Chapter 3
The Time Derivative

It is the aim of this chapter to define a derivative operator on a suitable L_2-space, which will be used as the derivative with respect to the temporal variable in our applications. As we want to deal with Hilbert space-valued functions, we start by introducing the concept of Bochner–Lebesgue spaces, which generalises the classical scalar-valued L_p-spaces to the Banach space-valued case.

3.1 Bochner–Lebesgue Spaces

Throughout, let (Ω, Σ, μ) be a σ-finite measure space and X a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We are aiming to define the spaces $L_p(\mu; X)$ for $1 \leqslant p \leqslant \infty$. This is the space of (equivalence classes of) measurable functions attaining values in X, which are p-integrable (if $p < \infty$), or essentially bounded (if $p = \infty$) with respect to the measure μ. We begin by defining the space of simple functions on Ω with values in X and the notion of Bochner-measurability.

Definition For a function $f : \Omega \to X$ and $x \in X$ we set

$$A_{f,x} := f^{-1}[\{x\}].$$

A function $f : \Omega \to X$ is called *simple* if $f[\Omega]$ is finite and for each $x \in X \setminus \{0\}$ the set $A_{f,x}$ belongs to Σ and has finite measure. We denote the set of simple functions by $S(\mu; X)$. A function $f : \Omega \to X$ is called *Bochner-measurable* if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $S(\mu; X)$ such that

$$f_n(\omega) \to f(\omega) \quad (n \to \infty)$$

for μ-a.e. $\omega \in \Omega$.

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Remark 3.1.1 Let us comment on the definition of Bochner-measurability.

(a) For a simple function f we have

$$f = \sum_{x \in X} x \cdot \mathbb{1}_{A_{f,x}},$$

where the sum is actually finite, since $\mathbb{1}_{A_{f,x}} = 0$ for all $x \notin f[\Omega]$.

(b) If $X = \mathbb{K}$, then a function is Bochner-measurable if and only if it has a μ-measurable representative. Indeed, if f is Bochner-measurable, we find a sequence $(f_n)_n$ in $S(\mu; \mathbb{K})$ such that $f_n \to f$ pointwise μ-a.e. Hence, we find a μ-nullset $N \in \Sigma$ such that $g_n := \mathbb{1}_{\Omega \setminus N} f_n \to \mathbb{1}_{\Omega \setminus N} f =: g$ pointwise on all of Ω. Since g_n is μ-measurable and μ-measurable functions are stable under pointwise limits, g is μ-measurable itself. Since $f = g$ except for a μ-nullset, f has a μ-measurable representative. If, on the other hand, f has a μ-measurable representative, let g be this representative. Approximating real and imaginary parts separately, it suffices to treat the case $\mathbb{K} = \mathbb{R}$. Then consider for $n \in \mathbb{N}$

$$s_n := \sum_{k \in \mathbb{Z}} \frac{k+1}{n} \mathbb{1}_{M_n^k},$$

where $M_n^k := g^{-1}[(\frac{k}{n}, \frac{k+1}{n}]]$. It is easy to see that $\sup_{\omega \in \Omega} |s_n(\omega) - g(\omega)| \leqslant 1/n$ for all $\omega \in \Omega$. Hence,

$$\widetilde{s}_n := \sum_{k \in \mathbb{Z}, |k| \leqslant 2^n} \frac{k+1}{n} \mathbb{1}_{M_n^k} \in S(\mu; \mathbb{R})$$

converges pointwise everywhere to g. In consequence, f is Bochner-measurable.

(c) It is easy to check that $S(\mu; X)$ is a vector space and an $S(\mu; \mathbb{K})$-module; that is, for $f \in S(\mu; X)$ and $g \in S(\mu; \mathbb{K})$ we have $g \cdot f \in S(\mu; X)$.

(d) If $f : \Omega \to X$ is Bochner-measurable, then $\|f(\cdot)\|_X : \Omega \to \mathbb{R}$ is Bochner-measurable. Indeed, since

$$\|f(\cdot)\|_X = \lim_{n \to \infty} \|f_n(\cdot)\|_X$$

μ-a.e. and a sequence $(f_n)_{n \in \mathbb{N}}$ in $S(\mu; X)$, it suffices to show that $\|f_n(\cdot)\|_X$ is simple for all $n \in \mathbb{N}$. The latter follows since $A_{f_n,x} \cap A_{f_n,y} = \varnothing$ for $x \neq y$ and thus

$$\|f_n(\cdot)\|_X = \sum_{x \in f_n[\Omega]} \|x\|_X \cdot \mathbb{1}_{A_{f_n,x}}$$

is a real-valued simple function.

(e) If one deals with arbitrary measure spaces, the definition of simple functions has to be weakened by allowing the sets $A_{f,x}$ to have infinite measure. However, since in the applications to follow we only work with weighted Lebesgue measures, we restrict ourselves to σ-finite measure spaces.

Definition (Bochner–Lebesgue Spaces) For $p \in [1, \infty]$ we define

$$\mathcal{L}_p(\mu; X) := \left\{ f : \Omega \to X\,;\ f \text{ Bochner-measurable, } \|f(\cdot)\|_X \in \mathcal{L}_p(\mu) \right\},$$

as well as

$$L_p(\mu; X) := {}^{\mathcal{L}_p(\mu; X)}\!\big/_{\sim},$$

where $\sim$ denotes the usual equivalence relation of equality μ-almost everywhere. We equip $L_p(\mu; X)$ with the norm

$$\|f\|_p := \begin{cases} \left(\int_\Omega \|f(\omega)\|_X^p \, \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \operatorname{ess-sup}_{\omega \in \Omega} \|f(\omega)\|_X, & \text{if } p = \infty \end{cases} \qquad (f \in L_p(\mu; X)).$$

We first prove a density result.

Lemma 3.1.2 *The space $S(\mu; X)$ is dense in $L_p(\mu; X)$ for $p \in [1, \infty)$.*

Proof Let $f \in L_p(\mu; X)$. Then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $S(\mu; X)$ such that $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega \setminus N$ for some nullset $N \subseteq \Omega$. W.l.o.g. we may assume that $\|f_n(\cdot)\|_X$ and $\|f(\cdot)\|_X$ are μ-measurable on $\Omega \setminus N$ for each $n \in \mathbb{N}$. For $n \in \mathbb{N}$ we define the set

$$I_n := \left\{ \omega \in \Omega \setminus N\,;\ \|f_n(\omega)\|_X \leqslant 2\,\|f(\omega)\|_X \right\} \in \Sigma,$$

and set $\widetilde{f_n} := f_n \mathbb{1}_{I_n}$. Then $\widetilde{f_n} \in S(\mu; X)$ and we claim that $\widetilde{f_n}(\omega) \to f(\omega)$ for all $\omega \in \Omega \setminus N$. Indeed, if $f(\omega) = 0$ then $\widetilde{f_n}(\omega) = 0$ and the claim follows. If $f(\omega) \neq 0$, then there is some $n_0 \in \mathbb{N}$ such that $\|f_n(\omega)\|_X \leqslant 2\,\|f(\omega)\|_X$ for $n \geq n_0$, and hence $\omega \in \bigcap_{n \geqslant n_0} I_n$. Consequently $\widetilde{f_n}(\omega) = f_n(\omega) \to f(\omega)$. By dominated convergence, it now follows that

$$\int_\Omega \left\| \widetilde{f_n}(\omega) - f(\omega) \right\|_X^p \mathrm{d}\mu(\omega) \to 0 \quad (n \to \infty),$$

which proves the claim. □

As a consequence of the latter lemma, we can show that Bochner-measurability is preserved by pointwise convergence almost everywhere.

