Solving a Rational Eigenvalue Problem in Fluid-Structure Interaction

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Abstract

In this paper we consider a rational eigenvalue problem governing the vibrations of a tube bundle immersed in an inviscid compressible fluid. Taking advantage of eigensolutions of appropriate sparse linear eigenproblems the large nonlinear eigenvalue problem is projected to a much smaller one which is solved by inverse iteration.

1 Introduction

Vibrations of a tube bundle immersed in an inviscid compressible fluid are governed under some simplifying assumptions by an elliptic eigenvalue problem with non-local boundary conditions which can be transformed to a rational eigenvalue problem. Discretizing this problem by finite elements one obtains a rational matrix eigenvalue problem

$$T(\lambda)x := -Ax + \lambda Bx + \sum_{j=1}^{K} \frac{\rho_0 \lambda}{k_j - \lambda m_j} C_j x = 0 \tag{1}$$

where the matrices A, B and C_j are symmetric and positive (semi-) definite, and they are typically large and sparse.

For linear sparse eigenvalue problems one gets approximations to eigenvalues and eigenvectors by projection methods where a sequence of low dimensional spaces V_k is constructed by the Lanczos process or the Jacobi–Davidson method, e.g., and taking advantage of shift–and–invert or rational

Krylov techniques one gets approximate eigenvalues in the wanted part of the spectrum.

Generalizations of this approach to the nonlinear eigenvalue problem $T(\lambda)x = 0$ are contained in recent papers by Ruhe [7] and Hager and Wiberg [3], [4] who updated linear eigenvalue problems which approximate the projection of the nonlinear eigenproblem to a Krylov space of $T(\sigma)^{-1}T(\lambda)$ for some shift σ and varying λ, and for symmetric nonlinear problems having a Rayleigh functional by Betcke and the author [1] who constructed ansatz vectors for a projection method by a Jacobi-Davidson type approach and solved the low dimensional projected nonlinear eigenproblem by inverse iteration. Work on a variant for non-symmetric problems is in progress.

In this paper we exploit the special structure of the nonlinear eigenproblem in fluid-solid interaction in a projection method. Motivated by a min-max characterization of eigenvalues for the rational eigenproblem (1) and by comparison theorems for problem (1) we determine ansatz vectors from suitable linear eigenproblems. This method turns out to be much more efficient than the Jacobi-Davidson approach in [1]. Notice however, that the method in [1] applies to a much wider class of nonlinear eigenproblems.

The paper is organized as follows: Section 2 describes the model for the fluid-structure interaction problem under consideration, and Section 3 summarizes numerical methods for nonlinear eigenvalue problems. In Section 4 we present a projection method for large and sparse nonlinear eigenvalue problems, and Section 5 contains a numerical example.

2 A Spectral Problem in Fluid-Solid Structures

This section is devoted to the presentation of the mathematical model which describes the problem governing free vibrations of a tube bundle immersed in a slightly compressible fluid under the following simplifying assumptions: The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity. Considering small vibrations of the fluid (and the tubes) around the state of rest, it can also be assumed that the fluid is irrotational.

Mathematically this problem can be described in the following way (cf. [2]). Let $\Omega \subset \mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ. We assume that there exists a family $\Omega_j \neq \emptyset$, $j = 1, \ldots, K$, (the sections of the tubes) of simply connected open sets such that $\bar{\Omega}_j \subset \Omega$ for every j, $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $j \neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With these notations we set

$\Omega_0 := \Omega \setminus \bigcup_{j=1}^{K} \Omega_j$. Then the boundary of Ω_0 consists of $K+1$ connected components which are Γ and Γ_j, $j = 1, \ldots, K$.

We denote by $H^1(\Omega_0) = \{u \in L^2(\Omega_0) : \nabla u \in L^2(\Omega_0)^2\}$ the standard Sobolev space equipped with the usual scalar product. Then the eigenfrequencies and the eigenmodes of the fluid-solid structure are governed by the following variational eigenvalue problem (cf. [2])

Find $\lambda \in \mathbb{R}$ *and* $u \in H^1(\Omega_0)$ *such that for every* $v \in H^1(\Omega_0)$

$$c^2 \int_{\Omega_0} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega_0} uv \, dx + \sum_{j=1}^{K} \frac{\lambda \rho_0}{k_j - \lambda m_j} \int_{\Gamma_j} un \, ds \cdot \int_{\Gamma_j} vn \, ds. \tag{2}$$

Here u is the potential of the velocity of the fluid, c denotes the speed of sound in the fluid, ρ_0 is the specific density of the fluid, k_j represents the stiffness constant of the spring system supporting tube j, m_j is the mass per unit length of the tube j, and n is the outward unit normal on the boundary of Ω_0.

