ON THE NUMBER OF EIGENVALUES OF A RATIONAL EIGENPROBLEM

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Abstract. In this paper we determine the number of eigenvalues of a rational eigenvalue problem governing free vibrations of a plate with elastically attached masses or the mechanical vibrations of a fluid-solid structure.

Key words. rational eigenvalue problem, fluid-solid vibrations, vibrations of a plate with attached masses

AMS subject classification. 35P30, 49R50, 65N25, 74H45

1. Introduction.

In this paper we consider the rational eigenvalue problem

$$Ax = \lambda Bx + \sum_{j=1}^{p} \frac{\lambda}{\sigma_j - \lambda} C_j x \tag{1.1}$$

where A, B, C_j are linear, continuous and symmetric operators on a real Hilbert space H. A and B are positive definite, B is completely continuous and the operators C_j are positive semidefinite and have finite dimensional range. The poles σ_j of problem (1.1) are assumed to be positive and ordered by magnitude: $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_p$. Problems of this type govern eigenvibrations of plates with elastically attached loads, and mechanical vibrations of fluid-solid structures, e.g.

In [1], [6], [7] the second author studied iterative projection methods of Jacobi-Davidson and of Arnoldi type for the rational sparse matrix eigenproblem (1.1). These methods determine eigenvalues quite efficiently, however, it was an open question whether all eigenvalues in a given interval (in particular between consecutive poles) had been found or not.

The question can be easily answered for intervals $(\mu_1, \mu_2]$ such that no pole σ_k is contained in $[\mu_1, \mu_2]$ considering the parameter dependent linear eigenproblem

$$Ax + \sum_{i=1}^{j} \frac{\mu}{\mu - \sigma_i} C_i x = \lambda Bx + \sum_{i=j+1}^{p} \frac{\lambda}{\sigma_i - \mu} C_i x. \tag{1.2}$$

If $\lambda_n(\mu)$ denotes the n smallest eigenvalue of (1.2) then $\lambda_n : (\sigma_k, \sigma_{k+1}) \to \mathbb{R}_+$ is a monotonely decreasing and continuous function, and $\hat{\lambda} \in (\sigma_k, \sigma_{k+1})$ is an eigenvalue of the nonlinear eigenproblem (1.1) if and only if it is a fixed point of $\lambda_n(\cdot)$ (and it is an n–th eigenvalue of (1.1) using the enumeration of eigenvalues of nonlinear eigenproblems introduced in [9]). Hence, if $N(\mu) = \max\{j \in \mathbb{N} : \lambda_j(\mu) \le \mu\}$ denotes the number of eigenvalues less than or equal to μ then the interval $(\mu_1, \mu_2]$ contains $N(\mu_2) - N(\mu_1)$ eigenvalues of (1.1).

The question is more involved for the interval $(\sigma_k, \sigma_{k+1}]$ where additionally we have to define what it means that a pole of problem (1.1) is an eigenvalue. In this paper we study the limit behaviour of eigenvalues and eigenvectors of problem (1.2)

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for $\mu \to \sigma_j-$ and $\mu \to \sigma_j+$. In both cases $\lambda_n(\mu)$ converges to 0 or to an eigenvalue of the reduced linear eigenproblem

$$Ax + \sum_{i=1}^{j-1} \frac{\sigma_j}{\sigma_j - \sigma_i} C_i x = \lambda Bx + \sum_{i=j+1}^{p} \frac{\lambda}{\sigma_i - \sigma_j} C_i x, \quad C_j x = 0. \tag{1.3}$$

These results yield that problem (1.1) has exactly $N_k + r_k - N_{k-1}$ eigenvalues in $(\sigma_{k-1}, \sigma_k]$ where N_j denotes the number of eigenvalues of (1.2) in $(0, \sigma_j]$ and r_j is the dimension of the range of C_j.

We will make extensive use of variational characterizations of eigenvalues like

$$\min_{\dim V = j} \max_{v \in V} R(v)$$

where R denotes the Rayleigh quotient of some linear eigenproblem in a Hilbert space H. Then, without explicitly mentioning, V is a j dimensional subspace of H, and the maximum is evaluated on $V \setminus \{0\}$.

The paper is organized as follows. Section 2 introduces a variational form of the rational eigenproblem (1.1) which is more appropriate for our investigation, and recalls how the problems to determine the eigenfrequencies and eigenmodes of a plate with elastically attached loads and of a fluid–solid structure are covered by this problem. In section 3 we study the limit behaviour of the parameter dependent linear eigenproblem (1.2), and derive the formula for the number of eigenvalues of (1.1) in an interval. The paper closes with a numerical example in Section 4.

