Modal Masters and Component Mode Method

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Abstract: In the dynamic analysis of structures condensation is often used to reduce the number of degrees of freedom to manageable size. The approximation properties can be improved substantially taking advantage of the Rayleigh functional or if generalized masters are used. In this note we prove that substructuring with modal masters is equivalent to the component mode method with fixed boundaries. This suggests a further reduction of cost for the component mode method.

Keywords: eigenvalue problem, condensation, generalized masters, component mode method

1 Introduction

In the analysis of the dynamic response of structures using finite element methods very often prohibitively many degrees of freedom are needed to model the behaviour of the system sufficiently accurate. Static condensation is frequently employed to economize the computation of a selected group of eigenvalues and eigenvectors. These methods choose from the degrees of freedom a small number of master variables. Neglecting inertia terms the remaining variables (termed slaves) are eliminated leaving a much smaller eigenproblem for the master variables only.

It has frequently been noted in the literature that the approximation properties of static condensation is satisfactory only for a small part of the lower end of the spectrum, and several attempts have been made to improve them (cf. [5], [6], [11], e.g.). Most of them are very time consuming because any single eigenvalue has to be corrected by an iterative process, and every iteration step requires the solution of a large linear system.

In [12] we took advantage of the structure of the exactly condensed eigenvalue problem $T(\lambda)u = 0$ which is equivalent to the original problem and which is nonlinear with respect to the parameter λ. For T a Rayleigh functional p exists which has similar properties as the Rayleigh quotient for linear eigenproblems. In particular, the eigenvectors of $T(\cdot)$ are stationary points of p. Hence, a first order error of an eigenvector approximation $\tilde{u}$ yields a second order error of $p(\tilde{u})$ as an approximation of the corresponding eigenvalue. Since the eigenvectors u^j of the statically condensed problem are usually good approximations of the master portions of the eigenvectors the approximation properties of static condensation can be improved substantially.

A different enhancement was derived in [8] incorporating general masters into the condensation process. Especially, combining substructuring and modal masters turned out to be successful (c.f. [13]). In this paper we prove that condensation in the presence of modal masters is equivalent to the component mode method introduced by Hurty [3] and by Craig and Bampton [1]. Moreover, we demonstrate that the cost of the component mode method can be reduced to that one of static condensation taking advantage of the Rayleigh functional.

2 Static Condensation

We consider the general matrix eigenvalue problem

$$Kx = \lambda Mx \tag{1}$$

where $K \in \mathbb{R}^{(n,n)}$ and $M \in \mathbb{R}^{(n,n)}$ are symmetric and positive definite matrices which are usually the stiffness and mass matrix of a finite element model of a structure, and which are usually very large and sparse.

To reduce the dimension of the system to manageable size one chooses m master variables x_m with $m \ll n$, and (after reordering the rows and columns of K and M) rewrites (1) into block form

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{bmatrix} x_m \\ x_s \end{bmatrix} = \lambda \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{bmatrix} x_m \\ x_s \end{bmatrix} \tag{2}$$

where $x_s \in \mathbb{R}^s$ indicates the slave part of x to be eliminated.

Neglecting inertia terms in the second equation of (2), solving for x_s, and substituting x_s into the first equation one obtains the statically condensed eigenproblem

$$K_0 x_m = \lambda M_0 x_m \tag{3}$$

with

$$\begin{aligned} K_0 &:= K_{mm} - K_{ms}K_{ss}^{-1}K_{sm}, \\ M_0 &:= M_{mm} - K_{ms}K_{ss}^{-1}M_{sm} - M_{ms}K_{ss}^{-1}K_{sm} \\ &\quad + K_{ms}K_{ss}^{-1}M_{ss}K_{ss}^{-1}K_{sm} \end{aligned}$$

which was introduced by Guyan [2] and Irons [4].

Static condensation is known to be accurate only for a few of the smallest eigenvalues. To enhance the approximation properties Mackens and the author in [8] introduced general masters. To this end let $Z := (z_1, \dots, z_m)$ be a basis of the space of master vectors, and let $Y := (y_{m+1}, \dots, y_n)$ be such that (Z, Y) is a nonsingular matrix.

