

Chapter 7

Examples of Evolutionary Equations



This chapter is devoted to a small tour through a variety of evolutionary equations. More precisely, we shall look into the equations of poro-elastic media, (time-)fractional elasticity, thermodynamic media with delay as well as visco-elastic media. The discussion of these examples will be similar to that of the examples in the previous chapter in the sense that we shall present the equations first, reformulate them suitably and then apply the solution theory to them. The study of visco-elastic media within the framework of partial integro-differential equations will be carried out in the exercises section.

7.1 Poro-Elastic Deformations

In this section we will discuss the equations of poro-elasticity, which form a coupled system of equations. More precisely, the equations of (linearised) elasticity are coupled with the diffusion equation. Before properly writing these equations we introduce the following notation and differential operators.

Definition Let $\mathbb{K}_{\text{sym}}^{d \times d} := \{A \in \mathbb{K}^{d \times d} ; A = A^\top\} \subseteq \mathbb{K}^{d \times d}$ be the (closed) subspace of symmetric $d \times d$ matrices. Let $\Omega \subseteq \mathbb{R}^d$ be open. Then define

$$L_2(\Omega)_{\text{sym}}^{d \times d} := L_2(\Omega; \mathbb{K}_{\text{sym}}^{d \times d})$$

$$= \left\{ (\Phi_{jk})_{j,k \in \{1, \dots, d\}} \in L_2(\Omega)^{d \times d} ; \forall j, k \in \{1, \dots, d\} : \Phi_{jk} = \Phi_{kj} \right\}.$$

Analogously, we set $C_c^\infty(\Omega)_{\text{sym}}^{d \times d} := C_c^\infty(\Omega; \mathbb{K}_{\text{sym}}^{d \times d})$.

Note that the symmetry of a $d \times d$ matrix here means that the matrix elements are symmetric with respect to the main diagonal. For $\mathbb{K} = \mathbb{C}$, this does not

correspond to the symmetry of the associated linear operator (which would rather be selfadjointness).

Definition Let $\Omega \subseteq \mathbb{R}^d$ be open. Then we define

$$\begin{aligned} \text{Grad}_c : C_c^\infty(\Omega)^d \subseteq L_2(\Omega)^d &\rightarrow L_2(\Omega)_{\text{sym}}^{d \times d} \\ (\phi_j)_{j \in \{1, \dots, d\}} &\mapsto \frac{1}{2} (\partial_k \phi_j + \partial_j \phi_k)_{j, k \in \{1, \dots, d\}}, \end{aligned}$$

and

$$\begin{aligned} \text{Div}_c : C_c^\infty(\Omega)_{\text{sym}}^{d \times d} \subseteq L_2(\Omega)_{\text{sym}}^{d \times d} &\rightarrow L_2(\Omega)^d \\ (\Phi_{jk})_{j, k \in \{1, \dots, d\}} &\mapsto \left(\sum_{k=1}^d \partial_k \Phi_{jk} \right)_{j \in \{1, \dots, d\}}. \end{aligned}$$

Similarly to the definitions in the previous chapter, we put $\text{Grad} := -\text{Div}_c^*$, $\text{Div} := -\text{Grad}_c^*$ and $\text{Grad}_0 := -\text{Div}^*$, $\text{Div}_0 := -\text{Grad}^*$, where (analogously to the scalar-valued case) we observe that $\text{Grad}_c \subseteq -\text{Div}_c^*$ motivating the notation Grad and Grad_0 .

Remark 7.1.1 Note that in the literature $\text{Grad} u$ is also denoted by $\varepsilon(u)$ and is called the *strain tensor*. Due to the (obvious) similarity to the scalar case, we refrain from using ε in this context and prefer Grad instead. Again, the index 0 in the operators refers to generalised Dirichlet (for Grad_0) or Neumann (for Div_0) boundary conditions.

We are now properly equipped to formulate the equations of poro-elasticity; see also [69] and below for further details. In an elastic body $\Omega \subseteq \mathbb{R}^d$, the displacement field, $u : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$, and the pressure field, $p : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$, of a fluid diffusing through Ω satisfy the following two energy balance equations

$$\begin{aligned} \partial_t \rho \partial_t u - \text{grad} \partial_t \lambda \text{div} u - \text{Div} C \text{Grad} u + \text{grad} \alpha^* p &= f, \\ \partial_t (c_0 p + \alpha \text{div} u) - \text{div} k \text{grad} p &= g. \end{aligned}$$

The right-hand sides $f : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ and $g : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ describe some given external forcing. We assume homogeneous Neumann boundary conditions for the diffusing fluid as well as homogeneous Dirichlet (i.e. clamped) boundary conditions for the elastic body. The operator $\rho \in L(L_2(\Omega)^d)$ describes the density of the medium Ω (usually realised as a multiplication operator by a bounded, measurable, scalar function). The bounded linear operators $C \in L(L_2(\Omega)_{\text{sym}}^{d \times d})$ and $k \in L(L_2(\Omega)^d)$ are the elasticity tensor and the hydraulic conductivity of the medium, whereas $c_0, \lambda \in L(L_2(\Omega))$ are the porosity of the medium and the compressibility of the fluid, respectively. The operator $\alpha \in L(L_2(\Omega))$ is the so-

called Biot–Willis constant. Note that in many applications ρ , c_0 , λ and α are just positive real numbers, and C and k are strictly positive definite tensors or matrices.

