LOCATING REAL EIGENVALUES OF A SPECTRAL PROBLEM IN FLUID–SOLID TYPE STRUCTURES

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Exploiting minmax characterizations for nonoverdamped nonlinear eigenvalue problems we prove inclusion theorems for a rational spectral problem governing mechanical vibrations of a tube bundle immersed in a fluid. The fluid is assumed to be viscous and incompressible, and its velocity field and pressure satisfy the steady Stokes equations.

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1 Introduction

In this paper we consider the problem to determine the vibratory eigenfrequencies and eigenmotions of a tube bundle immersed in an incompressible viscous fluid. The fluid is assumed to be contained in a three-dimensional cylindrical cavity with rigid walls. It is assumed that the tubes are parallel to each other and to the longitudinal axis of the cavity, that they are perfectly rigid (i.e. that they do not allow deformations), and that they are elastically mounted such that they can only vibrate transversally, but they can not move in the direction perpendicular to their sections. The cave is assumed to be very long. Due to these assumptions three-dimensional effects can be neglected, and so the problem can be studied in any transversal section of the cavity.

Small vibrations of the fluid and the tubes around the state of rest were modelled by Conca, Duran and Planchard [3], and it was shown that the vibrations are governed by a non-classical eigenvalue problem involving the Stokes system of equations with non-local and nonlinear boundary conditions which model the fluid-solid interaction. Its variational formulation is a rational eigenvalue problem whose coefficients are selfadjoint linear operators acting on a Hilbert space. Reducing this problem to one of determining the characteristic values of a compact (non-selfadjoint) operator it was proved in [3] that there exists a countable set of eigenvalues which converge to infinity. Moreover, it was shown that the number of eigenvalues with nonvanishing imaginary part is finite, that they are all lying in semicircle about the origin in the left half plane. In [2] an upper bound of the number of non-real eigenvalues was provided and upper and lower bounds of the real eigenvalues were stated.

In this paper we take advantage of the selfadjointness of the operators in the rational formulation of the eigenvalue problem, and characterize the eigenvalues outside the semicircle mentioned in the last paragraph as minmax values of a Rayleigh functional p. Comparing p to the Rayleigh quotients of suitable linear eigenvalue problems we derive upper and lower bounds.

A cruical point when applying minmax or maxmin characterizations of eigenvalues for nonoverdamped problems is to enumerate the eigenvalues correctly. The natural ordering to denote the smallest eigenvalue the first one, the second smallest the second one, etc. is inappropriate, but each eigenvalue inherits its number from the location of the singular value 0 in the spectrum of a corresponding linear eigenproblem. Hence, our bounds do not immediately compare to the inclusions of the real eigenvalues stated in [2].

Our paper is organized as follows. Section 2 summarizes the minmax characterization of eigenvalues of nonoverdamped eigenproblems where the eigenparameter appears nonlinearly. Section 3 contains the rational eigenvalue problem governing small vibrations of a tube bundle immersed in an incompressible viscous fluid and collects the results in [3] and [2] on the number and location of the eigenvalues. In Section 4 we derive lower and upper bounds of real eigenvalues. The paper closes with a numerical example demonstrating the sharpness of our bounds. Moreover, it shows that the bounds derived in [2] are false.

2 Characterization of eigenvalues of nonlinear eigenproblems

We consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \tag{1}$$

where $T(\lambda)$ for every λ in an open real interval J is a selfadjoint and bounded operator on a Hilbert space H. As in the linear case $\lambda \in J$ is called an eigenvalue of problem (1) if equation (1) has a nontrivial solution $x \neq 0$. Such an x is called an eigenelement or eigenvector corresponding to λ.

We assume that

$$f : \begin{cases} J \times H & \to \quad \mathbb{R} \\ (\lambda, x) & \mapsto \quad \langle T(\lambda)x, x\rangle \end{cases} \tag{2}$$

is continuously differentiable, and that for every fixed $x \in H^0$, $H^0 := H \setminus \{0\}$, the real equation

$$f(\lambda, x) = 0 \tag{3}$$

has at most one solution in J. Then equation (3) implicitly defines a functional p on some subset D of H^0 which we call the Rayleigh functional.

