Chapter 6
Solution Theory for Evolutionary Equations

In this chapter, we shall discuss and present the first major result of the manuscript: Picard's theorem on the solution theory for evolutionary equations which is the main result of [82]. In order to stress the applicability of this theorem, we shall deal with applications first and provide a proof of the actual result afterwards. With an initial interest in applications in mind, we start off with the introduction of some operators related to vector calculus.

6.1 First Order Sobolev Spaces

Throughout this section let $\Omega \subseteq \mathbb{R}^d$ be an open set.

Definition We define

$$\begin{aligned} \operatorname{grad}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega) \subseteq L_2(\Omega) &\to L_2(\Omega)^d \\ \phi &\mapsto \left(\partial_j \phi\right)_{j\in\{1,\ldots,d\}}, \end{aligned}$$

$$\begin{aligned} \operatorname{div}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega)^d \subseteq L_2(\Omega)^d &\to L_2(\Omega) \\ \left(\phi_j\right)_{j\in\{1,\ldots,d\}} &\mapsto \sum_{j\in\{1,\ldots,d\}} \partial_j \phi_j, \end{aligned}$$

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and if $d = 3$,

$$\begin{aligned} \operatorname{curl}_c \colon C_c^\infty(\Omega)^3 \subseteq L_2(\Omega)^3 &\to L_2(\Omega)^3 \\ (\phi_j)_{j\in\{1,2,3\}} &\mapsto \begin{pmatrix} \partial_2\phi_3 - \partial_3\phi_2 \\ \partial_3\phi_1 - \partial_1\phi_3 \\ \partial_1\phi_2 - \partial_2\phi_1 \end{pmatrix}. \end{aligned}$$

Furthermore, we put

$$\operatorname{div} := -\operatorname{grad}_c^*, \quad \operatorname{grad} := -\operatorname{div}_c^*, \quad \operatorname{curl} := \operatorname{curl}_c^*$$

and

$$\operatorname{div}_0 := -\operatorname{grad}^*, \quad \operatorname{grad}_0 := -\operatorname{div}^*, \quad \operatorname{curl}_0 := \operatorname{curl}^*.$$

Proposition 6.1.1 *The relations* div, div_0, grad, grad_0, curl *and* curl_0 *are all densely defined, closed linear operators.*

Proof The operators grad_c, div_c and curl_c are densely defined by Exercise 6.3. Thus, div, grad and curl are closed linear operators by Lemma 2.2.7. Moreover, it follows from integration by parts that $\operatorname{grad}_c \subseteq \operatorname{grad}$, $\operatorname{div}_c \subseteq \operatorname{div}$ and $\operatorname{curl}_c \subseteq \operatorname{curl}$. Thus, div, grad and curl are also densely defined. This, in turn, implies that grad_c, div_c and curl_c are closable by Lemma 2.2.7 with respective closures grad_0, div_0 and curl_0 by Lemma 2.2.4. □

We shall describe the domains of these operators in more detail in the next theorem.

Theorem 6.1.2 *If $f \in L_2(\Omega)$ and $g = (g_j)_{j\in\{1,\ldots,d\}} \in L_2(\Omega)^d$ then the following statements hold:*

(a) $f \in \operatorname{dom}(\operatorname{grad})$ *and* $g = \operatorname{grad} f$ *if and only if*

$$\forall \phi \in C_c^\infty(\Omega), j \in \{1,\ldots,d\} \colon -\int_\Omega f\partial_j\phi = \int_\Omega g_j\phi.$$

(b) $f \in \operatorname{dom}(\operatorname{grad}_0)$ *and* $g = \operatorname{grad}_0 f$ *if and only if there exists $(f_k)_k$ in $C_c^\infty(\Omega)$ such that $f_k \to f$ in $L_2(\Omega)$ and* $\operatorname{grad} f_k \to g$ *in $L_2(\Omega)^d$ as $k \to \infty$.*

(c) $g \in \operatorname{dom}(\operatorname{div})$ *and* $f = \operatorname{div} g$ *if and only if*

$$\forall \phi \in C_c^\infty(\Omega) \colon -\int_\Omega g \cdot \operatorname{grad}\phi = \int_\Omega f\phi.$$

(d) $g \in \operatorname{dom}(\operatorname{div}_0)$ *and* $f = \operatorname{div}_0 g$ *if and only if there exists $(g_k)_k$ in $C_c^\infty(\Omega)^d$ such that $g_k \to g$ in $L_2(\Omega)^d$ and* $\operatorname{div} g_k \to f$ *in $L_2(\Omega)$ as $k \to \infty$.*

If $d = 3$ *and* $f, g \in L_2(\Omega)^3$ *then the following statements hold:*

(e) $f \in \mathrm{dom}(\mathrm{curl})$ *and* $g = \mathrm{curl}\, f$ *if and only if*

$$\forall \phi \in C_c^\infty(\Omega)^3 : \int_\Omega f \cdot \mathrm{curl}\, \phi = \int_\Omega g \cdot \phi.$$

(f) $f \in \mathrm{dom}(\mathrm{curl}_0)$ *and* $g = \mathrm{curl}_0\, f$ *if and only if there exists* $(f_k)_k$ *in* $C_c^\infty(\Omega)^3$ *such that* $f_k \to f$ *in* $L_2(\Omega)^3$ *and* $\mathrm{curl}\, f_k \to g$ *in* $L_2(\Omega)^3$ *as* $k \to \infty$.

All the statements in Theorem 6.1.2 are elementary consequences of the integration by parts formula, the definitions of the adjoint and Lemma 2.2.4. We ask the reader to prove these statements in Exercise 6.4.

