Chapter 10
Differential Algebraic Equations

Let H be a Hilbert space and $\nu \in \mathbb{R}$. We saw in the previous chapter how initial value problems can be formulated within the framework of evolutionary equations. More precisely, we have studied problems of the form

$$\begin{cases} \left(\partial_{t,\nu} M_0 + M_1 + A\right) U = 0 \quad \text{on } (0,\infty)\,, \\ M_0 U(0+) = M_0 U_0 \end{cases} \tag{10.1}$$

for $U_0 \in H$, $M_0, M_1 \in L(H)$ and $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint; that is, we have considered material laws of the form

$$M(z) \coloneqq M_0 + z^{-1} M_1 \quad (z \in \mathbb{C} \setminus \{0\}).$$

Here, the initial value is attained in a weak sense as an equality in the extrapolation space $H^{-1}(A)$. The first line is also meant in a weak sense since the left-hand side turned out to be a functional in $H_\nu^{-1}(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; H^{-1}(A))$. In Theorem 9.4.3 it was shown that the latter problem can be rewritten as

$$\left(\partial_{t,\nu} M_0 + M_1 + A\right) U = \delta_0 M_0 U_0.$$

In this chapter we aim to inspect initial value problems a little closer but in the particularly simple case when $A = 0$. However, we want to impose the initial condition for U and not just $M_0 U$. Thus, we want to deal with the problem

$$\begin{cases} \left(\partial_{t,\nu} M_0 + M_1\right) U = 0 \quad \text{on } (0,\infty)\,, \\ U(0+) = U_0 \end{cases} \tag{10.2}$$

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for two bounded operators M_0, M_1 and an initial value $U_0 \in H$. This class of differential equations is known as *differential algebraic equations* since the operator M_0 is allowed to have a non-trivial kernel. Thus, (10.2) is a coupled problem of a differential equation (on $(\ker M_0)^{\perp}$) and an algebraic equation (on $\ker M_0$). We begin by treating these equations in the finite-dimensional case; that is, $H = \mathbb{C}^n$ and $M_0, M_1 \in \mathbb{C}^{n\times n}$ for some $n \in \mathbb{N}$.

10.1 The Finite-Dimensional Case

Throughout this section let $n \in \mathbb{N}$ and $M_0, M_1 \in \mathbb{C}^{n\times n}$.

Definition We define the *spectrum of the matrix pair* (M_0, M_1) by

$$\sigma(M_0, M_1) := \{z \in \mathbb{C}\,;\ \det(zM_0 + M_1) = 0\},$$

and the *resolvent set of the matrix pair* (M_0, M_1) by

$$\rho(M_0, M_1) := \mathbb{C} \setminus \sigma(M_0, M_1).$$

Remark 10.1.1

(a) It is immediate that $\sigma(M_0, M_1)$ is closed since the mapping $z \mapsto \det(zM_0+M_1)$ is continuous.
(b) Note in particular that the spectrum (the set of eigenvalues) of a matrix A corresponds in this setting to the spectrum of the matrix pair $(1, -A)$.

In contrast to the case of the spectrum of one matrix, it may happen that $\sigma(M_0, M_1) = \mathbb{C}$ (for example we can choose $M_0 = 0$ and M_1 singular). More precisely, we have the following result.

Lemma 10.1.2 *The set $\sigma(M_0, M_1)$ is either finite or equals the whole complex plane $\mathbb{C}$. If $\sigma(M_0, M_1)$ is finite then* $\operatorname{card}(\sigma(M_0, M_1)) \leqslant n$.

Proof The function $z \mapsto \det(zM_0 + M_1)$ is a polynomial of order less than or equal to n. If it is constantly zero, then $\sigma(M_0, M_1) = \mathbb{C}$ and otherwise $\operatorname{card}(\sigma(M_0, M_1)) \leqslant n$. □

Definition The matrix pair (M_0, M_1) is called *regular* if $\sigma(M_0, M_1) \neq \mathbb{C}$.

The main problem in solving an initial value problem of the form (10.2) is that one cannot expect a solution for each initial value $U_0 \in \mathbb{C}^n$ as the following simple example shows.

Example 10.1.3 Let $M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $U_0 \in \mathbb{C}^2$. We assume that there exists a solution $U : \mathbb{R}_{\geqslant 0} \to \mathbb{C}^2$ satisfying (10.2); that is,

$$\begin{aligned} U_1'(t) + U_2'(t) + U_1(t) &= 0 \quad (t > 0), \\ U_2(t) &= 0 \quad (t > 0), \\ U(0+) &= U_0. \end{aligned}$$

The second and third equation yield that the second coordinate of U_0 has to be zero. Then, for $U_0 = (x, 0) \in \mathbb{C}^2$ the unique solution of the above problem is given by

$$U(t) = \big(U_1(t), U_2(t)\big) = (x\mathrm{e}^{-t}, 0) \quad (t \geqslant 0).$$

Definition We call an initial value $U_0 \in \mathbb{C}^n$ *consistent* for (10.2) if there exists $\nu > 0$ and $U \in C(\mathbb{R}_{\geqslant 0}; \mathbb{C}^n) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; \mathbb{C}^n)$ such that (10.2) holds. We denote the set of all consistent initial values for (10.2) by

$$\mathrm{IV}(M_0, M_1) := \left\{ U_0 \in \mathbb{C}^n \, ; \, U_0 \text{ consistent} \right\}.$$

Remark 10.1.4 It is obvious that $\mathrm{IV}(M_0, M_1)$ is a subspace of $\mathbb{C}^n$. In particular, $0 \in \mathrm{IV}(M_0, M_1)$.

It is now our goal to determine the space $\mathrm{IV}(M_0, M_1)$. One possibility for doing so uses the so-called *quasi-Weierstraß normal form*.