2. A variational rational eigenvalue problem. Let H be a real, separable Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. We consider the rational eigenvalue problem in its variational form
Find $\lambda \in \mathbb{R}$ and $u \in H$, $u \neq 0$, such that

$$a(u, v) = \lambda b(u, v) + \sum_{j=1}^{p} \frac{\lambda}{\sigma_j - \lambda} c_j(u, v) \ \forall v \in H. \tag{2.1}$$

Here a is an H-elliptic, continuous and symmetric bilinear form, i.e. there exist positive constants α_0 and K_a such that

$$\alpha_0 \|u\|^2 \leq a(u, u), \ |a(u, v)| \leq K_a \|u\| \cdot \|v\| \text{ for every } u, v \in H.$$

b is a symmetric, completely continuous and positive definite bilinear form on H, i.e. there exists K_b such that

$$0 < b(u, u) \text{ for every } u \in H, \ u \neq 0 \text{ and } |b(u, v)| \leq K_b \|u\| \cdot \|v\| \text{ for every } u, v \in H,$$

and if $\{u_n\}, \{v_n\} \subset H$ are weakly convergent sequences such that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ then $b(u_n, v_n) \to b(u, v)$.

Finally, for $j = 1, \dots, p$ the form c_j is symmetric, positive semidefinite, bilinear and of finite rank, i.e.

$$0 \leq c_j(u, u) \text{ for every } u \in H$$

and the codimension of $\{u \in H : c_j(u, v) = 0 \text{ for every } v \in H\}$ is finite. Clearly, c_j is bounded, i.e. there exists K_j such that

$$|c_j(u, v)| \leq K_j \|u\| \cdot \|v\|.$$

It is well known (cf. Weinberger [10]) that the equalities

$$b(u,v) = a(Bu,v),\ c_j(u,v) = a(C_ju,v) \quad \text{for every } u,v \in H$$

define linear operators B and C_j on H which satisfy the conditions in Section 1. Hence, with A the identity on H the variational eigenproblem (2.1) is equivalent to problem (1.1) in the introduction.

Problems of this type are governing eigenvibrations of mechanical structures with elastically attached loads. Consider for example the flexurable vibrations of an isotropic thin plate the middle surface of which is occupying the plane domain Ω. Denote by $\rho = \rho(x)$ the volume mass density, $D = Ed^3/12(1-\nu^2)$ the flexurable rigidity of the plate, $E = E(x)$ Young's modulus, $\nu = \nu(x)$ the Poisson ratio, and $d = d(x)$ the thickness of the plate at a point $x \in \Omega$. Assume that for $j = 1,\dots,p$ at points $x_j \in \Omega$ masses m_j are joined to the plate by elastic strings with stiffness coefficients k_j. Then the vertical deflection $w(x,t)$ of the plate at a point x at time t and the vertical displacements $\xi_j(t)$ of the load of mass m_j at time t satisfy the following equations

$$Lw(x,t) + \rho d w_{tt}(x,t) - \sum_{j=1}^{p} m_k(\xi_j)_{tt}\delta(x-x_j) = 0, x \in \Omega,\ t > 0 \tag{2.2}$$

$$Bw(x,t) = 0, x \in \partial\Omega,\ t > 0 \tag{2.3}$$

$$m_j(\xi_j)_{tt} + k_j(\xi_j(t) - w(x_j,t)) = 0, t > 0,\ j = 1,\dots,p. \tag{2.4}$$

Here B denotes some suitable boundary operator, $\delta(x)$ denotes Dirac's delta distribution, and L the plate operator

$$L = \partial_{11}D(\partial_{11} + \nu\partial_{22}) + \partial_{22}D(\partial_{22} + \nu\partial_{11}) + 2\partial_{12}D(1-\nu)\partial_{12}$$

where $\partial_{ij} = \partial_i\partial_j$ and $\partial_i = \partial/\partial x_i$.

The eigenmodes and eigenfrequencies obtained from the ansatz

$$w(x,t) = u(x)e^{i\omega t} \quad \text{and} \quad \xi_j(t) = c_j e^{i\omega t}$$

satisfy the eigenproblem

$$Lu(x) = \lambda\rho d u + \sum_{j=1}^{p} \frac{\lambda\sigma_j}{\sigma_j - \lambda} m_j \delta(x - x_j)u, x \in \Omega \tag{2.5}$$

$$Bu(x) = 0, x \in \partial\Omega \tag{2.6}$$

where $\lambda = \omega^2$ and $\sigma_j = k_j/m_j$.

Multiplying (2.5) by a test function v and taking advantage of Green's formula the eigenproblem (2.5), (2.6) can be rewritten in its variational form (2.1) where H is the set of all functions u in the Sobolev space $H^2(\Omega)$ satisfying the essential boundary conditions (cf. [2], [5]).

Another problem of type (2.1) is governing free vibrations of a tube bundle immersed in a slightly compressible fluid under the following simplifying assumptions: The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity. Considering small vibrations of the fluid (and the tubes) around the state of rest, it can also be assumed that the fluid is irrotational.

Mathematically this problem can be described in the following way (cf. [4], [3]). Let $\Omega \subset \mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ. We assume that there exists a family $\Omega_j \neq \emptyset$, $j = 1, \dots, p$, (the sections of the tubes) of simply connected open sets such that $\bar{\Omega}_j \subset \Omega$ for every j, $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $j \neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With these notations we set $\Omega_0 := \Omega \setminus \bigcup_{j=1}^{K} \bar{\Omega}_j$. Then the boundary of Ω_0 consists of $p+1$ connected components which are Γ and Γ_j, $j = 1, \dots, p$.