The reformulation of the equations for poro-elasticity involves several ‘tricks’. One of these is to introduce the matrix trace as the operator

$$\begin{aligned} \text{trace}: L_2(\Omega)_{\text{sym}}^{d \times d} &\rightarrow L_2(\Omega) \\ (\Phi_{jk})_{j,k \in \{1, \dots, d\}} &\mapsto \sum_{j=1}^d \Phi_{jj}. \end{aligned}$$

Note that the adjoint is given by $\text{trace}^* f = \text{diag}(f, \dots, f) \in L_2(\Omega)_{\text{sym}}^{d \times d}$. It is then elementary to obtain $\text{trace Grad} \subseteq \text{div}$ as well as $\text{grad} = \text{Div trace}^*$. Hence, we formally get

$$\begin{aligned} \partial_t \rho \partial_t u - \text{Div} \left((\partial_t \text{trace}^* \lambda \text{trace} + C) \text{Grad} u - \text{trace}^* \alpha^* p \right) &= f, \\ \partial_t (c_0 p + \alpha \text{trace Grad} u) - \text{div} k \text{grad} p &= g. \end{aligned}$$

Next, we introduce a new set of unknowns

$$\begin{aligned} v &:= \partial_t u, \\ T &:= C \text{Grad} u, \\ \omega &:= \lambda \text{trace Grad} v - \alpha^* p, \\ q &:= -k \text{grad} p. \end{aligned}$$

Here, v is the velocity, T is the stress tensor and q is the heat flux. The quantity ω is an additional variable, which helps to rewrite the system into the form of evolutionary equations.

In order to finalise the reformulation we shall assume some additional properties on the coefficients involved. Throughout the rest of this section, we assume that

$$\begin{aligned} \rho &= \rho^* \geq c, \\ c_0 &= c_0^* \geq c, \\ \text{Re } \lambda &\geq c, \\ \text{Re } k &\geq c, \text{ and} \\ C &= C^* \geq c \end{aligned}$$

for some $c > 0$, where all inequalities are thought of in the sense of positive definiteness (compare Chap. 6). As a consequence, we obtain

$$\text{trace Grad} v = \lambda^{-1} \omega + \lambda^{-1} \alpha^* p.$$

Rewriting the defining equations for T , ω , and q together with the two equations we started out with, we obtain the system

$$\begin{aligned} \partial_t \rho v - \operatorname{Div} (T + \operatorname{trace}^* \omega) &= f, \\ \partial_t c_0 p + \alpha \lambda^{-1} \omega + \alpha \lambda^{-1} \alpha^* p + \operatorname{div} q &= g, \\ \lambda^{-1} \omega + \lambda^{-1} \alpha^* p - \operatorname{trace} \operatorname{Grad} v &= 0, \\ \partial_t C^{-1} T - \operatorname{Grad} v &= 0, \\ k^{-1} q + \operatorname{grad} p &= 0. \end{aligned}$$

Note that at this stage of modelling we assumed that we can freely interchange the order of differentiation, so that $\operatorname{Grad} \partial_t u = \partial_t \operatorname{Grad} u$. Introducing

$$M_0 := \begin{pmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & c_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha \lambda^{-1} \alpha^* & \alpha \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} \alpha^* & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k^{-1} \end{pmatrix}, \quad (7.1)$$

$$V := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \operatorname{trace} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 0 & 0 & -\operatorname{Div} & 0 \\ 0 & 0 & 0 & 0 & \operatorname{div}_0 \\ 0 & 0 & 0 & 0 & 0 \\ -\operatorname{Grad}_0 & 0 & 0 & 0 & 0 \\ 0 & \operatorname{grad} & 0 & 0 & 0 \end{pmatrix}, \quad (7.2)$$

we obtain

$$(\partial_t M_0 + M_1 + V A V^*) \begin{pmatrix} v \\ p \\ \omega \\ T \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This perspective enables us to prove well-posedness for the equations of poroelasticity by applying Theorem 6.2.1.