Proposition 10.1.5 (Quasi-Weierstraß Normal Form) *Assume that* (M_0, M_1) *is regular. Then there exist invertible matrices* $P, Q \in \mathbb{C}^{n \times n}$ *such that*

$$PM_0Q = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}, \quad PM_1Q = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix},$$

where $C \in \mathbb{C}^{k \times k}$ *and* $N \in \mathbb{C}^{(n-k)\times(n-k)}$ *for some* $k \in \{0, \ldots, n\}$. *Moreover, the matrix* N *is nilpotent; that is, there exists* $\ell \in \mathbb{N}$ *such that* $N^\ell = 0$.

Proof Since (M_0, M_1) is regular we find $\lambda \in \mathbb{C}$ such that $\lambda M_0 + M_1$ is invertible. We set $P_1 := (\lambda M_0 + M_1)^{-1}$ and obtain

$$\begin{aligned} M_{0,1} &:= P_1 M_0 = (\lambda M_0 + M_1)^{-1} M_0, \\ M_{1,1} &:= P_1 M_1 = (\lambda M_0 + M_1)^{-1} M_1 = 1 - \lambda M_{0,1}. \end{aligned}$$

Let now $P_2 \in \mathbb{C}^{n \times n}$ such that

$$M_{0,2} := P_2 M_{0,1} P_2^{-1} = \begin{pmatrix} J & 0 \\ 0 & \widetilde{N} \end{pmatrix}$$

for some invertible matrix $J \in \mathbb{C}^{k\times k}$ and a nilpotent matrix $\widetilde{N} \in \mathbb{C}^{(n-k)\times(n-k)}$ (use the Jordan normal form of $M_{0,1}$ here). Then

$$M_{1,2} := P_2 M_{1,1} P_2^{-1} = \begin{pmatrix} 1-\lambda J & 0 \\ 0 & 1-\lambda\widetilde{N} \end{pmatrix}.$$

Now, by the nilpotency of $\widetilde{N}$, the matrix $(1-\lambda\widetilde{N})$ is invertible by the Neumann series. We set

$$P_3 := \begin{pmatrix} J^{-1} & 0 \\ 0 & (1-\lambda\widetilde{N})^{-1} \end{pmatrix}$$

and obtain

$$P_3 M_{0,2} = \begin{pmatrix} 1 & 0 \\ 0 & (1-\lambda\widetilde{N})^{-1}\widetilde{N} \end{pmatrix}, \quad \text{and} \quad P_3 M_{1,2} = \begin{pmatrix} J^{-1}-\lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $(1-\lambda\widetilde{N})^{-1}\widetilde{N}$ is nilpotent, since the matrices commute and $\widetilde{N}$ is nilpotent. Thus, the assertion follows with $N := (1-\lambda\widetilde{N})^{-1}\widetilde{N}$, $C := J^{-1}-\lambda$, $P = P_3 P_2 P_1$, and $Q = P_2^{-1}$. □

It is clear that the matrices P, Q, C and N in the previous proposition are not uniquely determined by M_0 and M_1. However, the size of N and C as well as the degree of nilpotency of N are determined by M_0 and M_1 as the following proposition shows.

Proposition 10.1.6 *Let $P, Q \in \mathbb{C}^{n\times n}$ be invertible such that*

$$PM_0Q = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}, \quad PM_1Q = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix},$$

where $C \in \mathbb{C}^{k\times k}$, $N \in \mathbb{C}^{(n-k)\times(n-k)}$ for some $k \in \{0,\ldots,n\}$, and N is nilpotent. Then (M_0, M_1) is regular and

(a) *k is the degree of the polynomial $z \mapsto \det(zM_0 + M_1)$.*
(b) *$N^\ell = 0$ if and only if*

$$\sup_{|z|\geqslant r} \left\| z^{-\ell+1}(zM_0+M_1)^{-1} \right\| < \infty$$

for one (or equivalently all) $r > 0$ such that $B(0,r) \supseteq \sigma(M_0, M_1)$.

Proof First, note that

$$\det(zM_0 + M_1) = \frac{1}{\det P \ \det Q} \det \begin{pmatrix} z + C & 0 \\ 0 & zN + 1 \end{pmatrix} = \frac{1}{\det P \ \det Q} \det(z + C)$$

for all $(z \in \mathbb{C})$. Hence, (M_0, M_1) is regular and

$$k = \deg \det((\cdot) + C) = \deg \det((\cdot)M_0 + M_1),$$

which shows (a). Moreover, we have $\rho(M_0, M_1) = \rho(-C)$ and

$$(zM_0 + M_1)^{-1} = Q \begin{pmatrix} (z + C)^{-1} & 0 \\ 0 & (zN + 1)^{-1} \end{pmatrix} P \quad (z \in \rho(M_0, M_1)),$$

and hence, for $r > 0$ with $B(0, r) \supseteq \sigma(M_0, M_1)$ we have

$$\left\| (zM_0 + M_1)^{-1} \right\| \leqslant K_1 \left\| (zN + 1)^{-1} \right\| \quad (|z| \geqslant r)$$

for some $K_1 \geqslant 0$, since $\sup_{|z| \geqslant r} \left\| (z + C)^{-1} \right\| < \infty$. Now let $\ell \in \mathbb{N}$ such that $N^\ell = 0$. Then

$$\left\| (zN + 1)^{-1} \right\| = \left\| \sum_{k=0}^{\ell-1} (-1)^k z^k N^k \right\| \leqslant K_2 |z|^{\ell-1} \quad (|z| \geqslant r)$$

for some constant $K_2 \geqslant 0$ and thus,

$$\left\| (zM_0 + M_1)^{-1} \right\| \leqslant K_1 K_2 |z|^{\ell-1} \quad (|z| \geqslant r).$$