The eigenvalue problem is non–standard in two respects: The eigenparameter λ appears in a rational way in the boundary conditions, and the boundary conditions are non-local.

In Conca et al. [2] it was shown that the eigenvalues are the characteristic values of a linear compact operator acting on a Hilbert space. The operator associated with this eigenvalue problem is not selfadjoint, but it can be symmetrized in the sense that one can prove the existence of a selfadjoint operator which has the same spectrum as the original operator. Hence, the set of eigenvalues is a countably infinite set of positive real numbers that converge to infinity. The same result was obtained in [9] in a less technical way from a minmax characterization of eigenvalues of nonlinear eigenvalue problems (cf. [11]).

Discretizing problem (2) by finite elements one obtains a rational matrix eigenvalue problem

Find $\lambda \in \mathbb{R}$ *and* $x \neq 0$ *such that*

$$T(\lambda)x := -Ax + \lambda Bx + \sum_{j=1}^{K} \frac{\rho_0 \lambda}{k_j - \lambda m_j} C_j x = 0 \tag{3}$$

where the matrices A, B and C_j are symmetric, B is positive definite, and A and C_j are positive semi–definite. In a similar way as for linear eigenvalue problems it follows from minmax arguments that the eigenvalues of the discrete problem (3) are upper bounds of the corresponding eigenvalues of the nonlinear problem (2).

3 Solving nonlinear eigenvalue problems

In this section we summarize numerical methods for finite dimensional nonlinear eigenvalue problems

$$T(\lambda)x = 0 \tag{4}$$

where $T(\lambda)$ is a family of real symmetric $n \times n$-matrices, and we assume that the dimension n of problem (4) is large.

For dense problems algorithms for problem (4) are investigated in [5], [6], [8] which are all variants of inverse iteration

$$x^{k+1} = \alpha_k T(\lambda_k)^{-1} T'(\lambda_k) x^k \tag{5}$$

where α_k is a suitable normalization factor and λ_k is updated in some way. Similarly as in the linear case inverse iteration is quadratically convergent for simple eigenvalues, and the convergence is even cubic if problem (4) is symmetric and if λ_k is updated by the Rayleigh functional.

An essential disadvantage of inverse iteration is the fact that each eigenvalue has to be determined individually by an iterative process, and that each step of this iteration requires the solution of a linear system. Moreover, the coefficient matrix $T(\lambda_k)$ of system (4) changes in each step, and in contrast to the linear case replacing (5) by $x^{k+1} = \alpha_k T(\sigma)^{-1} T'(\lambda_k) x^k$ with a fixed shift σ results in convergence to an eigenpair of the linear system $T(\sigma)x = \gamma T'(\tilde{\lambda})x$ ($\gamma \neq 0$ depending on the normalization condition) from which we can not recover an eigenpair of the nonlinear problem (4). A remedy against this wrong convergence was proposed by Neumaier [5] who introduced the so called residual inverse iteration which converges linearly with a fixed shift, and quadratically or cubically if λ_k is updated, and the coefficient matrix changes in every iteration step.

For dense problems inverse iteration is a very capable method, however, for large and sparse nonlinear eigenvalue problems inverse iteration is much too expensive.

For sparse linear eigenvalue problems the most efficient methods are iterative projection methods, where approximations of the wanted eigenvalues and corresponding eigenvectors are obtained from projections of the eigenproblem to subspaces which are expanded in the course of the algorithm. Methods of this type for symmetric problems are the Lanczos method, rational Krylov subspace methods, and the Jacobi-Davidson method, e.g.