Theorem 7.1.2 *Put $H := L_2(\Omega)^d \times L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega)^{d \times d}_{\operatorname{sym}} \times L_2(\Omega)^d$ and let $M_0, M_1, V \in L(H)$ and A be given as in (7.1) and (7.2). Then there exists $\nu_0 > 0$ such that for all $\nu \geq \nu_0$ the operator $\overline{\partial_{t,\nu} M_0 + M_1 + V A V^*}$ is continuously invertible on $L_{2,\nu}(\mathbb{R}; H)$. The inverse S_ν of this operator is causal and eventually independent of ν . Moreover, $\sup_{\nu \geq \nu_0} \|S_\nu\| < \infty$ and $F \in \operatorname{dom}(\partial_{t,\nu})$ implies $S_\nu F \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(V A V^*)$.*

We will provide two prerequisites for the proof. We ask for the details of the proof of Theorem 7.1.2 in Exercise 7.1.

Proposition 7.1.3 *Let H_0, H_1 be Hilbert spaces, $B: \text{dom}(B) \subseteq H_0 \rightarrow H_0$ skew-selfadjoint, $V \in L(H_0, H_1)$ bijective. Then $(VBV^*)^* = -VBV^*$.*

The proof of Proposition 7.1.3 is left as (part of) Exercise 7.1.

Proposition 7.1.4 *Let H be a Hilbert space, $N_0, N_1 \in L(H)$ with $N_0 = N_0^*$. Assume there exist $c_0, c_1 > 0$ such that $\langle x, N_0x \rangle \geq c_0 \|x\|^2$ for all $x \in \text{ran}(N_0)$ and $\text{Re} \langle y, N_1y \rangle \geq c_1 \|y\|^2$ for all $y \in \ker(N_0)$. Then for all $0 < c'_1 < c_1$ there exists $\nu_0 > 0$ such that for all $\nu \geq \nu_0$ we have that*

$$\nu N_0 + \text{Re } N_1 \geq c'_1.$$

Proof Note that by the selfadjointness of N_0 we can decompose $H = \overline{\text{ran}}(N_0) \oplus \ker(N_0)$, see Corollary 2.2.6. Let $z \in H$, and $x \in \overline{\text{ran}}(N_0)$, $y \in \ker(N_0)$ such that $z = x + y$. For $\varepsilon, \nu > 0$ we estimate

$$\begin{aligned} & \nu \langle x + y, N_0(x + y) \rangle + \text{Re} \langle x + y, N_1(x + y) \rangle \\ &= \nu \langle x, N_0x \rangle + \text{Re} \langle y, N_1y \rangle + \text{Re} \langle x, N_1x \rangle + \text{Re} \langle x, N_1y \rangle + \text{Re} \langle y, N_1x \rangle \\ &\geq \nu c_0 \|x\|^2 + c_1 \|y\|^2 - \|N_1\| \|x\|^2 - 2 \|N_1\| \|x\| \|y\| \\ &\geq \nu c_0 \|x\|^2 + c_1 \|y\|^2 - \|N_1\| \|x\|^2 - \frac{1}{\varepsilon} \|N_1\|^2 \|x\|^2 - \varepsilon \|y\|^2 \\ &= \left(\nu c_0 - \frac{1}{\varepsilon} \|N_1\|^2 - \|N_1\| \right) \|x\|^2 + (c_1 - \varepsilon) \|y\|^2, \end{aligned}$$

where we have used the Peter–Paul inequality (i.e., Young’s inequality for products of non-negative numbers). For $0 < c'_1 < c_1$ we find $\varepsilon > 0$ such that $c_1 - \varepsilon > c'_1$. Then we choose $\nu_0 > \frac{1}{c_0} \left(c'_1 + \frac{1}{\varepsilon} \|N_1\|^2 + \|N_1\| \right)$. With this choice of ν_0 we deduce for all $\nu \geq \nu_0$ that

$$\nu \langle z, N_0z \rangle + \text{Re} \langle z, N_1z \rangle \geq c'_1 \left(\|x\|^2 + \|y\|^2 \right) = c'_1 \|z\|^2,$$

which yields the assertion. □

7.2 Fractional Elasticity

Let $\Omega \subseteq \mathbb{R}^d$ be open. In order to better fit to the experimental data of visco-elastic solids (i.e., to incorporate solids that ‘memorise’ previous force applied to them) the equations of linearised elasticity need to be extended in some way. The balance law

for the momentum, however, is still satisfied; that is, for the displacement $u: \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ we still have that

$$\partial_t \rho \partial_t u - \operatorname{Div} T = f,$$

where $\rho \in L(L_2(\Omega)^d)$ models the density and $f: \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ is a given external forcing term. The stress tensor, $T: \mathbb{R} \times \Omega \rightarrow \mathbb{R}_{\text{sym}}^{d \times d}$, does *not* follow the classical Hooke's law, which, if it did, would look like

$$T = C \operatorname{Grad} u$$

for $C \in L(L_2(\Omega)_{\text{sym}}^{d \times d})$. Instead it is amended by another material dependent coefficient $D \in L(L_2(\Omega)_{\text{sym}}^{d \times d})$ and a fractional time derivative; that is,