Assume on the other hand that

$$\sup_{|z| \geqslant r} \left\| z^{-\ell+1} (zM_0 + M_1)^{-1} \right\| < \infty$$

for some $\ell \in \mathbb{N}$ and $r > 0$ with $\sigma(M_0, M_1) \subseteq B(0, r)$. Then there exist $\widetilde{K}_1, \widetilde{K}_2 \geqslant 0$ such that

$$\left\| (zN + 1)^{-1} \right\| \leqslant \left\| \begin{pmatrix} (z + C)^{-1} & 0 \\ 0 & (zN + 1)^{-1} \end{pmatrix} \right\| \leqslant \widetilde{K}_1 \left\| (zM_0 + M_1)^{-1} \right\| \leqslant \widetilde{K}_2 |z|^{\ell-1}$$

for all $z \in \mathbb{C}$ with $|z| \geqslant r$. Now, let $p \in \mathbb{N}$ be minimal such that $N^p = 0$. We show that $p \leqslant \ell$ by contradiction. Assume $p > \ell$. Then we compute

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{n^\ell}(nN+1)^{-1}N^{p-\ell-1} = \lim_{n\to\infty} \sum_{k=0}^{p-1}(-1)^k n^{k-\ell} N^{k+p-\ell-1} \\ &= \lim_{n\to\infty} \sum_{k=0}^{\ell-1}(-1)^k n^{k-\ell} N^{k+p-\ell-1} + (-1)^\ell N^{p-1} \\ &= (-1)^\ell N^{p-1}, \end{aligned}$$

which contradicts the minimality of p. □

Theorem 10.1.7 *Let (M_0, M_1) be regular and $P, Q \in \mathbb{C}^{n\times n}$ be chosen according to Proposition 10.1.5. Let $k = \deg\det((\cdot)M_0 + M_1)$. Then*

$$\mathrm{IV}(M_0, M_1) = \left\{U_0 \in \mathbb{C}^n\,;\ Q^{-1}U_0 \in \mathbb{C}^k \times \{0\}\right\}.$$

Moreover, for each $U_0 \in \mathrm{IV}(M_0, M_1)$ the solution U of (10.2) is unique and satisfies $U \in C(\mathbb{R}_{\geqslant 0}; \mathbb{C}^n) \cap C^1(\mathbb{R}_{>0}; \mathbb{C}^n)$ as well as

$$\begin{aligned} M_0U'(t) + M_1U(t) &= 0 \quad (t > 0), \\ U(0+) &= U_0. \end{aligned}$$

Proof Let $C \in \mathbb{C}^{k\times k}$ and $N \in \mathbb{C}^{(n-k)\times(n-k)}$ be nilpotent as in Proposition 10.1.5. Obviously U is a solution of (10.2) if and only if $V := Q^{-1}U$ both is continuous on $\mathbb{R}_{\geqslant 0}$ and solves

$$\begin{aligned} \left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & N\end{pmatrix} + \begin{pmatrix}C & 0\\ 0 & 1\end{pmatrix}\right) V &= 0 \quad \text{on } (0,\infty), \\ V(0+) &= Q^{-1}U_0 =: V_0. \end{aligned} \tag{10.3}$$

Clearly, if $Q^{-1}U_0 = (x, 0) \in \mathbb{C}^k \times \{0\}$ then V given by $V(t) := (\mathrm{e}^{-tC}x, 0)$ for $t \geqslant 0$ is a solution of (10.3) for $\nu > 0$ large enough. On the other hand, if V given by $V(t) = (V_1(t), V_2(t)) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$ $(t \geqslant 0)$ is a solution of (10.3) then we have

$$\partial_{t,\nu}NV_2 + V_2 = 0 \quad \text{on } (0,\infty).$$

Since N is nilpotent, there exists $\ell \in \mathbb{N}$ with $N^\ell = 0$. Hence,

$$N^{\ell-1}V_2(t) = -N^{\ell-1}\partial_{t,\nu}NV_2(t) = \partial_{t,\nu}N^\ell V_2(t) = 0 \quad (t > 0),$$

which in turn implies $\partial_{t,\nu} N^{\ell-1} V_2 = 0$ on $(0, \infty)$. Using again the differential equation, we infer $N^{\ell-2} V_2(t) = 0$ for $t > 0$. Inductively, we deduce $V_2(t) = 0$ for $t > 0$ and by continuity $V_2(0+) = 0$, which yields $V_0 = Q^{-1} U_0 \in \mathbb{C}^k \times \{0\}$. The uniqueness follows from Proposition 10.2.7 below. □

10.2 The Infinite-Dimensional Case

Let now $M_0, M_1 \in L(H)$. Again, it is our aim to determine the space of consistent initial values for the problem

$$\begin{cases} \left(\partial_{t,\nu} M_0 + M_1\right) U = 0 \quad \text{on } (0, \infty), \\ U(0+) = U_0. \end{cases} \tag{10.4}$$

Here, consistent initial values are defined as in the finite-dimensional setting:

Definition We call an initial value $U_0 \in H$ *consistent* for (10.4) if there exist $\nu > 0$ and $U \in C(\mathbb{R}_{\geqslant 0}; H) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ such that (10.4) holds. We denote the set of all consistent initial values for (10.4) by

$$\mathrm{IV}(M_0, M_1) := \{U_0 \in H\,;\ U_0 \text{ consistent}\}.$$

Before we try to determine $\mathrm{IV}(M_0, M_1)$ we prove a regularity result for solutions of (10.4).

