On Spectral Theory, Control, and Higher Regularity of Infinite-dimensional Operator Equations

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Fabian Nuraddin Alexander Gabel

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1. Gutachter: PD Dr. habil. Christian Seifert
2. Gutachter: Prof. Dr. Felix L. Schwenninger

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doi:10.15480/882.5197

ORCiD: https://orcid.org/0000-0002-8053-0284

Summary

This work studies different physically motivated mathematical models ranging from quantum physics and abstract non-autonomous Cauchy problems to mathematical fluid mechanics.

In the first part, we study the approximation theory of discrete periodic Schrödinger operators and the connections to spectral theory. In particular, we will study the applicability of the finite section method (FSM). This method aims to solve infinite-dimensional operator equations via a projection-based truncation technique. Under additional assumptions on the range of the potential of the Schrödinger operator and the period length, we show that the FSM is applicable if and only if the monodromy matrix of the Schrödinger operator fulfills a trace condition. The applicability analysis of the finite section method demands a study of the invertibility of Schrödinger operators that have been restricted to the half-line and carry a Dirichlet boundary condition. For these operators, we give a set of conditions in order to verify the operators' invertibility. These conditions allow for implementation in a computer algebra system in order to carry out a complete study of all periodic Schrödinger operators with $\{0, \lambda\}$-valued potentials. In the course of this algorithmic study, we also provide counterexamples to demonstrate the optimality of our results.

In the second part of this work, we study the control and observability theory of Banach-space-valued differential equations. These equations are often denominated "abstract Cauchy problems" and represent a generalization of ordinary differential equations to the framework of operators on Banach spaces. This general viewpoint allows to interpret partial differential equations as operator equations in infinite-dimensional vector spaces and to study their solution theory employing operator theory. If the Cauchy problem features not a single operator but a family of operators that changes as time progresses, we speak of a non-autonomous Cauchy problem. In this class of non-autonomous Cauchy problems, we study observation systems consisting of observation families that additionally depend on time. For this class of systems, we derive an abstract theorem to assure final-state observability. To this end, we extend the classical Lebeau–Robbiano strategy to the non-autonomous setting by providing suitable counterparts of an abstract uncertainty principle and dissipation estimate. We also study the concrete application of this result to non-autonomous diffusion problems induced by families of strongly elliptic operators. These operators are often found in the mathematical description of models featuring gradient-induced transport, such as diffusion. For the class of non-autonomous diffusion equations, we prove necessary and sufficient geometric conditions on the family of observation sets.

The last part of this work studies a fundamental model of mathematical fluid mechanics: the Navier–Stokes equations. We study this system of partial differential equations on planar Lipschitz domains intending to prove higher regularity of so-called weak Leray–Hopf solutions in L^p-spaces and spaces of distributions. Our approach solves this problem by thoroughly analyzing the linear part of the Navier–Stokes equations represented by the Stokes operator. To this end, we develop a functional analytic framework for the Stokes operator on Lebesgue spaces on bounded planar Lipschitz domains. In particular, we prove that the Stokes operator is $\mathcal{R}$-sectorial on L^p_σ in order to conclude its maximal L^q-regularity. Furthermore, we show that the Stokes operator on bounded Lipschitz domains has a bounded H^∞-calculus. Building on these results, we derive a characterization of the domains of fractional powers of the Stokes operator in terms of Bessel-potential spaces.

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Dedicated to my parents Mina and Martin Gabel.

Chapter 1

Introduction

Describing aspects of physical phenomena by forming abstract mathematical models is a common practice in scientific work: the mathematical formalism allows the permeation of the mathematical model as a means of creating insights and knowledge over the described real-world phenomenon. The balancing act of mathematical modeling is to find a level of complexity of the model that, on the one hand, makes it possible to analyze the model with mathematical rigor and, on the other hand, does not compromise too much of the expected real-world behavior. Throughout this thesis, we will visit different physically motivated mathematical models exhibiting different degrees of complexity and study their mathematical properties. In particular, we will deal with discrete and continuous models, time-dependent and time-independent models, linear and non-linear models.

Discrete Stationary Models

In the first part of this thesis, we deal with *discrete Schrödinger operators.* These operators are used to model physical systems on lattices and, therefore, play an important role in theoretical solid-state physics. At its core, one considers a so-called *Hamiltonian*

$$H = \Delta + V$$

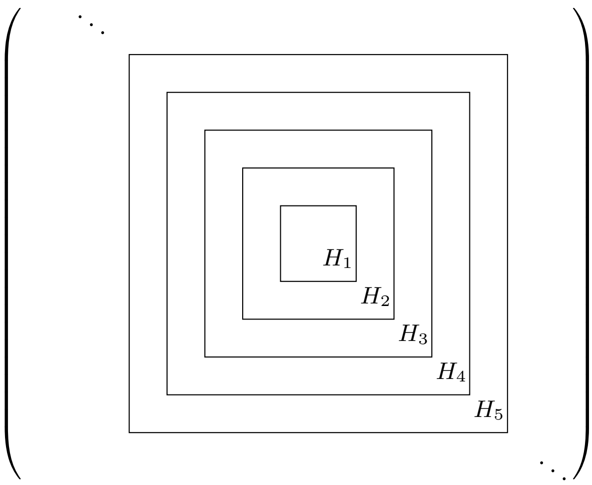
modeling the kinetic and potential energy of a single particle via the *discrete Laplacian* Δ and the *potential* V. We focus on *periodic* potentials V which describe a periodic *tight-binding model.* Periodicity is a phenomenon often encountered in materials with a translational symmetry which is usually found in crystalline structures like metals. Here, the spectrum of H is of particular interest as it allows to conclude the existence or absence of specific energy levels and corresponding eigenfunctions of the quantum physical system. As our model does not feature a time dependence, these eigenfunctions represent stationary equilibrium states that a time-dependent version of the system may attain after a long time.

The spectrum of H is also profoundly linked to the numerical analysis and approximation theory of finite-difference operators. More concretely, the discrete Hamiltonian H of the above tight-binding model falls into the large category of band operators for which a powerful approximation method exists: the *finite section method* (FSM). This method allows to study the approximate solvability of infinite-dimensional linear equations of the form

$$Hx = b$$

for $x, b \in \ell^p(\mathbb{Z})$ and asks if a suitably constructed sequence of finite-dimensional approximating equations $H_n x_n = b_n$ yields a well-defined sequence of approximate solutions x_n to the original problem. In this case, one calls the approximation method *applicable* to H. The following graphic illustrates the approximation process for the matrix representation of the Hamiltonian H.

$H =$



Our goal is to establish a link between properties of the potential V and the approximation theory of H. More precisely, we will derive a novel variety of subclasses of operators H with periodic potential V, guaranteeing the applicability of the FSM given that H is invertible. Furthermore, we demonstrate that the presented classes of potentials are sharp by providing counterexamples. The contents of this chapter appeared in the article [54], the preprint [53] and used the data and code repositories [52, 55].

Non-autonomous Linear Models

The second part of this thesis takes us from the discrete world to the continuous world and from time-independent models to models that feature evolution and time-dependent model parameters. More concretely, we will study *parabolic equations*—a generalized model for diffusion processes. *Diffusion* occurs with the distribution of heat in solid media or the concentration of substances when gradients of concentration and not convection dominate the motion. At the heart of our model of interest lies a family of elliptic differential operators

$$A(t) = \sum_{|\alpha| \leq m} a_\alpha(t) \partial_x^\alpha \cdot, \quad t \in I,$$

of order $m \geq 2$ parametrized over a finite time interval $I = [0, T]$. The family of elliptic operators $(A(t))_{t\in I}$ determines the distribution of the state u, e.g., the heat or concentration, and its *evolution* in time via the ordinary Banach space-valued differential equation

$$\dot{u} = -A(t)u.$$

Now additionally assume that we do not have access to the evolution of u directly but only to a *filtered* version $C(t)u(t)$ modeled via a family of observation operators $C(t)$ for $t \in I$. Based on the observations $C(t)u(t)$, we aim to deduce a conclusion about the *final state* $u(T)$ of our diffusion process in the form of a final-state *observability estimate*

$$\|u(T)\| \lesssim \int_0^T \|C(t)u(t)\| \, \mathrm{d}t. \tag{OBS}$$

Loosely speaking, this inequality represents a relation by which one can estimate the *total* energy $\|u(T)\|$ of the system at a fixed point in time T by sampling energy observation measurements $\|C(t)u(t)\|$ over time and calculating their average. The operator family $(C(t))_{t\in I}$ may be interpreted as a model to account for blind spots of the observation process, i.e., regions of the state space, where no measurements are sampled. Depending on the concrete Banach space in which $u(T)$ lies, it is also possible to interpret $\|u(T)\|$ as the total mass of the system or some higher moment if one considers stochastically motivated models.

Inequalities like (OBS) are interesting in the field of mathematical *control theory* as they are closely related to the problem of steering a system's state to a desired final state via a control function. In particular, one searches for sufficient conditions on the operators A and C such that the estimate (OBS) is valid. While criteria for *autonomous* versions of (OBS), i.e., when the operator families $A(\cdot)$ and $C(\cdot)$ are constant, are well-established, there exist only a few results for fully non-autonomous systems. Therefore, in this part of the thesis, we will extend a theorem about (OBS) for constant versions of the operator families A and C to the fully non-autonomous setting. More precisely, we will demonstrate a novel extension of the famous Lebeau–Robbiano strategy for proving final-state observability estimates that allows for non-autonomous families of operators $(A(t))$, $(C(t))$ and measurable observation sets $E \subseteq [0, T]$.

As an application of this abstract result, we will derive an observability estimate for non-autonomous elliptic differential operators and moving observation sets. This can be interpreted as estimating the "energy" of a diffusion system through a series of observations on time-dependent observation sets, i.e., observations that are subject to non-stationary blind spots. Furthermore, we relate observability to the geometric properties of a special class of observation operators $C(t)f := f|_{\Omega(t)}$ given via restriction to measurable sets $\Omega(t)$. The non-autonomy of the family $(C(t))$ gives rise to a new geometric property which we call *mean thickness* and which generalizes the notion of *thick* sets. We will prove this condition to be a necessary property for the existence of observation estimates. Some of the contents of this chapter are also part of the article [18].

Non-linear Models

The third part of this thesis takes us to *non-linear* systems of equations and studies a famous model for the description of fluid motion: the *Navier–Stokes equations*. Depending on the space dimension, this system consists of multiple conservation equations of momentum involving the velocity u and the pressure ϕ and one conservation equation of mass:

$$\begin{cases} u' - \Delta u + (u \cdot \nabla)u + \nabla\phi = f & \text{in } (0,\infty) \times \Omega, \\ \operatorname{div}(u) = 0 & \text{in } (0,\infty) \times \Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{NSE}$$

Proving the existence and uniqueness of classical solutions to (NSE) is considered one of the most important unsolved problems of our century and is therefore part of the famous millennium problems [76]. Nevertheless, the Navier–Stokes equations build the basis for a plethora of complex models that are used to simulate fluid flow phenomena, e.g., for calculating the drag of moving vehicles, the lift of an airplane, or for weather models, to name a few.

Compared to the models discussed in the previous paragraphs, we immediately see two factors of complexity in (NSE): first, instead of a single equation, our model consists of a *system* of partial differential equations; second, it brings a *non-linear* convective term $(u\cdot\nabla)u$ into the model. A suitable framework for the analysis of the Navier–Stokes equations builds on the thorough study of the linear part of (NSE), the Stokes equations.

$$\begin{cases} u' - \Delta u + \nabla\phi = f & \text{in } (0,\infty) \times \Omega, \\ \operatorname{div}(u) = 0 & \text{in } (0,\infty) \times \Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{SE}$$

More concretely, one studies the operator theoretic properties of the so-called *Stokes operator* which is associated with the part $-\Delta u + \nabla\phi$ in (SE) and aims to establish a semigroup theory for this operator. If the Stokes semigroup fulfills suitable smoothing properties, this opens the door to a variety of solution methods to (NSE), e.g., via iteration schemes demonstrated in the celebrated Fujita–Kato approach. A more abstract approach to the solution theory of (NSE) builds on the so-called *maximal regularity* of the Stokes operator and proving the existence of solutions via a fixed-point argument. Here the main idea is to treat the non-linear term $(u \cdot \nabla)u$ as an additional inhomogeneity and to deduce the solvability of the full Navier–Stokes equations by solving a related integral equation. All of these techniques form a well-established framework of tools to study (NSE) on domains Ω with *sufficiently smooth boundary* in the sense of differentiability. In real-world applications, however, fluid flow problems often involve less regular *rough* boundaries, e.g., Lipschitz domains like cylinders or more complicated geometries featuring edges, corners, and kinks. For these types of domains, less is known about the existence and regularity of solutions to (NSE).

We will study the regularity theory of solutions via the previously mentioned maximal regularity property of the Stokes operator. More precisely, we will show that the Stokes operator on bounded planar Lipschitz domains has the property of *maximal regularity* and a bounded H^∞-*calculus*. Furthermore, we will extend the currently known characterizations of domains of fractional powers of the Stokes operator. These results find their application when we transition to the full non-linear Navier–Stokes equations. Here, we will put them to work for the *regularity theory* of the Navier–Stokes equations on planar Lipschitz domains. More precisely, we will prove a theorem on the existence of *global strong solutions* in different regularity classes, improving the regularity of the classical Leray–Hopf solutions. The contents of this chapter appeared in the article [56].

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Chapter 2

Finite Sections of Discrete Schrödinger Operators

This chapter is based on the joint work with Dennis Gallaun, Julian Großmann, Marko Lindner, and Riko Ukena [52, 54, 55, 53].

2.1 Introduction

Consider the *one-dimensional discrete Schrödinger operator* H on $\ell^p(\mathbb{Z})$, $p \in [1, \infty]$, with *potential* $v\colon \mathbb{Z} \to \mathbb{C}$, acting on a two-sided infinite sequence $x\colon \mathbb{Z} \to \mathbb{C}$ via the relation

$$(Hx)_k = x_{k-1} + x_{k+1} + v(k)x_k\,, \quad k \in \mathbb{Z}\,. \tag{2.1}$$

Typically, x is an element of $\ell^p(\mathbb{Z})$, $p \in [1, \infty]$, and v is an element of $\ell^\infty(\mathbb{Z})$. Consequently, H acts as a bounded linear operator on every space $\ell^p(\mathbb{Z})$ and may be represented by a two-sided infinite matrix $(H_{ij})_{i,j\in\mathbb{Z}}$. If v is a *periodic* function, i.e., if there exists $K > 0$ such that $v(k + K) = v(k)$ for all $k \in \mathbb{Z}$, then H is called a *K-periodic Schrödinger operator*. We will often tacitly omit the period K and call H simply a *periodic* Schrödinger operator.

2.1.1 The Finite Section Method

We study the following problem in the numerical analysis of operator equations: consider the linear system $Hx = b$ in $\ell^p(\mathbb{Z})$ for some $p \in [1, \infty]$. If H is invertible, we want to approximate the unique solution x via a truncation technique that replaces the original infinite-dimensional system with a sequence of finite-dimensional linear systems $H_n x_n = b_n$, $n \in \mathbb{N}$. Suppose these systems are uniquely solvable for all but finitely many n. In that case, they give rise to a sequence of solutions $(x_n)_{n\in\mathbb{N}}$ that serves as a finite-dimensional approximation of x in the following sense: first, we embed each x_n into $\ell^p(\mathbb{Z})$ by extending the vectors of finite length with zero. Then we want the resulting sequence of vectors in $\ell^p(\mathbb{Z})$ to converge to x in the corresponding ℓ^p-norm, i.e.,

$$\lim_{n\to\infty} \left\| \begin{pmatrix} \vdots \\ 0 \\ x_n \\ 0 \\ \vdots \end{pmatrix} - x \right\|_p = 0 \,. \tag{2.2}$$

If (2.2) is valid, then we call the FSM *applicable* to H.

The following classical result regarding the applicability of the FSM, cf. [100, 126], summarizes all requirements for a successful FSM.

$$\begin{array}{l} \textit{The finite section method is applicable to } H \textit{ if and only if} \\ \quad \textit{(a) } H \textit{ is invertible, and} \\ \quad \textit{(b) all but finitely many } \text{finite sections } H_n := (H_{i,j})_{i,j=-n}^{n} \textit{ are invertible} \\ \quad \textit{(c) with uniformly bounded inverses } (H_n^{-1}). \end{array} \tag{2.3}$$

Some models also use *one-sided* or *half-line Schrödinger operators* H_+ on $\ell^p(\mathbb{Z}_+)$, $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. These operators can be seen as a restriction of H from (2.1) with a *Dirichlet boundary condition* for x_{-1}, given by

$$(H_+x)_k := x_{k-1} + x_{k+1} + v(k)x_k\,, \quad k \in \mathbb{Z}_+\,, \quad \text{where} \;\; x_{-1} := 0\,. \tag{2.4}$$

This operator is then represented by a one-sided infinite matrix $(H_{ij})_{i,j=0}^{\infty}$. For one-sided operators, the equivalence (2.3) holds analogously with H_+ instead of H.

As it turns out, the applicability question is strongly related to some spectral properties of H. In numerous works over the last decades, the class of periodic Schrödinger operators has been extensively studied, and several methods have been developed to understand their spectral properties. This includes *Floquet–Bloch theory*, see, e.g., [144, Chap. 7] and [127, Chap. XIII.16], the *transfer matrix formalism*, see, e.g., [28, Sec. 3] and [91], and more recently and generally the application of the dynamical systems formalism, see [27] and the references therein. In this chapter, we will employ a different tool to study the spectral theory of Schrödinger operators: *limit operators*, see, e.g., [99, 126]. This tool will allow us to tailor a framework suitable for the applicability analysis of the FSM. Using limit operator techniques, we will derive several spectral theoretic insights into two-sided and one-sided periodic Schrödinger operators and prove optimal results regarding the applicability of the FSM in the domain of periodic Schrödinger operators.

2.1.2 FSM-simple Operators

For some classes of operators, it turns out that one can skip the checks (b) and (c) in the equivalence (2.3) as the invertibility of H already implies the two other conditions. In other words, the *invertibility of H implies that the FSM is applicable to H*. We will call operators that fulfill the above dichotomy *FSM-simple*. If H is FSM-simple, either the FSM is applicable to H, or H is not invertible. The purpose of this chapter is to systematically study FSM-simplicity for the class of discrete periodic Schrödinger operators.

2.1.3 Outline

Let us outline the main results of this chapter. We derive sufficient conditions for a periodic Schrödinger operator to be FSM-simple. This study will also lead to novel insights into the spectral theory of one-sided periodic Schrödinger operators subject to a Dirichlet boundary condition. The following theorem summarizes our findings.

Theorem 2.1.1. *Let $p \in [1,\infty]$, $K \in \mathbb{N}$, and H be a discrete K-periodic Schrödinger operator on $\ell^p(\mathbb{Z})$ with potential v as defined in* (2.1). *The operator H is FSM-simple if one of the following conditions is fulfilled:*

(i) $K \in \mathbb{N}$ and $v(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

(ii) $K \leq 8$ and $v(n) \in \{0,\lambda\}$ for all $n \in \mathbb{Z}$ with a fixed $\lambda \in \mathbb{Q}$.

(iii) $K = 2$ and $v(n) \in \mathbb{R}$ for all $n \in \mathbb{Z}$.

If H_+ is a one-sided discrete K-periodic Schrödinger operator on $\ell^p(\mathbb{Z}_+)$ given by (2.4), *then statements (i)–(iii) analogously imply that H_+ is FSM-simple.*

The conditions formulated in Theorem 2.1.1 are optimal in the class of periodic Schrödinger operators.

Remark 2.1.2. (i) The 3-periodic Schrödinger operator having as potential the periodic continuation of $(2, \frac{1}{2}, \frac{1}{2})$ shows that Theorem 2.1.1(i) cannot include $\mathbb{Q}$-valued potentials and (iii) cannot include period three, see Example 2.4.8.

(ii) It is not possible to drop *rationality* of λ in Theorem 2.1.1(ii). The 5-periodic Schrödinger operator H having as potential the periodic continuation of $\frac{1}{\sqrt{2}}(1,1,0,1,0)$ is not FSM-simple, see Example 2.4.9.

(iii) The bound on the *period length* K in Theorem 2.1.1(ii) is optimal. The 9-periodic Schrödinger operator having the periodic continuation of $\frac{1}{2}(1,1,0,1,0,1,0,1,1)$ as potential is not FSM-simple. For a detailed study of this example, see Example 2.4.11 and the data analysis in [52].

A detailed discussion of the statements in Remark 2.1.2 and their optimality will be presented in Section 2.4.3.

For K-periodic Schrödinger operators H on $\ell^p(\mathbb{Z})$ with real-valued potential v, the invertibility is particularly easy to check, see, e.g., [123]: H is invertible if and only if at least one of the following monodromy matrices has its trace outside the interval $[-2,2]$:

$$M^{(j)} := \begin{pmatrix} -v(K-1+j) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -v(1+j) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -v(j) & -1 \\ 1 & 0 \end{pmatrix} \tag{2.5}$$

$$\widetilde{M}^{(j)} := \begin{pmatrix} -v(j) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -v(1+j) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -v(K-1+j) & -1 \\ 1 & 0 \end{pmatrix} \tag{2.6}$$

with $j \in \mathbb{Z}$. In particular, for FSM-simple operators, checking the trace of one of these matrices suffices to verify the applicability of the FSM. However, for a periodic Schrödinger operator that is possibly not FSM-simple, in addition to the trace condition mentioned above, one has to check whether the matrix entries $M_{2,1}$ and $M_{1,1}$ of the monodromy matrix M fulfill the condition

$$M_{2,1} \neq 0 \qquad \text{or} \qquad |M_{1,1}| > 1\,, \tag{2.7}$$

where M runs through the set of matrices (2.5) and (2.6). The following theorem summarizes this agenda for checking applicability of the FSM.

Theorem 2.1.3. *Let $p \in [1,\infty]$, $K \in \mathbb{N}$, H a discrete K-periodic Schrödinger operator on $\ell^p(\mathbb{Z})$ with real-valued potential v defined by* (2.1)*, and let H_+ be the one-sided restriction on $\ell^p(\mathbb{Z}_+)$ defined by* (2.4)*. In addition, let $M^{(j)}$ and $\widetilde{M}^{(j)}$ be given by* (2.5) *and* (2.6)*, respectively. Then the following holds:*

(i) The FSM is applicable to H if and only if $|\operatorname{tr}(M^{(0)})| > 2$ and

$$M^{(0)},\ldots,M^{(K-1)} \text{ and } \widetilde{M}^{(0)},\ldots,\widetilde{M}^{(K-1)} \text{ are subject to (2.7).}$$

(ii) The FSM is applicable to H_+ if and only if $|\operatorname{tr}(M^{(0)})| > 2$ and

$$M^{(0)} \text{ and } \widetilde{M}^{(0)},\ldots,\widetilde{M}^{(K-1)} \text{ are subject to (2.7)}\,.$$

In particular, the FSM is applicable to H_+ if it is applicable to its periodic extension H on $\ell^p(\mathbb{Z})$. Moreover, H_+ is FSM-simple if H is FSM-simple.

Remark 2.1.4. It may happen that the FSM is not applicable to H but to its one-sided restriction H_+. Indeed, this is the case for the 9-periodic Schrödinger operator H having as potential the periodic continuation of the vector $\frac{1}{\sqrt{2}}(1,1,1,0,1,1,0,1,0)$, see Example 2.4.12.

2.2 Approximation of Band Operators

This section briefly introduces the finite section method for band operators. We will state and prove several auxiliary results needed in the following sections.

2.2.1 Band Operators and the Finite Section Method

Given a discrete Schrödinger operator H as in (2.1), throughout this chapter, we will mostly think of it as being represented by a two-sided infinite tridiagonal matrix $A = (a_{ij})_{i,j\in\mathbb{Z}}$ defined by

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & v(0) & 1 & & \\ & 1 & v(1) & 1 & \\ & & 1 & v(2) & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{2.8}$$

The so-called *finite section method (FSM)* for H or A considers the sequence of finite submatrices

$$A_n = (a_{ij})_{i,j=-n}^{n}, \quad n \in \mathbb{N}\,, \tag{2.9}$$

and asks the following:

> *Are the matrices A and A_n invertible for all but finitely many n, and are their inverses strongly convergent to the inverse of A?*

In the case of a positive answer to the above question, we call the FSM *applicable* to A, respectively. This definition is compatible with the notion of applicability used in the introductory section 2.1.1. The convergence of the sequence of inverses (A_n^{-1}) to A^{-1} has to be understood as strong convergence of the embedded matrices A_n^{-1} into a two-sided infinite matrix, similar to (2.2), i.e.,

$$\begin{pmatrix} \ddots & & & & \iddots \\ & 0 & 0 & 0 & \\ \cdots & 0 & A_n^{-1} & 0 & \cdots \\ & 0 & 0 & 0 & \\ \iddots & & & & \ddots \end{pmatrix} \quad \to \quad A^{-1} \quad \text{strongly for } n \to \infty.$$

Now this approximation of A^{-1} can be used for solving equations $Ax = b$ approximately via the solutions of growing finite systems $A_n x_n = b_n$.

Remark 2.2.1. (i) The first rigorous treatments of this natural approximation via finite sections can be found in the works of Baxter [11] and Gohberg [67]. They studied finite sections of one-dimensional Wiener–Hopf equations which can be interpreted as continuous analogs to Toeplitz operator equations, i.e., equations that involve a convolution-like operator. The idea of taking finite sections to solve operator equations was subsequently extended to more general classes of operators as band-dominated operators with scalar and operator-valued coefficients. We refer the interested reader to [72, 126] for a state-of-the-art introduction.

(ii) In case the FSM is not applicable in the above sense, it may still be possible to establish an applicability result by modifying the shape of the finite sections, see [101, 125].

Operators like the Schrödinger operator H, whose infinite matrix representation only exhibits finitely many non-zero diagonals, are a well-known subject of investigation regarding the applicability of the FSM. Operators of this type are summarized in the class of *band operators*, which we introduce next.

Definition 2.2.2 (Band-width and Band Operator)**.** A finite sum

$$A \;=\; \sum_{k=-\omega}^{\omega} M_{a^{(k)}} S^k$$

of products of multiplication operators $(M_{a^{(k)}}x)_n = a_n^{(k)} x_n$ with $a^{(k)} \in \ell^\infty(\mathbb{Z})$ and powers of the *right shift operator*, $(Sx)_n = x_{n-1}$, is called a *band operator* with *band-width* $\omega \in \mathbb{N}$.

Per constructionem, a band operator A acts as a bounded operator on every $\ell^p(\mathbb{Z})$ with $p \in [1, \infty]$. For band operators with band-width ω, we have $a_{ij} = 0$ for all $|i - j| > \omega$. In the following, we will identify an operator A on $\ell^p(\mathbb{Z})$ with its usual *matrix representation* $(a_{ij})_{i,j\in\mathbb{Z}}$ with respect to the canonical basis in $\ell^p(\mathbb{Z})$.

Remark 2.2.3. Whenever necessary, we consider, for some index set $\mathbb{I} \subseteq \mathbb{Z}$, the space $\ell^p(\mathbb{I})$ as a subspace of $\ell^p(\mathbb{Z})$. The terms in Definition 2.2.2 naturally carry over to operators A on $\ell^p(\mathbb{I})$ by identifying them with their canonical extension by zero.

As we will see in this section, the invertibility of band operators lies at the core of the applicability analysis of the FSM. The following proposition shows that, when investigating the invertibility of a band operator on a space $\ell^q(\mathbb{Z})$, one can always fall back onto one's favorite $\ell^p(\mathbb{Z})$.

Proposition 2.2.4. *Let B be a band operator on $\ell^p(\mathbb{I})$ for some $\mathbb{I} \subseteq \mathbb{Z}$ and $p \in [1, \infty]$. If B is invertible, then B is also invertible as an operator on $\ell^q(\mathbb{I})$ for all $q \in [1, \infty]$.*

Proof. First, we identify the Banach space $\ell^p(\mathbb{Z})$ with the p-direct sum $\ell^p(\mathbb{J}) \oplus \ell^p(\mathbb{I})$, where $\mathbb{J} := \mathbb{Z} \setminus \mathbb{I}$ and the norm is given by

$$\|x \oplus y\| := \begin{cases} \left(\|x\|^p_{\ell^p(\mathbb{J})} + \|y\|^p_{\ell^p(\mathbb{I})} \right)^{\frac{1}{p}} & \text{if } p < \infty\,, \\ \max\left\{ \|x\|_{\ell^\infty(\mathbb{J})}, \|y\|_{\ell^\infty(\mathbb{I})} \right\} & \text{if } p = \infty\,. \end{cases}$$

Now, assume that B is invertible on $\ell^p(\mathbb{I})$. We extend the operator B to an invertible operator A on $\ell^p(\mathbb{Z})$ via

$$A := \begin{pmatrix} \mathbf{1}_{\mathbb{J}} & 0 \\ 0 & B \end{pmatrix}$$

with respect to the direct decomposition $\ell^p(\mathbb{Z}) = \ell^p(\mathbb{J}) \oplus \ell^p(\mathbb{I})$. Here, $\mathbf{1}_{\mathbb{J}}$ denotes the identity on $\ell^p(\mathbb{J})$.

Note that, as a band operator, A is an element of the so-called *Wiener algebra* $\mathcal{W}$, see, e.g., [99, Sec. 3.7.3]. Due to [89, p. 5.2.10], the algebra $\mathcal{W}$ is closed under taking inverses; a concise proof of this fact can also be found in [126, Cor. 2.5.4]. As an element of $\mathcal{W}$, the inverse A^{-1} acts boundedly on every space $\ell^q(\mathbb{Z})$, $q \in [1, \infty]$. Consequently, the operator A is invertible on all spaces $\ell^q(\mathbb{Z})$, $q \in [1, \infty]$. Moreover, we have

$$A^{-1} = \begin{pmatrix} \mathbf{1}_{\mathbb{J}} & 0 \\ 0 & B^{-1} \end{pmatrix}.$$

This shows that B^{-1} is a bounded operator on $\ell^q(\mathbb{I})$ for all $q \in [1, \infty]$. ☐

Assuming invertibility of A on $\ell^p(\mathbb{I})$, the applicability of the FSM is equivalent to the uniform boundedness of the inverses A_n^{-1}, a concept known as *stability.*

Definition 2.2.5 (Stability)**.** A sequence of operators $(A_n)_{n \in \mathbb{N}}$ defined on $\ell^p(\mathbb{I})$ for $p \in [1, \infty]$ is called *stable* if there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, the operators A_n are invertible and (A_n^{-1}) is uniformly bounded.

The basic result connecting the notions of applicability and stability is known as *Polski's theorem*, cf. [72, Thm. 1.4]:

> *The FSM is applicable to A if and only if the FSM-sequence (A_n) from* (2.9) *is stable and A is invertible.*

So Polski's theorem above precisely consists of the equivalence (2.3) that we presented in the introductory section.

Furthermore, stability of (A_n), and hence also applicability of the FSM, is closely connected to the entrywise limits

$$(a_{i+l_n,j+l_n})_{i,j=0}^{\infty} \xrightarrow{n\to\infty} L_+ \quad \text{and} \quad (a_{i+r_n,j+r_n})_{i,j=-\infty}^{0} \xrightarrow{n\to\infty} R_- \tag{2.10}$$

of one-sided infinite submatrices of A, where we consider sequences (l_n) and (r_n) with $\lim_{n\to\infty} l_n = -\infty$ and $\lim_{n\to\infty} r_n = \infty$ such that the limits in (2.10) exist.

The following result summarizes the connections between the concepts *applicability*, *stability*, and (2.10).

Lemma 2.2.6 ([22, Lem. 1.2], [125, Thm. 2.3])**.** *Let $p \in [1,\infty]$.*

(i) Let A be a band operator on $\ell^p(\mathbb{Z})$. Then the following are equivalent:

(a) The FSM is applicable to A.

(b) The FSM-sequence (A_n) with $A_n = (a_{ij})_{i,j=-n}^{n}$ is stable.

(c) The operator A and the limits L_+ and R_- from (2.10) *are invertible for all suitable sequences (l_n) and (r_n).*

(ii) Let A_+ be a band operator on $\ell^p(\mathbb{Z}_+)$. Then the following are equivalent:

(d) The FSM is applicable to A_+.

(e) The FSM-sequence (A_n) with $A_n = (a_{ij})_{i,j=0}^{n}$ is stable.

(f) The operator A_+ and the limits R_- from (2.10) *are invertible for all suitable sequences (r_n).*

2.2.2 Limit Operators

We now focus on the operator-theoretical tool of so-called *limit operators* in order to characterize conditions (c) and (f) of Lemma 2.2.6, cf. [100, 124, 126]. In the following, let $\mathbb{Z}_- := -\mathbb{Z}_+$. Note that the sets $\mathbb{Z}_-$ and $\mathbb{Z}_+$ include zero.

Definition 2.2.7 (Limit Operators and Compressions)**.** Let A be a band operator on $\ell^p(\mathbb{Z})$ for $p \in [1,\infty]$. An operator $B \in \ell^p(\mathbb{Z})$ with matrix representation $(b_{ij})_{i,j\in\mathbb{Z}}$ is called *a limit operator of* A if there is a sequence $h = (h_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ with $\lim_{n\to\infty} |h_n| = \infty$ and

$$a_{i+h_n,j+h_n} \xrightarrow{n\to\infty} b_{ij}$$

for all $i,j \in \mathbb{Z}$. In this case, we also write $A_h := B$ and say that h is the *corresponding sequence to* B. We denote the set of all limit operators of A by $\mathrm{Lim}(A)$. For $A_h \in \mathrm{Lim}(A)$, we write $A_h \in \mathrm{Lim}_+(A)$ or $A_h \in \mathrm{Lim}_-(A)$ if the corresponding sequence h tends to $+\infty$ or $-\infty$, respectively. If A is a band operator on $\ell^p(\mathbb{I})$ for $\mathbb{I} \neq \mathbb{Z}$, we define its sets of limit operators $\lim(A)$ and $\lim_\pm(A)$ to consist of the corresponding limit operators of the zero-extension of A to $\ell^p(\mathbb{Z})$.

Moreover, we call both operators

$$A_+ := (a_{ij})_{i,j=0}^{\infty} \quad \text{and} \quad A_- := (a_{ij})_{i,j=-\infty}^{0}$$

(one-sided) compressions of A.

Note that Definition 2.2.7 yields the identities $\operatorname{Lim}_\pm(A) = \operatorname{Lim}(A_\pm)$. In fact, the operators L_+ and R_- from (2.10) correspond to the one-sided compressions of the limit operators $A_{(l_n)}$ and $A_{(r_n)}$. Evidently, a limit operator A_h of a band operator A is again a band operator with the same band-width.

We now reformulate Lemma 2.2.6 in the language of limit operators.

Proposition 2.2.8 ([22, Lem 1.2], [125, Thm. 2.3])**.** *Let A be a band operator on $\ell^p(\mathbb{Z})$ for $p \in [1, \infty]$. Then the FSM is applicable to A if and only if*

(i) A is invertible on $\ell^p(\mathbb{Z})$,

(ii) for all $R \in \operatorname{Lim}_+(A)$, the compressions R_- are invertible on $\ell^p(\mathbb{Z}_-)$ and,

(iii) for all $L \in \operatorname{Lim}_-(A)$, the compressions L_+ are invertible on $\ell^p(\mathbb{Z}_+)$.

Now let A_+ be a band operator on $\ell^p(\mathbb{Z}_+)$ for $p \in [1, \infty]$. Then the FSM is applicable to A_+ if and only if

(iv) A_+ is invertible on $\ell^p(\mathbb{Z}_+)$ and,

(v) for all $R \in \operatorname{Lim}_+(A_+)$, the compressions R_- are invertible on $\ell^p(\mathbb{Z}_-)$.

The restriction to a particular choice of p in Proposition 2.2.8 can be dropped, due to Proposition 2.2.4. In subsequent parts of this chapter, this result will allow us to fall back onto the Hilbert space case $p = 2$ whenever needed.

2.2.3 Spectral and Fredholm Theory

The applicability analysis of the FSM heavily depends on spectral properties of the involved operators and their limit operators. After all, Proposition 2.2.8 requires us to verify invertibility not only of the operator itself but also of the one-sided compressions of its limit operators. This subsection aims to make the conditions in Proposition 2.2.8 more accessible by providing additional tools to study invertibility. More precisely, we will link our desired applicability property to the theory of Fredholm operators.

For a bounded linear operator $A\colon X \to Y$ between complex Banach spaces X and Y, we define its *spectrum* via

$$\sigma(A) := \{E \in \mathbb{C} : A - E \text{ is not invertible}\}.$$

The complement $\rho(A) := \mathbb{C} \setminus \sigma(A)$ is called *resolvent set* of A. Unraveling those spectral values $E \in \sigma(A)$ where the operator $A - E$ is "very far away from being invertible"—in the sense that they either have an infinite-dimensional kernel or they fail to be surjective by having an infinite-dimensional cokernel—leads to a property that generalizes invertibility: *Fredholmness*. Positively speaking, we call a bounded linear operator A a *Fredholm operator* if its *kernel* $\ker(A)$ and its *cokernel* $Y/\operatorname{Ran}(A)$ are *finite*-dimensional. Due to its finite-dimensional (co)-kernel, a Fredholm operator always has a closed range. Now the class of Fredholm operators gives rise to the set of spectral values

$$\sigma_{\text{ess}}(A) := \{E \in \mathbb{C} : A - E \text{ is not a Fredholm operator}\},$$

which we call the *essential spectrum* of A. Furthermore, let $\sigma_{\mathrm{dis}}(A) := \sigma(A) \setminus \sigma_{\mathrm{ess}}(A)$ denote the *discrete* spectrum of A. Clearly, $\sigma_{\mathrm{ess}}(A) \subseteq \sigma(A)$.

The following lemma establishes the equivalence of the Fredholmness of a band operator A, the invertibility of its limit operators, see [102, 124], and their injectivity on the space $\ell^\infty(\mathbb{Z})$, see [24, 23]. The lemma previously appeared in this form in [103].

Lemma 2.2.9 ([103, Lem. 2.6])**.** *Let $\mathbb{I} \in \{\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_-, \mathbb{N}\}$ and $p \in [1, \infty]$. For a band operator A on $\ell^p(\mathbb{Z})$, the following are equivalent:*

(a) A is a Fredholm operator on $\ell^p(\mathbb{I})$.

(b) All limit operators of A are invertible on $\ell^p(\mathbb{Z})$.

(c) All limit operators of A are injective on $\ell^\infty(\mathbb{Z})$.

As a consequence of Lemma 2.2.9, we derive the following set of relations between spectra, essential spectra, and limit operators, see also [102, Cor. 12].

Proposition 2.2.10. *Let A be a band operator on $\ell^p(\mathbb{I})$ with $\mathbb{I} \in \{\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_-, \mathbb{N}\}$. Then*

$$\sigma_{\mathrm{ess}}(A) = \bigcup_{B \in \mathrm{Lim}(A)} \sigma(B)\,.$$

In particular, we have:

(i) If $\sigma(B) = \sigma(B')$ for all $B, B' \in \mathrm{Lim}(A)$, then $\sigma_{\mathrm{ess}}(A) = \sigma(B)$.

(ii) If $B \in \mathrm{Lim}(A)$, then $\sigma(B) \subseteq \sigma_{\mathrm{ess}}(A) \subseteq \sigma(A)$.

(iii) If $\mathbb{I} = \mathbb{Z}$ and $A \in \mathrm{Lim}(A)$, then $\sigma(A) = \sigma_{\mathrm{ess}}(A)$.

Proof. Per definitionem, $E \in \sigma_{\mathrm{ess}}(A)$ if and only if $A - E$ is not Fredholm. By the equivalence (a)⇔(b) in Lemma 2.2.9, this is the case if and only if there exists some limit operator $B \in \mathrm{Lim}(A)$ such that $B - E$ is not invertible. But, by definition of the spectrum, this is equivalent to $E \in \sigma(B)$ for some $B \in \mathrm{Lim}(A)$. This proves the stated identity. □

As Proposition 2.2.10(iii) shows, the Fredholm property in Lemma 2.2.9(a) is particularly accessible for band operators A that additionally have the property $A \in \mathrm{Lim}(A)$. All subsequent examples of Schrödinger operators in this chapter will have this property, which is sometimes referred to as *self-similarity*, cf. [15, 24]. The combination of Proposition 2.2.4 with Lemma 2.2.9 leads to the following result that allows us to translate the *invertibility problem* of an operator into an *injectivity problem* of its limit operators.

Corollary 2.2.11. *Let A be a band operator on $\ell^p(\mathbb{Z})$ with $A \in \mathrm{Lim}(A)$. Then the following are equivalent:*

(a) All limit operators of A are injective on $\ell^\infty(\mathbb{Z})$.

(b) A is invertible on $\ell^p(\mathbb{Z})$ for all $p \in [1, \infty]$.

(c) A is invertible on $\ell^p(\mathbb{Z})$ for some $p \in [1, \infty]$.

Proof. The equivalence (b)⇔(c) is immediately given by Proposition 2.2.4. Since $A \in \mathrm{Lim}(A)$ by assumption, the implication (a)⇒(c) follows from Lemma 2.2.9(c)⇒(b). Now, as every invertible operator is in particular Fredholm, we also have the converse implication (c)⇒(a) as a consequence of Lemma 2.2.9(a)⇒(c). □

Note that Corollary 2.2.11 only handles operators that are defined on $\ell^p(\mathbb{Z})$. However, Lemma 2.2.6 also relies on the invertibility of one-sided compressions, i.e., operators that are only defined on $\ell^p(\mathbb{Z}_-)$ and $\ell^p(\mathbb{Z}_+)$. Therefore, the next corollary closes this gap and gives the corresponding result for compressions. For this, we consider only the case $p = 2$ in order to employ the *Hilbert space adjoint* A^* of an operator A on the Hilbert space $\ell^2(\mathbb{I})$ with $\mathbb{I} \subseteq \mathbb{Z}$.

Corollary 2.2.12. *Let B be a self-adjoint invertible band operator on $\ell^2(\mathbb{Z})$.*

(i) If the compression B_- is injective on $\ell^\infty(\mathbb{Z}_-)$, then B_- is invertible on $\ell^p(\mathbb{Z}_-)$ for all $p \in [1, \infty]$.

(ii) If the compression B_+ is injective on $\ell^\infty(\mathbb{Z}_+)$, then B_+ is invertible on $\ell^p(\mathbb{Z}_+)$ for all $p \in [1, \infty]$.

Proof. We only prove (i) since the proof of (ii) works completely analogously. For the proof of (i), we note that, due to Proposition 2.2.4, it suffices to consider $p = 2$. As B_- is injective on $\ell^\infty(\mathbb{Z}_-)$, it is also injective on the subset $\ell^2(\mathbb{Z}_-) \subseteq \ell^\infty(\mathbb{Z}_-)$. We will show that B_- has a dense and closed range making B_- also surjective.

The range of B_- is dense in $\ell^2(\mathbb{Z}_-)$. Since B is self-adjoint, its compression B_- is also self-adjoint. Therefore, the adjoint $(B_-)^*$ is also injective on $\ell^2(\mathbb{Z}_-)$ which implies that the range of B_- is dense in $\ell^2(\mathbb{Z}_-)$.

The range of B_- is closed in $\ell^2(\mathbb{Z}_-)$. Since B is invertible, it is, in particular, Fredholm on $\ell^2(\mathbb{Z})$. Consequently, Lemma 2.2.9(a)⇒(b) gives that all operators in $\mathrm{Lim}(B) \supseteq \mathrm{Lim}(B_-)$ are invertible on $\ell^2(\mathbb{Z})$. As all limit operators of B_- are invertible on $\ell^2(\mathbb{Z})$, the implication (b)⇒(a) in Lemma 2.2.9 gives that B_- is Fredholm. In particular, B_- has a closed range. □

The machinery of Fredholm theory provides us with results that can be put to direct use for the study of applicability of the finite section method. Most importantly, we saw that it suffices to study injectivity on ℓ^∞ instead of invertibility on ℓ^p.

Let us close this introduction with the following Fredholm theoretic consequence of Proposition 2.2.8 for a subclass of self-similar band operators.

Corollary 2.2.13. *Let A be a band operator on $\ell^p(\mathbb{Z})$ for $p \in [1, \infty]$. If A is FSM-simple and $A \in \mathrm{Lim}_+(A)$, then A_+ is FSM-simple.*

Proof. Let A be FSM-simple and $A_+ \in \ell^p(\mathbb{Z}_+)$ invertible. In particular A_+ is Fredholm so that, by Lemma 2.2.9(a)⇒(b), all limit operators $B \in \mathrm{Lim}(A_+)$ of A_+ are invertible. Furthermore A is invertible as $A \in \mathrm{Lim}_+(A) = \mathrm{Lim}(A_+)$. Now, in virtue of the fact that A is FSM-simple this gives that the FSM is applicable to A. In particular, Proposition 2.2.8(ii) gives the invertibility of all one-sided compressions $R_- \in \ell^p(\mathbb{Z}_-)$ of limit operators $R \in \mathrm{Lim}(A_+)$. Hence, the FSM is also applicable to A_+ as a consequence of Proposition 2.2.8(v), and we conclude that A_+ is an FSM-simple operator. □

2.3 Periodic Schrödinger Operators

In this section, we will analyze the spectral properties of periodic Schrödinger operators and their limit operators to lay the groundwork for our applicability study of the finite section method. We will always assume the hypothesis below.

Hypothesis 2.3.1. *Let $p \in [1, \infty]$ and $H\colon \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ be given by*

$$(Hx)_n \coloneqq x_{n+1} + x_{n-1} + v(n)x_n\,, \quad n \in \mathbb{Z}, \tag{2.11}$$

with the real-valued potential $v\colon \mathbb{Z} \to \mathbb{R}$. Assume in addition that v is periodic with period K, i.e., $v(n+K) = v(n)$ for all $n \in \mathbb{Z}$.

The potential v and H from Hypothesis 2.3.1 are simply called *K-periodic*. Many facts about such periodic Schrödinger operators are known, see, e.g., [42, 123, 144]. In this section, as preparation for Section 2.4, we will interpret these facts within the framework of limit operator theory. Thereby, we will uncover further structural insights into the class of periodic Schrödinger operators.

2.3.1 Limit Operators and Invertibility

As we saw in Proposition 2.2.10, the toolbox of limit operators opens a way into the study of spectral properties of an operator. Nevertheless, before diving into limit operator theory, let us first introduce a powerful representation of periodic Schrödinger operators via 2×2-matrices. This representation manages to encapsulate a lot of spectral properties as follows. Assume an operator H subject to Hypothesis 2.3.1. If we choose an energy $E \in \mathbb{R}$ and a vector $x \in \ker(H - E)$, the defining relation (2.11) yields the *eigenvalue equation*

$$0 = ((H-E)x)_n = x_{n+1} + x_{n-1} + (v(n) - E)x_n\,, \quad n \in \mathbb{Z}\,.$$

This scalar-valued three-term recurrence leads to the vector-valued two-term recursion

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} E - v(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{2.12}$$

We call the 2×2-matrix in (2.12) the *transfer matrix*

$$T(n,E) \coloneqq \begin{pmatrix} E - v(n) & -1 \\ 1 & 0 \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{2.13}$$

The transfer matrices (2.13) all belong to the *symplectic group* $\mathrm{Sp}(2, \mathbb{C})$, i.e., they satisfy the relation

$$T(n,E)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T(n,E) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As a consequence of this relation, every matrix in $\mathrm{Sp}(2, \mathbb{C})$ has determinant 1, which, in terms of the spectrum, tells us that the eigenvalues always form a pair of reciprocals $\{\lambda, \lambda^{-1}\}$.

The potential v of a K-periodic Schrödinger operator gives rise to K transfer matrices, and the multiplication of these matrices leads to the so-called *monodromy matrix*

$$M(E) \coloneqq T(K-1, E) \cdots T(1,E)\, T(0,E)\,, \tag{2.14}$$

which, similar to (2.12), establishes the K-step recursion

$$\begin{pmatrix} x_{(j+1)K} \\ x_{(j+1)K-1} \end{pmatrix} = M(E) \begin{pmatrix} x_{jK} \\ x_{jK-1} \end{pmatrix}, \quad j \in \mathbb{Z}. \tag{2.15}$$

In order to ease notation, we will employ the abbreviations $T(n) := T(n, 0)$ and $M := M(0)$ whenever the context excludes ambiguities. As we will see, the trace of a monodromy matrix already encodes all the information needed to study the spectral properties of the corresponding infinite-dimensional operator. This fact makes monodromy matrices particularly useful for numerical calculations.

The following lemma builds the bridge between the limit operator theory from Section 2.2.2 and periodic Schrödinger operators.

Lemma 2.3.2. *Assume Hypothesis 2.3.1.*

(i) Every limit operator of H is again a K-periodic Schrödinger operator and

$$\mathrm{Lim}(H) = \left\{S^{-k} H S^{k} : k = 0, \dots, K-1\right\} = \mathrm{Lim}_{+}(H) = \mathrm{Lim}_{-}(H),$$

where S denotes the right-shift operator, i.e., $(Sx)_n = x_{n-1}$ for all $n \in \mathbb{Z}$. In particular, the operator H is self-similar, *i.e., $H \in \mathrm{Lim}(H)$.*

(ii) $\sigma(B) = \sigma(H)$ for all $B \in \mathrm{Lim}(H)$.

(iii) For $E \in \mathbb{R}$, let $M_H(E)$ denote the monodromy matrix of H. If $B \in \mathrm{Lim}(H)$ with monodromy matrix $M_B(E)$, then $\mathrm{tr}(M_B(E)) = \mathrm{tr}(M_H(E))$ for all energies $E \in \mathbb{R}$.

Proof. Ad (i). Let $B \in \mathrm{Lim}(H)$ with corresponding sequence $(h_k)_{k\in\mathbb{N}}$. By Definition 2.2.7, the sequence of representation matrices $(H_k)_{k\in\mathbb{N}}$ with $H_k := S^{-h_k} H S^{h_k}$ converges entrywise to the representation matrix of B. As the diagonal of H consists of the periodic continuation of $w := (v(0), \dots, v(K-1))$, the diagonal of the shifted matrix H_k consists of a periodic continuation of the permutation $(v(-h_k), \dots, v(K-1-h_k))$ of w. Hence, in order to converge, the sequence (H_k) has to be eventually constant. This proves that $B = S^{-k} H S^{k}$ for some $k \in \{0, \dots, K-1\}$.

The other inclusion is straightforward: consider the sequence $h = (k + n \cdot K)_{n\in\mathbb{N}}$ with corresponding limit operator A_h. Then

$$S^{-k} H S^{k} = \lim_{n\to\infty} S^{-(k+n\cdot K)} H S^{k+n\cdot K} = A_h \in \mathrm{Lim}(H).$$

Ad (ii). This follows from the fact that the right-shift operator is an isomorphism on $\ell^p(\mathbb{Z})$.

Ad (iii). From part (i), it follows that the monodromy matrix for B is given by

$$M_B(E) = T(\tau(K-1), E) \cdots T(\tau(1), E)\, T(\tau(0), E),$$

where $(\tau(K-1), \dots, \tau(1), \tau(0))$ is just a cyclic permutation of $(K-1, \dots, 1, 0)$. Consequently, properties of the trace yield $\mathrm{tr}(M_B(E)) = \mathrm{tr}(M(E))$. □

Remark 2.3.3. As a consequence of Lemma 2.3.2(ii), H and all limit operators $B \in \mathrm{Lim}(H)$ are *simultaneously* invertible, the invertibility of one operator $B \in \mathrm{Lim}(H)$ is equivalent to the invertibility of all operators in $\mathrm{Lim}(H)$.

The complete description of the limit operators of a periodic Schrödinger operator from Lemma 2.3.2 now allows for a powerful characterization of invertibility.

Lemma 2.3.4. *Assume Hypothesis 2.3.1. The periodic Schrödinger operator H is invertible on $\ell^p(\mathbb{Z})$ for any $p \in [1,\infty]$ if and only if H is injective on $\ell^\infty(\mathbb{Z})$.*

Proof. Let H be injective on $\ell^\infty(\mathbb{Z})$. By Lemma 2.3.2(i), all limit operators of H are shifts of H and hence, also injective on $\ell^\infty(\mathbb{Z})$. In particular, by Lemma 2.3.2(i), we have $H \in \operatorname{Lim}(H)$. Now, Corollary 2.2.11(a)⇒(b) gives that H is invertible on $\ell^p(\mathbb{Z})$ for all $p \in [1,\infty]$. The converse implication follows from Corollary 2.2.11(c)⇒(a). □

The following proposition resembles a well-known result about the spectrum of periodic Schrödinger operators, which can be described by a trace condition for the monodromy matrix, see, e.g., [123]. We give a non-standard proof employing limit operator techniques.

