Chapter 4
Ordinary Differential Equations

In this chapter, we discuss a first application of the time derivative operator constructed in the previous chapter. More precisely, we analyse well-posedness of ordinary differential equations and will at the same time provide a Hilbert space proof of the classical Picard–Lindelöf theorem.[1] We shall furthermore see that the abstract theory developed here also allows for more general differential equations to be considered. In particular, we will have a look at so-called delay differential equations with finite or infinite delay; neutral differential equations are considered in the exercises section.

We start with some information on the time derivative and its domain.

4.1 The Domain of $\partial_{t,\nu}$ and the Sobolev Embedding Theorem

Let H be a Hilbert space. Readers familiar with the notion of Sobolev spaces might have already realised that the domain of $\partial_{t,\nu}$ can be described as $L_{2,\nu}(\mathbb{R}; H)$-functions with distributional derivative lying in $L_{2,\nu}(\mathbb{R}; H)$. We shall also use

$$H_\nu^1(\mathbb{R}; H) := \operatorname{dom}(\partial_{t,\nu}) \subseteq L_{2,\nu}(\mathbb{R}; H),$$

if we want to emphasise the target Hilbert space of the $\operatorname{dom}(\partial_{t,\nu})$-functions. In order to stress the distributional character of the derivative introduced, we include the following result. Later on, we have the opportunity to have a more detailed look at Sobolev spaces in more general contexts.

[1] There are different notions for this theorem. It is also called existence and uniqueness theorem for initial value problems for ordinary differential equations as well as Cauchy–Lipschitz theorem.

C. Seifert et al., *Evolutionary Equations*, Operator Theory: Advances and Applications 287, https://doi.org/10.1007/978-3-030-89397-2_4

Proposition 4.1.1 *Let $\nu \in \mathbb{R}$ and $f, g \in L_{2,\nu}(\mathbb{R}; H)$. Then the following conditions are equivalent:*

(i) $f \in \operatorname{dom}(\partial_{t,\nu})$ *and* $\partial_{t,\nu} f = g$.
(ii) *For all* $\phi \in C_c^\infty(\mathbb{R})$ *we have*

$$-\int_{\mathbb{R}} \phi' f = \int_{\mathbb{R}} \phi g,$$

where these integrals are Bochner integrals of the H*-valued functions* $t \mapsto \phi'(t) f(t)$ *and* $t \mapsto \phi(t) g(t)$*, respectively.*

Proof Assume that $f \in \operatorname{dom}(\partial_{t,\nu})$. By Proposition 3.2.4 and Corollary 3.2.6, we have that $\mathcal{D}_H = \operatorname{lin}\left\{\varphi \cdot x\,;\ \varphi \in C_c^\infty(\mathbb{R}),\ x \in H\right\} \subseteq \operatorname{dom}(\partial_{t,\nu}^*)$ (which also holds for $\nu = 0$) and

$$\left\langle \partial_{t,\nu} f, \psi \cdot x \right\rangle_{L_{2,\nu}} = \left\langle f, \left(-\psi' + 2\nu\psi\right) \cdot x \right\rangle_{L_{2,\nu}}$$

for all $x \in H$ and $\psi \in C_c^\infty(\mathbb{R})$. Hence, we obtain for all $\psi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} \left(-\psi' + 2\nu\psi\right) f \mathrm{e}^{-2\nu\cdot} = \int_{\mathbb{R}} \psi \partial_{t,\nu} f \mathrm{e}^{-2\nu\cdot};$$

putting $\phi := \mathrm{e}^{-2\nu\cdot}\psi$ and using that multiplication by $\mathrm{e}^{-2\nu\cdot}$ is a bijection on $C_c^\infty(\mathbb{R})$, we deduce the claimed formula with $g = \partial_{t,\nu} f$.

On the other hand, the equation involving g applied to $\phi = \mathrm{e}^{-2\nu\cdot}\psi$ for $\psi \in C_c^\infty(\mathbb{R})$ implies that

$$\int_{\mathbb{R}} \left(-\psi' + 2\nu\psi\right) f \mathrm{e}^{-2\nu\cdot} = \int_{\mathbb{R}} \psi g \mathrm{e}^{-2\nu\cdot}.$$

Testing this equation with $x \in H$ yields

$$\langle g, \psi \cdot x\rangle_{L_{2,\nu}} = \left\langle f, \left(-\psi' + 2\nu\psi\right) \cdot x\right\rangle_{L_{2,\nu}} = \left\langle f, \left(-\partial_{t,\nu}\psi \cdot x + 2\nu\psi \cdot x\right)\right\rangle_{L_{2,\nu}}.$$

Since $\mathcal{D}_H$ is dense in $\operatorname{dom}(\partial_{t,\nu})$ by Proposition 3.2.4, we infer that

$$\langle g, h\rangle_{L_{2,\nu}} = \left\langle f, \left(-\partial_{t,\nu} h + 2\nu h\right)\right\rangle_{L_{2,\nu}}$$

for all $h \in \operatorname{dom}(\partial_{t,\nu})$. Now, Corollary 3.2.6, yields

$$\langle g, h\rangle_{L_{2,\nu}} = \left\langle f, \partial_{t,\nu}^* h\right\rangle_{L_{2,\nu}} \quad (h \in \operatorname{dom}(\partial_{t,\nu}^*)).$$

Thus, $f \in \operatorname{dom}(\partial_{t,\nu}^{**}) = \operatorname{dom}(\partial_{t,\nu})$ and $\partial_{t,\nu} f = g$. $\square$

The next result is a version of the Sobolev embedding theorem. It particularly confirms that functions in the domain of $\partial_{t,\nu}$ are continuous. This result was announced in Exercise 3.7. Here, we make use of the explicit form of the domain of $\partial_{t,\nu}$ as being the range space of the integral operator I_ν. We define

$$C_\nu(\mathbb{R}; H) := \left\{ f : \mathbb{R} \to H\,;\ f \text{ continuous},\ \|f\|_{\nu,\infty} := \sup_{t\in\mathbb{R}} \left\| \mathrm{e}^{-\nu t} f(t) \right\|_H < \infty \right\}$$

and regard it as being endowed with the obvious norm.

Theorem 4.1.2 (Sobolev Embedding Theorem) *Let* $\nu \in \mathbb{R}$*. Then every* $f \in \operatorname{dom}(\partial_{t,\nu})$ *has a continuous representative, and the mapping*

$$\operatorname{dom}(\partial_{t,\nu}) \ni f \mapsto f \in C_\nu(\mathbb{R}; H)$$

is continuous.