Proposition 10.2.1 *Let $\nu > 0$, $U_0 \in H$ and $U \in C(\mathbb{R}_{\geqslant 0}; H) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ be a solution of (10.4). Then $M_0(U - \mathbb{1}_{[0,\infty)} U_0) \in H_\nu^1(\mathbb{R}; H)$ and*

$$\partial_{t,\nu} M_0 \left(U - \mathbb{1}_{[0,\infty)} U_0\right) + M_1 U = 0.$$

Proof We extend U to $\mathbb{R}$ by 0. First, observe that $M_0(U - \mathbb{1}_{[0,\infty)} U_0) \colon \mathbb{R} \to H$ is continuous, since U is continuous and $U(0+) = U_0$. By Lemma 9.4.2 (with $A = 0$), we obtain

$$U - \mathbb{1}_{[0,\infty)} U_0 \in \mathrm{dom}\left(\overline{\partial_{t,\nu} M_0 + M_1}\right) \text{ and } \left(\overline{\partial_{t,\nu} M_0 + M_1}\right)(U - \mathbb{1}_{[0,\infty)} U_0) = -M_1 U_0 \mathbb{1}_{[0,\infty)}.$$

Since $\partial_{t,\nu}$ is closed and M_0 is bounded, $\partial_{t,\nu} M_0$ is closed as well. Since M_1 is bounded, therefore also $\partial_{t,\nu} M_0 + M_1$ is closed. Thus, $U - \mathbb{1}_{[0,\infty)} U_0 \in \mathrm{dom}(\partial_{t,\nu} M_0 + M_1) = \mathrm{dom}(\partial_{t,\nu} M_0)$ and therefore $M_0(U - \mathbb{1}_{[0,\infty)} U_0) \in \mathrm{dom}(\partial_{t,\nu})$, and

$$\partial_{t,\nu} M_0(U - \mathbb{1}_{[0,\infty)} U_0) + M_1 U = 0.$$

□

We now come back to the space $\mathrm{IV}(M_0, M_1)$. Since we are now dealing with an infinite-dimensional setting, we cannot use normal forms to determine $\mathrm{IV}(M_0, M_1)$ without dramatically restricting the class of operators. Thus, we follow a different approach using so-called Wong sequences.

Definition We set

$$\mathrm{IV}_0 \coloneqq H$$

and for $k \in \mathbb{N}_0$ we set

$$\mathrm{IV}_{k+1} \coloneqq M_1^{-1}[M_0[\mathrm{IV}_k]].$$

The sequence $(\mathrm{IV}_k)_{k\in\mathbb{N}_0}$ is called the *Wong sequence* associated with (M_0, M_1).

Remark 10.2.2 By induction, we infer $\mathrm{IV}_{k+1} \subseteq \mathrm{IV}_k$ for each $k \in \mathbb{N}_0$.

As in the matrix case, we denote by

$$\rho(M_0, M_1) \coloneqq \left\{z \in \mathbb{C};\ (zM_0 + M_1)^{-1} \in L(H)\right\}$$

the *resolvent set of* (M_0, M_1).

Lemma 10.2.3 *Let $k \in \mathbb{N}_0$. Then:*

(a) $M_1(zM_0 + M_1)^{-1}M_0 = M_0(zM_0 + M_1)^{-1}M_1$ *for each* $z \in \rho(M_0, M_1)$.
(b) $(zM_0 + M_1)^{-1}M_0[\mathrm{IV}_k] \subseteq \mathrm{IV}_{k+1}$ *for each* $z \in \rho(M_0, M_1)$.
(c) *If* $x \in \mathrm{IV}_k$ *we find* $x_1, \ldots, x_{k+1} \in H$ *such that for each* $z \in \rho(M_0, M_1) \setminus \{0\}$

$$(zM_0 + M_1)^{-1}M_0x = \frac{1}{z}x + \sum_{\ell=1}^{k} \frac{1}{z^{\ell+1}}x_\ell + \frac{1}{z^{k+1}}(zM_0 + M_1)^{-1}x_{k+1}.$$

(d) *If* $\rho(M_0, M_1) \neq \varnothing$ *then* $M_1^{-1}[M_0[\overline{\mathrm{IV}_k}]] \in \overline{\mathrm{IV}_{k+1}}$.

Proof The proofs of the statements (a) to (c) are left as Exercise 10.6. We now prove (d). If $k = 0$ there is nothing to show. So assume that the statement holds for some $k \in \mathbb{N}_0$ and let $x \in M_1^{-1}\left[M_0\left[\overline{\mathrm{IV}_{k+1}}\right]\right]$. Since $\overline{\mathrm{IV}_{k+1}} \subseteq \overline{\mathrm{IV}_k}$, we infer $x \in M_1^{-1}\left[M_0\left[\overline{\mathrm{IV}_k}\right]\right] \subseteq \overline{\mathrm{IV}_{k+1}}$ by induction hypothesis. Hence, we find a sequence $(w_n)_{n\in\mathbb{N}}$ in IV_{k+1} with $w_n \to x$. Let now $z \in \rho(M_0, M_1)$. Then, by (b), we have $(zM_0 + M_1)^{-1}M_0w_n \in \mathrm{IV}_{k+2}$ for each $n \in \mathbb{N}$ and hence, $(zM_0 + M_1)^{-1}M_0x \in \overline{\mathrm{IV}_{k+2}}$. Moreover, since $M_1x \in M_0\left[\overline{\mathrm{IV}_{k+1}}\right]$, we find a sequence $(y_n)_{n\in\mathbb{N}}$ in IV_{k+1} with $M_0y_n \to M_1x$. Setting now

$$x_n \coloneqq (zM_0 + M_1)^{-1}zM_0x + (zM_0 + M_1)^{-1}M_0y_n \in \overline{\mathrm{IV}_{k+2}}$$

(where, again, we have used (b)) for $n \in \mathbb{N}$, we derive

$$\begin{aligned} x_n &= (zM_0 + M_1)^{-1} zM_0 x + (zM_0 + M_1)^{-1} M_0 y_n \\ &= x - (zM_0 + M_1)^{-1} (M_1 x - M_0 y_n) \to x \end{aligned}$$

as $n \to \infty$ and thus, $x \in \overline{\mathrm{IV}_{k+2}}$. □

The importance of the Wong sequence becomes apparent if we consider solutions of (10.4).