$$T = C \operatorname{Grad} u + D \partial_t^\alpha \operatorname{Grad} u,$$

for some $\alpha \in [0, 1]$, where $\partial_t^\alpha := \partial_t \partial_t^{\alpha-1}$, see Example 5.3.1(e). We shall simplify the present consideration slightly and refer to Exercise 7.2 instead for a more involved example. Throughout this section, we shall assume that

$$C = 0, \quad D = D^* \geq c, \quad \text{and} \quad \rho = \rho^* \geq c$$

for some $c > 0$. Thus, putting $v := \partial_t u$ and assuming the clamped boundary conditions again, we study well-posedness of

$$\partial_t \rho v - \operatorname{Div} T = f, \tag{7.3}$$

$$T = D \partial_t^\alpha \operatorname{Grad}_0 u. \tag{7.4}$$

In order to do that, we first rewrite the second equation. We will make use of the following proposition which will serve us to show bounded invertibility of ∂_t^α (in the space $L_{2,\nu}$), and which will also be employed to obtain well-posedness.

Proposition 7.2.1 *Let $\nu > 0$, $z \in \mathbb{C}_{\operatorname{Re} \geq \nu}$, $\alpha \in [0, 1]$. Then*

$$\operatorname{Re} z^\alpha \geq (\operatorname{Re} z)^\alpha \geq \nu^\alpha.$$

Proof Let us prove the first inequality. Note that without loss of generality, we may assume that $\operatorname{Re} z = 1$. Let $\varphi := \arg z \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Since $\ln \circ \cos$ is concave on $(-\frac{\pi}{2}, \frac{\pi}{2})$ (as $(\ln \circ \cos)' = -\tan$ is decreasing) and $(\ln \circ \cos)(0) = 0$, we obtain

$$\ln \cos(\alpha\varphi) = \ln \cos(\alpha\varphi + (1-\alpha)0) \geq \alpha \ln \cos(\varphi) + (1-\alpha) \ln \cos(0) = \ln (\cos(\varphi)^\alpha),$$

and therefore $\cos(\alpha\varphi) \geq \cos(\varphi)^\alpha$. Since $\operatorname{Re} z = 1$ implies $|z| = \frac{1}{\cos(\varphi)}$, we obtain

$$\operatorname{Re} z^\alpha = \frac{\cos(\alpha\varphi)}{(\cos \varphi)^\alpha} \geq 1 = (\operatorname{Re} z)^\alpha.$$

The second inequality follows from monotonicity of $x \mapsto x^\alpha$. \square

Applying Proposition 7.2.1 and noting that D is boundedly invertible we can reformulate (7.4) as

$$\partial_{t,v}^{-\alpha} D^{-1} T - \operatorname{Grad}_0 u = 0,$$

so that (7.4) and (7.3) read

$$\left(\partial_{t,v} \begin{pmatrix} \rho & 0 \\ 0 & \partial_{t,v}^{-\alpha} D^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{Div} \\ \operatorname{Grad}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ T \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

A solution theory for the latter equation, thus, reads as follows, where again $v := \partial_{t,v} u$.

Theorem 7.2.2 *Put $H := L_2(\Omega)^d \times L_2(\Omega)_{\operatorname{sym}}^{d \times d}$. Then for all $\nu > 0$ the operator*

$$\partial_{t,v} \begin{pmatrix} \rho & 0 \\ 0 & \partial_{t,v}^{-\alpha} D^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{Div} \\ \operatorname{Grad}_0 & 0 \end{pmatrix}$$

is densely defined and closable in $L_{2,\nu}(\mathbb{R}; H)$. The inverse of the closure is continuous, causal and eventually independent of ν .

Proof The proof rests on Theorem 6.2.1. Since $\begin{pmatrix} 0 & \operatorname{Div} \\ \operatorname{Grad}_0 & 0 \end{pmatrix}$ is skew-selfadjoint by Proposition 6.2.3(a), it suffices to confirm the positive definiteness condition for the material law. For this let $z \in \mathbb{C}_{\operatorname{Re} \geq \nu}$ and compute for $x \in L_2(\Omega)_{\operatorname{sym}}^{d \times d}$, using Proposition 7.2.1 and Proposition 6.2.3(b),

$$\operatorname{Re} \left\langle x, z z^{-\alpha} D^{-1} x \right\rangle = \operatorname{Re} \left\langle x, z^{1-\alpha} D^{-1} x \right\rangle \geq \nu^{1-\alpha} \left\langle x, D^{-1} x \right\rangle \geq \nu^{1-\alpha} \frac{c}{\|D\|^2} \|x\|^2.$$

This yields the assertion. \square

7.3 The Heat Equation with Delay

Let $\Omega \subseteq \mathbb{R}^d$ be open. In this section we concentrate on a generalisation of the heat equation discussed in the previous chapter. Although we keep the heat flux balance in the sense that

$$\partial_t \theta + \operatorname{div} q = Q,$$

with $q: \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ being the heat flux and $\theta: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ being the heat, we shall now modify Fourier's law to the extent that

$$q = -a \operatorname{grad} \theta - b\tau_{-h} \operatorname{grad} \theta$$

for some $a, b \in L(L_2(\Omega)^d)$ with $\operatorname{Re} a \geq c$ for some $c > 0$, and $h > 0$. We shall again assume homogeneous Neumann boundary conditions for q . Written in the now standard block operator matrix form, this modified heat equation reads

$$\left(\partial_{t,v} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (a + b\tau_{-h})^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix}.$$

In order to actually justify the existence of the operator $(a + b\tau_{-h})^{-1}$ as a bounded linear operator, we provide the following lemma.