Proposition 3.1.3 *Let $f_n, f : \Omega \to X$ for $n \in \mathbb{N}$. Moreover, assume that f_n is Bochner-measurable for each $n \in \mathbb{N}$ and $f_n(\omega) \to f(\omega)$ as $n \to \infty$ for μ-almost every $\omega \in \Omega$. Then f is Bochner-measurable.*

Proof Since $f_n \to f$ almost everywhere, we have $[f \neq 0]\setminus N \subseteq \bigcup_{n\in\mathbb{N}}[f_n \neq 0]\setminus N$ for some nullset $N \subseteq \Omega$. Moreover, since f_n is Bochner-measurable, the definition of simple functions yields that $\bigcup_{n\in\mathbb{N}}[f_n \neq 0] \subseteq \bigcup_{n\in\mathbb{N}} B_n$, where, for all $n \in \mathbb{N}$, B_n is measurable with $\mu(B_n) < \infty$. The latter implies that there exists a sequence of measurable sets $(A_n)_{n\in\mathbb{N}}$ such that $A_n \subseteq A_{n+1}$, $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ and

$$[f \neq 0] \setminus N \subseteq \bigcup_{n\in\mathbb{N}} A_n.$$

For $n \in \mathbb{N}$ we set $g_n := \mathbb{1}_{A_n \cap [\widetilde{f_n} \leqslant n]} f_n$, where $\widetilde{f_n} : \Omega \to \mathbb{R}$ is measurable and equals $\|f_n(\cdot)\|_X$ μ-almost everywhere (cp. Remark 3.1.1(d) and (b)). In this way we obtain a sequence of Bochner-measurable functions with $g_n \to f$ μ-almost everywhere. Moreover, $g_n \in L_1(\mu; X)$ for each $n \in \mathbb{N}$ and thus, for each $n \in \mathbb{N}$ we find a simple function h_n with $\|g_n - h_n\|_1 \leqslant 2^{-n}$ by Lemma 3.1.2. Then

$$\int_\Omega \sum_{n\in\mathbb{N}} \|g_n(\omega) - h_n(\omega)\|_X \, \mathrm{d}\mu(\omega) < \infty$$

and hence, $\sum_{n\in\mathbb{N}} \|g_n(\omega) - h_n(\omega)\|_X < \infty$ for μ-almost every $\omega \in \Omega$, which particularily implies $g_n - h_n \to 0$ μ-almost everywhere. Hence, $h_n \to f$ μ-almost everywhere, which shows the Bochner-measurability of f. □

We can now prove that the spaces $L_p(\mu; X)$ are actually Banach spaces.

Proposition 3.1.4 *Let $p \in [1, \infty]$. Then $(L_p(\mu; X), \|\cdot\|_p)$ is a Banach space and if $X = H$ is a Hilbert space, then so too is $L_2(\mu; H)$ with the scalar product given by*

$$\langle f, g\rangle_2 := \int_\Omega \langle f(\omega), g(\omega)\rangle_H \, \mathrm{d}\mu(\omega) \quad (f, g \in L_2(\mu; H)).$$

Proof We just show the completeness of $L_p(\mu; X)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L_p(\mu; X)$ such that $\sum_{n=1}^\infty \|f_n\|_p < \infty$. We set

$$g_n(\omega) := \|f_n(\omega)\|_X \quad (n \in \mathbb{N}, \omega \in \Omega).$$

Then $(g_n)_{n\in\mathbb{N}}$ is a sequence in $L_p(\mu)$ such that $\sum_{n=1}^\infty \|g_n\|_p < \infty$. By the completeness of $L_p(\mu)$ we infer that

$$g := \sum_{n=1}^\infty g_n$$

exists and is an element in $L_p(\mu)$. In particular, $g(\omega) < \infty$ for μ-a.e. $\omega \in \Omega$ and thus,

$$\sum_{n=1}^{\infty} \|f_n(\omega)\|_X = \sum_{n=1}^{\infty} g_n(\omega) < \infty$$

for μ-a.e. $\omega \in \Omega$. By the completeness of X we can define

$$f(\omega) := \sum_{n=1}^{\infty} f_n(\omega)$$

for μ-a.e. $\omega \in \Omega$. Note that f is Bochner-measurable by Proposition 3.1.3. We need to prove that $f \in L_p(\mu; X)$ and that $\sum_{n=1}^{k} f_n \to f$ in $L_p(\mu; X)$ as $k \to \infty$. For this, it suffices to prove that

$$\sum_{n=k}^{\infty} f_n \in L_p(\mu; X) \text{ and } \sum_{n=k}^{\infty} f_n \to 0 \text{ in } L_p(\mu; X) \text{ as } k \to \infty. \tag{3.1}$$

Indeed, this would imply both $f - \sum_{n=1}^{k} f_n \in L_p(\mu; X)$ and the desired convergence result. We prove (3.1) for $p < \infty$ and $p = \infty$ separately.

First, let $p = \infty$. For each $n \in \mathbb{N}$ we have $f_n \in L_\infty(\mu; X)$ and thus $\|f_n(\omega)\|_X \leqslant \|f_n\|_\infty$ for all $\omega \in \Omega \setminus N_n$ and some nullset $N_n \subseteq \Omega$. We set $N := \bigcup_{n=1}^{\infty} N_n$, which is again a nullset. For $k \in \mathbb{N}$ and $\omega \in \Omega \setminus N$ we then estimate

$$\left\| \sum_{n=k}^{\infty} f_n(\omega) \right\|_X \leqslant \sum_{n=k}^{\infty} \|f_n(\omega)\|_X \leqslant \sum_{n=k}^{\infty} \|f_n\|_\infty ,$$

which yields (3.1).

