



Classification of Codimension-1 Singular Bifurcations in Low-Dimensional DAEs

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The study of bifurcations of differential-algebraic equations (DAEs) is the topic of interest for many applied sciences, such as electrical engineering, robotics, etc. While some of them were investigated already, the full classification of such bifurcations has not been done yet. In this paper, we consider bifurcations of quasilinear DAEs with a singularity and provide a full list of all codimension-one bifurcations in lower-dimensional cases. Among others, it includes singularity-induced bifurcations (SIBs), which occur when an equilibrium branch intersects a singular manifold causing certain eigenvalues of the linearized problem to diverge to infinity. For these and other bifurcations, we construct the normal forms, establish the non-degeneracy conditions and give a qualitative description of the dynamics. Also, we study singular homoclinic and heteroclinic bifurcations, which were not considered before.

Keywords: differential algebraic equation (DAE), slow-fast dynamics, fold, singular induced bifurcation, bifurcation

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1. INTRODUCTION

Differential-algebraic equations (DAEs) play an important role in dynamical system modeling such as power systems (cf. [1, 2]), nonlinear-circuits [3–7], robotics (cf. [8]), flight control systems [9], multi-body systems [10], numeric PDEs ([11] and references in Kunkel and Mehrmann [12]).

We consider *quasilinear* DAEs of form

$$A(x, \alpha)\dot{x} = f(x, \alpha), \quad (x, \alpha) \in \mathbb{R}^n \times \mathbb{R}^m, \quad (1)$$

for smooth functions $A : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n}$, $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ of the phase variable $x \in \mathbb{R}^n$ and the parameter $\alpha \in \mathbb{R}^m$.

In the presence of *singularities*, that is, points (x, α) such that $\det A(x, \alpha)$ in Equation (1) vanishes, it is not possible to describe the local behavior of a DAE in terms of an explicit ODE. Regularization of such a singular DAE often leads to an ODE with higher dimensional manifolds of equilibria in phase space, which can manifest *bifurcations without parameters* (cf. [13]). System with singularities possessing the Hamiltonian structure were studied in [14–18].

In parametrized problems, a stability change due to the divergence of an eigenvalue was first analyzed by Venkatasubramanian [19, 20] and later addressed by many others [9, 20–25]. The change of stability, termed *singularity-induced bifurcations* (SIB), occurs when an equilibrium branch intersects a singular manifold, which results in the divergence of at least one eigenvalue through infinity.

Main efforts in studying such singularity-crossing equilibria have been given by trying to characterize the SIBs in terms of the *linearized* problems, such as using the matrix pencils

$\{A(x^*, \alpha^*), -D_x f(x^*, \alpha^*)\}$ associated to Equation (1) at the point of singularity (x^*, α^*) (cf. [20, 26]). Different sufficient conditions have been given in the framework of the tractability index (cf. [27, 28]) and the geometric index [26, 29, 30] among others. However, they may not provide a *necessary and sufficient characterization* of the local flow around such singularity-crossing equilibria.

Besides SIBs, there can be other singular behavior induced by the presence of singularities such as the change of the singularity surface itself (*fold*) or bifurcations of singular equilibria that change significantly the dynamics near the singularity.

Quasilinear DAEs (Equation 1) have a strong connection to another important class of dynamical systems—fast-slow systems. Indeed, a system

$$\varepsilon \dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

is a fast-slow system for small ε . Setting ε to zero we obtain the so-called *slow system*:

$$0 = f(x, y), \quad \dot{y} = g(x, y), \tag{2}$$

in which the first equation defines the slow manifold and the second one determines the dynamics, restricted onto it. System (2) is a DAE that can be brought to the form of an ODE system via time-differentiation of the algebraic equation and the substitution of \dot{y} from the second one:

$$f_x(x, y)\dot{x} = -f_y(x, y)g(x, y), \quad \dot{y} = g(x, y). \tag{3}$$

That is, the slow system can be reduced to the form (Equation 1), so our research here contributes also to the studies of bifurcations in slow-fast systems (see [31]). The correspondence between the terms and notions can be viewed in the following way:

- A singularity set in DAEs corresponds to a fold set of a slow manifold;
- a fold of the singularity surface corresponds to a cusp of a slow manifold.

In this paper, we will call bifurcations that are caused by the presence of singularities *singular bifurcations*, which is a more general consideration of possible scenarios, which may or may not involve equilibria directly. That is, we consider bifurcations caused by singularities including SIBs but *not exclusively* so.

To make a clear impression of the realm of all possible bifurcations, we focus on low-dimensional quasilinear DAEs of form (Equation 1) for which $x \in \mathbb{R}$ or $x \in \mathbb{R}^2$. Our goal is to provide a list of all possible singular bifurcations of codimension 1 in such systems.

The paper is organized as follows. In Section 2 the basic notions for the study of one-dimensional quasilinear DAEs are given, all possible codimension-one bifurcations are studied and the behavior in higher-codimension bifurcations is described. In Section 3 the main notions are given for a two-dimensional case, for which the full list of possible codimension-one bifurcations is provided. Section 4 contains the rigorous derivation of the dynamical behavior near some local singular bifurcations from Section 3.

2. QUASILINEAR DAEs: ONE-DIMENSIONAL CASE

Consider a quasilinear DAE (Equation 1) for $n = 1$, it is given by

$$g(x, \alpha)\dot{x} = f(x, \alpha), \quad (x, \alpha) \in \mathbb{R} \times \mathbb{R}^m, \tag{4}$$

for smooth functions $f: \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $g: \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$. Note that in this case, the singular set is precisely the set of zeros of g given by

$$\Sigma_\alpha = \{x \in \mathbb{R} : g(x, \alpha) = 0\}, \tag{5}$$

which under a regularity assumption on g , is composed of isolated points. We will call every such point a *singularity*. The set of zeros of f that are not zeros of g

$$E_\alpha = \{x \in \mathbb{R} : f(x, \alpha) = 0, g(x, \alpha) \neq 0\}, \tag{6}$$

will be called the *equilibrium set*. Under a similar regularity assumption, this set is also composed of isolated points. Each such point is an equilibrium of system (Equation 4).

Definition 2.1. A point $x \in \mathbb{R}$ is called a *singular equilibrium* if it lies in the intersection of the singular set Σ_α given by Equation (5) with the zeros set of $f(x, \alpha)$ for some $\alpha \in \mathbb{R}^m$.

Definition 2.2. For a fixed parameter value $\alpha = \alpha_0$, we will call the point x_0 a *simple equilibrium* if it is a simple zero of f and not a zero of g :

$$f(x_0, \alpha_0) = 0, f'_x(x_0, \alpha_0) \neq 0, g(x_0, \alpha_0) \neq 0.$$

Analogously, we will call x_0 a *simple singularity* if it is a simple zero of g and not a zero of f :

$$f(x_0, \alpha_0) \neq 0, g(x_0, \alpha_0) = 0, g'_x(x_0, \alpha_0) \neq 0.$$

Definition 2.3. A simple singularity point x_0 is called *incoming (outgoing)*, if there exists a small neighborhood U of x_0 such that for any initial condition $x \in U$ the solution $x(t)$ reaches x_0 in finite forward (backward) time.

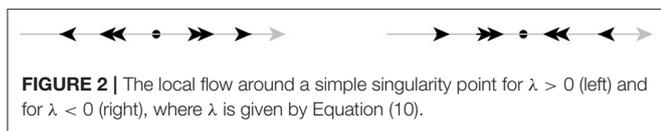
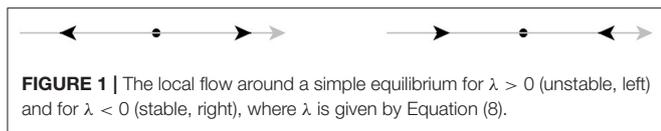
Remark 2.4. The incoming or outgoing simple singularities are known in the DAE literature as the *standard singular points* which was introduced in [32]. They behave like an *impasse point*, where solutions are no longer defined being either attractive or repelling (cf. [32, 33]).

For a simple equilibrium point x_0 there exists a small neighborhood $|x - x_0| < \varepsilon$ such that $g(x, \alpha_0) \neq 0$, and the system (Equation 4) can be rewritten as

$$\dot{x} = \frac{f(x, \alpha_0)}{g(x, \alpha_0)} := \tilde{f}(x, \alpha_0), \text{ where } \tilde{f}(x_0, \alpha_0) = 0. \tag{7}$$

Thus, the stability type of the equilibrium (x_0, α_0) is completely determined by the sign of the derivative

$$\lambda := \partial_x \tilde{f}|_{(x_0, \alpha_0)} = \frac{g \partial_x f - f \partial_x g}{g^2}|_{(x_0, \alpha_0)} = \frac{\partial_x f}{g}|_{(x_0, \alpha_0)} \neq 0, \tag{8}$$



which is non-zero by the assumption that x_0 is a simple equilibrium for $\alpha = \alpha_0$. If $\lambda < 0$, then the equilibrium x_0 is stable; if $\lambda > 0$, then it is unstable (see **Figure 1**).

For a simple singularity x_0 , there exists a small neighborhood in which $f(x, \alpha_0) \neq 0$ and the system (Equation 4) can be rewritten as

$$\tilde{g}\dot{x} = 1, \text{ for } \tilde{g} = \frac{g(x, \alpha)}{f(x, \alpha)} \text{ with } \tilde{g}(x_0, \alpha_0) = 0. \tag{9}$$

Thus, the following derivative

$$\lambda = \tilde{g}'_x|_{(x_0, \alpha_0)} = \frac{g_x f - g f_x}{f^2}|_{(x_0, \alpha_0)} = \frac{g_x}{f}|_{(x_0, \alpha_0)} \neq 0, \tag{10}$$

determines the type of the singularity at (x_0, α_0) . More precisely, if $\lambda > 0$, then the simple singularity point x_0 is outgoing; if $\lambda < 0$, then it is incoming. See **Figure 2**, we put here and further below the double arrow to reflect the fact that the trajectory reaches the singularity in finite time, and the velocity grows to infinity.