We assume that

$$\frac{\partial}{\partial\lambda} f(\lambda, x)|_{\lambda=p(x)} > 0 \quad \text{for every } x \in D. \tag{4}$$

Then it follows from the implicit function theorem that D is an open set and that p is continuously differentiable on D.

For the linear eigenvalue value problem $T(\lambda) := \lambda I - A$ where $A : H \to H$ is selfadjoint and continuous the assumptions above are fulfilled, p is the Rayleigh quotient and $D = H^0$. If A additionally is completely continuous then A has a countable set of eigenvalues which can be characterized as minmax and maxmin values of the Rayleigh quotient by the principles of Poincaré and of Courant, Fischer and Weyl (cf. [13]).

For the nonlinear case variational properties using the Rayleigh functional were proved for overdamped systems (i.e. if the Rayleigh functional is defined on H^0) by Duffin [5] and Rogers [8] for the finite dimensional case and by Hadeler [6], [7], Rogers [9], and Werner [14] for the infinite dimensional case. For nonoverdamped systems Werner and the author [12] proved a minmax characterization of Poincaré type, a maxmin characterization generalizing the principle of Courant, Fischer and Weyl is contained in [10]

In this section we assemble the results in [12] and [10] for the nonlinear nonoverdamped eigenvalue problem (1).

We denote by H_j the set of all j-dimensional subspaces of H and by $V_1 := \{v \in V : \|v\| = 1\}$ the unit sphere of the subspace V of H.

We already stressed the fact that the eigenvalues of problem (1) have to be enumerated appropriately to derive variational characterizations for nonoverdamped problems. To this end we assume that for every fixed $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that the linear operator $T(\lambda) + \nu(\lambda)I$ is completely continuous. Then the essential spectrum of $T(\lambda)$ contains only the point $-\nu(\lambda)$, and every eigenvalue $\mu > -\nu(\lambda)$ of $T(\lambda)$ can be characterized as maxmin value of the Rayleigh quotient of $T(\lambda)$. In particular, if λ is an eigenvalue of the nonlinear problem (1), then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $n \in \mathbb{N}$ such that

$$\mu_n(\lambda) := \max_{V \in H_n} \min_{v \in V_1} \langle T(\lambda)v, v\rangle = 0. \tag{5}$$

In this case we call λ an n-th eigenvalue of the nonlinear eigenvalue problem (1).

With this enumeration the following minmax characterization of the eigenvalues of problem (1) holds which was proved in [12].

THEOREM 2.1. *Under the conditions given above the following assertions hold:*

(i) For every $n \in \mathbb{N}$ there is at most one n-th eigenvalue of problem (1) which can be characterized by

$$\lambda_n = \min_{\substack{V \in H_n \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v). \tag{6}$$

The minimum is attained by the invariant subspace W of $T(\lambda_n)$ corresponding to the n largest eigenvalues of $T(\lambda_n)$, and $\sup_{v \in W \cap D} p(v)$ is attained by all eigenvectors of (1) corresponding to λ_n. The set of eigenvalues of (1) is at most countable.

(ii) If

$$\lambda_n = \inf_{\substack{V \in H_n \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v) \in J \tag{7}$$

for some $n \in \mathbb{N}$ then λ_n is the n-th eigenvalue of (1) and (6) holds.

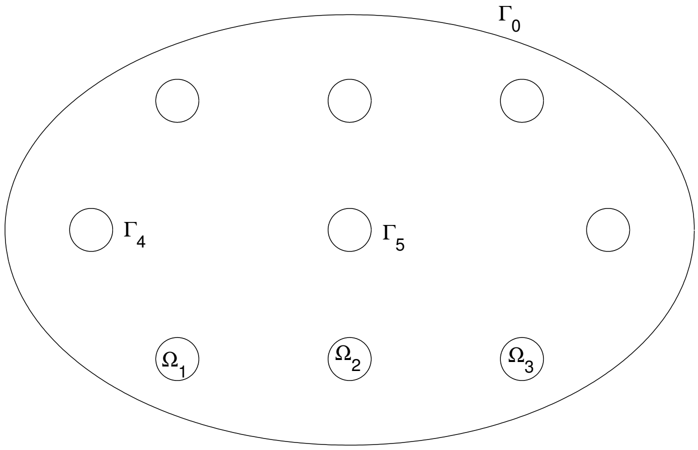

Fig. 1: Domain Ω

The characterization of the eigenvalues in Theorem 2 is a generalization of the minmax principle of Poincaré for linear eigenvalue problems. In a similar way as in [12] the maxmin characterization of Courant, Fischer and Weyl can be generalized to the nonlinear case (cf. [10]).