Proof We restrict ourselves to the case when $\nu > 0$; the remaining cases can be proved by invoking Corollary 3.2.5. Let $f \in \operatorname{dom}(\partial_{t,\nu})$. By definition, we find $g \in L_{2,\nu}(\mathbb{R}; H)$ such that $f = \partial_{t,\nu}^{-1} g = I_\nu g$. Then for all $t \in \mathbb{R}$ we compute

$$\begin{aligned}\int_{-\infty}^{t} \|g(\tau)\| \,\mathrm{d}\tau &= \int_{-\infty}^{t} \|g(\tau)\| \,\mathrm{e}^{-\nu\tau}\mathrm{e}^{\nu\tau} \,\mathrm{d}\tau \leqslant \sqrt{\int_{-\infty}^{t} \|g(\tau)\|^2 \,\mathrm{e}^{-2\nu\tau} \,\mathrm{d}\tau} \sqrt{\int_{-\infty}^{t} \mathrm{e}^{2\nu\tau} \,\mathrm{d}\tau} \\ &\leqslant \left\| \partial_{t,\nu} f \right\|_{L_{2,\nu}} \sqrt{\frac{1}{2\nu}} \mathrm{e}^{\nu t}.\end{aligned}$$

Thus, g is integrable on $(-\infty, t]$ for all $t \in \mathbb{R}$ and dominated convergence implies that

$$f = \left(t \mapsto \int_{-\infty}^{t} g(s) \,\mathrm{d}s \right)$$

is continuous. Moreover, for $t \in \mathbb{R}$ we obtain

$$\|f(t)\| \leqslant \int_{-\infty}^{t} \|g(\tau)\| \,\mathrm{d}\tau \leqslant \left\| \partial_{t,\nu} f \right\|_{L_{2,\nu}} \sqrt{\frac{1}{2\nu}} \mathrm{e}^{\nu t}$$

which yields the claimed continuity. □

Corollary 4.1.3 *For all* $f \in \operatorname{dom}(\partial_{t,\nu})$*, we have that* $\left\| \mathrm{e}^{-\nu t} f(t) \right\|_H \to 0$ *as* $t \to \pm\infty$.

The proof is left as Exercise 4.2.

4.2 The Picard–Lindelöf Theorem

The prototype of the Picard–Lindelöf theorem will be formulated for so-called uniformly Lipschitz continuous functions. We first need a preparation.

Definition Let X be a Banach space. Then we define

$$S_{\mathrm{c}}(\mathbb{R}; X) := \{f : \mathbb{R} \to X\,;\ f \text{ simple},\ \operatorname{spt} f \text{ compact}\}$$

to be the set of *simple functions from* $\mathbb{R}$ *to* X *with compact support*.

Lemma 4.2.1 *Let* X *be a Banach space and* $\nu, \eta \in \mathbb{R}$. *Then* $S_{\mathrm{c}}(\mathbb{R}; X)$ *is dense in* $L_{2,\nu}(\mathbb{R}; X) \cap L_{2,\eta}(\mathbb{R}; X)$*; that is, for all* $f \in L_{2,\nu}(\mathbb{R}; X) \cap L_{2,\eta}(\mathbb{R}; X)$ *there exists* $(f_n)_n$ *in* $S_{\mathrm{c}}(\mathbb{R}; X)$ *such that* $f_n \to f$ *in both* $L_{2,\nu}(\mathbb{R}; X)$ *and* $L_{2,\eta}(\mathbb{R}; X)$. *In particular,* $S_{\mathrm{c}}(\mathbb{R}; X)$ *is dense in* $L_{2,\nu}(\mathbb{R}; X)$.

Proof Let $f \in L_{2,\nu}(\mathbb{R}; X) \cap L_{2,\eta}(\mathbb{R}; X)$. Then for all $n \in \mathbb{N}$ we have that $\mathbb{1}_{[-n,n]} f \in L_{2,\nu}(\mathbb{R}; X) \cap L_{2,\eta}(\mathbb{R}; X)$ and $\mathbb{1}_{[-n,n]} f \to f$ in $L_{2,\nu}(\mathbb{R}; X)$ and in $L_{2,\eta}(\mathbb{R}; X)$ as $n \to \infty$. For $n \in \mathbb{N}$ let $(\widetilde{f}_{n,k})_k$ be in $S(\mu_{2,\nu}; X)$ such that $\widetilde{f}_{n,k} \to \mathbb{1}_{[-n,n]} f$ in $L_{2,\nu}(\mathbb{R}; X)$ as $k \to \infty$. We put $f_{n,k} := \mathbb{1}_{[-n,n]} \widetilde{f}_{n,k} \in S_{\mathrm{c}}(\mathbb{R}; X)$. Then $f_{n,k} \to \mathbb{1}_{[-n,n]} f$ in $L_{2,\nu}(\mathbb{R}; X)$ and in $L_{2,\eta}(\mathbb{R}; X)$ as $k \to \infty$. □

In order to define the notion of uniformly Lipschitz continuous functions, we first need the Lipschitz semi-norm.

Definition Let X_0, X_1 be normed spaces, and $F : X_0 \to X_1$ Lipschitz continuous. Then

$$\|F\|_{\mathrm{Lip}} := \sup_{\substack{x,y \in X_0 \\ x \neq y}} \frac{\|F(x) - F(y)\|}{\|x - y\|}$$

is the *Lipschitz semi-norm* of F.

Definition Let H_0, H_1 be Hilbert spaces, $\mu \in \mathbb{R}$. Then a function $F : S_{\mathrm{c}}(\mathbb{R}; H_0) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H_1)$ is called *uniformly Lipschitz continuous* if for all $\nu \geqslant \mu$ we have that F considered in $L_{2,\nu}(\mathbb{R}; H_0) \times L_{2,\nu}(\mathbb{R}; H_1)$ is Lipschitz continuous, and for the unique Lipschitz continuous extensions F^ν, $\nu \geqslant \mu$, we have that

$$\sup_{\nu \geqslant \mu} \left\| F^\nu \right\|_{\mathrm{Lip}} < \infty.$$

Remark 4.2.2 Another way to introduce uniformly Lipschitz continuous mappings is the following. Let H_0, H_1 be Hilbert spaces, $\mu \in \mathbb{R}$. Let $(F^\nu)_{\nu \geqslant \mu}$ be a family of Lipschitz continuous mappings $F^\nu : L_{2,\nu}(\mathbb{R}; H_0) \to L_{2,\nu}(\mathbb{R}; H_1)$ such that

$$\sup_{\nu \geqslant \mu} \left\| F^\nu \right\|_{\mathrm{Lip}} < \infty$$

and the mappings are consistent in the sense that for all $\nu, \eta \geqslant \mu$ and $f \in L_{2,\nu}(\mathbb{R}; H_0) \cap L_{2,\eta}(\mathbb{R}; H_0)$ we have

$$F^\nu(f) = F^\eta(f).$$

Then, for $\nu \geqslant \mu$ and $f \in S_c(\mathbb{R}; H_0)$ we have $F^\nu(f) \in \bigcap_{\eta \geqslant \mu} L_{2,\eta}(\mathbb{R}; H_1)$ and $F^\nu|_{S_c(\mathbb{R}; H_0)}$ is uniformly Lipschitz continuous.