Proposition 2.3.5. *Assume Hypothesis 2.3.1 and let $E \in \mathbb{R}$. Then $E \in \sigma(H)$ if and only if the* trace condition $|\operatorname{tr}(M(E))| \leq 2$ *holds.*

Proof. We prove the equivalence of the negative statements, i.e., we show that $H - E$ is invertible if and only if $|\operatorname{tr}(M(E))| > 2$. We divide the proof into three steps.

Step 1. Note that $H - E$ is invertible on $\ell^p(\mathbb{Z})$ if and only if, for all $(\alpha,\beta) \neq (0,0)$, the two-sided sequence

$$\left(M(E)^n \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right)_{n\in\mathbb{Z}} \tag{2.16}$$

is unbounded. Indeed, by Lemma 2.3.4, invertibility on $\ell^p(\mathbb{Z})$ is equivalent to $H - E$ being injective on $\ell^\infty(\mathbb{Z})$. Using (2.12) and the K-step recursion formula (2.15) for a solution to the eigenvalue equation $(H - E)x = 0$ gives (2.16). Injectivity of H now demands that the kernel sequence x is unbounded, i.e., $x \notin \ell^\infty(\mathbb{Z})$. Of course, x is unbounded if the subsequence (2.16) is unbounded. Conversely, if the subsequence (2.16) is bounded, so will be x as the missing entries of x can always be computed by applying at most $K - 1$ transfer matrices to the sequence (2.16).

Step 2. We show that the sequence (2.16) is unbounded for all pairs $(\alpha,\beta) \neq (0,0)$ if and only if $M(E)$ has two distinct real eigenvalues. Clearly, (2.16) is unbounded for all pairs $(\alpha,\beta) \neq (0,0)$ if and only if both eigenvalues λ_1, λ_2 of $M(E)$ fulfill the condition

$$|\lambda_i| \neq 1, \quad i \in \{1,2\}. \tag{2.17}$$

Indeed, as $M(E)$ is symplectic,

$$\det(M(E)) = \lambda_1 \cdot \lambda_2 = 1 \tag{2.18}$$

and the eigenvalues of $M(E)$ form a pair of reciprocals. This implies that, if both eigenvalues are distinct and real, the modulus of one of them needs to be strictly larger than 1, which proves that (2.16) will be unbounded.

Conversely, assume that both eigenvalues fulfill $\lambda_1 = \lambda_2 \in \mathbb{R}$. Then, by (2.18), $|\lambda_i| = 1$, $i \in 1,2$, contradicting the condition (2.17). Now assume that $M(E)$ has two proper complex eigenvalues. As $M(E) \in \mathbb{R}^{2\times 2}$, we have $\lambda_2 = \overline{\lambda_1}$ and hence again $|\lambda_i| = 1$, $i \in \{1,2\}$, contradicting (2.17).

Step 3. Lastly, note that $M(E)$ has two distinct real eigenvalues if the characteristic polynomial of the matrix $M(E)$

$$p_{M(E)}(\lambda) = \lambda^2 - \operatorname{tr}(M(E))\lambda + 1$$

has two distinct real solutions. However, having two distinct *real* solutions is equivalent to the discriminant of $p_{M(E)}$ being strictly positive, which is equivalent to the condition $\operatorname{tr}(M(E))^2 > 4$. ☐

Corollary 2.3.6. *Assume Hypothesis 2.3.1. Then H is invertible if and only if $|\operatorname{tr}(M)| > 2$. In this case, the eigenvalues λ_1, λ_2 of M fulfill the relation $|\lambda_1| > 1 > |\lambda_2|$.*

Proof. The stated equivalence follows from Proposition 2.3.5 by setting $E = 0$. The relation for the eigenvalues is a direct consequence of the considerations made in Step 2 of the proof of Proposition 2.3.5. ☐

In the following examples, we illustrate the use of the trace condition from Proposition 2.3.5 for determining the spectrum of periodic Schrödinger operators.

Example 2.3.7. (i) If the potential v has period $K = 1$ with $v(0) \in \mathbb{R}$, we get for the trace that $\operatorname{tr}(M(E)) = E - v(0)$ and $\sigma(H) = [-2 + v(0), 2 + v(0)]$.

(ii) If the potential v has period $K = 2$ with $v(0) = -1$ and $v(1) = 1$, we get for the trace that $\operatorname{tr}(M(E)) = E^2 - 3$ and $\sigma(H) = [-\sqrt{5}, -1] \cup [1, \sqrt{5}]$.

(iii) More generally, for a 2-periodic potential, we have

$$\operatorname{tr}(M(E)) = -2 + (E - v(0))(E - v(1)) = E^2 - E(v(0) + v(1)) + v(0)v(1) - 2\,.$$

This shows that we will have two spectral bands, namely

$$\begin{aligned}\sigma(H) = &\left[\frac{1}{2}(v(0) + v(1) - \sqrt{\delta}),\ \min\{v(0), v(1)\}\right] \\ &\cup \quad \left[\max\{v(0), v(1)\},\ \frac{1}{2}(v(0) + v(1) + \sqrt{\delta})\right],\end{aligned}$$

where $\delta := 16 + v(0)^2 - 2v(0)v(1) + v(1)^2 = 16 + (v(0) - v(1))^2$ denotes the discriminant of the polynomial $\operatorname{tr}(M(E)) - 2$. Observe that $\delta > 0$.

(iv) For a potential v with period 3 given by $v(0) = 0$, $v(1) = 1$, and $v(2) = 0$, we get $\operatorname{tr}(M(E)) = E^3 - E^2 - 3E + 1$. Therefore, the spectrum is given by

$$\sigma(H) = [-\sqrt{3}, -1] \cup [1 - \sqrt{2}, 1] \cup [1 + \sqrt{2}, \sqrt{3}].$$

The following corollary summarizes the observation of Example 2.3.7: the number of spectral bands is bounded by the period of the potential v.

Corollary 2.3.8. *Assume Hypothesis 2.3.1. The spectrum of H consists of at most K bands. More precisely, there are numbers $a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_K \le b_K$ with*

$$\sigma(H) = \bigcup_{i=1}^{K} [a_i, b_i]\,.$$

Proof. The trace of $M(E)$ gives us a polynomial p_K in the variable E with degree K. As Proposition 2.3.5 shows, a real number E lies in $\sigma(H)$ if and only if $-2 \le p_K(E) \le 2$. The real zeros of the polynomials $p_K + 2$ and $p_K - 2$ give us the numbers $\{a_i, b_i\}_{i=1,\dots,K}$. ☐

2.3.2 One-sided Periodic Schrödinger Operators

During the applicability analysis of the FSM, one also needs to check the invertibility of one-sided compressions of limit operators. Therefore, this section aims at gaining control over the spectrum of one-sided periodic Schrödinger operators. More precisely, we will derive a condition for one-sided operators similar to the trace condition from Corollary 2.3.6. This condition will solely work with monodromy matrices and allows us to check whether a one-sided compression is invertible.

If H is a periodic Schrödinger operator on $\ell^p(\mathbb{Z})$, we let H_+ denote its one-sided compression on $\ell^p(\mathbb{Z}_+)$ subject to the Dirichlet boundary condition $x_{-1} := 0$. Analogously, let H_- denote its one-sided compression on $\ell^p(\mathbb{Z}_-)$ subject to the Dirichlet boundary condition $x_1 := 0$. The matrix representation of $H_\pm$ coincides with the respective one-sided compressions from Definition 2.2.7.

Recall from Proposition 2.2.8 that checking applicability of the FSM involves compressions R_- on $\ell^p(\mathbb{Z}_-)$ and L_+ on $\ell^p(\mathbb{Z}_+)$. The following lemma shows that methods aiming to deduce invertibility of one-sided periodic Schrödinger operators only need to consider compressions on $\ell^p(\mathbb{Z}_+)$.

Lemma 2.3.9. *Assume Hypothesis 2.3.1. Let H^{R} denote the Schrödinger operator with the* reversed potential $v^{\mathrm{R}}(n) := v(-n)$, *$n \in \mathbb{Z}$. Then the following holds:*

(i) $\sigma(H) = \sigma(H^{\mathrm{R}})$ *and* $\sigma(H_-) = \sigma(H_+^{\mathrm{R}})$.

(ii) $\sigma(B) = \sigma(H)$ *for all* $B \in \mathrm{Lim}(H) \cup \mathrm{Lim}(H^{\mathrm{R}})$.

(iii) *If* $B \in \mathrm{Lim}(H) \cup \mathrm{Lim}(H^{\mathrm{R}})$ *with monodromy matrix* $M_B(E)$, *then* $\mathrm{tr}(M_B(E)) = \mathrm{tr}(M(E))$ *for all energies* $E \in \mathbb{R}$.

Proof. Ad (i). Consider the *flip operators*

$$\Phi\colon \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z}), \quad (x_n)_{n\in\mathbb{Z}} \mapsto (x_{-n})_{n\in\mathbb{Z}},$$

and

$$\Phi_-\colon \ell^p(\mathbb{Z}_+) \to \ell^p(\mathbb{Z}_-), \quad (x_n)_{n\in\mathbb{Z}_+} \mapsto (x_{-n})_{n\in\mathbb{Z}_-}.$$

Clearly, Φ and Φ_- are isomorphisms. The claims for the spectra of H and H_- follow from the identities $H = \Phi H^{\mathrm{R}} \Phi^{-1}$ and

$$H_- = \Phi_- H_+^{\mathrm{R}} \Phi_-^{-1}. \tag{2.19}$$

Indeed, for $x \in \ell^p(\mathbb{Z}_-)$, we have, on the one hand, for all $n \geq 0$,

$$(H_- x)_{-n} = \begin{cases} x_{-1} + v(0)x_0 & \text{if } n = 0, \\ x_{-n-1} + v(-n)x_{-n} + x_{-n+1} & \text{else.} \end{cases} \tag{2.20}$$

On the other hand, note that, for all $n \geq 0$,

$$(\Phi_- H_+^{\mathrm{R}} \Phi_-^{-1} x)_{-n} = (\Phi_-(H_+^{\mathrm{R}} \Phi_-^{-1} x))_{-n} = (H_+^{\mathrm{R}} \Phi_-^{-1} x)_n = (H_+^{\mathrm{R}}(\Phi_-^{-1} x))_n\,.$$

A case distinction shows

$$(\Phi_- H_+^{\mathrm{R}} \Phi_-^{-1} x)_{-n} = \begin{cases} (\Phi_-^{-1}x)_1 + v^{\mathrm{R}}(0)(\Phi_-^{-1}x)_0 & \text{if } n = 0, \\ (\Phi_-^{-1}x)_{n-1} + v^{\mathrm{R}}(n)(\Phi_-^{-1}x)_n + (\Phi_-^{-1}x)_{n+1} & \text{else.} \end{cases} \tag{2.21}$$

Using the definition of the potential v^{R} and the flip operator Φ_-, the comparison of (2.21) with (2.20) yields the claimed identity (2.19). A similar proof works for the identity involving Φ.

Ad (ii). If $B \in \mathrm{Lim}(H)$, this is a consequence of Lemma 2.3.2(ii). If $B \in \mathrm{Lim}(H^{\mathrm{R}})$, we have $\sigma(B) = \sigma(H^{\mathrm{R}})$ by Lemma 2.3.2(ii). As $\sigma(H) = \sigma(H^{\mathrm{R}})$ by part (i), the claim follows.

Ad (iii). Without loss of generality, it suffices to show the identity for $E = 0$. In accordance with our notation, we set $M_B \coloneqq M_B(0)$.

To prove the equality of traces note that, for $B \in \mathrm{Lim}(H)$, the claim follows from Lemma 2.3.2(iii). From the same result, it follows that, for $B \in \mathrm{Lim}(H^{\mathrm{R}})$, we have the equality $\mathrm{tr}(M_B) = \mathrm{tr}(M_{H^{\mathrm{R}}})$. Consequently our claim follows, once we show that $\mathrm{tr}(M_H) = \mathrm{tr}(M_{H^{\mathrm{R}}})$, where M_H and $M_{H^{\mathrm{R}}}$ denote the monodromy matrices of H and H^{R}, respectively.

To this end, we use the following relation between a transfer matrix $T(n)$ and its inverse

$$T(n)^{-1} = F^{-1}\, T(n)\, F \quad \text{with} \quad F^{-1} = F \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.22}$$

In terms of the monodromy matrices, relation (2.22) translates to

$$F^{-1}\, M_H\, F = T(K-1)^{-1} \cdots T(0)^{-1} = M_{H^{\mathrm{R}}}^{-1}\,.$$

Taking the trace above produces the identity

$$\mathrm{tr}(M_H) = \mathrm{tr}(F^{-1}\, M_H\, F) = \mathrm{tr}\left(M_{H^{\mathrm{R}}}^{-1}\right) = \mathrm{tr}\left(M_{H^{\mathrm{R}}}\right),$$

where the last equality holds for every real symplectic 2×2-matrix. □

Remark 2.3.10. Note that Lemma 2.3.9 extends the results of Lemma 2.3.2(ii) and (iii). Considering Lemma 2.3.9(iii), it is evident from Proposition 2.3.5 that the trace condition is simultaneously fulfilled for all operators in $\mathrm{Lim}(H) \cup \mathrm{Lim}(H^{\mathrm{R}})$ implying the equality of all spectra. Consequently, Lemma 2.3.9(iii) implies part (ii) and the same argument holds for the corresponding parts of Lemma 2.3.2.

In order to study the invertibility of one-sided compressions H_+, we will present a representation of the spectrum $\sigma(H_+)$. Recall that $H \in \mathrm{Lim}(H) = \mathrm{Lim}_+(H) = \mathrm{Lim}(H_+)$ thanks to Lemma 2.3.2(i). Therefore, by Proposition 2.2.10(ii), we have that $\sigma(H) \subseteq \sigma(H_+)$. As we will see in the next lemma, $\sigma(H) = \sigma_{\mathrm{ess}}(H_+)$ such that $\sigma(H_+) \setminus \sigma(H)$ consists of the *discrete spectrum* $\sigma_{\mathrm{dis}}(H_+)$.

Lemma 2.3.11. *Assume Hypothesis 2.3.1. Then* $\sigma_{\mathrm{ess}}(H_+) = \sigma(H)$.

Proof. Recall from Proposition 2.2.10 that the essential spectrum of H_+ equals the union of all spectra of limit operators $B \in \mathrm{Lim}(H_+)$. Due to Lemma 2.3.2(i) the limit operators of H_+ and H coincide such that

$$\sigma_{\mathrm{ess}}(H_+) = \bigcup_{B \in \mathrm{Lim}(H)} \sigma(B).$$

By Lemma 2.3.2(ii), $\sigma(B) = \sigma(H)$ for all limit operators $B \in \mathrm{Lim}(H)$. This gives the claimed identity. □

As Lemma 2.3.11 shows, passing from H to the one-sided compression H_+ may lead to eigenvalues in the spectrum. Therefore, our main goal during the applicability analysis of the FSM will be to determine whether 0 will become an eigenvalue of H_+.

Remark 2.3.12. Our proof of Lemma 2.3.11 uses Proposition 2.2.10, which is a general result about the essential spectrum of band operators employing its limit operators. However, in the case of *periodic Jacobi operators*, there exist more elementary proofs for the identity $\sigma_{\mathrm{ess}}(H_+) = \sigma(H)$, see, e.g., [135, Thm. 7.2.1].

Before we present a formula for the spectrum $\sigma(H_+)$, we introduce the following notation. Define

$$H_{a..b} \coloneqq \begin{pmatrix} v(a) & 1 & & & \\ 1 & v(a+1) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & v(b-1) & 1 \\ & & & 1 & v(b) \end{pmatrix} \tag{2.23}$$

for $a, b \in \mathbb{Z}_+$ with $a \le b$. Furthermore, recall the following recursion formula for tridiagonal matrices of the form (2.23)

$$\begin{aligned} \det(H_{a..b+1}) &= v(b+1)\det(H_{a..b}) - \det(H_{a..b-1}), \\ \text{with: } \det(H_{a..a-1}) &\coloneqq 1, \quad \det(H_{a..a-2}) \coloneqq 0, \end{aligned} \tag{2.24}$$

which is a direct consequence of Laplace's expansion formula.

The following proposition is based on Hagger's work [73, Thm. 4.42(i)] and gives the promised insight into the spectrum of one-sided compressions.

Proposition 2.3.13. *Assume Hypothesis 2.3.1 with period $K \ge 2$. Then*

$$\sigma(H_+) = \sigma(H) \cup \left\{E \in \mathbb{R} : M(E)_{2,1} = 0 \quad \text{and} \quad |M(E)_{1,1}| < 1\right\}. \tag{2.25}$$

In particular, H is invertible if H_+ is invertible. Furthermore, for the entries of $M(E)$, we have the representation formula

$$M(E) = (-1)^{K-1} \begin{pmatrix} -\det(H_{0..K-1} - E) & -\det(H_{1..K-1} - E) \\ \det(H_{0..K-2} - E) & \det(H_{1..K-2} - E) \end{pmatrix}. \tag{2.26}$$

Proof. We proceed in two steps.

Step 1. Note that formula (2.26) follows once we show for all $K \in \mathbb{N}$

$$M_{K-1}(E) = \begin{pmatrix} \det(E - H_{0..K-1}) & -\det(E - H_{1..K-1}) \\ \det(E - H_{0..K-2}) & -\det(E - H_{1..K-2}) \end{pmatrix}, \tag{2.27}$$

where we introduced the notation $M_{K-1}(E) \coloneqq T(K-1, E) \cdots T(0, E)$ in order to ease the upcoming induction proof of the next step. Indeed, (2.27) gives

$$M_{K-1}(E) = \begin{pmatrix} (-1)^K \det(H_{0..K-1} - E) & (-1)^K \det(H_{1..K-1} - E) \\ (-1)^{K-1} \det(H_{0..K-2} - E) & (-1)^{K-1} \det(H_{1..K-2} - E) \end{pmatrix}.$$

Step 2. We prove (2.27) via induction over $K \in \mathbb{N}$. Let $K = 1$. Then

$$M_0(E) = T(0, E) = \begin{pmatrix} E - v(0) & -1 \\ 1 & 0 \end{pmatrix}.$$

Now, as a consequence of (2.24), the representation (2.27) holds as $\det(E - H_{0..-1}) = \det(E - H_{1..0}) = 1$, $\det(H_{0..-2}) = 0$, and $\det(E - H_{0..0}) = E - v(0)$.

For $K+1$, the induction step, assume that (2.27) holds for some $K \in \mathbb{N}$. The induction hypothesis gives the decomposition

$$\begin{aligned} M_K(E) &= T(K, E) \cdot M_{K-1}(E) \\ &= \begin{pmatrix} E - v(K) & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \det(E - H_{0..K-1}) & -\det(E - H_{1..K-1}) \\ \det(E - H_{0..K-2}) & -\det(E - H_{1..K-2}) \end{pmatrix}. \end{aligned}$$

As

$$(E - v(K)) \det(E - H_{0..K-1}) - \det(E - H_{0..K-2}) = \det(E - H_{0..K})$$

and

$$-(E - v(K)) \det(E - H_{1..K-1}) + \det(E - H_{1..K-2}) = -\det(E - H_{1..K})$$

by the recursion formula (2.24), this gives the claim.

Step 3. Here, we prove the decomposition (2.25). Note that the original formulation in Hagger's work [73] states that, for a tridiagonal matrix representation of H with upper and lower diagonals having the constant value 1, we have the decomposition

$$\begin{aligned} \sigma(H_+) = \sigma_{\mathrm{ess}}(H_+) &\cup \{\text{bounded connected components of } \mathbb{C} \setminus \sigma_{\mathrm{ess}}(H_+)\} \\ &\cup \{E \in \sigma(H_{0..K-2}) : |\det(H_{0..K-1} - E)| < 1\}, \end{aligned} \tag{2.28}$$

cf. [73, Thm. 4.42(i)].

For the first set on the right-hand side of (2.28), recall that $\sigma_{\mathrm{ess}}(H_+) = \sigma(H)$ as a consequence of Lemma 2.3.11.

Next, note that the second set in the decomposition (2.28) is empty. Indeed, by Corollary 2.3.8, we know that $\sigma(H)$ equals a finite union of bounded intervals. This implies that $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(H_+) = \mathbb{C} \setminus \sigma(H)$ consists of a single unbounded connected component.

Finally, we have that $E \in \sigma(H_{0..K-2})$ if and only if $\det(E - H_{0..K-2}) = 0$. We use this fact to rewrite the third set in (2.28) via the representation formula (2.26). This proves the claimed identity (2.25). □

Remark 2.3.14. (i) As $\sigma(H) = \sigma_{\mathrm{ess}}(H_+) \subseteq \sigma(H_+)$ by Lemma 2.3.11, the spectrum of the compression H_+ on $\ell^p(\mathbb{Z}_+)$ can only be larger than the spectrum of H and the difference can only be given by eigenvalues of H_+. Proposition 2.3.13 shows that $\sigma_{\mathrm{dis}}(H_+) \subseteq \sigma(H_{0..K-2})$. Techniques that are part of the *Floquet–Bloch theory* allow to localize these eigenvalues, see, e.g., [31, Thm. 4.4.9], [123], and [144, Chap. 7]. In particular, for periodic band operators, one can show that these eigenvalues exhibit an *interlacing* property, i.e., there is always at most one so-called *Dirichlet eigenvalue* inside the gaps of $\sigma(H)$, see [54, Prop. 3.9].

(ii) Together with Corollary 2.3.6, formula (2.25) facilitates numerical calculations of the spectrum of one-sided periodic Schrödinger operators by reducing the computations to finding zeros of polynomials. We will exploit this fact in Section 2.4.2 in order to systematically study a large class of periodic potentials.

Example 2.3.15. (i) Consider the 2-periodic Schrödinger operator H from Example 2.3.7(iii). For the spectrum of H_+, we calculate according to Proposition 2.3.13

$$\sigma(H_{0..2-2}) = \sigma\big(v(0)\big) = \{v(0)\}$$

and

$$\det(H_{0..1} - v(0)) = \det\left(\begin{pmatrix} v(0) - v(0) & 1 \\ 1 & v(1) - v(0) \end{pmatrix}\right) = -1,$$

which implies $\sigma_{\mathrm{dis}}(H_+) = \emptyset$ and thus $\sigma(H_+) = \sigma(H)$.

(ii) For the 3-periodic Schrödinger operator from Example 2.3.7(iv), we find

$$\sigma(H_+) = \sigma(H) \cup \Big\{ -\frac{\sqrt{5}-1}{2} \Big\}.$$

As we saw in Corollary 2.3.6, the trace of the monodromy matrix M of a periodic Schrödinger operator H gives us a sufficient condition to decide whether $0 \in \sigma(H)$ and thereby control the invertibility of H on $\ell^p(\mathbb{Z})$. We also need to control whether $0 \in \sigma(H_+)$ to ensure the invertibility of the corresponding one-sided compression H_+ on $\ell^p(\mathbb{Z}_+)$. More precisely, we need to control whether 0 is a *Dirichlet eigenvalue* of H_+, i.e., an eigenvalue that stems from the transition from H to the compression H_+ by imposing a Dirichlet condition. The description of the spectrum of $\sigma(H_+)$ in Proposition 2.3.13 suggests that, in addition to the trace of M, we will need control over the matrix entries $M_{1,1}$ and $M_{2,1}$ in order to know about the Dirichlet eigenvalues of H_+. This relation is described in the following proposition.

Proposition 2.3.16. *Assume Hypothesis 2.3.1. Let H be invertible on $\ell^p(\mathbb{Z})$, and let $M := M(0)$ denote its monodromy matrix. Then the following are equivalent:*

(a) The compression H_+ is not injective on $\ell^\infty(\mathbb{Z}_+)$.

(b) The compression H_+ is not invertible on any $\ell^p(\mathbb{Z}_+)$.

(c) $\binom{1}{0}$ is an eigenvector of M with corresponding eigenvalue λ subject to $|\lambda| < 1$.

(d) $M_{2,1} = 0$ and $|M_{1,1}| < 1$.

Proof. From Proposition 2.2.4, we know that (b) is independent of the value $p \in [1, \infty]$. Choosing $p = \infty$, the statement (a)$\Rightarrow$(b) follows directly.

The implication (b)$\Rightarrow$(a) follows from Corollary 2.2.12(ii) for $p = 2$.

The equivalence (c)$\Leftrightarrow$(d) is straightforward. Indeed, the 2×2-matrix M has the eigenvector $\binom{1}{0}$ if and only if $M_{2,1} = 0$. The corresponding eigenvalue is then $\lambda = M_{1,1}$.

It remains to show the equivalence (a)$\Leftrightarrow$(c). To this end, assume that (a) holds. This means that $H_+x = 0$ has a bounded solution $x \in \ell^\infty(\mathbb{Z}_+) \setminus \{0\}$. Recall that $x_{-1} = 0$ due to the homogeneous Dirichlet boundary condition. The K-step recursion formula (2.15) for the eigenvalue equation of the one-sided Schrödinger operator gives us that

$$\begin{pmatrix} x_K \\ x_{K-1} \end{pmatrix} = M \begin{pmatrix} x_0 \\ x_{-1} \end{pmatrix} = M \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which inductively leads to the sequence

$$\begin{pmatrix} x_{nK} \\ x_{nK-1} \end{pmatrix} = M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n \in \mathbb{Z}_+. \tag{2.29}$$

By Corollary 2.3.6, the invertibility of H gives for the eigenvalues λ_1, λ_2 of M the relation $|\lambda_1| > 1 > |\lambda_2|$. Let furthermore (ξ_1, ξ_2) be a corresponding basis of eigenvectors of M. Expressing the initial vector $\binom{1}{0}$ in (2.29) through the basis (ξ_1, ξ_2) yields

$$\begin{pmatrix} x_{nK} \\ x_{nK-1} \end{pmatrix} = M^n(\alpha\xi_1 + \beta\xi_2) = \alpha\lambda_1^n\xi_1 + \beta\lambda_2^n\xi_2\,, \quad n \in \mathbb{Z}_+\,, \tag{2.30}$$

for some $\alpha, \beta \in \mathbb{R}$. By means of (2.30), we necessarily have $\alpha = 0$. Indeed, as x is bounded by assumption, the sequence on the left-hand side of (2.30) is bounded. Furthermore, $|\lambda_1| > 1$. Therefore, $\binom{1}{0}$ cannot have a component in the direction of ξ_1, i.e., $\alpha = 0$. Consequently, $\binom{1}{0}$ is an eigenvector of M to the eigenvalue λ_2 subject to $|\lambda_2| < 1$. This shows (c).

Finally, assume that (c) holds. Then, for all $0 \le j \le K-1$ and $n \in \mathbb{Z}_+$,

$$\begin{pmatrix} x_{nK+j} \\ x_{nK+j-1} \end{pmatrix} = T(j-1)\,\cdots\,T(0)\,M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda^n\,T(j-1)\,\cdots\,T(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

As $|\lambda| < 1$, this shows that the sequence $x = (x_j)_{j\in\mathbb{Z}_+}$ is bounded and (a) holds. □

Remark 2.3.17. The equivalence (b)⇔(d) in Proposition 2.3.16 can also be seen as a direct consequence of Proposition 2.3.13. Indeed, if $0 \in \sigma(H_+) \setminus \sigma(H)$, then $M_{2,1} = 0$ and $|M_{1,1}| < 1$ by (2.25).

Corollary 2.3.18. *Assume Hypothesis 2.3.1. Let H be invertible on $\ell^p(\mathbb{Z})$, and let $M := M(0)$ denote its monodromy matrix. Then the following are equivalent:*

(a) The compression H_+ is invertible on $\ell^p(\mathbb{Z}_+)$.

(b) $M_{2,1} \neq 0$ or $|M_{1,1}| > 1$.

If we consider a periodic Schrödinger operator with integer-valued potential, Proposition 2.3.16 leads to the following result about the invertibility of one-sided compressions H_+.

Proposition 2.3.19. *Assume Hypothesis 2.3.1 with $v(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. If H is invertible on $\ell^p(\mathbb{Z})$, then*

(i) H_+ is invertible on $\ell^p(\mathbb{Z}_+)$ and

(ii) H_- is invertible on $\ell^p(\mathbb{Z}_-)$.

Proof. Ad (i). Let H be invertible on $\ell^p(\mathbb{Z})$. In particular H is invertible on $\ell^\infty(\mathbb{Z})$ by Proposition 2.2.4. Let us show that H_+ is injective on $\ell^\infty(\mathbb{Z}_+)$. Indeed, for the sake of a contradiction, let us assume the contrary, i.e., let H_+ be not injective on $\ell^\infty(\mathbb{Z}_+)$. Then, by Proposition 2.3.16(a)⇔(d), $M_{2,1} = 0$ and $|M_{1,1}| < 1$. In particular, M is upper triangular with eigenvalues $M_{1,1}$ and $M_{2,2}$. However, as $v(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, we have $M \in \mathbb{Z}^{2\times 2}$, and this gives $M_{1,1} \in \mathbb{Z}$. Together with $|M_{1,1}| < 1$ this implies $M_{1,1} = 0$, which is impossible since $\det(M) = 1$ for symplectic matrices. So H_+ needs to be injective on $\ell^\infty(\mathbb{Z}_+)$. Finally, Corollary 2.2.12(ii) gives that H_+ is invertible on $\ell^p(\mathbb{Z}_+)$.

Ad (ii). Since H is invertible, H^{R} is invertible by Lemma 2.3.9(i). Applying (i) to H^{R} in place of H shows that H_+^{R} is invertible on $\ell^p(\mathbb{Z}_+)$. But, by the flip identity (2.19) from the proof of Lemma 2.3.9(i), H_- is invertible on $\ell^p(\mathbb{Z}_-)$. □

Remark 2.3.20. Note that it is possible to prove Proposition 2.3.19 without relying on the periodicity of the potential and only assuming Fredholmness of H instead of invertibility. The proof considers the realization of H on $\ell^2(\mathbb{Z})$ and shows invertibility of H_+ by Fredholm theory arguments, see [104, Thm. 1.1].

2.3.3 Periodic Schrödinger Operators with $\{0, \lambda\}$-valued Potentials

In the main results of Section 2.3.1 and Section 2.3.2, the potential v of the Schrödinger operator H was merely assumed to be periodic. In particular, a K-periodic potential was allowed to attain K different values. Let us introduce a subclass of periodic potentials based on the examples announced in Remark 2.1.2(ii) and (iii) and Remark 2.1.4. These examples share that the underlying periodic potential only takes the two values 0 and $\lambda > 0$. For this setting, consider the following modification of the defining equation (2.1)

$$(Hx)_k = x_{k-1} + x_{k+1} + \lambda\, v(k) x_k\,, \quad k \in \mathbb{Z}\,, \tag{2.31}$$

where we have factored out the so-called *coupling constant* $\lambda > 0$ such that v is merely a $\{0, 1\}$-valued potential. Given the parametrization of (2.31), we will study for which values of λ the FSM is applicable to the corresponding operator H in the upcoming section.

The interest in $\{0, \lambda\}$-valued potentials and operators of the form (2.31) is motivated by the numerical analysis of a subclass of *discrete aperiodic Schrödinger operators* for which the potential v is given as a *Sturmian potential*

$$v_{\alpha,\theta}(k) := \chi_{[1-\alpha,1)}(k\alpha + \theta \bmod 1)\,, \tag{2.32}$$

where the additional parameters $\alpha \in [0, 1] \setminus \mathbb{Q}$ and $\theta \in [0, 1)$ denote the irrational *slope* and *offset*, respectively, cf. [107, 131]. In the following, when we speak about a discrete *aperiodic* Schrödinger operator, we will always mean a one-dimensional discrete Schrödinger operator with potential $v_{\alpha,\theta}$ denoted by $H_{\lambda,\alpha,\theta}$.

As a Sturmian word's slope α is irrational, the corresponding potential $v_{\alpha,\theta}$ is not periodic. It was recently shown in [103] that the well-known *Fibonacci Hamiltonian*, which results for the choice of $\alpha = (\sqrt{5} - 1)/2$ and $\theta = 0$ in (2.32), is indeed FSM-simple. Studies aiming to extend this result to larger ranges of α led to the inspection of an additional natural approximation method relying on periodic Schrödinger operators of the form (2.31) as approximants to $H_{\lambda,\alpha,\theta}$, see, e.g., [53, Sec. 5]. Let us call this method *periodic*

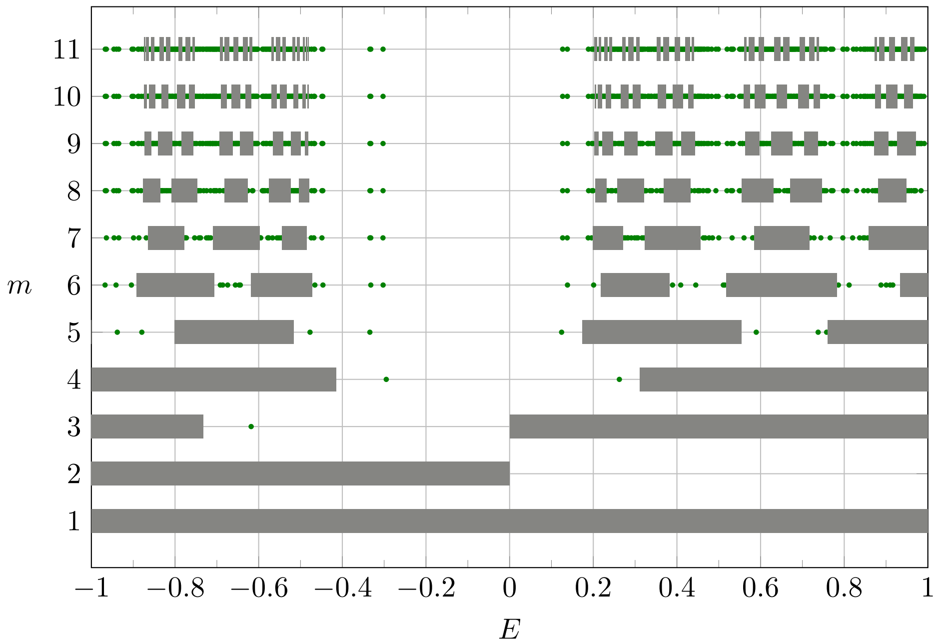

Figure 2.1: Approximation of the spectrum of the one-sided Fibonacci Hamiltonian and its limit operators restricted to energies $E \in [-1, 1]$. The band spectrum of the two-sided periodic Schrödinger approximants H_m, $m \in \mathbb{N}$, is represented by the color gray. The point spectrum of the one-sided periodic Schrödinger operators $B_+ \in \mathrm{Lim}(H_m) \cup \mathrm{Lim}(H_m^{\mathrm{R}})$, described by Theorem 2.3.13, is represented by the color green.

approximation. Figure 2.1 shows the spectra of the sequence of periodic approximants to the Fibonacci Hamiltonian and its one-sided compression.

Recall that the FSM described in Section 2.2 is an approximation method resembling the classical truncation strategy of using *finite-rank* operators to approximate an *infinite-rank* operator. In contrast to the FSM, the periodic approximation uses infinite-rank operators as approximants. Just as the FSM, periodic approximation can be interpreted as a truncation technique: *truncate the Sturmian potential at an index $K \in \mathbb{Z}_+$ and extend the truncation periodically. Then use the resulting periodic operators as approximants.* In fact, this truncation strategy resembles the operator theoretic analog of the approximation of irrational numbers via *truncated continued fractions* where one approximates $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ via $\alpha_m := \frac{p_m}{q_m}$ with suitably chosen q_m and p_m. We refer the interested reader to [83, 90] for details on the approximation of irrational numbers via continued fractions.

One can show, that periodic approximation creates a sequence of periodic operators converging pointwise to an aperiodic operator and that the invertibility of the periodic approximants implies the invertibility of the aperiodic limit. More precisely, we have the following classical result from Bellissard [16] for aperiodic Schrödinger operators. See also [53, Rem. 5.13].

Proposition 2.3.21 ([16, Prop. 4])**.** *Let $H_{\lambda,\alpha,\theta}$ be an aperiodic Schrödinger operator, and let $E \in \mathbb{R}$. Then $E \in \rho(H_{\lambda,\alpha,\theta})$ if and only if the condition*

$$\exists m \geq 0: \ \left|\operatorname{tr}\left(M_{\lambda,\alpha_m,0}(E)\right)\right| > 2 \text{ and } \left|\operatorname{tr}\left(M_{\lambda,\alpha_{m+1},0}(E)\right)\right| > 2 \tag{2.33}$$

holds, where $q_m \in \mathbb{N}$ is given by the m-th rational approximant $\frac{p_m}{q_m}$ to α for $m \in \mathbb{N}$ and $M_{\lambda,\alpha_m,0}(E)$ denotes the monodromy matrix corresponding to the approximating periodic Schrödinger operator $H_{\lambda,\alpha_m,0}$.

The following corollary connects Proposition 2.3.21 to the invertibility problem of all limit operators of a given aperiodic Schrödinger operator. For a proof, see [53, Cor. 5.16].

Corollary 2.3.22. *Let $H_{\lambda,\alpha,\theta}$ be an aperiodic Schrödinger operator. All limit operators of $H_{\lambda,\alpha,\theta}$ are invertible if and only if*

$$\exists m \geq 0: \ \left|\operatorname{tr}\left(M_{\lambda,\alpha_m,0}(0)\right)\right| > 2 \text{ and } \left|\operatorname{tr}\left(M_{\lambda,\alpha_{m+1},0}(0)\right)\right| > 2\,. \tag{2.34}$$

Example 2.3.23. Let $H_{\lambda,\alpha,0}$ be an aperiodic Schrödinger operator with $\alpha = (\sqrt{5}-1)/2$ and $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned}\operatorname{tr}(M_{\lambda,\alpha_6,0}(0)) &= \lambda(18\lambda^2 - 8)\,, \\ \operatorname{tr}(M_{\lambda,\alpha_7,0}(0)) &= \lambda(-108\lambda^4 + 84\lambda^2 - 13)\,.\end{aligned}$$

For $\lambda > \frac{1}{6}(3+\sqrt{3}) =: \lambda_0 \approx 0.788675$, we have

$$\operatorname{tr}\left(M_{\lambda,\alpha_6,0}(0)\right) > 2 \quad \text{and} \quad \operatorname{tr}\left(M_{\lambda,\alpha_7,0}(0)\right) < -2\,.$$

Hence, by Corollary 2.3.22, for $\lambda > \lambda_0$, all limit operators of the Fibonacci Hamiltonian $H_{\lambda,\alpha,0}$ are invertible. The bound λ_0 may be further improved by considering higher orders of approximation. Considering the polynomial $\operatorname{tr}(M_{\lambda,\alpha_8,0}(0))$, however, does not improve the result despite being algebraically solvable as well. Figure 2.2 visualizes the spectra of the periodic approximations for $m = 5, 6, 7, 8$ with special emphasis on the previously derived bound λ_0, see also [29, Sec. 7.1].

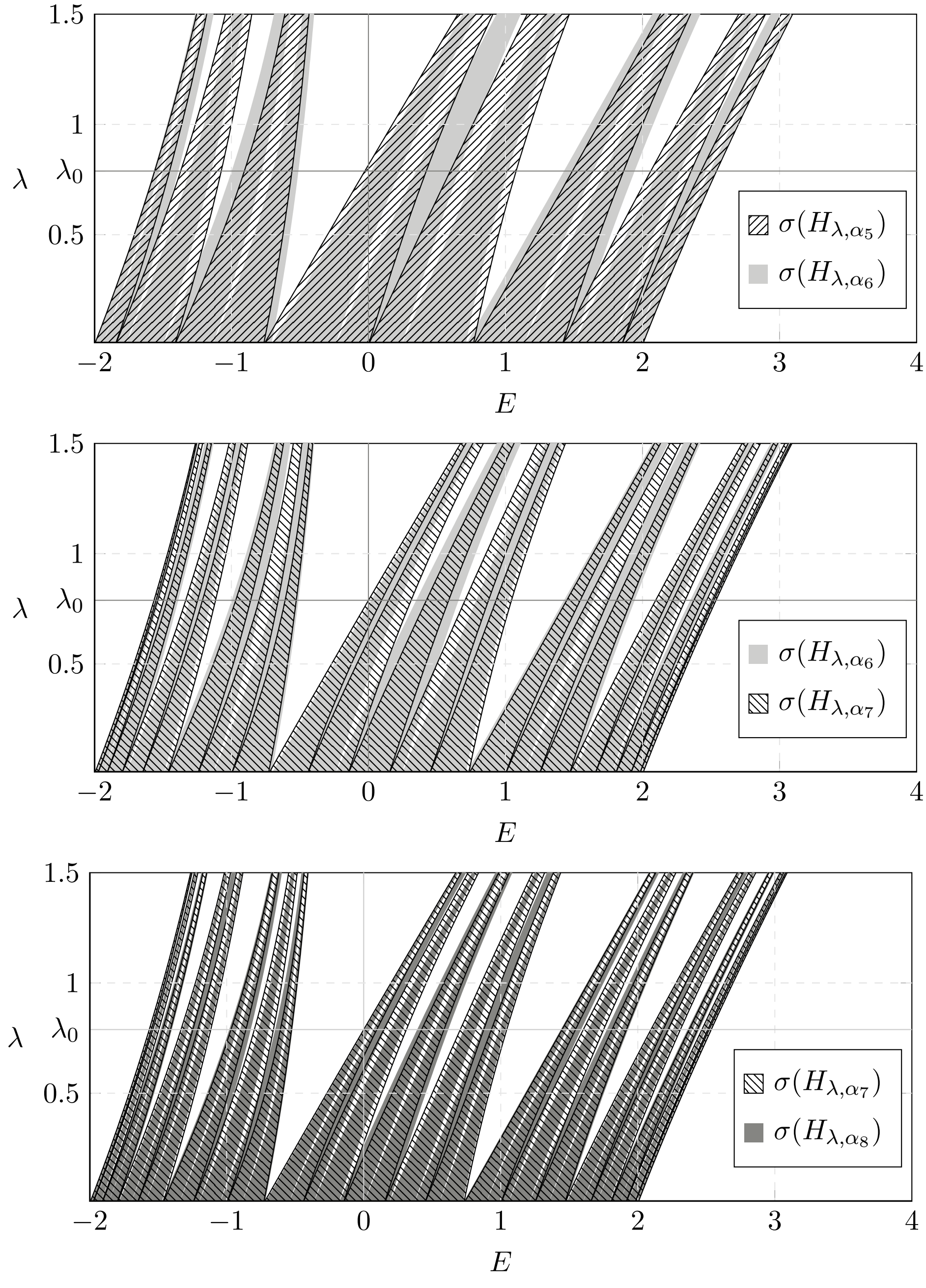

Figure 2.2: Spectra of the periodic approximations $H_{\lambda,\alpha_m} := H_{\lambda,\alpha_m,0}$ for $m = 5, 6, 7, 8$ of the scaled Fibonacci Hamiltonian $H_{\lambda,\alpha,0}$, where $\alpha = (\sqrt{5}-1)/2$ and $\lambda > 0$. If $\lambda > \lambda_0$, then $0 \notin \sigma(H_{\lambda,\alpha,0})$ as $\sigma(H_{\lambda,\alpha,0}) \subseteq \sigma(H_{\lambda,\alpha_m,0}) \cup \sigma(H_{\lambda,\alpha_{m+1},0})$ by Corollary 2.3.21 and all limit operators of $H_{\lambda,\alpha,0}$ are invertible by Corollary 2.3.22.

For more details on the periodic approximation of aperiodic Schrödinger operators, we refer the reader to the works [53] and [16].

Summing up, despite their simplicity, their crucial role in the approximation theory of aperiodic Schrödinger operators motivates the study of periodic Schrödinger operators with $\{0, \lambda\}$-valued potentials. The following section will therefore analyze this building block of the periodic approximation method by studying the invertibility properties of the approximants.

2.4 Finite Section Method for Periodic Schrödinger Operators

In this section, we dive into the applicability analysis of the FSM. The following proposition is a variant of the general strategy encoded in Proposition 2.2.8 adapted to its use in the setting of periodic Schrödinger operators.

Proposition 2.4.1. *Assume Hypothesis 2.3.1. Then the FSM is applicable to H if and only if the following operators are invertible:*

(i) H,

(ii) all L_+ *with* $L \in \mathrm{Lim}(H)$, *and*

(iii) all $\widetilde{L}_+$ *with* $\widetilde{L} \in \mathrm{Lim}(H^{\mathrm{R}})$.

Similarly, the FSM is applicable to H_+ *if and only if the following operators are invertible:*

(iv) H_+ *and*

(v) all $\widetilde{L}_+$ *with* $\widetilde{L} \in \mathrm{Lim}(H_+^{\mathrm{R}})$.

Proof. Note the identity $\mathrm{Lim}(H) = \mathrm{Lim}_+(H) = \mathrm{Lim}_-(H)$ which follows from Lemma 2.3.2. Therefore, comparing Proposition 2.4.1 with Proposition 2.2.8, the only thing to show is the equivalence of Proposition 2.4.1(iii) and Proposition 2.2.8(ii). But this is a direct consequence of Lemma 2.3.9(i) and the flip identity (2.19). With the same argument, the conditions for the one-sided operator H_+ follow. ☐

As a consequence of Proposition 2.4.1, one +an visualize the regions of applicability of the FSM by considering a union of spectra of limit operators. Let us fix a set decomposition suitable for visualization in the following corollary.

Corollary 2.4.2. *Assume Hypothesis 2.3.1. The FSM is applicable to H if and only if*

$$0 \notin \sigma(H) \;\cup\; \bigcup \left\{ \sigma_{\mathrm{dis}}(B_+) : B \in \mathrm{Lim}(H) \cup \mathrm{Lim}(H^{\mathrm{R}}) \right\}. \tag{2.35}$$

Proof. Proposition 2.4.1 implies that the FSM is applicable to H if and only if $0 \notin \sigma(H)$ and $0 \notin \sigma(B_+)$ for all $B \in \mathrm{Lim}(H) \cup \mathrm{Lim}(H^{\mathrm{R}})$. Recall from Lemma 2.3.9(ii) that all limit operators $B \in \mathrm{Lim}(H) \cup \mathrm{Lim}(H^{\mathrm{R}})$ have the same spectrum as H. In particular, this means that, for the FSM to be applicable to H, we need $0 \notin \sigma(H)$ and that 0 is not a Dirichlet eigenvalue of B_+ for all $B \in \mathrm{Lim}(H) \cup \mathrm{Lim}(H^{\mathrm{R}})$. This is precisely the statement (2.35). ☐

For the subclass of $\{0, \lambda\}$-valued potentials from Section 2.3.3, we will visualize the set (2.35) for different choices of v while varying the coupling constant $\lambda > 0$. Figure 2.3 visualizes (2.35) for the periodic continuation of $w = \lambda \cdot (1, 1, 0, 1)$, Figure 2.4 for the periodic continuation of $w = \lambda \cdot (1, 1, 0, 1, 0)$, and Figure 2.5 for the periodic continuation of $w = \lambda \cdot (1, 1, 0, 1, 0, 1, 0, 1, 1)$. All of the plots feature examples of potentials that can either be generalized to a larger class of similar potentials or serve as borderline cases illustrating the optimality of the results of Theorem 2.1.1 which we are going to prove in this section. We will revisit the latter two choices of w in Section 2.4.3. The underlying spectral data for the figures and the corresponding code for their calculation is available in [55].

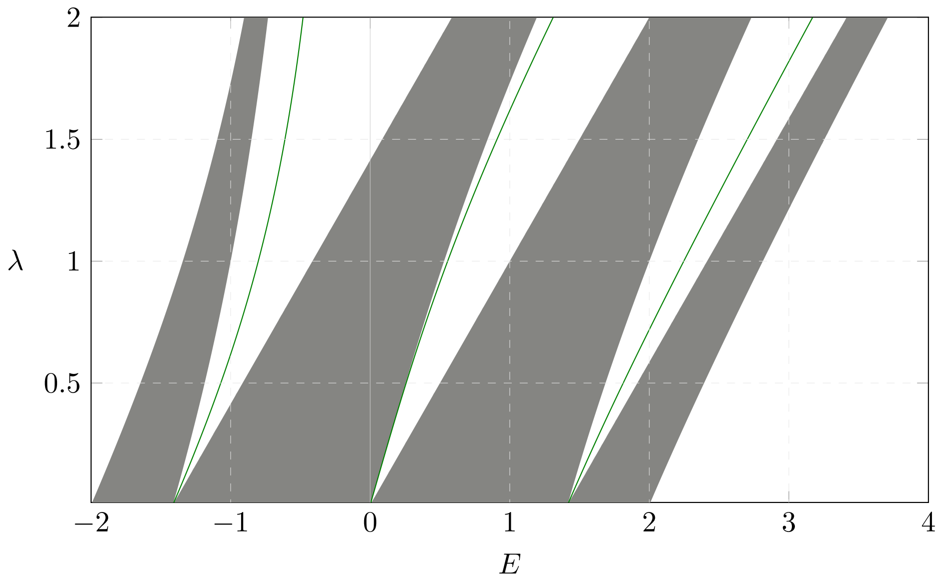

Figure 2.3: Union (2.35) of spectra for H with the 4-periodic potential v being the periodic extension of $w = \lambda \cdot (1, 1, 0, 1)$ while λ changes along the vertical axis. The spectral bands are shown in gray and the Dirichlet eigenvalues in green. Looking at the vertical line $E = 0$, the plot suggests that, whenever H is invertible, then all the B_+ and B_- are invertible. We will show in Section 2.4.2 that all 4-periodic operators with a $\{0, \lambda\}$-valued potential are in fact FSM-simple.

The outline of this section is as follows: we first prove Theorem 2.1.3 as this result will serve as agenda for the further applicability analyses that need to be carried out in order to study the concrete situation of integer-valued and $\{0, \lambda\}$-valued potentials in Theorem 2.1.1. We will also present the algorithm that allowed the systematic study of $\{0, \lambda\}$-valued potentials up to period 9. Lastly, we prove that our presented applicability results for periodic Schrödinger operators are optimal.

2.4.1 Applicability Analysis of the Finite Section Method

Recall from Proposition 2.4.1 that the applicability of the FSM builds on the invertibility of the operator H itself and selected one-sided compressions of limit operators of H and its *reversed* counterpart H^{R}. In Section 2.3, we learned how to characterize invertibility of

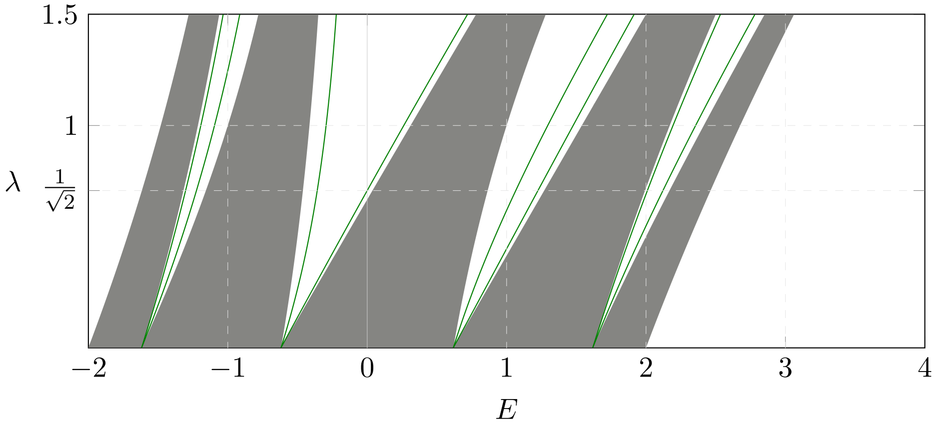

Figure 2.4: Union (2.35) of spectra for H with the 5-periodic potential v being the periodic extension of $w = \lambda \cdot (1, 1, 0, 1, 0)$ while λ changes along the vertical axis. The spectral bands are shown in gray and the Dirichlet eigenvalues in green. We see that one Dirichlet eigenvalue crosses the vertical line $E = 0$ at height $\lambda = \frac{1}{\sqrt{2}}$. For this λ, the operator H is invertible, but the FSM is not applicable. In Section 2.4.2, we detect this example algebraically, and, in Example 2.4.9, we prove that this operator is indeed not FSM-simple.

periodic Schrödinger operators via their monodromy matrices. Theorem 2.1.3 now builds the bridge between the abstract requirements for the applicability of the FSM and the concrete structure of periodic Schrödinger operators. More precisely, we will use the characterization of invertibility in terms of monodromy matrices as presented in Proposition 2.3.16.