We introduce the following notation:

$$\begin{aligned} H^1(\Omega) &:= \mathrm{dom}(\mathrm{grad}), \\ H_0^1(\Omega) &:= \mathrm{dom}(\mathrm{grad}_0), \\ H(\mathrm{div}, \Omega) &:= \mathrm{dom}(\mathrm{div}), \\ H(\mathrm{curl}, \Omega) &:= \mathrm{dom}(\mathrm{curl}). \end{aligned}$$

Following the rationale of appending zero as an index for $H_0^1(\Omega)$, we shall also use

$$\begin{aligned} H_0(\mathrm{div}, \Omega) &:= \mathrm{dom}(\mathrm{div}_0), \\ H_0(\mathrm{curl}, \Omega) &:= \mathrm{dom}(\mathrm{curl}_0). \end{aligned}$$

We caution the reader that other authors also use $H_0(\mathrm{div}, \Omega)$ and $H_0(\mathrm{curl}, \Omega)$ to denote the kernel of div and curl.

All the spaces just defined are so-called Sobolev spaces. We note that for $d = 3$ we clearly have $H^1(\Omega)^3 \subseteq H(\mathrm{div}, \Omega) \cap H(\mathrm{curl}, \Omega)$. On the other hand, note that $H(\mathrm{div}, \Omega)$ is neither a sub- nor a superset of $H(\mathrm{curl}, \Omega)$.

Remark 6.1.3 We emphasise that $H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{H^1(\Omega)} \subseteq H^1(\Omega)$ is a proper inclusion for many open Ω. The '0' in the index is a reminder of '0'-boundary conditions. In fact, the only difference between these two spaces lies in the behaviour of their elements at the boundary of Ω. The space H_0^1 signifies all H^1-functions vanishing at $\partial\Omega$ in a generalised sense. The corresponding statements are true for the inclusions $H_0(\mathrm{div}, \Omega) \subseteq H(\mathrm{div}, \Omega)$ and $H_0(\mathrm{curl}, \Omega) \subseteq H(\mathrm{curl}, \Omega)$. The space $H_0(\mathrm{div}, \Omega)$ describes $H(\mathrm{div}, \Omega)$-vector fields with vanishing normal component and to lie in $H_0(\mathrm{curl}, \Omega)$ provides a handy generalisation of vanishing tangential component. We will anticipate these abstractions when we apply the solution theory of evolutionary equations for particular cases. In a later chapter we will come back to this issue when we discuss inhomogeneous boundary value problems.

For later use, we record the following relationships between the vector-analytical operators introduced above.

Proposition 6.1.4 *Let $d = 3$. We have the following inclusions:*

$$\overline{\operatorname{ran}}(\operatorname{curl}_0) \subseteq \ker(\operatorname{div}_0),$$
$$\overline{\operatorname{ran}}(\operatorname{grad}_0) \subseteq \ker(\operatorname{curl}_0),$$
$$\overline{\operatorname{ran}}(\operatorname{curl}) \subseteq \ker(\operatorname{div}),$$
$$\overline{\operatorname{ran}}(\operatorname{grad}) \subseteq \ker(\operatorname{curl}).$$

Proof It is elementary to show that for given $\psi \in C_c^\infty(\Omega)^3$ and $\phi \in C_c^\infty(\Omega)$ we have $\operatorname{div}_0 \operatorname{curl}_0 \psi = 0$ as well as $\operatorname{curl}_0 \operatorname{grad}_0 \phi = 0$. Thus, we obtain $\operatorname{ran}(\operatorname{curl}_c) \subseteq \ker(\operatorname{div}_0)$ and $\operatorname{ran}(\operatorname{grad}_c) \subseteq \ker(\operatorname{curl}_0)$. Since $\ker(\operatorname{div}_0)$ and $\ker(\operatorname{curl}_0)$ are closed, and $C_c^\infty(\Omega)^3$ and $C_c^\infty(\Omega)$ are cores for curl_0 and grad_0 respectively, we obtain the first two inclusions. The last two inclusions follow from the first two by taking into account the orthogonal decompositions

$$L_2(\Omega)^3 = \overline{\operatorname{ran}}(\operatorname{grad}) \oplus \ker(\operatorname{div}_0) = \ker(\operatorname{curl}) \oplus \overline{\operatorname{ran}}(\operatorname{curl}_0)$$

and

$$L_2(\Omega)^3 = \overline{\operatorname{ran}}(\operatorname{grad}_0) \oplus \ker(\operatorname{div}) = \ker(\operatorname{curl}_0) \oplus \overline{\operatorname{ran}}(\operatorname{curl})$$

which follow from Corollary 2.2.6. □

6.2 Well-Posedness of Evolutionary Equations and Applications

The solution theory of evolutionary equations is contained in the next result, Picard's theorem. This result is central for all the derivations to come. In fact, with the notation of Theorem 6.2.1, we shall prove that for all (well-behaved) F there is a unique solution of

$$\big(\partial_{t,\nu} M(\partial_{t,\nu}) + A\big) U = F.$$

The solution U depends continuously and causally on the choice of F.

In order to formulate the result, for a Hilbert space H, $\nu \in \mathbb{R}$ and a given closed operator $A \colon \operatorname{dom}(A) \subseteq H \to H$ we define its extended operator in $L_{2,\nu}(\mathbb{R}; H)$, again denoted by A, by

$$\begin{aligned} L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A)) \subseteq L_{2,\nu}(\mathbb{R}; H) &\to L_{2,\nu}(\mathbb{R}; H) \\ f &\mapsto \big(t \mapsto Af(t)\big). \end{aligned}$$

We have collected some properties of extended operators in Exercises 6.1 and 6.2.