Theorem 4.2.3 (Picard–Lindelöf—Hilbert Space Version) *Let H be a Hilbert space, $\mu \in \mathbb{R}$ and $F\colon S_c(\mathbb{R}; H) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H)$ uniformly Lipschitz continuous with $L := \sup_{\nu \geqslant \mu} \|F^\nu\|_{\mathrm{Lip}}$. Then for all $\nu > \max\{L, \mu\}$ the equation*

$$\partial_{t,\nu} u_\nu = F^\nu(u_\nu)$$

admits a unique solution $u_\nu \in \mathrm{dom}(\partial_{t,\nu})$. Furthermore, for all $\nu > \max\{L, \mu\}$ the following properties hold:

(a) *If $F^\nu(u_\nu)$ is continuous in a neighbourhood of $a \in \mathbb{R}$, then u_ν is continuously differentiable in a neighbourhood of a.*
(b) *For all $a \in \mathbb{R}$, $\mathbb{1}_{(-\infty,a]} u_\nu$ is the unique fixed point $v \in L_{2,\nu}(\mathbb{R}; H)$ of $\mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^\nu$, that is, v uniquely solves*

$$v = \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^\nu(v).$$

(c) *For all $\eta \geqslant \nu$ we have that $u_\nu = u_\eta$.*
(d) *For all $f \in L_{2,\nu}(\mathbb{R}; H)$ the equation*

$$\partial_{t,\nu} v = F^\nu(v) + f$$

admits a unique solution $v_{\nu,f} \in \mathrm{dom}(\partial_{t,\nu})$, and if $f, g \in L_{2,\nu}(\mathbb{R}; H)$ satisfy $f = g$ on $(-\infty, a]$ for some $a \in \mathbb{R}$, then $v_{\nu,f} = v_{\nu,g}$ on $(-\infty, a]$.

Proof of Theorem 4.2.3—First Part Define $\Phi\colon L_{2,\nu}(\mathbb{R}; H) \to L_{2,\nu}(\mathbb{R}; H)$ by

$$\Phi(u) = \partial_{t,\nu}^{-1} F^\nu(u).$$

Since $\left\|\partial_{t,\nu}^{-1}\right\| \leqslant \frac{1}{\nu}$ and $\nu > L$ it follows that Φ is a contraction and thus admits a unique fixed point, which by definition solves the equation in question. Moreover, we have that $u_\nu = \Phi(u_\nu) = \partial_{t,\nu}^{-1} F^\nu(u_\nu) \in \mathrm{dom}(\partial_{t,\nu})$.

Differentiability of u_ν as in (a) follows from Exercise 4.1 and the continuity of $F^\nu(u_\nu)$.

For the unique existence asserted in (d), note that the unique existence of $v_{\nu,f}$ follows from the above considerations after realising that $\Psi(v) := \partial_{t,\nu}^{-1} F^\nu(v) + \partial_{t,\nu}^{-1} f$ defines a contraction in $L_{2,\nu}(\mathbb{R}; H)$. For the remaining statements in (d) and the statements in (b) and (c), we need some prerequisites. □

Definition Let H_0, H_1 be Hilbert spaces, $\nu \in \mathbb{R}$ and $F\colon L_{2,\nu}(\mathbb{R}; H_0) \to L_{2,\nu}(\mathbb{R}; H_1)$. Then, F is called *causal* if for all $a \in \mathbb{R}$ and all $f, g \in L_{2,\nu}(\mathbb{R}; H_0)$ with $f = g$ on $(-\infty, a]$, we have that $F(f) = F(g)$ on $(-\infty, a]$.

Remark 4.2.4 Let $\nu \in \mathbb{R}, a \in \mathbb{R}$. If $f \in L(L_{2,\nu}(\mathbb{R}; H))$ with $\operatorname{spt} f \subseteq (-\infty, a]$ then $f \in \bigcap_{\eta \leqslant \nu} L_{2,\eta}(\mathbb{R}; H)$ and

$$\|f\|_{L_{2,\eta}(\mathbb{R};H)} \leqslant \mathrm{e}^{(\nu-\eta)a} \|f\|_{L_{2,\nu}(\mathbb{R};H)} \quad (\eta \leqslant \nu).$$

Likewise, if $\operatorname{spt} f \subseteq [a, \infty)$, we get $f \in \bigcap_{\rho \geqslant \nu} L_{2,\rho}(\mathbb{R}; H)$ with

$$\|f\|_{L_{2,\rho}(\mathbb{R};H)} \leqslant \mathrm{e}^{(\nu-\rho)a} \|f\|_{L_{2,\nu}(\mathbb{R};H)} \quad (\rho \geqslant \nu).$$

Lemma 4.2.5 *Let H_0, H_1 be Hilbert spaces, $\mu \in \mathbb{R}$, $F\colon S_{\mathrm{c}}(\mathbb{R}; H_0) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H_1)$ uniformly Lipschitz continuous. Then the following statements hold:*

(a) *F^ν is causal for all $\nu \geqslant \mu$.*
(b) *The mapping $\partial_{t,\nu}^{-1} F^\nu$ is causal if $\nu \geqslant \max\{\mu, 0\}$ and $\nu \neq 0$.*
(c) *For all $\nu \geqslant \eta \geqslant \mu$, we have that $F^\nu = F^\eta$ on $L_{2,\nu}(\mathbb{R}; H_0) \cap L_{2,\eta}(\mathbb{R}; H_0)$.*

Proof (a) We divide the proof into three steps.

