INITIALIZING ITERATIVE PROJECTION METHODS FOR RATIONAL SYMMETRIC EIGENPROBLEMS

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Abstract. In this paper we consider sparse, symmetric eigenproblems which are rational perturbations of small rank of linear eigenproblems. Problems of this type arise in structural dynamics and in vibrations of fluid–solid structures. Taking advantage of the limit behaviour of certain parameter dependent linear eigenproblems we construct a suitable initial basis for iterative projection methods of Jacobi–Davidson or Arnoldi type.

Key words. rational eigenvalue problem, Jacobi–Davidson method, Arnoldi method, fluid-solid vibrations, vibrations of a plate with attached masses

AMS subject classification. 35P30, 49R50, 65N25, 74H45

1. Introduction. In this contribution we consider the rational eigenvalue problem

$$Kx = \lambda Mx + \sum_{j=1}^{p} \frac{\lambda}{\sigma_j - \lambda} C_j x \tag{1.1}$$

where $K \in \mathbb{R}^{n\times n}$ and $M \in \mathbb{R}^{n\times n}$ are sparse symmetric and positive definite matrices, and $C_j \in \mathbb{R}^{n\times n}$ are symmetric and positive semi-definite matrices of small rank, and $0 < \sigma_1 < \cdots < \sigma_p$ are positive numbers. Problems of this type arise in (finite element models of) vibrations of fluid–solid structures or plates with elastically attached masses (cf. [4]).

In [1] and [9] we discussed iterative projection methods of Jacobi–Davidson and of Arnoldi type for symmetric nonlinear eigenproblems $T(\lambda)x = 0$ which allow minmax characterizations of their eigenvalues. In this case the eigenvalues can be determined one after the other from projections to subspaces of small dimension which are expanded in the course of the algorithm. The projected eigenproblems inherit the symmetry, and its eigenvalues can be determined efficiently by safeguarded iteration. However, to start this method one needs to know a suitable initial space and the number of eigenvalues in the range of the Rayleigh functional.

For the rational eigenproblem (1.1) this is no problem for the eigenvalues in $[0, \sigma_1)$ since the smallest eigenvalue is a first eigenvalue, and the safeguarded iteration converges globally for any initial vector in the domain of definition of the Rayleigh functional. After convergence the method has collected enough information to find the second eigenvalue, and consecutively the further eigenvalues in $[0, \sigma_1)$. The only problem remains that we do not know in advance the number of eigenvalues of problem (1.1) in $[0, \sigma_1)$, and therefore we do not know whether the method has found all eigenvalues in this interval. For intervals $I_j := (\sigma_j, \sigma_{j+1})$, $j = 1, \ldots, p-1$ we neither know a suitable initial search space, nor the number of eigenvalues in I_j a priori. In this paper we take advantage of an embedding of problem (1.1) into a family of linear eigenproblems from which we gain the missing information.

The paper is organized as follows. Section 2 presents two examples yielding rational eigenproblems, and Section 3 summarizes the minmax characterization of

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eigenvalues for nonoverdamped problems and the safeguarded iteration to determine a particular eigenvalue. Section 4 outlines iterative projection methods for symmetric nonlinear eigenproblems, and Section 5 determines the number of eigenvalues in an interval I_j, and presents a suitable initial space for an iterative projection method. The paper closes with a numerical example from vibrations of fluid–solid structures.

2. Examples of rational eigenproblems. In this section we present two examples of rational eigenproblems of type (1.1).

2.1. Plates with elastically attached loads. Consider the flexurable vibrations of an isotropic thin plate the middle surface of which is occupying the plane domain Ω. Denote by $\rho = \rho(x)$ the volume mass density, $D = Ed^3/12(1-\nu^2)$ the flexurable rigidity of the plate, $E = E(x)$ Young's modulus, $\nu = \nu(x)$ the Poisson ratio, and $d = d(x)$ the thickness of the plate at a point $x \in \Omega$. Assume that for $j = 1, \dots, p$ at points $x_j \in \Omega$ masses m_j are joined to the plate by elastic strings with stiffness coefficients k_j. Then the vertical deflection $w(x,t)$ of the plate at a point x at time t and the vertical displacements $\xi_j(t)$ of the load of mass m_j at time t satisfy the following equations

$$Lw(x,t) + \rho d\frac{\partial^2}{\partial t^2}w(x,t) - \sum_{j=1}^{p} m_k \frac{\partial^2}{\partial t^2}\xi_j \delta(x - x_j) = 0, x \in \Omega,\ t > 0 \tag{2.1}$$

$$Bw(x,t) = 0, x \in \partial\Omega,\ t > 0 \tag{2.2}$$

$$m_j \frac{\partial^2}{\partial t^2}\xi_j + k_j(\xi_j(t) - w(x_j,t)) = 0, t > 0,\ j = 1, \dots, p. \tag{2.3}$$

Here B denotes some suitable boundary operator, $\delta(x)$ denotes Dirac's delta distribution, and L the plate operator

$$L = \partial_{11}D(\partial_{11} + \nu\partial_{22}) + \partial_{22}D(\partial_{22} + \nu\partial_{11}) + 2\partial_{12}D(1-\nu)\partial_{12}$$

where $\partial_{ij} = \partial_i\partial_j$ and $\partial_i = \partial/\partial x_i$.