Ruhe [7] and Hager and Wiberg [3], [4] generalized this approach to sparse nonlinear eigenvalue problems linearizing (4) by the secant method and solving the resulting linear eigenproblem by Arnoldi's method. In [4] Hager points out that the eigenpairs of (4) are determined one after another. After a Ritz value has converged only the approximate eigenvectors from previous Arnoldi runs, the just converged Ritz vector, and an approximation to a further eigenvector to be computed in the next Arnoldi run are kept, and the rest of the current Krylov space is purged. Hence, each eigenvalue of (4)

is determined by an individual approximate Arnoldi process essentially from scratch, and therefore the cost of the rational Krylov method for nonlinear problems is similar to the cost of inverse iteration.

A further generalization of projection methods to symmetric sparse nonlinear eigenvalue problems having real eigenvalues was proposed in [1] where an orthonormal basis of a suitable finite dimensional space is constructed by a Jacobi-Davidson approach. The projected nonlinear problem of small dimension then inherits the symmetry of the sparse problem, and can be solved by the safeguarded inverse iteration which converges cubically. Work on a variant for non-symmetric problems is in progress.

4 A Projection Method for Fluid-Solid Vibrations

In this section we propose a projection method for the eigenvalue problem (3). The choice of the ansatz vectors is motivated by comparison results which are obtained from a minmax characterization of eigenvalues of (3).

We consider the nonlinear eigenvalue problem (4) where $T(\lambda)$ is a family of real symmetric matrices for every λ in an open real interval J. We assume that $f(\lambda, x) := x^T T(\lambda)x$ is continuously differentiable on $J \times \mathbb{R}^n$, and that for every fixed $x \in \mathbb{R}^n \setminus \{0\}$ the real equation

$$f(\lambda, x) = 0 \tag{6}$$

has at most one solution in J. Then equation (6) implicitly defines a functional p on some subset D of $\mathbb{R}^n \setminus \{0\}$ which we call the Rayleigh functional. Moreover, we assume that $\frac{\partial}{\partial\lambda} f(\lambda, x)\big|_{\lambda=p(x)} > 0$ for every $x \in D$.

For a linear eigenvalue value problem $T(\lambda)x := (\lambda I - A)x = 0$ where A is a symmetric matrix the assumptions above are fulfilled, p is the Rayleigh quotient and $D = \mathbb{R}^n \setminus \{0\}$. In this case the eigenvalues can be characterized as minmax values of p.

For nonlinear eigenvalue problems variational properties using the Rayleigh functional were proved by Duffin, Rogers, Hadeler, and Werner if the problem is overdamped, i.e. if the Rayleigh functional p is defined in the entire space $\mathbb{R}^n \setminus \{0\}$. Nonoverdamped problems were studied by Werner and the author [11]. In this case the natural enumeration for which the smallest eigenvalue is the first one, the second smallest is the second one, etc. is not appropriate, but the number of an eigenvalue λ of the nonlinear problem (4) is obtained from the location of the eigenvalue 0 in the spectrum of the matrix $T(\lambda)$.

If $\lambda \in J$ is an eigenvalue of problem (4) then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $k \in \mathbb{N}$ such that

$$0 = \max_{V \in S_k} \min_{v \in V_1} v^T T(\lambda) v$$

where S_k denotes the set of all k–dimensional subspaces of $\mathbb{R}^n$ and $V^1 := \{v \in V : \|v\| = 1\}$ is the unit sphere in V. In this case we call λ a k-th eigenvalue of (4).

With this enumeration the following minmax characterization of the eigenvalues of the nonlinear eigenproblem (4) was proved in [11] (under an additional compactness condition even for the infinite dimensional case):

Theorem 1 *Under the conditions given above the following assertions hold:*

(i) For every $k \in \mathbb{N}$ there is at most one k-th eigenvalue of problem (4) which can be characterized by

$$\lambda_k = \min_{\substack{V \in S_k \\ V \cap D \neq \emptyset}} \max_{v \in V \cap D} p(v). \tag{7}$$

The set of eigenvalues of (4) in J is at most countable.

(ii) If

$$\lambda_k = \inf_{\substack{V \in S_k \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v) \in J$$

for some $k \in \mathbb{N}$ then λ_k is the k-th eigenvalue of (4) and (7) holds.