(i) Let $\nu \geqslant \mu$. In order to show causality of F^ν, we first note that it suffices to have $F^\nu(f) = F^\nu(g)$ on $(-\infty, a]$ for all $f, g \in S_{\mathrm{c}}(\mathbb{R}; H_0)$ with $f = g$ on $(-\infty, a]$. Indeed, let $f, g \in L_{2,\nu}(\mathbb{R}; H)$ with $f = g$ on $(-\infty, a]$ for some $a \in \mathbb{R}$. By Lemma 4.2.1 we find $(f_n)_n$ and $(\widetilde{g}_n)_n$ in $S_{\mathrm{c}}(\mathbb{R}; H_0)$ such that $f_n \to f$ and $\widetilde{g}_n \to g$ in $L_{2,\nu}(\mathbb{R}; H_0)$. Next, $\mathbb{1}_{(-\infty,a]} f_n \to \mathbb{1}_{(-\infty,a]} f = \mathbb{1}_{(-\infty,a]} g$ as $n \to \infty$ in $L_{2,\nu}(\mathbb{R}; H_0)$. Thus, putting $g_n := \mathbb{1}_{(-\infty,a]} f_n + \mathbb{1}_{(a,\infty)} \widetilde{g}_n$ for all $n \in \mathbb{N}$ we obtain that $g_n \to g$ in $L_{2,\nu}(\mathbb{R}; H_0)$. Since $F^\nu(f_n) = F^\nu(g_n)$ on $(-\infty, a]$ for all $n \in \mathbb{N}$ and $F^\nu\colon L_{2,\nu}(\mathbb{R}; H_0) \to L_{2,\nu}(\mathbb{R}; H_1)$ is continuous, taking the limit $n \to \infty$ yields $F^\nu(f) = F^\nu(g)$ on $(-\infty, a]$.

(ii) Let $a \in \mathbb{R}$, $c \geqslant 0$ and $f \in S_{\mathrm{c}}(\mathbb{R}; H_0)$ such that $f = 0$ on $(-\infty, a]$, $g \in \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H_1)$ such that $\|g\|_{L_{2,\nu}(\mathbb{R};H_1)} \leqslant c \|f\|_{L_{2,\nu}(\mathbb{R};H_0)}$ for all $\nu \geqslant \mu$. Then

$$\begin{aligned} \int_{-\infty}^{a} \|g(t)\|_{H_1}^2 \mathrm{e}^{2\nu(a-t)} \,\mathrm{d}t &\leqslant \int_{\mathbb{R}} \|g(t)\|_{H_1}^2 \mathrm{e}^{2\nu(a-t)} \,\mathrm{d}t \\ &\leqslant c^2 \int_{a}^{\infty} \|f(t)\|_{H_0}^2 \mathrm{e}^{2\nu(a-t)} \,\mathrm{d}t \to 0 \end{aligned}$$

as $\nu \to \infty$. Since $\mathrm{e}^{2\nu(a-t)} \to \infty$ as $\nu \to \infty$ for all $t < a$, the monotone convergence theorem implies $g = 0$ on $(-\infty, a]$.

(iii) Let $f, g \in S_{\mathrm{c}}(\mathbb{R}; H_0)$ such that $f = g$ on $(-\infty, a]$ for some $a \in \mathbb{R}$. Then $f - g = 0$ on $(-\infty, a]$. Since F is uniformly Lipschitz continuous, with $L := \sup_{\nu \geqslant \mu} \|F^\nu\|_{\mathrm{Lip}}$ we obtain $\|F^\nu(f) - F^\nu(g)\|_{L_{2,\nu}(\mathbb{R};H_1)} \leqslant L \|f - g\|_{L_{2,\nu}(\mathbb{R};H_0)}$ for all $\nu \geqslant \mu$. By (ii) we conclude $F^\nu(f) = F^\nu(g)$ on $(-\infty, a]$ for all $\nu \geqslant \mu$, which by (i) yields the assertion.

The statement in (b) directly follows from (a). Note that $\partial_{t,\nu}^{-1} F^{\nu}$ is uniformly Lipschitz continuous only for $\nu > 0$.

Let us prove (c). Since $F^{\nu}(f) = F(f) = F^{\eta}(f)$ for $f \in S_{\mathrm{c}}(\mathbb{R}; H_0)$, the set $S_{\mathrm{c}}(\mathbb{R}; H_0)$ is dense in $L_{2,\nu}(\mathbb{R}; H_0) \cap L_{2,\mu}(\mathbb{R}; H_0)$ by Lemma 4.2.1, and F^{ν} and F^{η} are Lipschitz-continuous, we obtain the assertion. □

Proof of Theorem 4.2.3—Second Part The remaining part in (d): Let $f, g \in L_{2,\nu}(\mathbb{R}; H)$ with $f = g$ on $(-\infty, a]$. Since $\nu > L \geqslant 0$, we compute using Lemma 4.2.5(b) and causality of $\partial_{t,\nu}^{-1}$ that

$$\begin{aligned} \mathbb{1}_{(-\infty,a]} v_{\nu,f} &= \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu}\left(v_{\nu,f}\right) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} f \\ &= \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu}\left(\mathbb{1}_{(-\infty,a]} v_{\nu,f}\right) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} \mathbb{1}_{(-\infty,a]} f \\ &= \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu}\left(\mathbb{1}_{(-\infty,a]} v_{\nu,f}\right) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} \mathbb{1}_{(-\infty,a]} g. \end{aligned}$$

The same computation also yields that

$$\mathbb{1}_{(-\infty,a]} v_{\nu,g} = \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu}\left(\mathbb{1}_{(-\infty,a]} v_{\nu,g}\right) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} \mathbb{1}_{(-\infty,a]} g.$$

It is easy to see that $u \mapsto \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu}(u) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} \mathbb{1}_{(-\infty,a]} g$ defines a contraction in $L_{2,\nu}(\mathbb{R}; H)$. Hence, the contraction mapping principle implies that $\mathbb{1}_{(-\infty,a]} v_{\nu,f} = \mathbb{1}_{(-\infty,a]} v_{\nu,g}$.

The statement in (b) follows from the fact that $u \mapsto \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu}(u)$ defines a contraction and Lemma 4.2.5(b).

For the proof of (c), we observe that for all $n \in \mathbb{N}$, we have $\mathbb{1}_{(-\infty,n]} u_{\eta} \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\eta}(\mathbb{R}; H)$. Hence, by (b) and Lemma 4.2.5(c), it follows that

$$\mathbb{1}_{(-\infty,n]} u_{\eta} = \mathbb{1}_{(-\infty,n]} \partial_{t,\eta}^{-1} F^{\eta}\left(\mathbb{1}_{(-\infty,n]} u_{\eta}\right) = \mathbb{1}_{(-\infty,n]} \partial_{t,\nu}^{-1} F^{\nu}\left(\mathbb{1}_{(-\infty,n]} u_{\eta}\right).$$

As $\mathbb{1}_{(-\infty,n]} u_{\nu}$ satisfies the same fixed point equation, we deduce $\mathbb{1}_{(-\infty,n]} u_{\eta} = \mathbb{1}_{(-\infty,n]} u_{\nu}$ for all $n \in \mathbb{N}$, which yields the assertion. □

As a first application of Theorem 4.2.3 we state and prove the classical version of the Theorem of Picard–Lindelöf.