We denote by $H^1(\Omega_0) = \{u \in L^2(\Omega_0) : \nabla u \in L^2(\Omega_0)^2\}$ the standard Sobolev space equipped with the usual scalar product

$$(u, v) := \int_{\Omega_0} (u(x)v(x) + \nabla u(x) \cdot \nabla v(x))\, dx.$$

Then the eigenfrequencies and the eigenmodes of the fluid-solid structure are governed by the following variational eigenvalue problem (cf. [4], [3])

Find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega_0)$ such that for every $v \in H^1(\Omega_0)$

$$c^2 \int_{\Omega_0} \nabla u \cdot \nabla v\, dx = \lambda \int_{\Omega_0} uv\, dx + \sum_{j=1}^{K} \frac{\lambda \rho_0}{k_j - \lambda m_j} \int_{\Gamma_j} un\, ds \cdot \int_{\Gamma_j} vn\, ds. \tag{2.7}$$

Here u is the potential of the velocity of the fluid, c denotes the speed of sound in the fluid, ρ_0 is the specific density of the fluid, k_j represents the stiffness constant of the spring system supporting tube j, m_j is the mass per unit length of the tube j, and n is the outward unit normal on the boundary of Ω_0.

The eigenvalue problem is non–standard in two respects: The eigenparameter λ appears in a rational way in the boundary conditions, and the boundary conditions are nonlocal.

Obviously $\lambda = 0$ is an eigenvalue of (2.7) with eigenfunction $u = \text{const}$. We reduce the eigenproblem (2.7) to the space

$$H := \{u \in H^1(\Omega_0) : \int_{\Omega_0} u(x)\, dx = 0\}$$

and consider the scalar product

$$\langle u, v \rangle := \int_{\Omega_0} \nabla u(x) \cdot \nabla v(x)\, dx.$$

on H which is known to define a norm on H which is equivalent to the norm induced by $(\cdot, \cdot)$.

3. Parameter dependent linear eigenproblems. In this section we consider parameter dependent linear eigenvalue problems. We assume that the poles of problem (2.1) are positive and are ordered by magnitude $0 =: \sigma_0 < \sigma_1 < \cdots < \sigma_p < \sigma_{p+1} := \infty$.

For fixed $\mu \in (\sigma_k, \sigma_{k+1})$ we consider the following linear eigenvalue problem: *Find* $\lambda \in \mathbb{R}$ *and* $u \in H$, $u \neq 0$, *such that*

$$a(u,v) + \sum_{j=1}^{k} \frac{\mu}{\mu - \sigma_j} c_j(u,v) = \lambda\Big(b(u,v) + \sum_{j=k+1}^{p} \frac{1}{\sigma_j - \mu} c_j(u,v)\Big) \ \forall v \in H. \quad (3.1\ \mu)$$

As a limit case of (3.1 μ) we consider as well:
Find $\lambda \in \mathbb{R}$ *and* $u \in H_k$, $u \neq 0$, *such that*

$$a(u,v) + \sum_{j=1}^{k-1} \frac{\sigma_k}{\sigma_k - \sigma_j} c_j(u,v) = \lambda\Big(b(u,v) + \sum_{j=k+1}^{p} \frac{1}{\sigma_j - \sigma_k} c_j(u,v)\Big) \ \forall v \in H_k \quad (3.2\ k)$$

where $H_k := \{u \in H \,:\, C_k u = 0\}$ denotes the kernel of the operator C_k.

Obviously, $\hat{\lambda} \in (\sigma_k, \sigma_{k+1})$ is an eigenvalue of the rational eigenproblem (2.1) if and only if it is a fixed point of the real functions $\lambda_n : (\sigma_k, \sigma_{k+1}) \to (0, \infty)$ where $\lambda_n(\mu)$ denotes the n-smallest eigenvalue of problem (3.1 μ).

In [8], using minmax theory for nonlinear eigenvalue problems, we took advantage of the parameter dependent eigenproblem (3.1 μ), and proved the following existence result for the rational eigenproblem (2.1):

THEOREM 3.1. *Assume that for some* $\mu \in (\sigma_k, \sigma_{k+1})$ *the linear eigenproblem (3.1* μ*) has an eigenvalue* $\lambda(\mu) \in (\sigma_k, \sigma_{k+1})$*. Then the nonlinear problem (2.1) has an eigenvalue* $\hat{\lambda} \in (\sigma_k, \sigma_{k+1})$ *for which the following inclusion holds*

$$\min(\mu, \lambda(\mu)) \leq \hat{\lambda} \leq \max(\mu, \lambda(\mu)).$$

Solov'ëv [5] studied problems (3.1 μ) and (3.2 k) for the plate-string-load eigenproblem (2.5), (2.6) with one mass attached to a clamped plate. In the following we generalize his results to the more general rational eigenproblem (2.1).

One of our main tools will be the following well known characterizations of eigenvalues of variational eigenvalue problems (cf. Weinberger [10]):

THEOREM 3.2. *Let* H *be an infinite dimensional real Hilbert space,* $a(\cdot,\cdot)$ *an* H*-elliptic, symmetric and bounded bilinear form and* $b(\cdot,\cdot)$ *a symmetric, positive definite, completely continuous bilinear form. Then the linear eigenproblem to find* $\lambda \in \mathbb{R}$ *and* $u \in H$, $u \neq 0$, *such that*

$$a(u,v) = \lambda b(u,v) \ \forall v \in H \quad (3.3)$$

has a countable set of eigenvalues λ_j *of finite multiplicity the only accumulation point of which is* ∞.