If we insert the unique represenation $x = Zx_m + Yx_s$ of $x \in \mathbb{R}^n$ into the original problem (1) and premultiply it by $(Z, Y)^T$ we obtain the following eigenvalue problem

$$\begin{bmatrix} K_{zz} & K_{zy} \\ K_{yz} & K_{yy} \end{bmatrix} \begin{bmatrix} x_m \\ x_s \end{bmatrix} = \lambda \begin{bmatrix} M_{zz} & M_{zy} \\ M_{yz} & M_{yy} \end{bmatrix} \begin{bmatrix} x_m \\ x_s \end{bmatrix} \tag{4}$$

where for $L \in \{K, M\}$

$$L_{zz} := Z^T LZ,\ L_{zy} := Z^T LY =: L_{yz}^T,\ L_{yy} := Y^T LY.$$

Therefore, the stiffness and the mass matrix have been decomposed with respect to the spaces Z and Y in a similar way as in (2).

In principle equation (4) could be employed to reduce the eigenvalue problem (1) using $\{z_1, \dots, z_m\}$ as master degrees of freedom. However, since in practice only the small set of masters is available, but the large set of slave vectors $\{y_{m+1}, \dots, y_n\}$ is definitely not the matrices $K_{zy}, K_{yy}, M_{zy}, M_{yy}$ are usually not at hand. Hence, the straightforward transfer of static condensation to perform the reduction in the presence of general masters does not apply. In [8] it has been shown how to generate the condensed problem corresponding to the decomposition (4) using the masters $z_1, \dots, z_m$ only, but not the complementary vectors $y_{m+1}, \dots, y_n$.

Theorem 1 *Let $V \in \mathbb{R}^{(n,n)}$ be a symmetric and positive definite metric matrix, and let $Z = (z_1, \dots, z_m) \in \mathbb{R}^{(n,m)}$ and $Y \in \mathbb{R}^{(n,n-m)}$ such that $Z^T VZ = I_m$ and $Z^T VY = O$.*

Then the condensed problem with general masters $z_1, \dots, z_m$ is given by

$$P^T KPx_m = \lambda P^T MPx_m$$

with the projection matrix

$$P = K^{-1}X \left[X^T K^{-1}X\right]^{-1},\ X := VZ. \tag{5}$$

Since $\left[X^T K^{-1} X\right]^{-1}$ is a nonsingular matrix the condensed problem is equivalent to the projection of the eigenproblem to the space spanned by the columns of $K^{-1}VZ$. Hence, choosing $V = M$ it is equivalent to one step of simultaneous inverse iteration.

Nodal condensation is often combined with substructuring, i.e. the structure under consideration is subdivided into substructures, and the masters are chosen to be the interface degrees of freedom. If the substructures are connected through the master variables only and if the slave variables are numbered appropriately, then the stiffness matrix is given by

$$K = \begin{bmatrix} K_{mm} & K_{ms1} & K_{ms2} & \dots & K_{msr} \\ K_{sm1} & K_{ss1} & O & \dots & O \\ K_{sm2} & O & K_{ss2} & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{smr} & O & O & \dots & K_{ssr} \end{bmatrix} \tag{6}$$

and the mass matrix M has the same block form. Hence the matrices K_{ss} and M_{ss} obtain block diagonal form and the cost of the condensation process can be reduced substantially. In particular it can be performed completely in parallel (cf. [10]).

For general masters condensation can be executed in parallel as well if in substructuring interface masters are accompanied only by general masters such that the support of each of them is contained in exactly one of the substructures (cf. [7]). This result suggests to use modal masters in a similar way as in component mode methods.

Assume that the original structure is subdivided into r components as above, and let K_{ssj} and M_{ssj} be the stiffness and mass matrix of the j-th substructure, respectively, where the boundaries of the substructures corresponding to interfaces between substructures are assumed to be fixed. For each substructure let the columns of the matrix Φ_j contain eigenvectors corresponding to a few of the smallest eigenvalues of

$$K_{ssj}\phi_j = \omega M_{ssj}\phi_j.$$

Then the general condensation process with mastervectors

$$Z = \begin{bmatrix} I_m & O & O & \dots & O \\ O & \Phi_1 & O & \dots & O \\ O & O & \Phi_2 & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & \Phi_r \end{bmatrix} =: \begin{bmatrix} I_m & O \\ O & \tilde{\Phi} \end{bmatrix}.$$

and scalar product defined by the matrix

$$V = \begin{bmatrix} I_m & O \\ O & M_{ss} \end{bmatrix}$$

is called modal condensation method. It is reasonable to choose the matrix V above since condensation can be interpreted as simultaneous inverse iteration (cf. (5)).