Theorem 4.2.6 (Picard–Lindelöf—Classical Version) *Let H be a Hilbert space, $\Omega \subseteq \mathbb{R} \times H$ be open, $f\colon \Omega \to H$ continuous, $(t_0, x_0) \in \Omega$. Assume there exists $L \geqslant 0$ such that for all $(t, x), (t, y) \in \Omega$ we have*

$$\|f(t, x) - f(t, y)\| \leqslant L \|x - y\|.$$

Then, there exists $\delta > 0$ *such that the initial value problem*

$$\begin{cases} u'(t) = f(t, u(t)) \quad (t \in (t_0, t_0 + \delta)), \\ u(t_0) = x_0, \end{cases} \tag{4.1}$$

admits a unique continuously differentiable solution, $u\colon [t_0, t_0 + \delta] \to H$*, which satisfies* $(t, u(t)) \in \Omega$ *for all* $t \in [t_0, t_0 + \delta]$.

Proof First of all we observe that we may assume, without loss of generality, that $x_0 = 0$. Indeed, to solve the initial value problem

$$\begin{cases} v'(t) = f(t, v(t) + x_0) \quad (t \in (t_0, t_0 + \delta)), \\ v(t_0) = 0, \end{cases}$$

for a continuously differentiable $v\colon [t_0, t_0 + \delta] \to H$ is equivalent to solving the problem in Theorem 4.2.6 for u by setting $u = v + \mathbb{1}_{[t_0, t_0+\delta]} x_0$. Appropriately shifting the time coordinate, we may also assume that $t_0 = 0$.

Thus, let $(0, 0) \in \Omega$. Then $[0, \delta'] \times B[0, \varepsilon] \subseteq \Omega$ for some $\delta', \varepsilon > 0$. Denote by $P\colon H \to H$ the projection onto $B[0, \varepsilon]$; that is, for $x \in H$, $Px \in B[0, \varepsilon]$ is the unique element satisfying

$$\|x - Px\|_H = \inf_{y \in B[0,\varepsilon]} \|x - y\|_H .$$

By Exercise 4.4, P is Lipschitz continuous with Lipschitz semi-norm bounded by 1. We then define

$$\begin{aligned} F\colon S_c(\mathbb{R}; H) &\to \bigcap_{\nu \geqslant 0} L_{2,\nu}(\mathbb{R}; H) \\ g &\mapsto \big(t \mapsto \mathbb{1}_{[0,\delta')}(t) f(t, P(g(t)))\big) \end{aligned}$$

and will prove that F is well-defined and uniformly Lipschitz continuous. Since the mapping $t \mapsto \mathbb{1}_{[0,\delta')}(t) f(t, 0)$ is supported on $\left[0, \delta'\right]$, we obtain for $\nu \geqslant 0$ that $F(0) \in L_{2,\nu}(\mathbb{R}; H)$. Moreover, for $\nu \geqslant 0$ and $g, h \in S_c(\mathbb{R}; H)$ we estimate

$$\begin{aligned} &\|F(g) - F(h)\|^2_{L_{2,\nu}(\mathbb{R};H)} \\ &= \int_{\mathbb{R}} \|F(g)(t) - F(h)(t)\|^2 \mathrm{e}^{-2\nu t}\, \mathrm{d}t = \int_0^{\delta'} \|f(t, P(g(t))) - f(t, P(h(t)))\|^2 \mathrm{e}^{-2\nu t}\, \mathrm{d}t \\ &\leqslant L^2 \int_0^{\delta'} \|P(g(t)) - P(h(t))\|^2 \mathrm{e}^{-2\nu t}\, \mathrm{d}t \leqslant L^2 \int_0^{\delta'} \|g(t) - h(t)\|^2 \mathrm{e}^{-2\nu t}\, \mathrm{d}t \\ &\leqslant L^2 \|g - h\|^2_{L_{2,\nu}(\mathbb{R};H)} , \end{aligned}$$

which shows that F is well-defined and uniformly Lipschitz continuous.

By Theorem 4.2.3, there exists $v \in \mathrm{dom}(\partial_{t,\nu})$ with $\nu > L$ such that

$$\partial_{t,\nu} v = F^{\nu}(v).$$

We read off from $v = \partial_{t,\nu}^{-1} F^{\nu}(v)$ that $v = 0$ on $(-\infty, 0]$, and that v is continuous by Theorem 4.1.2. Moreover, we obtain that

$$v(t) = \int_{-\infty}^{t} \mathbb{1}_{[0,\delta')}(\tau) f(\tau, P(v(\tau)))\,\mathrm{d}\tau = \int_{0}^{\min\{t,\delta'\}} f(\tau, P(v(\tau)))\,\mathrm{d}\tau,$$

from which we read off that v is continuously differentiable on $\left(0, \delta'\right)$ since f and P are also continuous. The same equality implies for $0 < t \leqslant \delta := \min\{\frac{\varepsilon}{M}, \delta'\}$, where $M := \sup_{(t,x)\in[0,\delta']\times B[0,\varepsilon]} \|f(t,x)\|$, that

$$\|v(t)\| \leqslant \int_0^t \|f(\tau, P(v(\tau)))\|\,\mathrm{d}\tau \leqslant M\delta \leqslant \varepsilon.$$

Thus, $(t, v(t)) \in \left[0, \delta'\right] \times B\,[0,\varepsilon] \subseteq \Omega$ for all $0 \leqslant t \leqslant \delta$ and so $Pv(t) = v(t)$ for $0 \leqslant t \leqslant \delta$. Thus, $u := v|_{[0,\delta]}$ satisfies (4.1).

Finally, concerning uniqueness, let $\widetilde{u}\colon [0,\delta] \to H$ be a continuously differentiable solution of (4.1). Let $\widetilde{v}$ be the extension of $\widetilde{u}$ by 0 to the whole of $\mathbb{R}$. Then we get that

$$\begin{aligned}
\mathbb{1}_{(-\infty,\delta]}\widetilde{v} &= \mathbb{1}_{(-\infty,\delta]} \int_0^{\cdot} \mathbb{1}_{[0,\delta')}(\tau) f(\tau, \widetilde{v}(\tau))\,\mathrm{d}\tau \\
&= \mathbb{1}_{(-\infty,\delta]} \int_{-\infty}^{\cdot} \mathbb{1}_{[0,\delta')}(\tau) f(\tau, P(\widetilde{v}(\tau)))\,\mathrm{d}\tau \\
&= \mathbb{1}_{(-\infty,\delta]} \partial_{t,\nu}^{-1} F^{\nu}(\mathbb{1}_{(-\infty,\delta]}\widetilde{v}).
\end{aligned}$$

Since $\mathbb{1}_{(-\infty,\delta]}v$ is the unique solution of the equation $w = \mathbb{1}_{(-\infty,\delta]}\partial_{t,\nu}^{-1}F^{\nu}(w)$, we obtain that $\mathbb{1}_{(-\infty,\delta]}\widetilde{v} = \mathbb{1}_{(-\infty,\delta]}v$, which yields $u = \widetilde{u}$. □

Remark 4.2.7 The reason for the proof of the classical Picard–Lindelöf theorem being seemingly complicated is two-fold. First of all, the Hilbert space solution theory is for L_2-functions rather than continuous (or continuously differentiable) functions. The second, maybe more important point is that the Hilbert space Picard–Lindelöf asserts a solution theory, which provides *global* existence in the time variable. The main body of the proof of the classical Picard–Lindelöf theorem presented here is therefore devoted to 'localisation' of the abstract theorem. Furthermore, note that the method of proof for obtaining uniqueness and the admittance of the initial value rests on causality. This effect will resurface when we discuss partial differential equations.