Theorem 6.2.1 (Picard) *Let $\nu_0 \in \mathbb{R}$ and H be a Hilbert space. Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law with $\mathrm{s_b}(M) < \nu_0$ and let $A\colon \operatorname{dom}(A) \subseteq H \to H$ be skew-selfadjoint. Assume that*

$$\operatorname{Re} \langle \phi, zM(z)\phi \rangle_H \geqslant c \|\phi\|_H^2 \quad (\phi \in H, z \in \mathbb{C}_{\operatorname{Re} \geqslant \nu_0})$$

for some $c > 0$. Then for all $\nu \geqslant \nu_0$ the operator $\partial_{t,\nu} M(\partial_{t,\nu}) + A$ is closable and

$$S_\nu := \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)^{-1} \in L(L_{2,\nu}(\mathbb{R}; H)).$$

Furthermore, S_ν is causal and satisfies $\|S_\nu\|_{L(L_{2,\nu})} \leqslant 1/c$, and for all $F \in \operatorname{dom}(\partial_{t,\nu})$ we have

$$S_\nu F \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(A).$$

Furthermore, for $\eta, \nu \geqslant \nu_0$ and $F \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\eta}(\mathbb{R}; H)$ we have that $S_\nu F = S_\eta F$.

The property that $S_\nu F = S_\eta F$ for all $F \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\eta}(\mathbb{R}; H)$ where $\eta, \nu \geqslant \nu_0$, for some $\nu_0 \in \mathbb{R}$, will be referred to as S_ν being *eventually independent of* ν in what follows.

Remark 6.2.2 If $F \in \operatorname{dom}(\partial_{t,\nu})$, then $U := S_\nu F \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(A)$ by Theorem 6.2.1. Since $M(\partial_{t,\nu})$ leaves the space $\operatorname{dom}(\partial_{t,\nu})$ invariant, this gives that $M(\partial_{t,\nu})U \in \operatorname{dom}(\partial_{t,\nu})$ and thus, U solves the evolutionary equation literally; that is,

$$(\partial_{t,\nu} M(\partial_{t,\nu}) + A)U = F,$$

while for $F \in L_{2,\nu}(\mathbb{R}; H)$, in general, we just have

$$\left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)U = F.$$

Definition Let H be a Hilbert space and $T \in L(H)$. If T is selfadjoint, we write $T \geqslant c$ for some $c \in \mathbb{R}$ if

$$\forall x \in H : \langle x, Tx \rangle_H \geqslant c \|x\|_H^2 .$$

Moreover, we define the *real part of* T by $\operatorname{Re} T := \frac{1}{2}(T + T^*)$.

Note that if H is a Hilbert space and $T \in L(H)$ then $\operatorname{Re} T$ is selfadjoint. Moreover,

$$\langle x, (\operatorname{Re} T)x \rangle_H = \operatorname{Re} \langle x, Tx \rangle_H \quad (x \in H).$$

Hence, in Theorem 6.2.1 the assumption on the material law can be rephrased as

$$\operatorname{Re} zM(z) \geqslant c \quad (z \in \mathbb{C}_{\operatorname{Re}\geqslant \nu_0}).$$

The following operators will be prototypical examples needed for the applications of the previous theorem.

Proposition 6.2.3 *Let H_0, H_1 be Hilbert spaces.*

(a) *Let $B\colon \operatorname{dom}(B) \subseteq H_0 \to H_1$, $C\colon \operatorname{dom}(C) \subseteq H_1 \to H_0$ be densely defined linear operators. Then*

$$\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix} \colon \operatorname{dom}(B) \times \operatorname{dom}(C) \subseteq H_0 \times H_1 \to H_0 \times H_1$$

$$(\phi, \psi) \mapsto (C\psi, B\phi)$$

is densely defined, and we have

$$\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & B^* \\ C^* & 0 \end{pmatrix}.$$

(b) *Let $a \in L(H_0)$, and $c > 0$. Assume $\operatorname{Re} a \geqslant c$. Then $a^{-1} \in L(H_0)$ with $\|a^{-1}\| \leqslant \frac{1}{c}$ and $\operatorname{Re} a^{-1} \geqslant c\,\|a\|^{-2}$.*

Proof The proof of the first statement can be done in two steps. First, notice that the inclusion $\begin{pmatrix} 0 & B^* \\ C^* & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^*$ follows immediately. If, on the other hand, $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \operatorname{dom}\left(\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^*\right)$ with $\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$ we get for all $x \in \operatorname{dom}(B)$ that

$$\begin{aligned} \langle Bx, \psi \rangle_{H_1} &= \left\langle \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_{H_0 \times H_1} = \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_{H_0 \times H_1} \\ &= \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle_{H_0 \times H_1} = \langle x, \xi \rangle_{H_0}. \end{aligned}$$

Hence, $\psi \in \operatorname{dom}(B^*)$ and $B^*\psi = \xi$. Similarly, we obtain $\phi \in \operatorname{dom}(C^*)$ and $C^*\phi = \zeta$.

For the second statement, we compute for all $\phi \in H_0$ using the Cauchy–Schwarz inequality

$$\|\phi\|_{H_0} \|a\phi\|_{H_0} \geqslant \left|\langle \phi, a\phi \rangle_{H_0}\right| \geqslant \operatorname{Re} \langle \phi, a\phi \rangle_{H_0} \geqslant c \,\langle \phi, \phi \rangle_{H_0} = c\,\|\phi\|_{H_0}^2 .$$

Thus, a is one-to-one. Since $\operatorname{Re} a = \operatorname{Re} a^*$ it follows that a^* is one-to-one, as well. Thus, we get that a has dense range by Theorem 2.2.5. The inequality

$$\|a\phi\|_{H_0} \geqslant c\,\|\phi\|_{H_0}$$

implies that a^{-1} is bounded with $\left\|a^{-1}\right\| \leqslant \frac{1}{c}$. Hence, as a^{-1} is closed, $\operatorname{dom}(a^{-1}) = \operatorname{ran}(a)$ is closed by Lemma 2.1.3 and hence, $\operatorname{dom}(a^{-1}) = H_0$; that is, $a^{-1} \in L(H_0)$. To conclude, let $\psi \in H_0$ and put $\phi := a^{-1}\psi$. Then $\|\psi\|_{H_0} = \left\|aa^{-1}\psi\right\|_{H_0} \leqslant \|a\|\left\|a^{-1}\psi\right\|_{H_0}$ and so

$$\begin{aligned}\operatorname{Re}\left\langle\psi, a^{-1}\psi\right\rangle_{H_0} &= \operatorname{Re}\langle a\phi,\phi\rangle_{H_0} = \operatorname{Re}\langle\phi, a\phi\rangle_{H_0} \geqslant c\,\langle\phi,\phi\rangle_{H_0} = c\left\langle a^{-1}\psi, a^{-1}\psi\right\rangle_{H_0}\\ &\geqslant c\frac{1}{\|a\|^2}\,\|\psi\|^2_{H_0}.\end{aligned}$$