Lemma 10.2.4 *Assume that $\rho(M_0, M_1) \neq \varnothing$. Let $\nu > 0$ and $U \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H) \cap C(\mathbb{R}_{\geqslant 0}; H)$ be a solution of (10.4). Then $U(t) \in \bigcap_{k\in\mathbb{N}_0} \overline{\mathrm{IV}_k}$ for each $t \geqslant 0$.*

Proof We prove the claim, $U(t) \in \overline{\mathrm{IV}_k}$ for all $t \geqslant 0$ and $k \in \mathbb{N}_0$, by induction. For $k = 0$ there is nothing to show. Assume now that $U(t) \in \overline{\mathrm{IV}_k}$ for each $t \geqslant 0$ and some $k \in \mathbb{N}_0$. By Proposition 10.2.1 we know that

$$\partial_{t,\nu} M_0 (U - \mathbb{1}_{[0,\infty)} U_0) + M_1 U = 0$$

and thus, in particular,

$$M_0 U(t) - M_0 U_0 + \int_0^t M_1 U(s)\,\mathrm{d}s = 0 \quad (t \geqslant 0).$$

Let now $t \geqslant 0$ and $h > 0$. Then we infer

$$M_0 U(t+h) - M_0 U(t) + M_1 \int_t^{t+h} U(s)\,\mathrm{d}s = 0$$

and hence,

$$\int_t^{t+h} U(s)\,\mathrm{d}s \in M_1^{-1}\big[M_0[\overline{\mathrm{IV}_k}]\big] \subseteq \overline{\mathrm{IV}_{k+1}}$$

by Lemma 10.2.3(d). Since U is continuous, the fundamental theorem of calculus implies $U(t) \in \overline{\mathrm{IV}_{k+1}}$, which yields the assertion. □

In particular, the space of consistent initial values has to be a subspace of $\bigcap_{k\in\mathbb{N}_0} \overline{\mathrm{IV}_k}$. We now impose an additional constraint on the operator pair (M_0, M_1), which is equivalent to being regular in the finite-dimensional setting (cf. Proposition 10.1.6).

Definition We call the operator pair (M_0, M_1) *regular* if there exists $\nu_0 \geqslant 0$ such that

(a) $\mathbb{C}_{\mathrm{Re}>\nu_0} \subseteq \rho(M_0, M_1)$, and
(b) there exist $C \geqslant 0$ and $\ell \in \mathbb{N}$ such that for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ we have $\left\|(zM_0 + M_1)^{-1}\right\| \leqslant C\,|z|^{\ell-1}$.

Moreover, we call the smallest $\ell \in \mathbb{N}$ satisfying (b) the *index of* (M_0, M_1), which is denoted by $\operatorname{ind}(M_0, M_1)$.

Remark 10.2.5 Note that for matrices M_0 and M_1 the index equals the degree of nilpotency of N in the quasi-Weierstraß normal form by Proposition 10.1.6.

From now on, we will require that (M_0, M_1) is regular. First, we prove an important result on the Wong sequence in this case.

Proposition 10.2.6 *Let* (M_0, M_1) *be regular,* $k \in \mathbb{N}_0$*, and* $k \geqslant \operatorname{ind}(M_0, M_1)$*. Then*

$$\overline{\mathrm{IV}_k} = \overline{\mathrm{IV}_{\operatorname{ind}(M_0,M_1)}}.$$

Proof We show that $\overline{\mathrm{IV}_k} = \overline{\mathrm{IV}_{k+1}}$ for each $k \geqslant \operatorname{ind}(M_0, M_1)$. Since the inclusion "$\supseteq$" holds trivially, it suffices to show $\mathrm{IV}_k \subseteq \overline{\mathrm{IV}_{k+1}}$. For doing so, let $k \geqslant \operatorname{ind}(M_0, M_1)$ and $x \in \mathrm{IV}_k$. By Lemma 10.2.3(c) we find $x_1, \ldots, x_{k+1} \in H$ such that

$$(zM_0 + M_1)^{-1} M_0 x = \frac{1}{z} x + \sum_{\ell=1}^{k} \frac{1}{z^{\ell+1}} x_\ell + \frac{1}{z^{k+1}} (zM_0 + M_1)^{-1} x_{k+1}$$

for each $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. Since $k \geqslant \operatorname{ind}(M_0, M_1)$, we derive

$$z(zM_0 + M_1)^{-1} M_0 x \to x \quad (\operatorname{Re} z \to \infty),$$

and since the elements on the left-hand side belong to IV_{k+1}, by Lemma 10.2.3(b), the assertion immediately follows. □

We now prove that in case of a regular operator pair (M_0, M_1) the solution of (10.4) for a consistent initial value U_0 is uniquely determined.

Proposition 10.2.7 *Let* (M_0, M_1) *be regular,* $U_0 \in \mathrm{IV}(M_0, M_1)$*, and* $\nu > 0$ *such that a solution* $U \in C(\mathbb{R}_{\geqslant 0}; H) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ *of (10.4) exists. Then this solution is unique. In particular*

$$(\mathcal{L}_\rho U)(t) = \frac{1}{\sqrt{2\pi}} \big((\mathrm{i}t + \rho) M_0 + M_1\big)^{-1} M_0 U_0 \quad (a.e.\ t \in \mathbb{R})$$

for each $\rho > \max\{\nu, \nu_0\}$.

