

RECURRENCE RELATIONS FOR THE EVEN AND ODD CHARACTERISTIC POLYNOMIALS OF A SYMMETRIC TOEPLITZ MATRIX

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Abstract. In a recent paper [12] Melman proved a recurrence relation of the even and odd characteristic polynomials of a real symmetric Toeplitz matrix T on which a symmetry exploiting method for computing the smallest eigenvalue of T can be based. In this note we present a proof of the recurrence relation which is less technical and more transparent.

Key words. symmetric Toeplitz matrix, eigenvalue, even/odd characteristic polynomial

1. Introduction. The problem of finding the smallest eigenvalue λ_1 of a real symmetric, positive definite Toeplitz matrix T is of considerable interest in signal processing. Given the covariance sequence of the observed data, Pisarenko [13] suggested a method which determines the sinusoidal frequencies from the eigenvector of the covariance matrix associated with its minimum eigenvalue. Several methods have been reported in the literature for computing the minimum eigenvalue of T , cf. [2], [7], [8], [9], [10], [12], [14], e.g., and the literature given therein.

In their seminal paper Cybenko and Van Loan [2] presented the following method: by bisection they determine an initial approximation $\mu_0 \in (\lambda_1, \omega_1)$, where ω_1 denotes the smallest pole of the secular equation f , and they improve μ_0 by Newton's method for f which converges monotonely and quadratically to λ_1 . This approach was improved considerably in [8] and [6] by replacing Newton's method by a more appropriate root finding methods for the secular equation. Taking advantage of the fact that the spectrum of a symmetric Toeplitz matrix can be divided into odd and even parts the methods based on the secular equation were accelerated in [14].

Mastronardi and Boley [10] suggested Newton's method for the characteristic polynomial χ_n of T which can be enhanced by a double step strategy and/or by Hermitean interpolation of the characteristic polynomial (cf. [9]). The advantage of this method upon Cybenko and Van Loan's approach is its conceptual simplicity, and its monotone convergence from below starting with any lower bound of λ_1 , for instance with the initial guess $\mu_0 = 0$. However, this convergence usually is slower than that of the root finding method mentioned in the last paragraph. In a recent paper Melman [12] proved that the even and odd factors of the characteristic polynomials of a real symmetric Toeplitz matrix (introduced by Delsarte and Genin [3], [4]) can be evaluated recursively. Based on this recurrence and the split Durbin algorithm [11] following the ideas how to exploit symmetry for the secular equation in [14] he defined an algorithm which enhances the method in [10].

The proof of the recurrence relations for the even and odd factors of the characteristic polynomial of T is quite technical. In this note we present a proof which is shorter and more transparent. We do not comment on resulting methods for computing the smallest eigenvalue of T . In particular we do not combine the symmetry exploiting method in [14] and the Newton method for the characteristic equation based on the even-odd recurrence to a hybrid method as was suggested in [7] for the basic methods. Details on numerical methods for computing the smallest eigenvalue of a symmetric Toeplitz matrix will be presented in the forthcoming thesis of the first author [5].

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2. Recurrence for the even–odd characteristic polynomials. Let $T_n = (t_{|i-j|})_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$ be a symmetric, positive definite Toeplitz matrix. We denote by J_n the exchange matrix containing ones in its southwest–northeast diagonal and zeros elsewhere.

If (x, λ) is an eigenpair of T_n then x is called symmetric and λ is called an even eigenvalue, if $x = J_n x$. If $x = -J_n x$, then x is called skew-symmetric and λ is an odd eigenvalue. Cantoni and Butler [1] showed that for every symmetric centrosymmetric matrix (and in particular for every real symmetric Toeplitz matrix) T_n there exists an orthonormal basis consisting of $\lfloor n/2 \rfloor$ skew-symmetric and $\lceil n/2 \rceil$ symmetric eigenvectors of T_n . Correspondingly, we split the eigenvalues of T_n into even and odd ones.

We denote by $\lambda_i^{(j)e}$, $i = 1, \dots, \lfloor j/2 \rfloor$, and $\lambda_i^{(j)o}$, $i = 1, \dots, \lfloor j/2 \rfloor$, the even and odd eigenvalues of the j -th principal submatrix T_j of T_n , respectively. Then the characteristic polynomial χ_j of T_j can be factorized into

$$\chi_j(\lambda) = \chi_j^e(\lambda) \chi_j^o(\lambda), \quad (2.1)$$

where

$$\chi_j^e(\lambda) := \prod_{i=1}^{\lfloor j/2 \rfloor} (\lambda_i^{(j)e} - \lambda) \quad \text{and} \quad \chi_j^o(\lambda) := \prod_{i=1}^{\lfloor j/2 \rfloor} (\lambda_i^{(j)o} - \lambda) \quad (2.2)$$

denote the even and odd characteristic polynomial of T_j .