The eigenmodes and eigenfrequencies obtained from the ansatz

$$w(x,t) = u(x)e^{i\omega t} \quad \text{and} \quad \xi_j(t) = c_j e^{i\omega t}$$

satisfy the eigenproblem

$$Lu(x) = \lambda\rho d u + \sum_{j=1}^{p} \frac{\lambda\sigma_j}{\sigma_j - \lambda} m_j \delta(x - x_j)u, x \in \Omega \tag{2.4}$$

$$Bu(x) = 0, x \in \partial\Omega \tag{2.5}$$

where $\lambda = \omega^2$ and $\sigma_j = k_j/m_j$, and discretizing by finite elements yields the rational matrix eigenvalue problem (1.1).

2.2. Fluid–structure interaction. Another rational eigenproblem of type (1.1) is governing free vibrations of a tube bundle immersed in a slightly compressible fluid under the following simplifying assumptions: The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional

effects are neglected, and so the problem can be studied in any transversal section of the cavity. Considering small vibrations of the fluid (and the tubes) around the state of rest, it can also be assumed that the fluid is irrotational.

Mathematically this problem can be described in the following way (cf. [6], [2]). Let $\Omega \subset \mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ. We assume that there exists a family $\Omega_j \neq \emptyset$, $j = 1, \ldots, p$, (the sections of the tubes) of simply connected open sets such that $\bar{\Omega}_j \subset \Omega$ for every j, $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $j \neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With these notations we set $\Omega_0 := \Omega \setminus \bigcup_{j=1}^{K} \bar{\Omega}_j$. Then the boundary of Ω_0 consists of $p+1$ connected components which are Γ and Γ_j, $j = 1, \ldots, p$.

We denote by $H^1(\Omega_0) = \{u \in L^2(\Omega_0) : \nabla u \in L^2(\Omega_0)^2\}$ the standard Sobolev space equipped with the usual scalar product

$$(u, v) := \int_{\Omega_0} (u(x)v(x) + \nabla u(x) \cdot \nabla v(x))\, dx.$$

Then the eigenfrequencies and the eigenmodes of the fluid-solid structure are governed by the following variational eigenvalue problem (cf. [6], [2])

Find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega_0)$ such that for every $v \in H^1(\Omega_0)$

$$c^2 \int_{\Omega_0} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega_0} uv \, dx + \sum_{j=1}^{p} \frac{\lambda \rho_0}{k_j - \lambda m_j} \int_{\Gamma_j} un \, ds \cdot \int_{\Gamma_j} vn \, ds. \tag{2.6}$$

Here u is the potential of the velocity of the fluid, c denotes the speed of sound in the fluid, ρ_0 is the specific density of the fluid, k_j represents the stiffness constant of the spring system supporting tube j, m_j is the mass per unit length of the tube j, and n is the outward unit normal on the boundary of Ω_0. Again, discretizing by finite elements yields a rational eigenproblem (1.1).

3. Minmax characterization for nonoverdamped problems. For $\lambda \in J$ in an open real interval J let $T(\lambda) \in \mathbb{R}^{n \times n}$ be a family of symmetric matrices the element of which are differentiable. We assume that for every $x \in \mathbb{R}^n \setminus \{0\}$ the real equation

$$f(\lambda, x) := x^T T(\lambda) x = 0 \tag{3.1}$$

has at most one solution $\lambda \in J$. Then equation (3.1) defines a functional p on some subset $D \subset \mathbb{R}^n$ which obviously generalizes the Rayleigh quotient for linear pencils $T(\lambda) = \lambda B - A$, and which we call the Rayleigh functional of the nonlinear eigenvalue problem

$$T(\lambda)x = 0. \tag{3.2}$$

We further assume that

$$x^T T'(p(x))x > 0 \quad \text{for every } x \in D \tag{3.3}$$

generalizing the definiteness requirement for linear pencils. By the implicit function theorem D is an open set, and differentiating the identity $x^T T(p(x))x = 0$ one obtains, that the eigenvectors of (3.2) are stationary points of p.

Under the conditions above we proved in [10] a minmax principle for the nonlinear eigenproblem (3.2) if the eigenvalues are enumerated appropriately. $\lambda \in J$ is an eigenvalue of (3.2) if and only if $\mu = 0$ is an eigenvalue of the matrix $T(\lambda)$, and by Poincaré's maxmin principle there exists $m \in \mathbb{N}$ such that

$$0 = \max_{\dim V = m} \min_{x \in V,\ x \neq 0} \frac{x^T T(\lambda) x}{\|x\|^2}.$$

Then we assign this m to λ as its number and call λ an m-th eigenvalue of problem (3.2).