Proof By Proposition 10.2.1 we have $M_0(U - \mathbb{1}_{[0,\infty)} U_0) \in H^1_\nu(\mathbb{R}; H)$ and

$$\partial_{t,\nu} M_0(U - \mathbb{1}_{[0,\infty)} U_0) + M_1 U = 0.$$

Applying the Fourier–Laplace transformation, $\mathcal{L}_\rho$, for $\rho > \max\{\nu, \nu_0\}$ we deduce

$$(\mathrm{i}t + \rho) M_0 \big(\mathcal{L}_\rho U(t) - \frac{1}{\sqrt{2\pi}} \frac{1}{\mathrm{i}t + \rho} U_0\big) + M_1 \mathcal{L}_\rho U(t) = 0 \quad (\text{a.e. } t \in \mathbb{R})$$

which in turn yields

$$\mathcal{L}_\rho U(t) = \frac{1}{\sqrt{2\pi}}\big((\mathrm{i}t+\rho)M_0 + M_1\big)^{-1} M_0 U_0 \quad (\text{a.e. } t \in \mathbb{R})$$

and, in particular, proves the uniqueness of the solution. □

Remark 10.2.8 Let U be a solution of (10.4) for a consistent initial value U_0. Then the formula in Proposition 10.2.7 shows that $U \in \bigcap_{\rho>\nu_0} L_{2,\rho}(\mathbb{R}; H)$ and hence, we also have $M_0(U - \mathbb{1}_{[0,\infty)}U_0) \in \bigcap_{\rho>\nu_0} H^1_\rho(\mathbb{R}; H)$. If $\nu_0 > 0$ then we even obtain $U \in L_{2,\nu_0}(\mathbb{R}; H)$ since $\sup_{\rho>\nu_0} \|U\|_{L_{2,\rho}(\mathbb{R};H)} = \sup_{\rho>\nu_0} \|\mathcal{L}_\rho U\|_{L_2(\mathbb{R};H)} < \infty$ (cp. Lemma 8.1.1), and therefore also $M_0(U - \mathbb{1}_{[0,\infty)}U_0) \in H^1_{\nu_0}(\mathbb{R}; H)$.

One interesting consequence of the latter proposition is the following.

Corollary 10.2.9 *Let (M_0, M_1) be regular. Then the operator $M_0 \colon \mathrm{IV}(M_0, M_1) \to H$ is injective.*

Proof Let $U_0 \in \mathrm{IV}(M_0, M_1)$ with $M_0U_0 = 0$. By Proposition 10.2.7, the solution U of (10.4) with $U(0+) = U_0$ satisfies

$$\mathcal{L}_\rho U(t) = \frac{1}{\sqrt{2\pi}}\big((\mathrm{i}t+\rho)M_0 + M_1\big)^{-1} M_0 U_0 = 0$$

and hence, $U = 0$, which in turn implies $U_0 = U(0+) = 0$. □

We now want to determine the space $\mathrm{IV}(M_0, M_1)$ in terms of the Wong sequence.

Proposition 10.2.10 *Let (M_0, M_1) be regular. Then*

$$\mathrm{IV}_{\mathrm{ind}(M_0,M_1)} \subseteq \mathrm{IV}(M_0, M_1) \subseteq \overline{\mathrm{IV}_{\mathrm{ind}(M_0,M_1)}}.$$

Proof The second inclusion follows from Lemma 10.2.4 and Proposition 10.2.6. Let now $U_0 \in \mathrm{IV}_{\mathrm{ind}(M_0,M_1)}$ and set

$$V(z) := \frac{1}{\sqrt{2\pi}}(zM_0 + M_1)^{-1} M_0 U_0 \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}).$$

Let $k := \mathrm{ind}(M_0, M_1)$. By Lemma 10.2.3(c) we find $x_1, \ldots, x_{k+1} \in H$ such that

$$V(z) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{z}U_0 + \sum_{\ell=1}^{k} \frac{1}{z^{\ell+1}} x_\ell + \frac{1}{z^{k+1}}(zM_0 + M_1)^{-1} x_{k+1}\right) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}).$$

In particular, we read off that $V \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ for all $\nu > \nu_0$. Now, let $\nu > \nu_0$. By the Theorem of Paley–Wiener (more precisely by Corollary 8.1.3) there exists $U \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ such that

$$\big(\mathcal{L}_\rho U\big)(t) = V(\mathrm{i}t + \rho) \quad (\text{a.e. } t \in \mathbb{R}, \rho > \nu).$$

Moreover,

$$zV(z) - \frac{1}{\sqrt{2\pi}}U_0 = \frac{1}{\sqrt{2\pi}}\left(\sum_{\ell=1}^{k}\frac{1}{z^\ell}x_\ell + \frac{1}{z^k}\left(zM_0 + M_1\right)^{-1}x_{k+1}\right) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu})$$

and hence $\left(z \mapsto zV(z) - \frac{1}{\sqrt{2\pi}}U_0\right) \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ as well. Since

$$\begin{aligned}\left(\mathcal{L}_\rho \partial_{t,\rho}(U - \mathbb{1}_{[0,\infty)}U_0)\right)(t) &= (\mathrm{i}t + \rho)\left(\mathcal{L}_\rho U\right)(t) - \frac{1}{\sqrt{2\pi}}U_0 \\ &= (\mathrm{i}t + \rho)V(\mathrm{i}t + \rho) - \frac{1}{\sqrt{2\pi}}U_0 \quad (\text{a.e. } t \in \mathbb{R}, \rho > \nu),\end{aligned}$$

we infer $U - \mathbb{1}_{[0,\infty)}U_0 \in H_\nu^1(\mathbb{R}; H)$ and, thus, $U - \mathbb{1}_{[0,\infty)}U_0$ is continuous by Theorem 4.1.2. Hence, $U \in C(\mathbb{R}_{\geqslant 0}; H)$ and since $\operatorname{spt} U \subseteq \mathbb{R}_{\geqslant 0}$ we derive $U(0+) = U_0$. Finally, by the definition of V,

$$M_0\left(zV(z) - \frac{1}{\sqrt{2\pi}}U_0\right) = -\frac{1}{\sqrt{2\pi}}M_1(zM_0 + M_1)^{-1}M_0U_0 = -M_1V(z)$$

for all $z \in \mathbb{C}_{\mathrm{Re}>\nu}$. Hence,

$$\partial_{t,\nu}M_0(U - \mathbb{1}_{[0,\infty)}U_0) + M_1U = 0,$$

from which we see that U solves (10.4). □

Finally, we treat the case when $\mathrm{IV}(M_0, M_1)$ is closed.

