is interval-concatenable. From the assumption that X is interval-linear it follows by 3.3.1(1) (interval-linear geometric interval spaces):

$$X \text{ is interval-convex.} \tag{5.2.1}$$

(5.2.1) and the assumption that X is interval-concatenable imply by 4.6.1 (medianity criterion for a compact geometric topological interval space) that X is median. □

PROPOSITION 5.2.2. *(branches in an arboric topological interval space) Let X be an arboric topological interval space. For $u \in X$ and B a u-branch,*

(1) *B is open.*
(2) *$B \cup \{u\}$ is closed.*

PROOF.

(1) There is a $b \in X$ such that
$$\begin{aligned} B &= \{x \in X | \text{not } \langle b,\, u,\, x\rangle\} \\ &= X \setminus \langle b,\, u,\, \cdot\rangle\ . \end{aligned} \tag{5.2.2}$$
By 2.4.4 (topological interval spaces),
$$\langle b,\, u,\, \cdot\rangle \text{ is closed.} \tag{5.2.3}$$
From (5.2.2) and (5.2.3) it follows that B is open.

(2) By 5.1.2(2) (neighbors relative to a point),
$$\text{The set of all } u\text{-branches is a partition of } X \setminus \{u\}\ . \tag{5.2.4}$$
(5.2.4) and the assumption that B is a u-branch imply
$$B \cup \{u\} = X \setminus \bigcup \{C | C \text{ is a } u\text{-branch and } C \neq B\}\ . \tag{5.2.5}$$
By (1), each u-branch is open. Therefore,
$$\bigcup \{C | C \text{ is a } u\text{-branch and } C \neq B\} \text{ is open.} \tag{5.2.6}$$

From (5.2.5) and (5.2.6) it follows that $B \cup \{u\}$ is closed.

$\square$

PROPOSITION 5.2.3. *(compact arboric topological interval spaces) Let X be a compact arboric topological interval space.*

(1) *For $u \in X$ and B a u-branch:*
  (a) *$(B, \langle u, \cdot, \cdot\rangle)$ has a maximal element.*
  (b) *There is an extremal point in X that belongs to B.*
  (c) *If $|B| \geq 2$, then $|B \cap \partial_M(X)| \geq 2$.*

(2) *For $a, b, p \in X$, if $\langle a, p, b\rangle$, then there is a $c \in \partial_M(X)$ such that $p = m(a, b, c)$.*

PROOF.

(1)
  (a) From the assumptions that X is arboric and B is a u-branch it follows by 5.2.2(2) (branches in an arboric topological interval space):

$$B \cup \{u\} \text{ is closed.} \tag{5.2.7}$$

Furthermore,

$$B \cup \{u\} \neq \emptyset\,. \tag{5.2.8}$$

The assumption that X is compact and (5.2.7) imply by 1.3.7 (compact topological spaces):

$$B \cup \{u\} \text{ is compact.} \tag{5.2.9}$$

(5.2.8) and (5.2.9) imply by 2.1.2 (compact topological posets) that there is a $b \in X$ such that

$$b \text{ is maximal in } (B \cup \{u\}, \langle u, \cdot, \cdot\rangle)\,. \tag{5.2.10}$$

The assumption that B is a u-branch entails $B \neq \emptyset$, i.e. there is a

$$b_0 \in B\,. \tag{5.2.11}$$

The assumption that B is a u-branch implies by 5.1.2(2) (neighbors relative to a point):

$$u \notin B\,. \tag{5.2.12}$$

From (5.2.12) and (5.2.11) it follows:

$$u \neq b_0\,. \tag{5.2.13}$$

From $\langle u, u, b_0\rangle$ and (5.2.13) it follows:

$$u \text{ is not maximal in } (B \cup \{u\}, \langle u, \cdot, \cdot\rangle)\,. \tag{5.2.14}$$

(5.2.10) and (5.2.14) imply:

$$b \neq u\,. \tag{5.2.15}$$

From (5.2.10) and (5.2.15) it follows that b is maximal in $(B, \langle u, \cdot, \cdot\rangle)$.

(b) By (1a), $(B, \langle u, \cdot, \cdot\rangle)$ has a maximal element m. By 5.1.5(2b) (arboric interval spaces), m is extremal.

(c) Case 1. For each $v \in B$, $\deg(v) \leq 2$. By 5.1.4 (median boundary of an arboric interval space), for each $v \in B$, $v \in \partial_M(X)$, i.e. $B \subseteq \partial_M(X)$. Therefore,

$$B \cap \partial_M(X) = B. \tag{5.2.16}$$

From (5.2.16) and the assumption $|B| \geq 2$ it follows that $|B \cap \partial_M(X)| \geq 2$.
Case 2. There is a $v \in B$ such that $\deg(v) \geq 3$. The assumptions that B is a u-branch and $v \in B$ imply $u \neq v$. Thus, the assumption $\deg(v) \geq 3$ can be expressed by saying that there are two v-branches C_1, C_2 different from $[u]_v$ such that $C_1 \neq C_2$. By (1b), there are extremal points c_1, c_2 such that

$$c_1 \in C_1 \text{ and } c_2 \in C_2. \tag{5.2.17}$$

In particular,

$$c_1, c_2 \in \partial_M(X). \tag{5.2.18}$$

The assumption that C_1, C_2 are v-branches different from $[u]_v$ implies:

$$u \notin C_1 \text{ and } u \notin C_2. \tag{5.2.19}$$

The assumptions that B is a u-branch, $v \in B$, C_1, C_2 are v-branches and (5.2.19) imply by 5.1.5(2g) (arboric interval spaces):

$$C_1 \subseteq B, C_2 \subseteq B. \tag{5.2.20}$$

From (5.2.17) and (5.2.20) it follows:

$$c_1, c_2 \in B. \tag{5.2.21}$$

(5.2.21) and (5.2.18) together say:

$$c_1, c_2 \in B \cap \partial_M(X). \tag{5.2.22}$$

The assumptions that C_1, C_2 are v-branches and $C_1 \neq C_2$ and (5.2.17) imply:

$$c_1 \neq c_2. \tag{5.2.23}$$

From (5.2.22) and (5.2.23) it follows that $|B \cap \partial_M(X)| \geq 2$.

(2) Case 1. $p \in \partial_M(X)$. From the assumption $\langle a, p, b\rangle$ it follows by 1.4.9(1) (median triangles) that $\begin{bmatrix} & p & \\ a & p & b \end{bmatrix}$ is a median triangle, i.e. $p = m(a, b, p)$.
Case 2. $p \notin \partial_M(X)$. This assumption and the assumption that X is arboric imply by 5.1.4 (median boundary of an arboric interval space) that $\deg(p) \geq 3$. Therefore, there is

$$B, \text{ a } p\text{-branch,} \tag{5.2.24}$$

such that

$$a, b \notin B. \tag{5.2.25}$$

From (5.2.24) it follows by (1b) that there is an extremal

$$c \in B\,. \tag{5.2.26}$$

In particular, c is median-extremal, i.e.

$$c \in \partial_M(X)\ .$$

(5.2.24), (5.2.25) and (5.2.26) imply:

$$\langle a,\, p,\, c\rangle\ , \tag{5.2.27}$$

$$\langle b,\, p,\, c\rangle\ . \tag{5.2.28}$$

The assumption $\langle a,\, p,\, b\rangle$, (5.2.27) and (5.2.28) together say that $p = m(a,\, b,\, c)$.

□

5.3. Arboric Metric Spaces

PROPOSITION 5.3.1. *(arboric metric spaces) Let X be an arboric metric space. For $u \in X$:*

(1) *For B a u-branch and finite $Y \subseteq X$, if the poset $(B,\, \langle u,\, \cdot,\, \cdot\rangle)$ has least element b , then:*
   (a) $\lambda_b^Y - \lambda_u^Y = (|Y \setminus B| - |Y \cap B|)\, d_{ub}$.
   (b) *If $|Y \cap B| < |Y \setminus B|$, then $\lambda_b^Y > \lambda_u^Y$.*
(2) *For $a,\, b \in X$, if u is pre-extremal, $a,\, b \in EN(u)$ and $a \neq b$, then:*
   (a) *For $x \in X$, if $x \notin \{a,\, b\}$, then $d_{xu} = d_{x,ab}$.*
   (b) *For $x \in X$, if $x \neq b$ and $a = u$, then $d_{xu} = d_{x,ab}$.*
   (c) *For finite $Y \subseteq X$, if each extremal point of X belongs to Y , then $\lambda_{ab}^Y = \lambda_u^Y$. In particular, if $\partial_M(X) \subseteq Y$, then $\lambda_{ab}^Y = \lambda_u^Y$.*

PROOF.

(1)
   (a) The assumption that $(B,\, \langle u,\, \cdot,\, \cdot\rangle)$ has least element b says that for $x \in B$, $\langle u,\, b,\, x\rangle$, i.e. $d_{ux} = d_{ub} + d_{bx}$. Therefore,

$$\begin{aligned}
\lambda_u^Y &= \sum_{x\in Y} d_{ux} \\
&= \left(\sum_{x\in Y\cap B} d_{ux}\right) + \left(\sum_{x\in Y\setminus B} d_{ux}\right) \\
&= \left(\sum_{x\in Y\cap B} (d_{ub} + d_{bx})\right) + \lambda_u^{Y\setminus B} \\
&= \left(\sum_{x\in Y\cap B} d_{ub}\right) + \left(\sum_{x\in Y\cap B} d_{bx}\right) + \lambda_u^{Y\setminus B} \\
&= |Y\cap B|\, d_{ub} + \lambda_b^{Y\cap B} + \lambda_u^{Y\setminus B}\,.
\end{aligned} \tag{5.3.1}$$

For $x \in Y \setminus B$, from the assumptions that $b \in B$ and B is a u-branch it follows that b is not a u-neighbor of x , i.e. $\langle b,\, u,\, x\rangle$, i.e. $d_{bx} = d_{bu} + d_{ux}$. Thus,

$$\begin{aligned}
\lambda_b^Y &= \sum_{x\in Y} d_{bx} \\
&= \left(\sum_{x\in Y\cap B} d_{bx}\right) + \left(\sum_{x\in Y\setminus B} d_{bx}\right) \\
&= \lambda_b^{Y\cap B} + \left(\sum_{x\in Y\setminus B} (d_{bu} + d_{ux})\right) \\
&= \lambda_b^{Y\cap B} + \left(\sum_{x\in Y\setminus B} d_{bu}\right) + \left(\sum_{x\in Y\setminus B} d_{ux}\right) \\
&= \lambda_b^{Y\cap B} + |Y\setminus B|\, d_{bu} + \lambda_u^{Y\setminus B} . \qquad (5.3.2)
\end{aligned}$$

(5.3.1) and (5.3.2) imply:

$$\begin{aligned}
\lambda_b^Y - \lambda_u^Y &= |Y\setminus B|\, d_{bu} - |Y\cap B|\, d_{ub} \\
&= |Y\setminus B|\, d_{ub} - |Y\cap B|\, d_{ub} \\
&= (|Y\setminus B| - |Y\cap B|)\, d_{ub} .
\end{aligned}$$

(b) follows from (1a).

(2)

(a) From the assumptions that u is pre-extremal, $a,\, b \in \text{EN}\,(u)$, $a \neq b$ and $x \notin \{a,\, b\}$ it follows by 5.1.6(1c) (extremal neighborhoods) that $u = m\,(x,\, a,\, b)$, i.e. $\begin{bmatrix} & b & \\ x & u & a \end{bmatrix}$ is a median triangle. By 1.4.20 (median triangles in a metric space), $d_{xu} = d_{x,ab}$.

(b) Case 1. $x \neq a$. This assumption and the assumption $x \neq b$ together say that $x \notin \{a,\, b\}$. By (2a), $d_{xu} = d_{x,ab}$.
Case 2. $x = a$. It is to be proved that $d_{au} = d_{a,ab}$. From the assumption $a = u$ it follows that $d_{au} = 0$, and by 1.8.2(4) (point-pair modular distance), $d_{a,ab} = 0$. Consequently, $d_{au} = d_{a,ab}$.