THEOREM 2.2. *If problem (1) has an n-th eigenvalue $\lambda_n \in J$ then*

$$\lambda_n = \max_{\substack{V \in H_{n-1} \\ V^\perp \cap D \neq \emptyset}} \inf_{v \in V^\perp \cap D} p(v) \tag{8}$$

3 A rational eigenvalue problem in fluid structure interaction

This section is devoted to the presentation of the mathematical model which describes the problem governing free vibrations of a tube bundle immersed in an incompressible viscous fluid whose velocity field and pressure satisfy the steady Stokes equations. The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity.

Considering small vibrations of the fluid (and the tubes) around the state of rest, and assuming that the fluid is viscous and incompressible, this is a non-classical eigenvalue problem involving the Stokes system of equations with nonlinear conditions on the boundaries of the tubes, which model the fluid-solid interaction. On the boundary of the cavity we assume the standard non-slip conditions.

Mathematically, the problem can be described in the following way (cf. [3]): Let $\Omega_0 \subset \mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ_0. We assume that there exists a family $\Omega_j \neq \emptyset$, $j = 1, \dots, K$, (the sections of the tubes) of simply connected open sets such that $\bar{\Omega}_j \subset \Omega_0$ for every j, $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $j \neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With these notations we set $\Omega := \Omega_0 \setminus \bigcup_{j=1}^K \bar{\Omega}_j$. Then the boundary Γ of Ω consists of $K+1$ connected components which are Γ_0 and Γ_j, $j = 1, \dots, K$.

If $\boldsymbol{u}(x)e^{-\omega t}$ is the velocity field of the fluid, $p(x)e^{-\omega t}$ denotes its pressure, and ν its kinematic viscosity then the eigenvalue problem governing the free vibrations of the fluid–solid structure which was derived by Conca, Duran and Planchard [3] obtains the following form

$$-2\nu \operatorname{div} e(\boldsymbol{u}) + \nabla p - \omega \boldsymbol{u} = 0 \quad \text{in } \Omega \tag{9}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega \tag{10}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_0 \tag{11}$$

$$\boldsymbol{u} = \frac{\omega}{k_i + \omega^2 m_i} \int_{\Gamma_i} \sigma(\boldsymbol{u}, p)\boldsymbol{n}\, ds \qquad \text{on } \Gamma_i. \tag{12}$$

Here m_i is the mass per unit length of the i-th tube, and k_i represents the stiffness constant of the spring system supporting the i-th tube. $e(\boldsymbol{u})$ is the linear strain tensor of the fluid defined by

$$2e(\boldsymbol{u}) = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T,$$

and $\sigma(\boldsymbol{u}, p)$ denotes its stress tensor satisfying the Stokes law

$$\sigma(\boldsymbol{u}, p) = -pI + 2\nu e(\boldsymbol{u}). \tag{13}$$

To rewrite problem (9) – (12) in variational form let

$$H^1(\Omega)^2 := \{\boldsymbol{v} \in L^2(\Omega)^2 \,:\, \nabla \boldsymbol{v} \in L^2(\Omega)^4\}$$

be the standard Sobolev space equipped with the usual scalar product. Then clearly

$$H := \{\boldsymbol{v} \in H^1(\Omega)^2 \,:\, \operatorname{div} \boldsymbol{v} = 0,\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0,\ \boldsymbol{v} \text{ constant on each } \Gamma_j,\ j = 1, \dots, K\}$$

is a closed subspace of $H^1(\Omega)^2$.

It is well known from Korn's inequality that the scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \int_\Omega e(\boldsymbol{u}) : e(\bar{\boldsymbol{v}})\, dx := \int_\Omega \sum_{i,j=1}^2 e_{ij}(\boldsymbol{u}) e_{ij}(\bar{\boldsymbol{v}})\, dx$$

defines a norm on H which is equivalent to the standard Sobolev norm. Hence, H equipped with this scalar product is a Hilbert space.

