QUADRATIC EIGENPROBLEMS OF RESTRICTED RANK — REMARKS ON A PAPER OF CONCA, DURAN AND PLANCHARD

HEINRICH VOSS

Abstract

In [2] Conca et al. stated two inclusion theorems for quadratic eigenvalue problems the proof of which are not complete. In this note we demonstrate by simple examples that the assertions as they stand are false. Taking advantage of an appropriate enumeration for eigenvalues of nonlinear eigenproblems we adjust the results.

Keywords. quadratic eigenproblem, inclusion theorem, fluid structure interaction
1991 Mathematics Subject Classification. 35P30

1 Introduction

In [3], [2], [1] Conca, Duran and Planchard studied the vibrations of a solid structure immersed in a viscous incompressible fluid whose velocity field and pressure satisfy the steady Stokes equations. The eigenfrequencies and eigenmodes satisfy a quadratic eigenvalue problem

$$Ru - \lambda u + \lambda^2 Su = 0 \tag{1}$$

where the zero order term R is a bounded selfadjoint positive operator acting on a Hilbert space of infinite dimension, and the second order term S is a selfadjoint operator of finite rank. In [1] it was proved that this problem has a countable sequence of eigenvalues which converge to zero, and only a finite number of eigenvalues can have a nonzero imaginary part. These nonreal eigenvalues play a very important rôle, since they correspond to unstable vibratory eigenmodes.

In [2] Conca et al. proved that the maximum number of nonreal eigenvalues can not exceed twice the rank of S, and [3] contains a numerical example demonstrating that this upper bound is actually attained. The proof is based on the fact that the set of eigenvalues of (1) is countable, and a comparison result for a finite dimensional analogue of (1)

$$Au - \lambda u + \lambda^2 Bu = 0 \tag{2}$$

where A and B are Hermitean $n \times n$-matrices, A is positive definite and B is non-negative with $r := \operatorname{rank} B < n$. Namely, they prove that each of the intervals $[\alpha_j, \alpha_{j+r}]$, $j = 1, \dots, n-r$, contains an eigenvalue λ_j of the quadratic eigenvalue problem (2), where $\alpha_1 \le \alpha_2 \le \dots \le \alpha_n$ denote the eigenvalues of the linear problem $Au = \alpha u$. Moreover, they claim that this λ_j is the j-smallest real eigenvalue of the quadratic eigenproblem (2), and that a corresponding inclusion holds for the infinite dimensional problem (1). However, the proves of these two statements are not complete, and indeed they are false.

In this note we demonstrate by examples that the inclusion theorems for (1) and (2) as stated in [2] are not correct. Taking advantage of an appropriate enumeration of the real eigenvalues of problems (1) and (2) that was exploited in [5] and [4] when proving variational characterizations of eigenvalues of nonlinear nonoverdamped eigenproblems we adjust the statements.

2 Inclusion results for quadratic eigenproblems

2.1 Finite dimensional case

Consider the finite dimensional quadratic eigenproblem

$$Au - \lambda u + \lambda^2 Bu = 0 \tag{3}$$

where A and B are Hermitean $n \times n$-matrices, A is positive definite and B is positive semidefinite with $r := \operatorname{rank} B < n$. Transforming (3) into a linear eigenproblem of dimension $2n$ it is easily seen that there exist exactly $n + r$ eigenvalues (not necessarily distinct) of problem (3).

Conca et al. [2] proved the existence of $n - r$ real eigenvalues of (3) in the following way. Denote by $\alpha_1 \le \alpha_2 \le \dots \le \alpha_n$ the eigenvalues of the linear problem $Au = \alpha u$, and for $\tau \ge 0$ the eigenvalues of $C_\tau := A + \tau B$ by $\mu_1(\tau) \le \mu_2(\tau) \le \dots \le \mu_n(\tau)$. Then for each j, $\mu_j(\tau)$ is a continuous, monotonely non-decreasing function of $\tau \in [0, \infty)$, and by the maxmin characterization of Courant and Fischer the following inclusions hold for every $\tau \ge 0$

$$\alpha_j \le \mu_j(\tau) \le \alpha_{j+r} \text{ for } j = 1, \dots, n-r, \quad \alpha_j \le \mu_j(\tau) \text{ for } j = n-r+1, \dots, n. \tag{4}$$

Since for every fixed $j = 1, \dots, n-r$ the function $\mu_j(\tau)$ is bounded, and $\mu_j(0) = \alpha_j > 0$, it is obvious that there exists $\hat\tau_j > 0$ such that $\mu_j(\hat\tau_j) = \sqrt{\hat\tau_j}$. Hence, if u denotes an eigenvector of $C_{\hat\tau_j}$ corresponding to $\mu_j(\hat\tau_j)$ then

$$(A + \hat\tau_j B)u = (A + \mu_j^2(\hat\tau_j)B)u = \mu_j(\hat\tau_j)u,$$

and $\mu_j(\hat\tau_j)$ is an eigenvalue of the quadratic problem (3) satisfying $\alpha_j \le \mu_j(\hat\tau_j) \le \alpha_{j+r}$.