Modal condensation was applied successfully to membrane and plate problems in [13] where the approximations of eigenvalues from nodal condensation were improved considerably. In the next section we will prove that modal condensation is identical to the component mode method with fixed interfaces.

3 Component Mode Method

Component mode methods have been used extensively in the dynamic analysis of complex structures during the last three decades. The basic idea is to treat the structure as an assembly of connected components of substructures each of which is analyzed separately. The space of physical degrees of freedom of each substructure is projected to a mode subspace spanned by a selected set of a few lower mode shapes and other supplementary modes yielding a reduced system of much smaller dimension than the original finite element model. Three types of component mode methods are in use depending on the type of boundary conditions at the interfaces: fixed-interface methods, free-interface methods and hybrid methods. In this paper we concentrate on fixed-interface methods which were introduced by Hurty [3] and by Craig and Bampton [1].

As in the previous section we consider the partition of the eigenvalue problem (1) given in (2) describing the free vibrations of a complex structure which is divided into r substructures, i.e. the stiffness matrix has the block form given in (6). As before x_m denotes the vector of degrees of freedom on the interfaces of the substructures and x_s corresponds to the interior degrees of freedom.

Let $\Phi \in \mathbb{R}^{(s,s)}$ be the modal matrix of the substructure eigenvalue problem

$$K_{ss}\phi = \omega M_{ss}\phi \tag{7}$$

normalized by $\Phi^T M_{ss} \Phi = I_s$, and denote by $\Omega := \text{diag}\{\omega_1, \ldots, \omega_s\}$ the diagonal matrix containing the eigenvalues of problem (7) in its diagonal. Then it holds that $\Phi^T K_{ss} \Phi = \Omega$, and it is easily seen that the variable transformation

$$\begin{bmatrix} x_m \\ x_s \end{bmatrix} = \begin{bmatrix} I_m & O \\ -K_{ss}^{-1} K_{sm} & \Phi \end{bmatrix} \begin{bmatrix} x_m \\ y_s \end{bmatrix}$$

transforms problem (1) to the equivalent eigenvalue problem

$$\begin{bmatrix} K_0 & O \\ O & \Omega \end{bmatrix} \begin{bmatrix} x_m \\ y_s \end{bmatrix} = \lambda \begin{bmatrix} M_0 & \tilde{M}_{ms} \\ \tilde{M}_{sm} & I_s \end{bmatrix} \begin{bmatrix} x_m \\ y_s \end{bmatrix} \tag{8}$$

where K_0 and M_0 are the reduced matrices in (3) obtained by static nodal condensation, and

$$\tilde{M}_{ms} = M_{ms}\Phi - K_{ms}K_{ss}^{-1}M_{ss}\Phi = \tilde{M}_{sm}^T. \tag{9}$$

Since the high frequencies of the subtructures do not influence the low frequencies of the entire structure very much the dimension of the eigenvalue problem (8) can be reduced considerably if we delete rows and columns corresponding to high frequencies of the slave eigenproblem (7). This is exactly the component mode method in [3] and [1]. Obviously, it is equivalent to applying the Rayleigh–Ritz method to problem (1) using as ansatz vectors the columns of

$$R = \begin{bmatrix} I_m & O \\ -K_{ss}^{-1}K_{sm} & \tilde{\Phi} \end{bmatrix} \tag{10}$$

where $\tilde{\Phi}$ contains only a small number of modes of the slave eigenvalue problem (7) corresponding to small eigenvalues.