(c) The assumption $a,\, b \in \text{EN}\,(u)$ says:

$$a,\, b \in \{u\} \cup \{x \in X | x \text{ is extremal and adjacent to } u\, .\} \, , \qquad (5.3.3)$$

Case 1. $a \neq u$ and $b \neq u$. From this assumption and (5.3.3) it follows that $a,\, b \in \{x \in X | x \text{ is extremal and adjacent to } u\, .\}$. In particular,

$$a,\, b \text{ are extremal.} \qquad (5.3.4)$$

(5.3.4) and the assumption that each extremal point of X belongs to Y imply:

$$a,\, b \in Y\, . \qquad (5.3.5)$$

From the assumptions that u is pre-extremal, $a \in \mathrm{EN}\,(u)$ and $a \neq b$ it follows by 5.1.6(1b) (extremal neighborhoods) that $\langle a,\, u,\, b\rangle$, i.e. $d_{au} + d_{ub} = d_{ab}$. Therefore,

$$d_{au} + d_{bu} = d_{ab}\,. \tag{5.3.6}$$

The assumptions that u is pre-extremal, $a,\, b \in \mathrm{EN}\,(u)$ and $a \neq b$ imply by (2a):

$$\text{For each } x \in Y \setminus \{a,\, b\}\ ,\ d_{xu} = d_{x,ab}\,. \tag{5.3.7}$$

From (5.3.5), (5.3.6) and (5.3.7) it follows:

$$\begin{aligned}
\lambda_u^Y &= \sum_{x \in Y} d_{xu} \\
&= d_{au} + d_{bu} + \sum_{x \in Y \setminus \{a,b\}} d_{xu} \\
&= d_{ab} + \sum_{x \in Y \setminus \{a,b\}} d_{x,ab} \\
&= \lambda_{ab}^Y\,.
\end{aligned}$$

Case 2. $a = u$ or $b = u$, without loss of generality $a = u$. Substituting the assumption $a = u$ into the assumption $b \neq a$,

$$b \neq u\,. \tag{5.3.8}$$

From (5.3.3) and (5.3.8) it follows that b is extremal. In particular,

$$b \text{ is extremal.} \tag{5.3.9}$$

(5.3.9) and the assumption that each extremal point of X belongs to Y imply:

$$b \in Y\,. \tag{5.3.10}$$

From the assumptions that u is pre-extremal, $a,\, b \in \mathrm{EN}\,(u)$, $a \neq b$ and $a = u$ it follows by (2b):

$$\text{For each } x \in Y \setminus \{b\}\ ,\ d_{xu} = d_{x,ab}\,. \tag{5.3.11}$$

By 1.8.2(4) (point-pair modular distance),

$$d_{b,ab} = 0\,. \tag{5.3.12}$$

From (5.3.10), the asumption that $a = u$, (5.3.11) and (5.3.12) it follows :

$$\begin{aligned}\lambda_u^Y &= \sum_{x\in Y} d_{xu} \\ &= d_{bu} + \sum_{x\in Y\setminus\{b\}} d_{xu} \\ &= d_{ba} + 0 + \sum_{x\in Y\setminus\{b\}} d_{x,ab} \\ &= d_{ab} + d_{b,ab} + \sum_{x\in Y\setminus\{b\}} d_{x,ab} \\ &= d_{ab} + \sum_{x\in Y} d_{x,ab} \\ &= \lambda_{ab}^Y \,.\end{aligned}$$

□

PROPOSITION 5.3.2. *(compact arboric metric spaces) Let X be a compact arboric metric space. For finite $Y \subseteq X$ and $a,\, b,\, u \in X$, if $\partial_M(X) \subseteq Y$, $a,\, b \in Y$, $\langle a,\, u \neq b\rangle$ and $\left|[b]_u\right| \geq 2$, then $\lambda_{ab}^Y < \lambda_u^Y$.*

PROOF. From the assumption $\left|[b]_u\right| \geq 2$ it follows by 5.2.3(1c) (compact arboric topological interval spaces) that

$$\left|[b]_u \cap \partial_M(X)\right| \geq 2\,. \tag{5.3.13}$$

The assumption $\partial_M(X) \subseteq Y$ implies:

$$[b]_u \cap \partial_M(X) \subseteq [b]_u \cap Y\,. \tag{5.3.14}$$

(5.3.14) and (5.3.13) imply $\left|[b]_u \cap Y\right| \geq 2$. Therefore, there is a

$$y \in [b]_u \cap Y \tag{5.3.15}$$

such that

$$y \neq b\,. \tag{5.3.16}$$

(5.3.15) says:

$$y \in [b]_u\,, \tag{5.3.17}$$

$$y \in Y\,. \tag{5.3.18}$$

The assumption that $\langle a,\, u,\, b\rangle$ says:

$$a \notin [b]_u\,. \tag{5.3.19}$$

From (5.3.17) and (5.3.19) it follows:

$$y \neq a\,. \tag{5.3.20}$$

(5.3.18), (5.3.20) and (5.3.16) together say:

$$y \in Y \setminus \{a,\, b\}\ . \tag{5.3.21}$$

(5.3.17) says:

$$\text{not } \langle y,\, u,\, b\rangle\ . \tag{5.3.22}$$

The assumptions $a,\, b \in Y$ and $\langle a,\, u,\, b\rangle$, (5.3.21) and (5.3.22) imply by 1.8.2(10c) (point-pair modular distance) that $\lambda^Y_{ab} < \lambda^Y_u$. □

The following theorem places the neighbor-joining method from [**41**] for reconstructing a weighted tree from the distances between its leaves in the conceptual framework of arboric metric spaces.

THEOREM 5.3.3. *(finite arboric metric spaces) Let X be a finite arboric metric space. For $Y \subseteq X$, if $\partial_M(X) \subseteq Y$, then:*

(1) *For $u \in X$, if u is non-extremal with greatest λ^Y_u , then u is pre-extremal.*
(2) *For $a,\, b \in X$, if $a,\, b \in Y$ and $a \neq b$ with greatest λ^Y_{ab} , then:*
 (a) *For $u \in X$, if u is non-extremal and $\langle a,\, u,\, b\rangle$, then u is non-extremal with greatest λ^Y_u .*
 (b) *If $|X| \geq 3$, then there is exactly one pre-extremal $u \in X$ such that $a,\, b \in EN(u)$.*
 (c) *If $|X| \geq 3$, then for the unique pre-extremal $u \in X$ such that $a,\, b \in EN(u)$, $EN(u) = \{u,\, a\} \cup \left\{y \in Y | \lambda^Y_{ay} = \lambda^Y_{ab}\right\}$.*

PROOF. From the assumptions that X is finite and $Y \subseteq X$ it follows:

$$Y \text{ is finite.} \tag{5.3.23}$$

(1) The assumption that u is non-extremal implies that it suffices to prove that there is at most one u-branch of size ≥ 2 . Seeking a contradiction, suppose that there are at least two u-branches of size ≥ 2 , i.e. there are a $u-$branch B with $|B| \geq 2$ with smallest $|Y \cap B|$ and a u-branch C with $C \neq B$ and $|C| \geq 2$. In particular,

$$|Y \cap C| \geq |Y \cap B|\ . \tag{5.3.24}$$

From the assumption that X is finite it follows:

$$B \text{ is finite.} \tag{5.3.25}$$

The assumption that B is a u-branch and (5.3.25) imply by 5.1.5(2d) (arboric interval spaces) that there is

$$b\,,\ \text{least element of } (B,\, \langle u,\, \cdot,\, \cdot\rangle)\ . \tag{5.3.26}$$

From the assumption $|B| \geq 2$ if follows that there is a

$$c \in B \tag{5.3.27}$$

such that

$$c \neq b\,. \tag{5.3.28}$$

In order to obtain a contradiction to the maximality of λ_u^X , it suffices to prove that b is non-extremal and $\lambda_b^Y > \lambda_u^Y$.
Step 1. Proof that b is non-extremal. (5.3.26) and (5.3.27) imply:

$$\langle u,\, b,\, c\rangle \ . \tag{5.3.29}$$

From the assumptions that b is an element of B and B is a u-branch it follows by 5.1.2(2) (neighbors relative to a point):

$$b \neq u\,. \tag{5.3.30}$$

(5.3.29), (5.3.30) and (5.3.28) imply that b is non-extremal.
Step 2. Proof that $\lambda_b^Y > \lambda_u^Y$. The assumption that B is a u-branch, (5.3.23) and (5.3.26) imply by 5.3.1(1b) (arboric metric spaces) that it suffices to prove $|Y \cap B| < |Y \setminus B|$.
Case 1. B and C are the only u-branches, i.e.

$$\deg(u) = 2\,. \tag{5.3.31}$$

By 5.1.4 (median boundary of an arboric interval space),

$$u \in \partial_M(X) \ . \tag{5.3.32}$$

From (5.3.32) and the assumption $Y \supseteq \partial_M(X)$ it follows that $u \in Y$, i.e. $Y \cap \{u\} = \{u\}$. Therefore,

$$|Y \cap \{u\}| = 1\,. \tag{5.3.33}$$

The assumptions that B, C are u-branches and $C \neq B$ imply by 5.1.2(2) (neighbors relative to a point) that B, C, $\{u\}$ are disjoint from each other. In particular, their respective subsets $Y \cap B$, $Y \cap C$, $Y \cap \{u\}$ are disjoint from each other, i.e.

$$Y \setminus B \supseteq (Y \cap C) \cup (Y \cap \{u\}) \ , \tag{5.3.34}$$

$$Y \cap C,\ Y \cap \{u\} \ \text{are disjoint.} \tag{5.3.35}$$

(5.3.34), (5.3.35), (5.3.24) and (5.3.33) imply:

$$\begin{aligned} |Y \setminus B| &\geq |Y \cap C| + |Y \cap \{u\}| \\ &\geq |Y \cap B| + 1\,. \end{aligned}$$

Consequently, $|Y \setminus B| > |Y \cap B|$.
Case 2. There is a u-branch D different from B and C . From the assumption that X is a finite metric space it follows:

$$\text{The topology determined by the metric of } X \text{ is discrete.} \tag{5.3.36}$$

The assumption that X is finite and arboric and (5.3.36) imply by 2.4.3 (discrete topological interval spaces):

$$X \text{ is a compact arboric topological interval space.} \tag{5.3.37}$$

From (5.3.37) and the assumption that D is a u-branch it follows by 5.2.3(1b) (compact arboric topological interval spaces) that there is an extremal $d \in X$ such that $d \in D$. Thus,

$$d \in \partial_M (X) \cap D . \tag{5.3.38}$$

(5.3.38) and the assumption $\partial_M (X) \subseteq Y$ imply $d \in Y \cap D$. Therefore,

$$|Y \cap D| \geq 1 . \tag{5.3.39}$$

The assumptions that $B,\, C,\, D$ are u-branches, $C \neq B$ and D is different from B and C imply that $B,\, C,\, D$ are disjoint from each other. In particular, their respective subsets $Y \cap B,\, Y \cap C,\, Y \cap D$ are disjoint from each other, i.e.

$$Y \setminus B \supseteq (Y \cap C) \cup (Y \cap D) \,, \tag{5.3.40}$$

$$Y \cap C,\, Y \cap D \text{ are disjoint.} \tag{5.3.41}$$

From (5.3.40), (5.3.41), (5.3.24) and (5.3.39) it follows:

$$\begin{aligned} |Y \setminus B| &\geq |Y \cap C| + |Y \cap D| \\ &\geq |Y \cap B| + 1 . \end{aligned}$$

Consequently, $|Y \setminus B| > |Y \cap B|$.

(2)

(a) The assumptions $a,\, b \in Y$ and $\langle a,\, u,\, b\rangle$ imply by 1.8.2(10a) (point-pair modular distance):

$$\lambda_{ab}^{Y} \leq \lambda_{u}^{Y} . \tag{5.3.42}$$

From the assumptions that u is non-extremal and X is finite it follows that there is a $v \in X$ such that

$$v \text{ is non-extremal with greatest } \lambda_{v}^{Y} . \tag{5.3.43}$$

By (1),

$$v \text{ is pre-extremal.} \tag{5.3.44}$$

The assumption that u is non-extremal and (5.3.43) imply:

$$\lambda_{u}^{Y} \leq \lambda_{v}^{Y} . \tag{5.3.45}$$

From (5.3.44) it follows by 5.1.6(1a) (extremal neighborhoods):

$$|\mathrm{EN}\,(v) \cap \partial_M (X)| \geq 2 ,$$

i.e. there are $c,\, d \in \mathrm{EN}\,(v) \cap \partial_M (X)$ such that

$$c \neq d . \tag{5.3.46}$$

In particular,

$$c,\, d \in \mathrm{EN}\,(v) \,, \tag{5.3.47}$$

$$c,\, d \in \partial_M (X) . \tag{5.3.48}$$

(5.3.44), (5.3.47), (5.3.46) and the assumption $\partial_M(X) \subseteq Y$ imply by 5.3.1(2c) (arboric metric spaces):

$$\lambda_v^Y = \lambda_{cd}^Y . \tag{5.3.49}$$

(5.3.48) and the assumption $\partial_M(X) \subseteq Y$ imply:

$$c,\, d \in Y . \tag{5.3.50}$$

From (5.3.50), (5.3.46) and the maximality of λ_{ab}^Y it follows:

$$\lambda_{cd}^Y \le \lambda_{ab}^Y . \tag{5.3.51}$$

(5.3.42), (5.3.45), (5.3.49) and (5.3.51) imply:

$$\lambda_v^Y = \lambda_u^Y . \tag{5.3.52}$$

From the assumption that u is non-extremal, (5.3.52) and (5.3.43) it follows that u is non-extremal with greatest λ_u^Y .