Even and odd characteristic polynomials for a symmetric Toeplitz matrix were introduced by Delsarte and Genin [3] who proved a recurrence relation which, however, couples the two types and the original characteristic polynomial of principle submatrices. The following recurrence was proved in [12]. We present a different proof which is less technical than the one given in [12].

THEOREM 2.1. *Let $T_j \in \mathbb{R}^{j \times j}$ be a symmetric and positive definite Toeplitz matrix, and assume that λ is not in the spectrum of T_{j-2} . Then the following recurrences hold for the even and odd characteristic polynomials*

$$\chi_j^e(\lambda) = \chi_{j-2}^e(\lambda) (t_0 + t_{j-1} - \lambda - 0.5(\tilde{t}_+^{(j-2)})^T (T_{j-2} - \lambda I_{j-2})^{-1} \tilde{t}_+^{(j-2)}) \quad (2.3)$$

$$\chi_j^o(\lambda) = \chi_{j-2}^o(\lambda) (t_0 - t_{j-1} - \lambda - 0.5(\tilde{t}_-^{(j-2)})^T (T_{j-2} - \lambda I_{j-2})^{-1} \tilde{t}_-^{(j-2)}) \quad (2.4)$$

where $\tilde{t}_\pm^{(j-2)} = t^{(j-2)} \pm J_{j-2} t^{(j-2)}$ and $t^{(j-2)} = (t_1, t_2, \dots, t_{j-2})^T$.

Proof. We first consider the case that the dimension $j = 2m$ is even. T_j can be written as

$$T_j = \begin{pmatrix} T_m & \tilde{H}_m \\ \tilde{H}_m^T & T_m \end{pmatrix} \quad \text{with} \quad \tilde{H}_m = \begin{pmatrix} t_m & t_{m+1} & \dots & t_{j-1} \\ t_{m-1} & t_m & \dots & t_{j-2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \dots & t_m \end{pmatrix}. \quad (2.5)$$

Let $x = \begin{pmatrix} \hat{x} \\ J_m \hat{x} \end{pmatrix}$, $\hat{x} = x(1:m)$, be a symmetric eigenvector of T_j corresponding to λ . Then it holds

$$T_j x = \begin{pmatrix} T_m & \tilde{H}_m \\ \tilde{H}_m^T & T_m \end{pmatrix} \begin{pmatrix} \hat{x} \\ J_m \hat{x} \end{pmatrix} = \begin{pmatrix} (T_m + \tilde{H}_m J_m) \hat{x} \\ (T_m + \tilde{H}_m^T J_m) J_m \hat{x} \end{pmatrix} = \lambda \begin{pmatrix} \hat{x} \\ J_m \hat{x} \end{pmatrix},$$

and the second equation is equivalent to

$$(J_m T_m J_m + J_m \tilde{H}_m^T) \hat{x} = (T_m + \tilde{H}_m J_m) \hat{x} = \lambda \hat{x}.$$

Hence, x is a symmetric eigenvector of T_j if and only if

$$(T_m + \tilde{H}_m J_m) \hat{x} =: (T_m + H_m) \hat{x} = \lambda \hat{x}. \quad (2.6)$$

Thus the even characteristic polynomial of T_j is

$$\begin{aligned} \chi_j^e(\lambda) &= \det \begin{pmatrix} t_0 + t_{j-1} - \lambda & t_1 + t_{j-2} & \cdots & t_{m-1} + t_m \\ t_1 + t_{j-2} & t_0 + t_{j-3} - \lambda & \cdots & t_{m-2} + t_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m-1} + t_m & t_{m-2} + t_{m-1} & \cdots & t_0 + t_1 - \lambda \end{pmatrix} \\ &= \det \begin{pmatrix} t_0 + t_{j-1} - \lambda & (r_1 + J_{m-1} r_2)^T \\ r_1 + J_{m-1} r_2 & T_{m-1} + H_{m-1} - \lambda I_{m-1} \end{pmatrix} \end{aligned} \quad (2.7)$$

where $r_1 = t^{(j-2)}(1 : m-1)$ and $r_2 = t^{(j-2)}(m : j-2)$, from which we obtain by block elimination

$$\chi_j^e(\lambda) = \chi_{j-2}^e(\lambda) (t_0 + t_{j-1} - \lambda - (r_1 + J_{m-1} r_2)^T (T_{m-1} + H_{m-1} - \lambda I_{m-1})^{-1} (r_1 + J_{m-1} r_2)).$$

Hence, the only thing that is left to prove is

$$\begin{aligned} &0.5(\tilde{t}_+^{(j-2)})^T (T_{j-2} - \lambda I_{j-2})^{-1} \tilde{t}_+^{(j-2)} \\ &= (r_1 + J_{m-1} r_2)^T (T_{m-1} + H_{m-1} - \lambda I_{m-1})^{-1} (r_1 + J_{m-1} r_2). \end{aligned} \quad (2.8)$$