Assume that the eigenvalues are ordered by magnitude according to their multiplicity

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \to \infty.$$

Then corresponding eigenelements u_j *can be chosen such that*

$$b(u_i, u_j) = \delta_{ij} \quad \text{and} \quad a(u_i, u_j) = \lambda_i \delta_{i,j},$$

and the following characterizations of the eigenvalues hold:

If $R(u) := a(u,u)/b(u,u)$ denotes the Rayleigh quotient of problem (3.3) then

$$\lambda_j = \min\{R(u) \,:\, u \in H \setminus \{0\},\ b(u,u_i) = 0\ \forall i = 1,\dots,j-1\} \tag{3.4}$$
$$= \max\{R(u) \,:\, u \in H \setminus \{0\},\ b(u,u_i) = 0\ \forall i > j\} \tag{3.5}$$
$$= \min_{\dim U = j} \max_{u \in U} R(u) \tag{3.6}$$
$$= \max_{w_1,\dots,w_{j-1} \in H} \min\{R(u) \,:\, b(u,w_i) = 0\ \forall i = 1,\dots,j-1\}. \tag{3.7}$$

(3.4) and (3.5) is called Rayleigh's principle, (3.5) is Poincaré's minmax characterization, and (3.6) is the maxmin characterization of Courant and Fischer.

If the eigenvalues $\lambda_n(\mu)$ of (3.1 μ) are ordered by magnitude according to their multiplicities then the characterizations in Theorem 3.2 yield that each function $\lambda_n : (\sigma_k, \sigma_{k+1}) \to \mathbb{R}$ is continuous and monotonely not increasing. Therefore, we obtain at once

THEOREM 3.3. *Let $\sigma_k < \mu_1 < \mu_2 < \sigma_{k+1}$, and set $N(\mu) = \max\{n \,:\, \lambda_n(\mu) \le \mu\}$. Then the nonlinear eigenvalue problem (1.1) has exactly $N(\mu_2) - N(\mu_1)$ eigenvalues in the interval $(\mu_1, \mu_2]$.*

We now study the limit behaviour of the spectra of (3.1 μ) for μ converging to a pole σ_k. For $\mu \in (\sigma_k, \sigma_{k+1})$ and $u \in H$, $u \ne 0$ we denote by

$$R(u;\mu) := \frac{a(u,u) + \sum_{j=1}^{k} \frac{\mu}{\mu - \sigma_j} c_j(u,u)}{b(u,u) + \sum_{j=k+1}^{p} \frac{1}{\sigma_j - \mu} c_j(u,u)} \tag{3.8}$$

the Rayleigh quotient of problem (3.1 μ), and for $u \in H_k$ by

$$R_k(u) := \frac{a(u,u) + \sum_{j=1}^{k-1} \frac{\sigma_k}{\sigma_k - \sigma_j} c_j(u,u)}{b(u,u) + \sum_{j=k+1}^{p} \frac{1}{\sigma_j - \sigma_k} c_j(u,u)} \tag{3.9}$$

the Rayleigh quotient of problem (3.2 k).

LEMMA 3.4. *$\kappa_n := \lim_{\mu \to \sigma_k+} \lambda_n(\mu)$ is the n-th eigenvalue $\hat\lambda_n$ of problem (3.2 k).*

Proof. For every $u \in H_k$ and $\mu \in (\sigma_k, \sigma_{k+1})$ it holds $R(u;\mu) \le R_k(u)$, and hence

$$\hat\lambda_n = \min_{\dim V = n, V \subset H_k} \max_{u \in V} R_k(u) \ge \min_{\dim V = n, V \subset H_k} \max_{u \in V} R(u;\mu)$$
$$\ge \min_{\dim V = n} \max_{u \in V} R(u;\mu) = \lambda_n(\mu) \quad \text{for every } \mu \in (\sigma_k, \sigma_{k+1}).$$

Therefore, by the monotonicity of $\lambda_n(\mu)$ there exists

$$\kappa_n = \lim_{\mu \to \sigma_k+} \lambda_n(\mu) \le \hat\lambda_n.$$

Let $u_n(\mu)$ be an eigenelement of (3.1 μ) corresponding to $\lambda_n(\mu)$ which is normalized by

$$b(u_n(\mu), u_n(\mu)) + \sum_{j=k+1}^{p} \frac{1}{\sigma_j - \mu} c_j(u_n(\mu), u_n(\mu)) = 1.$$

Then it holds that

$$a(u_n(\mu), u_n(\mu)) + \sum_{j=1}^{k} \frac{\mu}{\mu - \sigma_j} c_j(u_n(\mu), u_n(\mu)) = \lambda_n(\mu),$$

and from the ellipticity of a we obtain

$$\alpha_0\|u_n(\mu)\|^2 \le a(u_n(\mu),u_n(\mu)) \le a(u_n(\mu),u_n(\mu)) + \sum_{j=1}^{k}\frac{\mu}{\mu-\sigma_j}c_j(u_n(\mu),u_n(\mu)) \le \kappa_n.$$

Hence, the set $\{u_n(\mu)\}$ is bounded, and it contains a weakly convergent sequence $\{u_n(\mu_\ell)\}_{\ell=1,2,\dots}$.