□

The Heat Equation

The first example we will consider is the heat equation in an open subset $\Omega \subseteq \mathbb{R}^d$. Under a heat source, $Q\colon \mathbb{R}\times\Omega \to \mathbb{R}$, the heat distribution, $\theta\colon \mathbb{R}\times\Omega \to \mathbb{R}$, satisfies the so-called heat flux balance

$$\partial_t\theta + \operatorname{div} q = Q.$$

Here, $q\colon \mathbb{R}\times\Omega \to \mathbb{R}^d$ is the heat flux which is connected to θ via Fourier's law

$$q = -a\operatorname{grad}\theta,$$

where $a\colon \Omega \to \mathbb{R}^{d\times d}$ is the heat conductivity, which is measurable, bounded and uniformly strictly positive in the sense that

$$\operatorname{Re} a(x) \geqslant c$$

for all $x \in \Omega$ and some $c > 0$ in the sense of positive definiteness. Moreover, we assume that Ω is thermally isolated, which is modelled by requiring that the normal component of q vanishes at $\partial\Omega$; that is, $q \in \operatorname{dom}(\operatorname{div}_0)$ (see Remark 6.1.3). Written as a block matrix and incorporating the boundary condition, we obtain

$$\left(\partial_t\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & a^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}\right)\begin{pmatrix}\theta\\ q\end{pmatrix} = \begin{pmatrix}Q\\ 0\end{pmatrix}.$$

Theorem 6.2.4 *For all $\nu > 0$, the operator*

$$\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & a^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}$$

is densely defined and closable in $L_{2,\nu}\left(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d\right)$. The respective closure is continuously invertible with causal inverse being eventually independent of ν.

Proof The assertion follows from Theorem 6.2.1 applied to

$$M(z) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + z^{-1}\begin{pmatrix}0 & 0\\ 0 & a^{-1}\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix}0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}.$$

Note that M is a material law with $\mathrm{s_b}(M) = 0$ by Example 5.3.1. Moreover, for $(x, y) \in L_2(\Omega) \times L_2(\Omega)^d$ and $z \in \mathbb{C}_{\mathrm{Re}\geqslant\nu}$ with $\nu > 0$ we estimate

$$\begin{aligned}\operatorname{Re}\langle (x,y), zM(z)(x,y)\rangle_{L_2(\Omega)\times L_2(\Omega)^d} &\geqslant \operatorname{Re} z\,\|x\|^2_{L_2(\Omega)} + c\,\|a\|^{-2}\,\|y\|^2_{L_2(\Omega)^d}\\ &\geqslant \min\{\nu, c\,\|a\|^{-2}\}\,\|(x,y)\|^2_{L_2(\Omega)\times L_2(\Omega)^d},\end{aligned}$$

where we have used Proposition 6.2.3(b) in the first inequality. Moreover, A is skew-selfadjoint by Proposition 6.2.3(a). □

Remark 6.2.5 Assume that $Q \in \operatorname{dom}(\partial_{t,\nu})$. It then follows from Theorem 6.2.1 that

$$\begin{aligned}\begin{pmatrix}\theta\\ q\end{pmatrix} &:= \left(\overline{\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & a^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}}\right)^{-1}\begin{pmatrix}Q\\ 0\end{pmatrix}\\ &\in \operatorname{dom}\left(\partial_{t,\nu}\right) \cap \operatorname{dom}\left(\begin{pmatrix}0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}\right).\end{aligned} \tag{6.1}$$

Then, as in Remark 6.2.2, it follows that θ and q satisfy the heat flux balance and Fourier's law in the sense that $\theta \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(\operatorname{grad})$ and $q \in \operatorname{dom}(\operatorname{div}_0)$ and

$$\begin{aligned}\partial_t\theta + \operatorname{div}_0 q &= Q,\\ q &= -a\operatorname{grad}\theta.\end{aligned}$$

This regularity result is true even for $Q \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$; see [88] and Chap. 15, Theorem 15.2.3.

The Scalar Wave Equation

The classical scalar wave equation in a medium $\Omega \subseteq \mathbb{R}^d$ (think, for instance, of a vibrating string ($d = 1$) or membrane ($d = 2$)) consists of the equation of the balance of momentum where the acceleration of the (vertical) displacement, $u\colon \mathbb{R} \times$

$\Omega \to \mathbb{R}$, is balanced by external forces, $f : \mathbb{R} \times \Omega \to \mathbb{R}$, and the divergence of the stress, $\sigma : \mathbb{R} \times \Omega \to \mathbb{R}^d$, in such a way that

$$\partial_t^2 u - \operatorname{div} \sigma = f.$$

The stress is related to u via the following so-called stress-strain relation (here Hooke's law)

$$\sigma = T \operatorname{grad} u,$$

where the so-called elasticity tensor, $T : \Omega \to \mathbb{R}^{d \times d}$, is bounded, measurable, and satisfies

$$T(x) = T(x)^* \geqslant c$$

for some $c > 0$ uniformly in $x \in \Omega$. The quantity $\operatorname{grad} u$ is referred to as the strain. We think of u as being fixed at $\partial\Omega$ ("clamped boundary condition"). This is modelled by $u \in \operatorname{dom}(\operatorname{grad}_0)$.