It is clear from Definition 2.2, that simple equilibria and simple singularities persist under generic parametric perturbations. Indeed, the condition $f'_x(x_0, \alpha_0) \neq 0$ implies that by the Implicit Function Theorem, there exists locally a unique function $x^*(\alpha)$ with $x^*(\alpha_0) = x_0$, which fulfills the equation $f(x, \alpha) = 0$. Moreover, this equilibrium maintains the same stability type, as the exponent λ in Equation (8) preserves its sign. In a similar way, one can deduce the corresponding property for a simple singularity.

Theorem 2.5. *If in an open set $U \subset \mathbb{R}$ system (Equation 4) possesses a finite set of equilibria and a finite set of singularities and all of them are simple, then the system is structurally stable in U .*

Proof: Under sufficiently small perturbations, every simple equilibrium and every simple singularity stays in a small neighborhood of its initial position, remain simple, are distributed in the same order on the line and keep their stability types. Also, neither of these points reaches the boundary of U . The intervals bounded by these equilibria and singularities can be homeomorphically conjugated, with the direction of motion preserved.

However, when the conditions of Theorem 2.5 are violated, one may encounter *bifurcations*. There are three such possibilities of singular bifurcations¹:

- A1. A non-simple equilibrium: $f(x, \alpha) = 0, f'_x(x, \alpha) = 0$;
- A2. A non-simple singularity: $g(x, \alpha) = 0, g'_x(x, \alpha) = 0$;
- A3. A singular equilibrium: $f(x, \alpha) = 0, g(x, \alpha) = 0$.

These cases are not exclusive to each other. They can happen simultaneously, either at the same or different points, which increases the codimension of the problem. In the following, we assume that the bifurcation conditions occur at $a_0 = 0$ and $x_0 = 0$, which can be achieved by appropriate translation of coordinates and parameters.

We start with formulating the simplest possible cases, i.e. the cases of *codimension-1*.

- A1.1. A codimension one non-simple equilibrium: $f(0, 0) = 0, f'_x(0, 0) = 0, f''_{xx}(0, 0) \neq 0, g(0, 0) \neq 0$;
- A2.1. A codimension one non-simple singularity: $g(0, 0) = 0, g'_x(0, 0) = 0, g''_{xx}(0, 0) \neq 0, f(0, 0) \neq 0$;
- A3.0.0. A transcritical singularity (codimension-1 singular equilibrium²): $f(0, 0) = 0, g(0, 0) = 0, f'_x(0, 0) \neq 0, g'_x(0, 0) \neq 0$.

The case A1.1 is completely analogous to the usual fold bifurcation in dynamical systems without singularities. Indeed, since $g(x, \alpha) \neq 0$ in a small neighborhood of $(x_0, \alpha_0) = (0, 0)$, we can rewrite the system (Equation 4) using the function \tilde{f} as defined in Equation (7), where

$$\tilde{f}(0, 0) = \tilde{f}'_x(0, 0) = 0, \tilde{f}''_{xx}(0, 0) = \frac{f''_{xx}}{g}|_{(0,0)} \neq 0.$$

Proposition 2.6. *Assume that for system (Equation 4) the conditions of case A1.1 are fulfilled for $\alpha = 0$ at $x = 0$, and $f'_\alpha(0, 0) \neq 0$. Then for all small α by an invertible change of coordinate and parameter, the system can be brought near the origin to the following normal form:*

$$\dot{\eta} = \beta + s\eta^2 + O(\eta^3), \tag{11}$$

where $s = \text{sign} \frac{f''_{xx}}{g}|_{(0,0)}$ (cf. **Figure 3** left).

This result immediately follows from Kuznetsov [34], Theorem 3.2. Similarly, one can derive the normal form for the non-simple singularity in case A2.1.

Proposition 2.7. *Assume that for the system (Equation 4) the conditions of case A2.1 are fulfilled for $\alpha = 0$ at $x = 0$, and $g'_\alpha(0, 0) \neq 0$. Then, for all small α by an invertible change of coordinate and parameter, the system can be brought near the origin to the following normal form:*

$$(\beta + s\eta^2 + O(\eta^3))\dot{\eta} = 1, \tag{12}$$

¹We do not consider cases of infinite sets of equilibria or singularities, as it violates the assumption that functions f and g are smooth.

²Two zeros represent that the left- and right-hand sides of the equation are not degenerate. One codimension is added, because both functions vanish at the same point. The generalization of this case, case A3.m, n is given at Page 5.

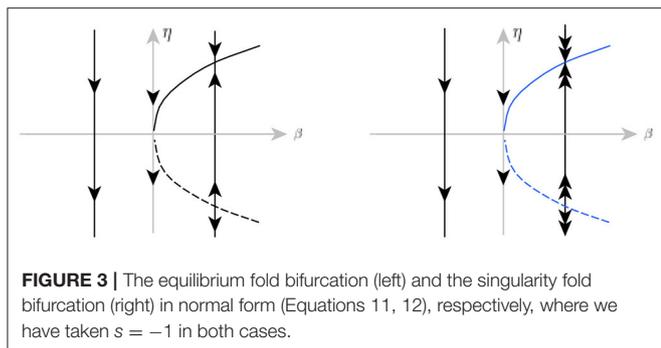


FIGURE 3 | The equilibrium fold bifurcation (left) and the singularity fold bifurcation (right) in normal form (Equations 11, 12), respectively, where we have taken $s = -1$ in both cases.

where $s = \text{sign} \frac{g''_{xx}}{f} \Big|_{(0,0)}$ (cf. **Figure 3** right).

Proof: In some small neighborhood of the origin, we have $f(x, \alpha) \neq 0$ for all small α . Then, the system can be rewritten in the form (Equation 9) with

$$\tilde{g}(0, 0) = \tilde{g}'_x(0, 0) = 0, \quad \tilde{g}''_{xx}(0, 0) = \frac{g''_{xx}}{f} \Big|_{(0,0)} \neq 0.$$

We expand \tilde{g} in Taylor series in x as $\tilde{g} = g_0(\alpha) + g_1(\alpha)x + g_2(\alpha)x^2 + O(x^3)$ with $g_0(0) = g_1(0) = 0$ and $g_2(0) = a \neq 0$. Using a parameter-dependent coordinate shift of the form $x = y + \delta(\alpha)$ with $\delta(0) = 0$, we can rewrite \tilde{g} as:

$$\begin{aligned} \tilde{g}(y + \delta(\alpha), \alpha) &= (g_0(\alpha) + g_1(\alpha)\delta(\alpha) + O(\alpha^2)) + (g_1(\alpha) \\ &\quad + 2g_2(\alpha)\delta(\alpha) + O(\alpha^2))y + (g_2(\alpha) + O(\alpha))y^2 \\ &\quad + O(y^3), \end{aligned}$$

where the linear term can be neglected by an appropriate choice of δ . Indeed, the coefficient of the linear term vanishes at $\alpha = \delta = 0$, and its derivative with respect to δ at zero is given by $2a \neq 0$. By the Implicit Function Theorem, there exists a function $\delta(\alpha) = \delta_1\alpha + O(\alpha^2)$ with $\delta_1 = -\frac{g'_{1,\alpha}(0)}{2a}$. Thus, we have

$$[g'_{0,\alpha}(0)\alpha + O(\alpha^2) + a(\alpha)y^2 + O(y^3)]\dot{y} = 1$$

with $a(0) = a$, which becomes (12) using the scaling

$$y = \frac{1}{|a(\alpha)|^{1/3}}\eta, \quad \beta = \frac{1}{|a(\alpha)|^{1/3}}(g'_{0,\alpha}(0)\alpha + O(\alpha^2)).$$

Moreover, as $g'_{0,\alpha}(0) = \frac{g'_{\alpha}}{f}(0, 0) \neq 0$, all the transformations are invertible.

The bifurcation occurs in the following way, if $s = -1$: for $\beta = 0$ there exists a locally unique non-simple singularity point, which disappears when $\beta < 0$ and is replaced by a pair of incoming and outgoing simple singularity points when $\beta > 0$ [see **Figure 3** (right)].

The following Proposition states the normal form of the transcritical singularity bifurcation in case A3.0.0.

Proposition 2.8. *If the system (Equation 4) satisfies the conditions of transcritical singularity, case A3.0.0, at $(x, \alpha) = (0, 0)$, and*

$$A := (g'_x f'_\alpha - f'_x g'_\alpha) \Big|_{(0,0)} \neq 0,$$

then there exists an invertible change of coordinate and parameter, which brings system (Equation 4) near the origin to the normal form

$$\eta\dot{\eta} = \beta + s\eta + O(\eta^2), \tag{13}$$

where

$$s = \text{sign} \frac{f'_x}{g'_x} \Big|_{(0,0)}.$$

(cf. **Figure 4**).

Proof: As $g'_x(0, 0) \neq 0$, the implicit equation $g(x, \alpha) = 0$ can be locally uniquely resolved with respect to x for small x and α . That is, there exists a smooth function $x^*(\alpha)$ such that $g(x^*(\alpha), \alpha) \equiv 0$ with $x^*(0) = 0$ and $x^*(\alpha) = -\frac{g'_\alpha}{g'_x} \Big|_{(0,0)}\alpha + O(\alpha^2)$. Consider a parameter-dependent shift of coordinate $x = x^*(\alpha) + y$. Then, the left-hand side of Equation (4) is transformed as

$$\begin{aligned} g(x^*(\alpha) + y, \alpha) &= g'_x(x^*(\alpha), \alpha)y + O(y^2) = (g_1 + O(\alpha))y + O(y^2) \\ &= (g_1 + O(\alpha))y(1 + O(y)), \end{aligned} \tag{14}$$

where $g_1 = g'_x(0, 0) \neq 0$. The right-hand side function f becomes

$$\begin{aligned} f(x^*(\alpha) + y, \alpha) &= f(x^*(\alpha), \alpha) + f'_x(x^*(\alpha), \alpha)y + O(y^2) \\ &= \frac{A}{g_1}\alpha + O(\alpha^2) + (f_1 + O(\alpha))y + O(y^2), \end{aligned} \tag{15}$$

where $f_1 = f'_x(0, 0)$.