For the nonlinear eigenproblem (3) the general conditions obviously are satisfied for every open interval $J \subset \mathbb{R}_+$ which does not contain k_j/m_j for $j = 1, \ldots, K$. Moreover for fixed $x \in \mathbb{R}^n$

$$f(\lambda, x) = -x^T Ax + \lambda x^T Bx + \sum_{j=1}^{K} \frac{\lambda \rho_0}{k_j - \lambda m_j} x^T C_j x \tag{8}$$

is monotonely increasing with respect to λ. Hence, every open interval J such that $k_j/m_j \notin J$ for $j = 1, \ldots, K$ contains at most countably many eigenvalues which can be characterized as minmax value of the Rayleigh functional p defined by $f(p(x), x) = 0$ where f is defined in (8).

We now assume that the quotients $\frac{k_j}{m_j}$ are ordered by magnitude

$$0 =: \frac{k_0}{m_0} < \frac{k_1}{m_1} \leq \frac{k_2}{m_2} \leq \ldots \leq \frac{k_K}{m_K} < \infty =: \frac{k_{K+1}}{m_{K+1}}.$$

If $\frac{k_{\ell-1}}{m_{\ell-1}} < \frac{k_\ell}{m_\ell}$ for some $\ell \in \{1, \ldots, K+1\}$ then problem(3) has a Rayleigh functional p_ℓ corresponding to the interval $J_\ell := (\frac{k_{\ell-1}}{m_{\ell-1}}, \frac{k_\ell}{m_\ell})$ which is defined in the domain of definition denoted by D_ℓ.

Comparing p_ℓ with the Rayleigh quotient of the linear eigenvalue problem

$$\Big(A + \sum_{j=1}^{\ell-1} \frac{\rho\kappa}{\kappa m_j - k_j} C_j\Big) x = \lambda \Big(B + \sum_{j=\ell}^{K} \frac{\rho_0}{k_j - \kappa m_j} C_j\Big) x. \tag{9}$$

for a parameter $\kappa \in J_\ell$ we obtained in [9] the following inclusion result for the eigenvalues in J_ℓ.

Theorem 2 *Let $\kappa \in J_\ell$, and assume that the m-th eigenvalue μ_m of the comparison problem (9) is contained in J_ℓ. Then the rational eigenvalue problem (3) has an m-th eigenvalue λ_m in J_ℓ, and the following inclusion holds*

$$\min(\mu_m, \kappa) \leq \lambda_m \leq \max(\mu_m, \kappa).$$

For the rational eigenvalue problem (3) the proof of the Inclusion Theorems 2 demonstrates that eigenvectors of the linear system (9) are good approximations to eigenvectors of the nonlinear problem, at least if the shift κ is close to the corresponding eigenvalue. This suggests the following projection method (which was already considered in [10] for the extreme intervals J_1 and J_{K+1}) if we are interested in eigenvalues of the nonlinear problem (3) in the interval J_ℓ.

Projection method

1. Choose a small number of shifts $\kappa_1, \ldots, \kappa_r \in J_\ell$.
2. For $j = 1, \ldots, r$ determine the eigenvectors u_{jk}, $k = 1, \ldots, s_j$, of the linear problem (9) with shift κ_j corresponding to eigenvalues in J_ℓ.
3. Let U be the matrix with columns u_{jk}, $j = 1, \ldots, r$, $k = 1, \ldots, s_j$. Determine the QR factorization with column pivoting which produces the QR factorization of UE where E denotes a permutation matrix such that the absolute values of the diagonal elements of R are monotonely decreasing.
4. For every j with $|r_{jj}| < \tau \cdot |r_{11}|$ drop the j-th column of Q where $\tau \in [0, 1)$ is a given tolerance, and denote by V the space that is spanned by the remaining columns of Q.
5. Project the nonlinear eigenvalue problem (3) to V and solve the projected problem by inverse iteration with variable shifts.

5 A numerical example

Consider the rational eigenvalue problem (2) where Ω is the ellipse with center $(0, 0)$ and length of semiaxes 8 and 4, and Ω_j, $j = 1, \ldots, 9$ are circles with radius 0.3 and centers $(-4, -2)$, $(0, -2)$, $(4, -2)$, $(-5, 0)$, $(0, 0)$, $(5, 0)$, $(-4, 2)$, $(0, 2)$ and $(4, 2)$. We assume $c = 1$, $\rho_0 = 1$, $k_j = 1$ for all j, $m_j = 5$ for the circles with centers $(-4, -2)$, $(0, -2)$ and $(4, -2)$ and $m_j = 1$ for all other circles in problem (2).