Lemma 7.3.1 *Let $h > 0$.*

- (a) *There exists $\nu_0 > 0$ such that for all $\nu \geq \nu_0$ the operator $a + b\tau_{-h}$ is continuously invertible on $L_{2,\nu}(\mathbb{R}; L_2(\Omega)^d)$.*
- (b) *For all $0 < c' < c/\|a\|^2$ there is $\nu_1 \geq \nu_0$ such that for all $z \in \mathbb{C}_{\operatorname{Re} \geq \nu_1}$ we have $\operatorname{Re}(a + be^{-zh})^{-1} \geq c'$.*

Proof Note that a is invertible with $\|a^{-1}\| \leq \frac{1}{c}$ and $\operatorname{Re} a^{-1} \geq \frac{c}{\|a\|^2}$ by Proposition 6.2.3(b).

- (a) By Example 5.3.4(c), for all $\nu > 0$ we obtain

$$\|b\tau_{-h}\|_{L(L_{2,\nu})} \leq \|b\|_{L(L_2(\Omega)^d)} \sup_{t \in \mathbb{R}} \left| e^{-(it+\nu)h} \right| = \|b\|_{L(L_2(\Omega)^d)} e^{-h\nu}.$$

Thus, we find $\nu_0 > 0$ such that for all $\nu \geq \nu_0$ we obtain $\|b\tau_{-h}a^{-1}\|_{L(L_{2,\nu})} \leq \frac{1}{c} \|b\tau_{-h}\|_{L(L_{2,\nu})} < 1$. Thus,

$$a + b\tau_{-h} = \left(1 + b\tau_{-h}a^{-1}\right)a$$

is continuously invertible by a Neumann series argument.

- (b) Let $0 < c' < c/\|a\|^2$, and set $d(z) := -be^{-zh}a^{-1}$. Moreover, we choose $\nu_1 \geq \nu_0$ such that $\|d(z)\|_{L(L_2(\Omega)^d)} \leq \min\{\frac{1}{2}, \varepsilon\}$ for all $z \in \mathbb{C}_{\operatorname{Re} \geq \nu_1}$, where $0 < \varepsilon \leq \frac{1}{2}c \left(\frac{c}{\|a\|^2} - c' \right)$. For $z \in \mathbb{C}_{\operatorname{Re} \geq \nu_1}$ we compute

$$\begin{aligned}
 \operatorname{Re} \left(a + be^{-zh} \right)^{-1} &= \operatorname{Re} a^{-1} (1 - d(z))^{-1} = \operatorname{Re} \left(a^{-1} \sum_{k=0}^{\infty} d(z)^k \right) \\
 &= \operatorname{Re} \left(a^{-1} + \sum_{k=1}^{\infty} a^{-1} d(z)^k \right) \\
 &\geq \frac{c}{\|a\|^2} - \left\| \sum_{k=1}^{\infty} a^{-1} d(z)^k \right\| \geq \frac{c}{\|a\|^2} - \frac{1}{c} \sum_{k=1}^{\infty} \|d(z)\|^k \\
 &= \frac{c}{\|a\|^2} - \frac{1}{c} \frac{\|d(z)\|}{1 - \|d(z)\|} \geq \frac{c}{\|a\|^2} - \frac{1}{c} 2\varepsilon \geq c'.
 \end{aligned}$$

□

With this lemma we are in the position to provide the well-posedness for the modified heat equation.

Theorem 7.3.2 *Let $H = L_2(\Omega) \times L_2(\Omega)^d$. There exists $\nu_0 > 0$ such that for all $\nu \geq \nu_0$ the operator*

$$\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (a + b\tau_{-h})^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}$$

is densely defined and closable with continuously invertible closure on $L_{2,\nu}(\mathbb{R}; H)$. The inverse of the closure is causal and eventually independent of ν .

Proof The proof rests on Theorem 6.2.1 and Lemma 7.3.1. □

7.4 Dual Phase Lag Heat Conduction

The last example is concerned with a different modification of Fourier's law. The heat flux balance

$$\partial_t \theta + \operatorname{div} q = Q \tag{7.5}$$

is accompanied by the modified Fourier's law

$$(1 + s_q \partial_t + \frac{1}{2} s_q^2 \partial_t^2) q = -(1 + s_\theta \partial_t) \operatorname{grad} \theta, \tag{7.6}$$

where $s_q \in \mathbb{R}$, $s_\theta > 0$ are given numbers, which are called 'phases'.