Theorem 2 *The component mode method with fixed boundaries and the static modal condensation method are identical, i.e. they yield the same approximations to eigenvalues and eigenvectors of problem (1).*

Proof: By Theorem 1 static modal condensation is nothing else but the Rayleigh – Ritz method with ansatz vectors

$$K^{-1}X(X^TK^{-1}X)^{-1},\ X := VZ,$$

and since $(X^TK^{-1}X)^{-1}$ is a nonsingular matrix this is equivalent to the projection of problem (1) to the subspace V_1 spanned by the columns of

$$K^{-1}VZ = K^{-1}\begin{bmatrix} I_m & O \\ O & M_{ss}\tilde{\Phi} \end{bmatrix}.$$

Let V_2 be the space spanned by the columns of R in (10). We are done if $V_1 = V_2$, and since both spaces are of the same dimension it suffices to show $V_1 \subset V_2$.

Taking advantage of the block structure of K in (2) it can easily be verified that

$$K^{-1}\begin{bmatrix} I_m \\ O \end{bmatrix} = \begin{bmatrix} I_m \\ -K_{ss}^{-1}K_{sm} \end{bmatrix} S$$

where $S := (K_{mm} - K_{ms}K_{ss}^{-1}K_{sm})^{-1}$ and

$$K^{-1}\begin{bmatrix} O \\ M_{ss}\tilde{\Phi} \end{bmatrix} = \begin{bmatrix} I_m \\ -K_{ss}^{-1}K_{sm} \end{bmatrix} T + \begin{bmatrix} O \\ \tilde{\Phi} \end{bmatrix} \tilde{\Omega}^{-1}$$

with

$$T = -SK_{ms}\tilde{\Phi}\tilde{\Omega}^{-1}$$

and $\tilde{\Omega}$ is a diagonal matrix containing the eigenvalues of (7) corresponding to the columns of $\tilde{\Phi}$ in its diagonal. Hence $V_1 \subset V_2$. q.e.d.

4 Rayleigh Functional

It is well known that often only very few of the eigenvalues at the lower end of the spectrum are approximated with reasonable accuracy using static condensation. Several attempts have been made to increase the accuracy most of them being iterative and therefore very time consuming (cf. [5], [11], e.g.).

In [12], [9] we took advantage of the properties of the exactly condensed problem which is obtained from problem (4) by solving the second equation of (4) for x_s and inserting x_s into the first equation. Obviously, one obtains a nonlinear eigenproblem

$$T(\lambda)x_m = 0.$$

It is well known that $T(\lambda)$ can be given a convenient form if modal properties of the slave problem are exploited.

Let $\Psi \in \mathbb{R}^{(s,s)}$ and $\Gamma := \operatorname{diag}\{\gamma_j\} \in \mathbb{R}^{(s,s)}$ be the modal matrix and the spectral matrix of the slave eigenvalue problem

$$K_{yy}\psi = \gamma M_{yy}\psi, \tag{11}$$

respectively, such that $\Psi^T M_{yy}\Psi = I_s$ and $\Psi^T K_{yy}\Psi = \Gamma$. Then $T(\lambda)$ can be rewritten as (cf. Leung [5])

$$T(\lambda) = -K_0 + \lambda M_0 + SD(\lambda)S^T \tag{12}$$

where K_0 and M_0 are the reduced stiffness and mass matrix of the statically condensed problem,

$$S := M_{zy}\Psi - K_{zy}\Psi\Gamma^{-1} \text{ and } D(\lambda) := \operatorname{diag}\left\{\frac{\lambda^2}{\gamma_j - \lambda}\right\}.$$

Let $\underline{\gamma}$ be the smallest eigenvalue of the slave eigenproblem (11), and let $J := (0, \underline{\gamma})$. Then for every fixed vector $u \in \mathbb{R}^m$, $u \neq 0$, the real valued function

$$f(\cdot, u) \ : \ J \to \mathbb{R}, \ \lambda \mapsto f(\lambda, u) := u^T T(\lambda) u,$$

i.e.

$$f(\lambda, u) = -u^T K_0 u + \lambda u^T M_0 u + \sum_{j=1}^{s} \frac{\alpha_j^2 \lambda^2}{\gamma_j - \lambda}$$

with

$$\alpha_j := \psi_j^T M_{yz} u - \frac{1}{\gamma_j} \psi_j^T K_{yz} u,$$

is strictly monotonely increasing. Hence, the nonlinear equation $f(\lambda, u) = 0$ has at most one solution. Therefore, it implicitly defines a functional

$$p \ : \ \mathbb{R}^m \supset D(p) \to J, \ f(p(u), u) = 0,$$

which is called the Rayleigh functional of the nonlinear eigenproblem (12).