In particular this proves the existence of $n-r$ real eigenvalues of (3), and thus problem (3) has at most $2r$ nonreal eigenvalues. However, it does not prove that $\lambda_j := \mu_j(\hat\tau_j)$ is the j-smallest eigenvalue of (3) as claimed by Conca et al. [2]. The following example demonstrates that this statement actually is false.

Example 1: Let

$$A = \text{diag}\,\{\frac{2}{3}, 3, 4\} \quad \text{and} \quad B = \text{diag}\,\{\frac{1}{3}, 0, 0\},$$

then $n = 3$, $r = 1$. The eigenvalues of problem (3) ordered by magnitude are $\lambda_j = j$, $j = 1, 2, 3, 4$, and the eigenvalues of $Au = \alpha u$ are $\alpha_1 = \frac{2}{3}$, $\alpha_2 = 3$ and $\alpha_3 = 4$. Hence, the inequality $3 = \alpha_2 \le \lambda_2 \le \alpha_3 = 4$ is violated.

The statement can be adjusted by employing an appropriate enumeration of the eigenvalues of (3). Every real eigenvalue $\lambda \in \mathbb{R}$ is a real eigenvalue of the linear Hermitean eigenproblem

$$(A + \lambda^2 B)u = \mu u \tag{5}$$

the eigenvalues of which are all real.

Definition: A real eigenvalue $\lambda \in \mathbb{R}$ of (3) is called a k-th eigenvalue, if λ is the k-smallest eigenvalue of the linear problem (5).

Theorem 1: *Let $\lambda \in \mathbb{R}$ be a k-th eigenvalue of the quadratic eigenproblem (3), and let $\alpha_1 \le \alpha_2 \le \ldots \le \alpha_n$ be the eigenvalues of the linear problem $Au = \alpha u$. Then it holds*

$$\alpha_k \le \lambda \le \alpha_{k+r} \quad (\textit{where } \alpha_m = \infty \textit{ for } m > n). \tag{6}$$

Proof: Since λ is the k-smallest eigenvalue of (5) the minmax characterization of Poincaré yields the existence of a k dimensional subspace W of $\mathbb{C}^n$ such that

$$\lambda = \min_{V \subset \mathbb{C}^n,\ \dim V = k} \max_{x \in V} \frac{x^H(A + \lambda^2 B)x}{x^H x} = \max_{x \in W} \frac{x^H(A + \lambda^2 B)x}{x^H x}.$$

Therefore

$$\alpha_k = \min_{V \subset \mathbb{C}^n,\ \dim V = k} \max_{x \in V} \frac{x^H A x}{x^H x} \le \max_{x \in W} \frac{x^H A x}{x^H x} \le \max_{x \in W} \frac{x^H(A + \lambda^2 B)x}{x^H x} = \lambda,$$

which proves the left inequality of (6).

Let $u_1, \ldots, u_{k-1}$ be an orthonormal system of eigenvectors of (5) corresponding to the $(k-1)$ smallest eigenvalues. Then it follows from the maxmin characterization of Courant and Fischer for $k + r \le n$

$$\begin{aligned}
\alpha_{k+r} &= \max_{\dim V \le k+r-1} \min \left\{ \frac{x^H A x}{x^H x} \ :\ x^H y = 0 \text{ for every } y \in V,\ x \ne 0 \right\} \\
&\ge \min \left\{ \frac{x^H A x}{x^H x} \ :\ x^H u_j = 0,\ j = 1, \ldots, k-1,\ Bx = 0,\ x \ne 0 \right\} \\
&= \min \left\{ \frac{x^H (A + \lambda^2 B) x}{x^H x} \ :\ x^H u_j = 0,\ j = 1, \ldots, k-1,\ Bx = 0,\ x \ne 0 \right\} \\
&\ge \min \left\{ \frac{x^H (A + \lambda^2 B) x}{x^H x} \ :\ x^H u_j = 0,\ j = 1, \ldots, k-1,\ x \ne 0 \right\} = \lambda.
\end{aligned}$$

In Example 1 we have $A - \lambda_1 I + \lambda_1^2 B = \text{diag}\,\{0, 2, 3\}$, $A - \lambda_2 I + \lambda_2^2 B = \text{diag}\,\{0, 1, 2\}$, $A - \lambda_3 I + \lambda_3^2 B = \text{diag}\,\{2/3, 0, 1\}$ and $A - \lambda_4 I + \lambda_4^2 B = \text{diag}\,\{2, -1, 0\}$. Hence, λ_1, λ_2 and λ_3 are first eigenvalues, and λ_4 is a second eigenvalue, and in correspondence with Theorem 1 it holds $\alpha_1 \le \lambda_j \le \alpha_2$ for $j = 1, 2, 3$ and $\alpha_2 \le \lambda_4 \le \alpha_3$.