We discretized this eigenvalue problem with linear elements obtaining a matrix eigenvalue problem

$$Ax = \lambda Bx + \frac{\lambda}{1 - 5\lambda} C_1 x + \frac{\lambda}{1 - \lambda} C_2 x \tag{10}$$

where C_1 and C_2 collect the contributions of all tubes corresponding to $m_j = 5$ and $m_j = 1$, respectively. Problem (10) of dimension $n = 36040$ has 10 eigenvalues $\lambda_1 \leq \ldots \leq \lambda_{10}$ in the interval $J_1 = [0, 0.2)$, 19 eigenvalues

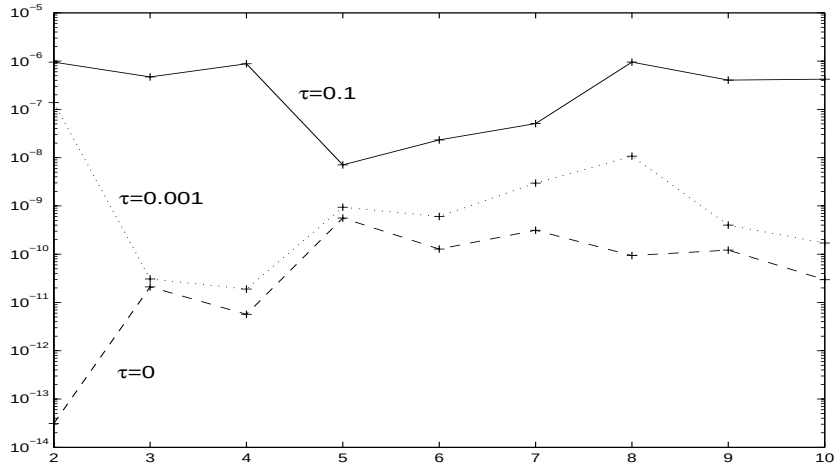

Fig. 1: relative errors; eigenvalues in (0,0.2); shifts 0.1e, 0.15, 0.175, 0.1875

$\tilde{\lambda}_5 \leq \ldots \leq \tilde{\lambda}_{23}$ in $J_2 := (0.2, 1)$, and 19 eigenvalues $\hat{\lambda}_{12} \leq \ldots \leq \hat{\lambda}_{30}$ in $J_3 = (1, 3)$.

To approximate the eigenvalues in J_1 we solved the linear eigenvalue problem

$$\left(B + \frac{1}{1-5\kappa}C_1 + \frac{1}{1-\kappa}C_2\right)x = \mu(A + \sigma B)x$$

by Lanczos' method with complete reorthogonalization for different parameters of κ obtaining approximations to eigenvectors of problem (9). We added σB on the right hand side with a small $\sigma > 0$ since A is singular.

With 4 parameters $\kappa_1 = 0.1$, $\kappa_2 = 0.15$, $\kappa_3 = 0.175$ and $\kappa_4 = 0.1875$ and tolerances $\tau_1 = 1e-1$, $\tau_2 = 1e-3$, and $\tau_3 = 0$ we obtained eigenvalue approximations to $\lambda_1, \ldots, \lambda_{10}$ the relative errors of which are displayed in Figure 1. The dimensions of the projected eigenvalue problems were 27, 36 and 45, respectively.

On an Intel Pentium 4 with 2 GHz and 1 GB RAM it took 28.70 seconds to solve the 4 linear eigenvalue problems, and 1.06 seconds for the QR factorization with column pivoting. To solve the projected nonlinear eigenvalue problems by safeguarded inverse iteration it took 2.14, 3.17 and 3.08 seconds, respectively.

To approximate the eigenvalues in J_2 we solved the linear problem

$$\left(B + \frac{1}{1-\kappa}C_2\right)x = \mu\left(A + \frac{\kappa}{5\kappa-1}C_1 + \sigma B\right)x$$

for 4 parameters $\kappa_1 = 0.6$, $\kappa_2 = 0.8$, $\kappa_3 = 0.9$, and $\kappa_4 = 0.95$, and with the same tolerances as before we obtained the relative errors in Figure 2. The dimensions of the projected problems are 39, 58 and 68, respectively. The CPU times in this run were 91.41 seconds for the linear eigenproblems,