Let w_n be the weak limit of this sequence. Then it follows from the complete continuity of b and c_j

$$\begin{aligned} 1 &= b(u_n(\mu_\ell),u_n(\mu_\ell)) + \sum_{j=k+1}^{p}\frac{1}{\sigma_j-\mu_\ell}c_j(u_n(\mu_\ell),u_n(\mu_\ell)) \\ &\to b(w_n,w_n) + \sum_{j=k+1}^{p}\frac{1}{\sigma_j-\sigma_k}c_j(w_n,w_n) = 1, \end{aligned}$$

and $w_n \ne 0$.

Moreover, from

$$0 < \lambda_n(\mu) = a(u_n(\mu),u_n(\mu)) + \sum_{j=1}^{k}\frac{\mu}{\mu-\sigma_j}c_j(u_n(\mu),u_n(\mu)) \le \hat\lambda_n$$

we get

$$\frac{\mu}{\mu-\sigma_k}c_k(u_n(\mu),u_n(\mu)) \le \hat\lambda_n,$$

and for $\mu=\mu_\ell\to\sigma_k$ it follows $c_k(w_n,w_n)=0$, i.e. $w_n \in H_k\setminus\{0\}$.

For every $v\in H_k$ it holds $c_k(u_n(\mu),v)=0$, and therefore

$$a(u_n(\mu),v)+\sum_{j=1}^{k-1}\frac{\mu}{\mu-\sigma_j}c_j(u_n(\mu),v) = \lambda_n(\mu)\Big(b(u_n(\mu),v)+\sum_{j=k+1}^{p}\frac{1}{\sigma_j-\mu}c_j(u_n(\mu),v)\Big),$$

from which we obtain for $\mu=\mu_\ell\to\sigma_k$

$$a(w_n,v)+\sum_{j=1}^{k-1}\frac{\sigma_k}{\sigma_k-\sigma_j}c_j(w_n,v) = \kappa_n\Big(b(w_n,v)+\sum_{j=k+1}^{p}\frac{1}{\sigma_j-\sigma_k}c_j(w_n,v)\Big)$$

for every $v\in H_k$. Thus, κ_n is an eigenvalue of problem (3.2 k), and w_n is a corresponding eigenelement.

The limits w_n and w_m are orthogonal for $n\ne m$ since the sequence $\{\mu_\ell\}$ can be chosen such that $u_n(\mu_\ell)\rightharpoonup w_n$ and $u_m(\mu_\ell)\rightharpoonup w_m$, from which we obtain

$$\begin{aligned} 0 &= b(u_n(\mu_\ell),u_m(\mu_\ell)) + \sum_{j=k+1}^{p}\frac{1}{\sigma_j-\mu_\ell}c_j(u_n(\mu_\ell),u_m(\mu_\ell)) \\ &\to b(w_n,w_m) + \sum_{j=k+1}^{p}\frac{1}{\sigma_j-\sigma_k}c_j(w_n,w_m) = 0. \end{aligned}$$

Hence, the dimension of $W := \text{span}\{w_1, \dots, w_n\}$ is n, and it holds

$$\hat\lambda_n = \min_{\dim V = n, V\subset H_k} \max_{u\in V} R_k(u) \le \max_{w\in W} R_k(w) \le \kappa_n \le \hat\lambda_n$$

which completes the proof. ∎

REMARK. Assume that the sequence $\{\mu_\ell\}$ is chosen such that $\mu_\ell \to \sigma_k+$ and $u_n(\mu_\ell) \rightharpoonup w_n$. Then the convergence of $u_n(\mu_\ell)$ to w_n is even strong. This follows from

$$\begin{aligned}
&\alpha_0\|u_n(\mu) - w_n\|^2 = a(u_n(\mu) - w_n, u_n(\mu) - w_n)\\
&\le a(u_n(\mu) - w_n, u_n(\mu) - w_n) + \sum_{j=1}^{k} \frac{\mu}{\mu-\sigma_j} c_j(u_n(\mu) - w_n, u_n(\mu) - w_n)\\
&= a(u_n(\mu), u_n(\mu)) + \sum_{j=1}^{k} \frac{\mu}{\mu-\sigma_j} c_j(u_n(\mu), u_n(\mu)) + a(w_n, w_n) + \sum_{j=1}^{k} \frac{\mu}{\mu-\sigma_j} c_j(w_n, w_n)\\
&\qquad -2\Big(a(u_n(\mu), w_n) + \sum_{j=1}^{k} \frac{\mu}{\mu-\sigma_j} c_j(u_n(\mu), w_n)\Big).
\end{aligned}$$

and by $c_k(w_n, v) = 0$ for every $v \in H$ and the normalization of $u_n(\mu)$ we may continue this estimate by

$$\begin{aligned}
&= \lambda_n(\mu) + a(w_n, w_n) + \sum_{j=1}^{k-1} \frac{\mu}{\mu-\sigma_j} c_j(w_n, w_n)\\
&\qquad -2\Big(a(u_n(\mu), w_n) + \sum_{j=1}^{k-1} \frac{\mu}{\mu-\sigma_j} c_j(u_n(\mu), w_n)\Big).
\end{aligned}$$

Letting $\mu = \mu_\ell \to \sigma_k+$ we obtain the strong convergence of $u_n(\mu_\ell)$ to w_n.