With this enumeration it holds (cf. [10]) that for every $m \in \{1, \ldots, n\}$ problem (3.2) has at most one m-th eigenvalue in J, which can be characterized by

$$\lambda_m = \min_{\dim V = m, D \cap V \neq \emptyset} \sup_{v \in D \cap V} p(v). \tag{3.4}$$

Conversely, if

$$\lambda_m := \inf_{\dim V = m, D \cap V \neq \emptyset} \sup_{v \in D \cap V} p(v) \in J, \tag{3.5}$$

then λ_m is an m-th eigenvalue of (3.2), and the characterization (3.4) holds. The minimum is attained by the invariant subspace of $T(\lambda_m)$ corresponding to its m largest eigenvalues, and the supremum is attained by any eigenvector of $T(\lambda_m)$ corresponding to $\mu = 0$.

To prove this characterization we took advantage of the following relation

$$\lambda \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \lambda_m \quad \Leftrightarrow \quad \mu_m(\lambda) := \max_{\dim V = m} \min_{x \in V,\ x \neq 0} \frac{x^T T(\lambda) x}{\|x\|^2} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0. \tag{3.6}$$

The enumeration of eigenvalues and the fact that the eigenvectors of (3.2) are the stationary vectors of the Rayleigh functional suggests the method in Algorithm 1 called safeguarded iteration for computing the m–th eigenvalue.

Algorithm 1 Safeguarded iteration

1: Start with an approximation σ_1 to the m-th eigenvalue of (3.2)
2: **for** $k = 1, 2, \ldots$ until convergence **do**
3: determine an eigenvector x_k corresponding to the m-largest eigenvalue of $T(\sigma_k)$
4: solve $x_k^T T(\sigma_{k+1}) x_k = 0$ for σ_{k+1}
5: **end for**

The following theorem contains the approximation properties of the safeguarded iteration. It was already proved in [11] but because this technical report is not easily available we repeat its proof here.

THEOREM 3.1.

(i) If $\lambda_1 := \inf_{x \in D} p(x) \in J$ and $\sigma_1 \in p(D)$ then the safeguarded iteration converges globally to λ_1.

(ii) If $\lambda_m \in J$ is a m-th eigenvalue of (3.2) which is simple then the safeguarded iteration converges locally and quadratically to λ_m.

(iii) Let $T(\lambda)$ be twice continuously differentiable, and assume that $T'(\lambda)$ is positive definite for $\lambda \in J$. If x_k in step 3. of Algorithm 1 is chosen to be an eigenvector corresponding to the m largest eigenvalue of the generalized eigenproblem $T(\sigma_k)x = \mu T'(\sigma_k)x$ then the convergence is even cubic.

Proof. (i): Assume that $x_{k-1} \in D$. Then $\sigma_k = p(x_{k-1}) \geq \lambda_1$, and (3.6) yields

$$\mu_1(\sigma_k) = \max_{x \neq 0} \frac{x^T T(\sigma_k) x}{x^T x} = \frac{x_k^T T(\sigma_k) x_k}{x_k^T x_k} \geq 0. \tag{3.7}$$

Suppose that $x_k \notin D$. Then it follows from (3.7) that $x_k^T T(\lambda) x_k > 0$ for every $\lambda \in J$.

Let $\tilde{x} \in D$ be an eigenvector of T corresponding to λ_1. Then we get from (3.3) $\tilde{x}^T T(\lambda)\tilde{x} < 0$ for every $\lambda \in J$, $\lambda < \lambda_1$. Hence for fixed $\lambda \in J$, $\lambda < \lambda_1$

$$q(t) := (\tilde{x} + t(x_k - \tilde{x}))^T T(\lambda)(\tilde{x} + t(x_k - \tilde{x})) = 0$$

has a solution $\tilde{t} \in (0, 1)$, i.e. $w := \tilde{x} + \tilde{t}(x_k - \tilde{x}) \in D$ and $p(w) = \lambda < \lambda_1$ contradicting (3.4).

The monotonicity of $\{\sigma_k\}$ follows directly from the definition of σ_{k+1}, (3.7) and (3.3). Let $\hat{\sigma} := \lim_{k\to\infty} \sigma_k$ and let $\{x_{k_j}\}$ be a convergent subsequence of $\{x_k\}$, $x_{k_j} \to \hat{x} \neq 0$.

Then by the continuity of $T(\lambda)$

$$0 = x_{k_j}^T T(\sigma_{k_j+1}) x_{k_j} \to \hat{x}^T T(\hat{\sigma}) \hat{x},$$

i.e. $\hat{x} \in D$ and $p(\hat{x}) = \hat{\sigma}$, and we get from the continuous dependence of $\mu_1(\sigma)$ on σ

$$T(\hat{\sigma})\hat{x} = \lim_{j\to\infty} T(\sigma_{k_j}) x_{k_j} = \lim_{j\to\infty} \mu_1(\sigma_{k_j}) x_{k_j} = \mu_1(\hat{\sigma})\hat{x}.$$

Multiplying this equation by $\hat{x}^T$ yields $\mu_1(\hat{\sigma}) = 0$, and hence $\hat{\sigma} = \lambda_1$.