4.3 Delay Differential Equations

In this section, our study will not be as in depth as done for the local Picard–Lindelöf theorem. Of course, the solution theory would not be a very good one if it was only applicable to, arguably, the easiest case of ordinary differential equations. We shall see next that the developed theory applies to more elaborate examples.

In what follows, let H be a Hilbert space over $\mathbb{K}$. We start out with a delay differential equation with so-called 'discrete delay'. For this, we introduce, for $h \in \mathbb{R}$, the *time-shift operator*

$$\begin{aligned} \tau_h \colon S_{\mathrm{c}}(\mathbb{R}; H) &\to \bigcap_{\nu\in\mathbb{R}} L_{2,\nu}(\mathbb{R}; H), \\ f &\mapsto f(\cdot + h). \end{aligned}$$

Lemma 4.3.1 *Let $\mu \in \mathbb{R}$. The mapping $\tau_h \colon S_{\mathrm{c}}(\mathbb{R}; H) \to \bigcap_{\nu\geqslant\mu} L_{2,\nu}(\mathbb{R}; H)$ is uniformly Lipschitz continuous if and only if $h \leqslant 0$. More precisely, for $\nu \in \mathbb{R}$ we have*

$$\|\tau_h\|_{L(L_{2,\nu}(\mathbb{R};H))} = \mathrm{e}^{h\nu}.$$

Proof Let $f \in S_{\mathrm{c}}(\mathbb{R}; H)$. Then for $\nu \in \mathbb{R}$ we compute

$$\begin{aligned} \|\tau_h f\|^2_{L_{2,\nu}(\mathbb{R};H)} &= \int_{\mathbb{R}} \|f(t+h)\|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t = \int_{\mathbb{R}} \|f(t)\|^2 \mathrm{e}^{-2\nu(t-h)}\,\mathrm{d}t \\ &= \|f\|^2_{L_{2,\nu}(\mathbb{R};H)} \mathrm{e}^{2\nu h}. \end{aligned}$$

Since $\sup_{\nu\geqslant\mu} \mathrm{e}^{2\nu h} < \infty$ if and only if $h \leqslant 0$ we obtain the equivalence. Moreover, the above equality also yields the norm of τ_h on $L_{2,\nu}(\mathbb{R}; H)$. □

We will reuse τ_h for the Lipschitz continuous extensions to $L_{2,\nu}(\mathbb{R}; H)$. The well-posedness theorem for delay equations with discrete delay is contained in the next theorem. We note here that we only formulate the respective result for right-hand sides that are globally Lipschitz continuous. With a localisation technique, as has already been carried out for the classical Picard–Lindelöf theorem, it is also possible to obtain local results.

Theorem 4.3.2 *Let H be a Hilbert space, $\mu \in \mathbb{R}$, $N \in \mathbb{N}$, $h_1, \ldots, h_N \in (-\infty, 0]$, and*

$$G \colon S_{\mathrm{c}}(\mathbb{R}; H^N) \to \bigcap_{\nu\geqslant\mu} L_{2,\nu}(\mathbb{R}; H)$$

uniformly Lipschitz. Then there exists an $\eta \in \mathbb{R}$ *such that for all* $\nu \geqslant \eta$ *the equation*

$$\partial_{t,\nu} u = G^{\nu}\left(\tau_{h_1} u, \ldots, \tau_{h_N} u\right)$$

admits a solution $u \in \operatorname{dom}(\partial_{t,\nu})$ *which is unique in* $\bigcup_{\nu \geqslant \eta} L_{2,\nu}(\mathbb{R}; H)$. *Moreover, for all* $a \in \mathbb{R}$ *the function* $u_a := \mathbb{1}_{(-\infty,a]} u$ *satisfies*

$$u_a = \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} G^{\nu}\left(\tau_{h_1} u_a, \ldots, \tau_{h_N} u_a\right).$$

Proof The assertion follows from Theorem 4.2.3 applied to $F := G \circ \big(\tau_{h_1}, \ldots, \tau_{h_N}\big)$ in conjunction with Lemma 4.3.1. □

Next, we formulate an initial value problem for a subclass of the latter type of equations.

Theorem 4.3.3 *Let* $h > 0$, $f\colon \mathbb{R}_{\geq 0} \times H \times H \to H$ *continuous, and* $f(\cdot, 0, 0) \in L_{2,\mu}(\mathbb{R}; H)$ *for some* $\mu > 0$. *Assume that there exists* $L \geqslant 0$ *with*

$$\|f(t,x,y) - f(t,u,v)\| \leqslant L \, \|(x,y) - (u,v)\| \quad \big((t,x,y),(t,u,v) \in \mathbb{R}_{\geq 0} \times H \times H\big).$$

Let $u_0 \in C\,([-h,0]; H)$. *Then the initial value problem*

$$\begin{cases} u'(t) = f(t, u(t), u(t-h)) & (t > 0), \\ u(\tau) = u_0(\tau) & (\tau \in [-h, 0]) \end{cases} \tag{4.2}$$

admits a unique continuous solution $u\colon\ [-h,\infty) \to H$, *continuously differentiable on* $(0,\infty)$.