Moreover, if $\hat\lambda_n$ is a simple eigenvalue then it follows in a standard way that the entire family $u_n(\mu)$ converges to w_n if the sign of $u_n(\mu)$ is chosen appropriately.

LEMMA 3.5. *Let codim $H_{k+1} = r$. Then*

$$\lim_{\mu\to\sigma_{k+1}-} \lambda_j(\mu) = 0 \quad \textit{for } j = 1, \dots, r. \tag{3.10}$$

Proof. Let $W \subset H$ such that $\dim W = r$ and $W \cap H_{k+1} = \{0\}$. Then it holds

$$\lambda_r(\mu) = \min_{\dim V = r} \max_{u\in V} R(u;\mu) \le \max_{u\in W} R(u;\mu). \tag{3.11}$$

Since c_{k+1} is positive definite on W there exists $\gamma > 0$ such that $c_{k+1}(u, u) \ge \gamma\|u\|^2$ from which we obtain for every $u \in W$

$$\begin{aligned}
R(u;\mu) &= \frac{a(u,u) + \sum_{j=1}^k \frac{\mu}{\mu-\sigma_j} c_j(u,u)}{b(u,u) + \sum_{j=k+1}^p \frac{1}{\sigma_j-\mu} c_j(u,u)} \le \frac{a(u,u) + \sum_{j=1}^k \frac{\mu}{\mu-\sigma_j} c_j(u,u)}{\frac{1}{\sigma_{k+1}-\mu} c_{k+1}(u,u)}\\
&\le \frac{K_a\|u\|^2 + \sum_{j=1}^k \frac{\mu}{\mu-\sigma_j} K_j\|u\|^2}{\frac{1}{\sigma_{k+1}-\mu}\gamma\|u\|^2} \le K(\sigma_{k+1} - \mu).
\end{aligned}$$

Thus, (3.11) implies

$$0 \le \lambda_1(\mu) \le \cdots \le \lambda_r(\mu) \to 0 \quad \text{for } \mu \to \sigma_{k+1} - .$$

□

LEMMA 3.6. *Let H_{k+1} have codimension r. Then for $n > r$*

$$\lim_{\mu \to \sigma_{k+1}-} \lambda_n(\mu) =: \tilde{\kappa}_n = \tilde{\lambda}_{n-r}, \tag{3.12}$$

where $\tilde{\lambda}_{n-r}$ is the $n-r$ smallest eigenvalue of (3.2 k+1).

Proof. Let V denote the invariant subspace of (3.1 μ) corresponding to the n smallest eigenvalues. Then Rayleigh's principle yields

$$\begin{aligned}\lambda_n(\mu) &= \max_{u \in V} R(u;\mu) \ge \max_{u \in V \cap H_{k+1}} R(u;\mu) \ge \max_{u \in V \cap H_{k+1}} R_{k+1}(u) \\ &\ge \min_{\dim W = n-r, W \subset H_{k+1}} \max_{u \in W} R_{k+1}(u) = \tilde{\lambda}_{n-r}\end{aligned}$$

since the dimension of $V \cap H_{k+1}$ is at least $n-r$. Hence,

$$\tilde{\kappa}_n \ge \tilde{\lambda}_{n-r}. \tag{3.13}$$

Let $\lambda_n(\mu)$ be the n smallest eigenvalue of (3.1 μ) and $u_n(\mu)$ be a corresponding eigenelement normalized by

$$a(u_n(\mu), u_m(\mu)) + \sum_{j=1}^{k} \frac{\mu}{\mu - \sigma_j} c_j(u_n(\mu), u_m(\mu)) = \delta_{nm} \tag{3.14}$$

such that

$$b(u_n(\mu), u_m(\mu)) + \sum_{j=k+1}^{p} \frac{1}{\sigma_j - \mu} c_j(u_n(\mu), u_m(\mu)) = \frac{1}{\lambda_n(\mu)} \delta_{nm}. \tag{3.15}$$

As in the proof of Lemma 3.4 the normalization (3.14) yields that the set $\{u_n(\mu)\}$ is bounded, and therefore it contains a weakly convergent sequence $u_n(\mu_\ell) \rightharpoonup w_n$. From (3.15) and the complete continuity of b and c_j we obtain

$$\begin{aligned}&\frac{1}{\sigma_{k+1} - \mu} c_{k+1}(u_n(\mu_\ell), u_n(\mu_\ell)) \\ &= \frac{1}{\lambda_n(\mu_\ell)} - b(u_n(\mu_\ell), u_n(\mu_\ell)) - \sum_{j=k+2}^{p} \frac{1}{\sigma_j - \mu} c_j(u_n(\mu_\ell), u_n(\mu_\ell)) \\ &\to \frac{1}{\tilde{\kappa}_n} - b(w_n, w_n) - \sum_{j=k+2}^{p} \frac{1}{\sigma_j - \sigma_{k+1}} c_j(w_n, w_n),\end{aligned} \tag{3.16}$$

and since $\tilde{\kappa}_n \ge \tilde{\lambda}_1 > 0$ for $n > r$, it follows $c_{k+1}(w_n, w_n) = 0$, i.e. $w_n \in H_{k+1}$.