Remark 7.4.1 The modified Fourier's law in (7.6) is an attempt to resolve the problem of infinite propagation speed which stems from a truncated Taylor series expansion of a model given by

$$\tau_{s_q} q = -\tau_{s_\theta} \operatorname{grad} \theta.$$

Note that it can be shown that such a model would even be ill-posed, see [34].

Let us turn back to the system (7.5) and (7.6). Notice, since $s_\theta > 0$, and due to a strictly positive real part of the derivative in our functional analytic setting, we deduce that $(1 + s_\theta \partial_{t,v})$ is continuously invertible for $v \geq 0$. Thus, we obtain

$$\partial_{t,v} (\partial_{t,v}^{-1} + s_q + \frac{1}{2} s_q^2 \partial_{t,v}) (1 + s_\theta \partial_{t,v})^{-1} q = -\operatorname{grad} \theta.$$

The block operator matrix formulation of the dual phase lag heat conduction model is thus

$$\left(\partial_{t,v} \begin{pmatrix} 1 & 0 \\ 0 & (\partial_{t,v}^{-1} + s_q + \frac{1}{2} s_q^2 \partial_{t,v}) (1 + s_\theta \partial_{t,v})^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix}.$$

Theorem 7.4.2 *Let $H = L_2(\Omega) \times L_2(\Omega)^d$. Assume $s_q \in \mathbb{R} \setminus \{0\}$, $s_\theta > 0$. Then there exists $v_0 > 0$ such that for all $v \geq v_0$ the operator*

$$\partial_{t,v} \begin{pmatrix} 1 & 0 \\ 0 & (\partial_{t,v}^{-1} + s_q + \frac{1}{2} s_q^2 \partial_{t,v}) (1 + s_\theta \partial_{t,v})^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}$$

is densely defined and closable with continuously invertible closure on $L_{2,v}(\mathbb{R}; H)$. The inverse of the closure is causal and eventually independent of v .

The proof of Theorem 7.4.2 is again based on Theorem 6.2.1. Thus, we shall only record the decisive observation in the next result. For this, we define

$$M(z) := \frac{z^{-1} + s_q + \frac{1}{2} s_q^2 z}{1 + s_\theta z} \in \mathbb{C} \quad (z \in \mathbb{C} \setminus \{0, -\frac{1}{s_\theta}\}).$$

Lemma 7.4.3 *Let $s_q \in \mathbb{R} \setminus \{0\}$, $s_\theta > 0$. Then there exist $v_0 \in \mathbb{R}$ and $c > 0$ such that for all $z \in \mathbb{C}_{\operatorname{Re} \geq v_0}$ we have*

$$\operatorname{Re} z M(z) \geq c.$$

Proof We put $\sigma := \frac{s_q}{s_\theta}$. Let $z \in \mathbb{C} \setminus \{0, -\frac{1}{s_\theta}\}$. We compute

$$z M(z) = \frac{1 + s_q z + \frac{1}{2} s_q^2 z^2}{1 + s_\theta z} = \frac{1}{2} s_q z \sigma + \sigma \left(1 - \frac{1}{2} \sigma\right) + \frac{1 - \sigma \left(1 - \frac{1}{2} \sigma\right)}{1 + s_\theta z}$$

and therefore

$$\operatorname{Re} z M(z) = \frac{1}{2} s_q \sigma \operatorname{Re} z + \sigma \left(1 - \frac{1}{2} \sigma \right) + \frac{\left(1 - \sigma \left(1 - \frac{1}{2} \sigma \right) \right) (1 + s_\theta \operatorname{Re} z)}{|1 + s_\theta z|^2}.$$

By assumption

$$0 < \frac{s_q^2}{s_\theta} = s_q \sigma,$$

and since

$$\frac{\left(1 - \sigma \left(1 - \frac{1}{2} \sigma \right) \right) (1 + s_\theta \operatorname{Re} z)}{|1 + s_\theta z|^2} \rightarrow 0$$

as $\operatorname{Re} z \rightarrow \infty$, we obtain

$$\operatorname{Re} z M(z) \geq \frac{1}{2} s_q \sigma \operatorname{Re} z - \delta$$

for some $\delta > 0$ and all $z \in \mathbb{C}$ with $\operatorname{Re} z$ large enough. □

7.5 Comments

The equations of poro-elasticity have been proposed in [69] and were mathematically studied in [63, 103].

Equations of fractional elasticity are discussed in [20, 73, 87, 134]. The well-posedness conditions stated here and in Exercise 7.2 can be generalised as it is outlined in [87] to the case where both C and D are non-negative, selfadjoint operators so that C and D satisfy the conditions imposed on N_1 and N_0 in Proposition 7.1.4. We refrained from presenting this argument here, as it seemed too technical for the time being. Note however that the proof is neither fundamentally different nor considerably less elementary.