Using $v := \partial_t u$ as an unknown, we can rewrite the balance of momentum and Hooke's law as 2×2-block-operator matrix equation

$$\left(\partial_t \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

The solution theory of evolutionary equations for the wave equation now reads as follows:

Theorem 6.2.6 *Let $\Omega \subseteq \mathbb{R}^d$ be open, and T as indicated above. Then, for all $\nu > 0$,*

$$\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}$$

is densely defined and closable in $L_{2,\nu}\left(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d\right)$. The respective closure is continuously invertible with causal inverse being eventually independent of ν.

Proof We apply Theorem 6.2.1 to $A = -\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}$, which is skew-selfadjoint by Proposition 6.2.3(a), and $M(z) = \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix}$, which defines a material law with $\mathrm{s_b}(M) = -\infty$. The positive definiteness constraint needed in Theorem 6.2.1 is satisfied by Proposition 6.2.3(b) on account of the selfadjointness of T, which implies the same for T^{-1}. Indeed, for $\nu_0 > 0$ and $z \in \mathbb{C}_{\mathrm{Re}\geqslant \nu_0}$ we estimate

$$\begin{aligned}
\mathrm{Re}\,\langle (x,y), zM(z)(x,y)\rangle_{L_2(\Omega)\times L_2(\Omega)^d} &= \mathrm{Re}\,\langle x, zx\rangle_{L_2(\Omega)} + \mathrm{Re}\left\langle y, zT^{-1}y\right\rangle_{L_2(\Omega)^d} \\
&\geqslant \nu_0 \|x\|^2_{L_2(\Omega)} + \nu_0 \frac{c}{\|T\|^2}\|y\|^2_{L_2(\Omega)^d} \\
&\geqslant \nu_0 \min\{1, c/\|T\|^2\}\, \|(x,y)\|^2_{L_2(\Omega)\times L_2(\Omega)^d}
\end{aligned}$$

for each $(x, y) \in L_2(\Omega) \times L_2(\Omega)^d$, where we used the selfadjointness of T^{-1} in the second line. □

Remark 6.2.7 Let $f \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$, $\nu > 0$, and define

$$\begin{pmatrix} u \\ \widetilde{\sigma} \end{pmatrix} = \left(\overline{\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & T^{-1}\end{pmatrix} - \begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad}_0 & 0 \end{pmatrix}}\right)^{-1} \begin{pmatrix} \partial_{t,\nu}^{-1} f \\ 0 \end{pmatrix}.$$

By Theorem 6.2.1, we obtain $\begin{pmatrix} u \\ \widetilde{\sigma} \end{pmatrix} \in \mathrm{dom}(\partial_{t,\nu}) \cap \mathrm{dom}\left(\begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad}_0 & 0 \end{pmatrix}\right)$. Hence, we have

$$\begin{aligned}
\partial_{t,\nu} u - \mathrm{div}\,\widetilde{\sigma} &= \partial_{t,\nu}^{-1} f \\
\partial_{t,\nu} T^{-1}\widetilde{\sigma} &= \mathrm{grad}_0\, u
\end{aligned}$$

or

$$\begin{aligned}
\partial_{t,\nu} u - \mathrm{div}\,\widetilde{\sigma} &= \partial_{t,\nu}^{-1} f \\
\widetilde{\sigma} &= T\partial_{t,\nu}^{-1}\,\mathrm{grad}_0\, u.
\end{aligned}$$

Thus, formally, after another time-differentiation and the setting of $\sigma = \partial_{t,\nu}\widetilde{\sigma}$ we obtain a solution of the wave equation, (u, σ). Notice, however, that differentiating $\mathrm{div}\,\widetilde{\sigma}$ cannot be done without any additional knowledge of the regularity of $\widetilde{\sigma}$. In fact, in order to arrive at the balance of momentum equation, one would need to have $\mathrm{div}\,\widetilde{\sigma} \in \mathrm{dom}(\partial_{t,\nu})$. However, one only has $\widetilde{\sigma} \in \mathrm{dom}(\partial_{t,\nu}) \cap \mathrm{dom}(\mathrm{div})$. It is an elementary argument, see [110, Lemma 4.6], that we in fact have $\mathrm{div}\,\partial_{t,\nu}^{-1} = \overline{\partial_{t,\nu}^{-1}\,\mathrm{div}}$, which suggests that, in general, $\mathrm{div}\,\widetilde{\sigma} \notin \mathrm{dom}(\partial_{t,\nu})$, see Exercise 6.6.

Maxwell's Equations

The final example in this chapter forms the archetypical evolutionary equation—Maxwell's equations in a medium $\Omega \subseteq \mathbb{R}^3$. In order to identify the particular choices of $M(\partial_{t,\nu})$ and A in the present situation (and to finally conclude the 2×2-block matrix formulation historically due to the work of [59, 64, 102]), we start out with Faraday's law of induction, which relates the unknown electric field, $E\colon \mathbb{R} \times \Omega \to$

$\mathbb{R}^3$, to the magnetic induction, $B\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$, via

$$\partial_t B + \operatorname{curl} E = 0.$$

We assume that the medium is contained in a perfect conductor, which is reflected in the so-called electric boundary condition which asks for the vanishing of the tangential component of E at the boundary. This is modelled by $E \in \operatorname{dom}(\operatorname{curl}_0)$. The next constituent of Maxwell's equations is Ampère's law

$$\partial_t D + j_c - \operatorname{curl} H = j_0,$$

which relates the unknown electric displacement, $D\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$, (free) current (density), $j_c\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$, and magnetic field, $H\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$, to the (given) external currents, $j_0\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$. Maxwell's equations are completed by constitutive relations specific to each material at hand. Indeed, the (bounded, measurable) dielectricity, $\varepsilon\colon \Omega\to\mathbb{R}^{3\times 3}$, and the (bounded, measurable) magnetic permeability, $\mu\colon \Omega\to\mathbb{R}^{3\times 3}$, are symmetric matrix-valued functions which couple the electric displacement to the electric field and the magnetic field to the magnetic induction via

$$D = \varepsilon E, \text{ and } B = \mu H.$$

Finally, Ohm's law relates the current to the electric field via the (bounded, measurable) electric conductivity, $\sigma\colon \Omega\to\mathbb{R}^{3\times 3}$, as

$$j_c = \sigma E.$$

All in all, in terms of (E, H), Maxwell's equations read

$$\left(\partial_t \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}\right)\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} j_0 \\ 0 \end{pmatrix}.$$