The Rayleigh functional has similar properties as the Rayleigh quotient for linear eigenproblems. In particular the eigenvectors of $T(\cdot)$ are stationary vectors of p. Thus, if $\bar{u} \in D(p)$ is a first order approximation of an eigenvector then $p(\bar{u})$ will be a second order approximation of the corresponding eigenvalue. Evaluating the Rayleigh functional at the eigenvectors of the statically condensed problem (which are actually contained in $D(p)$) therefore should improve eigenvalue approximation considerably.

A considerable saving of work can be made by the observation that usually the substructures are much stiffer than the entire structure. Hence, only very few substructure modes have to be considered in the evaluation of the Rayleigh functional to achieve an eigenvalue approximation of good accuracy. Consequently we need not solve the complete eigenvalue problems (11). Instead only a very small number of the smallest eigenvalues $\tilde{\gamma}_{ji}$ and corresponding eigenvectors $\tilde{\psi}_{ji}$ $(j = 1, \ldots, r, \ i = 1, \ldots, \tilde{s}_j)$ have to be determined. With these we compute the improved eigenvalue approximation $\tilde{p}(\tilde{u})$ as the root of the truncated rational function

$$\begin{aligned} \tilde{f}(\lambda, \tilde{u}) &:= -\tilde{u}^T \tilde{K}_0 \tilde{u} + \lambda \tilde{u}^T \tilde{M}_0 \tilde{u} \\ &\quad + \sum_{j=1}^{r} \sum_{i=1}^{\tilde{s}_j} \frac{\lambda^2}{\tilde{\gamma}_{ji} - \lambda} \Big(\tilde{\psi}_{ji}^T M_{smj} \tilde{u} - \frac{1}{\tilde{\gamma}_{ji}} \tilde{\psi}_{ji}^T K_{smj} \tilde{u} \Big)^2 \\ &=: \ -\kappa_0 + \lambda \kappa_1 + \sum_{j=1}^{r} \sum_{i=1}^{\tilde{s}_j} \sigma_{ji} \frac{\lambda^2}{\tilde{\gamma}_{ji} - \lambda}. \end{aligned} \tag{13}$$

For a given eigenvector approximation $\tilde{u}$ the function $\lambda \mapsto \tilde{f}(\lambda, \tilde{u})$ is monotonely increasing and convex, and therefore the solution $\tilde{p}(\tilde{u})$ of $\tilde{f}(\lambda, \tilde{u}) = 0$ can be computed easily with Newton's method.

5 Component Mode Method and Rayleigh Functional

Again we consider the eigenvalue problem (1) reduced by the component mode method to

$$\begin{bmatrix} K_0 & O \\ O & \Omega \end{bmatrix} \begin{bmatrix} x_m \\ y_s \end{bmatrix} = \lambda \begin{bmatrix} M_0 & \tilde{M}_{ms} \\ \tilde{M}_{sm} & I_s \end{bmatrix} \begin{bmatrix} x_m \\ y_s \end{bmatrix}. \qquad (14)$$

Applying exact condensation with interface masters x_m only we obtain

$$\tilde{T}(\lambda)x_m = (-K_0 + \lambda M_0 + \tilde{M}_{ms}\tilde{\Phi}\tilde{D}(\lambda)\tilde{\Phi}^T\tilde{M}_{sm})x_m = 0 \qquad (15)$$

where $\tilde{M}_{ms}$ is given in (9),

$$\tilde{D}(\lambda) = \mathrm{diag}\left\{\frac{\lambda^2}{\omega_j - \lambda}\right\},$$

and ω_j are the slave eigenvalues kept in the component mode method.