Multiplying equation (9) by $\bar{\boldsymbol{v}} \in H$ and integrating by parts one gets (cf. [3])

Find $\omega \in \mathbb{C}$ and $\boldsymbol{u} \in H$, $\boldsymbol{u} \neq \boldsymbol{0}$ such that for every $\boldsymbol{v} \in H$

$$2\nu \int_\Omega e(\boldsymbol{u}) : e(\bar{\boldsymbol{v}})\, dx = \omega \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\, dx + \sum_{j=1}^K \left(\omega m_j + \frac{k_j}{\omega} \right) \gamma_j(\boldsymbol{u}) \cdot \gamma_j(\bar{\boldsymbol{v}}), \tag{14}$$

where $\gamma_j(\boldsymbol{u})$ denotes the trace of $\boldsymbol{u}$ on Γ_j which by the definition of H is a constant vector. By standard arguments it can be shown that the eigenproblems (9) – (12) and (14) are equivalent in the following sense: If $(\boldsymbol{u}, p, \omega)$ solves the eigenproblem (9) – (12) then $(\boldsymbol{u}, \omega)$ is a solution of (14), and conversely, if $(\boldsymbol{u}, \omega)$ is a solution of (14) then there exists $p \in L^2(\Omega)$ such that $(\boldsymbol{u}, p, \omega)$ solves (9) – (12).

Conca, Duran and Planchard [3] multiplied the rational eigenproblem (14) by ω obtaining a quadratic problem. They proved that the eigenvalues of this problem are the characteristic values of a compact operator acting on a Hilbert space. Hence, they obtained that the set of eigenvalues of problem (14) is countable, and its only cluster point is ∞. Moreover, they proved the following location result.

THEOREM 3.1. *Let $(\omega, \boldsymbol{u})$ be a solution of the rational eigenvalue problem (14). Then the following assertions hold:*

(i) $Re(\omega) > 0$

(ii) If $Im(\omega) \neq 0$ then

$$|\omega|^2 < \frac{k}{m} := \max\left\{\frac{k_j}{m_j} \,:\, j = 1, \dots, k\right\}$$

and

$$Re(\omega) \geq \frac{1}{2}\mu, \ \sum_{j=1}^{K} k_j |\gamma_j(\boldsymbol{u})|^2 > 0,$$

where μ denotes the smallest eigenvalue of the linear eigenproblem:
Find $\mu \in \mathbb{C}$ and $\boldsymbol{v} \in H$, $\boldsymbol{v} \neq \boldsymbol{0}$ such that for every $\boldsymbol{w} \in H$

$$2\nu \int_\Omega e(\boldsymbol{v}) : e(\bar{\boldsymbol{w}})\, dx = \mu \left(\int_\Omega \boldsymbol{v} \cdot \bar{\boldsymbol{w}}\, dx + \sum_{j=1}^{K} m_j \gamma_j(\boldsymbol{v}) \cdot \gamma_j(\bar{\boldsymbol{w}}) \right). \tag{15}$$

From (i) it follows at once that problem (14) has only a finite number of non–real eigenvalues. In [2] Conca, Duran and Planchard proved that the maximum number of non–real eigenvalues is $4K$, and [1] contains a numerical example that demonstrates that this bound is sharp, which is approved by our numerical example in Section 5 as well.

4 Comparison Results

In this section we prove inclusion results for the real eigenvalues $\omega_j > \sqrt{\frac{k}{m}}$ taking advantage of the minmax characterization for these eigenvalues and comparing the Rayleigh functional with Rayleigh quotients R_1 of the linear eigenvalue problem (15) and R_2 of the linear problem:

Find $\omega \in \mathbb{C}$ and $\boldsymbol{v} \in H$ such that for every $\boldsymbol{w} \in H$

$$2\nu \int_\Omega e(\boldsymbol{v}) : e(\bar{\boldsymbol{w}})\, dx = \mu \left(\int_\Omega \boldsymbol{v} \cdot \bar{\boldsymbol{w}}\, dx + \sum_{j=1}^{K} \left(m_j + \frac{m}{k} k_j\right) \gamma_j(\boldsymbol{v}) \cdot \gamma_j(\bar{\boldsymbol{w}}) \right). \tag{16}$$