Now, let $p < \infty$. For $k \in \mathbb{N}$ we estimate

$$\begin{aligned}
\left(\int_\Omega \left(\left\| \sum_{n=k}^{\infty} f_n(\omega) \right\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} &\leqslant \left(\int_\Omega \left(\sum_{n=k}^{\infty} \|f_n(\omega)\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} \\
&= \left(\int_\Omega \lim_{m\to\infty} \left(\sum_{n=k}^{m} \|f_n(\omega)\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}}
\end{aligned}$$

$$= \lim_{m\to\infty} \left(\int\limits_{\Omega} \left(\sum_{n=k}^{m} \| f_n(\omega) \|_X \right)^p \, \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}}$$

$$\leqslant \lim_{m\to\infty} \sum_{n=k}^{m} \| f_n \|_p = \sum_{n=k}^{\infty} \| f_n \|_p ,$$

where we have used monotone convergence in the third line. This estimate yields (3.1). □

We now want to define an X-valued integral for functions in $L_1(\mu; X)$; the so-called Bochner-integral.

Proposition 3.1.5 *The mapping*[1]

$$\int_{\Omega} \mathrm{d}\mu \colon S(\mu; X) \subseteq L_1(\mu; X) \to X$$

$$f \mapsto \sum_{x\in X} x \cdot \mu(A_{f,x})$$

is linear and continuous, and thus has a unique continuous linear extension to $L_1(\mu; X)$*, called the* Bochner-integral. *Moreover,*

$$\left\| \int_{\Omega} f \, \mathrm{d}\mu \right\|_X \leqslant \| f \|_1 \quad (f \in L_1(\mu; X)),$$

and for $A \in \Sigma$, $f \in L_1(\mu; X)$ *we set*

$$\int_A f \, \mathrm{d}\mu := \int_{\Omega} f \cdot \mathbb{1}_A \, \mathrm{d}\mu.$$

Proof We first show linearity. Let $f, g \in S(\mu; X)$ and $\lambda \in \mathbb{K}$. Then, for $x \in X$ we have

$$A_{\lambda f+g,x} = (\lambda f+g)^{-1}[\{x\}] = \bigcup_{y\in X} \left(f^{-1}[\{y\}]\cap g^{-1}[\{x-\lambda y\}]\right) = \bigcup_{y\in X} A_{f,y}\cap A_{g,x-\lambda y},$$

[1] Note that the sum is indeed finite and all summands are well-defined if we set $0_X \cdot \infty := 0_X$.

and therefore $\mu(A_{\lambda f+g,x}) = \sum_{y\in X} \mu(A_{f,y} \cap A_{g,x-\lambda y})$. Thus, we compute

$$\begin{aligned}
\int_\Omega (\lambda f + g)\,\mathrm{d}\mu &= \sum_{x\in X} x \cdot \mu(A_{\lambda f+g,x}) = \sum_{x\in X}\sum_{y\in X} x \cdot \mu(A_{f,y} \cap A_{g,x-\lambda y}) \\
&= \sum_{y\in X}\sum_{x\in X} \lambda y \cdot \mu(A_{f,y} \cap A_{g,x-\lambda y}) \\
&\quad + \sum_{y\in X}\sum_{x\in X} (x-\lambda y) \cdot \mu(A_{f,y} \cap A_{g,x-\lambda y}) \\
&= \sum_{y\in X}\sum_{x\in X} \lambda y \cdot \mu(A_{f,y} \cap A_{g,x-\lambda y}) + \sum_{y\in X}\sum_{z\in X} z \cdot \mu(A_{f,y} \cap A_{g,z}),
\end{aligned}$$

where we interchanged the finite sums. Now,

$$\sum_{x\in X} \mu(A_{f,y} \cap A_{g,x-\lambda y}) = \mu\Big(A_{f,y} \cap \bigcup_{x\in X} A_{g,x-\lambda y}\Big) = \mu(A_{f,y})$$

as well as

$$\sum_{y\in X} \mu(A_{f,y} \cap A_{g,z}) = \mu\Big(\bigcup_{y\in X} A_{f,y} \cap A_{g,z}\Big) = \mu(A_{g,z}),$$

and therefore we conclude

$$\int_\Omega (\lambda f + g)\,\mathrm{d}\mu = \lambda \sum_{y\in X} y \cdot \mu(A_{f,y}) + \sum_{z\in X} z \cdot \mu(A_{g,z}) = \lambda \int_\Omega f\,\mathrm{d}\mu + \int_\Omega g\,\mathrm{d}\mu.$$

In order to prove continuity, let $f \in S(\mu; X)$. We estimate

$$\begin{aligned}
\left\| \int_\Omega f\,\mathrm{d}\mu \right\|_X &= \left\| \sum_{x\in f[\Omega]} x \cdot \mu(A_{f,x}) \right\|_X \leqslant \sum_{x\in f[\Omega]} \|x\|_X\, \mu(A_{f,x}) \\
&= \int_\Omega \sum_{x\in f[\Omega]} \|x\|_X\, \mathbb{1}_{A_{f,x}}\,\mathrm{d}\mu \\
&= \int_\Omega \|f(\cdot)\|_X\,\mathrm{d}\mu = \|f\|_1\,.
\end{aligned}$$

The remaining assertions now follow from Lemma 3.1.2 by continuous extension (see Corollary 2.1.5). □

The next proposition tells us how the Bochner-integral of a function behaves if we compose the function with a bounded or closed linear operator first. In what follows, let $X' := L(X, \mathbb{K})$ denote the dual space of X.

Proposition 3.1.6 *Let $f \in L_1(\mu; X)$, Y a Banach space.*

(a) *Let $B \in L(X, Y)$. Then $B \circ f \in L_1(\mu; Y)$ and*

$$\int_\Omega B \circ f \, \mathrm{d}\mu = B \int_\Omega f \, \mathrm{d}\mu.$$

(b) *If $X_0 \subseteq X$ is a closed subspace and $f(\omega) \in X_0$ for μ-a.e. $\omega \in \Omega$, then $\int_\Omega f \, \mathrm{d}\mu \in X_0$.*

(c) *(Theorem of Hille) Let $A\colon \mathrm{dom}(A) \subseteq X \to Y$ be a closed linear operator and assume that $f(\omega) \in \mathrm{dom}(A)$ for μ-a.e. $\omega \in \Omega$ and that $A \circ f \in L_1(\mu; Y)$. Then $\int_\Omega f \, \mathrm{d}\mu \in \mathrm{dom}(A)$ and*

$$A \int_\Omega f \, \mathrm{d}\mu = \int_\Omega A \circ f \, \mathrm{d}\mu.$$

Proof

(a) At first we observe that, if $f \in S(\mu; X)$, then

$$B \circ f = B \circ \sum_{x \in X \setminus \{0\}} x \cdot \mathbb{1}_{A_{f,x}} = \sum_{x \in X \setminus \{0\}} Bx \cdot \mathbb{1}_{A_{f,x}}.$$