(b) Step 1. Existence. The assumption that X is arboric implies by 1.7.5 (medianity of arboric interval spaces) that X is median. In particular,

$$X \text{ is modular.} \tag{5.3.53}$$

From (5.3.53), the assumptions $a \neq b$ and $|X| \ge 3$ it follows by 4.1.1(2) (modular interval spaces) that there is a $u \in X$ such that

$$u \text{ is non-extremal,} \tag{5.3.54}$$

$$\langle a,\, u,\, b\rangle \ . \tag{5.3.55}$$

(5.3.23), the assumptions that $a,\, b \in Y$ and $a \neq b$ with greatest λ_{ab}^Y, (5.3.54) and (5.3.55) imply by (2a) that u is non-extremal with greatest λ_u^Y . By (1),

$$u \text{ is pre-extremal.} \tag{5.3.56}$$

It suffices to prove $a,\, b \in \mathrm{EN}(u)$. Seeking a contradiction, assume $a \notin \mathrm{EN}(u)$ or $b \notin \mathrm{EN}(u)$, without loss of generality $b \notin \mathrm{EN}(u)$, i.e. $b \neq u$ and $\{b\}$ is not a u-branch, i.e.

$$|[b]_u| \ge 2 . \tag{5.3.57}$$

From the assumptions $\partial_M(X) \subseteq Y$ and $a,\, b \in Y$, (5.3.55), $b \neq u$ and (5.3.57) it follows by 5.3.2 (compact arboric metric spaces):

$$\lambda_{ab}^Y < \lambda_u^Y . \tag{5.3.58}$$

(5.3.56) implies by 5.1.6(1a) (extremal neighborhoods) that $|\mathrm{EN}(u) \cap \partial_M(X)| \ge 2$, i.e. there are $a',\, b' \in \mathrm{EN}(u) \cap \partial_M(X)$ such that

$$a' \neq b' . \tag{5.3.59}$$

In particular,

$$a',\, b' \in \mathrm{EN}(u) \ , \tag{5.3.60}$$

$$a',\, b' \in \partial_M(X) \ . \tag{5.3.61}$$

(5.3.56), (5.3.60), (5.3.59) and the assumption $\partial_M(X) \subseteq Y$ imply by 5.3.1(2c) (arboric metric spaces) that

$$\lambda_u^Y = \lambda_{a'b'}^Y . \tag{5.3.62}$$

Substituting (5.3.62) into (5.3.58),

$$\lambda_{ab}^Y < \lambda_{a'b'}^Y . \tag{5.3.63}$$

(5.3.61) and the assumption $\partial_M(X) \subseteq Y$ imply:

$$a',\, b' \in Y . \tag{5.3.64}$$

(5.3.63) together with (5.3.64) and (5.3.59) contradict the maximality of λ_{ab}^Y.
Step 2. Uniqueness follows by 5.1.6(2) (extremal neighborhoods).

(c) From the assumptions that u is pre-extremal, $a,\, b \in \mathrm{EN}(u)$ and $a \neq b$, (5.3.23) and the assumption $\partial_M(X) \subseteq Y$ it follows by 5.3.1(2c) (arboric metric spaces):

$$\lambda_{ab}^Y = \lambda_u^Y . \tag{5.3.65}$$

It is to be proved that $\mathrm{EN}(u) = \{u,\, a\} \cup \left\{y \in Y | \lambda_{ay}^Y = \lambda_{ab}^Y\right\}$.
Step 1. ($\subseteq$) For $y \in X$ it is to be proved: If $y \in \mathrm{EN}(u)$, then $y \in \{u,\, a\}$ or $\left(y \in Y \text{ and } \lambda_{ay}^Y = \lambda_{ab}^Y\right)$, i.e. $y \notin \{u,\, a\}$ implies $y \in Y$ and $\lambda_{ay}^Y = \lambda_{ab}^Y$.
Step 1.1. Proof that $y \in Y$. The assumption $y \notin \{u,\, a\}$ entails:

$$y \neq u . \tag{5.3.66}$$

The assumption $y \in \mathrm{EN}(u)$ says:

$$y \in \{u\} \cup \{x \in X | x \text{ is extremal and adjacent to } u .\} . \tag{5.3.67}$$

(5.3.67) and (5.3.66) imply that y is extremal. In particular,

$$y \in \partial_M(X) . \tag{5.3.68}$$

(5.3.68) and the assumption $\partial_M(X) \subseteq Y$ imply $y \in Y$.
Step 1.2. Proof that $\lambda_{ay}^Y = \lambda_{ab}^Y$. The assumption $y \notin \{u,\, a\}$ entails:

$$a \neq y . \tag{5.3.69}$$

From the assumptions that u is pre-extremal and $a,\, y \in \mathrm{EN}(u)$, (5.3.69), (5.3.23) and the assumption $\partial_M(X) \subseteq Y$ it follows by 5.3.1(2c) (arboric metric spaces):

$$\lambda_{ay}^Y = \lambda_u^Y . \tag{5.3.70}$$

(5.3.65) and (5.3.70) imply $\lambda_{ay}^Y = \lambda_{ab}^Y$.
Step 2. ($\supseteq$) For $y \in X$ it is to be proved that $y \in \{u,\, a\}$ or $\left(y \in Y \text{ and } \lambda_{ay}^Y = \lambda_{ab}^Y\right)$ implies $y \in \mathrm{EN}(u)$.
Case 1. $y \in \{u,\, a\}$. From $u \in \mathrm{EN}(u)$ and the assumption $a \in \mathrm{EN}(u)$ it follows that $y \in \mathrm{EN}(u)$.
Case 2. $y \in Y$ and $\lambda_{ay}^Y = \lambda_{ab}^Y$ and $y \notin \{u,\, a\}$. Seeking a contradiction, assume $y \notin \mathrm{EN}(u)$, i.e. $y \neq u$ and $\{y\}$ is not a u-branch, i.e.

$$\left|[y]_u\right| \geq 2 . \tag{5.3.71}$$

The assumption $y \notin \{u,\, a\}$ entails:

$$y \neq a\,. \tag{5.3.72}$$

From the assumption that u is pre-extremal, (5.3.72) and the assumption $a \in \mathrm{EN}\,(u)$ it follows by 5.1.6(1b) (extremal neighborhoods):

$$\langle a,\, u \neq y\rangle\ . \tag{5.3.73}$$

(5.3.73), the assumptions $\partial_M\,(X) \subseteq Y$, $a \in Y$ and $y \in Y$ and (5.3.71) imply by 5.3.2 (compact arboric metric spaces):

$$\lambda^Y_{ay} < \lambda^Y_u\ . \tag{5.3.74}$$

Substituting (5.3.65) into (5.3.74), $\lambda^Y_{ay} < \lambda^Y_{ab}$, contradicting the assumption $\lambda^Y_{ay} = \lambda^Y_{ab}$.

□

CHAPTER 6

Quadrimodular and Quadrimedian Spaces

In this chapter the following new concepts are introduced:

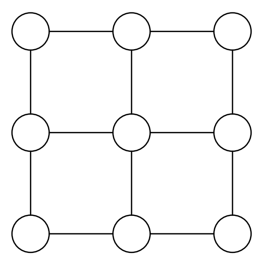
- ∘ quadrimodular interval space
- ∘ quadrimedian interval space
- ∘ subquadrimedian metric space
- ∘ geodesic quadrimedian closure of a subquadrimedian metric space

The following main new results are proved:

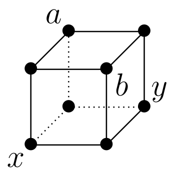
- ∘ 6.2.3 (quadrimodularity criterion for metric spaces)
- ∘ 6.4.1 (convex closure of the median boundary)
- ∘ 6.5.3 (existence and structural uniqueness of geodesic quadrimedian closure)
- ∘ 6.6.1 (median closure of the median boundary)
- ∘ 6.6.2 (compact arboric determination by the median boundary)

6.1. Quadrimodular and Quadrimedian Interval Spaces

Let X be an interval space. X is called *quadrimodular* iff for all $x,\ y,\ a,\ b \in X$, at least one of the three matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular. X is called *quadrimedian* iff it is quadrimodular and median. The following graph is quadrimedian.



The edge graph of a cube is median, but not quadrimodular: None of the three matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular.



When X is geometric, $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle in X and $Y := \{x,\, y,\, a,\, b,\, s,\, t,\, u,\, v\}$, then Y is quadrimedian, and the median closure of $\{x,\, y,\, a,\, b\}$ in X equals $\{x,\, y,\, a,\, b,\, s,\, t,\, u,\, v\}$. Further examples of quadrimedian and quadrimodular interval spaces and an example of a median metric space of size 8 that is not quadrimodular are provided by and presented after the next five propositions.

PROPOSITION 6.1.1. *(quadrimedianity of arboric interval spaces) Each arboric interval space is quadrimedian.*

PROOF. Let X be an arboric interval space. By 1.7.5 (medianity of arboric interval spaces), X is median. It remains to be proved that for $x,\, y,\, a,\, b \in X$, at least one of the three matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular. By 5.1.1 (arboricity criterion) , $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ or $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$ is quadrimodular. $\square$

For example, by 6.1.1 (quadrimedianity of arboric interval spaces) the metric space $(\mathbb{R},\, |\cdot - \cdot|)$ is quadrimedian, and each tree is quadrimedian.

6.1.1 (quadrimedianity of arboric interval spaces) may be used implicitly by applying results on quadrimodular and quadrimedian interval spaces to arboric interval spaces.

PROPOSITION 6.1.2. *(product of two arboric interval spaces) The product of two arboric interval spaces is geometric and quadrimedian.*

PROOF. Let $X_1,\, X_2$ be arboric interval spaces. In particular, $X_1,\, X_2$ are geometric, and by 1.7.5 (medianity of arboric interval spaces), $X_1,\, X_2$ are median. By 1.4.19 (product of geometric interval spaces) and 1.6.5 (product of median interval spaces), $X_1 \times X_2$ is geometric and median. It remains to be proved that $X_1 \times X_2$ is quadrimodular, i.e. for $x,\, y,\, a,\, b \in X_1 \times X_2$, at least one of the three matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular. The assumption that $X_1,\ X_2$ are arboric entails that in X_1 ,

$$\left(\left\langle \begin{matrix} a_1 \\ x_1 \end{matrix} - \begin{matrix} b_1 \\ y_1 \end{matrix} \right\rangle \text{ or } \left\langle \begin{matrix} a_1 \\ x_1 \end{matrix} \mid \begin{matrix} b_1 \\ y_1 \end{matrix} \right\rangle\right) \text{ or } \left\langle \begin{matrix} b_1 \\ x_1 \end{matrix} - \begin{matrix} a_1 \\ y_1 \end{matrix} \right\rangle ,$$

and in X_2 ,

$$\left(\left\langle \begin{matrix} a_2 \\ x_2 \end{matrix} - \begin{matrix} b_2 \\ y_2 \end{matrix} \right\rangle \text{ or } \left\langle \begin{matrix} a_2 \\ x_2 \end{matrix} \mid \begin{matrix} b_2 \\ y_2 \end{matrix} \right\rangle\right) \text{ or } \left\langle \begin{matrix} b_2 \\ x_2 \end{matrix} - \begin{matrix} a_2 \\ y_2 \end{matrix} \right\rangle .$$

Case 1. $\left(\left\langle \begin{matrix} a_1 \\ x_1 \end{matrix} - \begin{matrix} b_1 \\ y_1 \end{matrix} \right\rangle \text{ or } \left\langle \begin{matrix} a_1 \\ x_1 \end{matrix} \mid \begin{matrix} b_1 \\ y_1 \end{matrix} \right\rangle\right)$ and $\left(\left\langle \begin{matrix} a_2 \\ x_2 \end{matrix} - \begin{matrix} b_2 \\ y_1 \end{matrix} \right\rangle \text{ or } \left\langle \begin{matrix} a_2 \\ x_2 \end{matrix} \mid \begin{matrix} b_2 \\ y_2 \end{matrix} \right\rangle\right)$. In particular, $\begin{bmatrix} a_1 & b_1 \\ x_1 & y_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ x_2 & y_2 \end{bmatrix}$ are quadrimodular. By 2.2.7(2) (median quadrangles in a product of interval spaces), $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular.