Proof of Theorem 2.1.3. Ad (i). Without loss of generality, we assume that H is invertible. Indeed, if the FSM is applicable, this follows *per definitionem*, and, if $|\operatorname{tr}(M)| > 2$, this is a consequence of Corollary 2.3.6.

Let us now verify conditions (ii) and (iii) from Proposition 2.4.1. To this end, let $B \in \operatorname{Lim}(H) \cup \operatorname{Lim}(H^{\mathrm{R}})$ with monodromy matrix M. Due to Lemma 2.3.2(i), there exists $j \in \{0, \dots, K-1\}$ with $B = S^{-j}HS^{j}$ or $B = S^{j-1}H^{\mathrm{R}}S^{-j+1}$. In particular, $M = M^{(j)}$ or $M = \widetilde{M}^{(j)}$ in the formulation of Theorem 2.1.3.

Let us show that B_+ is invertible. Indeed, H is invertible by assumption, and $\sigma(B) = \sigma(H) = \sigma(H^{\mathrm{R}})$, as a consequence of Lemma 2.3.9(ii). Finally, under the assumption that the monodromy matrix M of B fulfills $M_{1,2} \neq 0$ or $|M_{1,1}| > 1$, Corollary 2.3.18(b)⇒(a) implies that B_+ is invertible on $\ell^p(\mathbb{Z}_+)$. As B was chosen arbitrarily, the proof for (i) follows from Proposition 2.4.1.

Ad (ii). It is no loss of generality to assume invertibility of H_+. Indeed, as in part (i), it follows *per definitionem* if the FSM is assumed to be applicable to H_+. For the converse implication, this follows from the trace condition $|\operatorname{tr}(M)| > 2$ and the assumption that the monodromy matrix M of H_+ is subject to the statement

$$\text{“ } M_{2,1} \neq 0 \quad \text{or} \quad |M_{1,1}| > 1 \text{ ”.}$$

In fact, the invertibility of H_+ now follows with the same argument as the invertibility of B_+ in part (i). By the characterization of limit operators from Lemma 2.3.2(i), we can

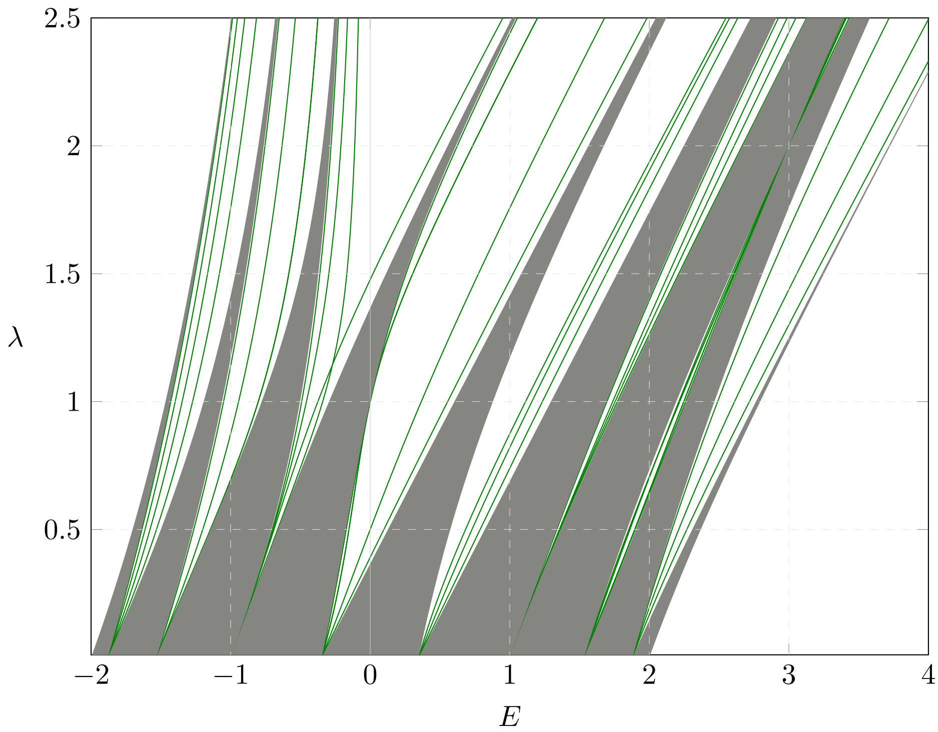

Figure 2.5: Union (2.35) of spectra for H with the 9-periodic potential v being the periodic extension of $w = \lambda \cdot (1, 1, 0, 1, 0, 1, 0, 1, 1)$ while λ changes along the vertical axis. The spectral bands are shown in gray and the Dirichlet eigenvalues in green. At $E = 3$ and $\lambda = 2$, two spectral bands merge into one before breaking up again. We observe that one Dirichlet eigenvalue crosses the vertical line $E = 0$ at height $\lambda = \frac{1}{2}$. Consequently, for $\lambda = \frac{1}{2}$, the operator H is invertible, but the FSM is not applicable. Unlike for periods of length $K < 9$, this crossing happens at a rational value of λ. In Section 2.4.2, we show how to detect this example algebraically, and, in Example 2.4.11, we prove that this operator is indeed not FSM-simple.

reuse this argument furthermore to conclude the invertibility of all compressions B_+ of limit operators $B_+ \in \mathrm{Lim}(H^{\mathrm{R}})$. This shows (iv) and (v) from Proposition 2.4.1 and thereby concludes the proof of (ii).

Finally, assume that H is FSM-simple. As $H \in \mathrm{Lim}(H) = \mathrm{Lim}_+(H)$ by Lemma 2.3.2(i), Corollary 2.2.13 gives that also H_+ needs to be FSM-simple. □

Given the overlap of requirements of the parts for H and H_+ in Proposition 2.4.1, it is natural to ask whether the applicability of the FSM to H_+ is indeed sufficient for the applicability to H. As we will see in Example 2.4.12 this is not always the case. However, note that the following additional structural assumption on the potential v guarantees simultaneous applicability of the FSM to H_+ and H.

Definition 2.4.3 (Palindrome)**.** Let $w \in \mathbb{R}^K$, $K \in \mathbb{N}$. If

$$w = (w_0, w_1, \dots, w_{K-2}, w_{K-1}) = (w_{K-1}, w_{K-2}, \dots, w_1, w_0) =: w^{\mathrm{R}},$$

we call w a *palindrome.*

Corollary 2.4.4. *Assume Hypothesis 2.3.1. If there exists $j \in \{0, \dots, K-1\}$ such that $S^{-j}\, v\, S^j$ is the periodic extension of a* palindrome*, then*

$$\{M^{(j)} : j = 0, \dots, K-1\} \;=\; \{\widetilde{M}^{(j)} : j = 0, \dots, K-1\}.$$

In particular, in this case, the FSM is applicable to H if and only if it is applicable to H_+.

Let us continue our applicability analysis of the FSM for periodic Schrödinger operators and prove Theorem 2.1.1.

Proof of Theorem 2.1.1. It suffices to analyze the case for the two-sided infinite operator H. Indeed, Theorem 2.1.3 implies that the one-sided compression H_+ is FSM-simple once the same holds for H. Our guideline will be to verify the conditions of Proposition 2.4.1.

Ad (i). Let the potential v be integer-valued, and assume that the corresponding Schrödinger operator H is invertible. Let $B \in \mathrm{Lim}(H) \cup \mathrm{Lim}(H^{\mathrm{R}})$. We want to show that B_+ is invertible. As H was assumed to be invertible, the same holds for H^{R} as a consequence of Lemma 2.3.9(i). Furthermore, the limit operator B is invertible as a consequence of Lemma 2.3.2(ii). As v is integer-valued, the invertibility of the one-sided compression B_+ now follows from Proposition 2.3.19(i). As B was chosen arbitrarily, this verifies conditions (ii) and (iii) from Proposition 2.4.1.

Ad (ii). This is proven via the algorithmic analysis carried out in [52]. We postpone the description of the algorithm to Section 2.4.2 below.

Ad (iii). Let H be a 2-periodic Schrödinger operator with real-valued potential v. Let $B \in \mathrm{Lim}(H) \cup \mathrm{Lim}(H^{\mathrm{R}})$. As a consequence of Lemma 2.3.2(i), B is again 2-periodic. Recall that Example 2.3.15(i) showed that the spectrum of one-sided compressions of 2-periodic Schrödinger operators does not add any Dirichlet eigenvalues. Hence, $\sigma(B_+) = \sigma(B)$. By Lemma 2.3.9(ii), we furthermore have $\sigma(B) = \sigma(H)$. Joining the former two spectral identities shows that if H is invertible, then B_+ is invertible. As B was chosen arbitrarily, this verifies conditions (ii) and (iii) from Proposition 2.4.1. □

For a 2-periodic potential, Figure 2.6 shows a plot of the critical region outside of which the trace condition is fulfilled and the FSM is applicable to H as a consequence of Theorem 2.1.1(iii).

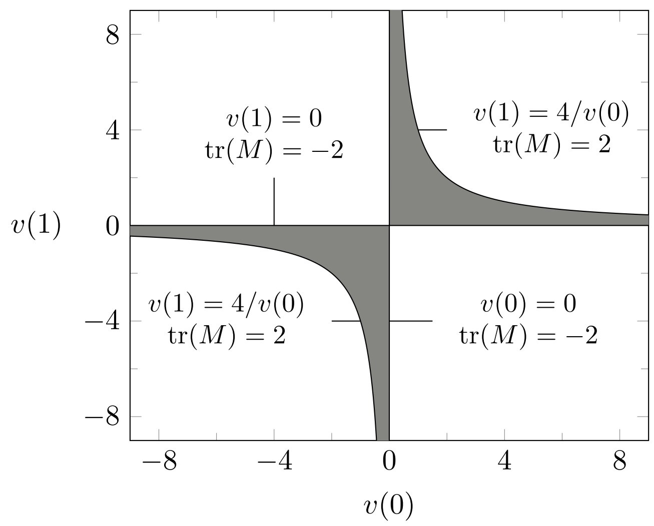

Figure 2.6: For a general 2-periodic potential v, Example 2.3.7(iii) showed that $\mathrm{tr}(M(0)) = -2 + v(0)v(1)$. For all points $(v(0), v(1))$ lying outside of the closed gray region, we have $|\,\mathrm{tr}(M(0))| > 2$. Hence, the FSM is applicable to the 2-periodic Schrödinger operator H with potential $(v(0), v(1))$ as a consequence of Theorem 2.1.1(iii).

2.4.2 Algorithmic Analysis of $\{0, \lambda\}$-valued Potentials

In this section, we present an algorithm for finding *non-FSM-simple* operators with $\{0, \lambda\}$-valued periodic potentials. As our algorithm performs an exhaustive search, its outcomes can be used to prove a positive result for specific periods $K \in \mathbb{N}$ by *not* finding such examples. Recall from Section 2.3.3 that, for $K \in \mathbb{N}$ and $w \in \{0, 1\}^K$, we consider the potential $v \in \{0, \lambda\}^{\mathbb{Z}}$ as the periodic extension of $\lambda \cdot w$ with corresponding Schrödinger operator H subject to Hypothesis 2.3.1.

The algorithm will build on Hagger's spectral representation formula (2.25) and, more specifically, on the invertibility result from Corollary 2.3.18 by analyzing the zeros of the entry $M_{2,1}$ of the monodromy matrix M for a given potential v. Before we go into details, let us note the following lemma about the structure of the matrix entry $M_{2,1}$.

Lemma 2.4.5. *Let $\lambda \in \mathbb{R}$, $w \in \{0, 1\}^K$, $K \in \mathbb{N}$, and v be the K-periodic extension of $\lambda \cdot w$. Let M be the corresponding monodromy matrix from formula* (2.14)*, and let $p_{M_{2,1}}(\lambda)$ denote the entry at position* $(2, 1)$ *of M as a polynomial in λ. If $K \le 9$ and* $\deg(p_{M_{2,1}}(\lambda)) \ge 1$*, then the polynomial equation*

$$p_{M_{2,1}}(\lambda) = 0$$

has a solution in radicals.

Proof. Recall from (2.14) that

$$M = \begin{pmatrix} -v(K-1) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -v(1) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -v(0) & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.36}$$

It is clear from (2.36) that each entry of M is a polynomial of degree at most K. If $K \leq 4$, this proves the claim by well-known results from algebra. For $K > 4$, we proceed in three steps.

Step 1. Let us prove that the claim follows once we can show that $M_{2,1}(\lambda)$ is always an *even* or an *odd* polynomial. Indeed, if $M_{2,1}(\lambda)$ is odd with $\deg(M_{2,1}(\lambda)) \in \{5, 7, 9\}$, we may factorize $M_{2,1}$ as $\lambda \cdot p(\lambda) = M_{2,1(\lambda)}$ with an even polynomial p having $\deg(p) \in \{4, 6, 8\}$. In particular, $q(\mu) := p(\mu^2)$ is a polynomial of degree 2, 3, or 4, respectively, such that one can determine the roots of q via the standard formulas. A similar argument works in the case that $\deg(M_{2,1}(\lambda)) \in \{6, 8\}$.

Step 2. In this step, we show that, for the verification of Step 1, it suffices to only work with the periodic continuation v of the constant word $\lambda \cdot w = (\lambda, \ldots, \lambda)$ of length K and show that, for this particular choice, the symmetry of the polynomial entries of the corresponding monodromy matrix M follows the pattern

$$M \sim \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix}, \quad \text{for } K \text{ odd}, \qquad M \sim \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}, \quad \text{for } K \text{ even}, \tag{2.37}$$

where the entries *even* or *odd* determine whether the corresponding polynomial entry of M is an even or an odd polynomial, respectively. Indeed, assume that (2.37) holds and we are given a non-constant $\{0, \lambda\}$-valued K-periodic potential v.

In order to avoid confusion, let M_v denote the monodromy matrix of the K-periodic Schrödinger operator with potential v and M_λ the monodromy matrix of the K-periodic Schrödinger operator with constant potential equal to the value λ. By (2.37), each entry of M_λ will be a linear combination of either even or odd monomials. Transitioning from M_λ to M_v, this means that a monomial λ^k in an entry of M_λ will correspond to a product $v(j_1) \cdots v(j_k)$ in the corresponding entry of M_v, where $0 \leq j_1 < j_2 < \cdots < j_k \leq K - 1$. Clearly, if $v(j_l) = 0$ for some $l \in \{1, \ldots, k\}$, the whole product will cancel, leaving the overall symmetry of the polynomial unchanged. The symmetry also remains unchanged if $v(j_l) = \lambda$ for all $l \in \{1, \ldots, k\}$.

Step 3. Given the constant potential $v(k) = \lambda$ for all $k \in \mathbb{Z}$, we will show that the symmetry pattern (2.37) holds via induction: for the base case, a short calculation reveals that (2.37) holds for $K = 2$ and $K = 3$. Indeed,

$$M = \begin{pmatrix} -\lambda & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{for } K = 1, \qquad M = \begin{pmatrix} \lambda^2 - 1 & \lambda \\ -\lambda & -1 \end{pmatrix}, \quad \text{for } K = 2.$$

For the induction step, assume that the claim holds for some $K \in \mathbb{N}$. If K is even, for $K + 1$, we decompose (2.36) as

$$M = \begin{pmatrix} -\lambda & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} = \begin{pmatrix} -\lambda \cdot \text{even} - \text{odd} & -\lambda \cdot \text{odd} - \text{even} \\ \text{even} & \text{odd} \end{pmatrix}.$$

If K is odd, for $K + 1$, we decompose (2.36) as

$$M = \begin{pmatrix} -\lambda & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix} = \begin{pmatrix} -\lambda \cdot \text{odd} - \text{even} & -\lambda \cdot \text{even} - \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}.$$

A comparison with (2.37) proves the claimed symmetry. $\square$

The overall goal of our algorithmic study is the following:

> *For a fixed period length K, find all $\{0,\lambda\}$-valued K-periodic potentials for which the corresponding monodromy matrix M fulfills $|\operatorname{tr}(M)| > 2$ and $M_{2,1} = 0$.*

If the algorithm *fails to find any such potential*, we conclude that the trace condition $|\operatorname{tr}(M)| > 2$ mutually excludes $M_{2,1} = 0$. In other words, *every* one-sided compression H_+ of a K-periodic Schrödinger operator H with $\{0,\lambda\}$-valued potential subject to $|\operatorname{tr}(M)| > 2$ will have a monodromy matrix with entry $M_{2,1} \neq 0$ and thus be invertible as a consequence of Corollary 2.3.18. In particular, this means that the FSM is applicable to each of these operators by Proposition 2.4.1.

Our algorithm now proceeds as follows: for a fixed period K and every $w \in \{0,1\}^K$, use formula (2.14) to compute the monodromy matrix $M = M(0)$ of the K-periodic Schrödinger operator having as potential v the periodic extension of $\lambda \cdot w$. The four entries $M_{i,j}$, $i,j \in \{1,2\}$, of the matrix M are given by polynomials $p_{M_{i,j}}(\lambda)$ in λ. In virtue of Lemma 2.4.5, we can compute the zeros of $p_{M_{2,1}}(\lambda)$ exactly by radicals up to the period length $K = 9$. Filter the list of all potentials by keeping only those with at least one zero $\lambda^* \in \mathbb{Q}$ and have the trace of M be greater than 2 in modulus. *Per constructionem*, the resulting list of potentials contains all $\{0,\lambda\}$-valued potentials of length K that lead to non-FSM-simple Schrödinger operators. We claim that this list remains empty for $K \leq 8$. The proof of this statement follows from the data in [52].

Example 2.4.6. We demonstrate the outcomes of the algorithm above for $\{0,\lambda\}$-valued potentials of period $K = 3$ in Table 2.1. We conclude that, for $K = 3$, all $w \in \{0,1\}^3$, and all $\lambda \in \mathbb{R}$, the implication

$$M_{2,1} = 0 \quad \overset{\text{Tab. 2.1}}{\Longrightarrow} \quad |\operatorname{tr}(M)| \leq 2 \quad \overset{\text{Prop. 2.3.5}}{\Longrightarrow} \quad H \text{ is not invertible} \tag{2.38}$$

is valid. The contraposition of (2.38) then shows

$$H \text{ invertible} \quad \overset{(2.38)}{\Longrightarrow} \quad M_{2,1} \neq 0 \quad \overset{\text{Cor. 2.3.18}}{\Longrightarrow} \quad H_+ \text{ is invertible.} \tag{2.39}$$

By Lemma 2.3.9(ii), all $B \in \operatorname{Lim}(H) \cup \operatorname{Lim}(H^{\mathrm{R}})$ are invertible if H is invertible. We apply the reasoning (2.39) with $B \in \operatorname{Lim}(H) \cup \operatorname{Lim}(H^{\mathrm{R}})$ in place of H and derive that all corresponding compressions B_+ are invertible if H is invertible. We conclude by Proposition 2.4.1 that the FSM is applicable to H if H is invertible. As Table 2.1 considers all possible $\{0,\lambda\}$-valued 3-periodic potentials this shows that all these potentials lead to FSM-simple operators.

For every period K, there is a total of 2^K potentials needed to check in order to replicate the procedure outlaid in Example 2.4.6. In order to handle this exponential complexity, we implemented the algorithm using the computer algebra system `SageMath` [129] and `Jupyter Notebooks` [84]. Recall that, thanks to Lemma 2.4.5, all relevant roots can be determined by analytic formulas which allows for purely symbolic computations throughout the whole program. The complete study up to 9-periodic potentials and the corresponding code have already been published in [52].

Remark 2.4.7. A careful study of the results provided in [52], shows that, if $K \in \{1,2,3,4\}$ and $\lambda \in \mathbb{R}$, then all discrete K-periodic Schrödinger operators with a $\{0,\lambda\}$-valued potential are FSM-simple. See also Example 2.4.9 for the optimality of this statement.

$\lambda \cdot w$	M	zeros of $p_{M_{2,1}}$	$\mathrm{tr}(M)$	at zeros
$(0,0,0)$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\emptyset$	0	
$(\lambda,0,0)$	$\begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}$	$\emptyset$	λ	
$(0,\lambda,0)$	$\begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}$	$\emptyset$	λ	
$(0,0,\lambda)$	$\begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}$	$\emptyset$	λ	
$(\lambda,\lambda,0)$	$\begin{pmatrix} \lambda & 1 \\ \lambda^2-1 & \lambda \end{pmatrix}$	$\{-1,1\}$	2λ	$\{-2,2\}$
$(\lambda,0,\lambda)$	$\begin{pmatrix} 2\lambda & 1 \\ -1 & 0 \end{pmatrix}$	$\emptyset$	2λ	
$(0,\lambda,\lambda)$	$\begin{pmatrix} \lambda & -\lambda^2+1 \\ -1 & \lambda \end{pmatrix}$	$\emptyset$	2λ	
$(\lambda,\lambda,\lambda)$	$\begin{pmatrix} -\lambda^3+2\lambda & -\lambda^2+1 \\ \lambda^2-1 & \lambda \end{pmatrix}$	$\{-1,1\}$	$-\lambda^3+3\lambda$	$\{-2,2\}$

Table 2.1: Systematic study of all $\{0,\lambda\}$-valued potentials with period $K = 3$. For every sequence $w \in \{0,1\}^3$, the monodromy matrix M corresponding to the periodic continuation v of $\lambda \cdot w$ is computed. Then all zeros of the entry $M_{2,1}$ as a polynomial $p_{M_{1,2}}(\lambda)$ are determined. Finally, in the last column, the trace of M is calculated yielding a function in λ which is then evaluated at each zero of $p_{M_{2,1}}(\lambda)$.

2.4.3 Optimality of the Applicability Results

In this section, we will prove that, in the setting of periodic Schrödinger operators, the results presented in Theorem 2.1.1 and Theorem 2.1.3 are optimal. In particular, we will provide counterexamples showing that neither the conditions on the period length K nor the set of values for λ can be relaxed. Moreover, we show that H and H_+ do not need to be simultaneously FSM-simple.

Optimality of Integer-valued Potentials

Example 2.4.8. Consider the 3-periodic Schrödinger operator H with potential $v(0) = 2$, $v(1) = \frac{1}{2}$, and $v(2) = \frac{1}{2}$. The operator and its limit operators have the monodromy matrices

$$M^{(0)} = \begin{pmatrix} 2 & \frac{3}{4} \\ 0 & \frac{1}{2} \end{pmatrix}, \quad M^{(1)} = \begin{pmatrix} 2 & 0 \\ -\frac{3}{4} & \frac{1}{2} \end{pmatrix}, \quad \text{and} \quad M^{(2)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}.$$

All matrices have trace $\frac{5}{2} > 2$. Therefore, H and its limit operators are invertible by Corollary 2.3.6. Note that

$$M^{(2)}_{2,1} = 0 \qquad \text{and} \qquad |M^{(2)}_{1,1}| = \frac{1}{2} < 1$$

so that $M^{(2)}$ fulfills condition (d) of Proposition 2.3.16. The equivalence of this condition with Proposition 2.3.16(b) gives that the one-sided compression B_+ with $B = S^{-2}HS^2 \in \operatorname{Lim}(H)$ is not invertible. Hence, H is not FSM-simple, and we see that we can neither drop the condition on integer-valuedness nor the restriction to 2-periodic potentials in Theorem 2.1.1(i) and (iii), respectively.

Note that $S^{-2}HS^2$ is the periodic extension of the palindrome $(\frac{1}{2}, 2, \frac{1}{2})$. As a consequence of Corollary 2.4.4, H_+ is not FSM-simple.

Optimality of $\{0, \lambda\}$-valued Potentials with Rational λ

Example 2.4.9. The 5-periodic Schrödinger operator H having as potential the periodic continuation of

$$\frac{1}{\sqrt{2}}(1, 1, 0, 1, 0)$$

has the monodromy matrix

$$M = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -1 \\ 0 & -\sqrt{2} \end{pmatrix},$$

see [52].

For the monodromy matrix, we have $|\operatorname{tr}(M)| > 2$ but also $M_{2,1} = 0$ and $|M_{1,1}| < 1$. Hence, H is invertible by Corollary 2.3.6 but H_+ is not invertible as a consequence of Proposition 2.3.16. This shows that H is not FSM-simple.

Remark 2.4.10. Another example similar to Example 2.4.9 can be found in the Bachelor thesis [157].

Optimality of 8-periodic $\{0, \lambda\}$-valued Potentials

Example 2.4.11. Consider the 9-periodic Schrödinger operator H having as potential the periodic continuation of

$$w := \frac{1}{2}(1, 1, 0, 1, 0, 1, 0, 1, 1)$$

has the monodromy matrix

$$M = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix},$$

see [52]. We have $|\operatorname{tr}(M)| = \frac{5}{2} > 2$, whence H is invertible by Corollary 2.3.6. But $M_{2,1} = 0$ such that H_+ is not invertible and the FSM is not applicable to H.

Simultaneous Applicability of the FSM

Example 2.4.12. Consider the 9-periodic Schrödinger operator H having the periodic continuation of

$$\frac{1}{\sqrt{2}}(1, 1, 1, 0, 1, 1, 0, 1, 0)$$

as its potential. One may check that, for the corresponding monodromy matrix $M^{(1)}$ of the limit operator $B = S^{-1}HS \in \mathrm{Lim}(H)$, one has that

$$M^{(1)} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 2 \\ 0 & -\sqrt{2} \end{pmatrix}$$

and therefore, the FSM is not applicable to H, as B_+ is not invertible. However, in contrast to Example 2.4.9, the results of [52] show that, for all monodromy matrices of operators $B \in \mathrm{Lim}(H^{\mathrm{R}})$, the conditions of Theorem 2.1.3(ii) are satisfied. Hence, the FSM is applicable to H_+.

Chapter 3

Observability and Control of Non-autonomous Cauchy Problems

This chapter is based on the joint work with C. Bombach, C. Seifert, and M. Tautenhahn [18].

3.1 Introduction

Let X and Y be Banach spaces, $T > 0$, $(A(t))_{t\in[0,T]}$ a family of operators on X

$$A(t)\colon \operatorname{Dom}(A(t)) \to X,$$

and $(C(t))_{t\in[0,T]}$ a family of bounded operators $C(t)\colon X \to Y$. Consider the following system of operator equations

$$\begin{aligned} \dot{x}(t) &= -A(t)x(t), \quad t \in (0,T], \quad x(0) = x_0, \\ y(t) &= C(t)x(t), \quad t \in [0,T], \end{aligned} \tag{3.1}$$

where the first equation in (3.1) describes the evolution of a state function x driven by the operators $A(t)$ and the second equation describes the observation function y as the linear image of the state function x under the operators $C(t)$. We are interested in the following question.

> *For a given measurable subset $E \subseteq [0,T]$ and $r \in [1,\infty]$, does there exist a constant $C_{\mathrm{obs}} \geq 0$ such that, for all initial values $x_0 \in X$, the estimate*
>
> $$\|x(T)\|_X \leq C_{\mathrm{obs}} \begin{cases} \left(\int_E \|y(t)\|_Y^r \,\mathrm{d}t\right)^{1/r}, & r \in [1,\infty), \\ \operatorname{ess\,sup}_{t\in E} \|y(t)\|_Y, & r = \infty, \end{cases} \tag{3.2}$$
>
> *holds?*

In case of a positive answer to the question above, we say that the system (3.1) *satisfies a final-state observability estimate in* $\mathrm{L}^r(E;Y)$. Loosely speaking, final-state observability allows one to retrieve information about the final state $x(T)$ by just observing the system through the measurements $y(t)$ at times $t \in E$.

In case the families $(A(t))_{t\in[0,T]}$ and $(C(t))_{t\in[0,T]}$ are constant, final-state observability for (3.1) has been studied thoroughly in the Hilbert space case both in abstract and in concrete situations. Autonomous self-adjoint Schrödinger operators A in $\mathrm{L}^2(\Omega)$ with bounded domain $\Omega \subseteq \mathbb{R}^d$ and a projection $Cu = u|_\omega$ for suitable subsets $\omega \subseteq \Omega$ were considered in [50, 94] as well as in [9, 68, 112, 113] for the case of unbounded domains. Moreover, second order elliptic operators A in $\mathrm{L}^2(\Omega)$ for bounded domains $\Omega \subseteq \mathbb{R}^d$ have been studied in [121], where also E in (3.2) is just a measurable subset of $[0, T]$. Of particular interest in understanding the case of unbounded domains is the specification of necessary and sufficient geometric conditions on Ω for observability, which were established in [41, 155] in the case of Hilbert spaces. In the Banach space case, a characterization of observability in terms of geometric conditions was given in [19, 61]. In the non-autonomous setting, results on Ornstein–Uhlenbeck operators on Hilbert spaces can be found in [12, 13].

3.1.1 Abstract Observability and Applications

Our approach to final-state observability is based on the *Lebeau–Robbiano strategy*, which originates in the works [78, 94, 95] and can be summarized as follows.

> *Given an evolution family generated by the non-autonomous operators $A(t)$ subject to an abstract* dissipation estimate *and a family of observation operators $C(t)$ subject to an* uncertainty estimate*, there exists a constant C_{obs} such that the final-state observability estimate* (3.2) *holds.*

All of the above will be made precise in Section 3.3. Originally, the Lebeau–Robbiano strategy was applied in the setting of C_0-semigroups on Hilbert spaces, see the references above, and subsequently generalized to C_0-semigroups on Banach spaces [19, 60, 61]. We will continue this path by extending the strategy of establishing a final-state observability estimate by proving dissipation and uncertainty in the setting of evolution families and non-autonomous Cauchy problems. The main application of our observability theorem is the following:

> *Consider an observation system* (3.1) *consisting of a parabolic equation in $\mathbb{R}^d$*
>
> $$\dot{u} = -A(t)u = -\sum_{|\alpha| \leq m} a_\alpha(t)\partial^\alpha u$$
>
> *with time-dependent uniformly elliptic differential operators $A(t)$ and observation operators $C(t)$ that are given via the restriction $u|_{\Omega(t)}$ of functions u on $\mathbb{R}^d$ to a time-dependent family of observability sets $\Omega(t) \subseteq \mathbb{R}^d$. Then, under suitable geometric assumptions on the family $(\Omega(t))_{t\in[0,T]}$, a final-state observability estimate* (3.2) *holds.*

More precisely, we analyze the connection between *final-state observability estimates* and the *geometry* of the observability sets. First and as an extension of the results known for the autonomous setting, in Theorem 3.4.4, we show that a *uniformly thick* family of observability sets $\Omega(t)$ guarantees the existence of a final-state observability estimate. Then, in Theorem 3.4.8, we derive a converse implication to Theorem 3.4.4 which builds upon a weaker notion of *thickness*.

3.1.2 Approximate Null-controllability and Duality

Before we step into the theory of our non-autonomous observability problem, let us describe the *dual* perspective on this problem. Despite not being crucial for the proofs of the results to follow, this will enrich the theory presented here by widening the perspective of applicability of the presented results.

Given a final time $T > 0$, consider the system

$$\dot{x}(t) = -A(t)x(t) + B(t)u(t), \quad t \in (0, T], \quad x(0) = x_0, \tag{3.3}$$

where $(B(t))_{t\in[0,T]}$ is a family of bounded linear operators $B(t)\colon U \to X$ for some Banach space U. For control systems (3.3), one may formulate the following question.

> *For a given measurable subset $E \subseteq [0, T]$ and $r \in [1, \infty]$, does there exist a function $u \in \mathrm{L}^r(0, T; U)$ with* $\operatorname{supp} u \subseteq E$ *such that $x(T) = 0$?*

In case the above question has a positive answer for all initial states $x_0 \in X$, we say that the system (3.3) *is null-controllable in $\mathrm{L}^r(0, T; U)$ with control function supported in E.* Problems like (3.3) for example appear in the field of controllability of partial differential equations, where $(A(t))_{t\in[0,T]}$ is a family of differential operators and each $B(t) = \mathbf{1}_{\Omega(t)}$ is a multiplication operator on $\mathrm{L}^p(\mathbb{R}^d)$ representing a family of control subsets $(\Omega(t))_{t\in[0,T]}$, cf. [12, 25, 93, 109].

In the case of a *reflexive* Banach space X, the system (3.3) is null-controllable in $\mathrm{L}^r(0, T; U)$ with control function supported in E if and only if the dual system for all initial values $x_0 \in X'$

$$\begin{aligned} \dot{x}(t) &= -A(t)'x(t), \quad t \in (0, T], \quad x(0) = x_0, \\ y(t) &= B(t)'x(t), \quad t \in [0, T], \end{aligned}$$

satisfies a final-state observability estimate in $\mathrm{L}^{r/(r-1)}(E; U')$, cf. [21, Rem. 2.1]. The above equivalence also holds for general Banach spaces X under the assumption that U is reflexive and $r \in (1, \infty)$, see [161, Thm. 2.1].

If X is not reflexive, or, if $r \in (1, \infty]$ and U is reflexive, then final-state observability is still equivalent to the following property:

> *For all measurable subsets $E \subseteq [0, T]$, there exists $C \geq 0$, such that, for all $\varepsilon > 0$ and all $x_0 \in X$, there exists $u \in \mathrm{L}^r(0, T; U)$ subject to the conditions* $\operatorname{supp} u \subseteq E$*, $\|u\|_{\mathrm{L}^r(0,T;U)} \leq C\|x_0\|_X$, and $\|x(T)\|_X < \varepsilon$.*

In fact, the above property is a variant of so-called *approximate null-controllability* and equivalent to null-controllability for reflexive Banach spaces. More precisely, the system (3.3) is approximately null-controllable if and only if the corresponding dual system satisfies a final-state observability estimate, see [21, 37, 153, 161].

3.1.3 Outline

Let us outline the content of this chapter. We start by introducing the abstract framework of evolution families for non-autonomous Cauchy problems in Section 3.2. Then we introduce the concrete class of elliptic differential operators that will form the example setting in which we apply our abstract result in Section 3.4. In Section 3.3, we derive sufficient conditions for observability of abstract non-autonomous systems. Building on this abstract result, we will verify these properties in the setting of differential operators and relate geometric conditions of the sets of observations to final-state observability in Section 3.4.

3.2 Evolution Families and Elliptic Operators

3.2.1 Abstract Non-autonomous Cauchy Problems

For Banach spaces X and Y, let $\mathcal{L}(X,Y)$ denote the set of *bounded linear operators* from X to Y. Similarly, set $\mathcal{L}(X) \coloneqq \mathcal{L}(X,X)$. Let $T > 0$, and let $A(t)\colon \operatorname{Dom}(A(t)) \to X$, $0 \le t \le T$, be a family of operators in X. Furthermore, let us always assume that the operators $A(t)$ are closed with common domain $\operatorname{Dom}(A(t)) =: D$ for all $t \in [0,T]$, where D densely embeds into X, and that the mapping $A\colon [0,T] \to \mathcal{L}(D,X)$ is strongly measurable. In particular, all graph norms $\|\cdot\|_{A(t)}$ are equivalent.

Consider the homogeneous initial value problem

$$\dot{x}(t) = -A(t)x(t), \quad t \in (0,T], \quad x(0) = x_0, \tag{NACP}$$

where $x_0 \in X$. We will call (NACP) the *non-autonomous Cauchy problem for A*. Cauchy problems generalize the idea of systems of ordinary differential equations to infinite-dimensional spaces while maintaining the core concept of a linear differential equation: a linear relation between a state function x and its time derivative $\dot{x}$. Several notions of solution to (NACP) exist in the literature that differ in the required amount of time regularity and space regularity of the sought state function, see, e.g., [108, Def. 4.1.1], [43, Sec. II.6]. Common to all notions of solution is that they all need to interpret the identities for $\dot{x}(t)$ and $x(0)$ in a well-defined sense. Throughout this chapter, we will make use of the following concept of solutions for (NACP); see [120, Chap. 4, Def. 2.8].

Definition 3.2.1 (Strong Solution)**.** A function $x\colon [0,T] \to X$ is said to be a *strong solution of* (NACP) if $x \in \mathrm{W}^{1,1}(0,T;X) \cap \mathrm{L}^1(0,T;D)$, $x(0) = x_0$, and $\dot{x}(t) = -A(t)x(t)$ for almost all $t \in (0,T)$.

In the case of a time-independent family of operators $A(t) = A$, the *operator semigroup approach* to Cauchy problems consists of a collection of techniques that aim at expressing solutions to (NACP) based on spectral properties of the operator A. The following definition provides a natural generalization of operator semigroups to the context of non-autonomous Cauchy problems.

Definition 3.2.2 (Evolution Family)**.** Let $T > 0$. A two-parameter family of bounded linear operators $(U(t,s))_{0 \le s \le t \le T}$ on X is called an *evolution family* if

(a) $U(s,s) = \mathrm{Id}$ and $U(t,s)U(s,r) = U(t,r)$ for $0 \le r \le s \le t \le T$.

If, additionally,

(b) $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \le s \le t \le T$,

then we say that the evolution family $(U(t,s))$ is *strongly continuous.*

An evolution family $(U(t,s))$ is called *exponentially bounded* if there exist $M \ge 1$ and $\omega \in \mathbb{R}$ such that $\|U(t,s)\|_{\mathcal{L}(X)} \le M\mathrm{e}^{\omega(t-s)}$ for all $0 \le s \le t \le T$.

An evolution family $(U(t,s))_{0 \le s \le t \le T}$ is called a *(strongly continuous) evolution family for A* if in addition to conditions (a) (and (b)) the following conditions are satisfied:

(c) For all $0 \leq s < T$ and $x_s \in D$, the function $x\colon [s,T] \to X$ defined by $x(t) = U(t,s)x_s$ is in $\mathrm{W}^{1,1}(s,T;X) \cap \mathrm{L}^1(s,T;D)$ and satisfies $\dot{x}(t) = -A(t)U(t,s)x_s$ for almost all $t \in (s,T)$.

(d) For all $0 < t \leq T$ and $x_T \in D$, the function $x\colon [t,T] \to X$ defined by $x(s) = U(T,s)x_T$ is in $\mathrm{W}^{1,1}(t,T;X) \cap \mathrm{L}^1(t,T;D)$ and satisfies $\dot{x}(s) = U(T,s)A(s)x_T$ for almost all $s \in (t,T)$.

Since the early works by Sobolevskiĭ [136] and Tanabe [141], non-autonomous Cauchy problems (NACP) and evolution families have been extensively studied by various authors, see, e.g. [2, 43, 108, 119, 120, 140, 160] and the references therein. Most of the aforementioned resources share the algebraic condition (a) in their definition of the evolution family $(U(t,s))$ but rely on other regularity assumptions (b), (c), and (d) and also assume other regularity properties of the operator family $(A(t))$. We chose the above definition for compatibility with the weak regularity assumptions on strong solutions in the sense of Definition 3.2.1. This will allow us to directly derive a closed formula for the evolution family to a family of differential operators in Section 3.4. A similar approach to deriving an evolution family was also taken in [12, Prop. 19].

The more general way to construct an evolution family for the initial value problem (NACP) is far more involved and usually demands further regularity assumptions on the family of operators $(A(t))$ in terms of time regularity and also compatibility of their resolvents. We refer the interested reader to [108, Chap. 6] and [120, Chap. 5].

Remark 3.2.3. (a) The condition in Definition 3.2.2(c) states that $u(t) \coloneqq U(t,0)u_0$ defines a strong solution of (NACP) on $[0,T]$ in the sense of Definition 3.2.1. Just as in the setting of autonomous Cauchy problems and one-parameter semigroups, the evolution family works as a "solution operator".

(b) Every strongly continuous evolution family is exponentially bounded. This fact is trivially true on the triangle $0 \leq s \leq t \leq T$. In contrast to the case of one-parameter semigroups, this fact remains no longer true once we allow $T = \infty$. Indeed, the evolution family $U(t,s) = \exp(t^2 - s^2)$ on $X = \mathbb{C}$ is not exponentially bounded, cf. [43, Sec. IV.9.6] as can be seen by estimating $U(t,t/2)$. Note, however, that one may have an exponentially bounded evolution family that is not strongly continuous. This is precisely the setting we will cover in our main result in Section 3.3.

(c) Every strongly continuous one-parameter semigroup $(S(t))_{0 \leq t \leq T}$ gives rise to a strongly continuous evolution family via $U_S(t,s) \coloneqq S(t-s)$ for $0 \leq s \leq t \leq T$.

(d) Throughout the literature, an evolution family is also referred to as an *evolution system*, *evolution operator*, *evolution process*, *propagator*, or *fundamental solution*.

In the following proposition, we relate the existence of an evolution family for A with the uniqueness of strong solutions for (NACP). The proof is an adaptation of [58, Prop. 3.3.4 and Cor. 3.3.5] and [59, Prop. 4.5] to the setting outlaid by Definition 3.2.1 and Definition 3.2.2.

Proposition 3.2.4. *Let $T > 0$ and $(U(t,s))_{0 \leq s \leq t \leq T}$ be an evolution family for A.*

(a) If (NACP) *has a strong solution $u \in \mathrm{W}^{1,1}(0,T;X) \cap \mathrm{L}^1(0,T;D)$, then it satisfies*

$$u(t) = U(t,s)u(s) \quad \textit{for} \quad 0 \leq s \leq t \leq T.$$

In particular, strong solutions are unique.

(b) If $(\widetilde{U}(t,s))_{0\le s\le t\le T}$ is a further evolution family for A, then $\widetilde{U} = U$.

Proof. Ad (a). We want to show that, for all $\psi \in \mathrm{C}_\mathrm{c}^\infty(0,T)$ and $0 \le s \le t \le T$, the identity

$$\int_s^t U(t,r)u(r)\,\psi'(r)\,\mathrm{d}r = 0 \tag{3.4}$$

holds, i.e., the mapping $r \mapsto U(t,r)u(r)$ is locally constant. This will then imply $u(t) = U(t,t)u(t) = U(t,s)u(s)$ which proves the claim. We carry out a density argument in three steps.

Step 1. We first prove the claim for $u \in \mathrm{C}^1(0,T)\otimes D = \operatorname{span}\{\varphi\cdot x \colon \varphi \in \mathrm{C}^1(0,T), x \in D\}$. Let $\varphi \in \mathrm{C}^1(0,T)$ and $x \in D$ such that $u(t) = \varphi(t)x$. Then, for all $\psi \in \mathrm{C}_\mathrm{c}^\infty(0,T)$, property (c) of evolution families reveals that

$$\begin{aligned}\int_0^T U(t,r)u(r)\dot{\psi}(r)\,\mathrm{d}r &= \int_0^T U(t,r)x\,\partial_r(\varphi\psi)(r)\,\mathrm{d}r - \int_0^T U(t,r)x\dot{\varphi}(r)\psi(r)\,\mathrm{d}r\\ &= -\int_0^T U(t,r)A(r)u(r)\psi(r)\,\mathrm{d}r - \int_0^T U(t,r)\dot{u}(r)\psi(r)\,\mathrm{d}r\\ &= 0,\end{aligned}$$

where ∂_r denotes a weak derivative and the last equality holds because u is a strong solution to (NACP) by assumption. Linearity of the integral yields (3.4).

Step 2. Assume that $u \in \mathrm{C}^1((0,T);D)$. By [149, Prop. 44.2], there exists a sequence (u_n) in $\mathrm{C}^1(0,T)\otimes D$ such that (u_n) and the sequence of time derivatives $(\dot{u}_n)$ converge uniformly to u and $\dot{u}$, respectively. Dominated convergence implies that (3.4) holds also for $u \in \mathrm{C}^1((0,T);D)$. Indeed, recall that the graph norms of $A(t)$ are equivalent by assumption, which gives rise to the estimate

$$\|U(t,r)A(r)u_n(r)\|_X \le \|U(t,r)\|_{\mathcal{L}(X)}\|A(r)u_n(r)\|_X \lesssim \|u_n(r)\|_D < \infty.$$

Step 3. Assume that $u \in \mathrm{W}^{1,1}(0,T;X)\cap \mathrm{L}^1(0,T;D)$. *First-order reflections*, cf. [44, Thm. 5.4.1], yield an extension $\widetilde{u} \in \mathrm{W}^{1,1}(\mathbb{R};X)\cap\mathrm{L}^1(\mathbb{R},D)$ with $\widetilde{u}(t) = u(t)$ for $t \in (0,T)$. Using a δ-sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathrm{C}^1(\mathbb{R})$, one can show that $(\varphi_n * \widetilde{u})|_{(0,T)} \in \mathrm{C}^1((0,T);D)$. Taking the limit $n\to\infty$ and possibly passing to a subsequence, we have that

$$\partial_t(\varphi_n * \widetilde{u})|_{(0,T)} \to \dot{u} \in \mathrm{L}^1(0,T;X) \quad\text{and}\quad (\varphi_n * \widetilde{u})|_{(0,T)} \to u \in \mathrm{L}^1(0,T;D)\,.$$

Thus, we observe that (3.4) also holds in the general case.

Ad (b). Let $(\widetilde{U}(t,s))$ be a further evolution family and $x_s \in D$. By Remark 3.2.3(a), the mapping $t \mapsto \widetilde{U}(t,s)x_s$ defines a strong solution to (NACP) on $[s,T]$. Furthermore, by part (a), strong solutions are unique. The representation formula for strong solutions gives

$$\widetilde{U}(t,s)x_s = U(t,s)x_s$$

for all $0 \le s \le t \le T$. Density of D in X now gives the claim. ∎

3.2.2 Properties of Non-autonomous Elliptic Operators

In this subsection, we apply the abstract theory of Section 3.2.1 to non-autonomous parabolic equations with time-dependent coefficients. We aim to define a family of non-autonomous differential operators and derive L^p-bounds for the associated evolution family. We will define these operators in terms of Fourier multipliers with symbols given via complex polynomials. To this end, we define the Fourier transformation $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ on the *Schwartz space* via

$$(\mathcal{F}u)(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\xi} u(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

As usual, we extend $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ to automorphisms of the space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^d)$. The inverse is given as

$$(\mathcal{F}^{-1}u)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi} u(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}^d.$$

In the following, we will tacitly identify $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ with a subspace of $\mathcal{S}'(\mathbb{R}^d)$ subject to the canonical embedding $u \mapsto \iota_u$ with

$$\iota_u(\varphi) := \int_{\mathbb{R}^d} u(x)\varphi(x)\,\mathrm{d}x, \quad \varphi \in \mathcal{S}(\mathbb{R}^d)\,.$$

Definition 3.2.5 (Non-autonomous Elliptic Homogeneous Polynomial)**.** Let $m \in \mathbb{N}$. For each multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$, let $a_\alpha\colon [0,T] \to \mathbb{C}$ be a complex-valued function. We call $\mathfrak{a}\colon [0,T] \times \mathbb{R}^d \to \mathbb{C}$ defined via

$$\mathfrak{a}(t,\xi) := \sum_{|\alpha|\le m} a_\alpha(t)(\mathrm{i}\xi)^\alpha, \quad t \in [0,T],\ \xi \in \mathbb{R}^d,$$

non-autonomous polynomial of degree m. The *principal symbol* $\mathfrak{a}_m$ of $\mathfrak{a}$ is given by

$$\mathfrak{a}_m(t,\xi) := \sum_{|\alpha|=m} a_\alpha(t)(\mathrm{i}\xi)^\alpha, \quad t \in [0,T],\ \xi \in \mathbb{R}^d.$$

We say that $\mathfrak{a}$ is *uniformly strongly elliptic with respect to t* if there exists an *ellipticity constant* $c > 0$ such that, for all $t \in [0,T]$ and $\xi \in \mathbb{R}^d$, the principal part $\mathfrak{a}_m$ fulfills the *ellipticity estimate*

$$\operatorname{Re}\mathfrak{a}_m(t,\xi) \ge c\,|\xi|^m. \tag{3.5}$$

Remark 3.2.6. (a) The presence of the factor i^α in the definition of $\mathfrak{a}$ and $\mathfrak{a}_m$ is motivated by our application of differential operators. In fact, we will see that one can easily associate the polynomial $\mathfrak{a}(t,\cdot)$ with a differential operator $A(t) := \sum_{|\alpha|\le m} a_\alpha(t)\partial_x^\alpha$ on $\mathrm{L}^p(\mathbb{R}^d)$ by means of the Fourier transform; cf. [43, Sec. IV.5].

(b) The principal part $\mathfrak{a}_m$ is *positive homogeneous* of degree m, i.e. $\mathfrak{a}_m(t,\lambda\xi) = \lambda^m\mathfrak{a}_m(t,\xi)$ for all $\xi \in \mathbb{R}^d$ and $\lambda > 0$. In fact, the above identity holds for all $\lambda \in \mathbb{R}$, but we refrain from calling $\mathfrak{a}_m\colon \mathbb{R}^d \to \mathbb{C}$ homogeneous as the domain and codomain are not vector spaces over the same field.

(c) Note that the uniform strong ellipticity of $\mathfrak{a}$ implies that m is even. In particular, we will always have $m \geq 2$ for the degree of the non-autonomous polynomial $\mathfrak{a}$. Indeed, note that, for all $\xi \in \mathbb{R}^d$ and $t \in [0, T]$,

$$\operatorname{Re} \mathfrak{a}_m(t, -\xi) = \begin{cases} \operatorname{Re} \mathfrak{a}_m(t, \xi) & \text{if } m \bmod 2 = 0, \\ -\operatorname{Re} \mathfrak{a}_m(t, \xi) & \text{if } m \bmod 2 \neq 0. \end{cases}$$

Assuming that the ellipticity estimate (3.5) holds, in the case $m \bmod 2 \neq 0$, we get that

$$-\operatorname{Re} \mathfrak{a}_m(t, \xi) = \operatorname{Re} \mathfrak{a}_m(t, -\xi) \geq c\, |\xi|^m \quad \text{and} \quad \operatorname{Re} \mathfrak{a}_m(t, \xi) \geq c\, |\xi|^m .$$

Adding both inequalities yields $c \leq 0$ for the ellipticity constant in contradiction to the assumption $c > 0$. Thus m needs to be even as the ellipticity estimate (3.5) enforces a positive sign of the term $\operatorname{Re} \mathfrak{a}_m(t, \xi)$.

As mentioned in Remark 3.2.6(a), strongly elliptic non-autonomous polynomials give rise to a certain class of differential operators if they are used as the symbol of a Fourier multiplier.

Definition 3.2.7. Let $\mathfrak{a}$ be a non-autonomous polynomial of degree $m \geq 2$. For $t \in [0, T]$, we define $A(t) \colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$A(t)u := \mathcal{F}^{-1}(\mathfrak{a}(t, \cdot)\mathcal{F}u) = \sum_{|\alpha| \leq m} a_\alpha(t) \partial^\alpha u .$$

We call the family $(A(t))_{t \in [0,T]}$ the *operator family associated with* $\mathfrak{a}$. If, furthermore, $\mathfrak{a}$ is uniformly strongly elliptic, we call $(A(t))_{t \in [0,T]}$ *elliptic*.

The following proposition summarizes the functional analytic properties of the operators $A(t)$ from Definition 3.2.7. For proofs of the results, we refer the interested reader to the monographs of Grafakos [69] and Haase [71].

Proposition 3.2.8. *Let* $\mathfrak{a} \colon [0, T] \times \mathbb{R}^d \to \mathbb{C}$ *be a uniformly strongly elliptic polynomial and* $A(t) \colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ *the operator family associated with* $\mathfrak{a}$*. Then the following statements hold.*

(a) $A(t)$ *leaves* $\mathcal{S}(\mathbb{R}^d)$ *invariant for all* $t \in [0, T]$*.*

For $p \in [1, \infty]$ *and* $t \in [0, T]$*, let* $A_p(t) := A(t)|_{\mathrm{L}^p(\mathbb{R}^d)}$ *denote the* part *of* $A(t)$ *in* $\mathrm{L}^p(\mathbb{R}^d)$*.*

(b) For $1 \leq p < \infty$ *and* $t \in [0, T]$*,* $A_p(t)$ *is a closed and densely defined operator.*

(c) For $1 < p < \infty$*, we have* $\operatorname{Dom}(A_p(t)) = \mathrm{W}^{p,m}(\mathbb{R}^d)$*.*

(d) For $p = 1$*, we have the inclusions* $\mathrm{W}^{1,m}(\mathbb{R}^d) \subseteq \operatorname{Dom}(A_1(t)) \subseteq \mathrm{W}^{1,m-1}(\mathbb{R}^d)$*.*

Let $m \in \mathbb{N}$ and $\mathfrak{a}$ a uniformly strongly elliptic polynomial, and let $(A(t))$ denote the associated operator family. Due to Proposition 3.2.8, we see that $\operatorname{Dom}(A_p(t))$ does not depend on t. Using the notation from Section 3.2.1, let write $D := \operatorname{Dom}(A_p(t))$, $t \in [0, T]$, in the following. For $p \in [1, \infty)$, we associate the non-autonomous Cauchy problem

$$\dot{u}(t) = -A_p(t)u(t), \quad t \in (0, T], \quad u(0) = u_0 \in \mathrm{L}^p(\mathbb{R}^d) \tag{3.6}$$

to the operator family $(A_p(t))_{t \in [0,T]}$. We will now define an evolution family for $(A_p(t))_{t \in [0,T]}$ and study its continuity properties.