Thus, $B \circ f \in S(\mu; Y)$ since $Bx \cdot \mathbb{1}_{A_{f,x}} \in S(\mu; Y)$, the sum is finite and $S(\mu; Y)$ is a vector space. Let now be $f \in L_1(\mu; X)$. Then there is $(f_n)_{n \in \mathbb{N}}$ a sequence in $S(\mu; X)$ such that $f_n \to f$ μ-a.e. Then $B \circ f_n \in S(\mu; Y)$ (see above) and due to the continuity of B we have that $B \circ f_n \to B \circ f$ μ-a.e., hence $B \circ f$ is Bochner-measurable. Moreover, $\|B \circ f(\cdot)\|_Y \leqslant \|B\| \, \|f(\cdot)\|_X$, which yields that $B \circ f \in L_1(\mu; Y)$. By continuity of both B and $\int_\Omega \mathrm{d}\mu$, it suffices to check the interchanging property for any $f \in S(\mu; X)$ alone. However, this is clear, since for a simple function f

$$B \circ f = B \left(\sum_{x \in X} x \cdot \mathbb{1}_{A_{f,x}} \right) = \sum_{x \in X} Bx \cdot \mathbb{1}_{A_{f,x}},$$

where the sum is actually finite and hence,

$$\begin{aligned} \int_\Omega B \circ f \, \mathrm{d}\mu &= \int_\Omega \sum_{x \in X} Bx \cdot \mathbb{1}_{A_{f,x}} \, \mathrm{d}\mu = \sum_{x \in X} \int_\Omega Bx \cdot \mathbb{1}_{A_{f,x}} \, \mathrm{d}\mu \\ &= \sum_{x \in X} Bx \cdot \mu(A_{f,x}) = B \left(\sum_{x \in X} x \cdot \mu(A_{f,x}) \right) = B \int_\Omega f \, \mathrm{d}\mu, \end{aligned}$$

where in the third equality we have used that $Bx \cdot \mathbb{1}_{A_{f,x}}$ is a simple function.

(b) Let $x' \in X'$ with $x'|_{X_0} = 0$. It follows from (a) that

$$x'\left(\int_\Omega f \, \mathrm{d}\mu\right) = \int_\Omega x' \circ f \, \mathrm{d}\mu = 0,$$

and since x' was arbitrary, it follows that $\int_\Omega f \, \mathrm{d}\mu \in X_0$ from the Theorem of Hahn–Banach.

(c) Consider the space $L_1(\mu; X \times Y)$. By assumption, it follows that

$$(f, A \circ f) \in L_1(\mu; X \times Y).$$

However, $(f, A \circ f)(\omega) = (f(\omega), (A \circ f)(\omega)) \in A \subseteq X \times Y$ for μ-a.e. $\omega \in \Omega$, and since A is closed we can use (b) to derive that

$$\int_\Omega (f, A \circ f) \, \mathrm{d}\mu \in A. \tag{3.2}$$

Let π_1, π_2 be the projection from $X \times Y$ to X and Y, respectively. It then follows from part (a) that

$$\pi_1\left(\int_\Omega (f, A \circ f) \, \mathrm{d}\mu\right) = \int_\Omega \pi_1(f, A \circ f) \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu,$$

and analogously for π_2. Using these equalities we derive from (3.2) that $\int_\Omega f \, \mathrm{d}\mu \in \mathrm{dom}(A)$ and that $A \int_\Omega f \, \mathrm{d}\mu = \int_\Omega A \circ f \, \mathrm{d}\mu$. □

As a consequence of the latter proposition, we derive the fundamental theorem of calculus for Banach space-valued functions.

Corollary 3.1.7 (Fundamental Theorem of Calculus) *Let $a, b \in \mathbb{R}, a < b$ and consider the measure space $([a, b], \mathcal{B}([a, b]), \lambda)$, where $\mathcal{B}([a, b])$ denotes the Borel-σ-algebra of $[a, b]$ and λ is the Lebesgue measure. Let $f : [a, b] \to X$ be continuously differentiable.*[2] *Then*

$$f(b) - f(a) = \int_{[a,b]} f' \, \mathrm{d}\lambda.$$

Proof Note first of all that continuous functions are Bochner-measurable (which can be easily seen using Theorem 3.1.10 below). Thus, the integral on the right-hand side is well-defined. Let $\varphi \in X'$. Then $\varphi \circ f : [a, b] \to \mathbb{K}$ is continuously differentiable, and $(\varphi \circ f)'(t) = \left(\varphi \circ f'\right)(t)$. Using Proposition 3.1.6 (a) together

[2] By this we mean that f is continuous on $[a, b]$, continuously differentiable on (a, b) and f' has a continuous extension to $[a, b]$.

with the fundamental theorem of calculus for the scalar-valued case we get

$$\varphi\left(\int_{[a,b]} f'\,\mathrm{d}\lambda\right) = \int_{[a,b]} \left(\varphi \circ f'\right)\,\mathrm{d}\lambda = \varphi\left(f(b)\right) - \varphi\left(f(a)\right) = \varphi\left(f(b) - f(a)\right).$$

Since this holds for all $\varphi \in X'$, the assertion follows from the Theorem of Hahn–Banach. □

Next we state a density result, which will be useful throughout the course.

Lemma 3.1.8 *Let $1 \leqslant p < \infty$, $\mathcal{D} \subseteq L_p(\mu)$ be total in $L_p(\mu)$ and X a Banach space. Then the set $\{\varphi(\cdot)x\,;\ x \in X,\ \varphi \in \mathcal{D}\}$ is total in $L_p(\mu; X)$.*

Proof By Lemma 3.1.2, we know that $S(\mu; X)$ is dense in $L_p(\mu; X)$. Thus, it suffices to approximate $\mathbb{1}_A x$ for some $A \in \Sigma$ with $\mu(A) < \infty$ and $x \in X$. For this, however, take a sequence $(\phi_n)_n$ in the linear hull of $\mathcal{D}$ with $\phi_n \to \mathbb{1}_A$ in $L_p(\mu)$ as $n \to \infty$. Then

$$\|\mathbb{1}_A x - \phi_n x\|_{L_p(\mu;X)} = \|x\|_X\,\|\mathbb{1}_A - \phi_n\|_{L_p(\mu)} \to 0 \quad (n \to \infty).$$

Thus, the claim follows. □

The following application of Lemma 3.1.8 also deals with a dense subset of X.

Lemma 3.1.9 *Let $1 \leqslant p < \infty$, $\mathcal{D} \subseteq L_p(\mu)$ be total in $L_p(\mu)$, X a Banach space, $D_0 \subseteq X$ total in X. Then $\{\varphi(\cdot)x\,;\ x \in D_0, \varphi \in \mathcal{D}\}$ is total in $L_p(\mu; X)$.*

Proof The proof follows upon realising that the set $\{\varphi(\cdot)x\,;\ x \in D_0,\ \varphi \in \mathcal{D}\}$ is total in the set $\{\varphi(\cdot)x\,;\ x \in X,\ \varphi \in \mathcal{D}\}$. From here we just apply Lemma 3.1.8. □

We conclude this section by stating and proving the celebrated Theorem of Pettis, which characterises Bochner-measurability in terms of weak measurability.