Next we prove that $w_n \ne 0$. To this end we decompose $u_n(\mu) := z_n(\mu) + y_n(\mu)$ where $z_n(\mu) \in H_{k+1}$ and $y_n(\mu) \perp H_{k+1}$. Then $z_n(\mu_\ell) \rightharpoonup w_n$ and $y_n(\mu_\ell) \rightharpoonup 0$, and since $H_{k+1}^{\perp}$ is finite dimensional we even have that $y_n(\mu_\ell)$ converges strongly to 0.

Hence,

$$\begin{aligned}
\frac{1}{\sigma_{k+1}-\mu}c_{k+1}(u_n(\mu),u_n(\mu)) &= \frac{1}{\sigma_{k+1}-\mu}c_{k+1}(u_n(\mu),y_n(\mu))\\
&= \frac{1}{\lambda_n(\mu)}\Big(a(u_n(\mu),y_n(\mu))+\sum_{j=1}^{k}\frac{\mu}{\mu-\sigma_j}c_j(u_n(\mu),y_n(\mu))\Big)\\
&\quad -b(u_n(\mu),y_n(\mu))-\sum_{j=k+2}^{p}\frac{1}{\sigma_j-\mu}c_j(u_n(\lambda),y_n(\lambda))\\
&\le \frac{1}{\lambda_n(\mu)}\Big(K_a\|u_n(\mu)\|\cdot\|y_n(\mu)\|+\sum_{j=1}^{k}\frac{\mu}{\mu-\sigma_j}c_j(u_n(\mu),y_n(\mu))\Big)\\
&\quad -b(u_n(\mu),y_n(\mu))-\sum_{j=k+2}^{p}\frac{1}{\sigma_j-\mu}c_j(u_n(\mu),y_n(\mu)).
\end{aligned}$$

For $\mu=\mu_\ell\to\sigma_{k+1}$ the right hand side converges to 0, and (3.16) yields

$$\frac{1}{\tilde\kappa_n}-b(w_n,w_n)-\sum_{j=k+2}^{p}\frac{1}{\sigma_j-\sigma_{k+1}}c_j(w_n,w_n)=0,$$

i.e. $w_n\ne 0$.

In the same way as in the proof of Lemma 3.4 it follows that $\tilde\kappa_n$ is an eigenvalue of problem (3.2 k+1) and w_n is a corresponding eigenelement, and by (3.13) it remains to show that $\tilde\kappa_n\le\tilde\lambda_{n-r}$.

Assume that there exists $m>r$ such that $\tilde\kappa_m>\tilde\lambda_{m-r}$ where m is chosen minimal, i.e. $\tilde\kappa_n=\tilde\lambda_{n-r}$ for $n=r+1,\dots,m-1$. Let z be an eigenelement of (3.2 k+1) corresponding to $\tilde\lambda_{m-r}$ satisfying $\tilde a(z,z;\sigma_{k+1})=1$ and $\tilde a(z,w_n;\sigma_{k+1})=0$ for $n=r+1,\dots,m-1$ where

$$\tilde a(u,v;\mu)=a(u,v)+\sum_{j=1}^{k}\frac{\mu}{\mu-\sigma_j}c_j(u,v),\ \ \mu\le\sigma_{k+1}.$$

For $\mu<\sigma_{k+1}$ let

$$y(\mu)=z-\sum_{i=1}^{m-1}\tilde a(z,u_i(\mu);\mu)u_i(\mu).$$

Then $\tilde a(y(\mu),u_i(\mu);\mu)=0$ for $i=1,\dots,m-1$, and Rayleigh's principle yields

$$R(y(\mu),\mu)\ge\lambda_m(\mu).$$

From Lemma 3.5 and $z\in H_{k+1}$ we obtain for $\mu\to\sigma_{k+1}$ and $i=1,\dots,r$

$$\tilde a(z,u_i(\mu);\mu)=\lambda_i(\mu)\Big(b(z,u_i(\mu))+\sum_{j=k+2}^{p}\frac{1}{\sigma_j-\mu}c_j(z,u_i(\mu))\Big)\to 0,$$

and for $i=r+1,\dots,m-1$

$$\tilde a(z,u_i(\mu);\mu)\to\tilde a(z,w_i,\sigma_{k+1})=0.$$

Hence, $y(\mu) \to z$ for $\mu \to \sigma_{k+1}$, and we finally obtain

$$\tilde{\lambda}_{m-r} = R_{k+1}(z) = \lim_{\mu \to \sigma_{k+1}} R(y(\mu), \mu) \geq \lim_{\mu \to \sigma_{k+1}} \lambda_m(\mu) = \tilde{\kappa}_m.$$

□

REMARK. Similarly as in the remark following Lemma 3.4 the sequence $u_n(\mu_\ell)$ can be shown to converge strongly to w_n, and again for simple eigenvalues $\tilde{\lambda}_{n-r}$ the entire family converges to w_n.