Lemma 3.2.9. *Let $m \geq 2$ and $\mathfrak{a}$ a uniformly elliptic non-autonomous polynomial with complex-valued coefficient functions $a_\alpha \colon [0,T] \to \mathbb{C}$, $|\alpha| \leq m$. Furthermore, let $c > 0$ denote the uniform ellipticity constant from (3.5). Assume that $a_\alpha \in \mathrm{L}^1(0,T)$ for all α. For all $c_0 \in (0,c)$, there exists $\omega \geq 0$ such that*

$$\int_s^t \operatorname{Re} \mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau \geq (t-s)c_0\,|\xi|^m - (t-s)\,\omega, \quad \xi \in \mathbb{R}^d,\ 0 \leq s < t \leq T. \tag{3.7}$$

In particular, $\mathrm{e}^{-\int_s^t \mathfrak{a}(\tau,\cdot)\,\mathrm{d}\tau} \in \mathcal{S}(\mathbb{R}^d)$.

Proof. We have $\operatorname{Re} \mathfrak{a}_m(t,\xi) \geq c\,|\xi|^m$ for all $t \in [0,T]$ and $\xi \in \mathbb{R}^d$. Thus, for $c_0 \in (0,c)$, we estimate

$$\int_s^t \operatorname{Re} \mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau \geq (t-s)c_0\,|\xi|^m + (t-s)(c-c_0)\,|\xi|^m + \sum_{|\alpha|<m} \int_s^t \operatorname{Re}(a_\alpha(\cdot)\,\mathrm{i}^\alpha)\,\mathrm{d}\tau\,\xi^\alpha.$$

Now consider the polynomial

$$\mathfrak{b}(\xi) := (t-s)(c-c_0)|\xi|^m + (t-s)\mathfrak{b}'(\xi) := (t-s)(c-c_0)|\xi|^m + (t-s)\sum_{|\alpha|<m} b_\alpha\,\xi^\alpha$$

with

$$b_\alpha := \frac{1}{t-s}\int_s^t \operatorname{Re}(a_\alpha(\tau)\mathrm{i}^\alpha)\,\mathrm{d}\tau\,.$$

Choose $R > 0$ such that

$$(c-c_0)\,|\xi|^m \geq -\sum_{|\alpha|<m} b_\alpha\,\xi^\alpha \quad \text{for all } |\xi| > R,$$

and let $\omega := \max\{|\mathfrak{b}'(\xi)| : \xi \in \mathbb{R}^d, |\xi| \leq R\} \geq 0$. We may now estimate

$$\int_s^t \operatorname{Re} \mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau \geq (t-s)c_0\,|\xi|^m - \omega$$

which proves estimate (3.7).

Now consider the function $\mathrm{e}^{-\int_s^t \mathfrak{a}(\tau,\cdot)\,\mathrm{d}\tau}$. Clearly, the function is smooth. For the decay property, note that, for all $\xi \in \mathbb{R}^d$ and $0 \leq s < t \leq T$,

$$\mathrm{e}^{-\int_s^t \mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau} \leq \mathrm{e}^{-(t-s)\omega}\ \mathrm{e}^{-(t-s)c_0|\xi|^m}.$$

This proves our claim. □

Under assumptions of Lemma 3.2.9, we may define a two-parameter family of operators $U(t,s) \colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $0 \leq s \leq t \leq T$, by

$$U(s,s)u := u, \qquad U(t,s)u := \mathcal{F}^{-1}\Big(\mathrm{e}^{-\int_s^t \mathfrak{a}(\tau,\cdot)\,\mathrm{d}\tau}\mathcal{F}u\Big), \quad t > s. \tag{3.8}$$

The next lemma collects several algebraic and functional analytic properties of the operator family $(U(t,s))_{0 \leq s \leq t \leq T}$.

Lemma 3.2.10. *Let $\mathfrak{a}$ be a uniformly elliptic non-autonomous polynomial with coefficients $a_\alpha \in \mathrm{L}^1(0,T)$. Let $(U(t,s))_{0\le s\le t\le T}$ be the operator family defined in* (3.8).

(a) For $0 \le s < t \le T$, the operator $U(t,s)$ is given as a convolution operator with kernel $p_{t,s} \in \mathcal{S}(\mathbb{R}^d)$ defined as

$$p_{t,s} \coloneqq \mathcal{F}^{-1}\mathrm{e}^{-\int_s^t \mathfrak{a}(\tau,\cdot)\,\mathrm{d}\tau}. \tag{3.9}$$

(b) For $0 \le r \le s \le t \le T$, we have that

$$U(s,s) = \mathrm{Id} \quad \text{and} \quad U(t,r) = U(t,s)U(s,r).$$

Moreover, $p_{t,r} = p_{t,s} * p_{s,r}$ for all $0 \le r < s < t \le T$.

(c) For $p \in [1,\infty]$ and $0 \le s \le t \le T$, the operator $U(t,s)$ leaves $\mathrm{L}^p(\mathbb{R}^d)$ invariant. In particular, $\|U(t,s)u\|_{\mathrm{L}^p(\mathbb{R}^d)} \le \|p_{t,s}\|_{\mathrm{L}^1(\mathbb{R}^d)}\|u\|_{\mathrm{L}^p(\mathbb{R}^d)}$ for all $u \in \mathrm{L}^p(\mathbb{R}^d)$.

Proof. Ad (a). This follows from Lemma 3.2.9 and the fact that the Fourier transform is an isomorphism on $\mathcal{S}(\mathbb{R}^d)$.

Ad (b). The proof of (b) is straightforward from the definitions of the operator family $(U(t,s))_{0\le s\le t\le T}$ in (3.8) and of its kernel in (3.9).

Ad (c). Statement (c) is a consequence of Young's inequality, see, e.g., [69, Thm. 1.2.10]. □

Let $(U(t,s))_{0\le s\le t\le T}$ be as in (3.8) and $p \in [1,\infty]$. For $0 \le s \le t \le T$, we define

$$U_p(t,s) \coloneqq U(t,s)|_{\mathrm{L}^p(\mathbb{R}^d)}. \tag{3.10}$$

By Lemma 3.2.10, the operators $(U_p(t,s))$ are bounded on $\mathrm{L}^p(\mathbb{R}^d)$ and defines an evolution family on $\mathrm{L}^p(\mathbb{R}^d)$ in the sense of Definition 3.2.2(a). In fact, the evolution family $(U_p(t,s))_{0\le s\le t\le T}$ can be seen as the solution operator to the non-autonomous Cauchy problem (3.6) induced by $(A_p(t))_{t\in[0,T]}$. The following theorem summarizes all of these properties.

Theorem 3.2.11. *Let $\mathfrak{a}$ be a uniformly strongly elliptic polynomial of degree $m \ge 2$ with coefficients $a_\alpha \in \mathrm{L}^\infty(0,T)$ for $|\alpha| \le m$. Let $(U(t,s))_{0\le s\le t\le T}$ be defined as in* (3.8).

(a) Let $p \in [1,\infty]$. Then $(U_p(t,s))_{0\le s\le t\le T}$ is an exponentially bounded evolution family.

(b) Let $p \in (1,\infty)$. Then $(U_p(t,s))_{0\le s\le t\le T}$ is the unique evolution family for the family of operators $(A_p(t))_{t\in[0,T]}$.

In order to prepare for the proof of Theorem 3.2.11, we will proceed in 3 Steps. First, we will show that the kernel $p_{t,s}$ is subject to *Gaussian bounds* that are uniform with respect to s,t. Then we will show that the family is exponentially bounded and strongly continuous. In the last step, we establish the announced relation between the evolution family (3.10) and the non-autonomous Cauchy problem (3.6).

Step 1. Kernel Estimates and Exponential Boundedness.

We will show that we can find Gaussian bounds for the kernel $p_{t,s}$ for $0 \le s < t \le T$ defined in (3.9).

Lemma 3.2.12. *Let $m \ge 2$ and $\mathfrak{a}$ a uniformly elliptic non-autonomous polynomial with complex-valued coefficient functions $a_\alpha\colon [0,T] \to \mathbb{C}$, $|\alpha| \le m$. Furthermore, let $c > 0$ denote the uniform ellipticity constant from* (3.5)*. Assume that $a_\alpha \in \mathrm{L}^\infty(0,T)$ for all $|\alpha| \le m$. Then the following statements hold.*

(a) For all $c_0 \in (0,c)$, there exists $\omega \ge 0$ depending only on c, c_0 and a_α, $|\alpha| < m$, such that, for all $0 \le s < t \le T$ and all $\xi \in \mathbb{R}^d$,

$$\int_s^t \operatorname{Re} \mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau \ge (t-s)\,c_0\,|\xi|^m - (t-s)\,\omega.$$

(b) There exist $C_1, C_2 \ge 0$ and $\omega \ge 0$ such that, for all $0 \le s < t \le T$ and all $x \in \mathbb{R}^d$,

$$|p_{t,s}(x)| \le C_1 \frac{1}{(t-s)^{d/m}}\,\mathrm{e}^{\omega(t-s)}\,\mathrm{e}^{-C_2\left(|x|^m/(t-s)\right)^{1/m-1}}.$$

In particular, for all $p \in [1,\infty]$ and $0 \le s < t \le T$, we have

$$\|U_p(t,s)\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))} = \|p_{t,s}\|_{\mathrm{L}^1(\mathbb{R}^d)} \le C_1\,\mathrm{e}^{\omega\,(t-s)} \int_{\mathbb{R}^d} \mathrm{e}^{-C_2|x|^{m/(m-1)}}\,\mathrm{d}x.$$

Proof. Ad (a). We proceed similarly to the proof of Lemma 3.2.9. By ellipticity, we have $\operatorname{Re}\mathfrak{a}_m(t,\xi) \ge c\,|\xi|^m$ for all $t \in [0,T]$ and $\xi \in \mathbb{R}^d$. Thus, for $c_0 \in (0,c)$, we estimate

$$\begin{aligned}\int_s^t \operatorname{Re}\mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau &\ge (t-s)c_0\,|\xi|^m + (t-s)(c-c_0)\,|\xi|^m + \sum_{|\alpha|<m} \int_s^t \operatorname{Re}(a_\alpha(\tau)\,\mathrm{i}^\alpha)\,\mathrm{d}\tau\,\xi^\alpha \\ &\ge (t-s)\Big(c_0\,|\xi|^m + (c-c_0)\,|\xi|^m + \sum_{|\alpha|<m} \xi^\alpha b_\alpha\Big),\end{aligned}$$

where

$$b_\alpha := \min\big\{\operatorname{Re}(a_\alpha(\tau)\,\mathrm{i}^\alpha) : \tau \in [0,T]\big\}, \quad |\alpha| < m.$$

Choose $R > 0$ such that

$$(c-c_0)\,|\xi|^m + \sum_{|\alpha|<m} b_\alpha\,\xi^\alpha \ge 0\,,$$

and let

$$\omega := \max\big\{\big|\sum_{|\alpha|<m} b_\alpha\,\xi^\alpha\big| : |\xi| \le R\big\} \ge 0\,.$$

We may now estimate

$$\int_s^t \operatorname{Re}\mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau \ge (t-s)\big(c_0\,|\xi|^m - \omega\big)$$

which proves our claim.

Ad (b). We follow the argument in [143, Prop. 1]. Note that, although $\mathfrak{a}(t,\cdot)$ is defined on $\mathbb{R}^d$, we can extend it to $\mathbb{C}^d$ for all $t \in [0,T]$. Let $0 \le s < t \le T$, $x \in \mathbb{R}^d$. Then, for $\eta \in \mathbb{R}^d$, we obtain via the change of variables formula

$$p_{t,s}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi} \mathrm{e}^{-x\cdot\eta} \mathrm{e}^{-\int_s^t \mathfrak{a}(\tau,\xi+\mathrm{i}\eta)\,\mathrm{d}\tau}\,\mathrm{d}\xi.$$

We decompose the polynomial $\mathfrak{a}$ as follows

$$\begin{aligned}\operatorname{Re}\mathfrak{a}(\tau,\xi+\mathrm{i}\eta) &= \operatorname{Re}\mathfrak{a}(\tau,\xi) + \operatorname{Re}\mathfrak{a}(\tau,\mathrm{i}\eta) + \operatorname{Re} a_0(\tau) \\ &\quad + \operatorname{Re}\sum_{\substack{|\alpha|\le m\\ \alpha\ne 0}} a_\alpha(\tau) \sum_{\substack{\beta\le\alpha\\ \beta\ne 0,\alpha}} \binom{\alpha}{\beta} (\mathrm{i}\xi)^\beta (-\eta)^{\alpha-\beta}.\end{aligned}$$

In view of part (a), there exist $c_0, c_1, c_2 > 0$ such that, for all $0 \le s < t \le T$ and all $\xi, \eta \in \mathbb{R}^d$, we have

$$\begin{aligned}&\int_s^t \operatorname{Re}\mathfrak{a}(\tau,\xi) + \operatorname{Re}\mathfrak{a}(\tau,\mathrm{i}\eta) + \operatorname{Re} a_0(\tau)\,\mathrm{d}\tau \\ &\qquad\qquad \ge (t-s)\left(c_0|\xi|^m + c_1|\eta|^m - c_2 - \|a_0(\cdot)\|_{\mathrm{L}^\infty(0,T)}\right).\end{aligned} \tag{3.11}$$

Furthermore, note that, for all $\xi \in \mathbb{R}^d$ and $1 \le k \le m$, we may estimate

$$|\xi|^k \le 1 + |\xi|^m.$$

Now Young's inequality for products gives that, for all $\varepsilon > 0$, $\xi, \eta \in \mathbb{R}^d$, and $1 \le k, l \le m$, we have

$$|\xi|^l |\eta|^{k-l} \le \varepsilon\,|\xi|^m + q(\varepsilon)\,|\eta|^m + \varepsilon + q(\varepsilon),$$

where $q(\varepsilon)$ depends continuously on ε. In particular, for each $\varepsilon > 0$, there exists $q = q(\varepsilon) > 0$ such that

$$\sum_{\substack{1\le k\le m\\ 1\le l\le k-1}} |\xi|^l |\eta|^{k-l} \le \varepsilon\,|\xi|^m + q\left(|\eta|^m + 1\right). \tag{3.12}$$

We introduce the constant

$$c_3 := 2 \sup_{\substack{|\alpha|\le m\\ \alpha\ne 0}} \|a_\alpha(\cdot)\|_{\mathrm{L}^\infty(0,T)} \sum_{\substack{\beta\le\alpha\\ \beta\ne 0,\alpha}} \binom{\alpha}{\beta}.$$

Now (3.12) with $\varepsilon = \frac{c_0}{2\,c_3}$ gives us the estimate

$$\begin{aligned}\operatorname{Re}\sum_{\substack{|\alpha|\le m\\ \alpha\ne 0}} a_\alpha(\tau) \sum_{\substack{\beta\le\alpha\\ \beta\ne 0,\alpha}} \binom{\alpha}{\beta} (\mathrm{i}\xi)^\beta (-\eta)^{\alpha-\beta} &\ge -c_3 \sum_{\substack{1\le k\le m\\ 1\le l\le k-1}} |\xi|^l\,|\eta|^{k-l} \\ &\ge -\frac{c_0}{2}|\xi|^m - c_3\,q\,(|\eta|^m + 1).\end{aligned} \tag{3.13}$$

Using (3.11) and (3.13) we arrive at

$$\int_s^t \operatorname{Re} \mathfrak{a}(\tau, \xi + \mathrm{i}\eta)\,\mathrm{d}\tau \geq (t-s)\Big(\frac{c_0}{2}|\xi|^m - \sigma|\eta|^m - \omega\Big),$$

where $\sigma := c_1 + c_3 q$ and $\omega = c_2 + \|a_0(\cdot)\|_{\mathrm{L}^\infty(0,T)} + c_3\, q$. Hence, we can estimate

$$\begin{aligned} |p_{t,s}(x)| &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-x\cdot\eta}\, \mathrm{e}^{-\int_s^t \operatorname{Re}\mathfrak{a}(\tau,\xi+\mathrm{i}\eta)\,\mathrm{d}\tau}\, \mathrm{d}\xi \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-x\cdot\eta}\, \mathrm{e}^{-(t-s)(\frac{c_0}{2}\,|\xi|^m - \sigma\,|\eta|^m - \omega)}\, \mathrm{d}\xi \\ &= C_1\, \frac{1}{(t-s)^{d/m}}\, \mathrm{e}^{-x\cdot\eta}\, \mathrm{e}^{\omega\,(t-s)}\, \mathrm{e}^{(t-s)\,\sigma\,|\eta|^m}, \end{aligned}$$

where $C_1 := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{c_0}{2}\,|\xi|^m}\,\mathrm{d}\xi$. Now, for $\eta := \frac{1}{2}\big(\frac{|x|}{\sigma\,(t-s)}\big)^{1/(m-1)} \frac{x}{|x|}$, we obtain

$$|p_{t,s}(x)| \leq C_1 \frac{1}{(t-s)^{d/m}}\, \mathrm{e}^{\omega\,(t-s)}\, \mathrm{e}^{-C_2\left(|x|^m/(t-s)\right)^{1/m-1}},$$

where $C_2 := \frac{2^{m-1}-1}{2^m}$. Thus, integration yields the assertion for $\|p_{t,s}\|_{\mathrm{L}^1(\mathbb{R}^d)}$.

For the description of the norm of $U_p(t,s)$, note that Lemma 3.2.10(a) gives that, for $p \in [1,\infty]$, the operator $U_p(t,s)$ is the convolution operator with kernel $p_{t,s}$, and Lemma 3.2.10(c) yields $\|U_p(t,s)\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))} = \|p_{t,s}\|_{\mathrm{L}^1(\mathbb{R}^d)}$. □

Step 2. Strong Continuity.

We now show that $(U_p(t,s))_{0\leq s\leq t\leq T}$ is strongly continuous for $p \in [1,\infty)$. Our approach extends a similar result for the classical setting of diffusion semigroups, see [43, Sec. II.2.13]. As a consequence of duality theory for evolution families, cf. [43, Sec. I.5.14] and [7, Sec. 7.2], the evolution family $(U_\infty(t,s))_{0\leq s\leq t\leq T}$ will be strongly continuous with respect to the weak*-topology.

We start with a subspace of $\mathcal{S}'(\mathbb{R}^d)$ that can be identified with a subset of $\mathrm{C}^\infty(\mathbb{R}^d)$. Let $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$ denote the *multiplier space*

$$\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d) := \big\{f \in \mathrm{C}^\infty(\mathbb{R}^d) : \forall g \in \mathcal{S}(\mathbb{R}^d),\ \alpha \in \mathbb{N}_0^d : \|f\|_{g,\alpha} < \infty\big\},$$

where the family of seminorms $(\|\cdot\|_{g,\alpha})_{g,\alpha}$ is defined via

$$\|f\|_{g,\alpha} := \sup_{x\in\mathbb{R}^d} |g(x)\,\partial^\alpha f(x)|\,, \quad f \in \mathrm{C}^\infty(\mathbb{R}^d),$$

and induces a locally convex topology on $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$, cf. [130, Chap. 7, §5, p. 243] and [86, Exmp. 5.3]. The following proposition shows that convergence on compact sets is compatible with the convergence on $\mathcal{O}_{\mathrm{M}}$.

Proposition 3.2.13. *Let $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{C}^\infty(\mathbb{R}^d)$ and $f \in \mathrm{C}^\infty(\mathbb{R}^d)$ such that, for all $\alpha \in \mathbb{N}_0^d$, we have $\sup_{n\in\mathbb{N}} \|\partial^\alpha f_n\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < \infty$ and $\partial^\alpha f_n \to \partial^\alpha f$ uniformly on compact sets. Then (f_n) is a sequence in $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$, $f \in \mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$, and $f_n \to f$ in $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$.*

Proof. First note that, since smooth functions with bounded derivatives belong to $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$, we obtain $f_n \in \mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. Indeed, if $\partial^\alpha f_n$ is bounded, then, in particular, it decays faster than the reciprocal of every polynomial, cf. [69, Rem. 2.2.3]. Note that, by assumption, we have that, for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$,

$$|\partial^\alpha f_n(x)| \leq \sup_{n\in\mathbb{N}} \|\partial^\alpha f_n\|_{\mathrm{L}^\infty(\mathbb{R}^d)} =: C_\alpha \,.$$

As $f_n \to f$ uniformly on compact sets, the convergence is also pointwise such that we also have $|\partial^\alpha f(x)| \leq C_\alpha$ for all $\alpha \in \mathbb{N}_0^d$ which yields $f \in \mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$.

Now let us show that the convergence of $f_n \to f$ is also in $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$. To this end, let $\alpha \in \mathbb{N}_0^d$ and $g \in \mathcal{S}(\mathbb{R}^d)$, and let $\varepsilon > 0$. Choose a compact set $K \subseteq \mathbb{R}^d$ such that

$$\sup_{x\notin K} |g(x)| \leq \frac{\varepsilon}{2\sup_{n\in\mathbb{N}} \|\partial^\alpha (f_n - f)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} + 1} \,.$$

Furthermore, choose $N \in \mathbb{N}$ such that, for all $n \geq N$, we have

$$\sup_{x\in K} |\partial^\alpha (f_n - f)(x)| \leq \frac{\varepsilon}{2\,\|g\|_{\mathrm{L}^\infty(\mathbb{R}^d)} + 1} \,.$$

Then we observe

$$\begin{aligned}
\|f_n - f\|_{g,\alpha} &= \sup_{x\in\mathbb{R}^d} |g(x)\,\partial^\alpha (f_n - f)(x)| \\
&\leq \sup_{x\in K} \Big|g(x)\,\partial^\alpha (f_n - f)(x)\Big| + \sup_{x\notin K} \Big|g(x)\,\partial^\alpha (f_n - f)(x)\Big| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \,.
\end{aligned}$$

As ε was chosen arbitrarily, we have $f_n \to f$ in $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$. □

Corollary 3.2.14. *If $a_\alpha(\cdot) \in \mathrm{L}^\infty(0,T)$ for all $|\alpha| \leq m$, the corresponding evolution family $(U(t,s))_{0\leq s\leq t\leq T}$ from (3.8) is strongly continuous on $\mathcal{S}(\mathbb{R}^d)$.*

Proof. Let $0 \leq s \leq t \leq T$. Let $((t_n, s_n))_{n\in\mathbb{N}}$ be a sequence of points in $[0,\infty) \times [0,\infty)$ subject to $0 \leq s_n \leq t_n \leq T$ for all $n \in \mathbb{N}$ with limit $(t_n, s_n) \to (t, s)$. For $n \in \mathbb{N}$, set

$$f_n := \mathrm{e}^{-\int_{s_n}^{t_n} \mathfrak{a}(\tau,\cdot)\,\mathrm{d}\tau} \in \mathrm{C}^\infty(\mathbb{R}^d).$$

Let furthermore $f := \lim_{n\to\infty} f_n$ denote the pointwise limit. Clearly,

$$f = \mathrm{e}^{-\int_{s}^{t} \mathfrak{a}(\tau,\cdot)\,\mathrm{d}\tau} \in \mathrm{C}^\infty(\mathbb{R}^d).$$

Let us show that $(f_n)_{n\in\mathbb{N}}$ fulfills the requirements of Proposition 3.2.13. First, note that the convergence of $(f_n)_{n\in\mathbb{N}}$ and its partial derivatives is also uniform on compact subsets of $\mathbb{R}^d$. Indeed, let $K \subseteq \mathbb{R}^d$ be a compact subset. By dominated convergence

$$\int_0^T \big(\mathbf{1}_{[s_n,t_n]}(\tau) - \mathbf{1}_{[s,t]}(\tau)\big)\,\mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau \to 0$$

for $n \to \infty$ uniformly for all $\xi \in K$ as we have

$$\int_0^T |\mathbf{1}_{[s_n,t_n]}(\tau) - \mathbf{1}_{[s,t]}(\tau)| \, \|\mathfrak{b}(\tau,\cdot)\|_{\mathrm{L}^\infty(K)} \, \mathrm{d}\tau \to 0,$$

where $\mathfrak{b}(\xi) \coloneqq \sum_{|\beta|\le m} \|a_\beta(\cdot)\|_{\mathrm{L}^\infty(0,T)} \xi^\beta$, $\xi \in \mathbb{R}^d$. This shows that

$$f_n(\xi) = \mathrm{e}^{-\int_{s_n}^{t_n} \mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau} \to \mathrm{e}^{-\int_s^t \mathfrak{a}(\tau,\xi)\,\mathrm{d}\tau} = f(\xi)$$

for $n \to \infty$ uniformly in K. Repeating this argument for partial derivatives $\partial^\alpha f_n$, one proves that

$$\partial^\alpha f_n \to \partial^\alpha f$$

for all $\alpha \in \mathbb{N}_0^d$ uniformly on compact subsets $K \subseteq \mathbb{R}^d$. As a consequence of Lemma 3.2.12(b), we furthermore have that $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < \infty$ for all $\alpha \in \mathbb{N}_0^d$.

Proposition 3.2.13 now implies that $f_n \to f$ in $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$. Let $g \in \mathcal{S}(\mathbb{R}^d)$. Then $\mathcal{F}g \in \mathcal{S}(\mathbb{R}^d)$ and the convergence $f_n \to f$ in $\mathcal{O}_{\mathrm{M}}(\mathbb{R}^d)$ implies $f_n \mathcal{F}g \to f\,\mathcal{F}g$ in $\mathcal{S}(\mathbb{R}^d)$. Thus,

$$U(t_n,s_n)g = \mathcal{F}^{-1}(f_n\,\mathcal{F}g) \to \mathcal{F}^{-1}(f\,\mathcal{F}g) = U(t,s)g$$

for $n \to \infty$ in $\mathcal{S}(\mathbb{R}^d)$ by continuity of $\mathcal{F}^{-1}$. □

Corollary 3.2.15. *Let $p \in [1,\infty)$. Then $(U_p(t,s))_{0\le s\le t\le T}$ is strongly continuous. Moreover, $(U_\infty(t,s))_{0\le s\le t\le T}$ is strongly continuous with respect to the weak*-topology.*

Proof. Since $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathrm{L}^p(\mathbb{R}^d)$ is dense for $p \in [1,\infty)$, Corollary 3.2.14 yields that the family $(U_p(t,s))_{0\le s\le t\le T}$ is strongly continuous for $p \in [1,\infty)$. Indeed, let $f \in \mathrm{L}^p(\mathbb{R}^d)$ and $(f_k)_{k\in\mathbb{N}}$ an approximating sequence in $\mathcal{S}(\mathbb{R}^d)$. By Lemma 3.2.12(b) we have $(U_p(t,s))_{0\le s\le t\le T}$ for all $p \in [1,\infty)$. In particular, there exists $C > 0$ such that

$$\|U(t,s)\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))} \le C$$

Now, given $\varepsilon > 0$, choose $k_0 \in \mathbb{N}$ such that $\|f - f_k\|_{\mathrm{L}^p(\mathbb{R}^d)} \le \frac{\varepsilon}{3C}$ for all $k \ge k_0$, and choose $n_0 \in \mathbb{N}$ such that $\|(U(t_n,s_n) - U(t,s))f_k\|_{\mathrm{L}^p(\mathbb{R}^d)} \le \frac{\varepsilon}{3}$ for all $n \ge n_0$. Then

$$\begin{aligned}\|U(t_n,s_n)f - U(t,s)f\|_{\mathrm{L}^p(\mathbb{R}^d)} &\le \|U(t_n,s_n)(f-f_k)\|_{\mathrm{L}^p(\mathbb{R}^d)} + \|(U(t,s) - U(t_n,s_n))f_k\|_{\mathrm{L}^p(\mathbb{R}^d)} \\ &\quad + \|(U(t,s)(f-f_k)\|_{\mathrm{L}^p(\mathbb{R}^d)} \\ &\le \varepsilon\,.\end{aligned}$$

As ε was chosen arbitrarily, we see that

$$U(t_n,s_n) \to U(t,s)$$

for $n \to \infty$ strongly in $\mathrm{L}^p(\mathbb{R}^d)$. This gives the first assertion.

For $0 \le s \le t \le T$, we have $U_\infty(t,s) = V_1(t,s)'$, where $(V_1(t,s))_{0\le s\le t\le T}$ is the L^1-realization of the evolution family $(V(t,s))_{0\le s\le t\le T}$ on $\mathcal{S}(\mathbb{R}^d)$ associated with the non-autonomous polynomial

$$\mathfrak{b}\colon [0,T]\times\mathbb{R}^d \to \mathbb{C}, (t,\xi) \mapsto \sum_{|\alpha|\le m} (-1)^{|\alpha|}\, \overline{a_\alpha}(t)\, (\mathrm{i}\xi)^\alpha$$

which inherits its uniform strong ellipticity from $\mathfrak{a}$ since m is even. Indeed,

$$\operatorname{Re}\mathfrak{b}_m(t,\xi) = \sum_{|\alpha|=m} (-1)^m \operatorname{Re}\overline{a_\alpha}(t)\ (\mathrm{i}\xi)^\alpha = \sum_{|\alpha|=m} \operatorname{Re} a_\alpha(t)\ (\mathrm{i}\xi)^\alpha = \operatorname{Re}\mathfrak{a}_m(t,\xi)\,.$$

Thus, the second assertion follows from the first assertion for $(V_1(t,s))$. □

Step 3. $(U_p(t,s))_{0\le s\le t\le T}$ as an Evolution Family for $(A_p(t))_{t\in[0,T]}$.

Now, we establish the relation between the evolution family $(U_p(t,s))_{0\le s\le t\le T}$ and the family of differential operators $(A_p(t))_{t\in[0,T]}$.

Proposition 3.2.16. *Let $p\in(1,\infty)$ and $u\in D=\mathrm{W}^{m,p}(\mathbb{R}^d)$.*

(a) Let $0\le s<T$. Then $U_p(\cdot,s)u\in \mathrm{W}^{1,1}(s,T;\mathrm{L}^p(\mathbb{R}^d))\cap \mathrm{L}^1(s,T;D)$ and, for almost all $t\in(s,T)$, we have $\partial_t(U_p(t,s)u) = -A_p(t)U_p(t,s)u$.

(b) Let $0<t\le T$. Then $U_p(t,\cdot)u\in \mathrm{W}^{1,1}(t,T;\mathrm{L}^p(\mathbb{R}^d))\cap \mathrm{L}^1(t,T;D)$ and, for almost all $s\in(t,T)$, we have $\partial_s(U_p(T,s)u) = U_p(T,s)A_p(s)u$.

Proof. Ad (a). By Young's inequality and the L^1-bound of the kernel from Lemma 3.2.12(b), we observe $U_p(\cdot,s)u\in \mathrm{L}^1(s,T;\mathrm{L}^p(\mathbb{R}^d))$. Note that $(s,T)\ni t\mapsto p_{t,s}$ is weakly differentiable and $\partial_t p_{t,s} = -A_p(t)p_{t,s}$ for all $t\in(s,T)$. For the weak derivative of $U_p(\cdot,s)u$, we have

$$\begin{aligned}\partial_t(U_p(t,s)u) &= \partial_t(p_{t,s}*u) = (\partial_t p_{t,s})*u = (-A_p(t)p_{t,s})*u\\ &= p_{t,s}*(-A_p(t)u) = -A_p(t)(p_{t,s}*u) = -A_p(t)U_p(t,s)u\end{aligned}$$

for almost all $t\in(s,T)$. In particular, the closedness of $A_p(t)$ implies $U_p(t,s)u\in D$ for almost all $t\in(s,T)$.

Recall that $D=\mathrm{W}^{p,m}(\mathbb{R}^d)$ and the Sobolev norm and the graph norms $\|\cdot\|_{A_p(t)}$ are equivalent, i.e., there exists $C>0$ such that

$$\frac{1}{C}\,\|v\|_{\mathrm{W}^{p,m}(\mathbb{R}^d)} \le \|v\|_{A_p(t)} \le C\,\|v\|_{\mathrm{W}^{p,m}(\mathbb{R}^d)}$$

for all $t\in[s,T]$ and $v\in D$ which follows from uniform strong ellipticity of $\mathfrak{a}$ and the boundedness of the coefficients a_α, see [71, Thm. 8.2.1 and Sec. 8.6]. In particular, $\|A_p(t)u\|_{\mathrm{L}^p(\mathbb{R}^d)} \le C\,\|u\|_{\mathrm{W}^{p,m}(\mathbb{R}^d)}$ for all $t\in[s,T]$, and therefore

$$\int_s^T \|A_p(t)u\|_{\mathrm{L}^p(\mathbb{R}^d)}\,\mathrm{d}t < \infty,$$

so, by Young's inequality and the kernel bound from Lemma 3.2.12(b), we observe that $\partial_t U_p(\cdot,s)u\in \mathrm{L}^1(s,T;\mathrm{L}^p(\mathbb{R}^d))$ as well as $U_p(\cdot,s)u\in \mathrm{L}^1(s,T;D)$.

Ad (b). The proof of statement (b) follows the same lines as the proof of (a). □

We are now in position to prove that the evolution family $(U_p(t,s))_{0\le s\le t\le T}$, $p\in(1,\infty)$, is indeed the unique evolution family for the family $(A(t))_{t\in[0,T]}$ of elliptic differential operators.

Proof of Theorem 3.2.11. Ad (a). By Lemma 3.2.10, $(U_p(t,s))_{0\le s\le t\le T}$ is an evolution family. Lemma 3.2.12(b) yields the exponential bound.

Ad (b). For $1<p<\infty$, Proposition 3.2.16 yields that $(U_p(t,s))_{0\le s\le t\le T}$ is an evolution family for $(A_p(t))_{t\in[0,T]}$. The uniqueness follows from Proposition 3.2.4. □

3.3 Observability for Evolution Families on Measurable Sets in Time

This section aims at presenting a novel extension of the Lebeau–Robbiano strategy in order to deduce observation estimates for evolution families on Banach spaces and time-dependent families of observation operators. To this end, we will extend the strategy presented in [111] to the non-autonomous setting. We want to emphasize that the approach given in this section is able to handle uncertainty principles that are only defined on a measurable set $E \subseteq [0, T]$ which is made possible by a fine-tuned version of the results presented in the author's joint work with Clemens Bombach, Christian Seifert, and Martin Tautenhahn. Our main result is similar to [12, Thm. 13] which in turn is based on [13, Thm. 2.1]. In contrast to the cited resources, our theorem works for evolution families on general Banach spaces and not only Hilbert spaces. In particular, projections are not needed, the evolution families only need to be bounded, not contractive, and the evolution family does not need to be continuous at all. It is natural to assume these less restrictive conditions as they mimic the autonomous case as shown in [61]. However, the non-autonomous generalization also comes with a further degree of freedom that allows for different restrictions to be made in terms of the uncertainty estimate. We want to emphasize that this proof will not rely on the validity of an interpolation estimate, contrasting the strategy presented in [156, Thm. 1.2]; see also the discussion in [61, Rem. 2.2]. More precisely, we will derive the interpolation estimate as a byproduct of our proof at the end of this section.

3.3.1 Density Point Induced Partitions of Measurable Sets

This section contains the measure theoretical foundations for the telescoping series argument to be applied in the proof of our main theorem in Section 3.3.2. Here and in the following, for a measurable set $E \subseteq \mathbb{R}$, let $|E|$ denote its Lebesgue measure.

Definition 3.3.1. Let $E \subseteq \mathbb{R}$ a measurable subset. We call $\ell \in E$ a *density point* of E, if

$$\lim_{r \to 0} \frac{|[\ell - r, \ell + r] \cap E|}{2r} = 1 .$$

Furthermore, we let $\mathfrak{D}(E)$ denote the set of all density points of E.

The following remark lists some easy observations and classical results regarding density points.

Remark 3.3.2. (a) Loosely speaking, a density point of E is a point whose small neighborhoods are almost entirely covered by E, see, e.g., [139, p. 106] and [138, Sec. I.2.1]. Sometimes a density point is also called *Lebesgue point* or *point of Lebesgue density* of E.

(b) If $\mathfrak{D}(E) \neq \emptyset$ then $|E| > 0$.

(c) For all $\ell \in \mathbb{R}$, we have

$$1 = \frac{\big|[\ell - r, \ell + r]\big|}{2r} = \frac{\big|[\ell - r, \ell + r] \cap E\big|}{2r} + \frac{\big|[\ell - r, \ell + r] \cap \complement E\big|}{2r} .$$

In particular, if $\ell \in \mathfrak{D}(E)$, then

$$\lim_{r\to 0} \frac{\big|[\ell - r, \ell + r] \cap \complement E\big|}{2r} = 0.$$

(d) As a consequence of *Lebesgue's density theorem*, almost every point $\ell \in E$ is a density point of E. In particular, $|\mathfrak{D}(E)| = |E|$.

(e) For an interval $I = [a, b]$, $a < b$, we have $\mathfrak{D}(E) = (a, b)$.

(f) If $E \subseteq F$ then $\mathfrak{D}(E) \subseteq \mathfrak{D}(F)$.

The following lemma generalizes the observation from Remark 3.3.2(e) to the case of measurable subsets.

Lemma 3.3.3. *Let $E \subseteq \mathbb{R}$ measurable and $\ell \in E$ a density point of E. Then*

$$\lim_{r\to 0} \frac{\big|[\ell, \ell + r] \cap E\big|}{r} = 1 \quad \textit{and} \quad \lim_{r\to 0} \frac{\big|[\ell - r, \ell] \cap E\big|}{r} = 1\,.$$

In particular, for $E \subseteq [a, b]$, $a < b$, we always have $\{a, b\} \cap \mathfrak{D}(E) = \emptyset$.

Proof. Indeed, monotonicity of the Lebesgue integral implies

$$\frac{|[\ell, \ell + r] \cap E|}{r} \le 1 \quad \text{and} \quad \frac{|[\ell - r, \ell] \cap E|}{r} \le 1 \quad \text{for all } r > 0. \tag{3.14}$$

Now assume that there exists $r_0 > 0$ and $\varepsilon > 0$ such that, for all $r \le r_0$, we have

$$\frac{|[\ell, \ell + r] \cap E|}{r} < 1 - \varepsilon.$$

As ℓ is a density point of E, there also exists $r_1 \le r_0$ such that $|[\ell-r_1, \ell+r_1] \cap E| > 2r\,(1-\frac{\varepsilon}{2})$. But then

$$\frac{|[\ell - r_1, \ell] \cap E|}{r_1} = \frac{|[\ell - r_1, \ell + r_1] \cap E|}{r_1} - \frac{|[\ell, \ell + r_1] \cap E|}{r_1} > 2 - \varepsilon - 1 + \varepsilon = 1\,,$$

which contradicts (3.14). This proves the first identity of Lemma 3.3.3. The second identity follows with the same argument. □

The following proposition is an improved version of [12, Prop. 14] and [154, Lem. 2.3]. See also [105, p. 256–257] for a similar statement. Our extension provides not only the explicit dependence on the involved parameters but also optimizes the outcome of the proof strategy presented in [12].

Proposition 3.3.4. *Let $T > 0$ and $E \subseteq [0, T]$ measurable with positive Lebesgue measure, and let*

$$0 < \varrho < 1 \quad \textit{and} \quad 0 < q \le \frac{\varrho^2}{4 - 2\varrho} < \frac{1}{2}\,. \tag{3.15}$$

Then, for almost every $\ell \in E$, there exists a sequence $(\ell_m)_{m\in\mathbb{N}}$ in $(\ell, T) \cap E$ with limit $\lim_{m\to\infty} \ell_m = \ell$ *and subject to $\ell < \ell_{m+1} < \ell_m < \cdots < \ell_1$ satisfying*

$$\ell_{m+1} - \ell_{m+2} \ge q\,(\ell_m - \ell_{m+1}) \tag{3.16}$$

and

$$\Big|\Big[\ell_{m+1},\ \ell_{m+1} + \frac{\ell_m - \ell_{m+1}}{2}\Big] \cap E\Big| \ge \varrho\,\frac{\ell_m - \ell_{m+1}}{2}\,, \quad m \in \mathbb{N}\,. \tag{3.17}$$

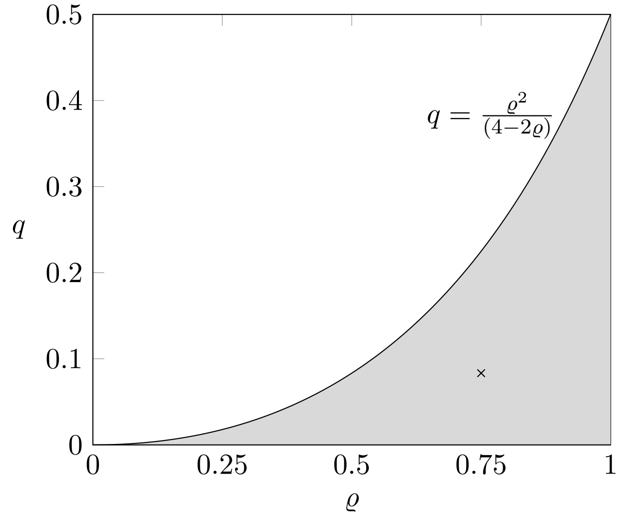

Figure 3.1: Allowed range of pairs (ϱ, q) from (3.15) such that Proposition 3.3.4 holds. There is a one-to-one correspondence between values $0 < q < \frac{1}{2}$ and the *optimal* $0 < \rho < 1$ given via the relation $q = \varrho^2/(4-2\varrho)$. The specific pair $(\varrho, q) = (\frac{3}{4}, \frac{1}{12})$ used in [12, Prop. 15] has been marked with a cross.

The region described by (3.15) is depicted in Figure 3.1.

Proof of Proposition 3.3.4. Let $0 < q < \frac{1}{2}$. We want to specify a range of possible values $\varrho \in (0, 1)$ such that the statements of Proposition 3.3.4 are valid. We divide our proof into three steps.

Step 1. Let us show that it suffices to prove the following statement.

For all density points $\ell \in \bigcup_{j\in\mathbb{N}} \mathfrak{D}(E_j)$, where

$$E_j := \Big\{\sigma \in E : \big|[\sigma, \sigma + r] \cap E\big| > \varrho\, r, \text{ for all } 0 < r < \frac{1}{j}\Big\}, \quad j \in \mathbb{N},$$

there exists $j_0 \in \mathbb{N}$ and a decreasing sequence $(\ell_m)_{m\in\mathbb{N}}$ in $(\ell, T) \cap E_{j_0}$ with $\lim_{m\to\infty} \ell_m = \ell$ and subject to the conditions (3.16) *and* (3.17).

Indeed, the given sequence $(\ell_m)_{m\in\mathbb{N}}$ lies in $(\ell, T) \cap E$ as $E_{j_0} \subseteq E$ *per constructionem* and, furthermore, fulfills the estimates (3.16) and (3.17) by assumption. Thus the only thing left to show is that $|E| = |\bigcup_{j\in\mathbb{N}} \mathfrak{D}(E_j)|$. Note that, *per constructionem*, we have the inclusions

$$E_j \subseteq E_{j+1} \quad \text{and} \quad \mathfrak{D}(E_j) \subseteq \mathfrak{D}(E_{j+1}), \quad j \in \mathbb{N}. \tag{3.18}$$

Furthermore, Lemma 3.3.3 implies that, for every $\ell \in \mathfrak{D}(E)$, there exists $j_0 \in \mathbb{N}$ such that $\ell \in E_j$ for all $j \geq j_0$. Therefore

$$\mathfrak{D}(E) \subseteq \bigcup_{j\in\mathbb{N}} E_j.$$

Lebesgue's density theorem now tells us that $|\mathfrak{D}(E)| = |E|$ and $|\mathfrak{D}(E_j)| = |E_j|$ for all $j \in \mathbb{N}$. Using monotonicity and *continuity from below*, it follows that

$$|E| = |\mathfrak{D}(E)| \le \Big|\bigcup_{j\in\mathbb{N}} E_j\Big| = \lim_{j\to\infty} |E_j| = \lim_{j\to\infty} |\mathfrak{D}(E_j)| = \Big|\bigcup_{j\in\mathbb{N}} \mathfrak{D}(E_j)\Big| \le |E|. \tag{3.19}$$

Inequality (3.19) now proves that $|E| = \big|\bigcup_{j\in\mathbb{N}} \mathfrak{D}(E_j)\big|$.

Step 2. In order to prove the initial statement of Step 1, let us fix $j \in \mathbb{N}$ and $\ell \in \mathfrak{D}(E_j)$. Since ℓ is a density point of E_j, Lemma 3.3.3 yields the existence of r_0 such that

$$\big|[\ell,\, \ell + r] \cap E\big| \ge \big|[\ell,\, \ell + r] \cap E_j\big| > \varrho\, r \quad \text{for all } 0 < r \le r_0. \tag{3.20}$$

It is no loss of generality, to additionally impose the condition $0 < r_0 < \min\{\ell - T, \frac{1}{j}\}$. Let us inductively construct a decreasing sequence $(\ell_m)_{m\in\mathbb{N}}$ with $\ell_m \in [\ell, \ell + r_0] \cap E_j$, $\lim_{m\to\infty} \ell_m = \ell$, and

$$\big|[\ell,\, \ell_m] \cap E\big| > \varrho\,(\ell_m - \ell) \quad \text{for all } m \in \mathbb{N}. \tag{3.21}$$

For the induction start, let $m = 1$ and note that the set $[\ell, \ell + r_0] \cap E_j$ has positive Lebesgue measure as a consequence of (3.20) with $r = r_0$. In particular, we may choose an arbitrary point $\ell_1 \in (\ell, \ell + r_0] \cap E_j$. Then ℓ_1 will fulfill (3.21) with $m = 1$. Indeed, this is a consequence of (3.20) with $r = \ell_1 - \ell \le r_0$.

For the rest of the proof, let us set

$$\kappa \coloneqq \min\Big\{\frac{2}{\varrho} - 3,\ 1\Big\} = \begin{cases} \frac{2}{\varrho} - 3 & \text{if } \frac{1}{2} \le \varrho\,, \\ 1 & \text{if } \varrho < \frac{1}{2}\,. \end{cases} \tag{3.22}$$

and carry out the induction step as a case distinction depending on the value of ϱ.

Step 3. Here we treat the case $\frac{1}{2} \le \varrho < 1$. Let $m \ge 1$, and assume we have the finite sequence $\ell < \ell_m < \ell_{m-1} < \cdots < \ell_1$ fulfilling (3.21). We want to choose our next sequence element

$$\ell_{m+1} \in S_m \cap E_j,$$

where we defined

$$S_m \coloneqq \Big[\ell + \frac{1-\varrho}{1+\kappa}\,(\ell_m - \ell)\,,\ \ \ell + \frac{\varrho+\kappa}{1+\kappa}\,(\ell_m - \ell)\Big], \quad m \in \mathbb{N}.$$

As a consequence of the definition of κ in (3.22), we have

$$0 < \frac{1-\varrho}{1+\kappa} < \frac{\varrho+\kappa}{1+\kappa} < 1 \tag{3.23}$$

implying $|S_m| > 0$ for all $m \in \mathbb{N}$. Indeed, note that $-1 < 1 - 2\varrho < 2/\varrho - 3 \le \kappa$ for all $\varrho \in [\frac{1}{2}, 1)$. Rearranging terms gives

$$1 - 2\varrho < \kappa \iff 1 - \varrho < \kappa + \varrho \iff \frac{1-\varrho}{1+\kappa} < \frac{\kappa+\varrho}{1+\kappa}$$

which shows the validity of (3.23).

Let us show that $|S_m \cap E_j| > 0$. Indeed, assuming $|S_m \cap E_j| = 0$, we get by decomposing the interval $[\ell, \ell_m]$ that

$$\begin{aligned}\big|[\ell, \ell_m] \cap E_j\big| &= \Big|\Big[\ell\,,\ \ell + \frac{1-\varrho}{1+\kappa}\,(\ell_m - \ell)\Big] \cap E_j\Big| + \Big|\Big[\ell + \frac{\varrho+\kappa}{1+\kappa}\,(\ell_m - \ell)\,,\ \ell_m\Big] \cap E_j\Big| \\ &\le \frac{1-\varrho}{1+\kappa}\,(\ell_m - \ell) + \Big(1 - \frac{\varrho+\kappa}{1+\kappa}\Big)\,(\ell_m - \ell) \\ &= 2\,\frac{1-\varrho}{1+\kappa}\,(\ell_m - \ell) \\ &\le \varrho\,(\ell_m - \ell),\end{aligned}$$

where the validity of the latest estimate is due to the definition of κ in (3.22). But this estimate now contradicts our induction hypothesis (3.21). Thus $S_m \cap E_j$ must have a non-zero measure. In particular, there exists a point $\ell_{m+1} \in S_m \cap E_j$. As a consequence of (3.23) and the definition of S_m, we have that $\ell < \ell_{m+1} < \ell_m < \ell_1$. In particular, the estimate $0 < \ell_{m+1} - \ell < \ell_1 - \ell \le r_0$ is valid. In virtue of (3.20) with $r = \ell - \ell_{m+1}$, this gives

$$\big|[\ell, \ell_{m+1}] \cap E\big| \ge \big|[\ell, \ell_{m+1}] \cap E_j\big| > \varrho\,(\ell_{m+1} - \ell).$$

This concludes our inductive definition in the case $\varrho \ge \frac{1}{2}$. *Per constructionem*, we have $\ell_{m+1} \in S_m$ for all $m \in \mathbb{N}$. The definition of S_m gives

$$0 \le \ell_{m+1} - \ell \le \frac{\rho+\kappa}{1+\kappa}\,(\ell_m - \ell) \le \cdots \le \Big(\frac{\rho+\kappa}{1+\kappa}\Big)^m\,(\ell_1 - \ell)\,.$$

As $\frac{\varrho+\kappa}{1+\kappa} < 1$, this shows $\lim_{m\to\infty} \ell_m = \ell$.

Let us show that the sequence $(\ell_m)_{m\in\mathbb{N}}$ fulfills the conditions (3.16) and (3.17). To this end, fix $m \in \mathbb{N}$ and recall that $\ell_m, \ell_{m+1} \in [\ell, \ell + r_0] \cap E_j$ *per constructionem*. This implies that

$$0 < r := \frac{\ell_m - \ell_{m+1}}{2} < \frac{r_0}{2}$$

and the definition of E_j now gives

$$\big|[\ell_{m+1},\, \ell_{m+1} + r] \cap E\big| > \varrho\,r$$

which proves (3.17). Furthermore, $\ell_{m+2} \in S_{m+1} \cap E_j$ implies

$$\frac{1-\varrho}{1+\kappa}(\ell_{m+1} - \ell) \le \ell_{m+2} - \ell \le \frac{\varrho+\kappa}{1+\kappa}\,(\ell_{m+1} - \ell)\,. \tag{3.24}$$

In particular, the second estimate in (3.24) implies that

$$-\ell_{m+2} \ge -\ell - \frac{\varrho+\kappa}{1+\kappa}\,(\ell_{m+1} - \ell) = -\frac{\varrho+\kappa}{1+\kappa}\,\ell_{m+1} + \frac{\varrho-1}{1+\kappa}\ell\,. \tag{3.25}$$

The first estimate in (3.24) for $\ell_{m+1} \in S_m \cap E_j$ gives that

$$\ell_m - \ell \ge \Big(\frac{1-\varrho}{\varrho+\kappa} + 1\Big)\,\ell_m - \frac{1+\kappa}{\varrho+\kappa}\,\ell_{m+1} = \frac{1+\kappa}{\varrho+\kappa}\,(\ell_m - \ell_{m+1})\,. \tag{3.26}$$

Using first estimate (3.25), then (3.24), and lastly (3.26), we get that

$$\begin{aligned}\ell_{m+2} - \ell_{m+1} &\geq \left(1 - \frac{\varrho + \kappa}{1+\kappa}\right)\ell_{m+2} - \frac{1-\varrho}{1+\kappa}\,\ell = \frac{1-\varrho}{1+\kappa}\,(\ell_{m+2} - \ell) \\ &\geq \left(\frac{1-\varrho}{1+\kappa}\right)^2 (\ell_{m+1} - \ell) \\ &\geq \frac{(1-\varrho)^2}{(1+\kappa)\,(\varrho+\kappa)}(\ell_{m+1} - \ell_{m+2}).\end{aligned}$$

Using that $\kappa = \frac{2}{\rho} - 3$, we get

$$\ell_{m+1} - \ell_{m+2} \geq \frac{(1-\varrho)^2}{(\frac{2}{\varrho} - 2)(\varrho + \frac{2}{\varrho} - 3)}\,(\ell_m - \ell_{m+1}) = \frac{\varrho^2}{4 - 2\varrho}\,(\ell_m - \ell_{m+1}) \geq q\,(\ell_m - \ell_{m+1})$$

for all $q \leq \frac{\varrho^2}{4-2\varrho}$. This proves our claim in the case $\varrho \geq \frac{1}{2}$.