Proof For $t < 0$ let $f(t,\cdot,\cdot) := 0$. We define $F\colon S_{\mathrm{c}}(\mathbb{R}; H) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H)$ by

$$\begin{aligned} &F(\phi)(t) \\ &:= f\big(t, \phi(t) + \mathbb{1}_{[0,\infty)}(t) u_0(0), \phi(t-h) + \mathbb{1}_{[0,\infty)}(t-h) u_0(0) + \mathbb{1}_{[0,h)}(t) u_0(t-h)\big) \end{aligned}$$

for all $t \in \mathbb{R}$. It is easy to see that F is uniformly Lipschitz continuous. Thus, by Theorem 4.2.3, we find $\eta \geqslant \mu$ such that for all $\nu \geqslant \eta$ the equation

$$\partial_{t,\nu} v = F^{\nu}(v)$$

admits a solution $v \in \bigcap_{\nu \geqslant \eta} \operatorname{dom}(\partial_{t,\nu})$ which is unique in $\bigcup_{\nu \geqslant \eta} L_{2,\nu}(\mathbb{R}; H)$. Note that $\operatorname{spt} F^{\nu}(v) \subseteq [0,\infty)$. Hence, $v = 0$ on $(-\infty, 0]$. By Theorem 4.1.2, we obtain that $v(0) = 0$. We claim that $u := v + \mathbb{1}_{[0,\infty)}(\cdot) u_0(0) + \mathbb{1}_{[-h,0)} u_0$ is a solution of (4.2). First of all note that u is continuous on $[-h,\infty)$. Next, for $0 < t < h$ we have that $t-h<0$ and thus $v(t-h)=0$ and so we see that

$$\begin{aligned}
&F^{\nu}(v)(t)\\
&\quad= f(t, v(t)+\mathbb{1}_{[0,\infty)}(t)u_0(0), v(t-h)+\mathbb{1}_{[0,\infty)}(t-h)u_0(0)+\mathbb{1}_{[0,h)}(t)u_0(t-h))\\
&\quad= f(t,u(t),u_0(t-h)).
\end{aligned}$$

Similarly, for $t \geqslant h$ we obtain

$$F^{\nu}(v)(t) = f(t,u(t),u(t-h))$$

and thus, by continuity of f, u_0 and u, it follows that v is continuously differentiable on $(0,\infty)$ and

$$u'(t) = v'(t) = \partial_{t,\nu}v(t) = f(t,u(t),u(t-h)).$$

It remains to show uniqueness. For this, let $w\colon [-h,\infty)\to H$ be a solution of (4.2). Then

$$w(t) = u_0(0) + \int_0^t f(s,w(s),w(s-h))\,\mathrm{d}s \quad (t\geqslant 0)$$

and $w(t) = u_0(t)$ if $t\in[-h,0]$. Extend w by 0 on $(-\infty,-h)$ and set $\widetilde{v} \coloneqq w - \mathbb{1}_{[0,\infty)}(\cdot)u_0(0) - \mathbb{1}_{[-h,0)}u_0$. We infer

$$\begin{aligned}
\widetilde{v}(t) &= \int_0^t f(s,w(s),w(s-h))\,\mathrm{d}s\\
&= \int_{-\infty}^t f\big(s,\widetilde{v}(s)+\mathbb{1}_{[0,\infty)}(s)u_0(0),\\
&\qquad\qquad \widetilde{v}(s-h)+\mathbb{1}_{[0,\infty)}(s-h)u_0(0)+\mathbb{1}_{[0,h)}(s)u_0(s-h)\big)\,\mathrm{d}s
\end{aligned}$$

for all $t\in\mathbb{R}$. For $a\in\mathbb{R}$ we set $\widetilde{v}_a \coloneqq \mathbb{1}_{(-\infty,a]}\widetilde{v} \in \bigcap_{\nu\in\mathbb{R}} L_{2,\nu}(\mathbb{R};H)$ and obtain, using the above formula for $\widetilde{v}$,

$$\widetilde{v}_a = \mathbb{1}_{(-\infty,a]}\partial_{t,\nu}^{-1}F^{\nu}(\widetilde{v}_a).$$

By uniqueness of the solution of

$$\mathbb{1}_{(-\infty,a]}v = \mathbb{1}_{(-\infty,a]}\partial_{t,\nu}^{-1}F^{\nu}\left(\mathbb{1}_{(-\infty,a]}v\right)$$

it follows that $\widetilde{v}_a = \mathbb{1}_{(-\infty,a]}v$ for all $a\in\mathbb{R}$ and, thus, $u=w$. □

The equation to come involves the whole history of the unknown; that is, the unknown evaluated at $(-\infty, 0]$. For a mapping $u\colon \mathbb{R} \to H$ and $t \in \mathbb{R}$ we define the 'history' of u up to time t as $u_t\colon \mathbb{R}_{\le 0} \to H$, $u_t(\theta) := u(t+\theta)$ for all $\theta \in \mathbb{R}_{\le 0}$. Moreover, we define the mapping

$$u_{(\cdot)}\colon \mathbb{R} \ni t \mapsto u_t,$$

which maps each $t \in \mathbb{R}$ to the history of u up to time t.

Lemma 4.3.4 *Let $\mu > 0$. Then*

$$\begin{aligned}\Theta\colon S_{\mathrm{c}}(\mathbb{R}; H) &\to \bigcap_{\nu \geqslant \mu} L_{2,\nu}\big(\mathbb{R}; L_2(\mathbb{R}_{\le 0}; H)\big)\\ u &\mapsto u_{(\cdot)}\end{aligned}$$

is uniformly Lipschitz continuous. More precisely, for all $\nu > 0$ we have

$$\left\|\Theta^\nu\right\| = \frac{1}{\sqrt{2\nu}}.$$

Proof Let $u \in S_{\mathrm{c}}(\mathbb{R}; H)$. Then $\Theta u(t) = u_t \in L_2(\mathbb{R}_{\le 0}; H)$ for all $t \in \mathbb{R}$ and we compute

$$\begin{aligned}\|\Theta u\|^2_{L_{2,\nu}\left(\mathbb{R}; L_2(\mathbb{R}_{\le 0}; H)\right)} &= \int_{\mathbb{R}} \int_{\mathbb{R}_{\le 0}} \|u(t+\theta)\|^2 \,\mathrm{d}\theta\, \mathrm{e}^{-2\nu t}\,\mathrm{d}t\\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_{\le 0}} \|u(t)\|^2\, \mathrm{e}^{-2\nu(t-\theta)}\,\mathrm{d}\theta\,\mathrm{d}t\\ &= \frac{1}{2\nu}\int_{\mathbb{R}} \|u(t)\|^2\,\mathrm{e}^{-2\nu t}\,\mathrm{d}t.\end{aligned}$$

□

Theorem 4.3.5 *Let H be a Hilbert space, $\mu \in \mathbb{R}$ and let $\Phi\colon S_{\mathrm{c}}\big(\mathbb{R}; L_2(\mathbb{R}_{\le 0}; H)\big) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H)$ be uniformly Lipschitz. Then, there exists $\eta > 0$ such that for all $\nu \geqslant \eta$ the equation*

$$\partial_{t,\nu} u = \Phi^\nu(u_{(\cdot)})$$

admits a solution $u \in \bigcap_{\nu \geqslant \eta} \operatorname{dom}(\partial_{t,\nu})$ unique in $\bigcup_{\nu \geqslant \eta} L_{2,\nu}(\mathbb{R}; H)$.