We are now in the position to determine the number of eigenvalues of the nonlinear eigenproblem (2.1) in an interval $(\sigma_k, \sigma_{k+1}]$ from the spectra of the linear problems (3.2 k) and (3.2 k+1). Lemmas 3.4 and 3.6 indicate that it is natural to call σ_k an eigenvalue of (2.1) if it is an eigenvalue of the linear problem (3.2 k). This is a natural continuation of problem (2.1) into its poles.

THEOREM 3.7. *Let $\lambda_n^{(k)}$ be the eigenvalues of the linear eigenproblem (3.2 k) ordered by magnitude and corresponding to their multiplicity.*

For $k = 1, \dots, p$ let $N_k := \max\{n \in \mathbb{N} : \lambda_n^{(k)} \leq \sigma_k\}$, and $r_k :=$ codim H_k, and set $N_0 := 0$, $N_{p+1} := \infty$, and $r_{k+1} = 0$.

Then the nonlinear eigenvalue problem (2.1) has exactly $N_k + r_k - N_{k-1}$ eigenvalues in the interval $(\sigma_{k-1}, \sigma_k]$ for $k = 1, 2, \dots, p, p+1$.

Proof. The proof follows immediately from the Lemmas 3.4 and 3.6. □

4. An example. Consider the simply supported plate occupying the domain $\Omega = (0,2) \times (0,1)$ with constant coefficients D, ν, ρ and d. We assume that 4 masses are attached to the plate at $x_1 = (0.4, 0.2)$, $x_2 = (1.6, 0.2)$, $x_3 = (0.4, 0.8)$ and $x_4 = (1.6, 0.8)$, where $\sigma_1 = \sigma_2 = \sigma_3 = 2000$ and $\sigma_4 = 4000$, and $m_1 = m_2 = m_3 = m_4 = 10^{-2}$. Then with $D = 1$ and $\rho d = 1$ the governing system obtains the form

$$\Delta^2 u = \lambda u + \frac{20}{2000 - \lambda}(\delta(x - x_1)u + \delta(x - x_2)u + \delta(x - x_3)u) + \frac{40}{4000 - \lambda}\delta(x - x_4)u,$$

$$u(x) = \Delta u(x) = 0, \ x \in \partial\Omega$$

We discretized the eigenproblem by Bogner-Fox-Schmit elements on a quadratic mesh with stepsize $h = 0.05$ which yielded a matrix eigenvalue problem

$$Kx = \lambda Mx + \frac{20\lambda}{2000 - \lambda}C_1 x + \frac{40\lambda}{4000 - \lambda}C_2 x$$

of dimension 3080. Here C_1 is a diagonal matrix of rank 3 which corresponds to the masses m_1, m_2 and m_3, and C_2 is a diagonal matrix of rank 1 corresponding to m_4. Figure 1 shows the eigencurves of the parameter dependent linear eigenproblems (3.1 μ).

The reduced problem

$$Kx = \lambda Mx + \frac{40}{2000 - 1000}C_2 x, \quad C_1 x = 0$$

has $N_1 = 3$ eigenvalues which are less than σ_1, and from $r_1 = 3$ it follows that the nonlinear problem has 6 eigenvalues in $(0, \sigma_1)$.

The reduced problem

$$Kx + \frac{20 \cdot 2000}{2000 - 1000}C_1 x = \lambda Mx, \quad C_2 x = 0$$

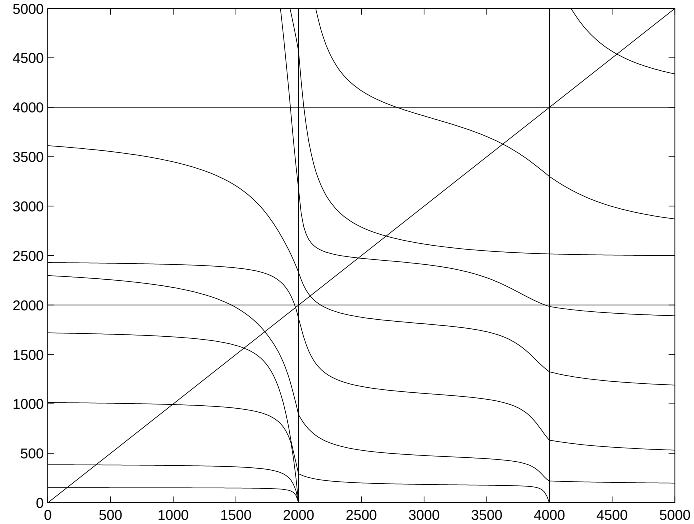

Fig. 1: eigencurves of problem (3.1 μ)

corresponding to σ_2 has $N_2 = 6$ eigenvalues less than σ_2, and from $r_2 = 1$ and $N_1 = 3$ it follows that the nonlinear problem has 4 eigenvalues in (σ_1, σ_2).

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