The heat equation with delay has also been studied in [55] with an entirely different strategy; the dual phase lag models have been dealt with in [68, 127].

Other ideas to rectify infinite propagation speed of the heat equation can be found in [3], where nonlinear models for heat conduction are being discussed.

The visco-elastic equations discussed in Exercise 7.6 are studied with convolution operators more general than below in [119]; see also [19, 27, 95, 116].

Exercises

Exercise 7.1 (Solutions to the Equations of Poro-Elasticity)

- (a) Prove Proposition 7.1.3.
- (b) Prove Theorem 7.1.2.
- (c) Let $\Omega \subseteq \mathbb{R}^d$ be open, $\nu > 0$, $f \in H^1_\nu(\mathbb{R}; L_2(\Omega)^d)$ and $g \in H^1_\nu(\mathbb{R}; L_2(\Omega))$. With the help of Theorem 7.1.2 show that for large enough $\nu > 0$ there exist a unique $u \in \text{dom}(\partial_{t,\nu}^2) \cap \text{dom}(\text{grad } \lambda \text{ div } \partial_{t,\nu}) \cap \text{dom}(\text{Div } C \text{ Grad}_0)$ and $p \in \text{dom}(\partial_{t,\nu}) \cap \text{dom}(\text{grad } \alpha^*) \cap \text{dom}(\text{div}_0 k \text{ grad})$ such that

$$\begin{aligned} \partial_{t,\nu} \rho \partial_{t,\nu} u - \text{grad } \lambda \text{ div } \partial_{t,\nu} u - \text{Div } C \text{ Grad}_0 u + \text{grad } \alpha^* p &= f \\ \partial_{t,\nu} c_0 p + \alpha \text{ div } \partial_{t,\nu} u - \text{div}_0 k \text{ grad } p &= g. \end{aligned}$$

Exercise 7.2 Let $\Omega \subseteq \mathbb{R}^d$ be open, $C, D \in L(L_2(\Omega)^{d \times d})$, $D = D^* \geq c$ for some $c > 0$ and $\alpha \in [\frac{1}{2}, 1]$. Show that there exists $\nu_0 > 0$ such that for all $\nu \geq \nu_0$ the system

$$\begin{aligned} \partial_{t,\nu} \rho v - \text{Div } T &= f, \\ T &= (C + D \partial_{t,\nu}^\alpha) \text{Grad}_0 u, \end{aligned}$$

where $v = \partial_{t,\nu} u$, admits a unique solution $(v, T) \in L_{2,\nu}(\mathbb{R}; L_2(\Omega)^d \times L_2(\Omega)^{d \times d})$ for all $f \in H^1_\nu(\mathbb{R}; L_2(\Omega)^d)$.

The following exercises are devoted to showing the well-posedness of certain equations in visco-elasticity, where the ‘viscous part’ is modelled by convolution with certain integral kernels. The proof of the positive definiteness property requires some preliminary results. We assume the reader to be equipped with the basics from the theory of functions of one complex variable.

For $U \subseteq \mathbb{C}$ open write $\tilde{U} := \{(x, y) \in \mathbb{R}^2; x + iy \in U\}$, and for $u: U \rightarrow \mathbb{C}$ holomorphic, define $f_{\text{Re } u}: \tilde{U} \rightarrow \mathbb{R}$ by $f_{\text{Re } u}(x, y) := \text{Re } u(x + iy)$ for $(x, y) \in \tilde{U}$. We put

$$H_{\text{Re}}(U) := \{f_{\text{Re } u}; u: U \rightarrow \mathbb{C} \text{ holomorphic}\}.$$

Exercise 7.3 Let $U \subseteq \mathbb{C}$ be open.

- (a) Let $f \in H_{\text{Re}}(U)$. Show that f satisfies the mean value property; that is, for all $(x, y) \in \tilde{U}$ and $r > 0$ with $\overline{B((x, y), r)} \subseteq \tilde{U}$ we have

$$f(x, y) = \frac{1}{2\pi} \int_0^{2\pi} f(x + r \cos \theta, y + r \sin \theta) d\theta.$$

- (b) Let $U := \mathbb{C}_{\text{Im}>0}$ and $f \in H_{\text{Re}}(U) \cap C(\mathbb{R} \times \mathbb{R}_{\geq 0})$. Moreover, assume that $f(x, 0) = 0$ for each $x \in \mathbb{R}$ and $f(x, y) \rightarrow 0$ as $|(x, y)| \rightarrow \infty$. Show that $f = 0$ on $\mathbb{R} \times \mathbb{R}_{\geq 0}$.