Theorem 3.1.10 (Theorem of Pettis) *Let $f\colon \Omega \to X$. Then f is Bochner-measurable if and only if*

(a) f *is* weakly Bochner-measurable*; that is, $x' \circ f\colon \Omega \to \mathbb{K}$ is Bochner-measurable for each $x' \in X'$, and*
(b) f *is* almost separably-valued*; that is, $\overline{\operatorname{lin} f[\Omega \setminus N_0]}$ is separable for some $N_0 \in \Sigma$ with $\mu(N_0) = 0$.*

Proof If f is Bochner-measurable, then clearly it is weakly Bochner-measurable. Further, as f is the almost everywhere limit of simple functions, it is almost separably-valued, since each simple function attains values in a finite-dimensional subspace of X.

Assume now conversely that f satisfies (a) and (b). We define $Y := \overline{\operatorname{lin} f[\Omega \setminus N_0]}$, which is a separable Banach space by (b). Thus, there exists a sequence $(x_n')_{n\in\mathbb{N}}$ in X' such that

$$\|y\| = \sup_{n\in\mathbb{N}} |x_n'(y)| \quad (y \in Y).$$

Since for each $n \in \mathbb{N}$ the function $g_n := |x_n' \circ f|$ is Bochner-measurable by (a) and Remark 3.1.1(d), we find a μ-nullset N_n and a measurable function $\widetilde{g}_n \colon \Omega \to \mathbb{R}$ such that $g_n = \widetilde{g}_n$ on $\Omega \setminus N_n$ by Remark 3.1.1(b). Then $\sup_{n\in\mathbb{N}} \widetilde{g}_n(\cdot)$ is measurable and

$$\|f(\omega)\| = \sup_{n\in\mathbb{N}} \widetilde{g}_n(\omega) \quad (\omega \in \Omega \setminus N),$$

where $N := \bigcup_{n\in\mathbb{N}_0} N_n$, which shows that $\|f(\cdot)\|$ is Bochner-measurable. Let $\varepsilon > 0$, $(y_n)_{n\in\mathbb{N}}$ a dense sequence in Y. Applying the previous argument to the function $f_k(\cdot) := f(\cdot) - y_k$ for $k \in \mathbb{N}$ we infer that $\|f_k(\cdot)\|$ is Bochner-measurable and hence, there is a μ-nullset N_k' and a measurable funtion $\widetilde{f}_k \colon \Omega \to \mathbb{R}$ such that $\|f_k\| = \widetilde{f}_k$ on $\Omega \setminus N_k'$. Consequently, the sets

$$E_k := [\widetilde{f}_k \leqslant \varepsilon] = \big\{\omega \in \Omega\,;\ \widetilde{f}_k(\omega) \leqslant \varepsilon\big\} \quad (k \in \mathbb{N})$$

are measurable. Moreover, by the density of $\{y_n\,;\ n \in \mathbb{N}\}$ in Y, we get that $\Omega \setminus N' \subseteq \bigcup_{k\in\mathbb{N}} E_k$ with $N' := \bigcup_{k=1}^{\infty} N_k' \cup N_0$. Setting $F_1 := E_1$ and $F_{n+1} = E_{n+1} \setminus \bigcup_{k=1}^{n} F_k$ for $n \in \mathbb{N}$, we obtain a sequence of pairwise disjoint measurable sets $(F_n)_{n\in\mathbb{N}}$ with $\Omega \setminus N' \subseteq \bigcup_{n\in\mathbb{N}} F_n$. We set

$$g := \sum_{k=1}^{\infty} y_k \mathbb{1}_{F_k}$$

and obtain $\|f(\omega) - g(\omega)\| \leqslant \varepsilon$ for each $\omega \in \Omega \setminus N'$. Hence, if g is Bochner-measurable, then f is Bochner-measurable as well. Indeed, we find a sequence of such functions converging to f μ-almost everywhere and so Proposition 3.1.3 applies. For showing the Bochner-measurability of g, let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of pairwise disjoint measurable sets such that $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$ and $\mu(\Omega_k) < \infty$ for each $k \in \mathbb{N}$. For $n \in \mathbb{N}$ we set

$$g_n := \sum_{k,j=1}^{n} y_k \mathbb{1}_{F_k \cap \Omega_j}.$$

Then $(g_n)_{n\in\mathbb{N}}$ is a sequence of simple functions with $g_n \to g$ pointwise as $n \to \infty$ and thus, g is Bochner-measurable. □

3.2 The Time Derivative as a Normal Operator

Now let H be a Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For $\nu \in \mathbb{R}$ and $p \in [1, \infty)$ we define the measure

$$\mu_{p,\nu}(A) := \int_A \mathrm{e}^{-p\nu t} \, \mathrm{d}\lambda(t)$$

for A in the Borel-σ-algebra, $\mathcal{B}(\mathbb{R})$, of $\mathbb{R}$. As our underlying Hilbert space for the time derivative we set

$$L_{2,\nu}(\mathbb{R}; H) := L_2(\mu_{2,\nu}; H).$$

In the same way we define

$$L_{p,\nu}(\mathbb{R}; H) := L_p(\mu_{p,\nu}; H)$$

for $p \in [1, \infty)$. If $H = \mathbb{K}$ we abbreviate $L_{p,\nu}(\mathbb{R}) := L_{p,\nu}(\mathbb{R}; \mathbb{K})$. Thus, $f \in L_{p,\nu}(\mathbb{R}; H)$ if and only if f is Bochner measurable and

$$\int_{\mathbb{R}} \|f(t)\|_H^p \, \mathrm{d}\mu_{p,\nu}(t) = \int_{\mathbb{R}} \|f(t)\|_H^p \, \mathrm{e}^{-p\nu t} \, \mathrm{d}t < \infty.$$

Our aim is to define the time derivative on $L_{2,\nu}(\mathbb{R}; H)$. For this, we define a suitable anti-derivative as an operator, which for $\nu \neq 0$ turns out to be one-to-one and bounded. Then we introduce the time derivative as the inverse of this anti-derivative. The reason for doing it that way is to easily get a formula for the adjoint for the time derivative using the boundedness of the anti-derivative.

We start our considerations with the definition of convolution operators in $L_{2,\nu}(\mathbb{R}; H)$.

Lemma 3.2.1 *Let $k \in L_{1,\nu}(\mathbb{R})$. We define the convolution operator*

$$k*\colon L_{2,\nu}(\mathbb{R}; H) \to L_{2,\nu}(\mathbb{R}; H)$$

by

$$(k * f)(t) := \int_{\mathbb{R}} k(s) f(t - s) \, \mathrm{d}s,$$

which exists for a.e. $t \in \mathbb{R}$. Then, $k*$ is linear and bounded with $\|k*\| \leqslant \|k\|_{L_{1,\nu}(\mathbb{R})}$.