Step 4. We now treat the case $\varrho < \frac{1}{2}$. Recall from (3.22) in Step 2 that $\kappa = 1$ now. Furthermore, recall that $\ell_1 \in (\ell, \ell + r_0] \cap E_j$ fulfills (3.21) for $m = 1$, i.e.,

$$\big|[\ell,\, \ell_1] \cap E\big| > \varrho\,(\ell_1 - \ell).$$

Now, let $m \geq 1$, and assume we have the finite sequence $\ell < \ell_m < \ell_{m-1} < \cdots < \ell_1$ fulfilling (3.21). We want to choose

$$\ell_{m+1} \in S_m \cap E_j\,,$$

where we defined

$$S_m := \Big[\ell + \frac{\varrho}{1+\kappa}\,(\ell_m - \ell),\ \ell + \frac{1-\varrho+\kappa}{1+\kappa}\,(\ell_m - \ell)\Big], \quad m \in \mathbb{N}.$$

As $\kappa = 1$, we have $0 < \frac{\varrho}{1+\kappa} < \frac{1-\varrho+\kappa}{1+\kappa} < 1$ such that $|S_m| > 0$ for all $m \in \mathbb{N}$.

Let us show that $|S_m \cap E_j| > 0$. Indeed, similar to the case $\varrho \geq \frac{1}{2}$ in Step 3, we get under the assumption $|S_m \cap E_j| = 0$ that

$$\big|[\ell, \ell_m] \cap E_j\big| \leq \frac{2\varrho}{1+\kappa}\,(\ell_m - \ell) = \varrho\,(\ell_m - \ell),$$

where the validity of the latest identity is due to $\kappa = 1$. But this estimate now contradicts (3.21), i.e., our induction hypothesis. Thus there exists a point $\ell_{m+1} \in S_m \cap E_j$. *Per constructionem*, we have $\ell_{m+1} < \ell_m$, and ℓ_{m+1} automatically fulfills the estimate $\ell_{m+1} - \ell < r_0$. In virtue of (3.20) with $r = \ell - \ell_{m+1}$, this gives

$$\big|[\ell, \ell_{m+1}] \cap E\big| \geq \big|[\ell, \ell_{m+1}] \cap E_j\big| > \varrho\,(\ell_{m+1} - \ell).$$

This concludes our inductive definition in the case $\varrho < \frac{1}{2}$. As in Step 3, one now shows that $\ell_{m+1} \in S_m$ for all $m \in \mathbb{N}$ implies $\lim_{m\to\infty} \ell_m = \ell$.

Let us show that the sequence $(\ell_m)_{m\in\mathbb{N}}$ fulfills the conditions (3.16) and (3.17). To this end, fix $m \in \mathbb{N}$ and recall that $\ell_m, \ell_{m+1} \in [\ell, \ell + r_0] \cap E_j$ *per constructionem*. This implies that

$$0 < r := \frac{\ell_m - \ell_{m+1}}{2} < \frac{r_0}{2}$$

and the definition of E_j now gives

$$\big|[\ell_{m+1},\, \ell_{m+1}+r] \cap E\big| \geq \varrho\, r$$

which proves (3.17). Furthermore, $\ell_{m+2} \in S_{m+1} \cap E_j$ implies

$$\frac{\varrho}{1+\kappa}\,(\ell_{m+1}-\ell) \leq \ell_{m+2}-\ell \leq \frac{1-\varrho+\kappa}{1+\kappa}\,(\ell_{m+1}-\ell)\,. \tag{3.27}$$

Note that this estimate has the same structure as the corresponding estimate (3.24) in Step 2. Using that $1-\varrho \geq \frac{1}{2}$, the calculation in Step 2 shows *mutatis mutandis* that, for the choice of $q \leq \frac{\varrho^2}{4-2\varrho}$,

$$\begin{aligned}\ell_{m+1}-\ell_{m+2} &\geq \frac{\varrho^2}{(1+\kappa)(1-\varrho+\kappa)}\,(\ell_m-\ell_{m+1}) = \frac{\varrho^2}{4-2\varrho}\,(\ell_m-\ell_{m+1})\\ &\geq q\,(\ell_m-\ell_{m+1})\,.\end{aligned}$$

This concludes our proof in the case $\varrho < \frac{1}{2}$. □

Let us also note the following proposition which is an adapted version of [121, Prop. 2.1]. See also [47, Lem. 2.1.5] for a similar statement.

Proposition 3.3.5. *Let $E \subseteq [0,T]$ be a measurable set with positive Lebesgue measure. For almost every point $\ell \in E$ and for each $0<q<1$, there exists a sequence $(\ell_m)_{m\in\mathbb{N}}$ in (ℓ,T) with limit $\lim_{m\to\infty}\ell_m = \ell$ and subject to $\ell < \ell_{m+1} < \ell_m < \cdots < \ell_1$ satisfying*

$$\ell_{m+1}-\ell_{m+2} = q\,(\ell_m-\ell_{m+1}) \tag{3.28}$$

and

$$\ell_m-\ell_{m+1} \leq 3\,|E \cap (\ell_{m+1},\ell_m)|\,, \quad m \in \mathbb{N}\,. \tag{3.29}$$

Remark 3.3.6. Observe that, due to the fixed step size in (3.28), in comparison with Proposition 3.3.4, the partitioning sequence $(\ell_m)_{m\in\mathbb{N}}$ cannot be chosen in E.

Proof of Proposition 3.3.5. As pointed out in Lemma 3.3.3, we have $E \subseteq (0,T)$. Let us fix a density point $\ell \in E$ and $q \in (0,1)$. We divide the proof into three steps.

Step 1. In this step, we define the sequence $(\ell_m)_{m\in\mathbb{N}}$ and derive identity (3.28). Furthermore, we will show that the sequence is strictly decreasing.

As ℓ is a density point, Remark 3.3.2(c) gives the existence of r_0 depending on q such that

$$|\complement E \cap (\ell-r,\ell+r)| < \frac{1-q}{2\,(1+q)}\,|E \cap (\ell-r,\ell+r)| \quad \text{for all } r < r_0. \tag{3.30}$$

Let us set $\widetilde{r}_0 := \min\{r_0, T-\ell\}$ and fix ℓ_1 subject to $\ell < \ell_1 < \ell+\widetilde{r}_0$. Now let $(\ell_m)_{m\in\mathbb{N}}$ denote the sequence defined via

$$\ell_{m+1} := \ell + q^m(\ell_1-\ell), \quad m \in \mathbb{N}. \tag{3.31}$$

As $q < 1$ by assumption, it is clear from the definition (3.31) that

$$\ell < \ell_{m+1} < \ell_m < \cdots < \ell_1. \tag{3.32}$$

The definition (3.31) also shows that

$$\ell_{m+1} - \ell_{m+2} = (q^m - q^{m+1})(\ell_1 - \ell) = q\,(q^{m-1} - q^m)(\ell_1 - \ell) = q\,(\ell_m - \ell_{m+1})$$

which verifies (3.28). For further usage, let us also note the following identity which follows directly from (3.31)

$$\begin{aligned}\ell_{m+1} - (2\ell - \ell_m) &= (q^m + q^{m-1})(\ell_1 - \ell) = \frac{1+q}{1-q}\,(1-q)q^{m-1}(\ell_1 - \ell)\\ &= \frac{1+q}{1-q}\,(q^{m-1} - q^m)(\ell_1 - \ell) = \frac{1+q}{1-q}\,(\ell_m - \ell_{m+1})\,.\end{aligned} \tag{3.33}$$

Step 2. In this step, we show that it suffices to prove that

$$|\complement E \cap (\ell_{m+1}, \ell_m)| < \frac{1}{2}\Big(|E \cap (\ell_{m+1}, \ell_m)| + |(\ell_{m+1}, \ell_m)|\Big) \quad \text{for all } m \in \mathbb{N}. \tag{3.34}$$

Indeed, partitioning the interval (ℓ_{m+1}, ℓ_m) and using (3.30) gives

$$\begin{aligned}\ell_m - \ell_{m+1} = |(\ell_{m+1}, \ell_m)| &= |E \cap (\ell_{m+1}, \ell_m)| + |\complement E \cap (\ell_{m+1}, \ell_m)|\\ &\le \frac{3}{2}\,|E \cap (\ell_{m+1}, \ell_m)| + \frac{1}{2}\big(\ell_m - \ell_{m+1}\big).\end{aligned}$$

Rearranging terms yields the desired estimate (3.29).

Step 3. Let us verify the estimate (3.34). Let $m \in \mathbb{N}$, and choose $r = \ell_m - \ell$. Note that, in virtue of the choice of ℓ_1 in Step 2 and the monotonicity (3.32), we have $r < r_0$. Furthermore, the monotonicity of the sequence $(\ell_m)_{m\in\mathbb{N}}$ implies

$$2\ell - \ell_m < \ell - (\ell_m - \ell) < \ell < \ell_{m+1}. \tag{3.35}$$

Now, using (3.35) to pass to a larger interval and $\theta = \ell_m - \ell$ in (3.30), we get

$$|\complement E \cap (\ell_{m+1}, \ell_m)| < |\complement E \cap (2\ell - \ell_m, \ell_m)| < \frac{1-q}{2\,(1+q)}\,|E \cap (2\ell - \ell_m, \ell_m)|. \tag{3.36}$$

Using estimate (3.35) to split up the interval $(2\ell - \ell_m, \ell_{m+1})$ leads us to the estimate

$$\begin{aligned}|E \cap (2\ell - \ell_m, \ell_m)| &\le |(2\ell - \ell_m, \ell_{m+1})| + |E \cap (\ell_{m+1}, \ell_m)|\\ &= \frac{1+q}{1-q}\,(\ell_m - \ell_{m+1}) + |E \cap (\ell_{m+1}, \ell_m)|,\end{aligned} \tag{3.37}$$

where the latest step is due to the identity (3.33). Plugging (3.37) into (3.36) yields (3.34) and concludes our proof. □

3.3.2 Non-autonomous Lebeau–Robbiano Strategy

In this section, we prove our main theorem on abstract observability of non-autonomous observation systems. To this end, we will adapt a well-established method known as the *Lebeau–Robbiano* strategy to our non-autonomous observation system (3.1). The name of this strategy attributes to the seminal works of Lebeau, Robbiano, Zuazua, and Jerison [78, 94, 95]. The core of this strategy is based on the validity of an abstract uncertainty principle, a dissipation estimate for a range of so-called *spectral parameters*, and a relation in their respective growth or decay rate. More precisely, in the setting of *autonomous* systems, i.e. $C(t) = C$, $t \in [0, T]$, and one-parameter semigroups $(S(t))_{t\in[0,T]}$, these estimates are of the form

$$\forall\lambda > 0\ \forall x \in X : \|P_\lambda x\|_X \lesssim \mathrm{e}^{\lambda^{\gamma_1}} \|CP_\lambda x\|_Y \tag{aUCP}$$

and

$$\forall\lambda > 0\ \forall 0 \le t \le T\ \forall x_0 \in X : \|(\mathrm{Id} - P_\lambda)S(t)x_0\|_X \lesssim \mathrm{e}^{-\lambda^{\gamma_2} t^{\gamma_3}} \|x_0\|_X, \tag{aDE}$$

see, e.g., [61]. It is a further essential ingredient of the Lebeau–Robbiano strategy to prove that the growth rate γ_1 in (aUCP) is strictly smaller than the decay rate γ_2 of the dissipation estimate (aDE).

For our non-autonomous adaptation of the Lebeau–Robbiano strategy, our observation system (3.1) will always be subject to the following hypothesis.

Hypothesis 3.3.7. *Let X and Y be Banach spaces, $T > 0$, $E \subseteq [0, T]$ measurable with positive Lebesgue measure, and $(U(t, s))_{0\le s\le t\le T}$ an exponentially bounded evolution family on X. Let $C\colon [0, T] \to \mathcal{L}(X, Y)$ be essentially bounded on E, and assume that $[0, T] \ni t \mapsto \|C(t)U(t, 0)x_0\|_Y$ is measurable for all $x_0 \in X$. Let $(P_\lambda)_{\lambda>0}$ be a family in $\mathcal{L}(X)$. Assume that there exist $d_0, d_1, \gamma_1 > 0$ such that*

$$\forall\lambda > 0\ \forall x \in X : \|P_\lambda x\|_X \le d_0\, \mathrm{e}^{d_1\lambda^{\gamma_1}} \operatorname{ess\,inf} \Big\{ E \ni \tau \mapsto \|C(\tau)P_\lambda x\|_Y \Big\} \tag{essUCP}$$

and $d_2 \ge 1$ and $d_3, \gamma_2, \gamma_3 > 0$ with $\gamma_2 > \gamma_1$ such that

$$\forall\lambda > 0\ \forall 0 \le s \le t \le T\ \forall x_s \in X : \|(\mathrm{Id} - P_\lambda)U(t, s)x_s\|_X \le d_2\, \mathrm{e}^{-d_3\lambda^{\gamma_2}(t-s)^{\gamma_3}} \|x_s\|_X. \tag{DE}$$

Remark 3.3.8. (a) In comparison to the autonomous case presented in [61], one notices several additions that may seem cumbersome. They are, however, strictly necessary if one evaluates the nature of the final-state observability estimate. Note that the final-state observability estimate involves integration over the set E. It is therefore natural to assume that the involved ingredients of the Lebeau–Robbiano strategy hold essentially uniformly on E as well.

(b) Due to the strict inequality $\gamma_2 > \gamma_1$ of decay and growth rate, the dissipation effect will asymptotically dominate the uncertainty effect introduced by the observation operators $C(t)$ for $\lambda \to \infty$. This will be the crucial ingredient in the proof of our main theorem as this will allow us to balance $\|U(t, 0)x_0\|_X$ and $\|C(t)U(t, 0)x_0\|_Y$ making available a telescoping series argument also known as *Miller's trick* [111].

(c) Note that the evolution family $(U(t,s))_{0\leq s\leq t\leq T}$ in Hypothesis 3.3.7 does not need to be explicitly related to a non-autonomous operator family. Indeed, as Theorem 3.3.9 will show, no differentiability properties of $(U(t,s))$ are assumed. However, in applications, one usually starts off with a non-autonomous Cauchy problem (NACP). This problem usually comes with a notion of *well-posedness* which in turn gives rise to an associated evolution family as the solution operator to the problem. While the existence of such an evolution family may be a delicate matter, Hypothesis 3.3.7 does not pose restrictions in terms of any type of continuity, contractivity, or differentiability on the involved evolution family. In particular, the hypothesis and the related Theorem 3.3.9 do not "see" any operators that may be associated with $(U(t,s))$. Only the algebraic property (a) of Definition 3.2.2 is required.

(d) A further generalization of the classical uncertainty estimate for the case of constant observation operators C is given in [18]. There, the authors used the following uncertainty principle that is *uniform* for all $t \in [0,T]$

$$\forall \lambda > 0 \; \forall x \in X : \|P_\lambda x\|_X \leq d_0\, \mathrm{e}^{d_1 \lambda^{\gamma_1}} \inf \Big\{ [0,T] \ni \tau \mapsto \|C(\tau) P_\lambda x\|_Y \Big\} \qquad \text{(uniUCP)}$$

Note, however, that this generalization is highly sensitive with regards to the properties of the family of observation operators $(C(t))_{t\in[0,T]}$. More precisely, estimate (uniUCP) involves the entire set $[0,T]$ despite the final-state observability estimate only involving integration over E. Furthermore, as the integral is not affected by deviations of $C(t)$ on sets of Lebesgue measure zero, one would expect the input to exhibit the same behavior. However, if $C(t) = 0$ for a single point $t \in [0,T] \setminus E$, then (uniUCP) would imply that $P_\lambda = 0$ for all $\lambda > 0$ and the proof of [18, Thm. 3.3] would no longer be accessible.

Theorem 3.3.9. *Assume Hypothesis 3.3.7, let $r \in [1,\infty]$, and let $E \subseteq [0,T]$ be measurable with positive Lebesgue measure. Then there exists $C_{\mathrm{obs}} \geq 0$ such that, for all $x_0 \in X$, we have the final-state observability estimate*

$$\|U(T,0)x_0\|_X \leq \begin{cases} C_{\mathrm{obs}} \Big(\int_E \|C(t)U(t,0)x_0\|_Y^r \,\mathrm{d}t \Big)^{\frac{1}{r}}, & r \in [1,\infty), \\ C_{\mathrm{obs}}\, \operatorname{ess\,sup}_{t\in E} \|C(t)U(t,0)x_0\|_Y, & r = \infty. \end{cases} \tag{3.38}$$

The main idea of the proof of Theorem 3.3.9 will be to construct a telescoping series argument based on the sequence $(\ell_m)_{m\in\mathbb{N}}$ from Proposition 3.3.5. The telescoping series argument has been the *leitmotif* in previous proofs of observability estimates building on methods developed by Miller [111], which were later used in the works by Phung and Wang [121, Thm. 1.1], Apraiz, Escauriaza, Wang, and Zhang [6, Thm. 7], Wang and Zhang [156, Thm. 1.2], and Beauchard, Egidi, and Pravda-Starov [12]. However, [6, 156] build their proofs on an interpolation result of the operators $U(t,s)$ and $C(t)U(t,s)$ which is not needed as the following streamlined proof shows. See Section 3.3.3 on further information regarding interpolation estimates.

Proof of Theorem 3.3.9. Let us fix $x_0 \in X$. We divide the proof into four steps.

Step 1. Let us show that it suffices to prove the final-state observability estimate (3.38) for the case $r = 1$. Indeed, assuming that (3.38) holds for $r = 1$, let $r \in (1, \infty]$. Then, for all $x_0 \in X$, we estimate by Hölder's inequality

$$\int_E \|C(t)U(t,0)x_0\|_Y \,\mathrm{d}t \le \begin{cases} |E|^{1-\frac{1}{r}} \, \left(\int_E \|C(t)U(t,0)x_0\|_Y^r \,\mathrm{d}t\right)^{\frac{1}{r}} & \text{for } r < \infty, \\ |E| \ \operatorname{ess\,sup}_{t\in E} \|C(t)U(t,0)x_0\|_Y & \text{for } r = \infty. \end{cases} \tag{3.39}$$

Consequently, the theorem's statement follows by multiplying the observability constant C_{obs} for $r = 1$ with the corresponding power of $|E|$.

Step 2. Before we continue with the proof, let us introduce shorthand notation for recurring expressions. For all $0 \le t \le T$ and $\lambda > 0$, we define for the state function

$$\begin{aligned} F(t) \coloneqq \|U(t,0)x_0\|_X\,, \quad F_\lambda(t) &\coloneqq \|P_\lambda U(t,0)x_0\|_X\,, \\ F_\lambda^\perp(t) &\coloneqq \|(\mathrm{Id} - P_\lambda)U(t,0)x_0\|_X \end{aligned} \tag{3.40}$$

and analogously for the observations

$$\begin{aligned} G(t) \coloneqq \|C(t)U(t,0)x_0\|_Y\,, \quad G_\lambda(t) &\coloneqq \|C(t)P_\lambda U(t,0)x_0\|_Y\,, \\ G_\lambda^\perp(t) &\coloneqq \|C(t)(\mathrm{Id} - P_\lambda)U(t,0)x_0\|_Y\,. \end{aligned} \tag{3.41}$$

With this notation, we need to prove that there exists $C_{\mathrm{obs}} \ge 0$ such that

$$F(T) \le C_{\mathrm{obs}} \int_E G(t)\,\mathrm{d}t\,. \tag{3.42}$$

Step 3. In this step, we show that there exist constants $\widetilde{c}_1, \widetilde{c}_2$ such that, for all $0 \le s < t \le T$, $t \in E$, and all $\varepsilon \in (0,1)$, we have the estimate

$$F(t) \le \widetilde{c}_1 \exp\left(\widetilde{c}_2 \left(\frac{1}{t-s}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}\right) \Big(\varepsilon^{-1}G(t) + \varepsilon F(s)\Big). \tag{3.43}$$

By transition to a subset $\widetilde{E} \subseteq E$ with $|\widetilde{E}| = |E|$, it is no loss of generality to assume that (essUCP) holds with inf instead of ess inf and that $(C(t))_{t\in E}$ is uniformly bounded. Note that the uncertainty estimate (essUCP) and uniform boundedness of $(C(t))_{t\in E}$ in $\mathcal{L}(X,Y)$ give

$$F_\lambda(t) \le d_0\, \mathrm{e}^{d_1\lambda^{\gamma_1}} G_\lambda(t) \quad \text{and} \quad G_\lambda^\perp(t) \le \|C(\cdot)\|_{E,\infty}\, F_\lambda^\perp(t) \quad \text{for all } t \in E. \tag{3.44}$$

where $\|C(\cdot)\|_{E,\infty} \coloneqq \sup_{t\in E} \|C(t)\|_{\mathcal{L}(X,Y)} < \infty$. By the algebraic property (a) of evolution families in Definition 3.2.2 and the dissipation estimate (DE), we obtain for all $0 \le s \le t$ the estimate

$$F_\lambda^\perp(t) = \big\|(\mathrm{Id} - P_\lambda)U(t,s)U(s,0)x_0\big\|_X \le d_2\, \mathrm{e}^{-d_3\lambda^{\gamma_2}(t-s)^{\gamma_3}} F(s). \tag{3.45}$$

Then we estimate $G_\lambda(t)$, $t \in E$, using the triangle inequality and the second estimate in (3.44) via

$$G_\lambda(t) \le G(t) + G_\lambda^\perp(t) \le G(t) + \|C(\cdot)\|_{E,\infty}\, F_\lambda^\perp(t). \tag{3.46}$$

Using the triangle inequality and concatenating the estimates (3.44), (3.45), and (3.46), we obtain for all $0 \le s < t \le T$, $t \in E$, and all $\lambda > 0$

$$\begin{aligned} F(t) \le F_\lambda(t) + F_\lambda^\perp(t) &\le d_0\, \mathrm{e}^{d_1\lambda^{\gamma_1}} G_\lambda(t) + d_2\, \mathrm{e}^{-d_3\lambda^{\gamma_2}(t-s)^{\gamma_3}} F(s) \\ &\le d_0\, \mathrm{e}^{d_1\lambda^{\gamma_1}} G(t) + \big(d_0 \|C(\cdot)\|_{E,\infty} + 1\big)\, d_2\, \mathrm{e}^{d_1\lambda^{\gamma_1} - d_3\lambda^{\gamma_2}(t-s)^{\gamma_3}} F(s) \\ &\le \widetilde{c}_1\, \mathrm{e}^{f(\lambda)} \Big(\mathrm{e}^{\frac{1}{2}d_3\lambda^{\gamma_2}(t-s)^{\gamma_3}} G(t) + \mathrm{e}^{-\frac{1}{2}d_3\lambda^{\gamma_2}(t-s)^{\gamma_3}} F(s)\Big), \end{aligned} \tag{3.47}$$

where we defined

$$\widetilde{c}_1 := \max\big\{d_0, (d_0\|C(\cdot)\|_{E,\infty} + 1)\, d_2\big\} \ge 1 \quad \text{and} \quad f(\lambda) := d_1\lambda^{\gamma_1} - \frac{1}{2} d_3 \lambda^{\gamma_2}(t-s)^{\gamma_3}.$$

Let us optimize $f(\lambda)$ with respect to λ before we continue estimating (3.47). Recall that, by assumption, we have $0 \le s < t$ and $\gamma_2 > \gamma_1$. A straightforward calculation reveals that f is maximal for the choice

$$\lambda^* := \left(\frac{2\, d_1\gamma_1}{d_3\gamma_2}\right)^{\frac{1}{\gamma_2-\gamma_1}} \left(\frac{1}{t-s}\right)^{\frac{\gamma_3}{\gamma_2-\gamma_1}} > 0\,.$$

Consequently, f may be estimated via

$$f(\lambda) \le f(\lambda^*) = \widetilde{c}_2 \left(\frac{1}{t-s}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}, \tag{3.48}$$

where we introduced the constant

$$\widetilde{c}_2 := \left(\frac{2\, d_1\gamma_1}{d_3\gamma_2}\right)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} d_1 \left(1 - \frac{\gamma_1}{\gamma_2}\right).$$

Using estimate (3.48), we can further estimate $F(t)$, $t \in E$, in (3.47) via

$$F(t) \le \widetilde{c}_1 \exp\left(\widetilde{c}_2 \left(\frac{1}{t-s}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}\right) \Big(\mathrm{e}^{\frac{1}{2}d_3\lambda^{\gamma_2}(t-s)^{\gamma_3}} G(t) + \mathrm{e}^{-\frac{1}{2}d_3\lambda^{\gamma_2}(t-s)^{\gamma_3}} F(s)\Big). \tag{3.49}$$

Let us concentrate on the behavior of the linear factors in front of $G(t)$ and $F(s)$. Note that the left-hand side of inequality (3.49) and the constants $\widetilde{c}_1, \widetilde{c}_2$ are independent of λ. Now observe that, for all $\varepsilon \in (0,1)$, there exists $\lambda_* > 0$ such that

$$\varepsilon = \mathrm{e}^{-\frac{1}{2}d_3\lambda_*^{\gamma_2}(t-s)^{\gamma_3}}.$$

As the prefactor of $G(t)$ in (3.49) is precisely the reciprocal ε^{-1}, this shows the validity of (3.43) for all $\varepsilon \in (0,1)$.

Step 4. In this step, we show the final-state observability estimate employing a telescoping series argument. Since $(U(t,s))_{0\le s\le t\le T}$ is an exponentially bounded evolution family, there exist $M \ge 1$ and $\omega \in \mathbb{R}$ such that

$$F(\tau) = \|U(\tau,t)U(t,0)x_0\|_X \le M\mathrm{e}^{\omega(\tau-t)} F(t) \le M\mathrm{e}^{\omega_+ T} F(t) \quad \text{for all } 0 \le t \le \tau \le T,$$

where $\omega_+ \coloneqq \max\{\omega, 0\}$. We apply the estimate (3.43) from Step 3 and obtain for all $0 \le s < t < \tau \le T$, $t \in E$, and $\varepsilon \in (0,1)$

$$F(\tau) \le M\,\widetilde{c}_1 \exp\left(\widetilde{c}_2 \left(\frac{1}{t-s}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}} + \omega_+ T\right)\Big(\varepsilon^{-1} G(t) + \varepsilon F(s)\Big). \tag{3.50}$$

By Proposition 3.3.5 for a fixed density point $\ell \in E$ and $q \coloneqq \left(\frac{3}{4}\right)^{\frac{\gamma_2-\gamma_1}{\gamma_1\gamma_3}} < 1$, there exists a strictly decreasing sequence $(\ell_m)_{m\in\mathbb{N}}$ in $[0,T]$ with $\ell_m \to \ell$ such that, for all $m \in \mathbb{N}$, we have the relations

$$\delta_{m+1} = q\,\delta_m \quad \text{and} \quad \Big|(\ell_{m+1}, \ell_m) \cap E\Big| \ge \frac{\delta_m}{3}\,, \tag{3.51}$$

where $\delta_m \coloneqq \ell_m - \ell_{m+1}$, $m \in \mathbb{N}$. Let us fix $m \in \mathbb{N}$ and define

$$\xi \coloneqq \ell_{m+1} + \frac{\delta_m}{6}, \quad s = \ell_{m+1} \quad \text{and} \quad \tau = \ell_m.$$

Then, for all $t \in (\xi, \ell_m)$, we have $t - s \ge \xi - \ell_m \ge \delta_m/6$. Inserting this into (3.50), we obtain for all $t \in (\xi, \ell_m) \cap E$ and all $\varepsilon \in (0,1)$ that

$$F(\ell_m) \le \widetilde{c}_3 \exp\left(\frac{\widetilde{c}_4}{\delta_m^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right)\Big(\varepsilon^{-1} G(t) + \varepsilon F(\ell_{m+1})\Big), \tag{3.52}$$

where

$$\widetilde{c}_3 \coloneqq M\,\widetilde{c}_1 \exp(\omega_+ T) \ge 1 \quad \text{and} \quad \widetilde{c}_4 \coloneqq \widetilde{c}_2\, 6^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}} > 0.$$

Now choose

$$\varepsilon \coloneqq (\widetilde{c}_3)^{-1} q \exp\left(-\frac{2\,\widetilde{c}_4}{\delta_m^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) < 1.$$

For this choice of ε, estimate (3.52) reads

$$F(\ell_m) \le (\widetilde{c}_3)^2 q^{-1} \exp\left(\frac{3\,\widetilde{c}_4}{\delta_m^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) G(t) + q \exp\left(-\frac{\widetilde{c}_4}{\delta_m^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) F(\ell_{m+1}).$$

Rearranging terms yields

$$\exp\left(-\frac{3\,\widetilde{c}_4}{\delta_m^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) F(\ell_m) - q \exp\left(-\frac{4\,\widetilde{c}_4}{\delta_m^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) F(\ell_{m+1}) \le (\widetilde{c}_3)^2\, q^{-1}\, G(t). \tag{3.53}$$

Furthermore, by (3.51), we have the following estimate for integral means on $(\xi, \ell_m) \cap E$

$$\frac{1}{|(\xi, \ell_m) \cap E|} \int_{\xi}^{\ell_m} \mathbf{1}_E(t) G(t)\,\mathrm{d}t \le \frac{6}{\delta_m} \int_{\ell_{m+1}}^{\ell_m} \mathbf{1}_E(t) G(t)\,\mathrm{d}t. \tag{3.54}$$

Taking the integral mean in (3.53) with respect to t on $(\xi, \ell_m) \cap E$ and using (3.54), we obtain after multiplying both sides of the resulting inequality with δ_m

$$\delta_m \exp\left(-\frac{3\,\widetilde{c}_4}{\delta_m^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) F(\ell_m) - \delta_{m+1} \exp\left(-\frac{3\,\widetilde{c}_4}{\delta_{m+1}^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) F(\ell_{m+1})$$
$$\leq 6\,(\widetilde{c}_3)^2\, q^{-1} \int_{\ell_{m+1}}^{\ell_m} \mathbf{1}_E(t) G(t)\,\mathrm{d}t.$$

Note that the constants $\widetilde{c}_3$ and $\widetilde{c}_4$ do not depend on m. Summing over all $m \in \mathbb{N}$ in the inequality above and noting that $\delta_m \to 0$, $\ell_m \to \ell$, and $\sup_{m\in\mathbb{N}} F(\ell_m) < \infty$, a telescoping sum argument yields

$$\delta_1 \exp\left(-\frac{3\,\widetilde{c}_4}{\delta_1^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) F(\ell_1) \leq 6\,(\widetilde{c}_3)^2\, q^{-1} \int_{\ell}^{\ell_1} \mathbf{1}_E(t) G(t)\,\mathrm{d}t.$$

Now we may deduce the estimate

$$F(\ell_1) \leq 6\,(\widetilde{c}_3)^2\, q^{-1}\, \delta_1^{-1} \exp\left(\frac{3\,\widetilde{c}_4}{\delta_1^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) \int_E G(t)\,\mathrm{d}t.$$

Finally, exponential boundedness of the evolution family gives $F(T) \leq M\mathrm{e}^{\omega(T-\ell_1)} F(\ell_1)$. This shows that there exists $C_{\mathrm{obs}} \geq 0$ such that the final-state observability estimate (3.42) holds. □

Remark 3.3.10. (a) As Step 1 of the proof of Theorem 3.4.4 shows, it is possible to choose an observability constant C_{obs} that is uniform in $r \in [1, \infty]$. Indeed, let $C_{\mathrm{obs}}^{(r)}$ denote the observability constant for fixed r as shown in the proof. From (3.39), we see that we may estimate $C_{\mathrm{obs}}^{(r)} \leq (1 + |E|)\, C_{\mathrm{obs}}^{(1)}$ for all $r \in [1, \infty]$ in order to deduce a observability constant uniform in r.

(b) If $E = [0, T]$, we cannot rely on Proposition 3.3.5 for the sequence $(\ell_m)_{m\in\mathbb{N}}$. Instead, we directly define

$$\ell := 0, \quad \ell_{m+1} := q^m T \quad \text{for} \quad m \in \mathbb{N}_0 \quad \text{and} \quad q := \left(\frac{3}{4}\right)^{\frac{\gamma_2-\gamma_1}{\gamma_1\gamma_3}}$$

such that $\ell_1 = T$ and an improved version of relation (3.51)

$$\delta_{m+1} = q\,\delta_m \quad \text{and} \quad |(\ell_{m+1}, \ell_m) \cap E| = \delta_m$$

is still valid. Setting $\xi = \ell_{m+1} + \delta_m/2$, one can reproduce the proof of Theorem 3.3.9 *mutatis mutandis* and derive the following estimate on the constant C_{obs} in Theorem 3.3.9 which does not depend on a density point ℓ or the sequence (ℓ_m).

$$C_{\mathrm{obs}} \leq \frac{C_1}{T^{1/r}} \exp\left(\frac{C_2}{T^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}} + C_3 T\right),$$

where we employ the definition $T^{1/\infty} := 1$ and the constants

$$C_1 := \frac{2M^3(\widetilde{c}_1)^2}{q(1-q)}, \quad C_2 := 3\,\widetilde{c}_2\left(\frac{2}{1-q}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}, \quad C_3 := 3\,\omega_+$$

with

$$\widetilde{c}_1 := \max\left\{d_0, (1 + d_0\|C(\cdot)\|_\infty)d_2\right\} \quad \text{and} \quad \widetilde{c}_2 := \left(\frac{2d_1\gamma_1}{d_3\gamma_2}\right)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} d_1\left(1 - \frac{\gamma_1}{\gamma_2}\right).$$

Thus, Theorem 3.3.9 yields a non-autonomous version of the results in [61, Thm. 2.1] and [19, Thm. A.1].

3.3.3 Interpolation Estimate

Step 3 of the proof of Theorem 3.3.9 is also the main ingredient to proving the following interpolation estimate that is inspired by [156, Thm. 1.2] which in turn attributes the main idea of the proof to [6, Thm. 6]. In the cited sources, a similar interpolation estimate is proven in the case of Hilbert spaces and one-parameter semigroups. Here, we present a version that is tailored to its use in the setting of abstract Banach spaces and evolution families, keeping track of the dependency of the constants on the parameters of Hypothesis 3.3.7. This interpolation estimate can be interpreted as a multiplicative version of the ε-based additive balancing of F and G in estimate (3.43).

Proposition 3.3.11. *Assume Hypothesis 3.3.7. Then there exist $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3 \geq 0$ such that, for all $x_0 \in X$ and $0 \leq s < t \leq T$, we have that*

$$\|U(t,0)x_0\|_X \leq \widetilde{C}_1 \exp\left(\frac{\widetilde{C}_2}{(t-s)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}} + \widetilde{C}_3(t-s)\right)\|C(t)U(t,0)x_0\|_Y^{\frac{1}{2}}\,\|U(s,0)x_0\|_X^{\frac{1}{2}}.$$

with constants $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3 \geq 0$ depending only on the parameters of Hypothesis 3.3.7.

Remark 3.3.12. (a) Proposition 3.3.11 allows for more freedom in the interpolation. In fact, as shown in [18, Thm. 3.5], one may introduce an interpolation parameter $\theta \in (0,1)$ to balance between both factors $\|C(t)U(t,0)x_0\|_Y$ and $\|U(s,0)x_0\|_X$. However, this comes with the consequence of an additional θ-dependency in the constants $\widetilde{C}_1$ and $\widetilde{C}_2$.

(b) For autonomous systems on Hilbert spaces with self-adjoint generator A with purely discrete spectrum, the interpolation estimate in Proposition 3.3.11 is equivalent to the uncertainty principle (essUCP) with *spectral projectors* P_λ, see [122, Thm. 2.1]. Note that in this case, the dissipation estimate (DE) is a trivial fact for orthogonal projections.

We start by recalling a standard interpolation argument formulated in the following lemma. See also [128, p. 110] or [92, Lem. 5.2] for similar statements.

Lemma 3.3.13. *Let $F_1, F_2, G \geq 0$ and $\theta \in (0,1)$. If there exist constants $C, D \geq 0$ such that*

$$F_2 \leq DF_1 \tag{3.55}$$

and

$$F_2 \le C\left(\varepsilon^{-\frac{\theta}{1-\theta}} G + \varepsilon F_1\right) \quad \text{for all } \varepsilon \in (0,1]\,, \tag{3.56}$$

then

$$F_2 \le \max\left\{\frac{C}{\theta^\theta(1-\theta)^{1-\theta}},\, D\Big(\frac{\theta}{1-\theta}\Big)^{1-\theta}\right\} F_1^\theta G^{1-\theta}.$$

In particular, if there exist constants $C, D \ge 0$ such that

$$F_2 \le C\left(\varepsilon^{-1} G + \varepsilon F_1\right) \quad \text{for all } \varepsilon \in (0,1] \tag{3.57}$$

and (3.55) *hold, then*

$$F_2 \le \max\{2C,\, D\}\, F_1^{1/2} G^{1/2}.$$

Proof. If $F_1 = 0$ or $G = 0$, the statement is obvious. Indeed, if $F_1 = 0$, inequality (3.55) yields $F_2 = 0$. If $G = 0$, inequality (3.56) yields $F_2 \le C\varepsilon F_1$ for all $\varepsilon \in (0,1]$. Letting $\varepsilon \to 0$ gives the claim also in this case.

For the rest of the proof, let us assume $F_1, G > 0$. Interpreting the right-hand side of (3.56) as a function of $\varepsilon \in \mathbb{R}_+$ and optimizing with respect to ε, we see that this function is minimal for

$$\varepsilon_0 := \left(\frac{\theta G}{(1-\theta)F_1}\right)^{1-\theta} > 0\,.$$

If $\varepsilon_0 > 1$, by inequality in (3.55) and the definition of ε_0, we observe

$$F_2 \le D F_1^\theta F_1^{1-\theta} = D F_1^\theta \left(\frac{\theta G}{(1-\theta)\varepsilon_0^{1/(1-\theta)}}\right)^{1-\theta} < D\left(\frac{\theta}{1-\theta}\right)^{1-\theta} F_1^\theta G^{1-\theta}.$$

If $\varepsilon_0 \le 1$, inequality (3.56) holds and gives

$$F_2 \le C\left(\varepsilon_0^{-\frac{\theta}{1-\theta}} G + \varepsilon_0 F_1\right) = \frac{C}{\theta^\theta(1-\theta)^{1-\theta}} F_1^\theta G^{1-\theta}. \qquad \square$$

Proof of Proposition 3.3.11. Let $\lambda > 0$, $x_0 \in X$, and $0 \le s < t \le T$. Recall the notation introduced in (3.40) and (3.41). We want to verify the assumptions of Lemma 3.3.13 for

$$F_1 = F(t), \quad F_2 = F(s), \quad \text{and} \quad G = G(t).$$

From Step 3 of the proof of Theorem 3.3.9, recall the additive ε-balancing (3.43)

$$F(t) \le \widetilde{c}_1 \exp\left(\widetilde{c}_2\left(\frac{1}{t-s}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}\right)\left(\varepsilon^{-1}G(t) + \varepsilon F(s)\right) \quad \text{for all } \varepsilon \in (0,1], \tag{3.58}$$

with

$$\widetilde{c}_1 = \max\left\{d_0, (1 + d_0\|C(\cdot)\|_\infty)d_2\right\} \quad \text{and} \quad \widetilde{c}_2 = \left(\frac{2d_1\gamma_1}{d_3\gamma_2}\right)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} d_1\left(1 - \frac{\gamma_1}{\gamma_2}\right).$$

This establishes assumption (3.57) from Lemma 3.3.13 with

$$C = \widetilde{c}_1 \exp\left(\widetilde{c}_2 \left(\frac{1}{t-s}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}\right).$$

Let $\varepsilon \in (0,1)$. Note that assumption (3.55) is valid for $D = M\mathrm{e}^{\omega_+(t-s)}$ as a consequence of the additivity and exponential boundedness of $(U(t,s))$

$$F(t) = \|U(t,s)U(s,0)x_0\|_X \leq M\mathrm{e}^{\omega_+(t-s)} F(s) \tag{3.59}$$

where $\omega_+ := \max\{\omega, 0\}$.

Consequently, Lemma 3.3.13 is applicable and yields the estimate

$$F(t) \leq \max\{2C, D\}\, G(t)^{\frac{1}{2}} F(s)^{\frac{1}{2}}. \tag{3.60}$$

Note that

$$\max\{2C, D\} \leq 2M\widetilde{c}_1 \exp\left(\widetilde{c}_2 \left(\frac{1}{t-s}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}} + \omega_+(t-s)\right).$$

Setting $\widetilde{C}_1 := 2M\widetilde{c}_1$, $\widetilde{C}_2 := \widetilde{c}_2$, and $\widetilde{C}_3 := \omega_+$, the statement of the proposition follows. $\square$

3.4 Observability for Non-autonomous Elliptic Operators

In this section, we show an observability estimate for the evolution family $(U_p(s,t))_{0\leq s\leq t\leq T}$ from Subsection 3.2.2.

3.4.1 Geometric Conditions and Uncertainty Estimates

Loosely speaking, a thick subset is a set such that the portion of it in a hypercube is bounded away from zero no matter where the hypercube is located. In the following, given a measurable set $\Omega \subseteq \mathbb{R}^d$, let $|\Omega|$ denote its Lebesgue measure.

Definition 3.4.1 (Thick Set)**.** Let $L \in (0,\infty)^d$ and $\varrho > 0$.

(a) A set $\Omega \subseteq \mathbb{R}^d$ is called (L,ϱ)-*thick* if Ω is measurable and, for all $x \in \mathbb{R}^d$, we have

$$\left|\Omega \cap \left(\bigtimes_{i=1}^{d} (0, L_i) + x\right)\right| \geq \varrho \prod_{i=1}^{d} L_i.$$

(b) Let $T > 0$. A family $(\Omega(t))_{t\in[0,T]}$ of sets $\Omega(t) \subseteq \mathbb{R}^d$ is called *mean* (L,ϱ)-*thick on* $[0,T]$ if $\Omega(t)$ is measurable for all $t \in [0,T]$, the mapping $[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto \mathbf{1}_{\Omega(t)}(x)$ is measurable, and, for all $x \in \mathbb{R}^d$, we have

$$\frac{1}{T}\int_0^T \left|\Omega(t) \cap \left(\bigtimes_{i=1}^{d} (0, L_i) + x\right)\right| \mathrm{d}t \geq \varrho \prod_{i=1}^{d} L_i.$$

(c) Let $T > 0$. A family $(\Omega(t))_{t\in[0,T]}$ of sets $\Omega(t) \subseteq \mathbb{R}^d$ is called *uniformly (L, ϱ)-thick on* $[0,T]$ if $\Omega(t)$ is (L, ϱ)-thick for all $t \in [0,T]$ and the mapping $[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto \mathbf{1}_{\Omega(t)}(x)$ is measurable.

We call $\Omega \subseteq \mathbb{R}^d$ *thick* if there exist $L \in (0,\infty)^d$ and $\varrho > 0$ such that Ω is (L, ϱ)-thick. Likewise, $(\Omega(t))_{t\in[0,T]}$ is called *mean/uniformly thick* if it is mean/uniformly (L, ϱ)-thick on $[0,T]$ for some $L \in (0,\infty)^d$ and $\varrho > 0$.

Note that equivalent notions of (mean/uniform) thickness are obtained by replacing the hypercubes $\bigtimes_{i=1}^d (0, L_i)$ with balls $\mathrm{B}(0,R)$ with some radius $R > 0$.

Example 3.4.2. (a) Let $\Omega = \mathbb{R}$. Then Ω is (L, ϱ) thick for all $L, \varrho > 0$.

(b) Let $\Omega = \bigcup_{k\in\mathbb{Z}} [2k, 2k+1]$. Then Ω is (L, ϱ) thick for all $L > 1$ and $\varrho \leq \frac{L-1}{L}$.

(c) Let $\Omega = [0,\infty)$. Then Ω is not thick.

(d) Let $\Omega_1 := [0,\infty)$, $\Omega_2 := (-\infty, 0]$, and

$$\Omega(t) := \begin{cases} \Omega_1 & \text{if } t \in [0,1)\,, \\ \Omega_2 & \text{if } t \in [1,2]\,. \end{cases}$$

Then $(\Omega(t))_{t\in[0,2]}$ is mean $(L, 1/2)$-thick for all $L > 0$ but not uniformly thick.

3.4.2 Sufficient and Necessary Conditions for Observability

Lemma 3.4.3. *Let $\mathfrak{a}$ be a uniformly strongly elliptic polynomial of degree $m \geq 2$ with coefficients $a_\alpha \in \mathrm{L}^\infty(0,T)$ for $|\alpha| \leq m$ and $(U(t,s))_{0\leq s\leq t\leq T}$ be defined as in* (3.8)*. For each $t \in [0,T]$, let $\Omega(t) \subseteq \mathbb{R}^d$ be measurable, and assume that $[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto \mathbf{1}_{\Omega(t)}(x)$ is measurable. Let $p \in [1,\infty]$ and $u_0 \in \mathrm{L}^p(\mathbb{R}^d)$. Then $[0,T] \ni t \mapsto \|\mathbf{1}_{\Omega(t)} U_p(t,0)u_0\|_{\mathrm{L}^p(\mathbb{R}^d)}$ is measurable.*

Proof. By Corollary 3.2.15, $(U_p(t,s))_{0\leq s\leq t\leq T}$ is strongly continuous for $p \in [1,\infty)$ and strongly continuous w.r.t. the weak*-topology for $p = \infty$. For $p \in [1,\infty)$, this implies that $[0,T] \ni t \mapsto \|\mathbf{1}_{\Omega(t)} U_p(t,0)u_0\|_{\mathrm{L}^p(\mathbb{R}^d)}$ is measurable. For $p = \infty$, the measurability follows from the variational description of the L^∞-norm via the canonical pairing with L^1-elements and the strong continuity of $(U_\infty(t,s))_{0\leq s\leq t\leq T}$ w.r.t. the topology $\sigma(\mathrm{L}^\infty(\mathbb{R}^d), \mathrm{L}^1(\mathbb{R}^d))$. Indeed, one can show that the mapping

$$[0,T] \ni t \mapsto \|1_{\Omega(t)} U_\infty(t,0)u_0\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$$

is lower-semicontinuous. $\square$

Our first result shows that uniform thickness implies an observability estimate.

Theorem 3.4.4. *Let $\mathfrak{a}$ be a uniformly strongly elliptic polynomial of degree $m \geq 2$ with coefficients $a_\alpha \in \mathrm{L}^\infty(0,T)$ for $|\alpha| \leq m$. Let $(U(t,s))_{0\leq s\leq t\leq T}$ as in* (3.8)*. Let $(\Omega(t))_{t\in[0,T]}$ be uniformly thick on $[0,T]$. Let $E \subseteq [0,T]$ be measurable with positive Lebesgue measure and $r \in [1,\infty]$. Then there exists $C_{\mathrm{obs}} \geq 0$ such that, for all $p \in [1,\infty]$ and $u_0 \in \mathrm{L}^p(\mathbb{R}^d)$, we have*

$$\|U_p(T,0)u_0\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq C_{\mathrm{obs}} \begin{cases} \left(\int_E \|(U_p(t,0)u_0)|_{\Omega(t)}\|^r_{\mathrm{L}^p(\Omega(t))} \,\mathrm{d}t \right)^{1/r}, & r \in [1,\infty), \\ \operatorname{ess\,sup}_{t\in E} \|(U_p(t,0)u_0)|_{\Omega(t)}\|_{\mathrm{L}^p(\Omega(t))}, & r = \infty. \end{cases}$$

Remark 3.4.5. In the situation of Theorem 3.4.4, if $E = [0, T]$, we obtain

$$C_{\mathrm{obs}} \leq \frac{C_1}{T^{1/r}} \exp\Big(\frac{C_2}{T^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}} + C_3 T\Big)$$

for some $C_1, C_2, C_3 \geq 0$, $\gamma_1 = \gamma_3 = 1$, and $\gamma_2 = m$; cf. Remark 3.3.10.

Before we come to the proof of Theorem 3.4.4, let us note the following results that will be needed to derive the uncertainty principle and the dissipativity estimate for the application of Theorem 3.3.9.

We start by introducing a family of *smooth frequency cutoffs*. Let $\eta \in \mathrm{C}_{\mathrm{c}}^{\infty}([0, \infty))$ with $0 \leq \eta \leq 1$ such that $\eta(r) = 1$ for $r \in [0, 1/2]$ and $\eta(r) = 0$ for $r \geq 1$. For $\lambda > 0$, we define

$$\chi_\lambda \colon \mathbb{R}^d \to \mathbb{R}, \quad \chi_\lambda(\xi) \coloneqq \eta\big(\frac{|\xi|}{\lambda}\big).$$

Since $\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$ for all $\lambda > 0$, we have $\mathcal{F}^{-1}\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$. For $\lambda > 0$, we define

$$P_\lambda \colon \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d), \quad P_\lambda f \coloneqq (\mathcal{F}^{-1}\chi_\lambda) * f.$$

Then, for all $\lambda > 0$, the operator P_λ is a bounded linear operator, the family $(P_\lambda)_{\lambda>0}$ is uniformly bounded by $\|\mathcal{F}^{-1}\chi_1\|_{\mathrm{L}^1(\mathbb{R}^d)}$. Furthermore, for all $f \in \mathcal{S}(\mathbb{R}^d)$, we have $P_\lambda f \in \mathcal{S}(\mathbb{R}^d)$, $\mathcal{F}P_\lambda f = \chi_\lambda \mathcal{F} f \in \mathcal{S}(\mathbb{R}^d)$, and $\operatorname{supp} \mathcal{F}P_\lambda f \subseteq \{y \in \mathbb{R}^d \colon |y| \leq \lambda\} \subseteq [-\lambda, \lambda]^d$, see [61, Thm. 3.3] for details.

The next result gives a dissipativity estimate for evolution families associated with the principal part of a polynomial.

Proposition 3.4.6 ([19, Prop. 3.2]). *Let $m \geq 2$ and*

$$V(t, s)u \coloneqq \mathcal{F}^{-1}\mathrm{e}^{-(t-s)|\cdot|^m}\mathcal{F}u, \quad u \in \mathcal{S}'(\mathbb{R}^d), 0 \leq s \leq t \leq T.$$

Then there exists $K_{m,d} \geq 0$ such that, for all $p \in [1, \infty]$, $\lambda > 0$, and $u \in \mathrm{L}^p(\mathbb{R}^d)$,

$$\|(\mathrm{Id} - P_\lambda)V(t, s)u\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq K_{m,d}\, \mathrm{e}^{-2^{-m-3}(t-s)\lambda^m} \|u\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

For the uncertainty principle, we will use the seminal Logvinenko–Sereda theorem in its quantitative version due to Kovrijkine, cf. [85, Thm. 3] and [106].

Theorem 3.4.7 (Logvinenko, Sereda, Kovrijkine). *There exists $K \geq 1$ such that, for all $p \in [1, \infty]$, $\lambda > 0$, $\varrho \in (0, 1]$, $L \in (0, \infty)^d$, (L, ϱ)-thick sets $\Omega \subseteq \mathbb{R}^d$, and $u \in \mathrm{L}^p(\mathbb{R}^d)$ subject to* $\operatorname{supp} \mathcal{F}u \subseteq [-\lambda, \lambda]^d$, *we have*

$$\|u\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq d_0\, \mathrm{e}^{d_1 \lambda} \|u\|_{\mathrm{L}^p(\Omega)}$$

with the constants

$$d_0 = \mathrm{e}^{Kd \ln\left(\frac{K^d}{\rho}\right)} \quad \textit{and} \quad d_1 = 2|L|_1 \ln\big(\frac{K^d}{\rho}\big).$$

We are now in the position to prove Theorem 3.4.4.