Proof This is another application of Theorem 4.2.3. □

4.4 Comments

In a way, the proof of Theorem 4.2.6 is standard PDE-theory in a nutshell; a solution theory for L_p-spaces is used to deduce existence and uniqueness of solutions and a posteriori regularity theory provides more information on the properties of the solution.

Note that—of course—other proofs are available for the Picard–Lindelöf theorem. We chose, however, to present this proof here in order to provide a perspective on classical results. Furthermore, we mention that in order to obtain unique existence for the solution, it suffices to assume that f satisfies a uniform Lipschitz condition with respect to the second variable and that f is measurable. Continuity of f is needed in order to obtain C^1-solutions.

A more detailed exposition and more examples of the theory applied to delay differential equations can be found in [52] and—in a Banach space setting—[85].

There is also a way of dealing with delay differential equations by expanding the state space the problem is formulated in. In this case, it is possible to make use of the rich theory of C_0-semigroups. We refer to [10] for this.

Causality is one of the main concepts for evolutionary equations. We have provided this notion for mappings defined on $L_{2,\nu}$-type spaces only. The situation becomes different if one considers merely densely defined mappings. Then it is a priori unclear, whether for a Lipschitz continuous mapping the continuous extension is also causal. For this we refer to Exercise 4.7 below and to [51, 131], and [138, Chapter 2] as well as to references mentioned there.

Exercises

Exercise 4.1

(a) Let X be a Banach space, $u\colon\ [a,b]\to X$ continuous. Show that $v\colon\ (a,b)\to X$ given by

$$v(t)=\int_a^t u(\tau)\,\mathrm{d}\tau$$

is continuously differentiable with $v'(t)=u(t)$ for all $t\in(a,b)$.

(b) Let H be a Hilbert space, and $\nu\in\mathbb{R}$. Let $u\in\operatorname{dom}(\partial_{t,\nu})$ with $\partial_{t,\nu}u$ continuous. Show that u is continuously differentiable and $u'=\partial_{t,\nu}u$.

Exercise 4.2 Prove Corollary 4.1.3.

Exercise 4.3 Let H be a Hilbert space. Show that

$$\operatorname{dom}(\partial_{t,\nu})\hookrightarrow C_\nu^{1/2}(\mathbb{R};H):=\left\{f\in C_\nu(\mathbb{R};H)\,;\ \mathrm{e}^{-\nu\cdot}f \text{ is } \tfrac{1}{2}\text{-Hölder continuous}\right\},$$

where a function $g\colon \mathbb{R} \to H$ is said to be $\frac{1}{2}$-*Hölder continuous* if

$$\sup_{\substack{s,t\in\mathbb{R}\\ t\neq s}} \frac{\|g(t)-g(s)\|}{|t-s|^{1/2}} < \infty.$$

Exercise 4.4 Let H be a Hilbert space, $C \subseteq H$ non-empty, closed and convex. Show that the projection, P, of H onto C defines a Lipschitz continuous mapping with Lipschitz semi-norm bounded by 1, where for $x \in H$, $Px \in C$ is the unique element satisfying

$$\|x - Px\|_H = \inf_{y\in C} \|x-y\|_H\,.$$

Exercise 4.5 Let $h\colon \mathbb{R} \times \mathbb{R}_{\leq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous satisfying

$$\|h(t,s,x) - h(t,s,y)\| \leqslant L\,\|x-y\|$$

with $h(\cdot,\cdot,0) = 0$. Let $R > 0$ and $u_0 \in C(\mathbb{R}_{\leq 0}; \mathbb{R}^n)$ have compact support. Show that the initial value problem

$$\begin{cases} u'(t) = \int_{-R}^{0} h(t,s,u_{(t)}(s))\,\mathrm{d}s & (t>0),\\ u(t) = u_0(t) & (t \leqslant 0)\end{cases}$$

admits a unique continuous solution $u\colon \mathbb{R} \to \mathbb{R}^n$, which is continuously differentiable on $\mathbb{R}_{>0}$.
Hint: Modify Θ from Lemma 4.3.4.

Exercise 4.6 Let H be a Hilbert space. Show that for a uniformly Lipschitz continuous $\Phi\colon S_{\mathrm{c}}\big(\mathbb{R}; L_2(\mathbb{R}_{\leq 0}; H)^2\big) \to \bigcap_{\nu\geqslant\mu} L_{2,\nu}(\mathbb{R}; H)$ the equation

$$\partial_{t,\nu} u = \Phi^{\nu}\left(u_{(\cdot)}, \left(\partial_{t,\nu}u\right)_{(\cdot)}\right)$$

admits a unique solution $u \in \mathrm{dom}(\partial_{t,\nu})$ for ν large enough.

Exercise 4.7 Let $D \subseteq L_2(\mathbb{R})$ be dense and suppose that $F\colon D \subseteq L_2(\mathbb{R}) \to L_2(\mathbb{R})$ admits a Lipschitz continuous extension F^0.

(a) Show that F^0 is causal if and only if for all $\phi \in S_{\mathrm{c}}(\mathbb{R};\mathbb{R})$ and all $a \in \mathbb{R}$ there exists $L \geqslant 0$ such that

$$\left|\left\langle \mathbb{1}_{(-\infty,a]} \cdot (F(f) - F(g)), \phi\right\rangle_{L_2(\mathbb{R})}\right| \leqslant L\,\left\|\mathbb{1}_{(-\infty,a]} \cdot (f-g)\right\|_{L_2(\mathbb{R})}$$

for all $f, g \in D$; that is, the mapping

$$\left(D, \left\|\mathbb{1}_{(-\infty,a]} \cdot (\cdot - \cdot)\right\|_{L_2(\mathbb{R})}\right) \ni f \mapsto F(f) \in \left(L_2(\mathbb{R}), \left|\left\langle \mathbb{1}_{(-\infty,a]} \cdot (\cdot - \cdot), \phi\right\rangle_{L_2(\mathbb{R})}\right|\right)$$

is Lipschitz continuous.

(b) For $a \in \mathbb{R}$ let $\operatorname{dom}(F) \cap \operatorname{dom}(F\mathbb{1}_{(-\infty,a]})$ be dense in $L_2(\mathbb{R})$ and if $f, g \in D = \operatorname{dom}(F)$ and $f = g$ on $(-\infty, a]$ then also $F(f) = F(g)$ on $(-\infty, a]$. Show that F^0 is causal.

(c) Assume for all $f, g \in D$ and $a \in \mathbb{R}$ that $f = g$ on $(-\infty, a]$ implies that $F(f) = F(g)$ on $(-\infty, a]$. Show that this is not sufficient for F^0 to be causal. *Hint:* Find a dense subspace $D = \operatorname{dom}(F)$ so that the first condition in (b) is not satisfied.

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