Problem (14) fulfills the conditions of the minmax theory for the interval $J := (\sqrt{\frac{k}{m}}, \infty)$ since for

$$F(\omega, \boldsymbol{u}) := -2\nu \int\limits_\Omega e(\boldsymbol{u}) : e(\bar{\boldsymbol{u}})\, dx + \omega \int\limits_\Omega |\boldsymbol{u}|^2\, dx + \sum_{j=1}^K (\omega m_j + \frac{k_j}{\omega}) |\gamma_j(\boldsymbol{u})|^2 \qquad (17)$$

we have

$$\frac{\partial}{\partial\omega} F(\omega, \boldsymbol{u}) = \int\limits_\Omega |\boldsymbol{u}|^2\, dx + \sum_{j=1}^K (m_j - \frac{k_j}{\omega^2}) |\gamma_j(\boldsymbol{u})|^2 > 0, \qquad (18)$$

if

$$m_j - \frac{k_j}{\omega^2} > 0 \text{ for every } j, \text{ i.e. } \omega^2 > \max_{j=1,\ldots,K} \frac{k_j}{m_j} = \frac{k}{m}.$$

Hence, all eigenvalues $\omega_j \in J$ of problem (14) can be characterized by

$$\omega_j = \min_{\substack{V \in H_j \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(\boldsymbol{v}). \qquad (19)$$

where the Rayleigh functional p is defined by $F(\omega, \boldsymbol{u}) = 0$, and F is given in (17). By D we denote the domain of definition of p.

LEMMA 4.1. *Let R_1 be the Rayleigh quotient of the linear eigenproblem (15). Then it holds*

$$p(\boldsymbol{u}) \leq R_1(\boldsymbol{u}) \quad \textit{for every } \boldsymbol{u} \in D. \qquad (20)$$

Proof. For every $\boldsymbol{u} \in H$, $\boldsymbol{u} \neq \boldsymbol{0}$ it holds

$$\begin{aligned} & F(R_1(\boldsymbol{u}), \boldsymbol{u}) \\ = & -2\nu \int\limits_\Omega e(\boldsymbol{u}) : e(\bar{\boldsymbol{u}})\, dx + R_1(\boldsymbol{u}) \int\limits_\Omega |\boldsymbol{u}|^2\, dx + \sum_{j=1}^K (R_1(\boldsymbol{u}) m_j + \frac{k_j}{R_1(\boldsymbol{u})}) |\gamma_j(\boldsymbol{u})|^2 \\ = & \frac{1}{R_1(\boldsymbol{u})} \sum_{j=1}^K k_j |\gamma_j(\boldsymbol{u})|^2 \geq 0. \end{aligned}$$

Hence, if $\boldsymbol{u} \in D$, i.e. $F(\omega, \boldsymbol{u}) = 0$ has a solution $p(\boldsymbol{u}) \in J$, then it follows from (4) that $p(\boldsymbol{u}) \leq R_1(\boldsymbol{u})$. □

LEMMA 4.2. *Let R_2 denote the Rayleigh quotient of the linear eigenproblem (16). If $R_2(\boldsymbol{u}) \in J$ for some $\boldsymbol{u} \in H^0$, then $\boldsymbol{u} \in D$, and $p(\boldsymbol{u}) \geq R_2(\boldsymbol{u})$.*

Proof. For $\boldsymbol{u} \in H^0$ such that $R_2(\boldsymbol{u}) > \sqrt{\frac{k}{m}}$

$$\begin{aligned} & F(R_2(\boldsymbol{u}), \boldsymbol{u}) \\ = & -2\nu \int\limits_\Omega e(\boldsymbol{u}) : e(\bar{\boldsymbol{u}})\, dx + R_2(\boldsymbol{u}) \int\limits_\Omega |\boldsymbol{u}|^2\, dx + \sum_{j=1}^K (R_2(\boldsymbol{u}) m_j + \frac{k_j}{R_2(\boldsymbol{u})}) |\gamma_j(\boldsymbol{u})|^2 \\ = & \sum_{j=1}^K \left(\frac{1}{R_2(\boldsymbol{u})} - \frac{m}{k} R_2(\boldsymbol{u}) \right) k_j |\gamma_j(\boldsymbol{u})|^2 < 0, \end{aligned}$$

and

$$\lim_{\omega\to\infty} F(\omega,\boldsymbol{u}) = \infty.$$

Thus, $\boldsymbol{u} \in D$, and $p(\boldsymbol{u}) \geq R_2(\boldsymbol{u})$. □

We are now in the position to proof the inclusion theorem for problem (14).
THEOREM 4.3.