Proof of Theorem 3.4.4. This proof consists of two parts. In the first part, we will show a dissipativity estimate, and, in the second part, we will derive an abstract uncertainty estimate. As both estimates do not depend on the value of p, it follows from Theorem 3.3.9 that also the observability constant C_{obs} can be chosen independently of p.

Since $\mathfrak{a}$ is uniformly strongly elliptic, there exists $c > 0$ such that, for all $t \in [0,T]$ and all $\xi \in \mathbb{R}^d$, we have $\operatorname{Re} \mathfrak{a}_m(t,\xi) \geq c\,|\xi|^m$. We define the uniformly strongly elliptic polynomials $\mathfrak{b}, \widetilde{\mathfrak{b}} \colon [0,T] \times \mathbb{R}^d \to \mathbb{C}$ by

$$\mathfrak{b}(t,\xi) := |\xi|^m \quad \text{and} \quad \widetilde{\mathfrak{b}}(t,\xi) := \frac{c}{2}\,|\xi|^m$$

and set $\widetilde{\mathfrak{a}} := \mathfrak{a} - \widetilde{\mathfrak{b}}$. Note that $\widetilde{\mathfrak{a}}$ is also uniformly strongly elliptic with ellipticity constant $\frac{c}{2}$.

Let $(V(t,s))_{0\leq s\leq t\leq T}$, $(\widetilde{V}(t,s))_{0\leq s\leq t\leq T}$, and $(\widetilde{U}(t,s))_{0\leq s\leq t\leq T}$ be the evolution families as in (3.8) for the polynomials $\mathfrak{b}$, $\widetilde{\mathfrak{b}}$, and $\widetilde{\mathfrak{a}}$, respectively. Note that $\widetilde{V}(t,s) = V(\frac{c}{2}\,t, \frac{c}{2}\,s)$ for all $0 \leq s \leq t \leq T$. Indeed, we have, according to the definition (3.8),

$$V\Big(\frac{c}{2}\,t, \frac{c}{2}\,s\Big) = \mathcal{F}^{-1}\mathrm{e}^{-(\frac{c}{2}t - \frac{c}{2}s)|\xi|^m}\mathcal{F} = \mathcal{F}^{-1}\mathrm{e}^{-(t-s)\frac{c}{2}|\xi|^m}\mathcal{F} = \widetilde{V}(t,s)\,.$$

Let $p \in [1,\infty]$. For $f \in \mathrm{L}^p(\mathbb{R}^d)$ and $0 \leq s \leq t \leq T$, we have by definition

$$\begin{aligned} U_p(t,s)f &= \mathcal{F}^{-1}\Big(\mathrm{e}^{-\int_s^t (\widetilde{\mathfrak{b}}(\tau,\cdot)+\widetilde{\mathfrak{a}}(\tau,\cdot))\,\mathrm{d}\tau}\mathcal{F}f\Big) \\ &= \mathcal{F}^{-1}\Big(\mathrm{e}^{-\int_s^t \widetilde{\mathfrak{b}}(\tau,\cdot)\,\mathrm{d}\tau}\mathcal{F}\mathcal{F}^{-1}\Big(\mathrm{e}^{-\int_s^t \widetilde{\mathfrak{a}}(\tau,\cdot)\,\mathrm{d}\tau}\mathcal{F}f\Big)\Big) = \widetilde{V}_p(t,s)\widetilde{U}_p(t,s)f. \end{aligned}$$

By Proposition 3.4.6, we infer that there exists $K_{m,d} \geq 0$, depending only on m and d, such that, for all $\lambda > 0$, all $f \in \mathrm{L}^p(\mathbb{R}^d)$, and all $0 \leq s \leq t \leq T$, we have

$$\|(\mathrm{Id} - P_\lambda)V_p(t,s)f\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq K_{m,d}\,\mathrm{e}^{-2^{-m-3}(t-s)\lambda^m}\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

Thus, we also conclude

$$\|(\mathrm{Id} - P_\lambda)\widetilde{V}_p(t,s)f\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq K_{m,d}\,\mathrm{e}^{-2^{-m-4}c(t-s)\lambda^m}\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

Moreover, by Theorem 3.2.11, there exist $\widetilde{M} \geq 1$ and $\widetilde{\omega} \in \mathbb{R}$ depending on $\widetilde{\mathfrak{a}}$ and therefore on $\mathfrak{a}$ such that $\|\widetilde{U}_p(t,s)\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))} \leq \widetilde{M}\mathrm{e}^{\widetilde{\omega}(t-s)}$ for all $0 \leq s \leq t \leq T$. Note that we can choose $\widetilde{\omega} = \omega$, where ω is an exponential growth rate for $(U_p(t,s))_{0\leq s\leq t\leq T}$. Indeed, this follows by choosing the same $c_0 = \frac{c}{4}$ in Lemma 3.2.12(a) and observing that the constant c_3 in the proof of Lemma 3.2.12(b) only depends on a uniform upper bound on the coefficients' modulus. Thus, for $\lambda > \lambda^* := (2^{m+4}\max\{\omega,0\}/c)^{1/m}$, $f \in \mathrm{L}^p(\mathbb{R}^d)$, and $0 \leq s \leq t \leq T$, we arrive at

$$\begin{aligned} \|(\mathrm{Id} - P_\lambda)U_p(t,s)f\|_{\mathrm{L}^p(\mathbb{R}^d)} &= \|(\mathrm{Id} - P_\lambda)\widetilde{V}_p(t,s)\,\widetilde{U}_p(t,s)f\|_{\mathrm{L}^p(\mathbb{R}^d)} \\ &\leq K_{m,d}\,\mathrm{e}^{-2^{-m-4}\,c\,(t-s)\lambda^m}\,\widetilde{M}\,\mathrm{e}^{\omega(t-s)}\|f\|_{\mathrm{L}^p(\mathbb{R}^d)} \\ &\leq K_{m,d}\,\widetilde{M}\,\mathrm{e}^{-(t-s)\,2^{-m-3}\,c\,\lambda^m}\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}. \end{aligned} \tag{3.61}$$

Let $L \in (0,\infty)^d$ and $\varrho > 0$ such that $(\Omega(t))_{t\in[0,T]}$ is uniformly (L,ϱ)-thick, $f \in \mathrm{L}^p(\mathbb{R}^d)$, and $\lambda > 0$. Since $\operatorname{supp}\mathcal{F}P_\lambda f \subseteq [-\lambda,\lambda]^d$, the Logvinenko–Sereda theorem, Theorem 3.4.7, implies

$$\|P_\lambda f\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \mathrm{e}^{-Kd\ln(\varrho/K^d)}\,\mathrm{e}^{-2K|L|_1\ln(\varrho/K^d)\lambda}\,\|(P_\lambda f)|_{\Omega(t)}\|_{\mathrm{L}^p(\Omega(t))} \tag{3.62}$$

for all $t \in [0,T]$, where $K \geq 0$ is a universal constant.

By (3.61), (3.62), Theorem 3.2.11(a), and Lemma 3.4.3, we conclude that Hypothesis 3.3.7 is satisfied with $Y = \mathrm{L}^p(\mathbb{R}^d)$ and $C(t)$ the restriction operator on $\Omega(t)$ for $t \in [0,T]$. Therefore, Theorem 3.3.9 yields the assertion. □

The following theorem will show a partial converse of Theorem 3.4.4, namely that a final-state observability estimate implies that the family $(\Omega(t))_{t\in[0,T]}$ is mean thick. For the pure Laplacian on $\mathrm{L}^2(\mathbb{R}^d)$ and time-independent set of observability, such a result has first been shown in [41, 155]. In the autonomous case, this has been generalized to strongly elliptic operators in $\mathrm{L}^p(\mathbb{R}^d)$ in [61, Thm. 3.3]. We also refer to [12, Thm. 5], where a similar result is shown for the non-autonomous Ornstein–Uhlenbeck equation.

Theorem 3.4.8. *Let $m \geq 2$ and $\mathfrak{a}$ be a uniformly strongly elliptic polynomial of degree m with coefficients $a_\alpha \in \mathrm{L}^\infty(0,T)$ for $|\alpha| \leq m$. Let $(\Omega(t))_{t\in[0,T]}$ such that $\Omega(t) \subseteq \mathbb{R}^d$ is measurable for all $t \in [0,T]$, and $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \mathbf{1}_{\Omega(t)}(x)$ is measurable. Let $(U(t,s))_{0\leq s\leq t\leq T}$ be as in (3.8). Let $p,r \in [1,\infty)$, and assume there exists $C_{\mathrm{obs}} \geq 0$ such that, for all $u_0 \in \mathrm{L}^p(\mathbb{R}^d)$, we have*

$$\|U_p(T,0)u_0\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq C_{\mathrm{obs}}\Big(\int_0^T \|U_p(t,0)u_0|_{\Omega(t)}\|^r_{\mathrm{L}^p(\Omega(t))}\,\mathrm{d}t\Big)^{1/r}.$$

Then the family $(\Omega(t))_{t\in[0,T]}$ is mean thick.

Proof. Our proof is inspired by the ideas in [12, 41, 155]. Let us show the contrapositive: assume that the family $(\Omega(t))_{t\in[0,T]}$ is not mean thick. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ such that, for all $n \in \mathbb{N}$, we have

$$\frac{1}{T}\int_0^T |\Omega(t)\cap \mathrm{B}(x_n,n)|^{r/p}\,\mathrm{d}t < \frac{1}{n}. \tag{3.63}$$

Let $f \in \mathcal{S}(\mathbb{R}^d)$, $\|f\|_{\mathrm{L}^p(\mathbb{R}^d)} = 1$, and set $f_n := f(\cdot - x_n)$ for $n \in \mathbb{N}$. Let $t \in (0,T)$ and $n \in \mathbb{N}$. Then $U_p(t,0)f_n = p_{t,0} * f_n = p_{t,0} * f(\cdot - x_n)$. Moreover,

$$\begin{aligned}
&\|(U_p(t,0)f_n)|_{\Omega(t)}\|^p_{\mathrm{L}^p(\Omega(t))} = \|\mathbf{1}_{\Omega(t)}\,U_p(t,0)f_n\|^p_{\mathrm{L}^p(\mathbb{R}^d)}\\
&= \|\mathbf{1}_{\Omega(t)}\,p_{t,0} * f(\cdot - x_n)\|^p_{\mathrm{L}^p(\mathbb{R}^d)} = \|\mathbf{1}_{(\Omega(t)-x_n)}\,p_{t,0} * f\|^p_{\mathrm{L}^p(\mathbb{R}^d)}\\
&= \|\mathbf{1}_{(\Omega(t)-x_n)\cap\mathrm{B}(0,n)}\,p_{t,0} * f\|^p_{\mathrm{L}^p(\mathbb{R}^d)} + \|\mathbf{1}_{(\Omega(t)-x_n)}(1-\mathbf{1}_{\mathrm{B}(0,n)})\,p_{t,0} * f\|^p_{\mathrm{L}^p(\mathbb{R}^d)}.
\end{aligned} \tag{3.64}$$

We first estimate the first summand on the right-hand side of (3.64). As a consequence of Lemma 3.2.12(b), there exists $C \geq 0$ such that $\|p_{t,0}\|^p_{\mathrm{L}^{p'}(\mathbb{R}^d)} \leq C$ for all $t \in (0,T)$, where $\frac{1}{p'} + \frac{1}{p} = 1$. By Hölder's and Young's inequality, we estimate

$$\begin{aligned}
\|\mathbf{1}_{(\Omega(t)-x_n)\cap\mathrm{B}(0,n)}\,p_{t,0} * f\|^p_{\mathrm{L}^p(\mathbb{R}^d)} &\leq |(\Omega(t)-x_n)\cap\mathrm{B}(0,n)|\,\|p_{t,0}\|^p_{\mathrm{L}^{p'}(\mathbb{R}^d)}\\
&\leq C\,|\Omega(t)\cap\mathrm{B}(x_n,n)|.
\end{aligned}$$

For the second summand on the right-hand side of (3.64), we have by Lemma 3.2.12(b), Hölder's inequality, and the Fubini–Tonelli theorem

$$\begin{aligned}
&\|\mathbf{1}_{(\Omega(t)-x_n)}(1-\mathbf{1}_{\mathrm{B}(0,n)})\,p_{t,0}*f\|^p_{\mathrm{L}^p(\mathbb{R}^d)} \leq \|(1-\mathbf{1}_{\mathrm{B}(0,n)})\,p_{t,0}*f\|^p_{\mathrm{L}^p(\mathbb{R}^d)}\\
&\leq \int_{\complement\mathrm{B}(0,n)}\Bigl(\int_{\mathbb{R}^d} C_1\,\mathrm{e}^{\omega t}\,\mathrm{e}^{-C_2|z|^{m/(m-1)}}\left|f(x-t^{1/m}z)\right|\,\mathrm{d}z\Bigr)^p\,\mathrm{d}x\\
&= C_1^p\mathrm{e}^{p\omega+T}\int_{\complement\mathrm{B}(0,n)}\Bigl(\int_{\mathbb{R}^d}\mathrm{e}^{-C_2|z|^{m/(m-1)}}\,\mathrm{d}z\Bigr)^{p/p'}\int_{\mathbb{R}^d}\mathrm{e}^{-C_2|z|^{m/(m-1)}}\left|f(x-t^{1/m}z)\right|^p\,\mathrm{d}z\,\mathrm{d}x\\
&\leq C_1^p\mathrm{e}^{p\omega+T}\Bigl(\int_{\mathbb{R}^d}\mathrm{e}^{-C_2|z|^{m/(m-1)}}\,\mathrm{d}z\Bigr)^{p/p'}\int_{\mathbb{R}^d}\int_{\complement\mathrm{B}(0,n)}\mathrm{e}^{-C_2|z|^{m/(m-1)}}\left|f(x-t^{1/m}z)\right|^p\,\mathrm{d}x\,\mathrm{d}z.
\end{aligned}$$

Let us focus on estimating the double integral over $\mathbb{R}^d\times\complement\mathrm{B}(0,n)$ in the previous calculation by splitting it up. To this end, let $\varepsilon>0$. Then there exist $n_0\in\mathbb{N}$ and $R>0$ such that

$$\int_{\complement\mathrm{B}(0,n_0)}|f(y)|^p\,\mathrm{d}y\leq\varepsilon\quad\text{and}\quad\int_{\complement\mathrm{B}(0,R)}\mathrm{e}^{-C_2|z|^{m/(m-1)}}\,\mathrm{d}z\leq\varepsilon\,.$$

Consequently, for $n\geq n_0+T^{1/m}R$, we have $\complement\mathrm{B}(0,n)-t^{1/m}\mathrm{B}(0,R)\subseteq\complement\mathrm{B}(0,n_0)$ and

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\complement\mathrm{B}(0,n)}\mathrm{e}^{-C_2|z|^{m/(m-1)}}\left|f(x-t^{1/m}z)\right|^p\,\mathrm{d}x\,\mathrm{d}z\\
&\qquad=\int_{\mathrm{B}(0,R)}\int_{\complement\mathrm{B}(0,n)}\mathrm{e}^{-C_2|z|^{m/(m-1)}}\left|f(x-t^{1/m}z)\right|^p\,\mathrm{d}x\,\mathrm{d}z\\
&\qquad\quad+\int_{\complement\mathrm{B}(0,R)}\int_{\complement\mathrm{B}(0,n)}\mathrm{e}^{-C_2|z|^{m/(m-1)}}\left|f(x-t^{1/m}z)\right|^p\,\mathrm{d}x\,\mathrm{d}z\\
&\qquad\leq\varepsilon\int_{\mathrm{B}(0,R)}\mathrm{e}^{-C_2|z|^{m/(m-1)}}\,\mathrm{d}z+\varepsilon\|f\|^p_{\mathrm{L}^p(\mathbb{R}^d)}.
\end{aligned}$$

Thus,

$$\sup_{t\in[0,T]}\|\mathbf{1}_{(\Omega(t)-x_n)}\,(1-\mathbf{1}_{\mathrm{B}(0,n)})p_{t,0}*f\|^p_{\mathrm{L}^p(\mathbb{R}^d)}\to 0 \tag{3.65}$$

as n tends to ∞. By (3.63), (3.64), and (3.65) we obtain

$$\int_0^T\|(U_p(t,0)f_n)|_{\Omega(t)}\|^r_{\mathrm{L}^p(\Omega(t))}\,\mathrm{d}t\to 0$$

as n tends to ∞. Since, $\|U_p(T,0)f_n\|_{\mathrm{L}^p(\mathbb{R}^d)}=\|p_{T,0}*f\|_{\mathrm{L}^p(\mathbb{R}^d)}>0$ for all $n\in\mathbb{N}$, an observability estimate does not hold. □

Remark 3.4.9. (a) Combining Theorem 3.4.4 and Theorem 3.4.8, we observe that uniformly thick observability sets allow for a final-state observability estimate, while such an estimate only implies that the observation sets are mean thick. It is an interesting question whether it is possible to close this gap, either by finding a suitable condition on the observation sets which is equivalent to a final-state observability estimate, or by proving that an observability estimate holds for mean thick sets. Even in the setting of Hilbert spaces and for autonomous problems, i.e., $p = r = 2$ and $A(\cdot)$ time-independent, an answer to this question is still open. However, for a certain class of non-autonomous diffusive evolution equations governed by the Ornstein–Uhlenbeck operator, it has recently been proven in [3] that the corresponding equation is cost-uniform approximately null-controllable, if and only if the family $(\Omega(t))_{t\in[0,T]}$ is mean thick.

(b) If the family of sets $(\Omega(t))_{t\in[0,T]}$ does not depend on t, then uniform thickness is equivalent to mean thickness. In this case, one can prove the statement of Theorem 3.4.8 for $r = \infty$ as well.

Chapter 4

Strong Solutions to the Navier–Stokes Equations in Planar Lipschitz Domains

This chapter is based on the joint work with Patrick Tolksdorf [56].

4.1 Introduction

Consider the *Stokes resolvent problem*

$$\begin{cases} \lambda u - \Delta u + \nabla \phi = f & \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$. Here, u denotes the velocity field and ϕ denotes the pressure. The resolvent parameter λ is supposed to lie in a *sector*

$$\Sigma_\theta \coloneqq \big\{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \pi - \theta \big\}, \quad \theta \in (0, \pi)\,,$$

in the complex plane, see Figure 4.1. Note there are different ways to define sectors depending on the meaning of the *opening angle* θ. In this chapter, we will follow the convention used by Shen in [134].

In the discipline of mathematical fluid mechanics, the Stokes resolvent problem has been the subject of a plethora of studies. Let us mention a few results to catch a glimpse of the rich theory of the Stokes operator. For results on smooth domains, one may consult the classical work by Giga [65] for the case of bounded domains. The Stokes operator on infinite layers $\Omega = \mathbb{R}^{d-1} \times (-1, 1)$ was discussed by Abels in [1]. Non-smooth domains and a possibly non-compact boundary are treated in [5], [17], [46], and [63]. For results on the regularity theory of the Stokes operator on Lipschitz domains, one may consult the seminal works [134] and [88], and, for an application to the Navier–Stokes equations on Lipschitz domains, [147]. The Stokes operator on exterior domains was discussed in, e.g., [20] and [148].

A natural tool in the study of the Stokes problem is the *Helmholtz decomposition* of the function spaces one aims to analyze. More precisely, this decomposition allows one to

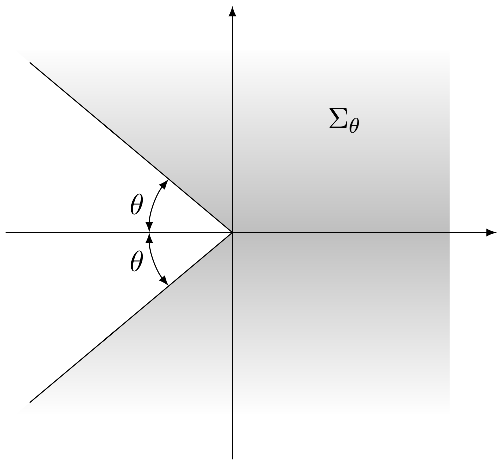

Figure 4.1: Sketch of the sector $\Sigma_\theta \coloneqq \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \pi - \theta\}, \theta \in (0, \pi)$, in the complex plane.

eliminate the pressure ϕ from the equation system and thus reduce the number of unknowns. We say that the Helmholtz decomposition of $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ exists if there is a decomposition

$$\mathrm{L}^p(\Omega; \mathbb{C}^d) = \mathrm{L}^p_\sigma(\Omega) \oplus \mathrm{G}_p(\Omega),$$

where $\mathrm{L}^p_\sigma(\Omega)$ denotes the closure in $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ of

$$\mathrm{C}^\infty_{\mathrm{c},\sigma}(\Omega) \coloneqq \{\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega; \mathbb{C}^d) : \operatorname{div}(\varphi) = 0\}$$

and where

$$\mathrm{G}_p(\Omega) \coloneqq \{g \in \mathrm{L}^p(\Omega; \mathbb{C}^d) : g = \nabla\Phi \text{ for some } \Phi \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)\}.$$

In other words, if the Helmholtz decomposition of $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ exists, then the closed subspace $\mathrm{L}^p_\sigma(\Omega)$ is a complemented subspace and the spaces $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ and $\mathrm{L}^p_\sigma(\Omega) \oplus \mathrm{G}_p(\Omega)$ are isomorphic Banach spaces. The bounded projection onto the space $\mathrm{L}^p_\sigma(\Omega)$ is called the *Helmholtz projection* and is denoted by $\mathbb{P}_p$.

In the case $p = 2$, the Helmholtz projection $\mathbb{P}_2$ is an *orthogonal projection*, and $\mathrm{G}_2(\Omega)$ is the orthogonal complement to $\mathrm{L}^2_\sigma(\Omega)$ with respect to the L^2-inner product. In particular, the Helmholtz decomposition of $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ exists for all open sets Ω, see, e.g., [137]. However, the situation for $p \neq 2$ is different and much more sensitive to the underlying geometry and regularity of $\partial\Omega$. Even smoothness of $\partial\Omega$ does not imply for certain unbounded domains the existence of the Helmholtz decomposition as the classical example of Maslennikova and Bogovskiĭ [110] shows. Also, irregularity of the boundary destroys the existence of the Helmholtz decomposition in certain L^p-spaces. This is shown in the works of Fabes, Mendez, and M. Mitrea [45] and of D. Mitrea [114]. Indeed, in [45] it is proved that, for each bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, there exists a constant $\varepsilon > 0$ such that the Helmholtz decomposition exists on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ whenever

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \frac{1}{6} + \varepsilon. \tag{4.2}$$

In particular, it is shown that this range of numbers p is sharp, i.e., for each $1 < p < \infty$ that does not lie in the interval $[3/2, 3]$, a bounded Lipschitz domain is constructed such that the Helmholtz decomposition of $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ fails. An analogous result was proved in [114, Thm. 1.1] for bounded planar Lipschitz domains. Here, the Helmholtz decomposition exists on $\mathrm{L}^p(\Omega; \mathbb{C}^2)$ whenever

$$\Big| \frac{1}{p} - \frac{1}{2} \Big| < \frac{1}{4} + \varepsilon. \tag{4.3}$$

While the existence of the Helmholtz decomposition may indicate a rich functional analytic theory of the Stokes operator, see, e.g., [62], it is by no means necessary as the article [17] shows.

However, based on this heuristic, the results in [45] led Taylor to conjecture in [142] that, for each three-dimensional bounded Lipschitz domain, there exists $\varepsilon > 0$ such that, for each p in the range (4.2), the Stokes operator gives rise to a bounded analytic semigroup. In 2001, one year after Taylor formulated this conjecture, Deuring [34] constructed bounded Lipschitz domains such that, for p large enough, the Stokes operator indeed fails to generate even a strongly continuous semigroup on $\mathrm{L}^p_\sigma(\Omega)$. Compare also the discussion of Deuring's result in the article of Monniaux and Shen [118, Sect. 6].

M. Mitrea and Monniaux were able to prove Taylor's conjecture for the Stokes operator but with *Neumann type boundary conditions* in 2009, see [115]. In 2012, Taylor's original conjecture was finally settled in the affirmative by Shen [134]. Later Shen's result was extended by Dikland to the case of weighted L^p-spaces [35]. Analogously to Taylor, D. Mitrea [114] conjectured that the Stokes operator gives rise to a bounded analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$ in *two-dimensional* bounded Lipschitz domains Ω if p is subject to (4.3). In the author's joint work with Patrick Tolksdorf [56], an affirmative solution of the conjecture stated in [114, Conj. 1.2] was given thereby proving the following theorem.

Theorem 4.1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain and $\theta \in (0, \pi/2)$. Then there exists $\varepsilon > 0$ such that, for all p that satisfy* (4.3)*, there exists a constant $C \geq 0$ such that, for all $f \in \mathrm{L}^p_\sigma(\Omega) \cap \mathrm{L}^2_\sigma(\Omega)$ and $\lambda \in \Sigma_\theta$, the unique weak solutions $u \in \mathrm{W}^{1,2}_0(\Omega; \mathbb{C}^2)$ and $\phi \in \mathrm{L}^2(\Omega)$ with $\int_\Omega \phi \; \mathrm{d}x = 0$ to* (4.1) *satisfy*

$$(1 + |\lambda|) \|u\|_{\mathrm{L}^p_\sigma(\Omega)} \leq C \, \|f\|_{\mathrm{L}^p_\sigma(\Omega)}. \tag{4.4}$$

Here, ε depends on θ, the Lipschitz character of Ω, and $\mathrm{diam}(\Omega)$ *and C depends on θ, p, the Lipschitz character of Ω, and* $\mathrm{diam}(\Omega)$.

This result opens the door to further functional analytic studies of the Stokes operator on planar Lipschitz domains as we will show in this chapter.

4.1.1 The Stokes Operator and the Stokes Semigroup on $\mathrm{L}^p_\sigma(\Omega)$

To put Theorem 4.1.1 into a functional analytic context, this section aims to briefly introduce the Stokes operator on the L^p-scale. Before we start, let us introduce the scale of solenoidal Sobolev functions

$$\mathrm{W}^{1,p}_{0,\sigma}(\Omega) \coloneqq \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_{\mathrm{W}^{1,p}(\Omega;\mathbb{C}^2)}}, \qquad 1 < p < \infty.$$

The Stokes Operator on $\mathrm{L}^2_\sigma(\Omega)$

On the Hilbert space $\mathrm{L}^2_\sigma(\Omega)$, define the Stokes operator as the realization of the sesquilinear form

$$\mathfrak{a}\colon \mathrm{W}^{1,2}_{0,\sigma}(\Omega) \times \mathrm{W}^{1,2}_{0,\sigma}(\Omega) \to \mathbb{C}, \quad (u,v) \mapsto \int_\Omega \nabla u \cdot \overline{\nabla v} \;\mathrm{d}x := \sum_{\alpha,\beta=1}^{2} \int_\Omega \partial_\alpha u_\beta \overline{\partial_\alpha v_\beta} \;\mathrm{d}x.$$

Then the domain of the Stokes operator A_2 on $\mathrm{L}^2_\sigma(\Omega)$ is defined as

$$\mathrm{Dom}(A_2) := \Big\{ u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega) : \, \exists f \in \mathrm{L}^2_\sigma(\Omega) : \forall v \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega) \text{ it holds } \mathfrak{a}(u,v) = \int_\Omega f \cdot \overline{v} \;\mathrm{d}x \Big\}$$

and, for $u \in \mathrm{Dom}(A_2)$ with corresponding function $f \in \mathrm{L}^2_\sigma(\Omega)$, we set

$$A_2 u := f.$$

With this definition, A_2 is densely defined, closed, and self-adjoint on $\mathrm{L}^2_\sigma(\Omega)$, see, e.g., [82]. Sometimes the Stokes operator is introduced differently focusing more on the operator participants of the Stokes problem (4.1). More precisely, we have the following equivalent definition of the Stokes operator on bounded Lipschitz domains, cf. [116, Thm. 4.7].

Proposition 4.1.2. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then the domain of the Stokes operator A_2 can equivalently be described via*

$$\mathrm{Dom}(A_2) = \Big\{ u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega) : \exists \phi \in \mathrm{L}^2(\Omega) : -\Delta u + \nabla \phi \in \mathrm{L}^2_\sigma(\Omega) \Big\},$$

where $-\Delta u + \nabla \phi \in \mathrm{L}^2_\sigma(\Omega)$ denotes a regular distribution with representative in $\mathrm{L}^2_\sigma(\Omega)$. In particular,

$$A_2 u = -\Delta u + \nabla \phi$$

for all $u \in \mathrm{Dom}(A_2)$ and $\phi \in \mathrm{L}^2(\Omega)$ such that $-\Delta u + \nabla \phi \in \mathrm{L}^2_\sigma(\Omega)$.

As a consequence of the *Lax–Milgram lemma* and *Poincaré's inequality*, we see that A_2 is invertible on $\mathrm{L}^2_\sigma(\Omega)$. Indeed, note that with the constant $c > 0$ from *Poincaré's* inequality,

$$|\mathfrak{a}(u,u)| = \|\nabla u\|^2_{\mathrm{L}^2(\Omega,\mathbb{C}^{2\times 2})} \geq \frac{1}{2}\|\nabla u\|^2_{\mathrm{L}^2(\Omega,\mathbb{C}^{2\times 2})} + \frac{1}{2\,c^2}\|u\|^2_{\mathrm{L}^2(\Omega,\mathbb{C}^2)},$$

which shows that the form $\mathfrak{a}$ is coercive. Additionally, $\mathfrak{a}$ is continuous as a consequence of Hölder's inequality. So, by the Lax–Milgram lemma, for each $f \in \mathrm{L}^2_\sigma(\Omega)$, there exists a unique $u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$ such that, for all $v \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$, we have

$$\mathfrak{a}(u,v) = \int_\Omega f \cdot \overline{v} \,\mathrm{d}x\,.$$

In particular, $u \in \mathrm{Dom}(A_2)$ and $A_2 u = f$. This gives $0 \in \rho(A_2)$.

Furthermore, for each $\theta \in (0,\pi)$, the Lax–Milgram lemma applied to the form

$$\mathfrak{a}_\lambda\colon \mathrm{W}^{1,p}_{0,\sigma}(\Omega) \times \mathrm{W}^{1,p}_{0,\sigma}(\Omega) \to \mathbb{C}, \quad (u,v) \mapsto \lambda \int_\Omega u \cdot \overline{v} \;\mathrm{d}x + \int_\Omega \nabla u \cdot \overline{\nabla v} \;\mathrm{d}x$$

together with the inverse triangle inequality for sectors gives that each $\lambda \in \Sigma_\theta$ is contained in the resolvent set $\rho(-A_2)$. Then, testing the resolvent equation (4.1) with the solution $u \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$ and applying Poincaré's and Hölder's inequality yields the bound

$$(1 + |\lambda|)\|(\lambda + A_2)^{-1} f\|_{\mathrm{L}^2_\sigma(\Omega)} \leq C\,\|f\|_{\mathrm{L}^2_\sigma(\Omega)}, \qquad \lambda \in \Sigma_\theta,\ f \in \mathrm{L}^2_\sigma(\Omega), \tag{4.5}$$

with $C \geq 0$ only depending on θ and $\operatorname{diam}(\Omega)$. Indeed, Poincaré's inequality and the reverse triangle inequality for sectors yield the coercivity estimate

$$|\mathfrak{a}_\lambda(u,u)| \geq C_1\big(|\lambda|\,\|u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^2)} + \|\nabla u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^{2\times 2})}\big) \geq C_2\,(|\lambda|+1)\,\|u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^2)} \tag{4.6}$$

with $C_1 > 0$ depending on θ and $C_2 > 0$ depending on θ and $\operatorname{diam}(\Omega)$. For the estimate (4.5), let $f \in \mathrm{L}^2_\sigma(\Omega)$ and $u \in \operatorname{Dom}(A_2)$ such that $(\lambda + A_2)u = f$. Then Hölder's inequality gives

$$|\mathfrak{a}_\lambda(u,u)| = \Big|\int_\Omega \overline{(\lambda + A_2)u}\cdot u \,\mathrm{d}x\Big| \leq \|u\|_{\mathrm{L}^2_\sigma(\Omega)}\|f\|_{\mathrm{L}^2_\sigma(\Omega)}. \tag{4.7}$$

Now the estimate (4.5) follows by plugging (4.6) and (4.7) together. In particular, this shows that we have $\{0\} \cup \Sigma_\theta \subseteq \rho(-A_2)$ with a resolvent estimate for the case of $p = 2$.

Theorem 4.1.1 now tells us that the operator $(\lambda + A_2)^{-1}$ restricts/extends (depending on whether $p \geq 2$ or $p < 2$) to a bounded operator on $\mathrm{L}^p_\sigma(\Omega)$ for p satisfying (4.3) and that this restriction/extension $(\lambda + A_2)^{-1}_p$ satisfies the bound

$$(1 + |\lambda|)\|(\lambda + A_2)^{-1}_p f\|_{\mathrm{L}^p_\sigma(\Omega)} \leq C\|f\|_{\mathrm{L}^p_\sigma(\Omega)}, \qquad \lambda \in \Sigma_\theta,\ f \in \mathrm{L}^p_\sigma(\Omega). \tag{4.8}$$

The Stokes Operator on $\mathrm{L}^p_\sigma(\Omega)$

Let us now introduce the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ for $p \neq 2$. For $p > 2$ satisfying (4.3), the realization of A_2 on $\mathrm{L}^p_\sigma(\Omega)$ is denoted by A_p and given as the *part* of A_2 in $\mathrm{L}^p_\sigma(\Omega)$, i.e.,

$$\operatorname{Dom}(A_p) := \big\{u \in \operatorname{Dom}(A_2) \cap \mathrm{L}^p_\sigma(\Omega) : A_2 u \in \mathrm{L}^p_\sigma(\Omega)\big\}, \quad A_p u := A_2 u \quad \text{for} \quad u \in \operatorname{Dom}(A_p).$$

With this definition, the estimate (4.8) now implies that $\lambda \in \rho(-A_p)$ and that $(\lambda + A_p)^{-1} = (\lambda + A_2)^{-1}_p$ holds. In particular, as $\mathrm{C}^\infty_{c,\sigma}(\Omega) \subseteq \operatorname{Dom}(A_p)$, the operator A_p is densely defined and, furthermore, closed.

For $p < 2$ satisfying (4.3) and $1/p + 1/p' = 1$, define A_p to be the L^p_σ-adjoint of the operator $A_{p'}$. This implies that A_p is closed and densely defined as well. Moreover, as A_2 is self-adjoint, we find that $(\lambda + A_p)^{-1} = (\lambda + A_2)^{-1}_p$.

With these definitions of the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$, Theorem 4.1.1 ensures that A_p is an invertible and sectorial operator. In particular, we have the following corollary to Theorem 4.1.1 which forms the departure point for the journey in the next sections.

Corollary 4.1.3. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that, for all p that satisfy* (4.3)*, the operator $-A_p$ generates an exponentially stable analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$. The constant ε depends only on the Lipschitz character of Ω and* $\operatorname{diam}(\Omega)$.

4.1.2 Functional Analytic Consequences and Outline

Let us outline the functional analytic consequences of the resolvent estimates for the Stokes operator and the analyticity of the Stokes semigroup. Rigorous definitions of the involved function spaces and operators will follow in the respective sections, together with the proofs of the outlined results.

In the case $p = 2$, the domains of the *fractional powers* $\mathrm{Dom}(A_2^\alpha)$ for $0 \leq \alpha < 3/4$ were characterized in terms of suitable *Bessel-potential spaces* $\mathrm{H}^{2\alpha,2}_{0,\sigma}(\Omega)$ by M. Mitrea and Monniaux [116, Thm. 5.1]. The following theorem gives an L^p_σ-version of this result. This generalizes the results of Giga [66, Thm. 3] from the smooth situation to the situation of planar and bounded Lipschitz domains.

Theorem 4.1.4. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exist $\varepsilon > 0$ and $\delta \in (0, 1]$ such that, for all $1 < p < \infty$ and $0 < \theta < 1$ satisfying* (4.3) *and either*

$$\theta < \frac{1}{2} + \frac{1}{2p} \quad \text{if} \quad \frac{1}{2} - \frac{1}{p} \leq \frac{\delta}{2} \qquad \text{or} \qquad \theta < \frac{1}{p} + \frac{1+\delta}{4} \quad \text{if} \quad \frac{1}{2} - \frac{1}{p} > \frac{\delta}{2}\,,$$

we have with equivalent norms that

$$\mathrm{Dom}(A_p^\theta) = \mathrm{H}^{2\theta,p}_{0,\sigma}(\Omega).$$

In particular, Theorem 4.1.4 characterizes the domain of the *square root* of A_p via $\mathrm{Dom}(A_p^{\frac{1}{2}}) = \mathrm{W}^{1,p}_{0,\sigma}(\Omega)$. As a corollary of Theorem 4.1.4 and the *Gagliardo–Nirenberg* inequality, see, e.g., [49] for a detailed proof, we obtain the following L^p–L^q*-smoothing estimates* for the Stokes semigroup.

Corollary 4.1.5. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that, for all $1 < p \leq q < \infty$ satisfying* (4.3), *there exists $C \geq 0$ such that*

$$\|\mathrm{e}^{-tA_p} f\|_{\mathrm{L}^q_\sigma(\Omega)} \leq C t^{-(\frac{1}{p} - \frac{1}{q})} \|f\|_{\mathrm{L}^p_\sigma(\Omega)}, \qquad t > 0,\ f \in \mathrm{L}^p_\sigma(\Omega),$$

and

$$\|\nabla \mathrm{e}^{-tA_p} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^{2\times 2})} \leq C t^{-\frac{1}{2} - (\frac{1}{p} - \frac{1}{q})} \|f\|_{\mathrm{L}^p_\sigma(\Omega)}, \qquad t > 0,\ f \in \mathrm{L}^p_\sigma(\Omega).$$

In order to prove Theorem 4.1.4, we show that the H^∞*-calculus* of A_p is bounded. Here, we say that an injective operator A on a Banach space X that is sectorial of some angle $\omega \in [0, \pi)$ has a bounded H^∞-calculus if, for some $\theta \in (0, \pi - \omega)$, there exists $C \geq 0$ such that, for all bounded analytic functions f on a sector Σ_θ, the estimate

$$\|f(A)\|_{\mathcal{L}(X)} \leq C \sup_{z \in \Sigma_\theta} |f(z)|$$

holds. The expression $f(A)$ is to be understood in the sense of a *regularization* of the *natural functional calculus* for sectorial operators, cf. [71, Sect. 2.3].

Theorem 4.1.6. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that, for all p that satisfy* (4.3), *the H^∞-calculus of A_p is bounded.*

The boundedness of the H^∞-calculus will be deduced by using a comparison principle due to Kunstmann and Weis [88] in which the Stokes operator is compared to the Dirichlet-Laplacian. A crucial ingredient for this comparison principle is the *$\mathcal{R}$-sectoriality* of the Stokes operator—a property that is well-known to be equivalent to *maximal* L^q*-regularity*.

Theorem 4.1.7. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that, for all p that satisfy* (4.3) *and for all $1 < q < \infty$, the Stokes operator A_p has maximal* L^q*-regularity.*

Additionally to this theorem, we employ the *square root property* $\mathrm{Dom}(A_p^{1/2}) = \mathrm{W}^{1,p}_{0,\sigma}(\Omega)$ as a special case of Theorem 4.1.4 to transfer the maximal regularity property from the ground space $X = \mathrm{L}^p_\sigma(\Omega)$ to the ground space $X = \mathrm{W}^{-1,p}_\sigma(\Omega) := [\mathrm{W}^{1,p'}_{0,\sigma}(\Omega)]^*$. Here we also establish the maximal L^q-regularity property for the *weak* Stokes operator defined on the space $\mathrm{W}^{-1,p}_\sigma(\Omega)$.

Finally, these properties will be used to investigate regularity properties of *Leray–Hopf weak solutions* to the Navier–Stokes equations in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^2$.

$$\begin{cases} u' - \Delta u + (u \cdot \nabla) u + \nabla \phi = f = f_0 + \mathbb{P}_2 \operatorname{div}(F) & \text{in } (0,\infty) \times \Omega, \\ \operatorname{div}(u) = 0 & \text{in } (0,\infty) \times \Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{4.9}$$

Weak solutions in the *Leray–Hopf class*

$$\mathrm{LH}_\infty(\Omega) := \mathrm{L}^\infty(0,\infty; \mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0,\infty; \mathrm{W}^{1,2}_{0,\sigma}(\Omega))$$

are known to exist since the seminal works of Leray [96, 97, 98] and Hopf [75]. In particular, in the two-dimensional case and if Ω is smooth enough, e.g., if the boundary is $\mathrm{C}^{1,1}$-regular, then Leray–Hopf weak solutions u to (4.9) with $F = 0$ and u_0 and f_0 regular enough, are known to be unique and regular, i.e., we have that

$$u \in \mathrm{L}^\infty(0,\infty; \mathrm{W}^{1,2}_{0,\sigma}(\Omega)) \cap \mathrm{L}^2(0,\infty; \mathrm{W}^{2,2}(\Omega; \mathbb{C}^2)) \cap \mathrm{W}^{1,2}(0,\infty; \mathrm{L}^2_\sigma(\Omega)),$$

see, e.g., [137, Thm. V.1.8.1]. If Ω is merely Lipschitz regular, then such regularity properties break down in general. The final theorem of this chapter establishes suitable regularity properties of Leray–Hopf weak solutions in this geometric setting.

Theorem 4.1.8. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$, depending only on the Lipschitz geometry of Ω, such that the following statements are valid.*

(a) *For all $1 < s < 2$ and $1 < p < 2$ that satisfy*

$$1 - \frac{1}{s} = \frac{1}{p} - \frac{1}{2} < \frac{1}{4} + \varepsilon$$

and all Leray–Hopf weak solutions

$$u \in \mathrm{L}^\infty(0,\infty; \mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0,\infty; \mathrm{W}^{1,2}_{0,\sigma}(\Omega))$$

to (4.9) *with initial data* u_0 *and force* $f = f_0$ *satisfying*

$$u_0 \in \big(\mathrm{L}^p_\sigma(\Omega), \mathrm{Dom}(A_p)\big)_{1-\frac{1}{s},s} \quad \text{and} \quad f_0 \in \mathrm{L}^s(0,\infty;\mathrm{L}^p_\sigma(\Omega)),$$

one has that

$$u \in \mathrm{W}^{1,s}(0,\infty;\mathrm{L}^p_\sigma(\Omega)) \cap \mathrm{L}^s(0,\infty;\mathrm{Dom}(A_p)).$$

(b) *For all* $1 < p < \infty$ *that satisfy* (4.3), *all* $1 < s < \infty$ *that satisfy*

$$\frac{1}{p} + \frac{1}{s} = 1,$$

and all Leray–Hopf weak solutions

$$u \in \mathrm{L}^\infty(0,\infty;\mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0,\infty;\mathrm{W}^{1,2}_{0,\sigma}(\Omega))$$

to (4.9) *with initial data* u_0 *and force* $f = \mathbb{P}_2 \operatorname{div}(F)$ *satisfying*

$$u_0 \in \big(\mathrm{W}^{-1,p}_\sigma(\Omega), \mathrm{W}^{1,p}_{0,\sigma}(\Omega)\big)_{1-\frac{1}{s},s} \quad \text{and} \quad F \in \mathrm{L}^s(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^{2\times 2})),$$

one has that

$$u \in \mathrm{W}^{1,s}(0,\infty;\mathrm{W}^{-1,p}_\sigma(\Omega)) \cap \mathrm{L}^s(0,\infty;\mathrm{W}^{1,p}_{0,\sigma}(\Omega)).$$

4.2 The L^2-Dirichlet Problem for the Stokes Resolvent System

Before we dive into the proofs of the results outlined in the previous section, let us briefly discuss the groundwork laid by our extension of Shen's resolvent estimates of the Stokes operator. In $d \geq 3$, Shen's proof of the resolvent estimates fundamentally bases on the resolution of the L^2-Dirichlet problem for the Stokes resolvent system [134, Thm. 1.1]. For the case of $d = 2$ this problem was resolved in the author's master thesis [51]. An overview of the full proof can be found in [56]. To formulate this problem, we state the following lemma and definition, cf. [133, p. 208].

Lemma and Definition 4.2.1. *Let* $\Xi \subseteq \mathbb{R}^d$, $d \geq 2$, *be a bounded Lipschitz domain. There exists* $\alpha > 1$ *depending only on* d *and the Lipschitz character of* Ξ *such that each of the sets*

$$\begin{aligned} \gamma^{\mathrm{in}}_\alpha(q) &:= \big\{x \in \Xi : |x - q| < \alpha \operatorname{dist}(x, \partial\Xi)\big\}, \qquad q \in \partial\Xi, \\ \gamma^{\mathrm{ex}}_\alpha(q) &:= \big\{x \in \overline{\Xi}^c : |x - q| < \alpha \operatorname{dist}(x, \partial\Xi)\big\}, \qquad q \in \partial\Xi, \end{aligned}$$

contains a cone of fixed height and aperture with vertex at q. *In this case, we call the family* $\{\gamma^{\mathrm{in}}_\alpha(q) : q \in \partial\Xi\}$ *an interior and* $\{\gamma^{\mathrm{ex}}_\alpha(q) : q \in \partial\Xi\}$ *an exterior family of non-tangential approach regions.*

In the following, we fix a value of $\alpha > 1$ subject to Lemma and Definition 4.2.1. The notions of the non-tangential maximal function and non-tangential convergence are introduced as follows.

Definition 4.2.2. Let $\Xi \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $\{\gamma_\alpha^{\mathrm{in}}(q) : q \in \partial\Xi\}$ a corresponding family of non-tangential approach regions. For a function $u \colon \Xi \to \mathbb{C}^m$, $m \in \mathbb{N}$, the *interior non-tangential maximal function of u* is defined as

$$(u)_{\mathrm{in}}^*(q) \coloneqq \sup\big\{|u(x)| : x \in \gamma_\alpha^{\mathrm{in}}(q)\big\}, \qquad q \in \partial\Xi.$$

Similarly, for a function $v \colon \overline{\Xi}^c \to \mathbb{C}^m$ and an exterior family of non-tangential approach regions $\{\gamma_\alpha^{\mathrm{ex}}(q) : q \in \partial\Xi\}$, the *exterior non-tangential maximal function of v* is defined analogously and denoted by $(v)_{\mathrm{ex}}^*$.

For a function $f \colon \partial\Xi \to \mathbb{C}^m$, we say that $u = f$ *in the sense of non-tangential convergence from the inside* if

$$\lim_{\substack{x \in \gamma_\alpha^{\mathrm{in}}(q) \\ x \to q}} u(x) = f(q), \qquad \text{a.e. } q \in \partial\Xi,$$

and we call f the *non-tangential limit of u inside of Ξ*. Analogously, we say that $v = f$ *in the sense of non-tangential convergence from the outside* if

$$\lim_{\substack{x \in \gamma_\alpha^{\mathrm{ex}}(q) \\ x \to q}} v(x) = f(q), \qquad \text{a.e. } q \in \partial\Xi,$$

and we call f the *non-tangential limit of v outside of Ξ*.

For a bounded Lipschitz domain $\Xi \subseteq \mathbb{R}^d$, $d \geq 2$, with connected boundary and $\lambda \in \Sigma_\theta$, consider the Dirichlet problem

$$\begin{cases} \lambda u - \Delta u + \nabla \phi = 0 & \text{in } \Xi, \\ \operatorname{div}(u) = 0 & \text{in } \Xi, \\ u = g & \text{on } \partial\Xi, \\ (u)_{\mathrm{in}}^* \in \mathrm{L}^2(\partial\Xi), \end{cases} \tag{Dir}$$

where the equality $u = g$ on $\partial\Xi$ is to be understood in the sense of non-tangential convergence from the inside. Due to the condition $\operatorname{div}(u) = 0$, the divergence theorem implies that the normal component of g, i.e., $g \cdot n$, has average zero on $\partial\Xi$. Thus, the definition of the boundary space

$$\mathrm{L}_n^2(\partial\Xi) \coloneqq \Big\{g \in \mathrm{L}^2(\partial\Xi; \mathbb{C}^d) : \int_{\partial\Xi} g \cdot n \, \mathrm{d}\sigma = 0\Big\}$$

seems natural. The key ingredient that allowed Shen to prove the counterpart to Theorem 4.1.1 in three and more dimensions is the following resolution of the L^2-Dirichlet problem:

Theorem 4.2.3 (L^2-Dirichlet problem)**.** *Let $\Xi \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain with connected boundary and let $\theta \in (0, \pi/2)$. To every resolvent parameter $\lambda \in \Sigma_\theta$ and every function $g \in \mathrm{L}_n^2(\partial\Xi)$, there exists a unique smooth function $u \colon \Xi \to \mathbb{C}^d$ that satisfies $(u)_{\mathrm{in}}^* \in \mathrm{L}^2(\partial\Xi)$ and a smooth function $\phi \colon \Xi \to \mathbb{C}$ that is unique up to the addition of constants such that* (Dir) *is satisfied. Moreover, there exists a constant $C \geq 0$ such that*

$$\|(u)_{\mathrm{in}}^*\|_{\mathrm{L}^2(\partial\Xi)} \leq C \, \|g\|_{\mathrm{L}^2(\partial\Xi)}. \tag{4.10}$$

The constant C depends only on d, θ, the Lipschitz character of Ξ, and on constants $\alpha, \beta \geq 0$ that satisfy $\alpha \leq \operatorname{diam}(\Xi) \leq \beta$.

The case $d \geq 3$ of Theorem 4.2.3 was already proved by Shen [134, Thm. 5.5]. In order to make the main idea of the proof accessible for the case $d = 2$, one needs to derive estimates on fundamental solutions of the Stokes resolvent problem, which have representations as so-called *double layer potentials*. For details on the proof, we refer the interested reader to [134] and to [56] for a detailed analysis of the peculiarities of the two-dimensional case. For an introduction to the method of layer potentials, we refer to [10, 133, 152].

4.3 Functional Analytic Properties of the Stokes Operator

In this section, we will prove several functional analytic properties of the Stokes operator.

4.3.1 $\mathcal{R}$-Sectoriality and Maximal Regularity

Let us recall the definition of the sector

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \pi - \theta\}, \quad \theta \in (0, \pi).$$

In this section, we want to prove that the Stokes operator on a bounded planar Lipschitz domain admits the property of maximal L^q-regularity which will be a crucial ingredient in our treatment of solutions to the Navier–Stokes equations in Section 4.4.

Recall that, for all $x \in X$ and $f \in \mathrm{L}^q(0, \infty; X)$, $1 < q < \infty$, the unique *mild solution* to the abstract Cauchy problem induced by an operator A

$$\begin{cases} \dot{u}(t) + Au(t) = f(t), & t \geq 0, \\ \qquad u(0) = x \end{cases} \tag{ACP}$$

is given by *Duhamel's formula*

$$u(t) = \mathrm{e}^{-tA}x + \int_0^t \mathrm{e}^{-(t-s)A} f(s) \, \mathrm{d}s \,,$$

cf. [8, Prop. 3.1.16].

Definition 4.3.1. For $1 < q < \infty$, we say that $A \colon \mathrm{Dom}(A) \subseteq X \to X$ *has maximal* L^q*-regularity* if, for $x = 0$ and all $f \in \mathrm{L}^q(0, \infty; X)$, the unique mild solution to (ACP) satisfies $u(t) \in \mathrm{Dom}(A)$ for almost every $t > 0$ and $Au \in \mathrm{L}^q(0, \infty; X)$.

Remark 4.3.2. In the case of Definition 4.3.1, u is also weakly differentiable with respect to t and satisfies $\dot{u} \in \mathrm{L}^q(0, \infty; X)$. By employing the closed graph theorem, the mere fact that $\dot{u}$ and Au lie in $\mathrm{L}^q(0, \infty; X)$ implies the existence of $C > 0$ such that, for all $f \in \mathrm{L}^q(0, \infty; X)$, the stability estimate

$$\|\dot{u}\|_{\mathrm{L}^q(0,\infty;X)} + \|Au\|_{\mathrm{L}^q(0,\infty;X)} \leq C \, \|f\|_{\mathrm{L}^q(0,\infty;X)}$$

holds. Note that Definition 4.3.1 only takes into account homogeneous initial values $x = 0$. Furthermore, it is well known, see, e.g., [38, Thm. 2.1], that $u \in \mathrm{L}^q(0, \infty; X)$ if and only if $0 \in \rho(A)$. In this case, one has a stability estimate of the form

$$\|u\|_{\mathrm{L}^q(0,\infty;X)} + \|\dot{u}\|_{\mathrm{L}^q(0,\infty;X)} + \|Au\|_{\mathrm{L}^q(0,\infty;X)} \leq C \, \|f\|_{\mathrm{L}^q(0,\infty;X)}. \tag{4.11}$$

This estimate (4.11) and with it the regularity of u can be extended to mild solutions with inhomogeneous initial data $u(0) = x$ for all x coming from the real interpolation space $(X, \mathrm{Dom}(A))_{1-1/q,q}$, cf. [146, Prop. 2.2.3]. In this case, the stability estimate generalizes to

$$\|u\|_{\mathrm{L}^q(0,\infty;X)} + \|\dot{u}\|_{\mathrm{L}^q(0,\infty;X)} + \|Au\|_{\mathrm{L}^q(0,\infty;X)} \leq C \big(\|f\|_{\mathrm{L}^q(0,\infty;X)} + \|x\|_{(X,\mathrm{Dom}(A))_{1-\frac{1}{q},q}}\big).$$

Notice further that the property that A has maximal L^q-regularity is *independent* of q, see [38, Thm. 4.2] or [39, Thm. 7.1] and the references given there.