We choose the neighborhood small enough for term $(1 + O(y))$ in formula (Equation 14) to stay always positive. Then, we reparametrize time by formula $dt/(1 + O(y)) = d\tau$, and also divide by a non-zero coefficient $(g_1 + O(\alpha))$, system (Equation 4) is transformed as:

$$y\dot{y} = \frac{A}{g_1}\alpha + O(\alpha^2) + \left(\frac{f_1}{g_1} + O(\alpha)\right)y + O(y^2). \tag{16}$$

It leads to Equation (13) by scaling $y \rightarrow \left|\frac{f_1}{g_1} + O(\alpha)\right|\eta$ and setting

$$\beta = \frac{A}{f_1^2}\alpha + O(\alpha^2). \tag{17}$$

Notice that as $A \neq 0$, the new small parameter β diffeomorphically depends on α .

Example 2.9. The following systems demonstrate examples of cases A1.1, A2.1 and A3.0.0, respectively.

$$(x + 1)\dot{x} = x^2 + \alpha, \quad x, \alpha \in \mathbb{R} \tag{18}$$

$$(x^2 + \alpha)\dot{x} = x + 1, \quad x, \alpha \in \mathbb{R} \tag{19}$$

$$(x + x^2 + \alpha)\dot{x} = x - x^2 + 2\alpha, \quad x, \alpha \in \mathbb{R}. \tag{20}$$

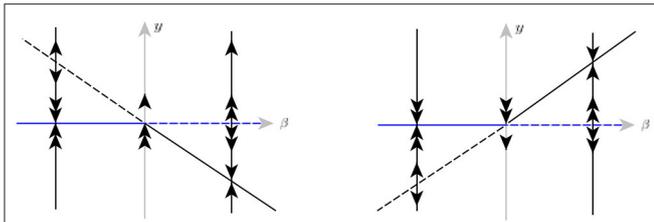


FIGURE 4 | Transcritical singularity bifurcation of Equation (13): $s > 0$ (left); $s < 0$ (right). Both feature a transition from an incoming to an outgoing singularity as β changes from the negative to the positive, where β changes according to Equation (17). The dashed (solid) lines indicate unstable (stable) equilibrium and singularity points.

More precisely, consider the system (Equation 18) with $f(x, \alpha) = x^2 + \alpha$ and $g(x, \alpha) = x + 1$ at $(x, \alpha) = (0, 0)$. It satisfies the non-simple equilibrium conditions, case A1.1:

$$f(0, 0) = f'_x(0, 0) = 0, \quad f''_{xx}(0, 0) = 2 \neq 0, \quad g(0, 0) = 1 \neq 0, \\ f'_\alpha(0, 0) = 1 \neq 0.$$

By Proposition 2.6, the normal form of this bifurcation is given by Equation (11) with $s = \text{sign} \frac{f''_{xx}}{g} \Big|_{(0,0)} = 1$.

For the system (Equation 19) at the non-simple singularity point $(x, \alpha) = (0, 0)$, one has

$$g(0, 0) = g'_x(0, 0) = 0, \quad g''_{xx}(0, 0) = 2 \neq 0, \quad f(0, 0) = 1 \neq 0, \\ g'_\alpha(0, 0) = 1 \neq 0.$$

By Proposition 2.7, the normal form here is (12) with $s = \text{sign} \frac{g''_{xx}}{f} \Big|_{(0,0)} = 1$.

The system (Equation 20) at point $(x, \alpha) = (0, 0)$ satisfies the conditions of the transcritical singularity:

$$f(0, 0) = g(0, 0) = 0, \quad f'_x(0, 0) = 1 \neq 0, \quad g'_x(0, 0) = 1 \neq 0, \\ \begin{vmatrix} g'_x & g'_\alpha \\ f'_x & f'_\alpha \end{vmatrix} \Big|_{(0,0)} = 1 \neq 0.$$

By Proposition 2.8, the normal form of this bifurcation is Equation (13) with $s = \text{sign} \frac{f'_x}{g'_x} \Big|_{(0,0)} = 1$.

Besides the codimension-1 bifurcations listed by A1.1–A3.0,0, one can describe generic unfoldings of bifurcations of higher codimension in a similar way. These bifurcations admit invertible changes of coordinates (shifts), that bring them to the corresponding normal forms. We do not give a proof here, because it is straightforward: similar to cases A1.1 and A2.1 there always exists the parameter-dependent shift of coordinates $x \rightarrow y + \delta(\alpha)$ such that the term that vanishes at $\alpha = 0$ and has the highest power in the Taylor expansion (y^m or y^n below), is eliminated for all small α . We distinguish the following bifurcations:

A1.m. A codimension- m equilibrium: $f(0, 0) = 0, f_x^{(i)}(0, 0) = 0$ for $1 \leq i \leq m$ and $f_x^{(m+1)}(0, 0) \neq 0, g(0, 0) \neq 0$. The normal

form is:

$$\dot{y} = \beta_0 + \beta_1 y + \beta_2 y^2 + \dots + \beta_{m-1} y^{m-1} + s y^{m+1} + O(y^{m+2}) \quad (21)$$

A2.n. A codimension- n singularity: $g(0, 0) = 0, g_x^{(i)}(0, 0) = 0$ for $1 \leq i \leq n$ and $g_x^{(n+1)}(0, 0) \neq 0, f(0, 0) \neq 0$. The normal form of this bifurcation is:

$$(\alpha_0 + \alpha_1 y + \alpha_2 y^2 + \dots + \alpha_{n-1} y^{n-1} + s y^{n+1} + O(y^{n+2})) \dot{y} = 1 \quad (22)$$

A3.m, n. A codimension- $(1 + m + n)$ singular equilibrium: $f(0, 0) = 0, f_x^{(i)}(0, 0) = 0$ for $1 \leq i \leq m, g(0, 0) = 0, g_x^{(j)}(0, 0) = 0$ for $1 \leq j \leq n$ and $f_x^{(m+1)}(0, 0) \neq 0, g_x^{(n+1)}(0, 0) \neq 0$. Its normal form is:

$$(\alpha_0 + \alpha_1 y + \dots + \alpha_{n-1} y^{n-1} + s y^{n+1}) \dot{y} = \beta_0 + \beta_1 y + \dots + \beta_m y^m + y^{m+1}. \quad (23)$$

In the formulas above coefficient s is equal to either $+1$ or -1 , and all α_i and β_i are small unfolding parameters.

Lemma 2.10. In one-dimensional system (Equation 4) the higher-order bifurcations occur in the way that under small perturbations the following dynamics is observed, depending on the case A1–A3.

Case A1.m: For any combination of integers $\{a_1, a_2, \dots, a_k\}$, such that $A = \sum_{i=1}^k a_i \leq m + 1$ and A has the same parity with $m + 1$, there exists a small perturbation of normal form (21), such that it has locally k equilibria with coordinates $x_1 < x_2 < \dots < x_k$, and every x_i is simple if $a_i = 1$ or non-simple of codimension $a_i - 1$, if $a_i > 1$.

Case A2.n: For any combination of integers $\{b_1, b_2, \dots, b_l\}$, such that $B = \sum_{j=1}^l b_j \leq n + 1$ and A has the same parity with n , there exists a small perturbation of normal form (22), such that it has locally l singularities with coordinates $y_1 < y_2 < \dots < y_l$, and every y_j is simple if $b_j = 1$ or non-simple of codimension $b_j - 1$, if $b_j > 1$.

Case A3.m, n: For any two combinations of integers $\{a_1, a_2, \dots, a_k\}$ and $\{b_1, b_2, \dots, b_l\}$ described above, and any two sets of local coordinates: $x_1 < x_2 < \dots < x_k$ and $y_1 < y_2 < \dots < y_l$, there exists a small perturbation of normal form (23), such that it has:

- 1) an equilibrium at the point x_i , if x_i does not coincide with any of y_j ; the equilibrium is simple if $a_i = 1$ or non-simple of codimension $a_i - 1$, if $a_i > 1$.
- 2) a singularity at the point y_j , if y_j does not coincide with any of x_i ; the singularity is simple if $b_j = 1$ or non-simple of codimension $b_j - 1$, if $b_j > 1$.
- 3) a singular equilibrium at point x_i if $x_i = y_j$ for some j ; this point is degenerate of codimension $a_i + b_j + 1$.

Proof: First, we consider cases A1.m and A2.n. Take a set of integers $\{a_1, a_2, \dots, a_k\}$ as described above in the respective case,

and select small $x_1 < x_2 < \dots < x_k$ (this means, that $|x_i| < \varepsilon$ for some ε). Then, construct the following polynomial:

$$P(y) = (y - x_1)^{a_1}(y - x_2)^{a_2} \dots (y - x_k)^{a_k}. \tag{24}$$

It has k roots at coordinates x_i with multiplicity a_i each. If we open the parentheses in formula (24), the resulting polynomial will be a small perturbation of the order- $(m + 1)$ polynomial standing in the right-hand side of normal form (21) or the order- $(n + 1)$ polynomial in the left-hand side of normal form (22). The statement of Lemma on equilibria or singularities respectively, directly follows.