(ii): If λ_m is a simple eigenvalue of T then it is an easy consequence of the implicit function theorem that for $|\lambda - \lambda_m|$ small enough the function $\lambda \to x(\lambda)$ is defined and continuously differentiable, where $x(\lambda)$ denotes the suitably normalized eigenvector of $T(\lambda)$ corresponding to the m-largest eigenvalue. Because D is an open set, $h(\lambda) := p(x(\lambda))$ is defined in a neighbourhood of λ_m, and since the eigenvalues of T are the stationary values of p, we get

$$h'(\lambda_m) = p'(x(\lambda_m))x'(\lambda_m)) = 0.$$

This proves the quadratic convergence of $\sigma_k = h(\sigma_{k-1})$ to λ_m.

(iii): Let $T'(\lambda)$ be positive definite and denote by $\mu(\lambda)$ the m-largest eigenvalue of the generalized eigenproblem $T(\lambda)x = \mu T'(\lambda)x$ and by $x(\lambda)$ a corresponding eigenvector which is suitably normalized such that $x(\cdot)$ is continuous. If λ_m is an m-th eigenvalue of $T(\cdot)$ then $\mu(\lambda_m) = 0$, and differentiating $T(\lambda)x(\lambda) = \mu(\lambda)T'(\lambda)x(\lambda)$ yields

$$T'(\lambda_m)x(\lambda_m) + T(\lambda_m)x'(\lambda_m) = \mu'(\lambda_m)T'(\lambda_m)x(\lambda_m).$$

Multiplying by $x(\lambda_m)^T$ from the left we get $\mu'(\lambda_m) = 1$, and therefore

$$T(\lambda_m)x'(\lambda_m) = 0. \tag{3.8}$$

If we define h analogously to part (ii) by $h(\lambda) = p(x(\lambda))$ then as before $h'(\lambda_m) = 0$, and from

$$h''(\lambda_m) = -2\frac{x'(\lambda_m)^T T(p(x(\lambda_m)))x'(\lambda_m)}{x(\lambda_m)^T T'(p(x(\lambda_m)))x(\lambda_m)}$$

and (3.8) it follows $h''(\lambda_m) = 0$, i.e. the safeguarded iteration converges cubically. □

4. Iterative projection methods. For sparse linear eigenvalue problems

$$Ax = \lambda x \tag{4.1}$$

iterative projection methods are very efficient. Here the dimension of the eigenproblem is reduced by projection to a subspace of much smaller dimension, and the reduced problem is handled by a fast technique for dense problems. The subspaces are expanded in the course of the algorithm in an iterative way with the aim that some of the eigenvalues of the reduced matrix become good approximations of some of the wanted eigenvalues of the given large matrix.

Generalizations to nonlinear eigenproblems are discussed in [1], [7], [8] or [9]. A typical example is the nonlinear symmetric Arnoldi method in Algorithm 2. Actually, the underlying idea is not to construct a Krylov space or an Arnoldi recursion, but similarly as in the Jacobi–Davidson method the search space is expanded by a direction which has a high approximation potential for the eigenvector wanted next (namely the improvement by the residual inverse iteration [5]). However, if it is applied to a linear problem then the preconditioned Arnoldi method results, and therefore the approach is called Arnoldi method.

Algorithm 2 Nonlinear Arnoldi Method

start with an initial pole σ and an initial basis V, $V^HV = I$;
determine preconditioner $M \approx T(\sigma)^{-1}$, σ close to first wanted eigenvalue
$k = 1$
while $m \leq$ number of wanted eigenvalues **do**
  compute appropriate eigenvalue μ and corresponding eigenvector y of the projected problem $T_V(\mu)y := V^HT(\mu)Vy = 0$.
  determine Ritz vector $u = Vy$ and residual $r_k = T(\mu)u$
  if $\|r_k\|/\|u\| < \epsilon$ **then**
    PRINT $\lambda_m = \mu$, $x_m = u$,
    if $m ==$ number of wanted eigenvalues **then**
      STOP
    end if
    $m = m + 1$
    if $(k > 1)$ & $(\|r_{k-1}\|/\|r_k\| >$ tol) **then**
      choose new pole σ
      determine new preconditioner $M \approx T(\sigma)^{-1}$
    end if
    restart if necessary
    choose approximations μ and u to next eigenvalue and eigenvector
    determine residual $r = T(\mu)u$
    $k = 0$
  end if
  $v = Mr$
  $v = v - VV^Hv$,$\tilde{v} = v/\|v\|$, $V = [V, \tilde{v}]$
  reorthogonalize if necessary
  update projected problem $T_V(\mu) = V^HT(\mu)V$
  $k = k + 1$
end while

A crucial point in iterative projection methods for general nonlinear eigenvalue

problems when approximating more than one eigenvalue is to inhibit the method to converge to the same eigenvalue repeatedly. In the linear case this is no problem. Krylov subspace solvers construct an orthogonal basis of the ansatz space not aiming at a particular eigenvalue, and one gets approximations to extreme eigenvalues without replication (at least if reorthogonalization is employed). If several eigenvalues are computed by the Jacobi–Davidson method then one determines an incomplete Schur factorization thus preventing the method from approaching an eigenvalue which was already obtained previously (c.f. [3]). For nonlinear problems a similar normal form does not exist.