Our proof of the maximal L^q-regularity of the Stokes operator will go through establishing another property of operators that we introduce next.

Definition 4.3.3. Let X and Y denote Banach spaces over $\mathbb{C}$ and $A\colon \mathrm{Dom}(A) \subseteq X \to X$ a closed linear operator.

(i) The operator A is said to be *sectorial* of angle $\omega \in [0, \pi)$ if

$$\sigma(A) \subseteq \overline{\Sigma_{\pi-\omega}}$$

and if, for all $\omega < \theta < \pi$, the family $\big(\lambda(\lambda + A)^{-1}\big)_{\lambda\in\Sigma_\theta} \subseteq \mathcal{L}(X)$ is bounded.

(ii) A family of operators $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is said to be *$\mathcal{R}$-bounded from X to Y* if there exists a constant $C > 0$ such that, for all $k_0 \in \mathbb{N}$, $(T_k)_{k=1}^{k_0} \subseteq \mathcal{T}$, and $(x_k)_{k=1}^{k_0} \subseteq X$, the inequality

$$\Big\| \sum_{k=1}^{k_0} r_k(\cdot)\, T_k x_k \Big\|_{\mathrm{L}^2(0,1;Y)} \leq C \Big\| \sum_{k=1}^{k_0} r_k(\cdot)\, x_k \Big\|_{\mathrm{L}^2(0,1;X)} \tag{4.12}$$

holds. Here, $r_k(t) \coloneqq \mathrm{sgn}(\sin(2^k \pi t))$ are the *Rademacher functions.* We also define the *$\mathcal{R}$-bound* of $\mathcal{T}$ as $\mathcal{R}(\mathcal{T}) \coloneqq \inf\{C > 0 : \text{(4.12) holds}\}$.

(iii) The operator A is said to be *$\mathcal{R}$-sectorial* of angle $\omega \in [0, \pi)$ if

$$\sigma(A) \subseteq \overline{\Sigma_{\pi-\omega}}$$

and if, for all $\omega < \theta < \pi$, the family $\big(\lambda(\lambda + A)^{-1}\big)_{\lambda\in\Sigma_\theta} \subseteq \mathcal{L}(X)$ is $\mathcal{R}$-bounded.

Remark 4.3.4. (i) By taking $k_0 = 1$, one sees that $\mathcal{R}$-boundedness implies boundedness of a family of operators. If X and Y are isomorphic to a Hilbert space, then $\mathcal{R}$-boundedness is equivalent to the boundedness of the family of operators, see [33, Rem. 3.2].

(ii) If X is a subspace of $\mathrm{L}^p(\Omega; \mathbb{C}^m)$ for some $1 < p < \infty$, $m \in \mathbb{N}$, and $\Omega \subseteq \mathbb{R}^d$ Lebesgue-measurable, then there exists $C > 0$ such that, for all $k_0 \in \mathbb{N}$ and $(f_k)_{k=1}^{k_0} \subseteq X$, it holds that

$$\frac{1}{C} \Big\| \sum_{k=1}^{k_0} r_k(\cdot) f_k \Big\|_{\mathrm{L}^2(0,1;X)} \leq \Big\| \Big[\sum_{k=1}^{k_0} |f_k|^2 \Big]^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\Omega)} \leq C \Big\| \sum_{k=1}^{k_0} r_k(\cdot) f_k \Big\|_{\mathrm{L}^2(0,1;X)}. \tag{4.13}$$

This means that $\mathcal{R}$-boundedness in L^p-spaces is equivalent to so-called *square function estimates*, see [87, Rem. 2.9].

(iii) The operator $-A$ generates a strongly continuous bounded analytic semigroup on X if and only if A is densely defined and sectorial of angle $\omega \in [0, \pi/2)$, see [43, Thm. II.4.6]. Moreover, this semigroup is exponentially stable if $0 \in \rho(A)$. A short calculation reveals that the condition $0 \in \rho(A)$ follows if one can show the boundedness of the family of operators $\big((1+\lambda)(\lambda + A)^{-1}\big)_{\lambda \in (0,\infty)}$, cf. [43, Prop. II.5.24].

If X is a closed subspace or quotient of a reflexive L^p-space, then the question of $\mathcal{R}$-sectoriality is intimately related to the question of the maximal L^q-regularity of the generator $-A\colon \mathrm{Dom}(A) \subseteq X \to X$ of a bounded analytic semigroup through a seminal result of Weis [159, Thm. 4.2].

Theorem 4.3.5 (Weis)**.** *Let $1 < p < \infty$ and X a closed subspace or quotient of $\mathrm{L}^p(\Omega; \mathbb{C}^m)$. If $-A\colon \mathrm{Dom}(A) \subseteq X \to X$ is the generator of a bounded analytic semigroup, then A has maximal L^q-regularity for $1 < q < \infty$ if and only if A is $\mathcal{R}$-sectorial of angle ω for some $\omega \in [0, \pi/2)$.*

The presented version of Theorem 4.3.5 is a special version of the original result by Weis, which holds for the class of so-called UMD-*spaces.* However, we will not go into details with the definition of UMD-spaces as the reflexive L^p-spaces as well as all closed subspaces and quotient spaces of these L^p-spaces have the UMD-property, see, e.g., [32], and these will be the only spaces that appear in our investigation.

Motivated by the relation in Theorem 4.3.5, we now aim at establishing $\mathcal{R}$-sectoriality of angle $\omega = 0$ for the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ for suitable p. By virtue of Remark 4.3.4(i) and (iii) and Theorem 4.3.5, this will readily prove Theorem 4.1.1, Corollary 4.1.3, and Theorem 4.1.7.

Proof of the $\mathcal{R}$-sectoriality of the Stokes Operator. The proof is decomposed into five steps.

Step 1: the case $p = 2$. Let $\theta \in (0, \pi)$. Recall from Section 4.1.1 that the Lax–Milgram lemma directly implies that $\{0\} \cup \Sigma_\theta \subseteq \rho(-A_2)$. Furthermore, for any $f \in \mathrm{L}^2(\Omega; \mathbb{C}^2)$, define $u := (\lambda + A_2)^{-1}\mathbb{P}_2 f$. Then, testing the resolvent equation with u and applying Poincaré's inequality, one directly obtains the inequality

$$(1 + |\lambda|)\, \|u\|_{\mathrm{L}^2_\sigma(\Omega)} \leq C\, \|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^2)}. \tag{4.14}$$

The constant $C \geq 0$ depends only on θ, and $\mathrm{diam}(\Omega)$. By virtue of Remark 4.3.4(i), this implies that the family $\big((1+|\lambda|)(\lambda + A_2)^{-1}\mathbb{P}_2\big)_{\lambda \in \Sigma_\theta} \subseteq \mathcal{L}(\mathrm{L}^2(\Omega;\mathbb{C}^2), \mathrm{L}^2_\sigma(\Omega))$ is $\mathcal{R}$-bounded. As $\mathbb{P}_2$ acts as the identity on $\mathrm{L}^2_\sigma(\Omega)$, this readily settles the $\mathcal{R}$-sectoriality in the case $p = 2$.

Step 2: reformulation into an ℓ^2-valued boundedness estimate in the case $p \geq 2$. Let $p \geq 2$. A combination of Definition 4.3.3(ii) and Remark 4.3.4(ii) shows the validity of the following statement, cf. [146, Prop. 2.3.4]:

The family $\big((1+|\lambda|)(\lambda + A_2)^{-1}\mathbb{P}_2|_{\mathrm{L}^p}\big)_{\lambda \in \Sigma_\theta}$ is $\mathcal{R}$-bounded from $\mathrm{L}^p(\Omega;\mathbb{C}^2)$ into $\mathrm{L}^p_\sigma(\Omega)$ if and only if there exists $C \geq 0$ such that, for all $k_0 \in \mathbb{N}$ and all $(\lambda_k)_{k=1}^{k_0} \subseteq \Sigma_\theta$, the operator

$$T_{(\lambda_k)_{k=1}^{k_0}} : \mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^2)) \to \mathrm{L}^p(\Omega; \ell^2(\mathbb{C}^2)),$$

$$f = (f_k)_{k\in\mathbb{N}} \mapsto \begin{pmatrix} (1+|\lambda_1|)(\lambda_1 + A_2)^{-1}\mathbb{P}_2|_{\mathrm{L}^p} f_1 \\ \vdots \\ (1+|\lambda_{k_0}|)(\lambda_{k_0} + A_2)^{-1}\mathbb{P}_2|_{\mathrm{L}^p} f_{k_0} \\ 0 \\ \vdots \end{pmatrix}$$

is well-defined and satisfies

$$\|T_{(\lambda_k)_{k=1}^{k_0}} f\|_{\mathrm{L}^p(\Omega;\ell^2(\mathbb{C}^2))} \leq C\, \|f\|_{\mathrm{L}^p(\Omega;\ell^2(\mathbb{C}^2))}. \tag{4.15}$$

In other words, the family of all operators that can be formed by the procedure above is uniformly bounded.

Step 3: verification of (4.15) *for certain values of* $p > 2$. To verify (4.15), one employs the following vector-valued version of the L^p-extrapolation theorem of Shen [145, Thm. 4.1], see [132, Thm. 3.3] for the scalar-valued version. Note that, in comparison to Shen's result [132, Thm. 3.3] this result does not assume any type of regularity of the involved sets other than measurability.

Theorem 4.3.6 (Shen, Tolksdorf)**.** *Let X be a Banach space, $\Upsilon \subseteq \mathbb{R}^2$ be Lebesgue-measurable and bounded, $\mathcal{M} \geq 0$, and let $T \in \mathcal{L}(\mathrm{L}^2(\Upsilon; X))$ with $\|T\|_{\mathcal{L}(\mathrm{L}^2(\Upsilon;X))} \leq \mathcal{M}$.*

Suppose that there exist constants $p > 2$, $R_0 > 0$, $\alpha_2 > \alpha_1 > 1$, and $\mathcal{C} \geq 0$ such that the following statement is valid: for all balls $\mathrm{B}(x_0, r)$ with $0 < r < R_0$, which are either centered on $\partial\Upsilon$, i.e., $x_0 \in \partial\Upsilon$, or satisfy $\mathrm{B}(x_0, \alpha_2 r) \subseteq \Upsilon$, and all compactly supported $f \in \mathrm{L}^\infty(\Upsilon; X)$ with $f = 0$ in $\Upsilon \cap \mathrm{B}(x_0, \alpha_2 r)$, the estimate

$$\left(\frac{1}{r^2}\int_{\Upsilon\cap\mathrm{B}(x_0,r)} \|Tf\|_X^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \mathcal{C}\left(\frac{1}{r^2}\int_{\Upsilon\cap\mathrm{B}(x_0,\alpha_1 r)} \|Tf\|_X^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \tag{4.16}$$

holds. Then, for each $2 < q < p$, the operator T restricts to a bounded linear operator on $\mathrm{L}^q(\Upsilon; X)$, with operator norm bounded by a constant depending on p, q, α_1, α_2, $\mathcal{C}$, $\mathcal{M}$, R_0, and $\operatorname{diam}(\Upsilon)$.

See Figure 4.2, for an accessible visualization of the main playgrounds in Theorem 4.3.6.

Remark 4.3.7. (i) We say that an operator satisfies a *weak reverse Hölder estimate* if it satisfies (4.16). The original version of Theorem 4.3.6 comes with an additional term on the right-hand side of (4.16) which will not be needed in our case.

(ii) One interpretation of (4.16) is to see it as a quantification of the *non-locality* of the operator T in the following sense. By assumption, f is compactly supported in Υ and vanishes on $\mathrm{B}(x_0, \alpha_2 r) \cap \Upsilon$. For a local operator, i.e., an operator that preserves the locality of the input in terms of the support of the output, a weak reverse Hölder estimate like (4.16) would trivially hold as both sides of the inequality would be identically zero. For a non-local operator, like, e.g., a Fourier transformation, however, the support of the output may drastically change making (4.16) non-trivial.

We will need the next lemma on a *reverse* trace-estimate, whose proof can be found in Wei and Zhang [158, Lem. 3.3]. Wei and Zhang originally formulated the result for Lipschitz domains $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, but the proof works *mutatis mutandis* also for the case $d = 2$ for which we state the result here. See also [10, Lem. 3.3].

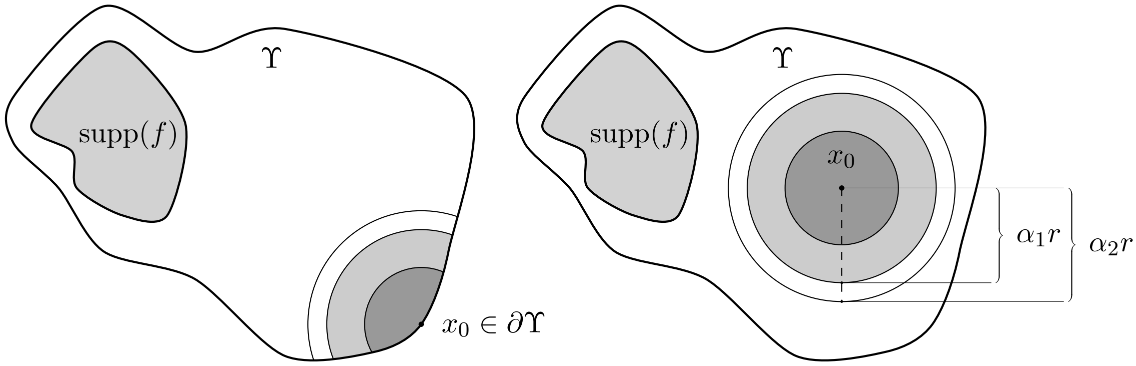

Figure 4.2: Illustration of the two cases in Shen's extrapolation Theorem 4.3.6 for which one needs to verify the weak reverse Hölder estimate (4.16). The left part considers $x_0 \in \partial\Upsilon$ and the right part x_0 with $\mathrm{B}(x_0, \alpha_2 r) \subseteq \Upsilon$. The weak reverse Hölder estimate now aims at estimating the L^p-norm of Tf inside $\Upsilon \cap \mathrm{B}(x_0, r)$, i.e., the innermost dark disc, versus the L^2-norm of Tf inside $\Upsilon \cap \mathrm{B}(x_0, \alpha_1 r)$, i.e., the middle gray disc for all compactly supported functions f with $\mathrm{supp}(f) \cap \mathrm{B}(x_0, \alpha_2, r) = \emptyset$.

Lemma 4.3.8. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists a constant $C \geq 0$ depending only on the Lipschitz constant of Ω such that, for all measurable functions $h \colon \Omega \to \mathbb{C}^N$ with interior non-tangential maximal function $(h)^*_{\mathrm{in}}$, we have the estimate*

$$\|h\|_{\mathrm{L}^4(\Omega;\mathbb{C}^N)} \leq C \, \|(h)^*_{\mathrm{in}}\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^N)} \, . \tag{4.17}$$

Remark 4.3.9. Note that Lemma 4.3.8 will fix the maximal value p such that the Stokes operator A_p will be $\mathcal{R}$-sectorial. In particular, for $p = 4$ we have $p' = \frac{4}{3}$, such that we see (4.3) reflected in this estimate. This also shows that the technique employed in the following can neither yield better results for domains that are more regular than Lipschitz nor for domains that have a value p that does not fulfill (4.3) while still having a Helmholtz projection.

Verification of the weak reverse Hölder estimate (4.16) *and application of Theorem 4.3.6.* We apply Theorem 4.3.6 as follows: first of all, we fix some notation regarding the bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^2$. Let $M \geq 0$ denote the Lipschitz constant of Ω. Define for $r > 0$ the "cylinder"

$$\mathrm{D}(r) \coloneqq \big\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < r \text{ and } |x_2| < (1 + M)r\big\}.$$

Note that the height $(1 + M)r$ is chosen such that each cylinder $\mathrm{D}(r)$ also contains the cone-like sets $\gamma^{\mathrm{in}}_\alpha(0) \subseteq \mathrm{D}(r)$ and $\gamma^{\mathrm{ex}}_\alpha(0) \subseteq \mathrm{D}(r)$ for a suitably chosen α. Furthermore, let $R_0 > 0$ be such that, for each $x_0 \in \partial\Omega$, there exists a Lipschitz continuous function $\eta \colon \mathbb{R} \to \mathbb{R}$ with $\eta(0) = 0$ and $\|\eta'\|_{\mathrm{L}^\infty(\mathbb{R})} \leq M$ such that, for all $0 < r < R_0$, one has after a possible rotation of the coordinate system that

$$\mathrm{D}(r) \cap [\Omega - \{x_0\}] = \big\{(y_1, y_2) \in \mathbb{R}^2 : |y_1| < r \text{ and } \eta(y_1) < y_2 < (1 + M)r\big\} \eqqcolon \mathrm{D}_\eta(r)$$

and

$$\mathrm{D}(r) \cap [\partial\Omega - \{x_0\}] = \big\{(y_1, y_2) \in \mathbb{R}^2 : |y_1| < r \text{ and } y_2 = \eta(y_1)\big\} \eqqcolon \mathrm{I}_\eta(r).$$

Notice that

$$\mathrm{B}(0,r) \subseteq \mathrm{D}(r) \quad \text{and} \quad \mathrm{D}(2r) \subseteq \mathrm{B}(0,\alpha_1 r) \tag{4.18}$$

with

$$\alpha_1 := 2\sqrt{1+(1+M)^2}\,.$$

Furthermore, let us remark that the *Lipschitz cylinders* $\mathrm{D}_\eta(r)$ are in fact Lipschitz domains with Lipschitz constant comparable to M. Additionally, the *Lipschitz character* is uniform with respect to all parameters if the radius r is bounded from below. For a detailed proof of the remarked facts, see, e.g., [146, Lem. 1.3.25].

Now, choose $X := \ell^2(\mathbb{C}^2)$, $\Upsilon := \Omega$, and $p_0 := 4$. Let further $\theta \in (0,\pi/2)$, $k_0 \in \mathbb{N}$, and let $(\lambda_k)_{k=1}^{k_0} \subseteq \Sigma_\theta$. For T being defined by $T := T_{(\lambda_k)_{k=1}^{k_0}}$, we know by Steps 1 and 2 that the operator T is bounded on $\mathrm{L}^2(\Omega;\ell^2(\mathbb{C}^2))$ and that the operator norm is bounded by the constant C from (4.14). As this constant is uniform with respect to k_0 and all choices of $(\lambda_k)_{k=1}^{k_0} \subseteq \Sigma_\theta$, we choose $\mathcal{M} := C$.

Now, if we can establish the validity of (4.16) uniformly in those parameters as well, the family of all such operators T restricts to a bounded family of operators in $\mathrm{L}^q(\Omega;\ell^2(\mathbb{C}^2))$ for all q subject to $2 \leq q < p$. Here, the parameter p still has to be fixed.

We concentrate first on verifying (4.16) for points in $x_0 \in \partial\Omega$. Let $\alpha_2 := 3\sqrt{1+(1+M)^2}$, and let $0 < 2r < R_0$. Finally, let $f \in \mathrm{L}^\infty(\Omega;\ell^2(\mathbb{C}^2))$ have compact support with $f = 0$ in $\Omega \cap \mathrm{B}(x_0,\alpha_2 r)$.

For $1 \leq k \leq k_0$ define $u_k := (\lambda_k + A_2)^{-1}\mathbb{P}_2 f_k$ and notice that $u_k \in \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$. We remark that, by virtue of [116, Cor. 5.7], u_k is even Hölder continuous in $\overline{\Omega}$. Thus, e.g., by [137, Thm. III.2.1.1(b)], there exists a unique pressure $\pi_k \in \mathrm{L}^2(\Omega)$ with average zero such that

$$\begin{cases} \lambda_k u_k - \Delta u_k + \nabla \pi_k = \mathbb{P}_2 f_k & \text{in } \Omega, \\ \operatorname{div}(u_k) = 0 & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the Helmholtz decomposition of $\mathrm{L}^2(\Omega;\mathbb{C}^2)$ in order to write $\mathbb{P}_2 f_k = f_k - \nabla h_k$ with some $h_k \in \mathrm{G}_2(\Omega)$, we find, since f_k vanishes in $\Omega \cap \mathrm{B}(x_0,\alpha_2 r)$, with $\phi_k := \pi_k + h_k$ that

$$\begin{cases} \lambda_k u_k - \Delta u_k + \nabla \phi_k = 0 & \text{in } \Omega \cap \mathrm{B}(x_0,\alpha_2 r), \\ \operatorname{div}(u_k) = 0 & \text{in } \Omega \cap \mathrm{B}(x_0,\alpha_2 r), \\ u_k = 0 & \text{on } \partial\Omega \cap \mathrm{B}(x_0,\alpha_2 r). \end{cases} \tag{4.19}$$

Inner regularity, see, e.g., [57, Thm. IV.4.3], now implies that the solutions u_k and ϕ_k are smooth in $\Omega \cap \mathrm{B}(x_0,\alpha_2 r)$.

By translation, rotation, and rescaling of Ω, we may assume that $x_0 = 0$, that $r = 1$, and that u_k and ϕ_k solve (4.19) in $\mathrm{D}_\eta(2)$. Let $1 < s < 2$, and let $g_{k,s} := u_k|_{\partial \mathrm{D}_\eta(s)}$. By Theorem 4.2.3, there exist $v_{k,s}$ and $\vartheta_{k,s}$ solving the Stokes resolvent problem with resolvent parameter λ_k in the Lipschitz domain $\Xi := \mathrm{D}_\eta(s)$ and boundary data $g_{k,s}$. Notice that the boundary datum is, in fact, the trace of u_k itself. Moreover, since u_k is Hölder continuous in $\overline{\mathrm{D}_\eta(s)} \subseteq \overline{\Omega}$, its interior non-tangential maximal function $(u_k)^*_{\mathrm{in}}$ lies in $\mathrm{L}^2(\partial \mathrm{D}_\eta(s))$ so that, by uniqueness of the solution, we must have that $u_k = v_{k,s}$.

Finally, notice that Lemma 4.3.8 gives us for every complex-valued measurable function h the validity of the estimate

$$\Big(\int_{\mathrm{D}_\eta(s)} |h|^4 \,\mathrm{d}x\Big)^{\frac{1}{4}} \le C\Big(\int_{\partial\mathrm{D}_\eta(s)} |(h)^*_{\mathrm{in}}|^2 \,\mathrm{d}\sigma(x)\Big)^{\frac{1}{2}}, \tag{4.20}$$

where the constant $C \ge 0$ only depends on the Lipschitz constant M. Thus, by virtue of inequality (4.20) and Theorem 4.2.3, we find that

$$\begin{aligned}\Big(\int_{\mathrm{D}_\eta(1)} \Big[\sum_{k=1}^{k_0} \big|(1+|\lambda_k|)\,u_k\big|^2\Big]^{\frac{4}{2}} \,\mathrm{d}x\Big)^{\frac{2}{4}} &\le C \int_{\partial\mathrm{D}_\eta(s)} \Big\{\Big(\Big[\sum_{k=1}^{k_0} \big|(1+|\lambda_k|)\,u_k\big|^2\Big]^{\frac{1}{2}}\Big)^*_{\mathrm{in}}\Big\}^2 \,\mathrm{d}\sigma(x)\\ &\le C \int_{\partial\mathrm{D}_\eta(s)} \sum_{k=1}^{k_0} |(1+|\lambda_k|)\,(u_k)^*_{\mathrm{in}}|^2 \,\mathrm{d}\sigma(x)\\ &\le C \int_{\partial\mathrm{D}_\eta(s)\setminus\mathrm{I}_\eta(s)} \sum_{k=1}^{k_0} \big|(1+|\lambda_k|)\,u_k\big|^2 \,\mathrm{d}\sigma(x).\end{aligned}$$

Notice that the constant C is independent of s. Integrating this inequality over all slices in $s \in (1,2)$, the *co-area formula*, cf. [48, Thm. 3.2.12], yields

$$\begin{aligned}\Big(\int_{D_\eta(1)} \Big[\sum_{k=1}^{k_0} |(1+|\lambda_k|)u_k|^2\Big]^{\frac{4}{2}} \,\mathrm{d}x\Big)^{\frac{2}{4}} &\le C \int_1^2 \int_{\partial\mathrm{D}_\eta(s)\setminus\mathrm{I}_\eta(s)} \sum_{k=1}^{k_0} \big|(1+|\lambda_k|)\,u_k\big|^2 \,\mathrm{d}\sigma(x)\,\mathrm{d}s\\ &\le C \int_{D_\eta(2)} \Big[\sum_{k=1}^{k_0} |(1+|\lambda_k|)u_k|^2\Big]^{\frac{2}{2}} \,\mathrm{d}x,\end{aligned}$$

which is, after taking the square root and using the inclusion relations (4.18), already the weak reverse Hölder estimate (4.16) with $p = p_0$.

The same strategy can be employed to establish this weak reverse Hölder estimate on balls $\mathrm{B}(x_0, 1)$ such that $\mathrm{B}(x_0, \alpha_2) \subseteq \Omega$. Indeed, note that, in this case, the localized Stokes resolvent system (4.19) reads

$$\begin{cases} \lambda_k u_k - \Delta u_k + \nabla \phi_k = 0 & \text{in } \mathrm{B}(x_0, \alpha_2 r),\\ \operatorname{div}(u_k) = 0 & \text{in } \mathrm{B}(x_0, \alpha_2 r).\end{cases}$$

Again, by translation, rotation, and rescaling of Ω, we may assume that $x_0 = 0$, $r = 1$ and u_k and ϕ_k are solutions of resolvent system on $\mathrm{B}(0, \alpha_2)$. In particular, by inner regularity, u_k and ϕ_k are smooth and the trace of u_k on $\partial\mathrm{B}(0, s)$ is well-defined for all $1 \le s \le \alpha_2$. Furthermore, Theorem 4.2.3 on the uniqueness of the solution to the L^2-Dirichlet problem with boundary data $g_{k,s} := u_k|_{\partial\mathrm{B}(0,s)}$ gives that $(u_k)^*_{\mathrm{in}} \in \mathrm{L}^2(\partial\mathrm{B}(0, s))$. Now we may calculate using [158, Lem 3.3] on balls $\mathrm{B}(0, s)$

$$\begin{aligned}\Big(\int_{\mathrm{B}(0,1)} \Big[\sum_{k=1}^{k_0} \big|(1+|\lambda_k|)\,u_k\big|^2\Big]^{\frac{4}{2}} \,\mathrm{d}x\Big)^{\frac{2}{4}} &\le C \int_{\partial\mathrm{B}(0,s)} \Big\{\Big(\Big[\sum_{k=1}^{k_0} \big|(1+|\lambda_k|)\,u_k\big|^2\Big]^{\frac{1}{2}}\Big)^*_{\mathrm{in}}\Big\}^2 \,\mathrm{d}\sigma(x)\\ &\le C \int_{\partial\mathrm{B}(0,s)} \sum_{k=1}^{k_0} |(1+|\lambda_k|)\,u_k|^2 \,\mathrm{d}\sigma(x)\end{aligned}$$

which, after an integration over all $s \in (1, \alpha_2)$, gives

$$\Big(\int_{\mathrm{B}(0,1)} \Big[\sum_{k=1}^{k_0} |(1+|\lambda_k|)u_k|^2 \Big]^{\frac{4}{2}} \, \mathrm{d}x \Big)^{\frac{2}{4}} \leq C \int_{\mathrm{B}(0,\alpha_2)} \Big[\sum_{k=1}^{k_0} |(1+|\lambda_k|)u_k|^2 \Big]^{\frac{2}{2}} \, \mathrm{d}x. \tag{4.21}$$

After taking the square root on both sides of (4.21), the weak reverse Hölder estimate (4.16) with $p = p_0$ and balls with an open α_2-neighborhood still contained in Ω follows.

We could now apply Theorem 4.3.6 directly and obtain the result that $T_{(\lambda_k)_{k=1}^{k_0}} \in \mathcal{L}(\mathrm{L}^2(\Omega, \ell^2(\mathbb{C}^2)))$ restrict to bounded linear operators on $\mathrm{L}^q(\Omega, \ell^2(\mathbb{C}^2))$, $2 < q < 4$, subject to the estimate

$$\|T_{(\lambda_k)_{k=1}^{k_0}} f\|_{\mathrm{L}^q(\Omega;\ell^2(\mathbb{C}^2))} \leq \mathcal{C} \, \|f\|_{\mathrm{L}^q(\Omega;\ell^2(\mathbb{C}^2))}$$

with $\mathcal{C}$ only depending on q, α_1, α_2, θ_0, R_0, and $\operatorname{diam}(\Omega)$. However, the range of possible q can be further improved by the so-called *self-improving property* of weak reverse Hölder estimates, see, e.g., [64, Thm. 6.38]. More precisely, having these weak reverse Hölder estimates at hand, the *Vitali covering lemma* gives the validity of these weak reverse Hölder estimates on *all* balls $\mathrm{B}(x_0, r)$ such that $\Omega \cap \mathrm{B}(x_0, \alpha_2 r) \neq \emptyset$, cf. the proof of [145, Lem. 4.2]. Then, the self-improving property yields the validity of (4.16) with $p = p_0 + \varepsilon$ for some $\varepsilon > 0$ depending only on the Lipschitz character of Ω and θ. As all parameters are uniform with respect to $(\lambda_k)_{k=1}^{k_0}$, we conclude that (4.15) holds for all $2 < p < 4 + \varepsilon$ with a uniform constant $C > 0$.

Step 4: the case $p < 2$. Let $1 < p < 2$ be such that its Hölder conjugate exponent p' satisfies (4.3). Since A_p is defined to be the adjoint of $A_{p'}$, the $\mathcal{R}$-sectoriality follows by duality, see [81, Lem. 3.1].

Step 5: conclusion. Having established the $\mathcal{R}$-sectoriality of A_p on $\mathrm{L}^p_\sigma(\Omega)$ for $2 \leq p < 4 + \varepsilon$ via square function estimates in Steps 1, 2, and 3 as well for $(4+\varepsilon)' < p \leq 2$ via duality in Step 4, we conclude the proof of Theorem 4.1.1, Corollary 4.1.3, and Theorem 4.1.7. ☐

Remark 4.3.10. At first sight, the choice of $p_0 = 4$ may seem arbitrary, but it is actually very important for the proof to go through, cf. Remark 4.3.9.

4.3.2 Boundedness of the H^∞-calculus

To introduce the boundedness of the H^∞-calculus, define for $\theta \in (0, \pi)$ the so-called *Dunford–Riesz class*

$$\mathrm{H}^\infty_0(\Sigma_\theta) := \Big\{ f \colon \Sigma_\theta \to \mathbb{C} : f \text{ holomorphic and } \exists\, \varepsilon, C \geq 0 : \forall z \in \Sigma_\theta : |f(z)| \leq \frac{C\,|z|^\varepsilon}{(1+|z|)^{2\varepsilon}} \Big\}.$$

Let $A \colon \operatorname{Dom}(A) \subseteq X \to X$ be a sectorial operator of angle $\omega \in [0, \pi)$ on a complex Banach space X. Then for any $\theta \in (0, \pi - \omega)$ and $\vartheta \in (\theta, \pi - \omega)$ one defines for $f \in \mathrm{H}^\infty_0(\Sigma_\theta)$

$$f(A) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \Sigma_\vartheta} f(\lambda)(\lambda - A)^{-1} \, \mathrm{d}\lambda.$$

Here, the path $\partial\Sigma_\vartheta$ is understood to be running counterclockwise. Using the sectoriality of A and the fact that $f \in \mathrm{H}^\infty_0(\Sigma_\theta)$ it is clear that $f(A) \in \mathcal{L}(X)$. If A is densely defined and has a dense range, then the question of whether there exists $C \geq 0$ such that, for all $f \in \mathrm{H}_0^\infty(\Sigma_\theta)$, one has

$$\|f(A)\|_{\mathcal{L}(X)} \leq C \sup_{z \in \Sigma_\theta} |f(z)|$$

is the question of the boundedness of the $\mathrm{H}^\infty(\Sigma_\theta)$-calculus of A [71, Prop. 5.3.4]. Our interest in the H^∞-calculus is for the connection of domains of fractional powers of A and the complex interpolation spaces between X and $\mathrm{Dom}(A)$. Indeed, given the boundedness of the $\mathrm{H}^\infty(\Sigma_\theta)$-calculus of a sectorial operator A one finds with equivalent norms that

$$\mathrm{Dom}(A^s) = \big[X, \mathrm{Dom}(A)\big]_s, \qquad s \in (0,1), \tag{4.22}$$

cf. [71, Thm. 6.6.9]. That, for $\theta \in (0, \pi)$, the $\mathrm{H}^\infty(\Sigma_\theta)$-calculus of the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ for

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| \leq \frac{1}{2d} + \varepsilon$$

and $d \geq 3$ is indeed bounded, is a result of Kunstmann and Weis [88, Thm. 16]. In the following, we will review their proof to confirm that their result stays valid in the two-dimensional case. The proof will be divided into three steps.

Step 1: the density of the domain and the range. The density of the domain of the Stokes operator was already discussed in Section 4.1.1.

Let us show that the range of the Stokes operator is dense. Since $0 \in \rho(A_p)$, we know that A_p is injective. Moreover, by the results from Subsection 4.3.1, we know that A_p is sectorial. The sectoriality combined with the reflexivity of $\mathrm{L}^p_\sigma(\Omega)$ implies the validity of the following algebraic and topological decomposition

$$\mathrm{L}^p_\sigma(\Omega) = \ker(A_p) \oplus \overline{\mathrm{ran}(A_p)},$$

see [71, Prop. 2.1.1. h)]. Here, $\ker(A_p)$ denotes the *kernel* of A_p and $\mathrm{ran}(A_p)$ its *range*. Since $\ker(A_p) = \{0\}$, this gives that $\mathrm{ran}(A_p)$ is dense in $\mathrm{L}^p_\sigma(\Omega)$.

Step 2: the comparison principle of Kunstmann and Weis. The boundedness of the $\mathrm{H}^\infty(\Sigma_\theta)$-calculus of A_p shall be deduced by that of the Dirichlet-Laplacian $-\Delta_p$ on $\mathrm{L}^p(\Omega; \mathbb{C}^2)$. That the $\mathrm{H}^\infty(\Sigma_\theta)$-calculus of $-\Delta_p$ is indeed bounded follows for example by combining the facts that the semigroup $(\mathrm{e}^{t\Delta_p})_{t\geq 0}$ satisfies heat kernel bounds, see, e.g., Davies [30, Cor. 3.2.8], and that $-\Delta_2$ has a bounded $\mathrm{H}^\infty(\Sigma_\theta)$-calculus on $\mathrm{L}^2(\Omega; \mathbb{C}^2)$ as a consequence of the spectral theorem for self-adjoint operators with a result of Duong and Robinson [40, Thm. 3.1].

The *comparison principle* of Kunstmann and Weis now reads as follows, see [88, Thm. 9].

Theorem 4.3.11 (Comparison Principle)**.** *Let X and Y be Banach spaces. Let $R\colon Y \to X$ and $S\colon X \to Y$ be bounded linear operators satisfying $RS = \mathrm{Id}_X$. Let B have a bounded $\mathrm{H}^\infty(\Sigma_\sigma)$-calculus in Y for some $\sigma \in (0, \pi)$, and let A be $\mathcal{R}$-sectorial in X. Assume that there are functions $\varphi, \psi \in \mathrm{H}_0^\infty(\Sigma_\nu) \setminus \{0\}$ where $\nu \in (0, \sigma)$ and $C_1, C_2 \geq 0$ such that, for some $\beta > 0$ and all $\ell \in \mathbb{Z}$,*

$$\sup_{1 \leq s,t \leq 2} \mathcal{R}\big\{\varphi(s2^{j+\ell}A)\, R\, \psi(t2^j B) : j \in \mathbb{Z}\big\} \leq C_1\, 2^{-\beta|\ell|} \quad \textit{and} \tag{4.23}$$

$$\sup_{1 \leq s,t \leq 2} \mathcal{R}\left\{\varphi(s2^{j+\ell}A)' S' \psi(t2^j B)' : j \in \mathbb{Z}\right\} \leq C_2\, 2^{-\beta|\ell|}. \tag{4.24}$$

Then A has a bounded $\mathrm{H}^\infty(\Sigma_\nu)$-calculus on X.

Remark 4.3.12. The original result of Kunstmann and Weis is even stronger as they actually do not assume A to be $\mathcal{R}$-sectorial but only *almost $\mathcal{R}$-sectorial*.

Kunstmann and Weis provided in [88, Prop. 10 and Prop. 11] also tools for establishing (4.23) and (4.24) which are summarized below.

Proposition 4.3.13. *In the setting of Theorem 4.3.11, suppose that there exist $\alpha_0 > 0$ and $C \geq 0$ such that, for $\alpha = \pm\alpha_0$, we have*

$$R\,\mathrm{Dom}(B^\alpha) \subseteq \mathrm{Dom}(A^\alpha), \qquad \|A^\alpha R y\|_X \leq C\,\|B^\alpha y\|_Y \quad \text{for all} \quad y \in \mathrm{Dom}(B^\alpha), \tag{4.25}$$

and

$$S\,\mathrm{Dom}(A^\alpha) \subseteq \mathrm{Dom}(B^\alpha), \qquad \|B^\alpha S x\|_Y \leq C\,\|A^\alpha x\|_X \quad \text{for all} \quad x \in \mathrm{Dom}(A^\alpha). \tag{4.26}$$

Then Condition (4.23) *and Condition* (4.24) *hold for the choice $C_1 = C_2 = C$, $\beta = \alpha_0$, and $\varphi(\lambda) = \psi(\lambda) = \lambda^{2\alpha_0}(1+\lambda)^{-4\alpha_0}$.*

Let us assume for a moment that the assumptions of Proposition 4.3.13 are verified for the choice $X = \mathrm{L}^2_\sigma(\Omega)$, $Y = \mathrm{L}^2(\Omega;\mathbb{C}^2)$, $R = \mathbb{P}_2$, S being the inclusion of $\mathrm{L}^2_\sigma(\Omega)$ into $\mathrm{L}^2(\Omega;\mathbb{C}^2)$, $A = A_2$, and $B = -\Delta_2$. Let $\varphi, \psi \in \mathrm{H}^\infty_0(\Sigma_\nu) \setminus \{0\}$ denote the functions provided by Proposition 4.3.13. Then we would find constants $C_1, C_2 > 0$ and some $\beta > 0$ such that

$$\sup_{1 \leq s,t \leq 2} \mathcal{R}\big\{\varphi(s2^{j+\ell}A_2)R\psi(-t2^j\Delta_2) : j \in \mathbb{Z}\big\} \leq C_1 2^{-\beta|\ell|} \quad \text{and} \tag{4.27}$$

$$\sup_{1 \leq s,t \leq 2} \mathcal{R}\big\{\varphi(s2^{j+\ell}A_2)'S'\psi(-t2^j\Delta_2)' : j \in \mathbb{Z}\big\} \leq C_2 2^{-\beta|\ell|} \tag{4.28}$$

for all $\ell \in \mathbb{Z}$. Let further p satisfy

$$0 < \frac{1}{2} - \frac{1}{p} < \frac{1}{4} + \varepsilon,$$

where $\varepsilon > 0$ is small enough such that A_p is $\mathcal{R}$-sectorial on $\mathrm{L}^p_\sigma(\Omega)$. Notice that also $-\Delta_p$ is $\mathcal{R}$-sectorial on $\mathrm{L}^p(\Omega;\mathbb{C}^2)$ due to the Gaussian upper bounds of the heat semigroup, see [74, Thm. 3.1]. Now, the $\mathcal{R}$-sectoriality of these two operators together with [80, Lem. 3.3] implies that the two sets

$$\big\{\varphi(s2^{j+\ell}A_p) : s > 0, j, \ell \in \mathbb{Z}\big\} \subseteq \mathcal{L}(\mathrm{L}^p_\sigma(\Omega))$$

and

$$\big\{\psi(-t2^j\Delta_p) : t > 0, j \in \mathbb{Z}\big\} \subseteq \mathcal{L}(\mathrm{L}^p(\Omega;\mathbb{C}^2))$$

are $\mathcal{R}$-bounded. Next, since singletons of bounded operators are always $\mathcal{R}$-bounded and since products of $\mathcal{R}$-bounded sets of operators are $\mathcal{R}$-bounded as well, cf. [33, Prop. 3.4], we find that also

$$\big\{\varphi(s2^{j+\ell}A_p)R\psi(-t2^j\Delta_p) : s, t > 0, j, \ell \in \mathbb{Z}\big\} \subseteq \mathcal{L}(\mathrm{L}^p(\Omega;\mathbb{C}^2), \mathrm{L}^p_\sigma(\Omega))$$

is $\mathcal{R}$-bounded. Finally, since $\mathcal{R}$-boundedness implies uniform boundedness, there exists $C \geq 0$ such that

$$\sup_{\ell \in \mathbb{Z}} \sup_{1 \leq s,t \leq 2} \mathcal{R}\big\{\varphi(s2^{j+\ell}A_p)R\psi(-t2^j\Delta_p) : j \in \mathbb{Z}\big\} \leq C.$$

Using the interpolation result in [80, Prop. 3.7] together with (4.27), one finds a constant $C \geq 0$ such that, for $2 < q < p$ with

$$\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{p} \quad \text{for some} \quad 0 < \theta < 1,$$

the following estimate holds for all $\ell \in \mathbb{Z}$

$$\sup_{1 \leq s,t \leq 2} \mathcal{R}\big\{\varphi(s2^{j+\ell}A_q)\, R\, \psi(-t2^j\Delta_q) : j \in \mathbb{Z}\big\} \leq C\, 2^{-(1-\theta)\beta|\ell|}.$$

Consequently, with the definitions $X = \mathrm{L}^q_\sigma(\Omega)$, $Y = \mathrm{L}^q(\Omega;\mathbb{C}^2)$, $R = \mathbb{P}_q$, S being the inclusion of $\mathrm{L}^q_\sigma(\Omega)$ into $\mathrm{L}^q(\Omega;\mathbb{C}^2)$, $A = A_q$, and $B = -\Delta_q$, Condition (4.23) of Theorem 4.3.11 is satisfied.

Condition (4.24) follows in a similar fashion by noticing that, due to self-adjointness, we have that

$$\psi(-t2^j\Delta_p)' = \psi(-t2^j\Delta_p') \simeq \psi(-t2^j\Delta_{p'})$$

and

$$\psi(t2^jA_p)' = \psi(t2^jA_p') \simeq \psi(t2^jA_{p'}).$$

Moreover, the dual operator S' of S is identified with an operator on $\mathrm{L}^{p'}(\Omega;\mathbb{C}^2)$ as follows. Let $\mathbf{f} \in \mathrm{L}^p(\Omega;\mathbb{C}^2)'$, and let $f \in \mathrm{L}^{p'}(\Omega;\mathbb{C}^2)$ denote its canonical identification. Then, for $g \in \mathrm{L}^{p'}_\sigma(\Omega)$, one calculates

$$\langle S'\mathbf{f}, g\rangle_{(\mathrm{L}^p_\sigma)',\mathrm{L}^p_\sigma} = \langle \mathbf{f}, Sg\rangle_{(\mathrm{L}^p_\sigma)',\mathrm{L}^p_\sigma} = \langle f, g\rangle_{\mathrm{L}^{p'},\mathrm{L}^p} = \langle f, \mathbb{P}_p g\rangle_{\mathrm{L}^{p'},\mathrm{L}^p} = \langle \mathbb{P}_{p'} f, g\rangle_{\mathrm{L}^{p'}_\sigma,\mathrm{L}^p_\sigma}.$$

Consequently, the operator

$$\varphi(s2^{j+\ell}A_p)'\, S'\, \psi(-t2^j\Delta_p)'$$

may be identified with the operator

$$\varphi(s2^{j+\ell}A_{p'})\, \mathbb{P}_{p'}\, \psi(-t2^j\Delta_{p'}).$$

Now, the same argument leading to (4.23) can be used to establish (4.24). The only difference is to use the $\mathcal{R}$-sectoriality of $A_{p'}$ on $\mathrm{L}^{p'}_\sigma(\Omega)$ and of $-\Delta_{p'}$ on $\mathrm{L}^{p'}(\Omega;\mathbb{C}^2)$. Hence, besides the verification of the conditions of Proposition 4.3.13 on the L^2-scale, this establishes the boundedness of the H^∞-calculus of the Stokes operator on the L^p_σ-scale and thus proves Theorem 4.1.6.

Step 3: verification of the conditions from Proposition 4.3.13. Let us briefly introduce a suitable scale of function spaces of smoothness $s \in \mathbb{R}$ and integrability $p \in \mathbb{R}$, the so-called *Bessel-potential spaces*, see [14, 70, 77, 116, 150, 151] for proofs of the stated results and for further information. Bessel-potential spaces can be interpreted as a generalization of Sobolev spaces $\mathrm{W}^{k,p}(\mathbb{R}^2;\mathbb{C}^2)$ where the order k is allowed to be non-integer. Consider the Fourier multiplier

$$\mathcal{F}^{-1}(1+|\cdot|^2)^{s/2}\mathcal{F}(f), \quad f \in \mathcal{S}(\mathbb{R}^2;\mathbb{C}^2). \tag{4.29}$$

Since the function $\xi \mapsto (1+|\xi|^2)^{s/2}$ is smooth and grows at most polynomially for $|\xi| \to \infty$, the multiplier (4.29) is well defined, cf. [70, Sec. 6.2.1]. If $f_s := \mathcal{F}^{-1}(1+|\cdot|^2)^{s/2}\mathcal{F}f \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C}^2)$, then one can show that f must have been in $\mathrm{L}^p(\mathbb{R}^2;\mathbb{C}^2)$ as well. Indeed,

$$f(x) = \big(\mathcal{F}^{-1}(1+|\cdot|^2)^{-s/2}\mathcal{F}f_s\big)(x) = \mathcal{J}_s(f_s)(x),$$

where $\mathcal{J}_s$ is the so-called *Bessel-potential* which is known to map $\mathrm{L}^p(\mathbb{R}^2;\mathbb{C}^2)$ to $\mathrm{L}^p(\mathbb{R}^2;\mathbb{C}^2)$, see [70, Cor. 6.1.6(a)]. One way to define Bessel-potential spaces $\mathrm{H}^{s,p}$, for $s > 0$ would be to set it as the space of all functions $f \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C}^2)$ such that there exists $f_0 \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C}^2)$ with $\mathcal{J}_s(f_0) = f$. We will however use a different approach to the definition which directly includes all $s \in \mathbb{R}$. For $s \in \mathbb{R}$ and $1 < p < \infty$, we define the *Bessel-potential space* on $\mathbb{R}^2$ by

$$\mathrm{H}^{s,p}(\mathbb{R}^2;\mathbb{C}^2) := \left\{ f \in \mathcal{S}(\mathbb{R}^2;\mathbb{C}^2)' : \mathcal{F}^{-1}(1+|\cdot|^2)^{s/2}\mathcal{F}(f) \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C}^2) \right\},$$

with the norm

$$\|f\|_{\mathrm{H}^{s,p}(\mathbb{R}^2;\mathbb{C}^2)} := \|\mathcal{F}^{-1}(1+|\cdot|^2)^{s/2}\mathcal{F}(f)\|_{\mathrm{L}^p(\mathbb{R}^2;\mathbb{C}^2)}.$$

Note that for $s \in \mathbb{N}_0$, we have $\mathrm{H}^{s,p}(\mathbb{R}^2,\mathbb{C}^2) = \mathrm{W}^{s,p}(\mathbb{R}^2,\mathbb{C}^2)$, i.e., the scale of Bessel-potential spaces extends the classical scale Sobolev spaces. The counterpart of $\mathrm{H}^{s,p}(\mathbb{R}^2;\mathbb{C}^2)$ on domains $\Omega \subseteq \mathbb{R}^2$ is defined via restriction

$$\mathrm{H}^{s,p}(\Omega;\mathbb{C}^2) := \left\{ \mathfrak{R}_\Omega(g) : g \in \mathrm{H}^{s,p}(\mathbb{R}^2;\mathbb{C}^2) \right\},$$

where $\mathfrak{R}_\Omega$ restricts distributions to Ω and the corresponding norm is given by the natural quotient norm

$$\|f\|_{\mathrm{H}^{s,p}(\Omega;\mathbb{C}^2)} := \inf_{\substack{g \in \mathrm{H}^{s,p}(\mathbb{R}^2;\mathbb{C}^2) \\ \mathfrak{R}_\Omega(g)=f}} \|g\|_{\mathrm{H}^{s,p}(\mathbb{R}^2;\mathbb{C}^2)}.$$

Indeed, also here, we have for $s \in \mathbb{N}_0$ the identification $\mathrm{H}^{s,p}(\Omega;\mathbb{C}^2) = \mathrm{W}^{s,p}(\Omega;\mathbb{C}^2)$. To incorporate traces that vanish at the boundary of Ω, one defines

$$\mathrm{H}^{s,p}_0(\Omega;\mathbb{C}^2) := \left\{ \mathfrak{R}_\Omega(g) : g \in \mathrm{H}^{s,p}(\mathbb{R}^2;\mathbb{C}^2),\ \operatorname{supp} g \subseteq \overline{\Omega} \right\}$$

with the quotient norm

$$\|f\|_{\mathrm{H}^{s,p}_0(\Omega;\mathbb{C}^2)} := \inf_{\substack{g \in \mathrm{H}^{s,p}(\mathbb{R}^2;\mathbb{C}^2) \\ \operatorname{supp} g \subseteq \overline{\Omega}}} \|g\|_{\mathrm{H}^{s,p}(\mathbb{R}^2;\mathbb{C}^2)}. \tag{4.30}$$

The spaces $\mathrm{H}^{s,p}_0(\Omega;\mathbb{C}^2)$ and $\mathrm{H}^{s,p}(\Omega;\mathbb{C}^2)$ coincide if $-1+1/p < s < 1/p$. Thus, if this condition applies, we may also write $\mathrm{H}^{s,p}_0(\Omega;\mathbb{C}^2)$ for $\mathrm{H}^{s,p}(\Omega;\mathbb{C}^2)$ if this simplifies the notation. Moreover, $\mathrm{C}^\infty_c(\Omega;\mathbb{C}^2)$ is dense in $\mathrm{H}^{s,p}_0(\Omega;\mathbb{C}^2)$ for all $s \in \mathbb{R}$ and $1 < p < \infty$. Finally, for $s > 0$, the space $\mathrm{H}^{s,p}_0(\Omega;\mathbb{C}^2)$ is reflexive. In particular, it holds for $1/p + 1/p' = 1$ that

$$\mathrm{H}^{s,p}_0(\Omega;\mathbb{C}^2)' = \mathrm{H}^{-s,p'}(\Omega;\mathbb{C}^2) \quad \text{and} \quad \mathrm{H}^{-s,p'}(\Omega;\mathbb{C}^2)' = \mathrm{H}^{s,p}_0(\Omega;\mathbb{C}^2).$$

If we consider Bessel-potential spaces as subspaces of scalar-valued tempered distributions $\mathcal{S}(\mathbb{R}^2)'$, we just write $\mathrm{H}^{s,p}(\mathbb{R}^2)$, $\mathrm{H}^{s,p}(\Omega)$, and $\mathrm{H}^{s,p}_0(\Omega)$.

The solenoidal counterparts of these spaces are defined for $s > -1 + 1/p$ as the *Stokes scale* associated with the Lipschitz domain Ω and are given by

$$\mathrm{H}^{s,p}_{0,\sigma}(\Omega;\mathbb{C}^2) := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_{\mathrm{H}^{s,p}_0(\Omega;\mathbb{C}^2)}}.$$

We remark that this definition is in line with the scale used by Mitrea and Monniaux due to their result in [116, Prop. 2.10].