Case 2. $\left(\left\langle \begin{matrix} a_1 \\ x_1 \end{matrix} - \begin{matrix} b_1 \\ y_1 \end{matrix} \right\rangle \text{ or } \left\langle \begin{matrix} a_1 \\ x_1 \end{matrix} \mid \begin{matrix} b_1 \\ y_1 \end{matrix} \right\rangle\right)$ and $\left\langle \begin{matrix} b_2 \\ x_2 \end{matrix} - \begin{matrix} a_2 \\ y_2 \end{matrix} \right\rangle$.

Case 2.1. $\left\langle \begin{matrix} a_1 \\ x_1 \end{matrix} - \begin{matrix} b_1 \\ y_1 \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} b_2 \\ x_2 \end{matrix} - \begin{matrix} a_2 \\ y_2 \end{matrix} \right\rangle$. By 2.2.8(2) (product of two interval spaces), $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular.

Case 2.2. $\left\langle \begin{matrix} a_1 \\ x_1 \end{matrix} \mid \begin{matrix} b_1 \\ y_1 \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} b_2 \\ x_2 \end{matrix} - \begin{matrix} a_2 \\ y_2 \end{matrix} \right\rangle$. By 2.2.4(2) (symmetries of quadrimodularity properties), $\left\langle \begin{matrix} b_1 \\ x_1 \end{matrix} \mid \begin{matrix} a_1 \\ y_1 \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} b_2 \\ x_2 \end{matrix} - \begin{matrix} a_2 \\ y_2 \end{matrix} \right\rangle$. In particular, $\begin{bmatrix} b_1 & a_1 \\ x_1 & y_1 \end{bmatrix}$ and $\begin{bmatrix} b_2 & a_2 \\ x_2 & y_2 \end{bmatrix}$ are quadrimodular. By 2.2.7(2) (median quadrangles in a product of interval spaces), $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$ is quadrimodular.

Case 3. $\left\langle \begin{matrix} b_1 \\ x_1 \end{matrix} - \begin{matrix} a_1 \\ y_1 \end{matrix} \right\rangle$ and $\left(\left\langle \begin{matrix} a_2 \\ x_2 \end{matrix} - \begin{matrix} b_2 \\ y_2 \end{matrix} \right\rangle \text{ or } \left\langle \begin{matrix} a_2 \\ x_2 \end{matrix} \mid \begin{matrix} b_2 \\ y_2 \end{matrix} \right\rangle\right)$. This case is analogous to case 2.

Case 4. $\left\langle \begin{matrix} b_1 \\ x_1 \end{matrix} - \begin{matrix} a_1 \\ y_1 \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} b_2 \\ x_2 \end{matrix} - \begin{matrix} a_2 \\ y_2 \end{matrix} \right\rangle$. This case is analogous to case 1. □

PROPOSITION 6.1.3. *(quadrimedianity of the plane)* $(\mathbb{R}^2,\ \|\cdot - \cdot\|_1)$ *is quadrimedian.*

PROOF. By 1.7.3(2) (arboricity of the real line),

$$(\mathbb{R},\ |\cdot - \cdot|) \text{ is arboric.} \tag{6.1.1}$$

By 1.4.23(2) (sum metric),

$$\left(\mathbb{R}^2,\ \langle\cdot,\ \cdot,\ \cdot\rangle_{\|\cdot - \cdot\|_1}\right) \text{ is the product of } \left(\mathbb{R},\ \langle\cdot,\ \cdot,\ \cdot\rangle_{|\cdot - \cdot|}\right) \text{ with itself.} \tag{6.1.2}$$

From (6.1.1) and (6.1.2) it follows by 6.1.2 (product of two arboric interval spaces) that $(\mathbb{R}^2,\ \|\cdot - \cdot\|_1)$ is quadrimedian. □

PROPOSITION 6.1.4. *(quadrimodularity of injective metric spaces) Each injective metric space is quadrimodular.*

PROOF. Let Y be an injective metric space. For $x,\, y,\, a,\, b \in Y$ it is to be proved that at least one of the matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular. Let without loss of generality $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$. It suffices to prove that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular. Setting

$$A := \{x,\, y,\, a,\, b\} \ ,$$

from the assumption

$$\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya} \, .$$

it follows by 2.6.2 (quadrimodular matrix representation) that there is

$$i\,,\ \text{an isometric map from } A \text{ into } \left(\mathbb{R}^2,\, \|\cdot - \cdot\|_1\right) \ , \tag{6.1.3}$$

such that

$$\left\langle \begin{matrix} i\,(a) & i\,(b) \\ i\,(x) & i\,(y) \end{matrix} \right\rangle \text{ is quadrimodular,} \tag{6.1.4}$$

i.e. there are $s',\, t',\, u',\, v' \in \mathbb{R}^2$ such that $\begin{bmatrix} i\,(a) & & & i\,(b) \\ & u' & v' & \\ & s' & t' & \\ i\,(x) & & & i\,(y) \end{bmatrix}$ is a median quadrangle in $(\mathbb{R}^2,\, \|\cdot - \cdot\|_1)$. Setting

$$T' := \{i\,(x)\,,\, i\,(y)\,,\, i\,(a)\,,\, i\,(b)\,,\, s',\, t',\, u',\, v'\} \ ,$$

by 2.2.2(3) (median quadrangles),

$$i\,(A) \text{ is an interval-spanning set in } (T',\, \|\cdot - \cdot\|_1) \ . \tag{6.1.5}$$

From (6.1.3) it follows:

$$i^{-1} \text{ is an isometric map from } (i\,(A)\,,\, \|\cdot - \cdot\|_1) \text{ into } Y \, . \tag{6.1.6}$$

The assumption that Y is injective, (6.1.5) and (6.1.6) imply by 1.4.27 (isometric maps into an injective metric space) that there is a

$$g\,,\ \text{an isometric map from } (T',\, \|\cdot - \cdot\|_1) \text{ into } Y \tag{6.1.7}$$

such that g extends i^{-1} , i.e.:

$$\text{For each } z \in A\,,\ g\,(i\,(z)) = z\, . \tag{6.1.8}$$

From (6.1.7) it follows by 1.4.12 (isometric maps) that g is an embedding from $\left(T',\, \langle\cdot,\,\cdot,\,\cdot\rangle_{\|\cdot - \cdot\|_1}\right)$ into $(Y,\, \langle\cdot,\,\cdot,\,\cdot\rangle_d)$. In particular,

$$g \text{ is a homomorphism of } \left(T',\, \langle\cdot,\,\cdot,\,\cdot\rangle_{\|\cdot - \cdot\|_1}\right) \text{ into } (Y,\, \langle\cdot,\,\cdot,\,\cdot\rangle_d) \ . \tag{6.1.9}$$

(6.1.9) and (6.1.4) imply by 2.2.6(2) (homomorphic image of a median quadrangle):

$$\begin{bmatrix} g(i(a)) & g(i(b)) \\ g(i(x)) & g(i(y)) \end{bmatrix} \text{ is quadrimodular.} \tag{6.1.10}$$

Substituting (6.1.8) into (6.1.10), $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular. □

For example, for $n \in \mathbb{Z}_{\geq 1}$, the metric space $(\mathbb{R}^n, \|\cdot - \cdot\|_\infty)$ is quadrimodular.

The following proposition provides an example of a median metric space of size 8 that is not quadrimodular.

PROPOSITION 6.1.5. *(median nonquadrimodular metric space) The metric space* $X := \left(\{0, 1\}^3, \|\cdot - \cdot\|_1\right)$ *has the following properties:*

(1) X *is median.*
(2) X *is not quadrimodular.*

PROOF.

(1) follows by 1.6.7(2) (binary Hamming spaces) with $Q = [3]$.
(2) It suffices to prove that for $x := (0, 0, 0)$, $y := (1, 1, 0)$, $a := (1, 0, 1)$ and $b := (0, 1, 1)$, none of the matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular.
Seeking a contradiction, assume at least one of the matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular.
Case 1. $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular, i.e. there are $s, t, u, v \in Y$ such that $\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix}$ is a median quadrangle. In particular

$$\langle x, s, t, y \rangle , \tag{6.1.11}$$

and by 4.4.2 (median quadrangles in a median interval space), $s = m(x, y, a) = (1, 0, 0)$ and $t = m(x, y, b) = (0, 1, 0)$. Therefore, $d_{xs} + d_{st} + d_{ty} = 1 + 2 + 1 = 4$, while $d_{xy} = 2$. By 1.4.6 (aligned sequences in a metric space), not $\langle x, s, t, y \rangle$, contradicting (6.1.11).
Case 2. $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$ or $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular. These cases are similar.

□

PROPOSITION 6.1.6. *(quadrimodular interval spaces) Each quadrimodular interval space is modular.*

PROOF. Let X be a quadrimodular interval space. For $x,\, y,\, a \in X$ it is to be proved that $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular. The assumption that X is quadrimodular entails that $\begin{bmatrix} a & a \\ x & y \end{bmatrix}$ or $\begin{bmatrix} a & y \\ x & a \end{bmatrix}$ is quadrimodular.

Case 1. $\begin{bmatrix} a & a \\ x & y \end{bmatrix}$ is quadrimodular. By 2.2.5(2) (quadrimodularity properties), $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular.

Case 2. $\begin{bmatrix} a & y \\ x & a \end{bmatrix}$ is quadrimodular. By 2.2.5(2) (quadrimodularity properties), $\begin{bmatrix} & y & \\ x & & a \end{bmatrix}$ is modular. By 1.4.10 (modular matrices), $\begin{bmatrix} & a & \\ x & & y \end{bmatrix}$ is modular. ☐

6.1.6 (quadrimodular interval spaces) may be used implicitly by applying results on modular interval spaces to quadrimodular interval spaces.

PROPOSITION 6.1.7. *(quadrimedian interval spaces) Let X be a quadrimedian interval space. For $A \subseteq X$, if A is median, then:*

(1) *For $Q = \begin{bmatrix} a & b \\ x & y \end{bmatrix}$ a matrix in A, if Q is quadrimodular in X, then Q is quadrimodular in A.*
(2) *A is a quadrimedian interval space.*

PROOF.

(1) The assumption that Q is quadrimodular in X says that there are $s,\, t,\, u,\, v \in X$ such that

$$\begin{bmatrix} a & & & b \\ & u & v & \\ & s & t & \\ x & & & y \end{bmatrix} \text{ is a median quadrangle.} \tag{6.1.12}$$

It suffices to prove $s,\, t,\, u,\, v \in A$. (6.1.12) implies by 4.4.2 (median quadrangles in a median interval space):

$$s = m(x,\, y,\, a)\ , \tag{6.1.13}$$
$$t = m(x,\, y,\, b)\ , \tag{6.1.14}$$
$$u = m(x,\, a,\, b)\ , \tag{6.1.15}$$
$$v = m(y,\, a,\, b)\ . \tag{6.1.16}$$

The assumptions that A is median, $x,\, y,\, a,\, b \in A$, (6.1.13), (6.1.14), (6.1.15) and (6.1.16) imply $s,\, t,\, u,\, v \in A$.

(2) From the assumption that A is median it follows that it suffices to prove that A is quadrimodular, i.e. for $x,\, y,\, a,\, b \in A$, in A at least one of the matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular. The assumption that X is quadrimedian entails that at least one of them is quadrimodular in X and consequently, by (1), quadrimodular in A.

□

6.2. Quadrimodular Metric Spaces

PROPOSITION 6.2.1. *(sets) For X a set and $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ families of sets, if for each $i \in I$, $A_i \subseteq B_i \subseteq X$, $\bigcup_{k\in I} A_k = X$ and $(B_i)_{i\in I}$ is disjoint, then for $i \in I$, $A_i = B_i$.*

PROOF. From the assumption that for each $i \in I$, $A_i \subseteq B_i$ it follows that it suffices to prove that for $i \in I$, $B_i \subseteq A_i$. For $x \in B_i$, it is to be proved that $x \in A_i$. The assumptions $x \in B_i$ and $B_i \subseteq X$ imply:

$$x \in X\,. \tag{6.2.1}$$

From (6.2.1) and the assumption $\bigcup_{k\in I} A_k = X$ it follows that there is a $j \in I$ such that

$$x \in A_j\,. \tag{6.2.2}$$

(6.2.2) and the assumption $A_j \subseteq B_j$ imply:

$$x \in B_j\,. \tag{6.2.3}$$

From the assumption $x \in B_i$, (6.2.3) and the assumption that $(B_i)_{i\in I}$ is disjoint it follows:

$$j = i\,. \tag{6.2.4}$$

Substituting (6.2.4) into (6.2.2), $x \in A_i$. □

PROPOSITION 6.2.2. *(criteria for quadrimodularity properties) Let X be a quadrimodular metric space. For $x,\, y,\, a,\, b \in X$:*

(1) (a) $\left\langle \begin{matrix} a & \cdot & b \\ x & & y \end{matrix} \right\rangle$ *iff* $d_{x,ya} = d_{x,yb}$ *and* $d_{x,ya} = d_{x,ab}$.
  (b) $\left\langle \begin{matrix} a & : & b \\ x & & y \end{matrix} \right\rangle$ *iff* $d_{x,ya} = d_{x,yb}$ *and* $d_{x,ya} < d_{x,ab}$.
  (c) $\left\langle \begin{matrix} a & \cdot\cdot & b \\ x & & y \end{matrix} \right\rangle$ *iff* $d_{x,ya} < d_{x,yb}$ *and* $d_{x,ya} = d_{x,ab}$.
  (d) $\left\langle \begin{matrix} a & :: & b \\ x & & y \end{matrix} \right\rangle$ *iff* $d_{x,ya} < d_{x,yb}$ *and* $d_{x,ya} < d_{x,ab}$.