However, if $T(\lambda)$ is a family of symmetric matrices allowing a minmax characterization of its eigenvalues then the projected problems inherit this property. The eigenvalues can be determined one after the other by safeguarded iteration, and approximating the m-th eigenvalue usually enough information about the next eigenvector is gathered to compute the $(m+1)$-th eigenvalue safely. This approach has the advantage that it is most unlikely that the method converges to an eigenvalue that has already been found previously. The only problem is to initialize this process, i.e. to determine the number of the smallest eigenvalue of the problem in the interval J under consideration and a suitable initial space V, and to make sure that all eigenvalues in J have been found. For the rational eigenproblem (1.1) this will be the subject of the next section.

In the following we comment briefly on some of the other steps. A broader discussion of Algorithm 2 is contained in [9].

2. In our numerical examples we used the LU factorization of $T(\sigma)$ if this could be determined inexpensively and otherwise an incomplete LU factorization, but every other preconditioner is fine.
3. k counts the number of iterations for fixed m. This is only needed to measure the speed of convergence and to decide whether a new preconditioner is recommended in condition 13.
13. The residual inverse iteration with fixed pole σ is known to converge linearly (cf. [5]), and the contraction rate satisfies $\mathcal{O}(|\sigma - \lambda_m|)$. We therefore update the preconditioner if the convergence measured by the quotient of the last two residual norms has become too slow.
    In our numerical examples it happened that the condition in step 7. was fulfilled in the first step after having increased m. In this case the quotient of the last two residual norms does not say anything about the speed of convergence, and we do not update the preconditioner.
17. As the subspaces expand in the course of the algorithm the increasing storage or the computational cost for solving the projected eigenvalue problems may make it necessary to restart the algorithm and purge some of the basis vectors. Since a restart destroys information on the eigenvectors and particularly on the one the method is just aiming at we restart only if an eigenvector has just converged.
23. v is orthogonalized with respect to the current search space V by classical Gram–Schmidt. It may be replaced by modified Gram–Schmidt for stability reasons. Notice, however, that the classical Gram-Schmidt procedure is able to use BLAS3, and thus can be faster than classical Gram–Schmidt by a better use of the cache.

5. Locating eigenvalues of rational eigenproblems. To determine the number of eigenvalues between two consecutive poles σ_k and σ_{k+1} of problem (1.1) we

consider for $\mu \in (\sigma_j, \sigma_{j+1})$ the parameter dependent linear eigenvalue problem

$$\Big(K + \sum_{j=1}^{k} \frac{\mu}{\mu - \sigma_j} B_j\Big)x = \lambda\Big(M + \sum_{j=k+1}^{p} \frac{1}{\sigma_j - \mu} B_j\Big)x. \tag{5.1}$$

We denote by $\lambda_m(\mu)$ the m-smallest eigenvalue. Then $\hat\lambda$ is an eigenvalue of the rational eigenproblem (1.1) if and only if $\hat\lambda$ is a fixed point of $\lambda_m : (\sigma_k, \sigma_{k+1}) \to \mathbb{R}$, and it is easily checked that it is an m-th eigenvalue.

For the Rayleigh quotient $R_\mu(x)$ of problem (5.1) it holds $R_{\mu_1}(x) \geq R_{\mu_2}(x)$ for $\mu_1 \leq \mu_2$ and every $x \neq 0$, and therefore each of the functions $\lambda_m(\cdot)$ is monotonely nonincreasing. Hence, if $N(\mu)$ for $\mu \in (\sigma_k, \sigma_{k+1})$ denotes the number of eigenvalues of problem (5.1) which are less than μ, then for $\sigma_k < \alpha < \beta < \sigma_{k+1}$ the interval $[\alpha, \beta)$ contains $N(\beta) - N(\alpha)$ eigenvalues of the rational problem (1.1), and they are enumerated by $N(\alpha) + 1, N(\alpha) + 2, \ldots, N(\beta)$.

To determine the number of eigenvalues between the poles σ_k and σ_{k+1} we have to study the limit behaviour of the function $\lambda_m(\mu)$ for μ tending to the boundaries of the interval. In [4] we obtained the following results which were even proved for the infinite dimensional case.