2.2 Infinite dimensional case

Let $\mathcal{H}$ be a separable Hilbert space of infinite dimension. Assume that $R : \mathcal{H} \to \mathcal{H}$ is compact, selfadjoint and positive definite with eigenvalues $\xi_1 \geq \xi_2 \geq \ldots > 0$, and let $S : \mathcal{H} \to \mathcal{H}$ be a selfadjoint, continuous and non-negative operator of finite rank r. For the quadratic eigenvalue problem

$$Ru - \lambda u + \lambda^2 Su = 0 \tag{7}$$

Conca et al. [2] in a similar way as for the finite dimensional case proved that problem (7) admits at most $2r$ nonreal eigenvalues, and that for every j there exists an eigenvalue λ_j of (7) such that

$$\xi_j \leq \lambda_j \leq \xi_{j-r} \quad (\text{where } \xi_m = \infty \text{ for } m \leq 0).$$

Moreover, they claim that λ_j is the j-largest real eigenvalue of problem (7). Example 2 demonstrates that this statement is false, too.

Example 2: Let $u_1, u_2, \ldots$ be a Hilbert basis of $\mathcal{H}$, let R and S be defined by

$$R(u_j) = \frac{1}{j^2} u_j,\ j \in \mathbb{N}, \quad \text{and} \quad S(u_j) = \begin{cases} \frac{3}{16} u_1 & \text{for } j = 1 \\ 0 & \text{for } j > 1 \end{cases}.$$

Then $\xi_j = \frac{1}{j^2}$, and $\lambda_1 = 4$, $\lambda_2 = \frac{4}{3}$ and $\lambda_j = \frac{1}{(j-1)^2}$ for $j \geq 3$, and the inequality $\xi_2 \leq \lambda_1 \leq \xi_1$ is violated.

As in the finite dimensional case the statement can be adjusted if the number of a real eigenvalue λ of (7) is inherited from its number as an eigenvalue of the linear problem $(R + \lambda^2 S)u = \mu u$. Namely, $\lambda \in \mathbb{R}$ is defined to be a k-th eigenvalue of the quadratic problem (7) if

$$\lambda = \max_{V \subset \mathcal{H},\ \dim V = k} \min_{x \in V} \frac{\langle (R + \lambda^2 S)x, x \rangle}{\|x\|^2}.$$

With this enumeration the following inclusion theorem can be proved analogously to Theorem 1.

Theorem 2: *If λ is a k-th eigenvalue of the quadratic problem (7) then it holds*

$$\xi_k \leq \lambda \leq \xi_{k-r} \quad (\textit{where } \xi_m = \infty \textit{ for } m \leq 0).$$

References

[1] C. Conca, M. Duran and J. Planchard, A quadratic eigenvalue problem involving Stokes equation. Comput. Meth. Appl. Mech. Engrg. **100**, 295 – 313 (1992)

[2] C. Conca, M. Duran and J. Planchard, A bound for the number of non real solutions of a quadratic eigenvalue problem. Adv. Math. Sci. Appl. **1**, 229 – 249 (1992)

[3] C. Conca and M. Duran, A numerical study of a spectral problem in solid-fluid type structures. Numer. Meth. Part. Diff. Equ. **11**, 423 – 444 (1995)

[4] H. Voss, A Maxmin Principle for Nonlinear Eigenvalue Problems with Application to a Rational Spectral Problem in Fluid–Solid Vibration. Techn. Rep. **48**, Dept. Math., TU Hamburg-Harburg 2002. Submitted to Applications of Mathematics

[5] H. Voss and B. Werner, A Minimax Principle for Nonlinear Eigenvalue Problems with Applications to Nonoverdamped Systems. Math. Meth. Appl. Sci. **4**, 415 – 424 (1982)

Address of author: Heinrich Voss, Department of Mathematics, Technical University of Hamburg – Harburg, D–21071 Hamburg, Germany
E-mail address: voss@tu-harburg.de, URL: www.tu-harburg.de/mat/hp/voss