Proof Let $f \in L_{2,\nu}(\mathbb{R}; H)$. We first prove that $s \mapsto k(s)f(t-s) \in L_1(\mathbb{R}; H)$ for a.e. $t \in \mathbb{R}$. The Bochner-measurability is clear since k and f are both Bochner-measurable. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|k(s)f(t-s)\|_H \, \mathrm{d}s \right)^2 \mathrm{e}^{-2\nu t} \, \mathrm{d}t \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |k(s)|^{\frac{1}{2}} \mathrm{e}^{-\frac{\nu}{2}s} \, |k(s)|^{\frac{1}{2}} \mathrm{e}^{-\frac{\nu}{2}s} \, \|f(t-s)\|_H \, \mathrm{e}^{-\nu(t-s)} \, \mathrm{d}s \right)^2 \mathrm{d}t \\
&\leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |k(s)| \, \mathrm{e}^{-\nu s} \, \mathrm{d}s \right) \left(\int_{\mathbb{R}} |k(s)| \, \mathrm{e}^{-\nu s} \, \|f(t-s)\|_H^2 \, \mathrm{e}^{-2\nu(t-s)} \, \mathrm{d}s \right) \mathrm{d}t \\
&= \|k\|_{L_{1,\nu}(\mathbb{R})} \int_{\mathbb{R}} |k(s)| \int_{\mathbb{R}} \|f(t-s)\|^2 \, \mathrm{e}^{-2\nu(t-s)} \, \mathrm{d}t \, \mathrm{e}^{-\nu s} \, \mathrm{d}s \\
&= \|k\|_{L_{1,\nu}(\mathbb{R})}^2 \, \|f\|_{L_{2,\nu}(\mathbb{R};H)}^2 \,,
\end{aligned}$$

which on the one hand proves that

$$\int_{\mathbb{R}} \|k(s)f(t-s)\|_H \, \mathrm{d}s < \infty$$

for a.e. $t \in \mathbb{R}$ and on the other hand shows the norm estimate, once we have shown the Bochner-measurability of $k*f$. For proving the latter, we apply Theorem 3.1.10. Since f is Bochner-measurable, we find a nullset N such that $H_0 := \overline{\operatorname{lin} f[\mathbb{R} \setminus N]}$ is separable. Hence, for almost every $t \in \mathbb{R}$ we have

$$(k*f)(t) = \int_{\mathbb{R}} k(s)f(t-s) \, \mathrm{d}s = \int_{\mathbb{R}\setminus N} k(t-s)f(s) \, \mathrm{d}s \in H_0$$

by Proposition 3.1.6(b). Thus, $k*f$ is almost separably-valued. Moreover, for $x' \in H'$ we have by Proposition 3.1.6(a)

$$x' \circ (k*f) = k*(x' \circ f)$$

almost everywhere and thus, the weak Bochner-measurability follows from the fact that the convolution of two measurable scalar-valued functions is measurable. Since the linearity of $k*$ is clear the proof is done. □

Definition For $\nu \neq 0$ we define the operator

$$I_\nu \colon L_{2,\nu}(\mathbb{R}; H) \to L_{2,\nu}(\mathbb{R}; H)$$

by

$$I_\nu := \begin{cases} \mathbb{1}_{[0,\infty)}*, & \text{if } \nu > 0, \\ -\mathbb{1}_{(-\infty,0]}*, & \text{if } \nu < 0. \end{cases}$$

Note that, by Lemma 3.2.1, I_ν is bounded with $\|I_\nu\| \leqslant \frac{1}{|\nu|}$.

Remark 3.2.2 For $\nu > 0$, $f \in L_{2,\nu}(\mathbb{R}; H)$ we have

$$I_\nu f(t) = \mathbb{1}_{[0,\infty)} * f(t) = \int_0^\infty f(t-s)\,\mathrm{d}s = \int_{-\infty}^t f(s)\,\mathrm{d}s \quad (\text{a.e. } t \in \mathbb{R}).$$

Analogously, for $\nu < 0$, $f \in L_{2,\nu}(\mathbb{R}; H)$ we have

$$I_\nu f(t) = -\int_t^\infty f(s)\,\mathrm{d}s \quad (\text{a.e. } t \in \mathbb{R}).$$

Proposition 3.2.3 *Let $\nu \neq 0$. Then I_ν is one-to-one and $C_\mathrm{c}^1(\mathbb{R}; H)$, the space of continuously differentiable, compactly supported functions on $\mathbb{R}$ with values in H, is in the range of I_ν.*

Proof We just prove the assertion for the case when $\nu > 0$. Let $f \in L_{2,\nu}(\mathbb{R}; H)$ satisfy $I_\nu f = 0$. In particular, we obtain for all $t \in \mathbb{R} \setminus N$ that $0 = I_\nu f(t) = \int_{-\infty}^t f(s)\,\mathrm{d}s$ for some Lebesgue nullset, $N \subseteq \mathbb{R}$. Then for $a, b \in \mathbb{R} \setminus N$ with $a < b$ and $x \in H$ we have that

$$\begin{aligned}\left\langle f, \mathrm{e}^{2\nu(\cdot)}\mathbb{1}_{[a,b]} \cdot x \right\rangle_{L_{2,\nu}(\mathbb{R};H)} &= \int_\mathbb{R} \left\langle f(t), \mathrm{e}^{2\nu t}\mathbb{1}_{[a,b]}(t) \cdot x \right\rangle_H \mathrm{e}^{-2\nu t}\,\mathrm{d}t \\ &= \left\langle \int_a^b f(t)\,\mathrm{d}t, x \right\rangle_H \\ &= \langle (I_\nu f)(b) - (I_\nu f)(a), x \rangle_H = 0.\end{aligned}$$

Thus $f = 0$. Indeed, since $\mathbb{R} \setminus N$ is dense in $\mathbb{R}$, $\left\{\mathrm{e}^{2\nu(\cdot)}\mathbb{1}_{[a,b]}\,;\ a, b \in \mathbb{R} \setminus N\right\}$ is total in $L_{2,\nu}(\mathbb{R})$. Hence, $\left\{\mathrm{e}^{2\nu(\cdot)}\mathbb{1}_{[a,b]} \cdot x\,;\ a, b \in \mathbb{R} \setminus N,\ x \in H\right\}$ is total in $L_{2,\nu}(\mathbb{R}; H)$ by Lemma 3.1.8. This proves the injectivity of I_ν. Moreover, if $\varphi \in C_\mathrm{c}^1(\mathbb{R}; H)$ then by Corollary 3.1.7 we have

$$\varphi(t) = \int_{-\infty}^t \varphi'(s)\,\mathrm{d}s = \left(I_\nu \varphi'\right)(t) \quad (\text{a.e. } t \in \mathbb{R}).$$ □

Definition For $\nu \neq 0$ we define the *time derivative*, $\partial_{t,\nu}$, on $L_{2,\nu}(\mathbb{R}; H)$ by

$$\partial_{t,\nu} := I_\nu^{-1}.$$

Note that by Lemma 3.2.1 and Proposition 3.2.3, $\partial_{t,\nu}$ is a closed linear operator for which $C_c^1(\mathbb{R}; H) \subseteq \mathrm{dom}(\partial_{t,\nu})$. Since

$$C_c^1(\mathbb{R}; H) \supseteq \mathrm{lin}\left\{\varphi \cdot x\,;\ \varphi \in C_c^1(\mathbb{R}),\ x \in H\right\}$$

we infer that $\partial_{t,\nu}$ is densely defined by Lemma 3.1.8 and Exercise 3.2. Moreover, since $I_\nu \varphi' = \varphi$ for $\varphi \in C_c^1(\mathbb{R}; H)$ we get that

$$\partial_{t,\nu}\varphi = \varphi';$$

that is, $\partial_{t,\nu}$ extends the classical derivative of continuously differentiable functions. We shall discuss the actual domain of $\partial_{t,\nu}$ in the next chapter.