LEMMA 5.1.

$$\kappa_m := \lim_{\mu \to \sigma_k +} \lambda_m(\mu)$$

is the m-th eigenvalue of the reduced problem

Find $\lambda \in \mathbb{R}$ and $x \in H_k := \{x \in H : C_k x = 0\}$ such that

$$\Big(K + \sum_{j=1}^{k-1} \frac{\sigma_k}{\sigma_k - \sigma_j} C_j\Big)x = \lambda\Big(M + \sum_{j=k+1}^{p} \frac{1}{\sigma_j - \sigma_k} C_j\Big)x, \tag{5.2}$$

If κ_m is a simple eigenvalue then the eigenvectors $x_m(\mu)$ corresponding to $\lambda_m(\mu)$ (and normalized suitably) converge to an eigenvector corresponding to κ_m

LEMMA 5.2. *Let $r_{k+1} = rank(C_{k+1})$. Then*

$$\lim_{\mu \to \sigma_{k+1} -} \lambda_j(\mu) = 0 \quad \text{for } j = 1, \ldots, r_{k+1}.$$

For $m > r_{k+1}$

$$\lim_{\mu \to \sigma_{k+1} -} \lambda_m(\mu) =: \tilde\kappa_m = \tilde\lambda_{m-r},$$

where $\tilde\lambda_{m-r}$ is the $m - r$ smallest eigenvalue of the reduced problem

Find $\lambda \in \mathbb{R}$ and $x \in H_{k+1} := \{x \in H : C_{k+1} x = 0\}$ such that

$$\Big(K + \sum_{j=1}^{k} \frac{\sigma_{k+1}}{\sigma_{k+1} - \sigma_j} C_j\Big)x = \lambda\Big(M + \sum_{j=k+2}^{p} \frac{1}{\sigma_j - \sigma_{k+1}} C_j\Big)x,$$

These results demonstrate that it is reasonable to call a pole σ_j an m-th eigenvalue of the rational eigenproblem (1.1) if and only if it is the m-smallest eigenvalue of the restricted problem (5.2).

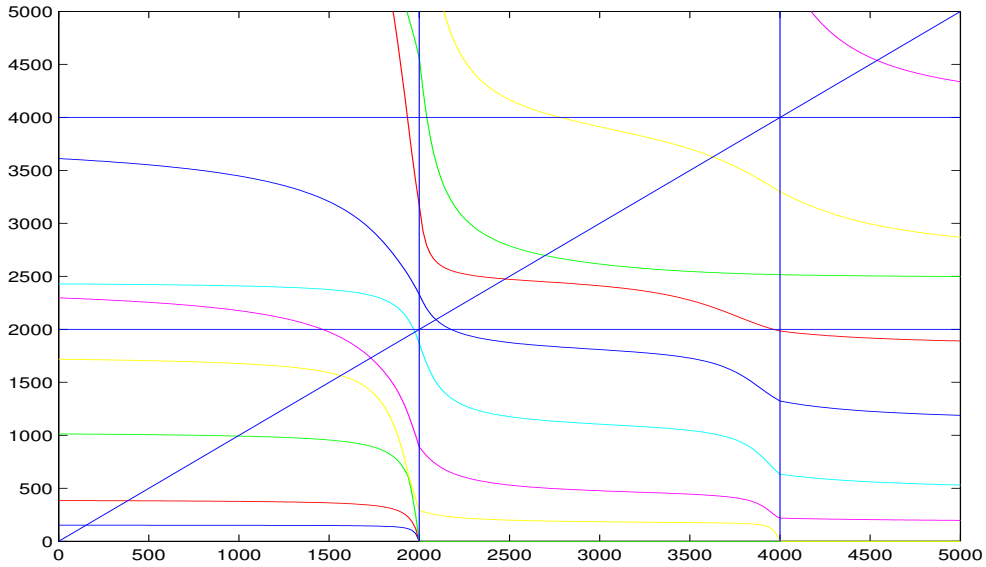

Fig. 1: Eigencurves of (5.1) for a plate with 4 masses

If we connect the $m + r_{k+1}$-th eigencurve in (σ_k, σ_{k+1}) to the m-th eigencurve in $(\sigma_{k+1}, \sigma_{k+2})$ then we obtain a joint curve which is continuous at σ_{k+1}. Figure 1 contains the eigencurves for a plate to which 3 identical masses and one further mass are attached.

For the number of eigenvalue of (1.1) between two consecutive poles we obtain from Lemmas 5.1 and 5.2.

THEOREM 5.3. *Let m_k be the number of eigenvalues λ_j of the reduced problem (5.2) satisfying $\lambda_j \le \sigma_k$, and let r_k be the rank of C_k.*

Then the rational eigenproblem

$$Kx = \lambda\Big(M + \sum_{j=1}^{p} \frac{1}{\sigma_j - \lambda} C_j\Big)x,$$

has $n_{k+1} + r_{k+1} - n_k$ eigenvalues in $(\sigma_k, \sigma_{k+1}]$ enumerated by $n_k + 1, n_k + 2, \dots, n_{k+1} + r_{k+1}$.