Now take case A3. m, n and the respective sets of integers $\{a_1, a_2, \dots, a_k\}$ and $\{b_1, b_2, \dots, b_l\}$ and coordinates: $x_1 < x_2 < \dots < x_k$ and $y_1 < y_2 < \dots < y_l$. We construct two polynomials:

$$\begin{aligned} P(y) &= (y - x_1)^{a_1}(y - x_2)^{a_2} \dots (y - x_k)^{a_k}, \\ Q(y) &= (y - y_1)^{b_1}(y - y_2)^{b_2} \dots (y - y_l)^{b_l}. \end{aligned} \tag{25}$$

Polynomial $P(y)$ is the small perturbation of the right-hand side and polynomial $Q(y)$ is the small perturbation of the left-hand side of normal form (23). At the same time, this system possesses equilibria, singularities and singular equilibria exactly as described in the Lemma.

3. QUASILINEAR DAEs: TWO-DIMENSIONAL CASE

Consider the two-dimensional quasilinear DAEs of form (Equation 1), where A is everywhere nonsingular except on the singular set Σ_α . The simplest possible form of such DAEs is given by (cf. [35])

$$\begin{cases} g(x, y, \alpha)\dot{x} = f_1(x, y, \alpha) \\ \dot{y} = f_2(x, y, \alpha), \end{cases} \tag{26}$$

for $(x, y) \in \mathbb{R}^2, \alpha \in \mathbb{R}^m$ and $g: \mathbb{R}^2 \times \mathbb{R}^m \rightarrow \mathbb{R}, f_1, f_2: \mathbb{R}^2 \times \mathbb{R}^m \rightarrow \mathbb{R}$ are smooth functions.

In this case, the singular set

$$\Sigma = \{(x, y) : g(x, y, \alpha) = 0\}$$

is the zero curve of g . This curve is the boundary of two domains:

$$\Sigma^+ = \{(x, y) : g(x, y, \alpha) > 0\}, \quad \Sigma^- = \{(x, y) : g(x, y, \alpha) < 0\}.$$

We assume that g is such that g_x has finitely many zeros on Σ . This means that ∇g is zero also at finitely many points of Σ . Every point of Σ with $\nabla g \neq 0$ belongs to the closure of both Σ^+ and Σ^- .

In order to describe the dynamics of such a two-dimensional system, we introduce the basic dynamical elements, such as special points and cycles.

Definition 3.1. A point (x, y) is called

- an *equilibrium* if $f_1(x, y, \alpha) = f_2(x, y, \alpha) = 0$; and $g(x, y, \alpha) \neq 0$;

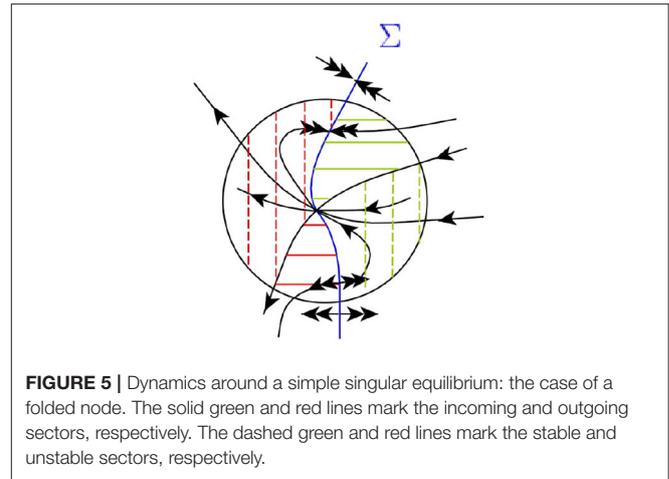


FIGURE 5 | Dynamics around a simple singular equilibrium: the case of a folded node. The solid green and red lines mark the incoming and outgoing sectors, respectively. The dashed green and red lines mark the stable and unstable sectors, respectively.

- a *singular equilibrium* if $g(x, y, \alpha) = f_1(x, y, \alpha) = 0$.
- a *fold point* or a *fold*, if $g(x, y, \alpha) = g_x(x, y, \alpha) = 0$;

Remark 3.2. The fold points given by Definition 3.1 are also referred to as *non-standard algebraic singular points* in DAE terminology (cf. [32, 33]). The singular equilibria are also called as non-standard in some literature, e.g., in [32], or standard (as extended in [33]) geometric singular points.

Making time transformation $d\tau := \frac{dt}{g}$ for $g \neq 0$ in Equation (26), one obtains the *desingularized* system:

$$\begin{cases} \dot{x} = f_1(x, y, \alpha) \\ \dot{y} = f_2(x, y, \alpha)g(x, y, \alpha). \end{cases} \tag{27}$$

The correspondence between the systems is the following: every trajectory Γ of system (Equation 27) contains one or more trajectories of system (Equation 26). The intersection points of Γ with singularity curve $g = 0$ (if any) split Γ into connected pieces $\Gamma_1, \Gamma_2, \dots$ that are trajectories of Equation (26) with the same direction of time as Γ for every $\Gamma_i \subset \Sigma^+$ and with the opposite direction of time if $\Gamma_i \subset \Sigma^-$. It is clear that both equilibria and singular equilibria of system (Equation 26), given by Definition 3.1, are equilibria of the ODE system (Equation 27) in the usual sense.

Remark 3.3. Transforming the time by formula $d\tau := \frac{dt}{-g}$ creates another desingularized system, in which the time flows in the opposite direction on every trajectory of Equation (27). The trajectories of system (Equation 26) follow the trajectories of this system in Σ^- and flow in the opposite direction in Σ^+ .

Consider a point $M \in \Sigma$ and trajectory $\Gamma \ni M$ of ODE system (Equation 27), see [36] for details. The trajectory will intersect Σ transversely if it is not an equilibrium state and if its tangent vector $(f_1, f_2 \cdot g)^T|_M = (f_1, 0)^T$ is not orthogonal to $\nabla g = (g_x, g_y)^T$, i.e. $f_1 \cdot g_x \neq 0$. Locally, Γ is split by M into two components, Γ_1 and Γ_2 such that $M = \Gamma(0), \Gamma_1 \subset \Gamma(t)$ for $t < 0$ and $\Gamma_2 \subset \Gamma(t)$ for $t > 0$. If $f_1 \cdot g_x > 0$, then Γ crosses Σ from Σ^-

to Σ^+ , $\Gamma_2 \subset \Sigma^+$ is the trajectory of Equation (26), and $\Gamma_1 \subset \Sigma^-$ is the trajectory of Equation (26) with time reversal. In this case we call point M *outgoing*. When $f_1 \cdot g_x < 0$, the direction of time is preserved on $\Gamma_1 \subset \Sigma^+$ and reversed on $\Gamma_2 \subset \Sigma^-$, and point M is called *incoming*. Inequalities $f_1 \cdot g_x > 0$ and $f_1 \cdot g_x < 0$ are open conditions, thus curve Σ consists of incoming Σ^{inc} and outgoing Σ^{out} zones, separated by points where $f_1 \cdot g_x = 0$, those are either singular equilibria $f_1 = 0$ or fold points $g_x = 0$.

Definition 3.4. A limit cycle of system (Equation 27) is called a *limit cycle* of system (Equation 26), if it has no intersections with the singularity curve Σ . Otherwise, it is called a *folded limit cycle*.

The limit cycle is a periodic orbit of system (Equation 26). A folded limit cycle consists of more than one orbit of system (Equation 26).

3.1. Structurally Stable Objects

Definition 3.5. An equilibrium of system (Equation 26) is called *simple* or *hyperbolic*, if the linearization matrix of desingularized system (Equation 27) in this point does not have eigenvalues on the imaginary axis.

Definition 3.6. A singular equilibrium (x_0, y_0) of system (Equation 26) is called *simple*, if the following inequalities are fulfilled:

$$f_2(x_0, y_0, \alpha) \neq 0, \quad g_x(x_0, y_0, \alpha) \neq 0, \quad \det \frac{\partial(f_1, g)}{\partial(x, y)} \Big|_{(x_0, y_0, \alpha)} \neq 0, \tag{28}$$

and the linearization matrix of desingularized system (Equation 27) in this point does not have eigenvalues on the imaginary axis.

Definition 3.7. A fold (x_0, y_0) of system (Equation 26) is called *simple*, if the following inequalities are fulfilled in it:

$$f_1(x_0, y_0, \alpha) \neq 0, \quad g_y(x_0, y_0, \alpha) \neq 0, \quad g_{xx}(x_0, y_0, \alpha) \neq 0. \tag{29}$$

3.1.1. Simple Equilibria

Simple equilibria lie outside the singularity curve. A topological type of an equilibrium M is determined by eigenvalues λ_1 and λ_2 of the linearization matrix

$$A_{EQ} = \begin{pmatrix} f_{1x} & f_{1y} \\ g_{2x} & g_{2y} \end{pmatrix}. \tag{30}$$

For real λ_1 and λ_2 , M is saddle if $\lambda_1 \lambda_2 < 0$ and node if $\lambda_1 \lambda_2 > 0$. If the eigenvalues are a complex-conjugate pair, the equilibrium is a focus. A node or a focus M is stable if

$$M \in \Sigma^+, \operatorname{Re} \lambda_{1,2} < 0 \quad \text{or} \quad M \in \Sigma^-, \operatorname{Re} \lambda_{1,2} > 0, \tag{31}$$

and unstable if

$$M \in \Sigma^+, \operatorname{Re} \lambda_{1,2} > 0 \quad \text{or} \quad M \in \Sigma^-, \operatorname{Re} \lambda_{1,2} < 0. \tag{32}$$

Simple equilibria persist under small perturbations, because they remain equilibria and retain their topological type in the desingularized system. Thus, in the original system, they lie outside the singularity curve Σ , and their topological type also does not change.