Theorem 10.2.11 *Let (M_0, M_1) be regular and $\mathrm{IV}(M_0, M_1)$ closed. Then the operator $S\colon \mathrm{IV}(M_0, M_1) \to C(\mathbb{R}_{\geqslant 0}; H)$, which assigns to each initial state, $U_0 \in \mathrm{IV}(M_0, M_1)$, its corresponding solution, $U \in C(\mathbb{R}_{\geqslant 0}; H)$, of (10.4) is bounded in the sense that*

$$S_n\colon \mathrm{IV}(M_0, M_1) \to C([0, n]; H), \quad U_0 \mapsto SU_0|_{[0,n]}$$

is bounded for each $n \in \mathbb{N}$.

Proof By Proposition 10.2.10 we infer that $\mathrm{IV}(M_0, M_1) = \overline{\mathrm{IV}_k}$ with $k := \operatorname{ind}(M_0, M_1)$. Let $\nu > \nu_0 \geqslant 0$. By Proposition 10.2.7 and Corollary 8.1.3, there exists $C \geqslant 0$ such that

$$\begin{aligned}\sqrt{2\pi}\left\|\partial_{t,\nu}^{-k}SU_0\right\|_{L_{2,\nu}(\mathbb{R}_{\geq 0}; H)} &= \left\|\left(z \mapsto z^{-k}(zM_0 + M_1)^{-1}M_0U_0\right)\right\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)} \\ &\leqslant C\sqrt{\frac{\pi}{\nu}}\,\|M_0U_0\|_H\end{aligned}$$

for each $U_0 \in \mathrm{IV}(M_0, M_1)$, where we have used the regularity of (M_0, M_1) and

$$\left\| (z \mapsto z^{-1} M_0 U_0) \right\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)} = \sqrt{\frac{\pi}{\nu}} \, \| M_0 U_0 \|_H \, .$$

In particular, $S \colon \mathrm{IV}(M_0, M_1) \to H^{-1}(\partial_{t,\nu}^k)$ is bounded. Since $L_{2,\nu_0}(\mathbb{R}_{\geqslant 0}; H) \hookrightarrow H^{-1}(\partial_{t,\nu}^k)$ continuously, we infer that $S \colon \mathrm{IV}(M_0, M_1) \to L_{2,\nu_0}(\mathbb{R}_{\geqslant 0}; H)$ is bounded by the closed graph theorem. Hence, also

$$S_n \colon \mathrm{IV}(M_0, M_1) \to L_2([0, n]; H), \quad U_0 \mapsto SU_0|_{[0,n]}$$

is bounded for each $n \in \mathbb{N}$ and since $C([0, n]; H) \hookrightarrow L_2([0, n]; H)$ continuously, we infer that S_n is bounded with values in $C([0, n]; H)$ again by the closed graph theorem. □

Remark 10.2.12 The variant of the closed graph theorem used in the proof above is the following: Let X, Y be Banach spaces and Z a Hausdorff topological vector space (e.g. a Banach space) such that $Y \hookrightarrow Z$ continuously. Let $T \colon X \to Z$ be linear and continuous with $T[X] \subseteq Y$. Then $T \in L(X, Y)$. Indeed, by the closed graph theorem it suffices to show that $T : X \to Y$ is closed. For doing so, let $(x_n)_n$ be a sequence in X with $x_n \to x$ and $T x_n \to y$ for some $x \in X, y \in Y$. Then $T x_n \to T x$ in Z by the continuity of T and $T x_n \to y$ in Z by the continuous embedding. Hence, $y = T x$ and thus, T is closed.

10.3 Comments

The theory of differential algebraic equations in finite dimensions is a very active field. The main motivation for studying these equations comes from the modelling of electrical circuits and from control theory (see e.g. [28] and Exercise 10.5). The main reference for the statements presented in the first part of this chapter is the book by Kunkel and Mehrmann [57]. Of course, also in the finite-dimensional case Wong sequences can be used to determine the consistent initial values, see Exercise 10.1. For instance, in [13] the connection between Wong sequences and the quasi-Weierstraß normal form for matrix pairs is studied. Of course, the theory is not restricted to linear and homogeneous problems. Indeed, in the non-homogeneous case it turns out that the set of consistent initial values also depends on the given right-hand side.

The theory of differential algebraic equations in infinite dimensions is less well studied than the finite-dimensional case. We refer to [114], where the theory of C_0-semigroups is used to deal with such equations. Moreover, we refer to [97, 98], where sequences of projectors are used to decouple the system. Moreover, there exist several references in the Russian literature, where the equations are called Sobolev type equations (see e.g. [111]). The results on infinite-dimensional problems presented here are based on [121, 124, 125]. In [124] the focus was on systems with index 0 with an emphasis on exponential stability and dichotomy.