Theorem 5.3 answers the question how to initialize the Arnoldi method for the rational eigenproblem (1.1). For the interval $J_1 := (0, \sigma_1)$ the infimum of the Rayleigh functional is contained in J_1, and due to the global convergence of the safeguarded iteration we can start with any one dimensional space V such that $p(V) \in J_1$, and can compute the eigenvalues in J_1 one after the other until the method leaves the interval J_1. Let $\tilde{N}_1$ be the number of the largest eigenvalue $\lambda_{\tilde{N}_1}$ of (1.1) found in J_1.

To start the method for $J_2 := (\sigma_1, \sigma_2)$ we choose $\hat{\mu} = \sigma_1 + \varepsilon$, $\varepsilon > 0$ small, and determine the eigenvalues of the linear problem (5.1) for $\mu = \hat{\mu}$ which are less than $\hat{\mu}$. We assume that these are $\tilde{n}_1$. If $\tilde{n}_1 + r_1 = N_1$ then all eigenvalues in J_1 have been found, no eigenvalue exist in $(\sigma_1, \hat{\mu})$, and we can start the Arnoldi method for J_2 with an orthonormal basis of the eigenspace of problem (5.1) corresponding to the $\tilde{n}_1 + 1$ smallest eigenvalues. Otherwise we have to explore the intervals $(\lambda_{\tilde{N}_1}, \sigma_1)$ and $(\sigma_1, \hat{\mu})$ for further eigenvalues. For the subsequent intervals we can proceed in the analogous way.