(i) Assume that the j-th eigenvalue

$$\mu_j := \min_{V\in H_j} \max_{u\in V^0} R_2(\boldsymbol{u}) > \frac{k}{m}. \tag{21}$$

of problem (16) is contained in J. Then the nonlinear eigenproblem (14) has a j-th eigenvalue $\omega_j \in J$, and μ_j is a lower bound of ω_j

$$\mu_j \leq \omega_j \tag{22}$$

(ii) If (14) has a j-th eigenvalue $\omega_j \in J$, then

$$\omega_j \leq \eta_j := \min_{V\in H_j} \max_{u\in V^0} R_1(\boldsymbol{u}). \tag{23}$$

Proof. (i): For $V \in H_j$ let $\boldsymbol{u}_V \in V$ such that $R_2(\boldsymbol{u}_V) = \max_{v\in V^0} R_2(\boldsymbol{v})$. Then

$$R_2(\boldsymbol{u}_V) \geq \min_{W\in H_j} \max_{w\in W^0} R_2(\boldsymbol{w}) = \mu_j > \frac{k}{m},$$

and Lemma 4 yields

$$\boldsymbol{u}_V \in D \quad \text{and} \quad p(\boldsymbol{u}_V) \geq R_2(\boldsymbol{u}_V).$$

In particular $V \cap D \neq \emptyset$ for every $V \in H_j$.

Moreover,

$$\begin{aligned} \mu_j &= \min_{V\in H_j} \max_{v\in V^0} R_2(\boldsymbol{v}) = \min_{V\in H_j} R_2(\boldsymbol{u}_V) \\ &\leq \min_{V\in H_j} p(\boldsymbol{u}_V) \leq \min_{V\in H_j} \sup_{u\in V\cap D} p(\boldsymbol{u}). \end{aligned}$$

By Theorem 2, (ii), the nonlinear eigenvalue problem (14) has a j-th eigenvalue ω_j, and $\mu_j \leq \omega_j$.

(ii): Since $V \cap D \neq \emptyset$ for every $V \in H_j$ we obtain from Lemma 4

$$\begin{aligned} \omega_j &= \min_{\substack{V\in H_j \\ V\cap D\neq\emptyset}} \sup_{v\in V\cap D} p(\boldsymbol{v}) \leq \min_{\substack{V\in H_j \\ V\cap D\neq\emptyset}} \sup_{v\in V\cap D} R_1(\boldsymbol{v}) \\ &\leq \min_{\substack{V\in H_j \\ V\cap D\neq\emptyset}} \max_{v\in V^0} R_1(\boldsymbol{v}) = \min_{V\in H_j} \max_{v\in V^0} R_1(\boldsymbol{v}) = \eta_j. \qquad \square \end{aligned}$$

REMARK. Multiplying the nonlinear eigenproblem (14) by ω and considering the resulting quadratic eigenproblem:

Find $\rho := \frac{1}{\omega} \in \mathbb{C}$ *and* $\boldsymbol{v} \in H$ *such that for every* $\boldsymbol{w} \in H$

$$\Big(\int_\Omega \boldsymbol{v}\cdot\bar{\boldsymbol{w}}\,dx + \sum_{j=1}^K m_j\gamma_j(\boldsymbol{v})\cdot\gamma_j(\bar{\boldsymbol{w}})\Big) \;-\; \rho\Big(2\nu\int_\Omega e(\boldsymbol{v}):e(\bar{\boldsymbol{w}})\,dx\Big) + \rho^2\sum_{j=1}^K k_j\gamma_j(\boldsymbol{v})\cdot\gamma_j(\bar{\boldsymbol{w}}) = 0. \qquad (24)$$

as positive perturbation of finite range of the linear eigenproblem (15) Conca, Duran and Planchard claimed the following bounds.