We will need the following result about the Helmholtz projection on Bessel-potential spaces [116, Prop. 2.16 (II)].

Proposition 4.3.14 (Mitrea, Monniaux)**.** *For $|s| < \frac{1}{2}$, the Helmholtz projection $\mathbb{P}$ acts as a bounded linear projection on $\mathrm{H}^{s,2}(\Omega;\mathbb{C}^2)$ and yields the following topological direct sum:*

$$\mathrm{H}^{s,2}(\Omega;\mathbb{C}^2) = \mathrm{H}^{s,2}_{0,\sigma}(\Omega) \oplus \nabla \mathrm{H}^{s+1,2}(\Omega)\,.$$

Furthermore, $\mathrm{H}^{s,2}_{0,\sigma}(\Omega)$ is the range of $\mathbb{P}$ and is reflexive with $\mathrm{H}^{s,2}_{0,\sigma}(\Omega)' = \mathrm{H}^{-s,2}_{0,\sigma}(\Omega)$.

In the next step, we establish a relation between the scales of Bessel-potential spaces from Proposition 4.3.14 and suitable interpolation and extrapolation scales of Banach spaces that are induced by fractional powers of A_2 and $-\Delta_2$, see [87, Def. 15.21].

For a given Banach space X, a sectorial operator $A\colon \mathrm{Dom}(A) \to X$ with $0 \in \rho(A)$, and $\alpha \in \mathbb{R}$, we define

$$\dot{X}_{\alpha,A} := (\mathrm{Dom}(A^\alpha), \|A^\alpha \cdot \|_X)^\sim$$

to be the completion of the domain $\mathrm{Dom}(A^\alpha)$ with respect to the *homogeneous graph norm.*

Remark 4.3.15.
(i) For X reflexive, we have a natural isomorphism $\big(\dot{X}_{\alpha,A}\big)' = \big(X'\big)^{\boldsymbol{\cdot}}_{-\alpha,A'}$, see [87, Prop. 15.23]. In particular, if X is a Hilbert space and A is self-adjoint, then $\big(\dot{X}_{\alpha,A}\big)' = \dot{X}_{-\alpha,A}$ via the usual identification of A with A' via the Riesz isomorphism.

(ii) For sectorial operators with $0 \in \rho(A)$, the scale $\big(\dot{X}_{\alpha,A}\big)$ coincides with the usual *extrapolated fractional power scale of order 1* or the *interpolation-extrapolation scale* according to [4, Thm. V.1.3.8 and Thm. V.1.5.4]. In particular, for $\alpha > 0$, the domain $\mathrm{Dom}(A^\alpha)$ is already complete with respect to the homogeneous graph norm as a consequence of the closed graph theorem.

The following proposition, whose first part is due to Mitrea and Monniaux [116, Thm. 5.1], characterizes the fractional power domains of the Stokes operator A_2 and the Dirichlet-Laplacian $-\Delta_2$ on $\mathrm{L}^2(\Omega;\mathbb{C}^2)$ in terms of the Bessel-potential spaces from Proposition 4.3.14.

Proposition 4.3.16. *Let $|s| < \frac{1}{2}$. Then*

$$\big(\mathrm{L}^2_\sigma(\Omega)\big)^{\boldsymbol{\cdot}}_{s/2,A_2} = \mathrm{H}^{s,2}_{0,\sigma}(\Omega) \quad \textit{and} \quad \big(\mathrm{L}^2(\Omega;\mathbb{C}^2)\big)^{\boldsymbol{\cdot}}_{s/2,-\Delta_2} = \mathrm{H}^{s,2}_0(\Omega;\mathbb{C}^2). \tag{4.31}$$

In particular, for $s > 0$,

$$\mathrm{Dom}(A_2^{s/2}) = \mathrm{H}^{s,2}_{0,\sigma}(\Omega) \quad \textit{and} \quad \mathrm{Dom}((-\Delta_2)^{s/2}) = \mathrm{H}^{s,2}_0(\Omega;\mathbb{C}^2). \tag{4.32}$$

Proof. We only prove the facts for the Stokes operator. The corresponding facts for the Dirichlet-Laplacian follow, e.g., from [79, Sect. 7].

We start with the case $s > 0$. By virtue of the invertibility of A_2 and of Remark 4.3.15 (ii), we only have to calculate $\mathrm{Dom}(A_2^{s/2})$. Since A_2 is self-adjoint, its H^∞-calculus on $\mathrm{L}^2_\sigma(\Omega)$ is bounded, see, e.g., [33, Sect. 2.4]. Thus, employing (4.22), one finds that

$$\mathrm{Dom}(A_2^{s/2}) = \big[\mathrm{L}^2_\sigma(\Omega), \mathrm{Dom}(A_2^{1/2})\big]_s.$$

Furthermore, it is known that $\mathrm{Dom}(A_2^{1/2}) = \mathrm{W}^{1,2}_{0,\sigma}(\Omega)$, see, e.g., [137, Lem. III.2.2.1] and that the arising interpolation space is computed as

$$\big[\mathrm{L}^2_\sigma(\Omega), \mathrm{W}^{1,2}_{0,\sigma}(\Omega)\big]_s = \mathrm{H}^{s,2}_{0,\sigma}(\Omega),$$

see [116, Thm. 2.12].

Now, let $s < 0$. Using Proposition 4.3.14, the fact that A_2 is self-adjoint, the isomorphism from Remark 4.3.15 (i), and the result for the case $s > 0$ yield

$$\mathrm{H}^{s,2}_{0,\sigma}(\Omega) = \mathrm{H}^{-s,2}_{0,\sigma}(\Omega)' = \big((\mathrm{L}^2_\sigma(\Omega))^{\boldsymbol{\cdot}}_{-s/2,A_2}\big)' = \big(\mathrm{L}^2_\sigma(\Omega)\big)^{\boldsymbol{\cdot}}_{s/2,A_2}$$

which completes the proof of the statement. □

For $B_2 \coloneqq -\Delta_2$ and $s > 0$, the following diagram summarizes the interplay of Proposition 4.3.14 (vertical arrows) and Proposition 4.3.16 (horizontal arrows).

$$\begin{tikzcd}
\mathrm{H}^{s,2}_0 \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "\mathbb{P}"] & \mathrm{Dom}(B_2^{s/2}) \arrow[r, "B_2^{s/2}"] & \mathrm{L}^2 \arrow[d, "\mathbb{P}_2"] \\
\mathrm{H}^{s,2}_{0,\sigma} \arrow[r, leftrightarrow, "\simeq"] & \mathrm{Dom}(A_2^{s/2}) \arrow[r, "A_2^{s/2}"] & \mathrm{L}^2_\sigma
\end{tikzcd}$$

We have now gathered all the prerequisites needed to verify the conditions from Proposition 4.3.13. As in Step 2, we let $X = \mathrm{L}^2_\sigma(\Omega)$, $Y = \mathrm{L}^2(\Omega;\mathbb{C}^2)$, $R = \mathbb{P}_2$, and use for $S\colon X \to Y$ the inclusion map. Furthermore, we fix some $0 < \alpha_0 < 1/4$, and we carry out the proof in two separate cases depending on the sign of the parameter α.

The case $\alpha = \alpha_0 > 0$. In this case,

$$R\,\mathrm{Dom}(B_2^\alpha) = R\,\mathrm{H}^{2\alpha,2}_0(\Omega;\mathbb{C}^2) = \mathrm{H}^{2\alpha,2}_{0,\sigma}(\Omega) = \mathrm{Dom}(A_2^\alpha)$$

by (4.32), Proposition 4.3.14, and the characterization of the fractional power domains of the Dirichlet-Laplacian. Moreover, for all $y \in \mathrm{H}^{2\alpha,2}_0(\Omega;\mathbb{C}^2)$, one estimates

$$\|A_2^\alpha R y\|_{\mathrm{L}^2_\sigma(\Omega)} \lesssim \|Ry\|_{\mathrm{H}^{2\alpha,2}_{0,\sigma}(\Omega)} \lesssim \|y\|_{\mathrm{H}^{2\alpha,2}(\Omega;\mathbb{C}^2)} \lesssim \|B_2^\alpha y\|_{\mathrm{L}^2(\Omega;\mathbb{C}^2)}\,. \tag{4.33}$$

This gives Condition (4.25) for $\alpha > 0$. Similarly, we verify that

$$S\,\mathrm{Dom}(A_2^\alpha) = \mathrm{H}^{2\alpha,2}_{0,\sigma}(\Omega) \subseteq \mathrm{H}^{2\alpha,2}_0(\Omega;\mathbb{C}^2) = \mathrm{Dom}(B_2^\alpha)$$

and calculate for every $x \in \mathrm{H}^{2\alpha,2}_{0,\sigma}(\Omega)$

$$\|B_2^\alpha S x\|_{\mathrm{L}^2(\Omega;\mathbb{C}^2)} = \|B_2^\alpha x\|_{\mathrm{L}^2(\Omega;\mathbb{C}^2)} \lesssim \|x\|_{\mathrm{H}^{2\alpha,2}_{0,\sigma}(\Omega)}\,. \tag{4.34}$$

This gives Condition (4.26) for $\alpha > 0$.

The case $\alpha = -\alpha_0 < 0$. For the case of negative exponents, the inclusions on the left-hand side of (4.25) and (4.26) are straightforward since $\mathrm{Dom}(A_2^\alpha) = \mathrm{L}^2_\sigma(\Omega)$ and $\mathrm{Dom}(B_2^\alpha) = \mathrm{L}^2(\Omega;\mathbb{C}^2)$ as sets. The first inclusion follows from the classical mapping properties of the Helmholtz projection on L^2, see [137, Lem. 2.5.2], while the second one is trivial.

The boundedness estimate in Condition (4.25) follows via duality from (4.34). Indeed, let $y \in \mathrm{L}^2(\Omega;\mathbb{C}^2)$ and $g \in \mathrm{L}^2_\sigma(\Omega)$. Then

$$\langle A_2^\alpha R B_2^{-\alpha} y, g\rangle_{\mathrm{L}^2_\sigma,\mathrm{L}^2_\sigma} = \langle y, B_2^{-\alpha} S A_2^\alpha g\rangle_{\mathrm{L}^2,\mathrm{L}^2},$$

and the claim follows by taking the supremum over all g. For the remaining part of Condition (4.26), note that, since $\alpha < 0$, we have $A_2^{-\alpha} x \in \mathrm{Dom}(A_2^\alpha) = \mathrm{L}^2_\sigma(\Omega)$ implying the identity $SA_2^{-\alpha} = RA_2^{-\alpha}$. Now, the desired estimate follows via duality from (4.33) as, for $x \in \mathrm{L}^2_\sigma(\Omega)$ and $h \in \mathrm{L}^2(\Omega;\mathbb{C}^2)$, we have

$$\langle B_2^\alpha S A_2^{-\alpha} x, h\rangle_{\mathrm{L}^2,\mathrm{L}^2} = \langle x, A_2^{-\alpha} R B_2^\alpha h\rangle_{\mathrm{L}^2_\sigma,\mathrm{L}^2_\sigma}.$$

4.3.3 Domains of Fractional Powers

This section deals with the calculation of domains of fractional powers for the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$. This extends the results from the case $p = 2$ established in [116, Thm. 5.1]. The same approach could also be used to extend the results in three and higher dimensions in the L^p_σ-situation, where currently only the domains of A_p^θ are characterized for $0 \le \theta \le 1/2$. As a preparation, we state a regularity result for the Poisson problem for the Stokes system with homogeneous Dirichlet boundary conditions, initially formulated by Dindoš and Mitrea [36, Thm. 5.6] and later improved by Mitrea and Wright [117, Thm. 10.6.2]. We remark that this theorem is formulated in terms of *Besov spaces* $\mathrm{B}^s_{p,q}$ and *Triebel–Lizorkin spaces* $\mathrm{F}^s_{p,q}$ and that we present the particular case of *Bessel-potential spaces* satisfying the relation $\mathrm{H}^{s,p} = \mathrm{F}^s_{p,2}$.

Theorem 4.3.17 (Mitrea, Wright)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $0 < \delta \le 1$ depending only on Ω such that, for all $1 < p < \infty$ and $0 < s < 1$ satisfying either*

$$0 < \frac{1}{p} < s + \frac{1+\delta}{2} \quad \text{and} \quad 0 < s \le \frac{1+\delta}{2} \tag{4.35}$$

or

$$-\frac{1+\delta}{2} < \frac{1}{p} - s < \frac{1+\delta}{2} \quad \text{and} \quad \frac{1+\delta}{2} < s < 1\,, \tag{4.36}$$

the Stokes system

$$\begin{cases} -\Delta u + \nabla\phi = f & \text{in } \Omega, \\ \mathrm{div}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has for all $f \in \mathrm{H}^{s+1/p-2,p}(\Omega;\mathbb{C}^2)$ unique solutions $u \in \mathrm{H}^{s+1/p,p}(\Omega;\mathbb{C}^2)$ and $\phi \in \mathrm{H}^{s+1/p-1,p}(\Omega)$ with ϕ unique up to the addition of constants. Moreover, there exists a constant $C \ge 0$ depending only on p, s, and Ω such that the following estimate holds

$$\|u\|_{\mathrm{H}^{s+\frac{1}{p},p}(\Omega;\mathbb{C}^2)} + \|\phi\|_{\mathrm{H}^{s+\frac{1}{p}-1,p}(\Omega)} \le C\,\|f\|_{\mathrm{H}^{s+\frac{1}{p}-2,p}(\Omega;\mathbb{C}^2)}.$$

Remark 4.3.18. Let $1 < p < \infty$. If $f \in \mathrm{L}^p_\sigma(\Omega)$, then $f \in \mathrm{H}^{s+1/p-2,p}(\Omega;\mathbb{C}^2)$ for all $s \in (0,1)$. Indeed, this follows from the embedding property $\mathrm{H}^{s_1,p}(\Omega;\mathbb{C}^2) \subseteq \mathrm{H}^{s_2,p}(\Omega;\mathbb{C}^2)$, $-\infty < s_2 < s_1 < \infty$ of Bessel-potential spaces, cf. [116, Sec. 2.1]. Thus, for all $f \in \mathrm{H}^{s+1/p-2,p}(\Omega;\mathbb{C}^2)$, with s subject to (4.35) or (4.36), Theorem 4.3.17 gives the existence of a unique solution $u \in \mathrm{H}^{s+1/p,p}(\Omega;\mathbb{C}^2) \subseteq \mathrm{L}^p_\sigma(\Omega)$. In particular $u \in \mathrm{Dom}(A_p)$ such that

$$\mathrm{Dom}(A_p) \subseteq \bigcap_s \mathrm{H}^{s+\frac{1}{p},p}(\Omega;\mathbb{C}^2),$$

where the intersection is taken over all $s \in (0,1)$ that either satisfy (4.35) or (4.36). If $\delta = 1$, then (4.36) is void what implies that

$$\mathrm{Dom}(A_p) \subseteq \bigcap_{t<1+\frac{1}{p}} \mathrm{H}^{t,p}(\Omega;\mathbb{C}^2).$$

If $\delta \in (0,1)$, the first inequality in (4.36) implies that s must satisfy

$$s < \min\Big\{1,\ \frac{1}{p} + \frac{1+\delta}{2}\Big\}.$$

A calculation of the minimum reveals that

$$\mathrm{Dom}(A_p) \subseteq \bigcap_{t<1+\frac{1}{p}} \mathrm{H}^{t,p}(\Omega;\mathbb{C}^2) \quad \text{if} \quad \frac{1}{2} - \frac{1}{p} \leq \frac{\delta}{2} \tag{4.37}$$

and that

$$\mathrm{Dom}(A_p) \subseteq \bigcap_{t<\frac{2}{p}+\frac{1+\delta}{2}} \mathrm{H}^{t,p}(\Omega;\mathbb{C}^2) \quad \text{if} \quad \frac{1}{2} - \frac{1}{p} > \frac{\delta}{2}. \tag{4.38}$$

Let us note the following embedding result for the domain of the Stokes operator. The following lemma is an adapted version of [147, Lem. 2.5] for the case of $d = 2$.

Lemma 4.3.19. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that, for all*

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \frac{1}{4} + \varepsilon,$$

we have the continuous embedding

$$\mathrm{W}^{2,p}_{0,\sigma}(\Omega) \subseteq \mathrm{Dom}(A_p).$$

In particular, the representation formula

$$A_p u = -\mathbb{P}_p\, \Delta u, \qquad u \in \mathrm{W}^{2,p}_{0,\sigma}(\Omega), \tag{4.39}$$

is valid.

Proof. The proof of this in three or more dimensions is presented in [147] and carries over *mutatis mutandis* to the case of $d = 2$. For completeness, we provide the short proof. Let $u \in \mathrm{W}^{2,p}_{0,\sigma}(\Omega)$. We divide the proof into two parts.

Step 1. Assume $p \geq 2$. In particular $\Delta u \in \mathrm{L}^p(\Omega; \mathbb{C}^2)$. As a consequence of the results for the Helmholtz projection on $\mathrm{L}^p(\Omega; \mathbb{C}^2)$ proved by D. Mitrea in [114, Thm. 4.4], there exists $\Phi \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$, such that

$$-\Delta u = -\mathbb{P}_p \Delta u - (\mathrm{Id} - \mathbb{P}_p)\Delta_u = -\mathbb{P}_p \Delta u - \nabla \Phi.$$

This proves (4.39).

Step 2. Let $p < 2$ and $u \in \mathrm{W}^{2,p}_{0,\sigma}(\Omega)$ and choose a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{C}^\infty_{c,\sigma}(\Omega)$ with $u_n \to u$ in $\mathrm{W}^{2,p}(\Omega; \mathbb{C}^2)$. In particular $u_n \in \mathrm{W}^{2,2}_{0,\sigma}(\Omega)$ for all $n \in \mathbb{N}$. As $\mathbb{P}_p$ and $\mathbb{P}_2$ are compatible, formula (4.39) for A_2 shows that $(A_2 u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^p_\sigma(\Omega)$. Recall that A_p is defined as the closure of A_2 in $\mathrm{L}^p_\sigma(\Omega)$. Consequently, we have that $u \in \mathrm{Dom}(A_p)$ and conclude using the boundedness of the Helmholtz projection $\mathbb{P}_p$ that

$$A_p u = \lim_{n\to\infty} A_2 u_n = \lim_{n\to\infty} -\mathbb{P}_2 \, \Delta u_n = -\mathbb{P}_p \lim_{n\to\infty} \Delta u_n = -\mathbb{P}_p \Delta u\,.$$

This proves the validity of the representation formula (4.39). □

We are now ready to prove the embedding of the Bessel-potential spaces into domains of fractional powers $\mathrm{Dom}(A_p^\theta)$ from Theorem 4.1.4.

Theorem 4.3.20. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that, for all $1 < p < \infty$ satisfying (4.3) and all $0 < \theta < 1$, the continuous embedding*

$$\mathrm{H}^{2\theta,p}_{0,\sigma}(\Omega) \subseteq \mathrm{Dom}(A_p^\theta) \tag{4.40}$$

holds. Furthermore, there exists $0 < \delta \leq 1$ such that, if θ and p additionally satisfy either

$$\theta < \frac{1}{2} + \frac{1}{2p} \quad \text{if} \quad \frac{1}{2} - \frac{1}{p} \leq \frac{\delta}{2} \tag{4.41}$$

or

$$\theta < \frac{1}{p} + \frac{1+\delta}{4} \quad \text{if} \quad \frac{1}{2} - \frac{1}{p} > \frac{\delta}{2}\,, \tag{4.42}$$

we have with equivalent norms that

$$\mathrm{H}^{2\theta,p}_{0,\sigma}(\Omega) = \mathrm{Dom}(A_p^\theta).$$

Proof. The boundedness of the H^∞-calculus of A_p by Theorem 4.1.6 and its consequence for complex interpolation (4.22) imply that

$$\big[\mathrm{L}^p_\sigma(\Omega), \mathrm{Dom}(A_p^\alpha)\big]_\theta = \mathrm{Dom}(A_p^{\alpha\theta}) \tag{4.43}$$

for $\alpha > 0$. Moreover, we find by Lemma 4.3.19 the continuous embedding

$$\big[\mathrm{L}^p_\sigma(\Omega), \mathrm{W}^{2,p}_{0,\sigma}(\Omega)\big]_\theta \subseteq \big[\mathrm{L}^p_\sigma(\Omega), \mathrm{Dom}(A_p)\big]_\theta. \tag{4.44}$$

Due to the interpolation result in [116, Thm. 2.12], it is known that the interpolation space on the left-hand side of (4.44) coincides with $\mathrm{H}^{2\theta,p}_{0,\sigma}(\Omega)$. Taking $\alpha = 1$ in (4.43) results in the continuous embedding stated in (4.40). Let us restate this embedding with r instead of p and s instead of θ for further reference:

For any $1 < r < \infty$ *satisfying* (4.3) *and any* $s \in (0, 1)$ *it holds that*

$$\mathrm{H}^{2s,r}_{0,\sigma}(\Omega) \subseteq \mathrm{Dom}(A_r^s). \tag{4.45}$$

Now we prove the converse inclusion to (4.45). To this end, let $\delta \in (0, 1]$ be given by Theorem 4.3.17. We assume first that θ fulfills both estimates (4.41) and (4.42) simultaneously and with an additional lower bound, i.e.,

$$\frac{1-\delta}{4} + \frac{1}{2p} < \theta < \min\Big\{\frac{1}{2} + \frac{1}{2p}, \frac{1}{p} + \frac{1+\delta}{4}\Big\}. \tag{4.46}$$

A straightforward calculation shows that the minimum can be calculated as

$$\min\Big\{\frac{1}{2} + \frac{1}{2p}, \frac{1}{p} + \frac{1+\delta}{4}\Big\} = \begin{cases} \frac{1}{2} + \frac{1}{2p} & \text{if } \frac{1}{2} - \frac{1}{p} \leq \frac{\delta}{2}, \\ \frac{1}{p} + \frac{1+\delta}{4} & \text{if } \frac{1}{2} - \frac{1}{p} > \frac{\delta}{2}. \end{cases}$$

We want to deduce the desired embedding (4.45) by establishing that the fractional power

$$A_p^{-\theta} \colon \mathrm{L}^p_\sigma(\Omega) \to \mathrm{H}^{2\theta,p}_{0,\sigma}(\Omega). \tag{4.47}$$

is well-defined and bounded. Indeed, *per definitionem*, for $u \in \mathrm{Dom}(A_p^\theta)$, we have $A_p^\theta u \in \mathrm{L}^p_\sigma(\Omega)$. The validity of (4.47) now implies $u = A_p^{-\theta} A_p^\theta u \in \mathrm{H}^{2\theta,p}_0(\Omega; \mathbb{C}^2)$.

Let us establish (4.47). Since

$$\mathrm{H}^{2\theta,p}_{0,\sigma}(\Omega) = \mathrm{H}^{2\theta,p}_0(\Omega; \mathbb{C}^2) \cap \mathrm{L}^p_\sigma(\Omega)$$

holds due to [116, Cor. 2.11] and since $A_p^{-\theta}$ is bounded from $\mathrm{L}^p_\sigma(\Omega)$ to $\mathrm{L}^p_\sigma(\Omega)$, it suffices to prove the boundedness of

$$A_p^{-\theta} \colon \mathrm{L}^p_\sigma(\Omega) \to \mathrm{H}^{2\theta,p}_0(\Omega; \mathbb{C}^2). \tag{4.48}$$

The following diagram summarizes the mapping properties of the operator $A_p^{-\theta}$, including the claim (4.48).

$$\begin{tikzcd} & \mathrm{L}^p_\sigma \arrow[dl, bend right, "A_p^{-\theta}"'] \arrow[dr, bend left, "A_p^{-\theta}"] & \\ \mathrm{L}^p_\sigma & \cap & \mathrm{H}^{2\theta}_0 \\ & = \mathrm{H}^{2\theta}_{0,\sigma} & \end{tikzcd}$$

In order to verify the boundedness property (4.48), we find by the self-adjointness of A_2 and of the projection $\mathbb{P}_2$ for $f \in \mathrm{C}^\infty_{\mathrm{c},\sigma}(\Omega)$ and $g \in \mathrm{C}^\infty_\mathrm{c}(\Omega;\mathbb{C}^2)$ that

$$\begin{aligned} \Big| \int_\Omega A_p^{-\theta} f \ \cdot \ \overline{g} \ \mathrm{d}x \Big| &= \Big| \int_\Omega \mathbb{P}_2 A_2^{-\theta} f \ \cdot \ \overline{g} \ \mathrm{d}x \Big| = \Big| \int_\Omega f \cdot \overline{A_{p'}^{-\theta}\, \mathbb{P}_{p'} g} \ \mathrm{d}x \Big| \\ &\leq \|f\|_{\mathrm{L}^p_\sigma(\Omega)} \, \|A_{p'}^{-\theta}\, \mathbb{P}_{p'} g\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^2)} \, . \end{aligned} \tag{4.49}$$

Since $\mathrm{H}^{-2\theta,p'}(\Omega;\mathbb{C}^2)' = \mathrm{H}^{2\theta,p}_0(\Omega;\mathbb{C}^2)$ and since $\mathrm{C}^\infty_\mathrm{c}(\Omega;\mathbb{C}^2)$ is dense in $\mathrm{H}^{-2\theta,p'}(\Omega;\mathbb{C}^2)$, cf. [150, Thm. 3.5(i)], the boundedness property (4.48) follows from (4.49) once it is shown that there exists $C \geq 0$ such that, for all $g \in \mathrm{C}^\infty_\mathrm{c}(\Omega;\mathbb{C}^2)$, it holds

$$\|A_{p'}^{-\theta} \mathbb{P}_{p'} g\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^2)} \leq C \, \|g\|_{\mathrm{H}^{-2\theta,p'}(\Omega;\mathbb{C}^2)}. \tag{4.50}$$

To this end, we find by virtue of the first inequality in (4.46) combined with Remark 4.3.18 that

$$\operatorname{ran}(A_{p'}^{-1}) = \operatorname{Dom}(A_{p'}) \subseteq \mathrm{H}^{2(1-\theta),p'}_{0,\sigma}(\Omega). \tag{4.51}$$

Indeed, if p is subject to (4.46), then the following calculation shows for $\delta \in (0,1]$

$$\frac{1}{2p} \leq \frac{1-\delta}{4} + \frac{1}{2p} \overset{(4.46)}{<} \theta \iff 1 - \frac{1}{p'} = \frac{1}{p} < 2\theta \iff 2(1-\theta) < 1 + \frac{1}{p'} \, .$$

This settles the embedding questions if $\frac{1}{2} - \frac{1}{p} \leq \frac{\delta}{2}$ via (4.37) if we choose $s = 2(1-\theta)$ and $p = p'$ in Remark 4.3.18. If p is subject to the condition $\frac{1}{2} - \frac{1}{p} > \frac{\delta}{2}$, we have that $\frac{1}{2p} < \frac{1-\delta}{4}$ and can further estimate

$$\frac{3}{2} - \frac{3}{2p'} = \frac{3}{2p} < \frac{1-\delta}{4} + \frac{1}{p} \overset{(4.46)}{<} \theta \iff 2 - \frac{4\theta}{3} < \frac{2}{p'} < \frac{2}{p'} + \frac{1+\delta}{2}$$

which shows that, for the choice of $s = 2(1-\theta)$ and $p = p'$ in Remark 4.3.18, we have the desired embedding via (4.38).

The embedding (4.51) enables us to use (4.45) with $r = p'$ and $s = 1 - \theta$ to deduce that

$$\|A_{p'}^{-\theta}\, \mathbb{P}_{p'} g\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^2)} = \|A_{p'}^{1-\theta}\, A_{p'}^{-1}\, \mathbb{P}_{p'} g\|_{\mathrm{L}^{p'}(\Omega;\mathbb{C}^2)} \leq C \, \|A_{p'}^{-1}\, \mathbb{P}_{p'} g\|_{\mathrm{H}^{2-2\theta,p'}_{0,\sigma}(\Omega;\mathbb{C}^2)}.$$

Define $u := A_{p'}^{-1} \mathbb{P}_{p'} g$ and let ϕ denote the associated pressure. Then u and ϕ solve the Stokes system given by

$$\begin{cases} -\Delta u + \nabla \phi = \mathbb{P}_{p'} g & \text{in } \Omega, \\ \quad \operatorname{div}(u) = 0 & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.52}$$

Now, by virtue of the Helmholtz decomposition of $\mathrm{L}^p(\Omega;\mathbb{C}^2)$, see [114, Thm. 4.4], the function $\mathbb{P}_{p'} g$ is given by $g + \nabla h$ for some $h \in \mathrm{W}^{1,p'}(\Omega)$. It follows that u and $\phi - h$ solve system (4.52) but with right-hand side g instead of $\mathbb{P}_{p'} g$. Now, define $s := 1 + 1/p - 2\theta$. Then the conditions imposed on θ in (4.46) imply that s and p' satisfy (4.35) so that Theorem 4.3.17 implies that

$$\|u\|_{\mathrm{H}^{2-2\theta,p'}_{0,\sigma}(\Omega)} \leq C \, \|g\|_{\mathrm{H}^{-2\theta,p'}(\Omega;\mathbb{C}^2)}$$

which in turn shows that (4.50) holds. This concludes the proof in the case where θ satisfies the lower bound in (4.46).

To get rid of the lower bound in (4.46), we proceed via interpolation. Note that the interpolation result on the Stokes scale [116, Thm. 2.12] gives

$$\Big[\mathrm{H}^{s_0,p_0}_{0,\sigma}(\Omega), \mathrm{H}^{s_1,p_1}_{0,\sigma}(\Omega)\Big]_{\vartheta} = \mathrm{H}^{s,p}_{0,\sigma}(\Omega) \tag{4.53}$$

for all $1 < p < \infty$, $-1 + \frac{1}{p_i} < s_i$, $i \in \{0,1\}$, $\vartheta \in [0,1]$, $\frac{1}{p} := \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}$, and $s := (1-\vartheta)s_0 + \vartheta s_1$.

Now, for general θ subject to (4.41) or (4.42), chose a *good* value α with

$$\alpha > 0 \text{ subject to the upper and lower bounds in (4.46) satisfying } 0 < \theta < \alpha. \tag{α gd.}$$

Then we calculate using (4.53), the general interpolation result for fractional powers (4.43), and the corresponding result for α

$$\begin{aligned}\mathrm{Dom}(A_p^{\theta}) &= \mathrm{Dom}(A_p^{\alpha\frac{\theta}{\alpha}}) \\ &\overset{(4.43)}{=} \Big[\mathrm{L}^p_\sigma(\Omega), \mathrm{Dom}(A^\alpha)\Big]_{\frac{\theta}{\alpha}} \overset{(\alpha\text{ gd.})}{=} \Big[\mathrm{H}^{0,p}_{0,\sigma}(\Omega), \mathrm{H}^{2\alpha,p}_{0,\sigma}(\Omega)\Big]_{\frac{\theta}{\alpha}} \overset{(4.53)}{=} \mathrm{H}^{2\alpha\frac{\theta}{\alpha},p}_{0,\sigma}(\Omega) = \mathrm{H}^{2\theta,p}_{0,\sigma}(\Omega).\end{aligned}$$

This proves the result also for general $\theta > 0$. □

4.3.4 The Weak Stokes Operator

In this section, we expand our functional analytic framework by a Stokes-like operator on spaces of regular distributions. This operator will prove to be helpful in Section 4.4 for establishing higher regularity of solutions to the Navier–Stokes equations with the right-hand side in divergence form, cf. [26, Sect. 7].

For the course of this section, let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain and $\varepsilon > 0$ such that, for all p that satisfy (4.3), the operator $-A_p$ generates a bounded analytic semigroup. Moreover, for $1/p + 1/p' = 1$, let $\Phi\colon \big[\mathrm{L}^{p'}_\sigma(\Omega)\big]^* \to \mathrm{L}^p_\sigma(\Omega)$ denote the canonical isomorphism between the *anti*dual $\big[\mathrm{L}^{p'}_\sigma(\Omega)\big]^*$ and $\mathrm{L}^p_\sigma(\Omega)$ with the duality pairing

$$\langle \Phi^{-1} u, v\rangle_{[\mathrm{L}^{p'}_\sigma]^*, \mathrm{L}^{p'}_\sigma} = \langle u, v\rangle_{\mathrm{L}^p_\sigma, \mathrm{L}^{p'}_\sigma} = \int_\Omega u \cdot \overline{v} \; \mathrm{d}x, \quad u \in \mathrm{L}^p_\sigma(\Omega),\, v \in \mathrm{L}^{p'}_\sigma(\Omega).$$

We regard Φ^{-1} also as the canonical inclusion of $\mathrm{L}^p_\sigma(\Omega)$ into $\mathrm{W}^{-1,p}_\sigma(\Omega)$ via

$$\langle \Phi^{-1} u, v\rangle_{\mathrm{W}^{-1,p}_\sigma, \mathrm{W}^{1,p'}_{0,\sigma}} = \langle u, v\rangle_{\mathrm{L}^p_\sigma, \mathrm{L}^{p'}_\sigma}, \quad u \in \mathrm{L}^p_\sigma(\Omega),\, v \in \mathrm{W}^{1,p'}_{0,\sigma}(\Omega).$$

In this sense, we define the *weak Stokes operator* $\mathcal{A}_p$ in $\mathrm{W}^{-1,p}_\sigma(\Omega)$ by $\mathrm{Dom}(\mathcal{A}_p) := \Phi^{-1}\mathrm{W}^{1,p}_{0,\sigma}(\Omega)$ and

$$\mathcal{A}_p\colon \mathrm{Dom}(\mathcal{A}_p) \subseteq \mathrm{W}^{-1,p}_\sigma(\Omega) \to \mathrm{W}^{-1,p}_\sigma(\Omega), \quad w \mapsto \Big[v \mapsto \int_\Omega \nabla\Phi w \cdot \overline{\nabla v} \; \mathrm{d}x\Big].$$

Recall that, by Theorem 4.3.20 and the invertibility of A_p', cf. Theorem 4.1.1 and Remark 4.3.4(iii), the square root of the Stokes operator is an isomorphism

$$A_{p'}^{\frac{1}{2}}\colon \mathrm{W}^{1,p'}_{0,\sigma}(\Omega) \to \mathrm{L}^{p'}_\sigma(\Omega)\,.$$

This fact allows to deduce the relation

$$\big[A_{p'}^{\frac{1}{2}}\big]^* \Phi^{-1} \in \mathrm{Isom}(\mathrm{L}^p_\sigma(\Omega), \mathrm{W}^{-1,p}_\sigma(\Omega))$$

and use the operator $\big[A_{p'}^{\frac{1}{2}}\big]^* \Phi^{-1}$ as a similarity transform that connects A_p and $\mathcal{A}_p$, namely

$$\mathcal{A}_p = \big[A_{p'}^{\frac{1}{2}}\big]^* \Phi^{-1} \circ A_p \circ A_p^{-\frac{1}{2}} \Phi = \big[A_{p'}^{\frac{1}{2}}\big]^* \Phi^{-1} \circ A_p \circ \Phi \big[A_{p'}^{-\frac{1}{2}}\big]^*. \tag{4.54}$$

The representation formulas (4.54) were derived in [26, Lem. 5.1] for the case $d = 3$. However, the presented proof carries over *mutatis mutandis* to the case $d = 2$.

Permanence properties of the class of $\mathcal{R}$-sectorial operators dictate that $\mathcal{A}_p$ inherits the $\mathcal{R}$-sectoriality of A_p, see, e.g., [33, Sect. 4.1]. In particular, the representation (4.54) induces a representation of the resolvents

$$(\lambda + \mathcal{A}_p)^{-1} = \big[A_{p'}^{\frac{1}{2}}\big]^* \Phi^{-1} \circ (\lambda + A_p)^{-1} \circ \Phi \big[A_{p'}^{-\frac{1}{2}}\big]^*, \tag{4.55}$$

see the proof of [33, Prop. 1.3 (iv)]. The following result is a corollary of (4.55) in terms of a similar semigroup and its implication on maximal regularity.

Proposition 4.3.21. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that, for all p that satisfy* (4.3)*, the following statements are valid.*

(i) $\rho(\mathcal{A}_p) = \rho(A_p)$ *and* $-\mathcal{A}_p$ *generates a bounded analytic semigroup* $(\mathrm{e}^{-t\mathcal{A}_p})_{t\geq 0}$ *on* $\mathrm{W}^{-1,p}_\sigma(\Omega)$.

(ii) *For* $u \in \mathrm{W}^{-1,p}_\sigma(\Omega)$ *and* $f \in \mathrm{L}^p_\sigma(\Omega)$, *the following two identities hold for all* $t \geq 0$

$$\mathrm{e}^{-t\mathcal{A}_p} u = \big[A_{p'}^{\frac{1}{2}}\big]^* \Phi^{-1} \mathrm{e}^{-tA_p} \Phi \big[A_{p'}^{-\frac{1}{2}}\big]^* u \quad \textit{and} \quad \Phi^{-1} \mathrm{e}^{-tA_p} f = \mathrm{e}^{-t\mathcal{A}_p} \Phi^{-1} f.$$

In particular, the weak Stokes semigroups are consistent on the $\mathrm{W}^{-1,p}_\sigma$*-scale.*

(iii) $\mathcal{A}_p$ *has maximal* L^q*-regularity for* $1 < q < \infty$.

4.4 Global Strong Solutions to the Navier–Stokes Equations in Planar Lipschitz Domains

This section is devoted to proving the regularity properties of Leray–Hopf weak solutions to the Navier–Stokes equations (4.9) stated in Theorem 4.1.8. Let

$$u \in \mathrm{LH}_\infty(\Omega) = \mathrm{L}^\infty(0,\infty; \mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0,\infty; \mathrm{W}^{1,2}_{0,\sigma}(\Omega))$$

be such a solution. Sufficient conditions on u_0 and f for the existence of a solution u are that $u_0 \in \mathrm{L}^2_\sigma(\Omega)$ and that $f = f_0 + \mathbb{P}_2 \operatorname{div}(F)$ with $f_0 \in \mathrm{L}^1(0,\infty; \mathrm{L}^2_\sigma(\Omega))$ and $F \in \mathrm{L}^2(0,\infty; \mathrm{L}^2(\Omega; \mathbb{C}^{2\times 2}))$, see, e.g., [137, Thm. V.3.1.1]. These solutions further satisfy the *energy inequality*

$$\begin{aligned} E_\infty(u) &:= \frac{1}{2}\|u\|^2_{\mathrm{L}^\infty(0,\infty;\mathrm{L}^2(\Omega;\mathbb{C}^2))} + \|\nabla u\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(\Omega;\mathbb{C}^{2\times 2}))} \\ &\leq 2\|u_0\|^2_{\mathrm{L}^2_\sigma(\Omega)} + 4\|F\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(\Omega;\mathbb{C}^{2\times 2}))} + 8\|f_0\|^2_{\mathrm{L}^1(0,\infty;\mathrm{L}^2_\sigma(\Omega))}, \end{aligned}$$

see, e.g., [137, Eq. V.(3.1.13)]. The key to establishing the higher regularity result for Leray–Hopf weak solutions lies in the following two nonlinear estimates whose proofs can be found in [137, Lem. V.1.2.1 and Rem. V.1.2.2].

Lemma 4.4.1. (i) *For all $1 \leq s < 2$, there exists a constant $C \geq 0$ depending only on s such that, for all $1 \leq p < 2$ satisfying*

$$\frac{1}{p} + \frac{1}{s} = \frac{3}{2}$$

and all

$$u \in \mathrm{L}^\infty(0, \infty; \mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0, \infty; \mathrm{W}^{1,2}_{0,\sigma}(\Omega)),$$

one has

$$\|(u \cdot \nabla)u\|_{\mathrm{L}^s(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^2))} \leq C E_\infty(u).$$

(ii) *For all $1 \leq s \leq \infty$, there exists a constant $C \geq 0$ depending only on s such that, for all $1 \leq p < \infty$ satisfying*

$$\frac{1}{p} + \frac{1}{s} = 1$$

and all

$$u \in \mathrm{L}^\infty(0, \infty; \mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0, \infty; \mathrm{W}^{1,2}_{0,\sigma}(\Omega)),$$

one has

$$\|u \otimes u\|_{\mathrm{L}^s(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^{2\times 2}))} \leq C E_\infty(u).$$

Relying on the maximal regularity results proved in Section 4.3.1, we are now in the position to prove Theorem 4.1.8 which we restate here for convenience.

Theorem 4.1.8. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$, depending only on the Lipschitz geometry of Ω, such that the following statements are valid.*

(a) *For all $1 < s < 2$ and $1 < p < 2$ that satisfy*

$$1 - \frac{1}{s} = \frac{1}{p} - \frac{1}{2} < \frac{1}{4} + \varepsilon$$

and all Leray–Hopf weak solutions

$$u \in \mathrm{L}^\infty(0, \infty; \mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0, \infty; \mathrm{W}^{1,2}_{0,\sigma}(\Omega))$$

to (4.9) with initial data u_0 and force $f = f_0$ satisfying

$$u_0 \in \big(\mathrm{L}^p_\sigma(\Omega), \mathrm{Dom}(A_p)\big)_{1-\frac{1}{s},s} \quad \textit{and} \quad f_0 \in \mathrm{L}^s(0, \infty; \mathrm{L}^p_\sigma(\Omega)),$$

one has that

$$u \in \mathrm{W}^{1,s}(0, \infty; \mathrm{L}^p_\sigma(\Omega)) \cap \mathrm{L}^s(0, \infty; \mathrm{Dom}(A_p)).$$

(b) *For all* $1 < p < \infty$ *that satisfy* (4.3), *all* $1 < s < \infty$ *that satisfy*

$$\frac{1}{p} + \frac{1}{s} = 1,$$

and all Leray–Hopf weak solutions

$$u \in \mathrm{L}^\infty(0,\infty;\mathrm{L}^2_\sigma(\Omega)) \cap \mathrm{L}^2(0,\infty;\mathrm{W}^{1,2}_{0,\sigma}(\Omega))$$

to (4.9) *with initial data* u_0 *and force* $f = \mathbb{P}_2 \operatorname{div}(F)$ *satisfying*

$$u_0 \in \big(\mathrm{W}^{-1,p}_\sigma(\Omega), \mathrm{W}^{1,p}_{0,\sigma}(\Omega)\big)_{1-\frac{1}{s},s} \quad \text{and} \quad F \in \mathrm{L}^s(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^{2\times 2})),$$

one has that

$$u \in \mathrm{W}^{1,s}(0,\infty;\mathrm{W}^{-1,p}_\sigma(\Omega)) \cap \mathrm{L}^s(0,\infty;\mathrm{W}^{1,p}_{0,\sigma}(\Omega)).$$

Proof. Ad (a). First of all, notice that, as a consequence of [137, Thm. IV.2.4.1], one has that

$$\int_0^t \mathrm{e}^{-(t-\tau)A_2} A_2^{-\frac{1}{2}} \mathbb{P}_2 \operatorname{div}(u(\tau)\otimes u(\tau)) \,\mathrm{d}\tau \in \operatorname{Dom}(A_2^{\frac{1}{2}})$$

for almost every $t > 0$ and the weak solution u fulfills the integral equation

$$u(t) = \mathrm{e}^{-tA_2} u_0 + \int_0^t \mathrm{e}^{-(t-\tau)A_2} f_0(\tau)\,\mathrm{d}\tau - A_2^{\frac{1}{2}} \int_0^t \mathrm{e}^{-(t-\tau)A_2} A_2^{-\frac{1}{2}} \mathbb{P}_2 \operatorname{div}(u(\tau)\otimes u(\tau))\,\mathrm{d}\tau,$$

see [137, Thm. V.1.3.1]. Since the Stokes semigroups are consistent on the L^p_σ-scale and since by assumption $u_0 \in \mathrm{L}^p_\sigma(\Omega)$ and $f_0 \in \mathrm{L}^s(0,\infty;\mathrm{L}^p_\sigma(\Omega))$, one can replace the L^2_σ-semigroups by the L^p_σ-semigroups. Notice further that $\operatorname{div}(u(\tau)\otimes u(\tau)) = (u(\tau)\cdot\nabla)u(\tau)$ since u is divergence-free. Now, Lemma 4.4.1(i) implies that $(u\cdot\nabla)u \in \mathrm{L}^s(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^2))$. In particular, for almost every $0 < \tau < t$, one has that $(u(\tau)\cdot\nabla)u(\tau) \in \mathrm{L}^p(\Omega;\mathbb{C}^2)$ which implies that

$$\mathrm{e}^{-(t-\tau)A_2} A_2^{-\frac{1}{2}} \mathbb{P}_2 \operatorname{div}\big(u(\tau)\otimes u(\tau)\big) = A_2^{-\frac{1}{2}} \mathrm{e}^{-(t-\tau)A_p} \mathbb{P}_p (u(\tau)\cdot\nabla)u(\tau). \tag{4.56}$$

Note that, in identity (4.56), the semigroup generated by $-A_2$ and the square root of A_2 commute as a consequence of the functional calculus for A_2. Furthermore, we also used the consistency of the Helmholtz projections $\mathbb{P}_p$ here which is a consequence of the uniqueness of the Helmholtz decomposition on $\mathrm{L}^p(\Omega;\mathbb{C}^2)$. Since $A_2^{-\frac{1}{2}}$ is a bounded operator on $\mathrm{L}^2_\sigma(\Omega)$, one can pull this operator in front of the integral, yielding

$$u(t) = \mathrm{e}^{-tA_p} u_0 + \int_0^t \mathrm{e}^{-(t-s)A_p} f_0(\tau)\,\mathrm{d}\tau - \int_0^t \mathrm{e}^{-(t-\tau)A_p} \mathbb{P}_p (u(\tau)\cdot\nabla)u(\tau)\,\mathrm{d}\tau.$$

Consequently, u is a mild solution to the *linear* equation

$$\begin{cases} u'(t) + A_p u(t) = f_0(t) - \mathbb{P}_p(u(t)\cdot\nabla)u(t), & t > 0, \\ \qquad\qquad u(0) = u_0. \end{cases}$$

Note that we have $f_0 - \mathbb{P}_p(u\cdot\nabla)u \in \mathrm{L}^s(0,\infty;\mathrm{L}^p_\sigma(\Omega))$. By Theorem 4.1.7, the Stokes operator on $\mathrm{L}^p_\sigma(\Omega)$ has maximal L^s-regularity and thus fulfills

$$u \in \mathrm{W}^{1,s}(0,\infty;\mathrm{L}^p_\sigma(\Omega)) \cap \mathrm{L}^s(0,\infty;\mathrm{Dom}(A_p)).$$

Ad (b). Let u be the Leray–Hopf weak solution corresponding to the initial value $u(0) = u_0$ and the right-hand side $f = \mathbb{P}_2 \operatorname{div}(F)$. In particular, u can be represented via the representation formula

$$u(t) = \mathrm{e}^{-tA_2}u_0 + A_2^{\frac{1}{2}} \int_0^t \mathrm{e}^{-(t-\tau)A_2} A_2^{-\frac{1}{2}}\, \mathbb{P}_2 \operatorname{div}\big(F(\tau) - u(\tau)\otimes u(\tau)\big)\, \mathrm{d}\tau, \tag{4.57}$$

for almost every $t > 0$. Now, Lemma 4.4.1(ii) implies that $u\otimes u \in \mathrm{L}^s(0,\infty;\mathrm{L}^p(\Omega;\mathbb{C}^{2\times 2}))$. As in Subsection 4.3.4, let again $\Phi\colon [\mathrm{L}^{p'}_\sigma(\Omega)]^* \to \mathrm{L}^p_\sigma(\Omega)$ denote the canonical isomorphism between $[\mathrm{L}^{p'}_\sigma(\Omega)]^*$ and $\mathrm{L}^p_\sigma(\Omega)$. We will need the identity

$$\Phi^{-1} A_2^{-\frac{1}{2}}\, \mathbb{P}_2 \operatorname{div}(F) = \big[A_2^{-\frac{1}{2}}\big]^*\, \mathbb{P}_2 \operatorname{div}(F) \tag{4.58}$$

on $\mathrm{W}^{-1,2}_\sigma(\Omega)$. Indeed, the characterization of $A_2^{-\frac{1}{2}}\, \mathbb{P}_2 \operatorname{div}$ by means of functionals as described in [137, Lem. III.2.6.2] gives that, for $F \in \mathrm{L}^2(\Omega;\mathbb{C}^{2\times 2})$ and $v \in \mathrm{C}^\infty_{0,\sigma}(\Omega)$,

$$\begin{aligned}
\big\langle\, \Phi^{-1} A_2^{-\frac{1}{2}}\, \mathbb{P}_2 \operatorname{div}(F), v \,\big\rangle_{\mathrm{W}^{-1,2}_\sigma, \mathrm{W}^{1,2}_{0,\sigma}} &= \big\langle\, A_2^{-\frac{1}{2}}\, \mathbb{P}_2 \operatorname{div}(F), v \,\big\rangle_{\mathrm{L}^2_\sigma, \mathrm{L}^2_\sigma} \\
&= \big\langle\, \mathbb{P}_2 \operatorname{div}(F), A_2^{-\frac{1}{2}} v \,\big\rangle_{\mathrm{W}^{-1,2}_\sigma, \mathrm{W}^{1,2}_{0,\sigma}} \\
&= \big\langle\, \big[A_2^{-\frac{1}{2}}\big]^*\, \mathbb{P}_2 \operatorname{div}(F), v \,\big\rangle_{\mathrm{W}^{-1,2}_\sigma, \mathrm{W}^{1,2}_{0,\sigma}}.
\end{aligned}$$

If we now bring the representation of the semigroup for the weak Stokes operator $\mathcal{A}_2$ from Proposition 4.3.21(ii) together with identity (4.58), we see that

$$\Phi^{-1} A_2^{\frac{1}{2}} \mathrm{e}^{-tA_2} A_2^{-\frac{1}{2}}\, \mathbb{P}_2 \operatorname{div}(F) = \mathrm{e}^{-t\mathcal{A}_2}\, \mathbb{P}_2 \operatorname{div}(F) \tag{4.59}$$

for all $F \in \mathrm{L}^2(\Omega;\mathbb{C}^{2\times 2})$ and $t > 0$. Following the idea of the proof of [26, Thm. 3.3], applying the embedding Φ^{-1} to equation (4.57), and using (4.59) and the consistency of the semigroups $(\mathrm{e}^{-t\mathcal{A}_p})_{t\geq 0}$, see Proposition 4.3.21(ii), yields

$$\Phi^{-1}u(t) = \mathrm{e}^{-t\mathcal{A}_p}\, \Phi^{-1}u_0 + \int_0^t \mathrm{e}^{-(t-\tau)\mathcal{A}_p}\, \mathbb{P}_p \operatorname{div}\big(F(\tau) - u(\tau)\otimes u(\tau)\big)\, \mathrm{d}\tau$$

for almost every $t > 0$. Consequently, $\Phi^{-1}u$ is a mild solution to the *linear* equation

$$\begin{cases} \mathcal{U}'(t) + \mathcal{A}_p \mathcal{U}(t) = \mathbb{P}_p \operatorname{div}\big(F(t) - u(t)\otimes u(t)\big), & t > 0, \\ \qquad\qquad \mathcal{U}(0) = \Phi^{-1}u_0. \end{cases}$$

Proposition 4.3.21(iii) on the maximal regularity of the weak Stokes operator now implies

$$\Phi^{-1}u \in \mathrm{W}^{1,s}(0,\infty;\mathrm{W}^{-1,p}_\sigma(\Omega)) \cap \mathrm{L}^s(0,\infty;\Phi^{-1}\mathrm{W}^{1,p}_{0,\sigma}(\Omega)).$$

We arrive at the desired regularity result by translating this result into terms of u. $\square$

Combining Theorem 4.1.8 with embeddings of $\mathrm{Dom}(A_p)$ into Bessel-potential spaces leads to the following corollary.

Corollary 4.4.2. *In the situation of Theorem 4.1.8*(a)*, the solution u satisfies*

$$u \in \mathrm{W}^{1,s}(0,\infty;\mathrm{L}^p_\sigma(\Omega)) \cap \mathrm{L}^s(0,\infty;\mathrm{H}^{\alpha,p}_{0,\sigma}(\Omega))$$

for any $0 < \alpha < 1 + 1/p$.

Proof. Notice that $\mathrm{Dom}(A_p)$ embeds for $1 < p < 2$ continuously into $\mathrm{H}^{\alpha,p}_{0,\sigma}(\Omega)$ by Remark 4.3.18. The rest follows by Theorem 4.1.8(a). □

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