Proposition 3.2.4 *Let $\nu \neq 0$. Then $\mathcal{D}_H := \mathrm{lin}\left\{\varphi \cdot x\,;\ \varphi \in C_c^\infty(\mathbb{R}),\ x \in H\right\}$ is a core for $\partial_{t,\nu}$. Here, $C_c^\infty(\mathbb{R})$ denotes the space of smooth functions on $\mathbb{R}$ with compact support.*

Proof We first prove that

$$\left\{\varphi'\,;\ \varphi \in C_c^\infty(\mathbb{R})\right\} \tag{3.3}$$

is dense in $L_{2,\nu}(\mathbb{R})$. As $C_c^\infty(\mathbb{R})$ is dense in $L_{2,\nu}(\mathbb{R})$ (see Exercise 3.2), it suffices to approximate functions in $C_c^\infty(\mathbb{R})$. For this, let $f \in C_c^\infty(\mathbb{R})$. We now define

$$\varphi_n(t) := \begin{cases} \int_{-\infty}^t f(s) - f(s-n)\,\mathrm{d}s & \text{if } \nu > 0, \\ \int_{-\infty}^t f(s) - f(s+n)\,\mathrm{d}s & \text{if } \nu < 0 \end{cases} \qquad (t \in \mathbb{R}, n \in \mathbb{N}).$$

Then $\varphi_n \in C_c^\infty(\mathbb{R})$ for each $n \in \mathbb{N}$ and

$$\varphi_n'(t) = \begin{cases} f(t) - f(t-n) & \text{if } \nu > 0, \\ f(t) - f(t+n) & \text{if } \nu < 0 \end{cases} \qquad (t \in \mathbb{R}, n \in \mathbb{N}).$$

Consequently,

$$\begin{aligned} \|\varphi_n' - f\|_{L_{2,\nu}(\mathbb{R})}^2 &= \begin{cases} \int_{\mathbb{R}} |f(t-n)|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t & \text{if } \nu > 0, \\ \int_{\mathbb{R}} |f(t+n)|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t & \text{if } \nu < 0 \end{cases} \\ &= \|f\|_{L_{2,\nu}(\mathbb{R})}^2\,\mathrm{e}^{-2|\nu|n} \to 0 \quad (n \to \infty), \end{aligned}$$

which shows the density of (3.3) in $L_{2,\nu}(\mathbb{R})$. By Lemma 3.1.8 we have that

$$\left\{\varphi' \cdot x\,;\ \varphi \in C_c^\infty(\mathbb{R}),\ x \in H\right\}$$

is total in $L_{2,\nu}(\mathbb{R}; H)$ and so $\partial_{t,\nu}[\mathcal{D}_H]$ is dense in $L_{2,\nu}(\mathbb{R}; H)$. Now let $f \in \operatorname{dom}(\partial_{t,\nu})$ and $\varepsilon > 0$. By what we have shown above there exists some $\varphi \in \mathcal{D}_H$ such that

$$\|\partial_{t,\nu}\varphi - \partial_{t,\nu} f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \varepsilon.$$

Since $\partial_{t,\nu}^{-1} = I_\nu$ is bounded with $\left\|\partial_{t,\nu}^{-1}\right\| \leqslant \frac{1}{|\nu|}$, the latter implies that

$$\|\varphi - f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \frac{\varepsilon}{|\nu|},$$

and hence, $\mathcal{D}_H$ is indeed a core for $\partial_{t,\nu}$. □

Corollary 3.2.5 *For $\nu \in \mathbb{R}$ the mapping*

$$\begin{aligned} \exp(-\nu \mathrm{m}) : L_{2,\nu}(\mathbb{R}; H) &\to L_2(\mathbb{R}; H) \\ f &\mapsto (t \mapsto \mathrm{e}^{-\nu t} f(t)) \end{aligned}$$

is unitary, and for $\nu, \mu \neq 0$ one has

$$\exp(-\nu \mathrm{m})(\partial_{t,\nu} - \nu)\exp(-\nu \mathrm{m})^{-1} = \exp(-\mu \mathrm{m})(\partial_{t,\mu} - \mu)\exp(-\mu \mathrm{m})^{-1}.$$

Proof The proof is left as Exercise 3.5. For this we recall that the equality to be proven is an equality of relations and particularly includes the equality of the (natural) domains of the operators involved. Furthermore, note that it suffices to show equality on $C_c^\infty(\mathbb{R}; H)$ and then to use an appropriate density result. □

By Corollary 3.2.5 we can now define $\partial_{t,0}$. Let $\nu \neq 0$. Then

$$\partial_{t,0} := \exp(-\nu \mathrm{m})(\partial_{t,\nu} - \nu)\exp(-\nu \mathrm{m})^{-1}.$$

Note that in view of Corollary 3.2.5, the assertion of Proposition 3.2.4 now also holds for $\nu = 0$.

Finally, we want to compute the adjoint of $\partial_{t,\nu}$.

Corollary 3.2.6 *Let $\nu \in \mathbb{R}$. The adjoint of $\partial_{t,\nu}$ is given by*

$$\partial_{t,\nu}^* = -\partial_{t,\nu} + 2\nu.$$

In particular, $\partial_{t,\nu}$ is a normal operator with $\operatorname{Re} \partial_{t,\nu} := \frac{1}{2}\left(\overline{\partial_{t,\nu} + \partial_{t,\nu}^*}\right) = \nu$, *and $\partial_{t,0}$ is skew-selfadjoint.*

Proof Let $\nu \neq 0$ first. Integrating by parts, one obtains

$$\begin{aligned}\int_{\mathbb{R}} \left\langle \partial_{t,\nu}\varphi(t), \psi(t)\right\rangle \mathrm{e}^{-2\nu t}\,\mathrm{d}t &= \int_{\mathbb{R}} \left\langle \varphi'(t), \psi(t)\right\rangle \mathrm{e}^{-2\nu t}\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \left\langle \varphi(t), -\psi'(t) + 2\nu\psi(t)\right\rangle \mathrm{e}^{-2\nu t}\,\mathrm{d}t\end{aligned}$$

for $\varphi, \psi \in C_c^\infty(\mathbb{R}; H)$. Since $C_c^\infty(\mathbb{R}; H)$ is a core for $\partial_{t,\nu}$ by Proposition 3.2.4, the latter shows

$$\partial_{t,\nu} \subseteq -\partial_{t,\nu}^* + 2\nu.$$

Since we know that $\partial_{t,\nu}$ is onto, it suffices to prove that $-\partial_{t,\nu}^* + 2\nu$ is one-to-one, since this would imply equality in the latter operator inclusion. For doing so, we apply Theorem 2.2.5 to compute

$$\ker(-\partial_{t,\nu}^* + 2\nu) = \operatorname{ran}(-\partial_{t,\nu} + 2\nu)^\perp.$$