3.1.2. Simple Singular Equilibria

In a similar way we classify simple singular equilibria using eigenvalues $\lambda_{1,2}$ of linearization matrix

$$A_{sEQ} = \begin{pmatrix} f_{1x} & f_{1y} \\ g_x f_2 & g_y f_2 \end{pmatrix}. \tag{33}$$

Definition 3.8. Let $\lambda_{1,2}$ be the eigenvalues of A_{sEQ} in Equation (33) evaluated at a simple singular equilibrium M of Equation (26). Then, M is called a *folded node*, if $\lambda_{1,2} \in \mathbb{R}$ and $\lambda_1 \lambda_2 > 0$; a *folded saddle*, if $\lambda_{1,2} \in \mathbb{R}$ and $\lambda_1 \lambda_2 < 0$; and a *folded focus*, if $\lambda_{1,2} \notin \mathbb{R}$.

The dynamics near a folded node and a folded saddle is determined by eigendirections corresponding to eigenvalues $\lambda_{1,2}$ and respective invariant manifolds. The following lemma states that these eigendirections are never tangent to Σ at simple singular equilibria.

Lemma 3.9. *In a folded node and a folded saddle, the eigendirections, corresponding to eigenvalues $\lambda_{1,2}$ are transverse to the singularity curve Σ .*

Proof: We prove the lemma by contradiction. Assume that for some eigenvalue λ_1 its eigenvector is tangent to Σ . Then, tangent to Σ vector $(g_y, -g_x)^T$ is the eigenvector of matrix A_{sEQ} :

$$\begin{pmatrix} f_{1x} - \lambda_1 & f_{1y} \\ g_x f_2 & g_y f_2 - \lambda_1 \end{pmatrix} \begin{pmatrix} g_y \\ -g_x \end{pmatrix} = \begin{pmatrix} f_{1x} g_y - f_{1y} g_x - \lambda_1 g_y \\ g_x \lambda_1 \end{pmatrix} = 0,$$

which implies either $g_x = 0$ or $\lambda_1 = 0$, both conditions contradict the assumption that the considered singular equilibrium is simple.

To describe the dynamical properties of all three types of singular equilibria, we consider a small neighborhood U of M . Locally, U is divided by curve Σ into two disconnected parts given by $\Sigma_{loc}^+ \subset \Sigma^+$ and $\Sigma_{loc}^- \subset \Sigma^-$.

Folded node. Consider a folded node M with $\lambda_2 < \lambda_1 < 0$ (for the case $0 < \lambda_1 < \lambda_2$ the statement will be the same with Σ^+ and Σ^- interchanged). It has a leading direction e^L defined by the eigenvector corresponding to λ_1 and a non-leading direction e^{nL} defined by the eigenvector corresponding to λ_2 . There exists a semi-stable smooth invariant manifold $W^{nL}(M)$ tangent to e^{nL} at point M . Its existence follows from the desingularized system (Equation 27), this system has at M a stable or a completely unstable node equilibrium, that possesses a smooth strong stable (unstable) manifold. After coming back to the original system, a part of this manifold lying in Σ^- changes the direction of time, so that the manifold becomes a semi-stable manifold $W^{nL}(M)$.

Manifold $W^{nL}(M)$ and Σ intersect transversely according to Lemma 3.9. They divide neighborhood U into four sectors, we will call them incoming, stable, outgoing and unstable. The **incoming** sector lies in Σ_{loc}^+ every initial condition from this sector reaches Σ^{inc} in forward time and leave U in backward time. The **stable** sector also lies in Σ_{loc}^+ and contains the leading direction e^L . The trajectories from this sector reach M tangent to e^L in forward time and reach Σ^{out} in backward time.

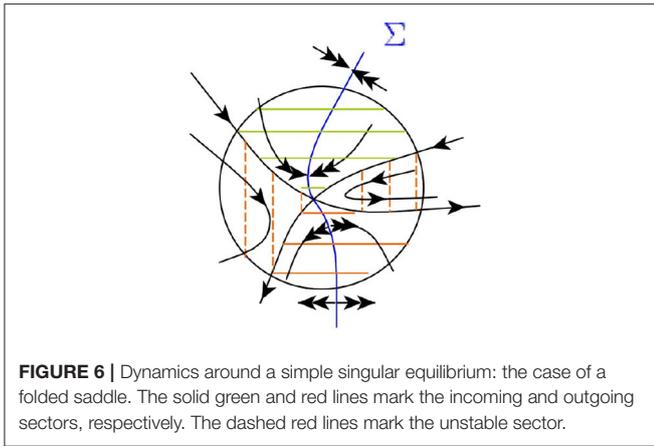


FIGURE 6 | Dynamics around a simple singular equilibrium: the case of a folded saddle. The solid green and red lines mark the incoming and outgoing sectors, respectively. The dashed red lines mark the unstable sector.

In a similar way, we describe the dynamics in Σ_{loc}^- : it is divided by $W^{nl}(M)$ into unstable and outgoing sectors. In the **outgoing** sector trajectories leave U in forward time and reach Σ^{out} in backward time. In the **unstable** sector the trajectories reach Σ^{inc} in forward time and point M in backward time. All four types of behavior are illustrated at **Figure 5**.

Folded saddle. In a folded saddle M the eigenvalues of the desingularized linearization matrix are $\lambda_1 < 0 < \lambda_2$. Point M belongs to two smooth invariant manifolds: W^- tangent to eigendirection e^- corresponding to λ_1 and W^+ tangent to eigendirection e^+ corresponding to λ_2 . They divide Σ^+ into three sectors: incoming, saddle and outgoing. The incoming sector is bounded by Σ^{inc} and W^- , all orbits from it reach Σ^{inc} in forward time and leave U in backward time. The saddle sector is bounded by W^+ and W^- , all orbits leave U in both directions of time. The outgoing sector is bounded by Σ^{out} and W^+ , the orbits in it leave U in forward time and reach Σ^{out} in backward time. All above describes the dynamics also in Σ^- , where the outgoing sector is bounded by Σ^{out} and W^- and the incoming sector is bounded by Σ^{inc} and W^+ . The stable manifold of M is $W^s(M) = M \cup (W^- \cap \Sigma^+) \cup (W^+ \cap \Sigma^-)$, the unstable is $W^u(M) = M \cup (W^+ \cap \Sigma^+) \cup (W^- \cap \Sigma^-)$, both manifolds are C^0 in M (see **Figure 6**).

Folded focus. Near a folded focus all orbits reach Σ^{out} in backward time and Σ^{inc} in forward time (see **Figure 7**).

Under small (smooth) perturbations, simple singular equilibria persist and retain their topological type. The reasons of it are that matrix (Equation 33) is non-degenerate at such point, and by the Implicit Function Theorem equation $g = 0, f_1 = 0$ has a unique solution for small α , provided that it exists for $\alpha = 0$. Also, as the topological type of such an equilibrium persists in the desingularized system, it also persists in the original one.

3.1.3. Simple Fold

Near the simple fold point $M(x_0, y_0)$, where $(g = g_x = 0, g_{xx} \neq 0, g_y \neq 0)$, equation $g(x, y) = 0$ of the singularity curve Σ can be locally explicitly resolved as a function $y = \psi(x) = -\frac{g_{xx}(M)}{g_y(M)}x^2 + O(x^3)$, $y_0 = \psi(x_0)$. The fold point is a local maximum or

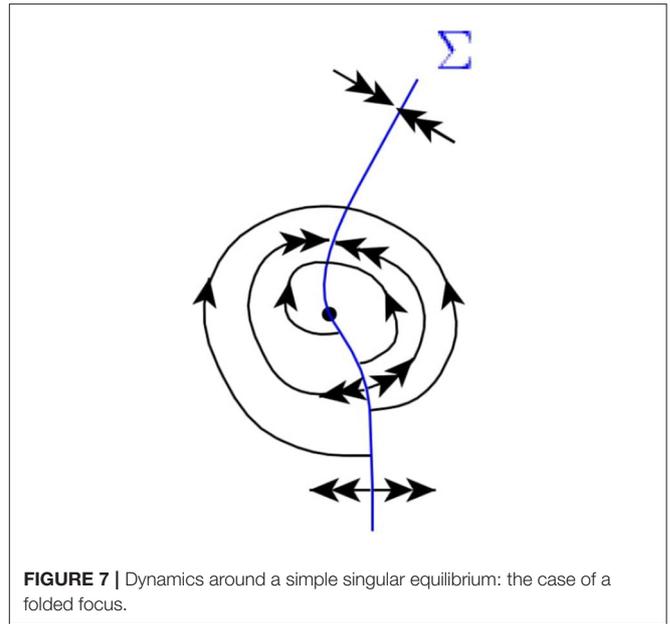


FIGURE 7 | Dynamics around a simple singular equilibrium: the case of a folded focus.

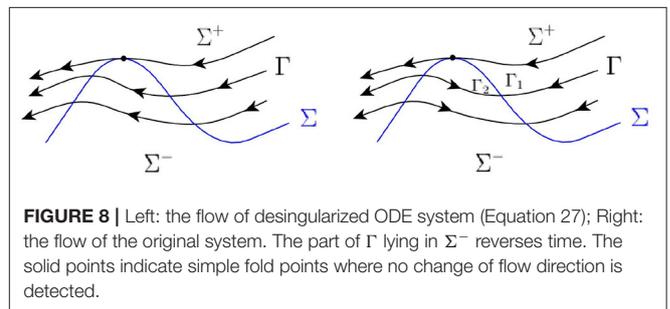


FIGURE 8 | Left: the flow of desingularized ODE system (Equation 27); Right: the flow of the original system. The part of Γ lying in Σ^- reverses time. The solid points indicate simple fold points where no change of flow direction is detected.

minimum of this function. Consider the desingularized system (Equation 27) and its solution with initial condition (x_0, y_0) . At the point M its y component $f_2(x_0, y_0)g(x_0, y_0)$ vanishes, thus the trajectory is tangent to Σ , see **Figure 8**.