From the decomposition (2.5) and

$$\tilde{t}_+^{(j-2)} = t^{(j-2)} + J_{j-2} t^{(j-2)} = \begin{pmatrix} r_1 + J_{m-1} r_2 \\ r_2 + J_{m-1} r_1 \end{pmatrix} = \begin{pmatrix} r_1 + J_{m-1} r_2 \\ J_{m-1} (r_1 + J_{m-1} r_2) \end{pmatrix}$$

we obtain similarly as (2.6)

$$(T_{j-2} - \lambda I_{j-2})^{-1} \tilde{t}_+^{(j-2)} = \begin{pmatrix} (T_{m-1} + H_{m-1} - \lambda I_{m-1})^{-1} (r_1 + J_{m-1} r_2) \\ J_{m-1} (T_{m-1} + H_{m-1} - \lambda I_{m-1})^{-1} (r_1 + J_{m-1} r_2) \end{pmatrix}$$

from which we immediately get (2.8).

Recurrence (2.4) for the odd characteristic polynomial follows in a completely analogous way using $(T_m - H_m) \hat{x} = \lambda \hat{x}$ for a skew-symmetric eigenvector $x = \begin{pmatrix} \hat{x} \\ -J_m \hat{x} \end{pmatrix}$.

If $j = 2m + 1$ is odd, then T_j can be written as

$$T_j = \begin{pmatrix} T_m & J_m \hat{t} & \tilde{D}_m \\ \hat{t}^T J_m & t_0 & \hat{t}^T \\ \tilde{D}_m^T & \hat{t} & T_m \end{pmatrix} \quad \text{with} \quad D_m = \begin{pmatrix} t_{m+1} & t_{m+2} & \cdots & t_{j-1} \\ t_m & t_{m+1} & \cdots & t_{j-2} \\ \vdots & \vdots & \ddots & \vdots \\ t_2 & t_3 & \cdots & t_{m+1} \end{pmatrix} \quad (2.9)$$

where $\hat{t} := (t_1, \dots, t_m)^T$.

If $x = -J_j x$ is a skew-symmetric eigenvector of T_j , then it holds $x = (\hat{x}^T, 0, -\hat{x}^T J_m)^T$ with $\hat{x} = x(1 : m)$, and recurrence (2.4) for the odd characteristic polynomial follows analogously to the case of even dimension j from $(T_m - D_m)\hat{x} = \lambda\hat{x}$ with $D_m = \hat{D}_m J_m$. For the symmetric case this proof does not work since the $(m + 1)$ -th component of a symmetric eigenvector in general is different from 0.

In the determinant defining the characteristic polynomial

$$\chi_j(\lambda) = \det \begin{pmatrix} t_0 - \lambda & (t^{(j-2)})^T & t_{j-1} \\ t^{(j-2)} & T_{j-2} - \lambda I_{j-2} & J_{j-2} t^{(j-2)} \\ t_{j-1} & (t^{(j-2)})^T J_{j-2} & t_0 - \lambda \end{pmatrix}$$

of T_j we add the last column to the first one, and block eliminate the elements 2 to $j - 1$ of the first column. We subtract the first row from the last one, thus having eliminated the last element of the first column, and obtain

$$\chi_j(\lambda) = (t_0 + t_{j-1} - \lambda - (t^{(j-2)})^T (T_{j-2} - \lambda I_{j-2})^{-1} \tilde{t}_+^{(j-2)}) \psi_j(\lambda) \quad (2.10)$$

where

$$\psi_j(\lambda) = \det \begin{pmatrix} T_{j-2} - \lambda I_{j-2} & J_{j-2} t^{(j-2)} \\ -(\tilde{t}_-^{(j-2)})^T & t_0 - t_{j-1} - \lambda \end{pmatrix}. \quad (2.11)$$

Block eliminating the last row of this determinant yields

$$\begin{aligned} \psi_j(\lambda) &= (t_0 - t_{j-1} - \lambda + (t^{(j-2)})^T J_{j-2} (T_{j-2} - \lambda I_{j-2})^{-1} \tilde{t}_-^{(j-2)}) \det(T_{j-2} - \lambda I_{j-2}) \\ &= (t_0 - t_{j-1} - \lambda + (t^{(j-2)})^T J_{j-2} (T_{j-2} - \lambda I_{j-2})^{-1} J_{j-2}^2 \tilde{t}_-^{(j-2)}) \det(T_{j-2} - \lambda I_{j-2}) \\ &= (t_0 - t_{j-1} - \lambda - (t^{(j-2)})^T (T_{j-2} - \lambda I_{j-2})^{-1} \tilde{t}_-^{(j-2)}) \chi_{j-2}^o(\lambda) \chi_{j-2}^e(\lambda), \end{aligned} \quad (2.12)$$

and from the recurrence (2.4) we obtain

$$\psi_j(\lambda) = \chi_j^o(\lambda) \chi_{j-2}^e(\lambda),$$

which together with (2.1) and (2.10) implies the validity of recurrence (2.3). \square

REMARK: The representation (2.7) of the even characteristic polynomial is already contained in [3]. We included its proof to enhance the readability of this note.

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