(2) *The following conditions are equivalent:*
  (a) $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ *is quadrimodular.*
  (b) $M\,(x,\, y,\, a) \cap [x,\, b] \neq \emptyset$.
  (c) $[y,\, a] \cap [x,\, b] \neq \emptyset$.
  (d) $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$.

PROOF.

(1) Define

$$Y := X^{2\times 2}\,.$$

It suffices to prove for the following families of sets $(A_i)_{i\in[7]}$ and $(B_i)_{i\in[7]}$ that for each $i \in [7]$, $A_i = B_i$.

$$A_1 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | \left\langle \begin{matrix} a & :: & b \\ x & & y \end{matrix} \right\rangle \right\} ,$$
$$A_2 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | \left\langle \begin{matrix} a & : & b \\ x & & y \end{matrix} \right\rangle \right\} ,$$
$$A_3 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | \left\langle \begin{matrix} a & \cdot\cdot & b \\ x & & y \end{matrix} \right\rangle \right\} ,$$
$$A_4 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | \left\langle \begin{matrix} a & \cdot & b \\ x & & y \end{matrix} \right\rangle \right\} ,$$
$$A_5 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | \left\langle \begin{matrix} b & :: & a \\ x & & y \end{matrix} \right\rangle \right\} ,$$
$$A_6 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | \left\langle \begin{matrix} b & \cdot\cdot & a \\ x & & y \end{matrix} \right\rangle \right\} ,$$
$$A_7 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | \left\langle \begin{matrix} b & :: & y \\ x & & a \end{matrix} \right\rangle \right\} .$$

and

$$B_1 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | d_{x,ya} < d_{x,yb} \text{ and } d_{x,ya} < d_{x,ab}\,. \right\} ,$$
$$B_2 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | d_{x,ya} = d_{x,yb} \text{ and } d_{x,ya} < d_{x,ab}\,. \right\} ,$$
$$B_3 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | d_{x,ya} < d_{x,yb} \text{ and } d_{x,ya} = d_{x,ab}\,. \right\} ,$$
$$B_4 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | d_{x,ya} = d_{x,yb} \text{ and } d_{x,ya} = d_{x,ab}\,. \right\} ,$$
$$B_5 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | d_{x,yb} < d_{x,ya} \text{ and } d_{x,yb} < d_{x,ab}\,. \right\} ,$$
$$B_6 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | d_{x,yb} < d_{x,ya} \text{ and } d_{x,yb} = d_{x,ab}\,. \right\} ,$$
$$B_7 := \left\{ \begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y | d_{x,ab} < d_{x,ya} \text{ and } d_{x,ab} < d_{x,yb}\,. \right\} .$$

By 6.2.1 (sets) it suffices to prove that for each $i \in [7]$, $A_i \subseteq B_i \subseteq Y$, $\bigcup_{i\in[7]} A_i = Y$ and $(B_i)_{i\in[7]}$ is disjoint.

Step 1. Proof that for each $i \in [7]$, $A_i \subseteq B_i \subseteq Y$. By 2.6.3(4), (2), (3) and (1), respectively (quadrimodular matrices in a metric space), $A_1 \subseteq B_1$, $A_2 \subseteq B_2$, $A_3 \subseteq B_3$ and $A_4 \subseteq B_4$. By 2.6.3(4) (quadrimodular matrices in a metric space) in connection with 1.8.2(2) (point-pair modular distance), $A_5 \subseteq B_5$ and $A_7 \subseteq B_7$. By 2.6.3(3) (quadrimodular matrices in a metric space) in connection with 1.8.2(2) (point-pair modular distance), $A_6 \subseteq B_6$.

Step 2. Proof that $\bigcup_{i\in[7]} A_i = Y$. The assumption that X is quadrimodular entails that for each $\begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y$, at least one of the matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular. By 2.2.5(1) (quadrimodularity properties), for each $\begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y$, at least one of the following conditions holds: $\left\langle\begin{array}{ccc} a & & b \\ & :: & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} a & & b \\ & : & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} a & & b \\ & \cdot\cdot & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} a & & b \\ & \cdot & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & a \\ & :: & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & a \\ & : & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & a \\ & \cdot\cdot & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & a \\ & \cdot & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & y \\ & :: & \\ x & & a \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & y \\ & : & \\ x & & a \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & y \\ & \cdot\cdot & \\ x & & a \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & y \\ & \cdot & \\ x & & a \end{array}\right\rangle$. By 2.2.4(5), (4) and (3) (symmetries of quadrimodularity properties), for each $\begin{bmatrix} a & b \\ x & y \end{bmatrix} \in Y$, at least one of the following conditions holds: $\left\langle\begin{array}{ccc} a & & b \\ & :: & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} a & & b \\ & : & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} a & & b \\ & \cdot\cdot & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} a & & b \\ & \cdot & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & a \\ & :: & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & a \\ & \cdot\cdot & \\ x & & y \end{array}\right\rangle$, $\left\langle\begin{array}{ccc} b & & y \\ & :: & \\ x & & a \end{array}\right\rangle$, i.e. $\bigcup_{i\in[7]} A_i = Y$.

Step 3. Proof that $(B_i)_{i\in[7]}$ is disjoint. It suffices to prove that the following families are disjoint:

$$(B_1 \cup B_2 \cup B_3 \cup B_4,\ B_5 \cup B_6,\ B_7)\ ,$$
$$(B_1,\ B_2,\ B_3,\ B_4)\ ,$$
$$(B_5,\ B_6)\ .$$

Step 3.1. Proof that $(B_1 \cup B_2 \cup B_3 \cup B_4,\ B_5 \cup B_6,\ B_7)$ is disjoint. If $\begin{bmatrix} a & b \\ x & y \end{bmatrix} \in B_1 \cup B_2 \cup B_3 \cup B_4$, then $d_{x,ya} \leq d_{x,yb}$, while if $\begin{bmatrix} a & b \\ x & y \end{bmatrix} \in B_5 \cup B_6$, then $d_{x,ya} > d_{x,yb}$, and the inequalities $d_{x,ya} \leq d_{x,yb}$ and $d_{x,ya} > d_{x,yb}$ exclude each other. If $\begin{bmatrix} a & b \\ x & y \end{bmatrix} \in B_1 \cup B_2 \cup B_3 \cup B_4$, then $d_{x,ya} \leq d_{x,ab}$, while if $\begin{bmatrix} a & b \\ x & y \end{bmatrix} \in B_7$, then $d_{x,ya} > d_{x,ab}$,

and the inequalities $d_{x,ya} \le d_{x,ab}$ and $d_{x,ya} > d_{x,ab}$ exclude each other. If $\begin{bmatrix} a & b \\ x & y \end{bmatrix} \in B_5 \cup B_6$, then $d_{x,yb} \le d_{x,ab}$, while if $\begin{bmatrix} a & b \\ x & y \end{bmatrix} \in B_7$, then $d_{x,yb} > d_{x,ab}$, and the inequalities $d_{x,yb} \le d_{x,ab}$ and $d_{x,yab} > d_{x,ab}$ exclude each other.
Step 3.2. The proofs that $(B_1,\, B_2,\, B_3,\, B_4)$ and $(B_5,\, B_6)$ are disjoint is similar to step 3.1.

(2) Step 1. (2a) $\Rightarrow$ (2b). From the assumption that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular it follows by 2.2.5(5) (quadrimodularity properties) that $M\,(x,\, y,\, a) \cap [x,\, b] \neq \emptyset$.
Step 2. (2b) $\Rightarrow$ (2c) follows from $M\,(x,\, y,\, a) \subseteq [y,\, a]$.
Step 3. (2c) $\Rightarrow$ (2d). The assumption $[y,\, a] \cap [x,\, b] \neq \emptyset$ implies by 1.8.2(9) (point-pair modular distance) that $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$.
Step 4. (2d) $\Rightarrow$ (2a). The assumption $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$ says
$(d_{x,ya} < d_{x,yb}$ and $d_{x,ya} < d_{x,ab})$ or $(d_{x,ya} = d_{x,yb}$ and $d_{x,ya} < d_{x,ab})$
or $(d_{x,ya} < d_{x,yb}$ and $d_{x,ya} = d_{x,ab})$ or $(d_{x,ya} = d_{x,yb}$ and $d_{x,ya} = d_{x,ab})$.
Case 1. $d_{x,ya} < d_{x,yb}$ and $d_{x,ya} < d_{x,ab}$. By (1d), $\left\langle \begin{matrix} a & & b \\ & :: & \\ x & & y \end{matrix} \right\rangle$.
Case 2. $d_{x,ya} = d_{x,yb}$ and $d_{x,ya} < d_{x,ab}$. By (1b), $\left\langle \begin{matrix} a & & b \\ & : & \\ x & & y \end{matrix} \right\rangle$.
Case 3. $d_{x,ya} < d_{x,yb}$ and $d_{x,ya} = d_{x,ab}$. By (1c), $\left\langle \begin{matrix} a & & b \\ & \cdot\cdot & \\ x & & y \end{matrix} \right\rangle$.
Case 4. $d_{x,ya} = d_{x,yb}$ and $d_{x,ya} = d_{x,ab}$. By (1a), $\left\langle \begin{matrix} a & & b \\ & \cdot & \\ x & & y \end{matrix} \right\rangle$.

$\square$

In 6.2.2(2) (criteria for quadrimodularity properties), 'quadrimodular metric space' cannot be replaced by 'modular metric space' or 'median metric space'. For example, define $Q := \{1,\, 2,\, 3\}$ and $X := \left(\{0,\, 1\}^Q,\, \|\cdot - \cdot\|_1\right)$, $x := (0,\, 0,\, 0)$, $y = (1,\, 1,\, 0)$, $a = (0,\, 1,\, 1)$ and $b = (1,\, 0,\, 1)$.

The assumption, with 'quadrimodular metric space' replaced by 'median metric space', is satisfied: By 1.6.7(2) (binary Hamming spaces), X is a median metric space. Furthermore, (2d) holds: $d_{x,ya} = 1$, $d_{x,yb} = 1$ and $d_{x,ab} = 1$. Therefore, $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$.

But (2c) does not hold. Therefore, because $M\,(x,\, y,\, a) \subseteq [y,\, a]$, (2b) does not hold either. And by 2.2.5(5) (quadrimodularity properties) it follows that (2a) also does not hold.

The following theorem characterizes quadrimodular metric spaces.

THEOREM 6.2.3. *(quadrimodularity criterion for metric spaces) Let X be a metric space. X is quadrimodular iff for $x,\, y,\, a,\, b \in X$, $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$ implies that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular.*

PROOF.

Step 1. ($\Rightarrow$) Suppose that X is quadrimodular. By 6.2.2(2) (criteria for quadrimodularity properties), $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}$ implies that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular.

Step 2. ($\Leftarrow$) Suppose that for $x,\, y,\, a,\, b \in X$,

$$\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya} \text{ implies that } \begin{bmatrix} a & b \\ x & y \end{bmatrix} \text{ is quadrimodular.} \tag{6.2.5}$$

For $x,\, y,\, a,\, b \in X$ it is to be proved that at least one of the matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular.

Case 1.

$$\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya}\,. \tag{6.2.6}$$

From (6.2.6) and (6.2.5) it follows that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular.

Case 2. $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,yb}$ .By 1.8.2(2) (point-pair modular distance),

$$\min\{d_{x,yb},\, d_{x,ya},\, d_{x,ba}\} = d_{x,yb}\,. \tag{6.2.7}$$

(6.2.7) and (6.2.5) imply that $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$ is quadrimodular.