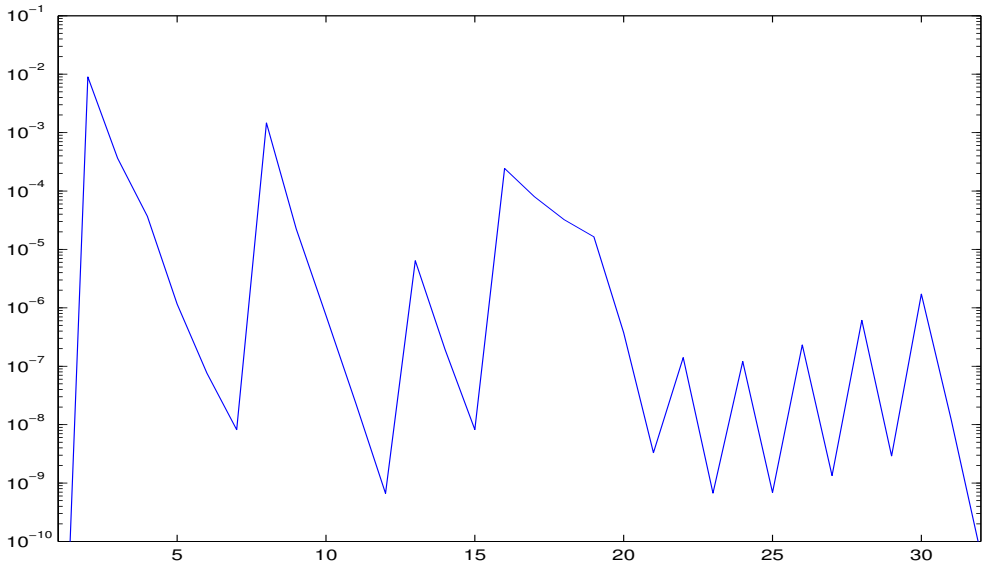

Fig. 2: Eigenvalues in [0,0.2); no restarts

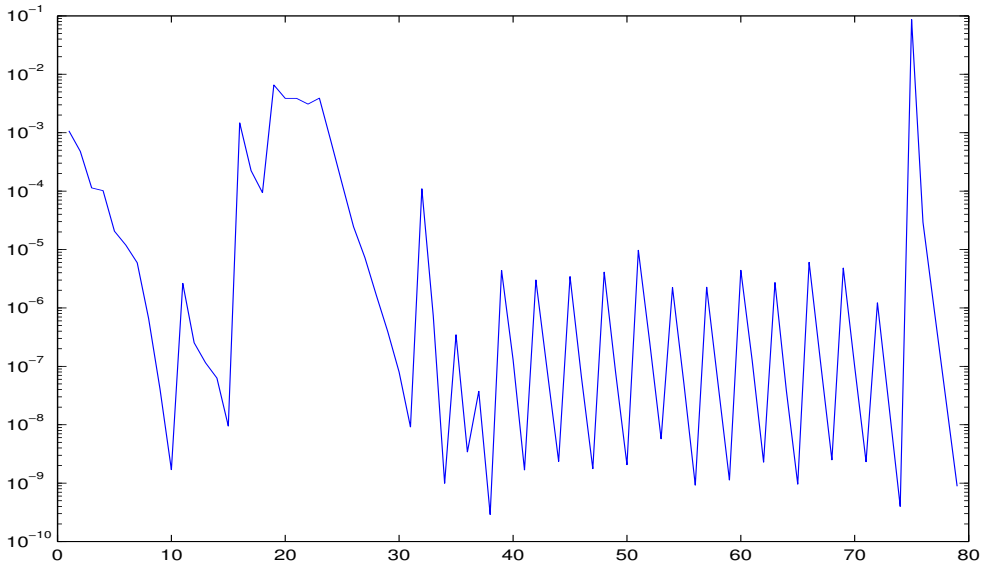

Fig. 3: Eigenvalues in (0.2,1); no restarts

6. Numerical example. We consider the rational eigenvalue problem (2.6) where Ω is the ellipse with center $(0,0)$ and length of semiaxes 8 and 4, and Ω_j, $j = 1,\dots,9$ are circles with radius 0.3 and centers $(-4,-2)$, $(0,-2)$, $(4,-2)$, $(-5,0)$, $(0,0)$, $(5,0)$, $(-4,2)$, $(0,2)$ and $(4,2)$. We assume that $k_j = 1$ for alle j, $m_j = 5$ for j=1,2,3, and $m_1 = 1$ for $j = 4,\dots,9$.

Discretizing problem (2.6) by finite elements one gets a rational matrix eigenvalue problem

$$T(\lambda)x := -Kx + \lambda Mx + \frac{\lambda}{0.2-\lambda}C_1x + \frac{\lambda}{1-\lambda}C_2x = 0 \tag{6.1}$$

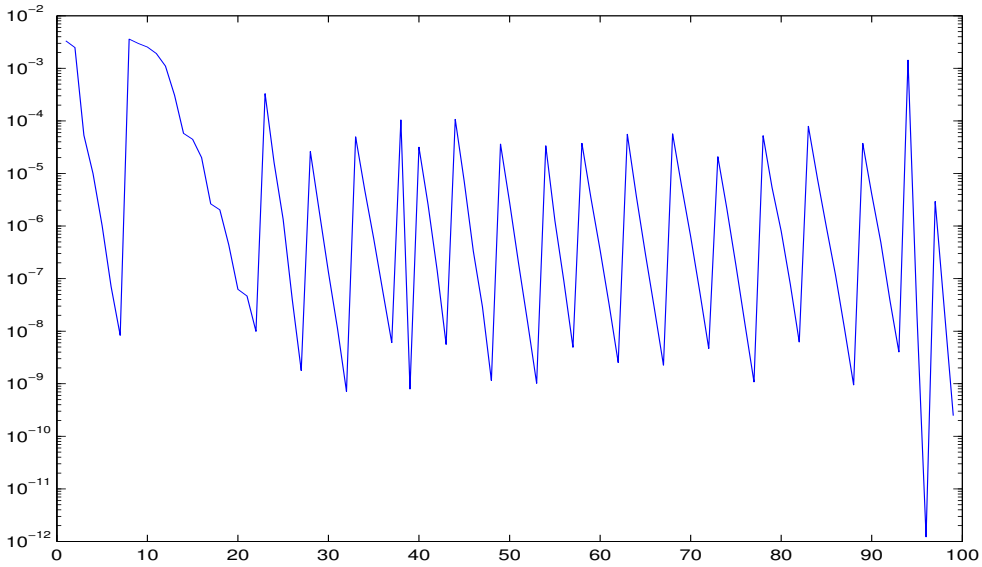

Fig. 4: Eigenvalues in (1,3); no restarts

where C_1 and C_2 collect the contributions of the first 3 and 6 remaining tubes,respectively. K, M, and C_1 and C_2 are symmetric matrices, K, C_2 and C_2 are positive semidefinite, and M is positive definite. In our example the dimension is $n = 36040$.

Since every tube contributing to C_j increases the rank of C_j by 2 we have $r_1 = 6$ and $r_2 = 12$, and since $m_1 = 4$ and $m_2 = 11$, there are 10 eigenvalues in $[0, 0.2)$ enumerated $1, \ldots, 10$ and 19 eigenvalues in $(0.2, 1)$ enumerated $5, 6, \ldots, 23$. The interval $(1, \infty)$ contains 36029 eigenvalues enumerated $12, 13, \ldots, 36040$, 19 of which are contained in the interval $(1, 3)$. The largest eigenvalue $\tilde{\lambda}_{23} = 0.9920$ in $(0.2, 1)$, and the smallest eigenvalue $\hat{\lambda}_{12} = 1.0044$ are very close to the pole $\sigma_2 = 1$, and we had to apply bisection to catch $\tilde{\lambda}_{23}$.

The experiments were run under MATLAB 6.5 on an Intel Centrino M processor with 1.7 GHz and 1 GB RAM. We preconditioned by the LU factorization of $T(\sigma)$, and terminated the iteration if the norm of the residual was less than 10^{-8}. Figures 2. – 4. show the convergence histories of Arnoldi's method for the three intervals $[0, 0.2)$, $(0.2, 1)$ and $(1, 3)$. The CPU times for the Arnoldi method are 30.0 seconds, 77.8 seconds and 94.4 seconds, respectively, including the respective CPU times 0.10, 1.56 and 2.56 seconds to solve all projected nonlinear eigenproblems. The methods needs some initial steps to collect enough information, after that every 3 – 4 iterations it finds the next eigenvalue.

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