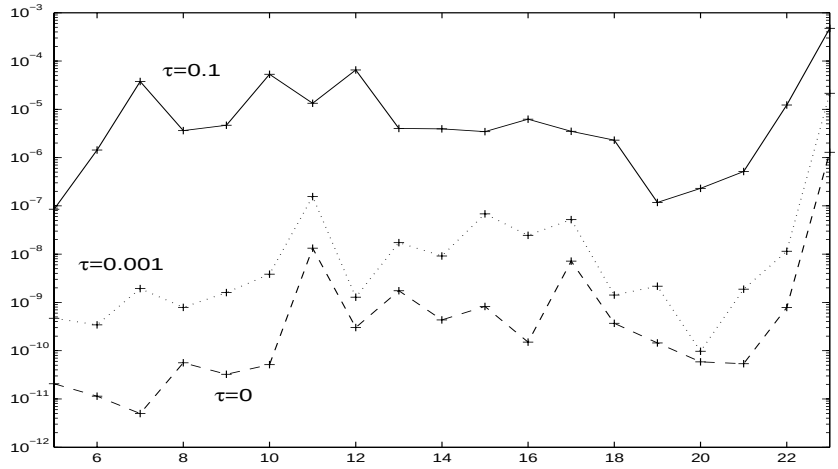

Fig. 2: relative errors; eigenvalues in (0.2,1); shifts 0.6, 0.8, 0.9, 0.95

2.63 seconds for the QR factorization, and 4.45, 8.14, and 7.61 seconds for inverse iteration.

Figure 3 shows the relative errors of eigenvalues $\hat{\lambda}_j$, $j = 12, \ldots, 30$, in the interval $J_3 = (1, 3)$ which were obtained with shift parameters $\kappa_1 = 1.25$ and $\kappa_2 = 1.5$, $\kappa_3 = 2$ and $\kappa_4 = 2.5$ in the linear problem

$$Bx = \mu\Big(A + \frac{\kappa}{5\kappa - 1}C_1 + \frac{\kappa}{\kappa - 1}C_2 + \sigma B\Big)x$$

and tolerances $\tau_1 = 1e - 1$, $\tau_2 = 1e - 3$ and $\tau_3 = 0$. The dimensions of the nonlinear projected problem are 46, 61, and 87, respectively. The CPU times in this run were 211.41 seconds for the linear eigenproblems, 3.83 second for the QR factorization, and 5.69, 9.00, and 13.82 seconds for inverse iteration.

The Jacobi-Davidson type method proposed in [1] achieved the same accuracy as our method for $\tau = 0$ for the eigenvalue approximations in J_1, J_2, and J_3, respectively, projecting problem (2) to a rational eigenproblem of dimension 31, 64, and 75, and requiring 296.38, 575.34, and 608.53 seconds, respectively. Hence, the method considered here is much more efficient than the method from [1] which on the other hand applies to a much wider class of nonlinear eigenproblems, including non-symmetric problems.

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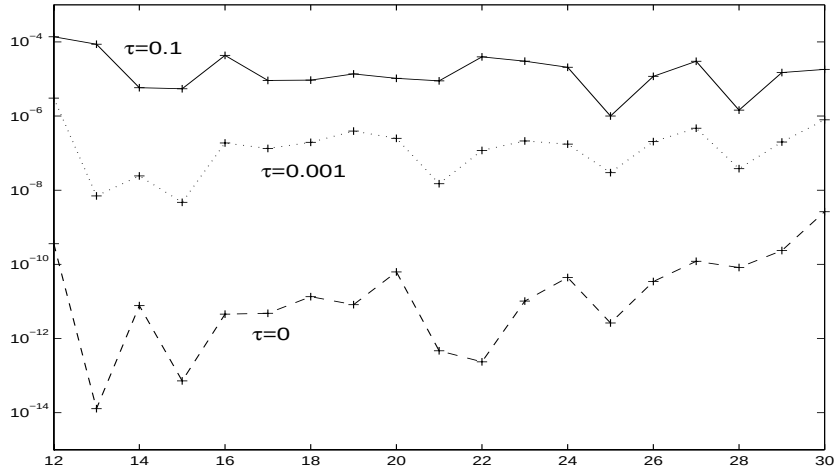

Fig. 3: relative errors; eigenvalues in (1,3); shifts 1.25, 1.5, 2.0, 2.5

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