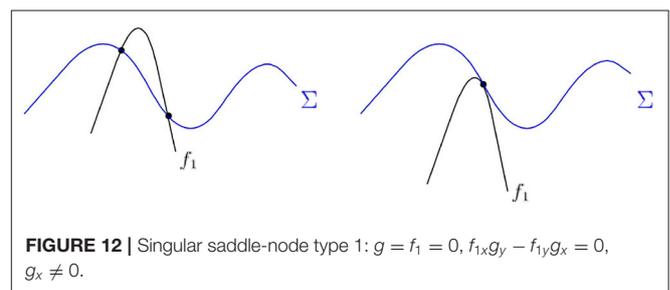
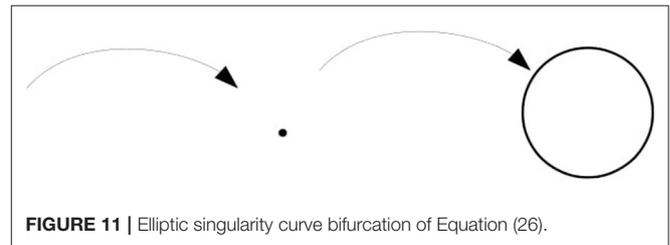
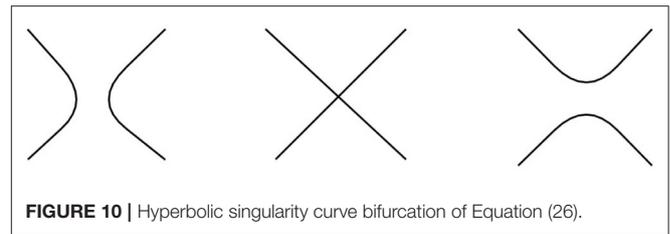
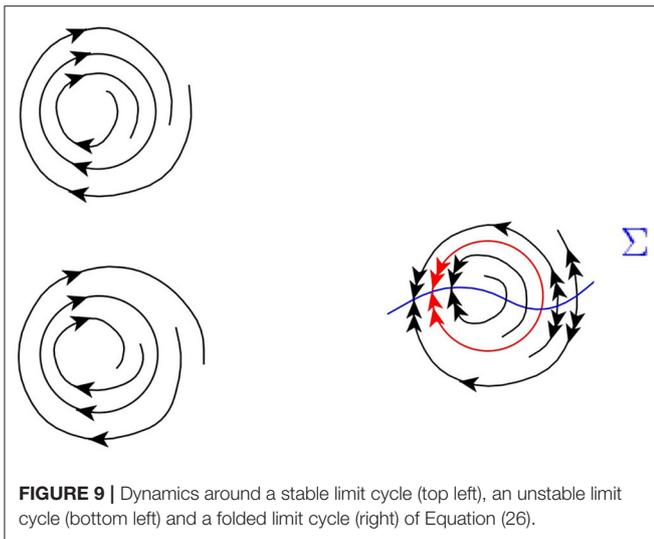
The simple fold is either Σ^+ -convex, when $g_{xx}(M)g_y(M) > 0$ and Σ^- -convex, when $g_{xx}(M)g_y(M) < 0$. Also, it persists under small perturbations. Indeed, for system of equations $g = g_x = 0$ we have

$$\frac{\partial(g, g_x)}{\partial(x, y)}(x_0, y_0) = \begin{pmatrix} 0 & g_y \\ g_{xx} & g_{xy} \end{pmatrix}(x_0, y_0) = -g_{xx}(x_0, y_0)g_y(x_0, y_0) \neq 0 \tag{34}$$

so that by the Implicit Function Theorem the system can be uniquely solved with respect to (x, y) for all small α . This solution gives a unique fold point in a small neighborhood of point M . In addition, condition $f_1(x, y) \neq 0$ is fulfilled in some small neighborhood of the fold point, also for small α , thus no other objects (regular or singular equilibria) appear there under small perturbations.

3.1.4. Regular and Folded Limit Cycles

The limit cycles, regular and folded, both correspond to limit cycles of the desingularized system (Equation 27). By standard



methods (the Poincaré crosssection) one defines the multiplier $\mu > 0$ of such an orbit. A regular cycle is simple (structurally stable), if its multiplier differs from one. A folded cycle is structurally stable also if $\mu \neq 1$, and, in addition, it intersects the singularity curve only transversely. The possible types of simple limit cycles are illustrated in Figure 9.

3.2. Bifurcations

The bifurcations in two-dimensional DAEs are divided in three main groups: geometric (bifurcations of the singularity set), local and global bifurcations.

3.2.1. Geometric Bifurcations

Geometric bifurcations are related to the reconstruction of the topology of the singularity set Σ . It means that after arbitrary small perturbations the set Σ is topologically not equivalent to itself at the initial parameter value. This happens when its branches appear, disappear or interact with each other. Among codimension-one bifurcations, there are those, related to failure of local existence of a unique branch of Σ , i.e. existence of a point (x, y) , where $\nabla g(x, y) = 0$. At the same time, the Hessian should be non-zero at the bifurcation moment, so that the codimension is not higher than one:

$$\nabla g(x, y) = 0, \det D^2g(x, y) \neq 0 \tag{35}$$

Depending on the sign of the Hessian, two cases are possible [37]:

T1. Hyperbolic bifurcation. $g_x = g_y = 0$ and $D^2g(x, y) < 0$. For example, $g(x, y, \alpha) = x^2 - y^2 - \alpha$ at $(0, 0, 0)$ (see Figure 10).

T2. Elliptic bifurcation. $g_x = g_y = 0$ and $D^2g(x, y) > 0$. For example, $g(x, y, \alpha) = x^2 + y^2 - \alpha$ (see Figure 11).

3.2.2. Local Bifurcations

Local bifurcations occur to simple objects described in subsection 3.1, when they lose their simple properties. From the list below, bifurcations **L1, L6, L8** occur outside the singularity curve Σ and thus they are unfolded as in regular ODEs. Bifurcations **L2, L7, L9** involve the curve Σ , but after the

desingularization procedure, they become bifurcations **L1, L6, L8** respectively, in the ODE system (Equation 27) and their unfolding can be described accordingly. The rest, bifurcations **L3–L5**, either are unfolded in a different way in regular systems or do not have regular analogs at all. The latter are studied in detail in Section 4.

L1. Saddle-node. This bifurcation occurs when at the equilibrium point M the eigenvalues of the linearization matrix (30) are $\lambda_1 = 0$ and $\lambda_2 \neq 0$, i.e. when $f_{1x}f_{2y} - f_{1y}f_{2x} = 0$. Like in a regular ODE system, a codimension-one saddle-node under small perturbations either disappears so that in some small neighborhood there are no equilibria, or splits into two simple equilibria, a saddle and a node.

L2. Singular saddle-node of type I. According to the classification by C. Kuehn [38]. This bifurcation corresponds to the existence of a singular equilibrium, for which linearization matrix (33) has eigenvalues $\lambda_1 = 0$ and $\lambda_2 \neq 0$. This happens when $f_{1x}g_y - f_{1y}g_x = 0$. A codimension-one singular equilibrium of type I under small perturbations either disappears or splits into two simple singular equilibria – a folded saddle and a folded node, see Figure 12.

L3. Singular saddle-node of type II. According to the classification by C. Kuehn [38]. This bifurcation occurs when a singular equilibrium point M ($f_1 = g = 0$) also satisfies regular equilibrium condition $f_2 = 0$. A small perturbation of a codimension-one singular saddle-node of type II leads to

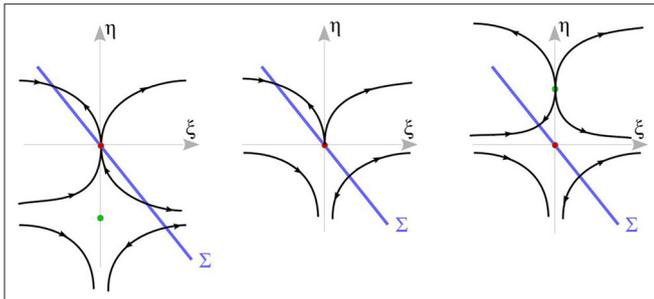


FIGURE 13 | The singular saddle-node bifurcation of type II at $(\xi^*, \eta^*, \alpha^*) = (0, 0, 0)$ for $\Delta_4 > 0$ and $f_{1x} > 0$. As α changes from negative to positive, the local flow changes from the left figure to right figure.

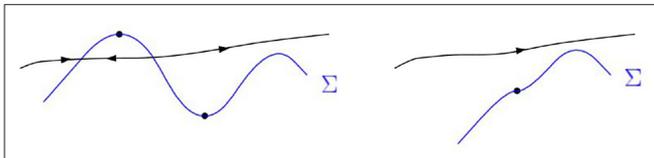


FIGURE 14 | A cubic fold bifurcation $g = g_x = g_{xx} = 0, f_1 \neq 0$.

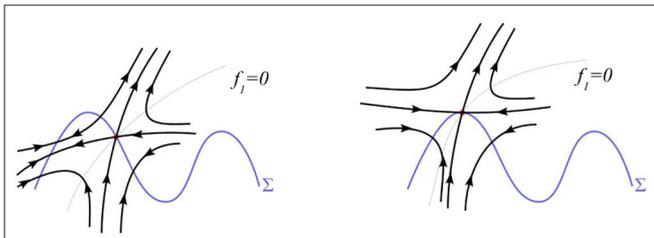


FIGURE 15 | A singular equilibrium-fold bifurcation $g = g_x = f_1 = 0, g_{xx} \neq 0$.

the appearance of a simple equilibrium and a simple singular equilibrium. They appear in combinations either node + folded saddle, or saddle + folded node. For the derivation of the normal form refer to Lemma 4.1 below. The bifurcation is illustrated on **Figure 13**, see also [9, 20–23, 39], where such a bifurcation was studied.