For the time being, we shall assume that there exist $c > 0$ and $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ we have

$$\nu\varepsilon(x) + \operatorname{Re}\sigma(x) \geqslant c, \quad \mu(x) \geqslant c \quad (x \in \Omega)$$

in the sense of positive definiteness. Note that the latter condition allows particularly for $\varepsilon = 0$ on certain regions, if $\operatorname{Re}\sigma$ compensates for this. To approximate small ε by 0 is referred to as the eddy current approximation in these regions. With the above preparations at hand, we may now formulate the well-posedness result concerning Maxwell's equations.

Theorem 6.2.8 *Let $\Omega \subseteq \mathbb{R}^3$ be open and $\nu \geqslant \nu_0$. Then*

$$\partial_{t,\nu}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix} + \begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -\operatorname{curl}\\ \operatorname{curl}_0 & 0\end{pmatrix}$$

is densely defined and closable in $L_{2,\nu}\left(\mathbb{R}; L_2(\Omega)^3 \times L_2(\Omega)^3\right)$. The respective closure is continuously invertible with causal inverse being eventually independent of ν.

Proof The assertion follows from Theorem 6.2.1 applied to the material law

$$M(z) = \begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix} + z^{-1}\begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix}$$

and the skew-selfadjoint operator

$$A = \begin{pmatrix}0 & -\operatorname{curl}\\ \operatorname{curl}_0 & 0\end{pmatrix}. \qquad \square$$

Remark 6.2.9 In the physics literature (see e.g. [40, Chapter 18]), Maxwell's equations are usually complemented by Gauss' law,

$$\operatorname{div}_0 B = 0,$$

as well as the introduction of the charge density, $\rho = \operatorname{div} \varepsilon E$, and the current, $j = j_0 - j_c$, by the continuity equation

$$\partial_t \rho = \operatorname{div} j.$$

We shall argue in the following that these equations are *automatically* satisfied if (E, H) is a solution to Maxwell's equation. Indeed, assuming $j_0 \in \operatorname{dom}(\partial_{t,\nu})$, then, as a consequence of Theorem 6.2.1, for

$$\begin{pmatrix}E\\ H\end{pmatrix} = \left(\overline{\partial_{t,\nu}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix} + \begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -\operatorname{curl}\\ \operatorname{curl}_0 & 0\end{pmatrix}}\right)^{-1}\begin{pmatrix}j_0\\ 0\end{pmatrix}$$

we observe $\begin{pmatrix}E\\ H\end{pmatrix} \in \operatorname{dom}\left(\partial_{t,\nu}\right) \cap \operatorname{dom}\left(\begin{pmatrix}0 & -\operatorname{curl}\\ \operatorname{curl}_0 & 0\end{pmatrix}\right)$. Reformulating the latter equation yields

$$B = \mu H = -\partial_{t,\nu}^{-1}\operatorname{curl}_0 E,$$

$$\varepsilon E = \partial_{t,\nu}^{-1}\left(-\sigma E + j_0 + \operatorname{curl} H\right) = \partial_{t,\nu}^{-1} j + \partial_{t,\nu}^{-1}\operatorname{curl} H.$$

Since $\operatorname{curl}_0 E \in \operatorname{ran}(\operatorname{curl}_0)$, we have by Proposition 3.1.6(b) that $\partial_{t,\nu}^{-1} \operatorname{curl}_0 E \in \overline{\operatorname{ran}}(\operatorname{curl}_0)$. Thus, by Proposition 6.1.4, we obtain

$$\operatorname{div}_0 B = \operatorname{div}_0 \left(-\partial_{t,\nu}^{-1} \operatorname{curl}_0 E\right) = 0.$$

Similarly, we deduce that

$$\rho = \operatorname{div} \varepsilon E = \operatorname{div} \partial_{t,\nu}^{-1} j.$$

If, in addition, we have that $j \in \operatorname{dom}(\operatorname{div})$, we recover the continuity equation. In general, the continuity equation is satisfied in the integrated sense just derived.

We shall keep the list of examples to that for now. In the course of this book, we will see more (involved) examples. Furthermore, we will study the boundary conditions more deeply and shall relate the conditions introduced abstractly here to more classical formulations involving trace spaces.

6.3 Proof of Picard's Theorem

In this section we shall prove the well-posedness theorem. For this, we recall an elementary result from functional analysis. It is remindful of the Lax–Milgram lemma.

Proposition 6.3.1 *Let H be a Hilbert space and $B\colon \operatorname{dom}(B) \subseteq H \to H$ densely defined and closed. Assume there exists $c > 0$ such that*

$$\operatorname{Re} \langle \phi, B\phi \rangle_H \geqslant c \,\|\phi\|_H^2 \quad (\phi \in \operatorname{dom}(B)),$$
$$\operatorname{Re} \left\langle \psi, B^* \psi \right\rangle_H \geqslant c \,\|\psi\|_H^2 \quad (\psi \in \operatorname{dom}(B^*)).$$

Then $B^{-1} \in L(H)$ and $\left\|B^{-1}\right\| \leqslant 1/c$.