Case 3. $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ab}$. By 1.8.2(2) (point-pair modular distance),

$$\min\{d_{x,by},\, d_{x,ba},\, d_{x,ya},\, \} = d_{x,ab}\,. \tag{6.2.8}$$

From (6.2.8) and (6.2.5), transformed by the cycle (yab) , it follows that $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular. □

PROPOSITION 6.2.4. *(existence of quadrimodular extension) For each metric space X , there is an isometric map from X into a quadrimodular metric space.*

PROOF. By 1.4.26 (existence of injective closure), there is an isometric map from X into an injective metric space Y . By 6.1.4 (quadrimodularity of injective metric spaces), Y is quadrimodular. □

6.3. Quadrimedian Geometric Interval Spaces

PROPOSITION 6.3.1. *(quadrimedian geometric interval spaces) Let X be a quadrimedian geometric interval space. For $b,\, c,\, s\, \in\, X$, if s is maximal in $\left(\langle b,\, c,\, \cdot\rangle\, ,\, \langle b,\, \cdot,\, \cdot\rangle\right)$, then $s\, \in\, \partial_M\left(X\right)$.*

PROOF. Seeking a contradiction, assume $s \notin \partial_M\left(X\right)$, i.e. there are $x,\, y,\, a \in X$ such that

$$s = m\left(x,\, y,\, a\right)\ . \tag{6.3.1}$$

and

$$s \notin \{x,\, y,\, a\}\ . \tag{6.3.2}$$

In particular,

$$x \neq s\, . \tag{6.3.3}$$

The assumption that X is quadrimedian entails that at least one of the matrices $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & a \\ x & y \end{bmatrix}$, $\begin{bmatrix} b & y \\ x & a \end{bmatrix}$ is quadrimodular. Suppose without loss of generality that $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ is quadrimodular. By 2.2.5(5) (quadrimodularity properties), $M\left(x,\, y,\, a\right) \cap \left[x,\, b\right] \neq \emptyset$, i.e.

$$m\left(x,\, y,\, a\right) \in \left[x,\, b\right]\ . \tag{6.3.4}$$

Substituting (6.3.1) into (6.3.4), $s \in \left[x,\, b\right]$, i.e. $\langle x,\, s,\, b\rangle$. Therefore,

$$\langle b,\, s,\, x\rangle\ . \tag{6.3.5}$$

The assumption $\langle b,\, c,\, s\rangle$ and (6.3.5) imply:

$$\langle b,\, c,\, x\rangle\ . \tag{6.3.6}$$

(6.3.6), (6.3.5) and (6.3.3) contradict the maximality of s in $\left(\langle b,\, c,\, \cdot\rangle\, ,\, \langle b,\, \cdot,\, \cdot\rangle\right)$. ☐

In 6.3.1 (quadrimedian geometric interval spaces), 'quadrimedian' cannot be replaced by 'median'. For example, define $Q\, :=\, \{1,\, 2,\, 3\}$ and $X\, :=\, \left(\{0,\, 1\}^Q\, ,\, \|\cdot - \cdot\|_1\right)$, $b\, :=\, (0,\, 0,\, 0)$, $c := (1,\, 1,\, 0)$ and $s := (1,\, 1,\, 1)$.

The assumptions, with 'quadrimedian' replaced by 'median', are satisfied: By 1.6.7(2) (binary Hamming spaces), X is a median metric space and therefore a median geometric interval space, $\langle b,\, c,\, \cdot\rangle = \{c,\, s\}$, and s is maximal in $\left(\langle b,\, c,\, \cdot\rangle\, ,\, \langle b,\, \cdot,\, \cdot\rangle\right)$.

But the claim of the proposition is not satisfied: $s \notin \partial_M\left(X\right)$. Proof: For $d := (1,\, 0,\, 1)$ and $e := (0,\, 1,\, 1)$, $s = m\left(c,\, d,\, e\right)$, but $s \notin \{c,\, d,\, e\}$.

6.4. Quadrimedian Geometric Topological Interval Spaces

Part (3) of the following theorem is for a compact quadrimedian geometric topological interval space what [**40**, 3.24] (Krein-Milman theorem) is for a compact convex set in a locally convex real topological vector space.

THEOREM 6.4.1. *(convex closure of the median boundary) Let X be a compact quadrimedian geometric topological interval space.*

(1) *For* $a,\, p \in X$:
   (a) *The poset* $(\langle a,\, p,\, \cdot\rangle\,,\, \langle a,\, \cdot,\, \cdot\rangle)$ *has a maximal element.*
   (b) *There is a* $b \in \partial_M(X)$ *such that* $\langle a,\, p,\, b\rangle$.
(2) *For* $p \in X$ *, there are* $a,\, b \in \partial_M(X)$ *such that* $\langle a,\, p,\, b\rangle$.
(3) $[\partial_M(X)] = X$.

PROOF.

(1)
   (a) By 2.4.4 (topological interval spaces),
   $$\langle a,\, p,\, \cdot\rangle \neq \emptyset\,, \tag{6.4.1}$$
   $$\langle a,\, p,\, \cdot\rangle \text{ is closed.} \tag{6.4.2}$$
   The assumption that X is compact and (6.4.2) imply by 1.3.7 (compact topological spaces):
   $$\langle a,\, p,\, \cdot\rangle \text{ is compact.} \tag{6.4.3}$$
   From (6.4.3) and (6.4.1) it follows by 2.1.2 (compact topological posets), the poset$(\langle a,\, p,\, \cdot\rangle\,,\, \langle a,\, \cdot,\, \cdot\rangle)$ has a maximal element.
   (b) By (1a), the poset$(\langle a,\, p,\, \cdot\rangle\,,\, \langle a,\, \cdot,\, \cdot\rangle)$ has a maximal element b. By 6.3.1 (quadrimedian geometric interval spaces), $b \in \partial_M(X)$.
(2) By two applications of (1b), there is an $a \in \partial_M(X)$ such that $\langle p,\, p,\, a\rangle$, and there is a $b \in \partial_M(X)$ such that $\langle a,\, p,\, b\rangle$.
(3) For $p \in X$ it is to be proved that p belongs to each convex superset of $\partial_M(X)$. This condition follows by (2).

□

In 6.4.1(3) (convex closure of the median boundary), 'quadrimedian' cannot be replaced by 'median'. For example, define $Q := \{1,\, 2,\, 3\}$ and $X := \left(\{0,\, 1\}^Q\,,\, \|\cdot - \cdot\|_1\right)$.

The assumptions, with 'quadrimedian' replaced by 'median', are satisfied: By 1.6.7(2) (binary Hamming spaces), X is a median metric space. X is finite, thus compact.

But the claim of the proposition is not satisfied: $[\partial_M(X)] \neq X$. Proof: It suffices to prove $[\partial_M(X)] = \emptyset$. Thus, it is sufficient to prove $\partial_M(X) = \emptyset$. This claim follows from the fact that each $p \in X$ is the median of the three points obtained by changing one coordinate of p at a time.

6.5. Quadrimedian Metric Spaces

Let X be metric space.

A *geodesic median closure* of X is a pair $(Y,\, i)$ such that Y is a median metric space, i is an isometric map from X into Y and the median closure of $i(X)$ in Y equals Y . In particular, when Y is a median metric space, X a subspace of Y with median closure Y in Y and i is the inclusion map of X into Y , then $(Y,\, i)$ is a geodesic median closure of X .

By 6.2.4 (existence of quadrimodular extension), there is an isometric map from X into a quadrimodular metric space. X is called *subquadrimedian* iff there is an isometric map from X into a quadrimedian metric space. Each subspace of a quadrimedian metric space is subquadrimedian. A *geodesic quadrimedian closure* of X is a pair $(Y,\, i)$ such that Y is a quadrimedian metric space and $(Y,\, i)$ is a geodesic median closure of X. In particular, when Y is a quadrimedian metric space, X a subspace of Y with median closure Y in Y and i is the inclusion map of X into Y, then $(Y,\, i)$ is a geodesic quadrimedian closure of X.

The following theorem is for a geodesic quadrimedian closure of a metric space what [**25**, theorem 9.23] is for an algebraic closure of a field.

THEOREM 6.5.1. *(extension to a geodesic quadrimedian closure) Let X be a metric space with a geodesic quadrimedian closure $(Y,\, i_Y)$. For i_Z an isometric map from X into a quadrimedian metric space Z, there is exactly one isometric map F from Y into Z such that $i_Z = F \circ i_Y$. In particular, if $X \subseteq Y$ with inclusion map $i_Y : X \to Y$, then each isometric map from X into a quadrimedian metric space Z has exactly one extension to an isometric map from Y into Z.*

$$\begin{tikzcd} Y \arrow[r, tail, dashed, "F"] & Z \\ X \arrow[u, tail, "i_Y"] \arrow[ur, tail, "i_Z"'] & \end{tikzcd}$$

PROOF. The assumption that $(Y,\, i_Y)$ is a geodesic quadrimedian closure entails:

$$\text{The median closure of } i_Y(X) \text{ in } Y \text{ equals } Y, \tag{6.5.1}$$

and that Y is quadrimedian, i.e.:

$$Y \text{ is median.} \tag{6.5.2}$$

$$Y \text{ is quadrimodular.} \tag{6.5.3}$$

Step 1. Existence of F. i_Y^{-1} is an isometric map from $i_Y(X)$ onto X. Therefore, $i_Z \circ i_Y^{-1}$ is an isometric map from $i_Y(X)$ into Z. The set of all isometric maps $f : S \to Z$ such that $i_Y(X) \subseteq S \subseteq Y$ and f extends $i_Z \circ i_Y^{-1}$, viewed as subsets of $S \times Z$, is a poset under set inclusion in which every chain C has an upper bound $\bigcup C$ if $C \neq \emptyset$ and $i_Z \circ i_Y^{-1}$ if $C = \emptyset$. By 1.2.5 (Zorn's lemma), this poset has a maximal element F. Setting $S_0 := \operatorname{dom} F$:

$$F \text{ extends } i_Z \circ i_Y^{-1}. \tag{6.5.4}$$

$$F \text{ is an isometric map from } (S_0,\, d) \text{ into } Z. \tag{6.5.5}$$

It suffices to prove $S_0 = Y$ and $F \circ i_Y = i_Z$.

Step 1.1. Proof that $S_0 = Y$. Seeking a contradiction, suppose $S_0 \subsetneq Y$. From (6.5.1) and the assumption $i_Y(X) \subseteq S_0 \subsetneq Y$ it follows that S_0 is not median, i.e. there are $x,\, y,\, a \in S_0$ such that $m(x,\, y,\, a) \in Y \setminus S_0$. Thus, $S_0 \subsetneq S_0 \cup \{m(x,\, y,\, a)\} \subseteq Y$. In order to obtain a contradiction to the maximality of F , it suffices to prove that F has an extension to an isometric map G from $((S_0 \cup \{m(x,\, y,\, a)\})\,,\, d)$ into Z . Define

$$G(b) := \begin{cases} F(b) & b \in S_0 \\ m(F(x)\,,\, F(y)\,,\, F(a)) & b = m(x,\, y,\, a) \end{cases} .$$

It remains to be proved that for each $b \in S_0$, $d_{G(m(x,y,a))G(b)} = d_{m(x,y,a)b}$, i.e. $d_{m(F(x),F(y),F(a))F(b)} = d_{m(x,y,a)b}$.

The assumption that Z is quadrimedian entails:

$$Z \text{ is median.} \tag{6.5.6}$$

$$Z \text{ is quadrimodular.} \tag{6.5.7}$$

Therefore, at least one of the matrices $\begin{bmatrix} F(a) & F(b) \\ F(x) & F(y) \end{bmatrix}$, $\begin{bmatrix} F(b) & F(a) \\ F(x) & F(y) \end{bmatrix}$, $\begin{bmatrix} F(b) & F(y) \\ F(x) & F(a) \end{bmatrix}$ is quadrimodular. Suppose without loss of generality that

$$\begin{bmatrix} F(a) & F(b) \\ F(x) & F(y) \end{bmatrix} \text{ is quadrimodular.} \tag{6.5.8}$$

By 2.6.3(5) (quadrimodular matrices in a metric space),

$$\min\left\{d_{F(x),F(y)F(a)},\, d_{F(x),F(y)F(b)},\, d_{F(x),F(a)F(b)}\right\} = d_{F(x),F(y)F(a)} . \tag{6.5.9}$$

(6.5.5) and (6.5.9) imply by 2.6.4 (isometric invariance of point-pair modular distance):

$$\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\} = d_{x,ya} . \tag{6.5.10}$$

From (6.5.3) and (6.5.10) it follows by 6.2.2(2) (criteria for quadrimodularity properties):

$$\begin{bmatrix} a & b \\ x & y \end{bmatrix} \text{ is quadrimodular in } Y . \tag{6.5.11}$$

From (6.5.6) and (6.5.8) it follows by 4.7.3 (distance from a median):

$$d_{m(F(x),F(y),F(a))F(b)} = d_{F(x)F(b)} - d_{F(x),F(y)F(a)} . \tag{6.5.12}$$

(6.5.2) and (6.5.11) imply by 4.7.3 (distance from a median):

$$d_{m(x,y,a)b} = d_{xb} - d_{x,ya} . \tag{6.5.13}$$

From (6.5.5) it follows by 2.6.4 (isometric invariance of point-pair modular distance):

$$d_{F(x)F(b)} - d_{F(x),F(y)F(a)} = d_{xb} - d_{x,ya} . \tag{6.5.14}$$

Substituting (6.5.12) and (6.5.13) into (6.5.14), $d_{m(F(x),F(y),F(a))F(b)} = d_{m(x,y,a)b}$.