14. Cubic fold. A fold point $g = g_x = 0, g_y \neq 0$, is non-simple of codimension one (a cubic fold) if $g_{xx} = 0$ and $g_{xxx} \neq 0$. Under small perturbations the cubic fold generically either disappears or splits into a pair of simple folds with the opposite convexity, as stated by Lemma 4.3. The bifurcation is illustrated in **Figure 14**.

15. Singular equilibrium-fold This bifurcation occurs when at a point M both conditions of a fold and a singular equilibrium are fulfilled, i.e. $g = g_x = f_1 = 0$ and $f_2 \neq 0, f_{1x} \neq 0, g_{xx} \neq 0, g_y \neq 0$. Under small perturbations, a singular equilibrium-fold splits into a simple fold and a simple singular equilibrium, see Lemma 4.4 for details. The bifurcation is illustrated at **Figure 15**.

16. Transition between folded node and folded focus. This bifurcation occurs at a singular equilibrium, when the two eigenvalues of the linearization matrix (Equation 33) coincide. In small perturbations they either become a real (folded node) or a complex-conjugated (folded focus) pair of different eigenvalues.

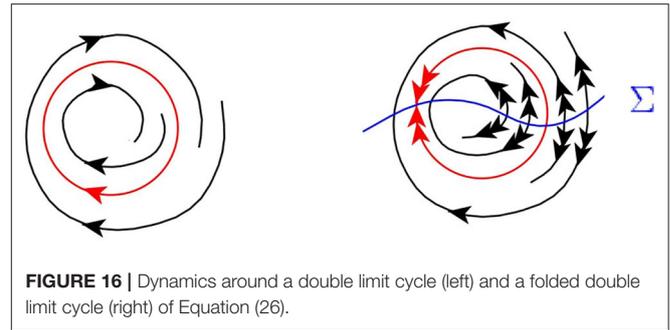


FIGURE 16 | Dynamics around a double limit cycle (left) and a folded double limit cycle (right) of Equation (26).

Note that the similar transition for a regular equilibrium is not considered a bifurcation—the local flows around a node and a focus can be topologically conjugated. However, in the case of a singular equilibrium, the local flows near a folded node and a folded focus are not similar: near the folded node there exists a subset of points such that their trajectories reach the singular equilibrium in forward or backward time, while in the neighborhood of a folded focus there are no such orbits (see Subsection 3.1.2 for details).

L7. Andronov-Hopf bifurcation. This standard bifurcation occurs when a stable (unstable) focus equilibrium has a pair of pure imaginary eigenvalues (of matrix Equation 30). Under small perturbations such a weak focus either becomes a simple stable (unstable) focus, or unstable (stable) focus and a stable (unstable) limit cycle is born.

L8. Folded Andronov-Hopf bifurcation. This bifurcation takes place, when linearisation matrix (Equation 33) has a pair of pure imaginary eigenvalues [40]. This bifurcation corresponds to a regular Andronov-Hopf in the desingularized system (Equation 27), in which a limit cycle is born from a focus equilibrium. In the original system (Equation 26) under small perturbations a folded limit cycle is born from a folded focus.

L9. Double limit cycle. Existence of a limit cycle with multiplier equal to +1. Under small perturbations this cycle either disappears or is split into stable and unstable simple limit cycles, see **Figure 16**.

L10. Folded double limit cycle. This bifurcation corresponds to the existence of a double limit cycle in the desingularized system (Equation 27), that intersects the singularity curve. Under small perturbations such a cycle either disappears or is split into a pair of simple folded limit cycles.

3.2.3. Global Bifurcations

This is the class of bifurcations, in which special orbits (homoclinic or heteroclinic) exist in the system. They are usually destroyed by small perturbations. Such orbits can be regular, when they have no intersections with the singularity curve Σ , or folded if such an intersection point exist.

G1. A homoclinic orbit to a saddle. In the desingularized system (Equation 27) there exists a homoclinic loop Γ to a saddle equilibrium, that is also an equilibrium in system (Equation 26).

G1a. Regular. This is the standard homoclinic bifurcation, when the image of Γ does not intersect Σ . Upon small perturbations it either just disappears, or disappears with the creation of a limit cycle.

G1b. Folded. The image of Γ in Equation (26) has intersections with Σ . Under small perturbations, such a homoclinic loop gives rise to a folded limit cycle.

G2. A homoclinic orbit to a folded saddle a homoclinic orbit Γ exists in the desingularized system (Equation 27). In the original system (Equation 26) this equilibrium lies at Σ .

G2a. Regular. In the original system (Equation 26) the image of Γ does not intersect Σ . Under small perturbations, when the folded saddle disappears, a regular limit cycle is born.

G2b. Folded. In the original system (Equation 26) the image of Γ has intersections with Σ . Under small perturbations, when the folded saddle disappears, a folded limit cycle is born.

G3. A homoclinic orbit to a fold point. In the desingularized system (Equation 27) there exists a simple limit cycle L . In the original system (Equation 26) the image of curve L is tangent to Σ at a simple fold point F .

G3a. Regular. The image of curve L does not have common points with Σ other than F . Under small perturbations, cycle L in system (Equation 27) persists, and in system (Equation 26) this curve becomes either a regular or a folded limit cycle.

G3b. Folded. The image of curve L have intersections with Σ other than F . Under small perturbations, cycle L in system (Equation 27) persists, and in system (Equation 26) it becomes a folded limit cycle. **G4. A homoclinic orbit to a saddle-node.** In the desingularized system (Equation 27) there exists a homoclinic orbit L to a saddle-node equilibrium M . In system (Equation 26) point M does not belong to singularity curve Σ . Upon the disappearance of the equilibrium a limit cycle is born in system (Equation 27).

G4a. Regular. The image of L in system (Equation 26) does not intersect the singularity curve Σ . Upon the disappearance of the equilibrium a limit cycle is born also in the original system (Equation 26).

G4b. Folded. The image of L in system (Equation 26) intersects transversely the singularity curve Σ . Upon the disappearance of the equilibrium a folded limit cycle is born.

G5. A homoclinic orbit to a singular saddle-node of type I. In the desingularized system (Equation 27) there exists a homoclinic orbit L to a saddle-node equilibrium M . In system (Equation 26) point M is a singular equilibrium (a singular saddle-node of type I). Upon the disappearance of the equilibrium a periodic orbit is born in system (Equation 27).

G5a. Regular. The image of L in system (Equation 26) does not intersect the singularity curve Σ . Upon the disappearance of the singular equilibrium a limit cycle is born in the original system (Equation 26).

G5b. Folded. The image of L in system (Equation 26) intersects transversely the singularity curve Σ . Upon the disappearance of the singular equilibrium a folded limit cycle is born.

G6. A heteroclinic connection. This bifurcation corresponds to the existence of such an orbit L in system (Equation 27) that

- passes through two different fold points of system (Equation 26). In system (Equation 26) a piece of L that lies between the fold points, is a heteroclinic connection of two folds (see Figure 17).

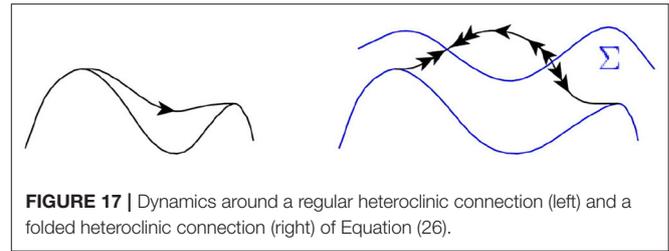


FIGURE 17 | Dynamics around a regular heteroclinic connection (left) and a folded heteroclinic connection (right) of Equation (26).

- passes through one fold point F of system (Equation 26) and tends to an equilibrium in forward or backward time, without passing through other fold points that differ from F . This is a heteroclinic connection of a fold and a regular or singular equilibrium in system (Equation 26)
- tends to different equilibria in both forward and backward time, without passing through any of folds of system (Equation 26). This is a heteroclinic connection between two equilibria, two singular equilibria or between an equilibrium and a singular equilibrium.

All the heteroclinic connections listed above can be either regular, when they do not intersect the singularity curve, or folded, when such an intersection exists. Under small perturbations heteroclinic connections are generically broken.

4. LOCAL BIFURCATIONS L3–L5

In this section the character of those local bifurcations, that do not have analogs in regular ODEs is studied in detail. These bifurcations are **L3**: Singular saddle-node of type II, **L4**: Cubic fold and **L5**: Singular equilibrium-fold from subsection 3.2.2.

We introduce the following notations:

$$\begin{aligned} \Delta_1 &= \det \frac{\partial(f_1, f_2)}{\partial(x, y)} \Big|_{(0,0,0)}, & \Delta_2 &= \det \frac{\partial(f_1, g)}{\partial(x, y)} \Big|_{(0,0,0)}, \\ \Delta_3 &= \det \frac{\partial(g, g_x)}{\partial(y, \alpha)} \Big|_{(0,0,0)}, & \Delta_4 &= \frac{\partial(f_1, f_2, g)}{\partial(x, y, \alpha)} \Big|_{(0,0,0)}, \\ \Delta_5 &= \frac{\partial(f_1, g, g_x)}{\partial(x, y, \alpha)} \Big|_{(0,0,0)}. \end{aligned} \quad (36)$$

4.1. L3: Singular Saddle-Node of Type II

This bifurcation occurs at point $M(0,0)$ when both conditions for an equilibrium and a singular equilibrium are fulfilled in it for $\alpha = 0$. This means that $g(0, 0, 0) = f_1(0, 0, 0) = f_2(0, 0, 0) = 0$. We assume that the following inequalities also hold, so that the codimension is equal to one and that the parametric family is transversal:

$$f_{1x}(0, 0, 0) \neq 0, \quad g_x(0, 0, 0) \neq 0, \quad \Delta_1 \neq 0, \quad \Delta_2 \neq 0, \quad \Delta_4 \neq 0. \quad (37)$$

Lemma 4.1. Assume that genericity (codimension one + transversality) conditions (37) are fulfilled at a singular saddle-node type II point M . Then in a generic unfolding point M splits

into a pair of structurally stable points, a regular and a singular equilibria. They are either a saddle and a folded node or a node and a folded saddle.