Let $0 < \tilde{\omega}_1 \le \tilde{\omega}_2 \le \dots$ *be the real eigenvalues of the nonlinear eigenproblem (14) ordered by magnitude and regarding their multiplicity, and let* $0 < \eta_1 \le \eta_2 \le \dots$ *be the eigenvalues of the linear problem (15). Then it holds that*

$$\tilde{\omega}_j \le \eta_j, \qquad \text{for} \quad j = 1,\dots,2K \qquad (25)$$

$$\eta_{j-2K} \le \tilde{\omega}_j \le \eta_j, \qquad \text{for} \quad j \ge 2K+1 \qquad (26)$$

where K *denotes the number of tubes.*

We already pointed out in [11] that the natural enumeration to call the smallest eigenvalue the first one, the second smallest the second on, etc. is not appropriate for the quadratic eigenvalue problem (24), and therefore the proof of these bounds is not correct. The numerical example in the next section demonstrates that the bounds (25) and (26) actually do not hold.

For those eigenvalues ω_j contained in J, the bounds (25) and (26) can be adjusted if we replace the natural ordering of the eigenvalues $\tilde{\omega}_j$ by the enumeration introduced in Section 2. The upper bound $\omega_j \le \eta_j$ is already contained in Theorem 4, (ii).

The lower bound is obtained from the maxmin characterization in Theorem 2. Let $W = \mathrm{span}\{\boldsymbol{u}_1,\dots,\boldsymbol{u}_{n-2K-1}\}$ denote the subspace of H spanned by the eigenelements of problem (15) corresponding to the $n-2K-1$ smallest eigenvalues, and let

$$Z = \{\boldsymbol{u}\in H \;:\; \sum_{j=1}^K k_j\gamma_j(\boldsymbol{u})\gamma_j(\bar{\boldsymbol{v}}) = 0 \text{ for every } \boldsymbol{v}\in H\}^\perp.$$

Then obviously $p(\boldsymbol{u}) = R_1(\boldsymbol{u})$ for every $\boldsymbol{u} \in D\cap Z$, and we obtain from Rayleigh's principle and the maxmin characterization in Theorem 2

$$\begin{aligned} \eta_{n-2K} &= \min_{u\in W^\perp} R_1(\boldsymbol{u}) \le \min_{u\in(W+Z)^\perp} R_1(\boldsymbol{u}) \\ &\le \inf_{u\in(W+Z)^\perp\cap D} p(\boldsymbol{u}) \le \max_{\dim V\le n-1}\ \inf_{u\in V^\perp\cap D} p(\boldsymbol{u}) = \omega_n. \end{aligned}$$

5 Numerical Experiments

While the variational form (14) was convenient for the theoretical study of problem (9) – (12) its numerical treatment requires to deal with the incompressibility condition div $\boldsymbol{u} =$

0 implicitly, and to use a mixed variational formulation, which reads (cf. [1], [4])
Find $(\boldsymbol{u}, p, \omega) \in H \times L^2(\Omega) \times \mathbb{C}$, $(\boldsymbol{u}, p) \neq (\boldsymbol{0}, 0)$ *such that for every* $(\boldsymbol{v}, q) \in H \times L^2(\Omega)$

$$2\nu \int_\Omega e(\boldsymbol{u}) : e(\bar{\boldsymbol{v}})\, dx + \int_\Omega p \operatorname{div} \bar{\boldsymbol{v}}\, dx \tag{27}$$

$$= \omega \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\, dx + \sum_{j=1}^{K} \left(\omega m_j + \frac{k_j}{\omega} \right) \gamma_j(\boldsymbol{u}) \cdot \gamma_j(\bar{\boldsymbol{v}}),$$

$$\int_\Omega \bar{q} \operatorname{div} \boldsymbol{u}\, dx = 0. \tag{28}$$

Here H denotes the space

$$H := \{\boldsymbol{v} \in H^1(\Omega)^2 \,:\, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0,\ \boldsymbol{v} \text{ constant on each } \Gamma_j,\ j = 1, \dots, K\}$$

which again is a closed subspace of $H^1(\Omega)^2$.