We also add the following remark concerning the result in Theorem 10.2.11. By Corollary 10.2.9 we know that $M_0\colon \operatorname{IV}(M_0, M_1) \to H$ is injective. If $\operatorname{IV}(M_0, M_1)$ is closed, it follows that the operator $C\colon \operatorname{dom}(C) \subseteq \operatorname{IV}(M_0, M_1) \to \operatorname{IV}(M_0, M_1)$ given by

$$\operatorname{dom}(C) := \{U_0 \in \operatorname{IV}(M_0, M_1)\,;\; M_1 U_0 \in M_0\,[\operatorname{IV}(M_0, M_1)]\}\,,$$
$$CU_0 := M_0^{-1} M_1 U_0 \quad (U_0 \in \operatorname{dom}(C))$$

is well-defined and closed. Using this operator, C, Theorem 10.2.11 states that if $\operatorname{IV}(M_0, M_1)$ is closed then $-C$ generates a C_0-semigroup on $\operatorname{IV}(M_0, M_1)$. The precise statement can be found in [121, Theorem 5.7]. Moreover, C is bounded if $\operatorname{IV}_{\operatorname{ind}(M_0,M_1)}$ is closed (cf. Exercise 10.7).

Exercises

Exercise 10.1 Let $M_0, M_1 \in \mathbb{C}^{n\times n}$ such that (M_0, M_1) is regular and define the Wong sequence $(\operatorname{IV}_j)_{j\in\mathbb{N}_0}$ associated with (M_0, M_1). Moreover, let $P, Q \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{k\times k}$, and $N \in \mathbb{C}^{(n-k)\times(n-k)}$ be as in the quasi-Weierstraß normal form for (M_0, M_1) with N nilpotent (cf. Proposition 10.1.5). We decompose a vector $x \in \mathbb{C}^n$ into $\check{x} \in \mathbb{C}^k$ and $\widehat{x} \in \mathbb{C}^{n-k}$ such that $x = (\check{x}, \widehat{x})$. Prove that

$$x \in \operatorname{IV}_j \Leftrightarrow \widehat{Q^{-1}x} \in \operatorname{ran} N^j \quad (j \in \mathbb{N}_0).$$

Moreover, show that for each $z \in \rho(M_0, M_1)$ we have

$$\operatorname{IV}_j = \operatorname{ran}\Big((zM_0 + M_1)^{-1} M_0\Big)^j \quad (j \in \mathbb{N}_0).$$

Exercise 10.2 Let $E \in \mathbb{C}^{n\times n}$. We set $k := \operatorname{ind}(E, 1)$, where 1 denotes the identity matrix in $\mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a *Drazin inverse of E* if the following properties hold:

- $EX = XE$,
- $XEX = X$,
- $XE^{k+1} = E^k$.

Prove that each matrix $E \in \mathbb{C}^{n\times n}$ has a unique Drazin inverse.
Hint: For the existence consider the quasi-Weierstraß form for $(E, 1)$.

Exercise 10.3 Let $M_0, M_1 \in \mathbb{C}^{n\times n}$ with (M_0, M_1) regular and $M_0M_1 = M_1M_0$. Denote by M_0^{D} the Drazin inverse of M_0 (see Exercise 10.2). Prove:

(a) $M_0^{\mathrm{D}}M_1 = M_1M_0^{\mathrm{D}}$,
(b) $\operatorname{ran} M_0^{\mathrm{D}}M_0 = \mathrm{IV}(M_0, M_1)$,
(c) For all $U_0 \in \mathrm{IV}(M_0, M_1)$ the solution U of (10.2) is given by

$$U(t) = \mathrm{e}^{-tM_0^{\mathrm{D}}M_1}U_0 \quad (t \geqslant 0).$$

Exercise 10.4 Let $M_0, M_1 \in \mathbb{C}^{n\times n}$ with (M_0, M_1) regular. Prove that there exist two matrices $E, A \in \mathbb{C}^{n\times n}$ with (E, A) regular and $EA = AE$ such that

- $\mathrm{IV}(E, A) = \mathrm{IV}(M_0, M_1)$,
- U solves the initial value problem (10.2) for the matrices M_0, M_1 if and only if U solves the initial value problem (10.2) for the matrices E, A with the same initial value $U_0 \in \mathrm{IV}(M_0, M_1)$.

Exercise 10.5 We consider the following electrical circuit (see Fig. 10.1) with a resistor with resistance $R > 0$, an inductor with inductance $L > 0$ and a capacitor with capacitance $C > 0$. We denote the respective voltage drops by v_R, v_L and v_C. Moreover, the current is denoted by i. The constitutive relations for resistor, inductor and capacitor are given by

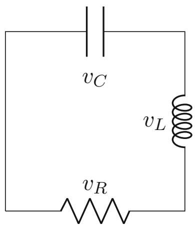
$$\begin{aligned} Ri &= v_R, \\ Li' &= v_L, \\ Cv_C' &= i, \end{aligned}$$

respectively. Moreover, by Kirchhoff's second law we have

$$v_R + v_C + v_L = 0.$$

Write these equations as a differential algebraic equation and compute the index and the space of consistent initial values. Moreover, compute the solution for each consistent initial value for $R = 2$ and $C = L = 1$.

Fig. 10.1 Electrical circuit



Exercise 10.6 Prove the assertions (a) to (c) in Lemma 10.2.3.

Exercise 10.7 Let $M_0, M_1 \in L(H)$.

(a) Assume that $\rho(M_0, M_1) \neq \varnothing$. Prove that for each $k \in \mathbb{N}$ the space IV_k is closed if and only if $M_0\,[\mathrm{IV}_{k-1}]$ is closed.

(b) Assume that (M_0, M_1) is regular with $\mathrm{ind}(M_0, M_1) \geqslant 1$. Prove that if $\mathrm{IV}_{\mathrm{ind}(M_0,M_1)}$ is closed then the operator

$$M_0|_{\mathrm{IV}_{\mathrm{ind}(M_0,M_1)}} \colon \mathrm{IV}_{\mathrm{ind}(M_0,M_1)} \to M_0\left[\mathrm{IV}_{\mathrm{ind}(M_0,M_1)-1}\right]$$

is an isomorphism.

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