Step 1.2. Proof that $F \circ i_Y = i_Z$. From (6.5.4) it follows for $x \in X$:

$$\begin{aligned} (F \circ i_Y)(x) &= F(i_Y(x)) \\ &= \left(i_Z \circ i_Y^{-1}\right)(i_Y(x)) \\ &= i_Z(x) \ . \end{aligned}$$

Consequently, $F \circ i_Y = i_Z$.

Step 2. Uniqueness of F . Let F' be another isometric map from Y into Z such that $i_Z = F' \circ i_Y$. Define

$$A := \{a \in Y | F(a) = F'(a)\} \ .$$

It is to be proved that $A = Y$. From the assumptions $i_Z = F \circ i_Y$ and $i_Z = F' \circ i_Y$ it follows that $F \circ i_Y = F' \circ i_Y$, i.e. for each $x \in X$, $F(i_Y(x)) = F'(i_Y(x))$, i.e.

$$A \supseteq i_Y(X) \ . \tag{6.5.15}$$

From (6.5.15) and (6.5.1) it follows that it suffices to prove that A is median. For $a,\, b,\, c \in X$ it is to be proved that $a,\, b,\, c \in A$ implies $m(a,\, b,\, c) \in A$. Define $u := m(a,\, b,\, c)$, i.e.

$$\begin{bmatrix} & c & \\ a & u & b \end{bmatrix} \text{ is a median triangle.} \tag{6.5.16}$$

It is to be proved that $u \in A$. The assumption $a,\, b,\, c \in A$ says:

$$F(a) = F'(a) \ , \tag{6.5.17}$$

$$F(b) = F'(b) \ , \tag{6.5.18}$$

$$F(c) = F'(c) \ . \tag{6.5.19}$$

From the assumption that F and F' are isometric maps from Y into Z it follows by 1.4.12 (isometric maps) that F and F' are embeddings of $(Y,\, \langle \cdot,\, \cdot,\, \cdot \rangle_d)$ into $(Z,\, \langle \cdot,\, \cdot,\, \cdot \rangle_d)$. In particular,

$$F \text{ and } F' \text{ are homomorphisms from } (Y,\, \langle \cdot,\, \cdot,\, \cdot \rangle_d) \text{ to } (Z,\, \langle \cdot,\, \cdot,\, \cdot \rangle_d) \tag{6.5.20}$$

(6.5.16) and (6.5.20) imply by 1.4.13(2) (homomorphisms of interval spaces) that

$$\begin{bmatrix} & F(c) & \\ F(a) & F(u) & F(b) \end{bmatrix} ,$$

$$\begin{bmatrix} & F'(c) & \\ F'(a) & F'(u) & F'(b) \end{bmatrix}$$

are median triangles, i.e.

$$F(u) \in M(F(a)\, ,\, F(b)\, ,\, F(c)) \ , \tag{6.5.21}$$

$$F'(u) \in M(F'(a)\, ,\, F'(b)\, ,\, F'(c)) \ . \tag{6.5.22}$$

Substituting (6.5.17), (6.5.18) and (6.5.19) into (6.5.22),

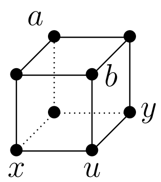
$$F'(u) \in M(F(a)\, ,\, F(b)\, ,\, F(c)) \ . \tag{6.5.23}$$

From (6.5.2), (6.5.21) and (6.5.23) it follows that $F(u) = F'(u)$, i.e. $u \in A$. $\square$

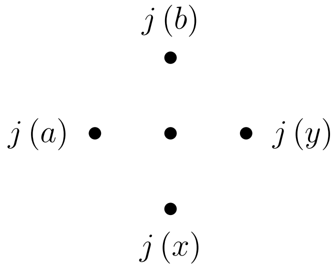
In 6.5.1 (extension to a geodesic quadrimedian closure), 'quadrimedian' cannot be replaced by 'median'. For example, in the next proposition, X is a metric space with a geodesic median closure $(Y,\, i_Y)$, $X \subseteq Y$ with inclusion map $i_Y : X \to Y$, and j is an isometric map from X into the median metric space Z . But j has no extension to an isometric map from Y into Z because $|Y| > |Z|$. Another example that shows that 6.5.1 (extension to a geodesic quadrimedian closure) with 'quadrimedian' replaced by 'median' does not hold has been provided in [**7**, remark 2.10. (1)].

PROPOSITION 6.5.2. *(structural non-uniqueness of median closure) Define* $Q := \mathbb{N}_3$ *and:*

- $Y := \left(\{0,\, 1\}^Q,\, \|\cdot - \cdot\|_1\right)$.
- $x := (0,\, 0,\, 0)$, $y = (1,\, 1,\, 0)$, $a = (0,\, 1,\, 1)$, $b = (1,\, 0,\, 1)$, $X := \{x,\, y,\, a,\, b\}$ *and* $i :\, X \to Y$ *the inclusion map.*



- $j :\, X \to \mathbb{R}^2$ *by* $j\,(x) = (0,\, -1)$, $j\,(y) = (1,\, 0)$, $j\,(a) = (-1,\, 0)$ *and* $j\,(b) = (0,\, 1)$, *and* $Z := (j\,(X) \cup \{(0,\, 0)\}\,,\, \|\cdot - \cdot\|_1)$.



Then:

(1) *Each* $p \in Y \setminus X$ *is the median of three elements of* X .
(2) $(Y,\, i)$ *is a median closure of* X *with* $|Y| = 8$.
(3) $(0,\, 0)$, *the only element of* $Z \setminus j\,(X)$, *is the median of three elements of* $j\,(X)$,
(4) $(Z,\, j)$ *is a median closure of* X *with* $|Z| = 5$.

PROOF.

(1) $(0,\, 0,\, 1) = m\,(x,\, a,\, b)$, $(0,\, 1,\, 0) = m\,(x,\, y,\, a)$, $(1,\, 0,\, 0) = m\,(x,\, y,\, b)$, and $(1,\, 1,\, 1) = m\,(y,\, a,\, b)$.
(2) By 1.6.7(2) (binary Hamming spaces), the metric space Y is median. By (1), $(Y,\, i)$ is a median closure of X .
(3) $(0,\, 0) = m\,(j\,(x)\,,\, j\,(y)\,,\, j\,(a))$.
(4) By (3) it suffices to prove that the metric space Z is median and that j is an isometric map from X into $(Z,\, \|\cdot - \cdot\|_1)$.
Step 1. Proof that Z is median. Z is an isometric copy of the tree $(N,\, E)$ with $N := j\,(X) \cup \{(0,\, 0)\}$ and with $(0,\, 0)$ adjacent to each of the other vertices and no other adjacency. By 1.7.4 (tree representation of finite arboric interval spaces), Z is arboric. By 1.7.5 (medianity of arboric interval spaces), Z is median.
Step 2. Proof that j is an isometric map from X into $(Z,\, \|\cdot - \cdot\|_1)$. For $p,\, q \in X$, if $p \neq q$, then $\|p - q\|_1 = 2$ and $\|j\,(p) - j\,(q)\|_1 = 2$.

$\square$

The following theorem is for the geodesic quadrimedian closure of a metric space what [**25**, 9.22] (theorem of Steinitz) is for the algebraic closure of a field.

THEOREM 6.5.3. *(existence and structural uniqueness of geodesic quadrimedian closure) Let X be a subquadrimedian metric space.*

(1) *X has a geodesic quadrimedian closure.*
(2) *Let $(Y,\, i_Y)$, $(Z,\, i_Z)$ be geodesic quadrimedian closures of X . Then there is exactly one isometric map F from Y onto Z such that $i_Z = F \circ i_Y$. In particular, Z is an isometric copy of Y .*

PROOF.

(1) The assumption that X is subquadrimedian says that there is
$$Y\,,\ \text{a quadrimedian metric space,} \tag{6.5.24}$$
and
$$i\,,\ \text{an isometric map from } X \text{ into } Y\,. \tag{6.5.25}$$
(6.5.24) entails that Y is median. Define
$$\bar{X} := \text{the median closure of } i\,(X) \text{ in } Y\,. \tag{6.5.26}$$
In particular:
$$\bar{X} \text{ is a median subspace of } Y\,. \tag{6.5.27}$$
$$i\,(X) \subseteq \bar{X}\,. \tag{6.5.28}$$
From (6.5.24) and (6.5.27) it follows by 6.1.7(2) (quadrimedian interval spaces):
$$\bar{X} \text{ is a quadrimedian metric space.} \tag{6.5.29}$$
(6.5.25) and (6.5.28) imply:
$$i \text{ is an isometric map from } X \text{ into } \bar{X}\,. \tag{6.5.30}$$
(6.5.30), (6.5.29) and (6.5.26) together say that $\left(\bar{X},\, i\right)$ is a geodesic quadrimedian closure of X .

(2) By 6.5.1 (extension to a geodesic quadrimedian closure), there is exactly one
$$F\,,\ \text{an isometric map from } Y \text{ into } Z \tag{6.5.31}$$
such that
$$i_Z = F \circ i_Y\,. \tag{6.5.32}$$
It suffices to prove $F\,(Y) = Z$. The assumption that $(Z,\, i_Z)$ is a geodesic quadrimedian closure says that the median closure of $i_Z\,(X)$ in Z equals Z , i.e. Z is the smallest median set in Z containing $i_Z\,(X)$. Therefore, it suffices to prove that $F\,(Y)$ is median and contains $i_Z\,(X)$.
Step 1. Proof that $F\,(Y)$ is median. From (6.5.31) it follows that F is an isometric map from the quadrimedian metric space Y onto $F\,(Y)$. By 1.4.12 (isometric maps), F is an isomorphism of $(Y,\, \langle\cdot,\,\cdot,\,\cdot\rangle_d)$ onto $(F\,(Y)\,,\, \langle\cdot,\,\cdot,\,\cdot\rangle_d)$. From this and the assumption

that $(Y, \langle \cdot, \cdot, \cdot \rangle_d)$ is quadrimedian it follows that $(F(Y), \langle \cdot, \cdot, \cdot \rangle_d)$ is quadrimedian. In particular, $F(Y)$ is median.
Step 2. Proof that $F(Y) \supseteq i_Z(X)$. The assumption that (Y, i_Y) is a geodesic quadrimedian closure of X entails $Y \supseteq i_Y(X)$. Therefore,

$$F(Y) \supseteq F(i_Y(X)) . \tag{6.5.33}$$

(6.5.32) implies:

$$i_Z(X) = F(i_Y(X)) . \tag{6.5.34}$$

Substituting (6.5.34) into (6.5.33), $F(Y) \supseteq i_Z(X)$.

□

In 6.5.3(2) (existence and structural uniqueness of geodesic quadrimedian closure), 'geodesic quadrimedian closure' cannot be replaced by 'geodesic median closure'. For example, in 6.5.2 (structural non-uniqueness of median closure), X is a subquadrimedian metric space. (Y, i) is a median closure of X with $|Y| = 8$, and (Z, j) is a median closure of X with $|Z| = 5$. Therefore, $|Z| \neq |Y|$. In particular, Z is not an isometric copy of Y .

6.6. Applications to Arboric Spaces

The following theorem answers the question when in a compact arboric topological interval space the median closure of a set equals the whole space.

THEOREM 6.6.1. *(median closure of the median boundary) Let X be a compact arboric topological interval space.*

(1) *For $p \in X$, there are $a, b, c \in \partial_M(X)$ such that $p = m(a, b, c)$.*
(2) *The median closure of $\partial_M(X)$ in X equals X .*
(3) *For $Y \subseteq X$, the median closure of Y in X equals X iff $\partial_M(X) \subseteq Y$.*

PROOF. The assumption that X is arboric implies by 6.1.1 (quadrimedianity of arboric interval spaces)

$$X \text{ is quadrimedian.} \tag{6.6.1}$$

(1) From (6.6.1) it follows by 6.4.1(2) (convex closure of the median boundary) that there are $a, b \in \partial_M(X)$ such that $\langle a, p, b \rangle$. By 5.2.3(2) (compact arboric topological interval spaces) there is a $c \in \partial_M(X)$ such that $p = m(a, b, c)$.
(2) For $p \in X$ it is to be proved that p belongs to each median set containing $\partial_M(X)$. This claim follows by (1).
(3) Step 1. (⇒) is entailed by 4.4.1(2) (median boundary of a median interval space).
Step 2. (⇐) follows from (2).