Proof Since B is not necessarily bounded here, the present argument requires a refinement of the one in Proposition 6.2.3. In fact, the first assumed inequality implies closedness of the range of B as well as continuous invertibility with $B^{-1}\colon \operatorname{ran}(B) \to H$. The fact that $\operatorname{ran}(B)$ is dense in H follows from the second inequality. □

Remark 6.3.2 In the proof of Theorem 6.2.1, we will apply Proposition 6.3.1 in a situation, where $\operatorname{dom}(B^*) \subseteq \operatorname{dom}(B)$. In this case, the condition

$$\operatorname{Re} \langle \phi, B\phi \rangle_H \geqslant c \,\|\phi\|_H^2 \quad (\phi \in \operatorname{dom}(B))$$

readily implies

$$\operatorname{Re}\left\langle \psi, B^*\psi \right\rangle_H \geqslant c\, \|\psi\|_H^2 \quad (\psi \in \operatorname{dom}(B^*)).$$

Next, we turn to the proof of Picard's theorem. For this, we recall that we do not notationally distinguish between the operator A defined on H and its extension to H-valued functions. We leave it to the context, which realisation of A is considered, which will always be obvious; see also Exercises 6.1 and 6.2.

Proof of Theorem 6.2.1 Let $\nu \geqslant \nu_0$ and $z \in \mathbb{C}_{\operatorname{Re}\geqslant \nu}$. Define $B(z) := zM(z) + A$. Since $M(z) \in L(H)$ it follows from Theorem 2.3.2 that $B(z)^* = (zM(z))^* - A$ and $\operatorname{dom}(B(z)) = \operatorname{dom}(B(z)^*) = \operatorname{dom}(A)$. Moreover, for all $\phi \in \operatorname{dom}(A)$ we have

$$\operatorname{Re}\langle \phi, B(z)\phi\rangle_H = \operatorname{Re}\langle \phi, (zM(z) + A)\,\phi\rangle_H = \operatorname{Re}\langle \phi, zM(z)\phi\rangle_H \geqslant c\,\|\phi\|_H^2\,,$$

due to the skew-selfadjointness of A. Thus, by Proposition 6.3.1 (see also Remark 6.3.2) applied to $B(z)$ instead of B, we deduce that

$$S\colon \mathbb{C}_{\operatorname{Re}\geqslant \nu} \ni z \mapsto B(z)^{-1}$$

is bounded and assumes values in $L(H)$ with norm bounded by $1/c$. By Exercise 6.5, we have that S is holomorphic. Thus, S is a material law and $\left\| S(\partial_{t,\nu}) \right\| \leqslant 1/c$ by Proposition 5.3.2. Moreover, Theorem 5.3.6 implies that $S(\partial_{t,\nu})$ is independent of ν and causal.

Next, if $f \in \operatorname{dom}(\partial_{t,\nu})$, it follows that $(\mathrm{i}m + \nu)\,\mathcal{L}_\nu f \in L_2(\mathbb{R}; H)$. Hence, for all $t \in \mathbb{R}$ we obtain

$$\begin{aligned} AS(\mathrm{i}t + \nu)\mathcal{L}_\nu f(t) &= A\big((\mathrm{i}t + \nu)\,M(\mathrm{i}t + \nu) + A\big)^{-1}\mathcal{L}_\nu f(t) \\ &= \mathcal{L}_\nu f(t) - (\mathrm{i}t + \nu)\,M(\mathrm{i}t + \nu)S(\mathrm{i}t + \nu)\mathcal{L}_\nu f(t). \end{aligned}$$

Thus, by the boundedness of M and S, we deduce $S(\mathrm{i}\cdot + \nu)\mathcal{L}_\nu f \in L_2(\mathbb{R}; \operatorname{dom}(A))$. This implies $S(\partial_{t,\nu})f \in L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A))$ by Exercise 6.2. Similarly, but more easily, it follows that $(\mathrm{i}\cdot + \nu)\,S(\mathrm{i}\cdot + \nu)\mathcal{L}_\nu f \in L_2(\mathbb{R}; H)$ also, which shows $S(\partial_{t,\nu})f \in \operatorname{dom}(\partial_{t,\nu})$.
We now define the operator $B(\mathrm{i}m + \nu)$ by

$$\begin{aligned} \operatorname{dom}(B(\mathrm{i}m + \nu)) := \big\{ f \in L_2(\mathbb{R}; H)\,;\, & f(t) \in \operatorname{dom}(A) \text{ for a.e. } t \in \mathbb{R}, \\ & (t \mapsto B(\mathrm{i}t + \nu)f(t)) \in L_2(\mathbb{R}; H)\big\} \end{aligned}$$

and

$$B(\mathrm{i}m + \nu)f := (t \mapsto B(\mathrm{i}t + \nu)f(t)) \quad (f \in \operatorname{dom}(B(\mathrm{i}m + \nu))).$$

Then one easily sees that $B(\mathrm{im}+\nu) = S(\mathrm{im}+\nu)^{-1}$ and since $S(\mathrm{im}+\nu)$ is closed, it follows that $B(\mathrm{im}+\nu)$ is closed as well. Moreover

$$(\mathrm{im}+\nu)M(\mathrm{im}+\nu) + A \subseteq B(\mathrm{im}+\nu)$$

and hence, the operator $(\mathrm{im}+\nu)M(\mathrm{im}+\nu) + A$ is closable, which also yields the closability of $\partial_{t,\nu}M(\partial_{t,\nu}) + A$ by unitary equivalence. To complete the proof, we have to show that

$$\overline{(\mathrm{im}+\nu)M(\mathrm{im}+\nu) + A} = B(\mathrm{im}+\nu),$$

as this equality implies $S(\partial_{t,\nu}) = \left(\overline{\partial_{t,\nu}M(\partial_{t,\nu}) + A}\right)^{-1}$ by unitary equivalence. For showing the asserted equality, let $f \in \mathrm{dom}(B(\mathrm{im}+\nu))$. For $n \in \mathbb{N}$ we define $f_n := \mathbb{1}_{[-n,n]}f$. Then $f_n \in \mathrm{dom}(\mathrm{im}+\nu) \cap \mathrm{dom}(A) \subseteq \mathrm{dom}\left((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A\right)$ for each $n \in \mathbb{N}$ and by dominated convergence, we have that $f_n \to f$ as $n \to \infty$ as well as

$$\begin{aligned}\left((\mathrm{im}+\nu)M(\mathrm{im}+\nu) + A\right)f_n &= B(\mathrm{im}+\nu)f_n \\ &= \mathbb{1}_{[-n,n]}B(\mathrm{im}+\nu)f \to B(\mathrm{im}+\nu)f\end{aligned}$$