Exercise 7.4 In this exercise we show a version of *Poisson's formula*. Let $U := \mathbb{C}_{\text{Im}>0}$ and $f \in H_{\text{Re}}(U) \cap C(\mathbb{R} \times \mathbb{R}_{\geq 0})$.

- (a) Assume that $f(\cdot, 0) \in L_p(\mathbb{R})$ for some $1 \leq p < \infty$. Show that $\mathbb{C}_{\text{Im}>0} \ni z \mapsto \frac{1}{\pi} \int_{\mathbb{R}} \frac{\text{Im} z' + i(\text{Re} z - x')}{(\text{Re} z - x')^2 + (\text{Im} z)^2} f(x', 0) dx'$ is holomorphic.
- (b) Assume that $f(\cdot, 0) \in L_{\infty}(\mathbb{R})$. Show that $\frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-x')^2 + y^2} f(x', 0) dx' \rightarrow f(x_0, 0)$ as $x \rightarrow x_0$ and $y \rightarrow 0+$.
- (c) (Poisson's formula) Assume that $f(\cdot, 0) \in L_p(\mathbb{R})$ for some $1 \leq p < \infty$ and $f(x, y) \rightarrow 0$ as $|(x, y)| \rightarrow \infty$ in $\mathbb{R} \times \mathbb{R}_{\geq 0}$. Show that

$$f(x, y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-x')^2 + y^2} f(x', 0) dx' \quad ((x, y) \in \mathbb{R} \times \mathbb{R}_{>0}).$$

Hint: Apply Exercise 7.3(b).

Exercise 7.5 Let $v_0 \in \mathbb{R}$ and $k \in L_{1, v_0}(\mathbb{R}; \mathbb{R})$ with $\text{spt } k \subseteq \mathbb{R}_{\geq 0}$.

- (a) Show that for all $(x, v) \in \mathbb{R} \times \mathbb{R}_{>v_0}$ we have

$$\text{Im}(\mathcal{L}k)(ix + v) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{v - v_0}{(x-x')^2 + (v-v_0)^2} \text{Im}(\mathcal{L}k)(ix' + v_0) dx'.$$

Hint: Approximate k by functions in $C_c^{\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$ and use Poisson's formula (see Exercise 7.4).

- (b) Assume there exists $d \geq 0$ such that for all $x \in \mathbb{R}$

$$x \text{Im}(\mathcal{L}k)(ix + v_0) \leq d.$$

Show that for all $v \geq v_0$ and $x \in \mathbb{R}$ we have

$$x \text{Im}(\mathcal{L}k)(ix + v) \leq 4d.$$

Hint: Use the formula in (a) and split the integral into positive and negative part of \mathbb{R} ; use symmetry of $(\mathcal{L}k)$ under conjugation due to the realness of k .

Exercise 7.6 Let $\Omega \subseteq \mathbb{R}^d$ be open, $v_0 \in \mathbb{R}$ and $k \in L_{1, v_0}(\mathbb{R}; \mathbb{R})$ with $\text{spt } k \subseteq \mathbb{R}_{\geq 0}$. Assume there exists $d \geq 0$ such that

$$x \text{Im}(\mathcal{L}k)(ix + v_0) \leq d \quad (x \in \mathbb{R}).$$

Show that there exists $\nu_1 \geq \nu_0$ such that for all $\nu \geq \nu_1$ the operator

$$\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & (1 - k*)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \text{Div} \\ \text{Grad}_0 & 0 \end{pmatrix}$$

is well-defined, densely defined and closable in $L_{2,\nu}(\mathbb{R}; H)$ with $H = L_2(\Omega)^d \times L_2(\Omega)_{\text{sym}}^{d \times d}$. Further, show that its closure is continuously invertible, and that the corresponding inverse is causal and eventually independent of ν .

Exercise 7.7 Let $\nu_0 \in \mathbb{R}$ and $k \in L_{1,\nu_0}(\mathbb{R}; \mathbb{R})$ with $\text{spt } k \subseteq \mathbb{R}_{\geq 0}$.

- (a) Assume that k is absolutely continuous with $k' \in L_{1,\nu_0}(\mathbb{R}; \mathbb{R})$. Show that there exist $\nu_1 \geq \nu_0$ and $d \geq 0$ with

$$x \text{Im}(\mathcal{L}k)(ix + \nu_1) \leq d \quad (x \in \mathbb{R}).$$

- (b) Assume that $k(t) \geq 0$ for all $t \in \mathbb{R}$ and that $k(t) \leq k(s)$, whenever $s \leq t$. Show that there exists $\nu_1 \geq \nu_0$ with

$$x \text{Im}(\mathcal{L}k)(ix + \nu_1) \leq 0 \quad (x \in \mathbb{R}).$$

Hint: For part (b) use the explicit formula for $\text{Im}(\mathcal{L}k)$ as an integral and the periodicity of \sin .

Remark: The condition in (a) is a standard assumption for convolution kernels in the framework of visco-elastic equations; the condition in (b) is from [95].

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