Moreover, we have that $-\partial_{t,\nu} + 2\nu$ is unitarily equivalent to $-\partial_{t,-\nu}$ by Corollary 3.2.5 and since $\partial_{t,-\nu}$ is onto, so is $-\partial_{t,\nu} + 2\nu$ and thus $\ker(-\partial_{t,\nu}^* + 2\nu) = L_{2,\nu}(\mathbb{R}; H)^\perp = \{0\}$, which yields the assertion.

The case $\nu = 0$ follows directly from the definition of $\partial_{t,0}$. □

3.3 Comments

Standard references for Bochner integration and related results are [6, 31].

Considering the derivative operator in an exponentially weighted space goes back (at least) to Morgenstern [67], where ordinary differential equations were considered in a classical setting. In fact, we shall return to this observation in the next chapter when we devote our study to some implications of the already developed concepts on ordinary and delay differential equations.

A first occurrence of the derivative operator in exponentially weighted L_2-spaces can be found in [83], where a corresponding spectral theorem has been focussed on. We will prove in a later chapter that the spectral representation of the time derivative as a multiplication operator can be realised by a shifted variant of the Fourier transformation—the so-called Fourier–Laplace transformation.

In an applied context, the time derivative operator discussed here has been introduced in [82].

Exercises

Exercise 3.1 A sequence $(\varphi_n)_n$ in $C_c^\infty(\mathbb{R}^d)$ is called a δ-*sequence* if

(a) $\varphi_n \geqslant 0$ for $n \in \mathbb{N}$,
(b) $\operatorname{spt}\varphi_n \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right]^d$ for $n \in \mathbb{N}$,
(c) $\int_{\mathbb{R}^d} \varphi_n = 1$ for $n \in \mathbb{N}$.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{spt}\varphi \subseteq [-1, 1]^d$, $\varphi \geqslant 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Prove that $(\varphi_n)_n$ given by $\varphi_n(x) := n^d\varphi(nx)$ for $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ defines a δ-sequence. Moreover, give an example for such a function φ.

Exercise 3.2 It is well-known that $\{\mathbb{1}_I\,;\ I\ d\text{-dimensional bounded interval}\}$ is total in $L_2(\mathbb{R}^d)$.

(a) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, $f \in L_2(\mathbb{R}^d)$. Define as usual

$$f * \varphi := \left(x \mapsto \int_{\mathbb{R}^d} f(x-y)\varphi(y)\,\mathrm{d}y\right).$$

Prove that $f * \varphi \in C^\infty(\mathbb{R}^d)$ with $\partial^\alpha (f * \varphi) = f * \partial^\alpha\varphi$ for all $\alpha \in \mathbb{N}_0^d$, where $\partial^\alpha\varphi = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}\varphi$. Moreover, prove that $\operatorname{spt} f * \varphi \subseteq \operatorname{spt} f + \operatorname{spt}\varphi$.

(b) Let $(\varphi_n)_n$ be a δ-sequence and $f \in L_2(\mathbb{R}^d)$. Show that $f * \varphi_n \to f$ in $L_2(\mathbb{R}^d)$ as $n \to \infty$.
Hint: Prove that $\mathbb{1}_I * \varphi_n \to \mathbb{1}_I$ in $L_2(\mathbb{R}^d)$ for all d-dimensional bounded intervals and use that $\|f * \varphi_n\|_2 \leqslant \|f\|_2$ (see also Lemma 3.2.1).

(c) Prove that $C_c^\infty(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$.

Exercise 3.3 Let $a < b$, X_0, X_1, X_2 be Banach spaces, $f\colon (a,b) \to X_0$ and $g\colon (a,b) \to X_1$ both continuously differentiable, $\ell\colon X_0 \times X_1 \to X_2$ bilinear and continuous. Prove that $h\colon (a,b) \to X_2$ given by

$$h(t) := \ell(f(t), g(t)) \quad (t \in (a,b))$$

is continuously differentiable with

$$h'(t) = \ell(f'(t), g(t)) + \ell(f(t), g'(t)) \quad (t \in (a,b)).$$

If f, f', g, g' have continuous extensions to $[a,b]$, prove the integration by parts formula:

$$\int_a^b \ell(f'(t), g(t))\,\mathrm{d}t = \ell(f(b), g(b)) - \ell(f(a), g(a)) - \int_a^b \ell(f(t), g'(t))\,\mathrm{d}t.$$

Exercise 3.4 For $\nu \neq 0$, show that $\|I_\nu\| = \frac{1}{|\nu|}$.

Exercise 3.5 Prove Corollary 3.2.5.

Exercise 3.6 Let $\nu \in \mathbb{R}$ and H be a complex Hilbert space. Prove that $\sigma(\partial_{t,\nu}) \subseteq \{\mathrm{i}t + \nu\,;\ t \in \mathbb{R}\}$, where $\partial_{t,0}$ is defined in Corollary 3.2.6.
Hint: For $f \in \operatorname{dom}(\partial_{t,\nu})$, $z \in \mathbb{C}$ compute $\operatorname{Re}\left\langle (z - \partial_{t,\nu})f, f\right\rangle_{L_{2,\nu}(\mathbb{R};H)}$ by using Corollary 3.2.6. For proving the surjectivity of $z - \partial_{t,\nu}$ for a suitable z, use the formula

$$\overline{\operatorname{ran}}(z - \partial_{t,\nu}) = \ker(z^* - \partial_{t,\nu}^*)^{\perp}.$$

Remark: Later we will see that, actually, $\sigma(\partial_{t,\nu}) = \{\mathrm{i}t + \nu\,;\ t \in \mathbb{R}\}$.

Exercise 3.7 Consider the differential equation

$$\left(\partial_{t,\nu}^2 - 1\right) u = \mathbb{1}_{[-1,1]}.$$

Since $\partial_{t,\nu}^2 - 1 = \left(\partial_{t,\nu} - 1\right)\left(\partial_{t,\nu} + 1\right)$, it follows by Exercise 3.6 that there is a unique $u \in L_{2,\nu}(\mathbb{R})$ solving this equation if $\nu \notin \{-1, 1\}$. Compute these solutions.
Hint: For $u \in \operatorname{dom}(\partial_{t,\nu})$ use the fact that u is necessarily continuous (which we shall establish in the next chapter).

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