□

THEOREM 6.6.2. *(compact arboric determination by the median boundary) Let X and Y be compact arboric metric spaces.*

(1) *Each isometric map from $\partial_M(X)$ onto $\partial_M(Y)$ has an extension to an isometric map from X onto Y .*
(2) *If $\partial_M(Y)$ is an isometric copy of $\partial_M(X)$, then Y is an isometric copy of X .*

PROOF.

(1) For f an isometric map from $\partial_M(X)$ onto $\partial_M(Y)$ it is to be proved that f has an extension to an isometric map from X onto Y . From the assumption that X and Y are compact and arboric it follows by 6.6.1(2) (median closure of the median boundary):
$$\text{The median closure of } \partial_M(X) \text{ in } X \text{ equals } X\,. \tag{6.6.2}$$
$$\text{The median closure of } \partial_M(Y) \text{ in } Y \text{ equals } Y\,. \tag{6.6.3}$$
From the assumption that X and Y are arboric it follows by 6.1.1 (quadrimedianity of arboric interval spaces):
$$X \text{ is quadrimedian.} \tag{6.6.4}$$
$$Y \text{ is quadrimedian.} \tag{6.6.5}$$
Let
$$i := \text{the inclusion map of } \partial_M(X) \text{ into } X\,. \tag{6.6.6}$$
(6.6.4) and (6.6.2) together say:
$$(X,\, i) \text{ is a quadrimedian closure of } \partial_M(X)\,. \tag{6.6.7}$$
The assumption that f is an isometric map from $\partial_M(X)$ onto $\partial_M(Y)$ says:
$$f \text{ is an isometric map from } \partial_M(X) \text{ into } Y\,. \tag{6.6.8}$$
$$f(\partial_M(X)) = \partial_M(Y)\,. \tag{6.6.9}$$
Substituting (6.6.9) into (6.6.3),
$$\text{The median closure of } f(\partial_M(X)) \text{ in } Y \text{ equals } Y\,. \tag{6.6.10}$$
(6.6.8), (6.6.5) and (6.6.10) together say:
$$(Y,\, f) \text{ is a quadrimedian closure of } \partial_M(X)\,. \tag{6.6.11}$$
(6.6.7) and (6.6.11) imply by 6.5.3(2) (existence and structural uniqueness of geodesic quadrimedian closure) that there is an isometric map F from X onto Y such that
$$f = F \circ i\,. \tag{6.6.12}$$
(6.6.6) and (6.6.12) imply that F is an extension of f .

(2) follows by (1).

□

From 6.6.2 (compact arboric determination by the median boundary) it follows that a weighted tree is structurally determined by the distances between its vertices of degrees 1 and 2 .

Conclusion

The following additions have been made to the broad spectrum of mathematical topics that can be treated in the conceptual framework of interval spaces.

Theorems 3.2.4 (antiexchange criterion for triangle-convex geometric interval spaces) and 3.3.2 (perspectivity relation) characterize a geometric property of an interval relation in terms of a fundamental property of a family of derived binary relations, namely the property of being a family of partial orders and the property of being a family of equivalence relations, respectively. This way, these results reinforce the choice of geometric axioms. This leads to the question which other results can be proved that characterize a geometric property by a fundamental combinatorial concept in the spirit of these two theorems.

From 4.3.1 (modular geometric topological interval spaces) it follows that in a modular metric space, each non-empty compact convex set is a Chebyshev set, thus opening a gate for approximation theory in modular metric spaces.

Proposition 4.6.1 (medianity criterion for a compact geometric topological interval space) generalizes [**31**, 3.1.7], a theorem on finite connected graphs. This raises the question which further results about finite connected graphs can be generalized to compact geometric topological interval spaces.

Theorem 4.7.2(1) (metrizability criterion), like [**32**, theorem 34.1] (Urysohn metrization theorem) for topological spaces, is a sufficient criterion for metrizability in terms of a separation property. The questions arise: How far does the class of submedian-metrizable finite interval spaces extend beyond the class of finite geometric interval spaces with point-interval separation? What can be said about submedian metrizability of infinite interval spaces? Is there a general principle behind both theorems?

Theorem 5.3.3 (2b), (2c) (finite arboric metric spaces) places the neighbor-joining method for reconstructing a weighted tree from the distances between its leaves in the conceptual framework of arboric metric spaces. Parts (2b) and (2c) in connection with 4.4.1(2) (median boundary of a median interval space) yield consistency of the neighbor-joining method for reconstructing a weighted tree from the distances between its vertices of degrees 1 and 2 . There is no restriction on the degrees of the vertices in the tree. This method was established in [**41**]. In [**45**], the expression that is minimized in the neighbor-joining method was simplified. In [**15**] consistency of that method is also treated, and section 4.3 contains the following remark regarding the degrees: “We would like to point out that the case [...] may occur when the node [...] is of a degree greater than 3 [...] However, this does not invalidate the proof, which holds when the degree of nodes internal to T is at least 3 . [...] The case where the internal nodes may be of degree 2 requires special treatment (as well as the redefinition of the notion of neighbor).” The concept of an extremal neighborhood entails such a redefinition, and the proof of 5.3.3(2b), (2c) (finite arboric metric spaces) entails such special treatment in the conceptual framework of arboric metric spaces and median boundaries. This brings up the question what further use can be made of interval spaces for the theory of neighbor-joining and for tree reconstruction in general, including approximate tree reconstruction.

By propositions 6.1.2 (product of two arboric interval spaces) and 6.1.7(2) (quadrimedian interval spaces), each median subspace of a product of two arboric interval spaces is geometric and quadrimedian. This leads to the question how far the class of quadrimedian geometric interval spaces extends beyond the class of median subspaces of products of two arboric interval spaces.

Theorem 6.2.3 (quadrimodularity criterion for metric spaces) characterizes quadrimodular metric spaces in terms of the expression $\min\{d_{x,ya},\, d_{x,yb},\, d_{x,ab}\}$. This expression can also be used for characterizing metric spaces that have an isometric map into an arboric metric space. It follows from [**14**, 3.38] that a metric space X has an isometric map into an arboric metric space iff for all $x,\, y,\, a,\, b \in X$, at least two of the three point-pair modular distances $d_{x,ya},\, d_{x,yb},\, d_{x,ab}$ equal their minimum. [**14**, 3.12] gives an equivalent condition in terms of point-point distances, the so-called four-point condition.

Theorem 6.4.1 (3) (convex closure of the median boundary) is for a compact quadrimedian geometric topological interval space what [**40**, 3.24] (Krein-Milman theorem) is for a compact convex set in a locally convex real topological vector space. The question arises which applications of the Krein-Milman theorem have interesting counterparts for compact quadrimedian geometric topological interval spaces. It implies theorem 6.6.1 (median closure of the median boundary).

Theorem 6.5.3 (existence and structural uniqueness of geodesic quadrimedian closure) is for the geodesic quadrimedian closure of a metric space what [**25**, 9.22] (theorem of Steinitz) is for the algebraic closure of a field. It raises the question which metric spaces are subquadrimedian. Together with theorem 6.6.1 (median closure of the median boundary) it implies 6.6.2 (compact arboric determination by the median boundary). This latter result entails that a weighted tree is determined up to isometry by the distances between its vertices of degrees 1 and 2 . It follows that it is determined up to isomorphism of weighted trees by these vertices. Thus, if it has no vertices of degree 2 , then it is determined up to isomorphism by its leaves. This brings up the question which well-known results on trees can be generalized to compact arboric spaces.

Thus, the two structure theorems 6.4.1 (3) (convex closure of the median boundary) and 6.5.3 (existence and structural uniqueness of geodesic quadrimedian closure), which are valid for quadrimedian spaces, but unvalid for median spaces, are analogous to two central structure theoremes of analysis and algebra, the Krein-Milman theorem for a compact convex set in a locally convex real topological vector space and the theorem of Steinitz on the algebraic closure of a field. Therefore, the concept of a quadrimedian interval space seems to be a natural sharpening of the concept of a median interval space.

Here are two further examples of questions on arboric and quadrimedian spaces: Are all injective closures of a quadrimedian metric space isometric copies of each other? Do arboricity and quadrimedianity carry over to injective closures?

Thus, some evidence has been added that interval spaces provide a solid conceptual framework for a broad range of mathematical topics.

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Nomenclature

$[A]$ convex closure , page 49
$\langle\cdot,\, a,\, b\rangle$ 1-section of a ternary relation, page 34
$\langle\cdot,\, a,\, \cdot\rangle$ $(1,\, 3)$-section of a ternary relation, page 33
$\langle\cdot,\, \cdot,\, a\rangle$ $(1,\, 2)$-section of a ternary relation, page 33
$\langle\cdot,\, \cdot,\, \cdot\rangle_d$ interval relation of a metric space, page 36
$\langle\cdot,\, \cdot,\, \cdot\rangle$ ternary relation, page 33
$\langle a,\, b,\, \cdot\rangle$ 3-section of a ternary relation, page 34
$\langle a,\, \cdot,\, b\rangle$ 2-section of a ternary relation, page 34
$\langle a,\, \cdot,\, \cdot\rangle$ $(2,\, 3)$-section of a ternary relation, page 33
$\langle a_0,\, a_1,\, ...,\, a_k\rangle$ alignment of a sequence, page 38
$[a,\, b]$ interval, page 34
$[a,\, b]_d$ geodesic interval in a metric space, page 36
C_n cycle, page 24
$d\,(A,\, B)$ distance between sets, page 32
$d\,(a,\, B)$ distance from a point to a set, page 32
$d\,(a,\, b)$ distance between points, page 32
d_{AB} distance between sets, page 32
d_{aB} distance from a point to a set, page 32
d_{ab} distance between points, page 32
$\deg\,(x)$ degree, page 129
$\partial_M\,(X)$ median boundary, page 84
$\mathrm{dom}\ f$ domain of a map, page 23
$\downarrow y$ principal down-set, page 26
$d_w\,(a,\, b)$ shortest path distance, page 65
$d_{x,ya}$ Gromov product, page 65
$d_{x,ya}$ point-pair modular distance, page 65
$E\,(a,\, b)$ set of edges of a path, page 25
$\mathrm{EN}\,(u)$ extremal neighborhood, page 133
$\|f\|_1$ 1-norm, page 61
$f\,(A_0)$ image of a set under a map, page 23
$f^{-1}\,(B_0)$ preimage of a set under a map, page 23
$g \circ f$ composite of two maps, page 23
$K_{m,n}$ complete-bipartite graph, page 24
λ_{ab}^{Y} augmented modular distance sum, page 66

λ_u^Y distance sum, page 66
$l\,(w)$ length of a walk, page 24
$[m]$ initial segment of the set of positive integers, page 23
$\langle M \rangle$ modularity of a partial matrix, page 43
$M\,(a,\,b,\,c)$ set of medians, page 42
$m\,(a,\,b,\,c)$ median, page 59
$\neg\,\langle \cdot,\,u,\,\cdot \rangle$ u-neighbor relation, page 127
p_{ab} unique path, page 25
$\prod_{q\in Q}\left(X_q,\,\langle \cdot,\,\cdot,\,\cdot \rangle_q\right)$ product of a family of interval spaces, page 44
p_{ab}^T unique path, page 25
$\langle q_1 - q_2 \rangle$ horizontal quadrimodularity of a 2×2-matrix of points with columns $q_1,\,q_2$, page 79
$\langle q_1 \cdot\cdot q_2 \rangle$ proper horizontal quadrimodularity of a 2×2-matrix of points with columns $q_1,\,q_2$, page 79
$\langle q_1 \cdot q_2 \rangle$ star-quadrimodularity of a 2×2-matrix of points with columns $q_1,\,q_2$, page 79
$\langle q_1 :: q_2 \rangle$ proper quadrimodularity of a 2×2-matrix of points with columns $q_1,\,q_2$, page 79
$\langle q_1 : q_2 \rangle$ proper vertical quadrimodularity of a 2×2-matrix of points with columns $q_1,\,q_2$, page 79
$\langle q_1 q_2 \rangle$ quadrimodularity of a 2×2-matrix of points with columns $q_1,\,q_2$, page 79
S_m set of permutations, page 23
$\uparrow a$ principal up-set, page 26
$X_{\geq a}$ principal up-set, page 26
$X_{\leq a}$ principal down-set, page 26
$X^{m\times n}$ set of matrices, page 23
$\|x\|_p$ p-norm, page 31
$[x]_u$ u-branch, page 128

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