$n \to \infty$. This shows that $f \in \mathrm{dom}\left(\overline{(\mathrm{im}+\nu)M(\mathrm{im}+\nu) + A}\right)$ and hence, the assertion follows. □

Remark 6.3.3 Note that Theorem 6.2.1 can partly be generalised in the following way (with the same proof). Let $M\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to L(H)$ be holomorphic and A a closed, densely defined operator in H such that $zM(z) + A$ is boundedly invertible for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ and that $\sup_{z\in\mathbb{C}_{\mathrm{Re}>\nu_0}} \left\|(zM(z)+A)^{-1}\right\|_{L(H)} < \infty$. Then $S_\nu \in L(L_{2,\nu}(\mathbb{R}; H))$ is causal and eventually independent of ν.

Remark 6.3.4 As the proof of Theorem 6.2.1 shows, for $\nu \geqslant \nu_0$ we have that $S\colon \mathbb{C}_{\mathrm{Re}\geqslant\nu} \ni z \mapsto (zM(z)+A)^{-1} \in L(H)$ is a material law and $S_\nu = S(\partial_{t,\nu})$. Thus, the solution operator is a material law operator, and by Remark 5.3.3 applied to S and $z \mapsto \frac{1}{z}1_H$ we obtain

$$S_\nu \partial_{t,\nu} \subseteq \partial_{t,\nu} S_\nu.$$

6.4 Comments

The proof of Theorem 6.2.1 here is rather close to the strategy originally employed in [82], at least where existence and uniqueness are concerned. The causality part is a consequence of some observations detailed in [52, 131]. The original process of proving causality used the Theorem of Paley and Wiener, which we shall discuss later on.

The eddy current approximation has enjoyed great interest in the mathematical and physical community, in particular for the case when $\varepsilon = 0$ everywhere. The reason being that then Maxwell's equations are merely of parabolic type. We shall refer to [79] and the references therein for an extensive discussion.

Both Proposition 6.3.1 and the Lax–Milgram lemma have been put into a general perspective in [89].

Exercises

Exercise 6.1 Let (Ω, Σ, μ) be a σ-finite measure space and let H_0, H_1 be Hilbert spaces. Let $A\colon \operatorname{dom}(A) \subseteq H_0 \to H_1$ be densely defined and closed. Show that the operator

$$\begin{aligned} A_\mu\colon L_2(\mu; \operatorname{dom}(A)) \subseteq L_2(\mu; H_0) &\to L_2(\mu; H_1) \\ f &\mapsto \big(\omega \mapsto Af(\omega)\big) \end{aligned}$$

is densely defined and closed. Moreover, show that $\big(A_\mu\big)^* = (A^*)_\mu$.

Exercise 6.2 In the situation of Exercise 6.1, if $(\Omega_1, \Sigma_1, \mu_1)$ is another σ-finite measure space and $\mathcal{F}\colon L_2(\mu) \to L_2(\mu_1)$ is unitary, show that for $j \in \{0, 1\}$ there exists a unique unitary operator $\mathcal{F}_{H_j}\colon L_2(\mu; H_j) \to L_2(\mu_1; H_j)$ such that

$$\mathcal{F}_{H_j}(\phi x) = (\mathcal{F}\phi)x \quad (\phi \in L_2(\mu), x \in H_j).$$

Furthermore, prove that

$$\mathcal{F}_{H_1} A_\mu \mathcal{F}_{H_0}^* = A_{\mu_1}.$$

Exercise 6.3 Show that for $\Omega \subseteq \mathbb{R}^d$ open, the set $C_c^\infty(\Omega) \subseteq L_2(\Omega)$ is dense.

Exercise 6.4 Prove Theorem 6.1.2.

Exercise 6.5 Let H be a Hilbert space, $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint, and $c > 0$. Moreover, let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be holomorphic with

$$\operatorname{Re} M(z) \geqslant c \quad (z \in \operatorname{dom}(M)).$$

Show that $\operatorname{dom}(M) \ni z \mapsto (M(z) + A)^{-1}$ is holomorphic.

Exercise 6.6 Let $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be a densely defined and closed linear operator acting in Hilbert spaces H_0 and H_1. For $\nu > 0$ show that

$$\overline{\partial_{t,\nu}^{-1} C} = C\partial_{t,\nu}^{-1}.$$

Hint: Apply Exercise 6.2 and show $\overline{(\mathrm{im} + \nu)^{-1} C} = C(\mathrm{im} + \nu)^{-1}$ with a suitable approximation argument.

Exercise 6.7 Let $\Omega \subseteq \mathbb{R}^d$ be open.

(a) Compute $H_0^1(\Omega)^\perp$ where the orthogonal complement is computed in $H^1(\Omega)$.
(b) Assume that

$$D := \left\{\phi \in H^1(\Omega)\,;\ \operatorname{grad}\phi \in \operatorname{dom}(\operatorname{div}), \phi = \operatorname{div}\operatorname{grad}\phi\right\} \subseteq C^\infty(\Omega).$$

and show that $C^\infty(\Omega) \cap H^1(\Omega) \subseteq H^1(\Omega)$ is dense.

Remark The regularity assumption in (b) always holds and is known as Weyl's Lemma, see e.g. [45, Corollary 8.11], where the more general situation of an elliptic operator with smooth coefficients is treated. See also [32, p.127], where the regularity is shown for harmonic distributions.

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