We discretized this problem by finite elements using piecewise quadratic ansatz functions on a regular triangulation of Ω for the velocity field, and piecewise linear functions on the same triangulation for the pressure yielding a rational matrix eigenvalue problem which can be reduced to a general matrix eigenvalue problem and solved using standard numerical software. The convergence properties of this approach are studied in [4].

We consider problem (27), (28) where $\Omega_0 = (0, 3) \times (0, 3)$ is the section of the cave, and four structures are contained in it with sections $\Omega_1 = (0.8, 1.0) \times (0.8, 1.0)$ $\Omega_2 = (2.0, 2.2) \times (0.8, 1.0)$, $\Omega_3 = (0.8, 1.0) \times (2.0, 2.2)$ and $\Omega_4 = (2.0, 2.2) \times (2.0, 2.2)$. In all experiments we chose $\nu = 1$ and $m := m_j = 1$, $j = 1, 2, 3, 4$, and we assumed that all $k_j =: k$ are identical.

For $k \geq 30.82$ the discrete version of (27), (28) has non-real eigenvalues, and for $k \geq 106.03$ there exist 16 non-real eigenvalues demonstrating that the bound $4K$ on the number of non-real eigenvalues is exact.

For $k = 400$ the smallest real eigenvalue is $\tilde{\omega}_1 = 13.478$, whereas the smallest eigenvalue of (15) is $\eta_1 = 9.671$ demonstrating that (25) does not hold. Finally, $k = 1$ contradicts the lower bound in (26), since $\tilde{\omega}_9 = 9.605$, whereas $\eta_1 = 9.672$.

The following table contains the smallest eigenvalues of the linear problems (16) and (15) which for $m = 1$ and identical k_j are bounds for eigenvalues greater than $\sqrt{k}$. In columns 4 and 5 we added the smallest real eigenvalues of the rational eigenproblem for $k = 1$ and $k = 400$ satisfying $\omega_j > \sqrt{k}$ where these eigenvalues are enumerated in the way introduced in Section 2.

j	μ_j	η_j	$\omega_j(k=1)$	$\omega_j(k=400)$
1	$5.5273441e+00$	$9.6715372e+00$	$9.6051792e+00$	
2	$6.3743399e+00$	$1.1103915e+01$	$1.1055398e+01$	
3	$6.4656339e+00$	$1.2237010e+01$	$1.2164933e+01$	
4	$7.1377728e+00$	$1.2907175e+01$	$1.2899003e+01$	
5	$8.6717113e+00$	$1.3728925e+01$	$1.3663594e+01$	
6	$9.5210984e+00$	$1.4538864e+01$	$1.4516629e+01$	
7	$1.0069363e+01$	$1.5059600e+01$	$1.5026228e+01$	
8	$1.0290363e+01$	$1.5874023e+01$	$1.5872178e+01$	
9	$1.3605487e+01$	$1.7630409e+01$	$1.7588566e+01$	
10	$1.3716557e+01$	$1.9280833e+01$	$1.9246874e+01$	
11	$1.5190485e+01$	$1.9647977e+01$	$1.9604771e+01$	
12	$1.5870909e+01$	$1.9893125e+01$	$1.9848079e+01$	
13	$2.2897653e+01$	$2.3661814e+01$	$2.3655921e+01$	$2.2948292e+01$
14	$3.2789694e+01$	$3.3880279e+01$	$3.3877702e+01$	$3.3216627e+01$
15	$3.5497102e+01$	$3.6107300e+01$	$3.6106314e+01$	$3.5808040e+01$
16	$3.5548471e+01$	$3.6185088e+01$	$3.6183766e+01$	$3.5828882e+01$
17	$3.7479680e+01$	$3.7852470e+01$	$3.7851786e+01$	$3.7663430e+01$
18	$3.7929796e+01$	$3.8388335e+01$	$3.8387384e+01$	$3.8142407e+01$
19	$3.8082108e+01$	$3.8535861e+01$	$3.8535044e+01$	$3.8307897e+01$
20	$4.3651778e+01$	$4.3695701e+01$	$4.3695639e+01$	$4.3677373e+01$
21	$4.9197938e+01$	$4.9379084e+01$	$4.9378915e+01$	$4.9323111e+01$

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