Proof: First of all, we note that equations $f_1 = f_2 = g = 0$ have a solution $x = y = \alpha = 0$, and by the transversality condition $\Delta_4 \neq 0$ no other solution exists nearby, thus for small $\alpha \neq 0$ no other singular saddle-nodes of type II exist in some small neighborhood of the origin.

Condition $\Delta_1 \neq 0$ implies that for all small α equations $f_1 = f_2 = 0$ have a unique equilibrium solution

$$(x_e(\alpha), y_e(\alpha)) = \frac{1}{\Delta_1} \left(\det \frac{\partial(f_1, f_2)}{\partial(y, \alpha)} \Big|_{(0,0,0)}, -\det \frac{\partial(f_1, f_2)}{\partial(x, \alpha)} \Big|_{(0,0,0)} \right) \alpha + O(\alpha^2). \tag{38}$$

Similarly, as $\Delta_2 \neq 0$, equations $f_1 = g = 0$ have a unique singular equilibrium solution:

$$(x_s(\alpha), y_s(\alpha)) = \frac{1}{\Delta_2} \left(\det \frac{\partial(f_1, g)}{\partial(y, \alpha)} \Big|_{(0,0,0)}, -\det \frac{\partial(f_1, g)}{\partial(x, \alpha)} \Big|_{(0,0,0)} \right) \alpha + O(\alpha^2). \tag{39}$$

At the bifurcation moment $\alpha = 0$ the linearization matrix (33) at the singular saddle node has eigenvalues $\lambda_1 = f_{1x} \neq 0$ and $\lambda_2 = 0$. Upon a small perturbation, linearization matrices (30) and (33) of, respectively, the equilibrium and the singular equilibrium, will have real eigenvalues $\lambda_1 = f_{1x}(0, 0, 0) + O(\alpha)$ and $\lambda_2 = O(\alpha)$, so they have either node or saddle type. The sign of the first eigenvalue at each point is given for small α by the sign of derivative $f_{1x}(0, 0, 0)$. The product of eigenvalues is equal to the determinant of the linearization matrix. At the equilibrium the determinant is:

$$\det A_{EQ} = g(x^e(\alpha), y^e(\alpha), \alpha)(\Delta_1 + O(\alpha)) = \Delta_4 \alpha + O(\alpha^2),$$

and at the singular equilibrium:

$$\det A_{sEQ} = f_2(x^s(\alpha), y^s(\alpha), \alpha)(\Delta_2 + O(\alpha)) = -\Delta_4 \alpha + O(\alpha^2).$$

Thus, the bifurcation occurs in the following way:

When $\Delta_4 \alpha > 0$, the equilibrium is a node (stable if $f_{1x} < 0$ and unstable if $f_{1x} > 0$), and the singular equilibrium is a folded saddle;

When $\Delta_4 \alpha < 0$, the equilibrium is a saddle, and the singular equilibrium is a folded node.

Example 4.2. Consider

$$\begin{cases} (x - x^3)\dot{x} = y - x + \alpha \\ \dot{y} = y \end{cases}. \tag{40}$$

Then, for $\alpha = 0$ we have $f_1 = f_2 = g = 0$, i.e. a singular saddle-node of type II at the origin $(x, y) = (0, 0)$. By formulas (36), it follows that $f_{1x} = -1, g_x = 1, \Delta_1 = -1, \Delta_2 = -1$ and $\Delta_4 = -1$, the conditions (37) are fulfilled. For small α this bifurcation point unfolds into an equilibrium EQ: $(x^e, y^e) = (\alpha, 0) + O(\alpha^2)$ and a singular equilibrium sEQ: $(x^s, y^s) = (0, -\alpha) + O(\alpha^2)$. For $\alpha > 0$ EQ is a stable node and sEQ is a folded saddle. For $\alpha < 0$, EQ is a saddle and sEQ is a folded node.

4.2. L4: Cubic Fold

The cubic fold bifurcation occurs at point $M(0, 0)$ for $\alpha = 0$ if conditions $g = g_x = g_{xx} = 0$ are fulfilled at it. In addition, to keep the codimension of the problem equal to one and to construct a transversal parametric family, we assume the following inequalities to hold:

$$f_1 \neq 0, \quad g_y \neq 0, \quad g_{xxx} \neq 0, \quad \Delta_3 \neq 0 \tag{41}$$

Lemma 4.3. Assume that genericity (codimension one + transversality) conditions (Equation 41) are fulfilled at a cubic fold point M . Then in a generic unfolding point M splits into a pair of simple folds, or disappears.

Proof: For system of equations $g = g_x = 0$ the Implicit Function Theorem are not fulfilled, because $\frac{\partial(g, g_x)}{\partial(x, y)} \Big|_{(0,0,0)} = 0$. Then we look for a solution of this system in the form

$$x = \beta_1 \alpha^{\delta_1} + o(\alpha^{\delta_1}), \quad y = \beta_2 \alpha^{\delta_2} + o(\alpha^{\delta_2})$$

for positive α , and

$$x = \beta_1 (-\alpha)^{\delta_1} + o((-\alpha)^{\delta_1}), \quad y = \beta_2 (-\alpha)^{\delta_2} + o((-\alpha)^{\delta_2})$$

for negative α . The equations take form

$$\begin{aligned} 0 &= g_y \beta_2 \alpha^{\delta_2} + g_\alpha \alpha + \frac{1}{6} g_{xxx} \beta_1^3 \alpha^{3\delta_1} + h.o.t. \\ 0 &= g_{xy} \beta_2 \alpha^{\delta_2} + g_{x\alpha} \alpha + \frac{1}{2} g_{xxx} \beta_1^2 \alpha^{2\delta_1} + h.o.t. \end{aligned} \tag{42}$$

and respectively

$$\begin{aligned} 0 &= g_y \beta_2 (-\alpha)^{\delta_2} - g_\alpha (-\alpha) + \frac{1}{6} g_{xxx} \beta_1^3 (-\alpha)^{3\delta_1} + h.o.t. \\ 0 &= g_{xy} \beta_2 (-\alpha)^{\delta_2} - g_{x\alpha} (-\alpha) + \frac{1}{2} g_{xxx} \beta_1^2 (-\alpha)^{2\delta_1} + h.o.t. \end{aligned} \tag{43}$$

For them to be solvable it is required that $\delta_1 = 1/2, \delta_2 = 1$. Then for $\alpha > 0$ from (42) we have

$$\beta_1^2 = \frac{2\Delta_3}{g_y g_{xxx}}, \quad \beta_2 = -\frac{g_\alpha}{g_y} \tag{44}$$

and for $\alpha < 0$ from (43):

$$\beta_1^2 = -\frac{2\Delta_3}{g_y g_{xxx}}, \quad \beta_2 = \frac{g_\alpha}{g_y}. \tag{45}$$

Then, two simple folds exist for perturbations $\frac{2\Delta_3}{g_y g_{xxx}} \alpha > 0$, and the cubic fold disappears, and no folds exist locally, when $\frac{2\Delta_3}{g_y g_{xxx}} \alpha < 0$.

4.3. L5. Singular Equilibrium-Fold

The singular equilibrium-fold bifurcation occurs at point $M(0, 0)$ for $\alpha = 0$ if conditions $g = g_x = f_1 = 0$ are fulfilled at it. In addition, to keep the codimension of the problem equal to one and to construct a transversal parametric family, we assume the following inequalities to hold:

$$g_y \neq 0, \quad g_{xx} \neq 0, \quad \Delta_2 \neq 0, \quad \Delta_3 \neq 0, \quad \Delta_5 \neq 0 \tag{46}$$

Lemma 4.4. Assume that genericity (codimension one + transversality) conditions (Equation 46) are fulfilled at a singular equilibrium-fold point M . Then in a generic unfolding point M splits into a simple folds and a simple singular equilibrium.

Proof: By the transversality condition $\Delta_5 \neq 0$, system of equations $g = g_x = f_1 = 0$ has locally no solutions for $\alpha \neq 0$, then the singular equilibrium-fold disappears under such small perturbations.

The genericity condition $\Delta_2 \neq 0$ implies that system of equations $f_1 = g = 0$ has locally a unique singular equilibrium solution for small α :

$$(x_s(\alpha), y_s(\alpha)) = \frac{1}{\Delta_2} \left(\det \frac{\partial(f_1, g)}{\partial(y, \alpha)} \Big|_{(0,0,0)}, - \det \frac{\partial(f_1, g)}{\partial(x, \alpha)} \Big|_{(0,0,0)} \right) \alpha + O(\alpha^2). \quad (47)$$

Also, the genericity condition $\Delta_3 \neq 0$ implies that a unique fold solution of system $g = g_x = 0$ exists for small α :

$$(x_f(\alpha), y_f(\alpha)) = \frac{1}{\Delta_3} \left(\det \frac{\partial(g, g_x)}{\partial(y, \alpha)} \Big|_{(0,0,0)}, - \det \frac{\partial(g, g_x)}{\partial(x, \alpha)} \Big|_{(0,0,0)} \right) \alpha + O(\alpha^2). \quad (48)$$

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DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

HR wrote the Introduction section, prepared the figures and the examples. IO contributed to the basic definitions and explanation of the dynamics, as well as the preparation of the list of bifurcations. Both authors contributed to the conception, design of the study, computation of the normal forms, manuscript revisions, read, and approved the submitted version.

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