Obviously, the Rayleigh functional of (15) is the truncation of the Rayleigh functional of problem (12) considered in (13). This observation suggests the following modification: Instead of solving the eigenvalue problem (14) obtained by the component mode method one should solve the corresponding statically condensed eigenvalue problem (3) (which is much smaller) and evaluate the curtailed Rayleigh functional (13) at the eigenvectors of problem (3). The loss of accuracy should be marginal.

To demonstrate that the loss of accuracy replacing the component mode method by the static nodal condensation combined with the evaluation of the truncated Rayleigh functional actually is not very severe we consider the free vibrations of a uniform thin clamped plate covering the rectangular region $\Omega := (0,4) \times (0,3)$ which are governed by the eigenvalue problem

$$\Delta^2 u = \lambda u \quad \text{in } \Omega, \;\; u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \qquad (16)$$

We discretized this problem by Bogner-Fox-Schmidt elements (with node variables u, u_x, u_y and u_{xy}) on a quadratic mesh of meshsize $h = 0.1$ and obtained a discrete problem of dimension $n = 4524$. Dividing Ω into twelve identical substructures each of them being a square of sidelength 1 and choosing all interface degrees of freedom as masters we obtained a reduced problem of dimension $m = 636$.

Table 1 contains the relative errors of the 10 smallest eigenvalues if we add 1, 2, 4, 8 and 16 modal masters in each substructure obtaining problems of dimension 648, 660, 684, 732 and 828, respectively. Table 2 shows the relative errors if we improve the approximations of the static nodal condensation by the curtailed Rayleigh functional taking into account 1, 2, 4, 8 and 16 modes of each substructure.

Table 1: Clamped plate: componend mode method

# of modes					
0	1	2	4	8	16
3.7e-3	2.1e-4	1.4e-4	1.1e-4	1.6e-5	9.6e-6
9.6e-3	8.3e-4	6.5e-4	2.6e-4	5.4e-5	2.4e-5
1.4e-2	2.5e-3	6.4e-4	4.5e-4	1.3e-4	4.7e-5
1.8e-2	3.9e-3	3.6e-3	5.8e-4	2.1e-4	6.2e-5
2.2e-2	4.2e-3	1.7e-3	6.4e-4	2.3e-4	6.8e-5
2.9e-2	8.8e-3	5.9e-3	9.5e-4	5.0e-4	1.2e-4
9.3e-2	4.3e-3	2.7e-3	1.4e-3	2.2e-4	9.2e-5
1.0e-1	3.7e-3	2.4e-3	1.5e-3	1.9e-4	9.6e-5
1.2e-1	7.6e-3	6.2e-3	1.6e-3	3.7e-4	1.1e-4
1.1e-2	1.5e-2	3.2e-3	1.6e-3	6.6e-4	1.4e-4

References

[1] R.R. Craig and M.C.C. Bampton. Coupling of substructures for dynamic analysis. *AIAA Journal*, 6:1313 – 1319, 1968.

[2] R.J. Guyan. Reduction of stiffness and mass matrices. *AIAA Journal*, 3:380, 1965.

[3] W.C. Hurty. Dynamic analysis of structural systems using component modes. *AIAA Journal*, 3:678 – 685, 1965.

[4] B.M. Irons. Structural eigenvalue problems: elimination of unwanted variables. *AIAA Journal*, 3:961–962, 1965.

[5] Y.-T. Leung. An accurate method of dynamic condensation in structural analysis. *Internat. J. Numer. Meth. Engrg*, 12:1705 – 1715, 1978.

[6] Y.-T. Leung. An accurate method of dynamic substructuring with simplified computation. *Internat. J. Numer. Meth. Engrg*, 14:1241 – 1256, 1979.

[7] W. Mackens and H. Voss. General masters in parallel condensation of eigenvalue problems. *Parallel Computing*, 25, 1999.

[8] W. Mackens and H. Voss. Nonnodal condensation of eigenvalue problems. *ZAMM*, 79:243 – 255, 1999.

Table 2: Clamped plate: Rayleigh functional

# of modes					
0	1	2	4	8	16
3.7e-3	2.1e-4	1.4e-4	1.1e-4	1.6e-5	9.7e-6
9.6e-3	8.3e-4	6.6e-4	2.6e-4	5.6e-5	2.6e-5
1.4e-2	2.5e-3	6.5e-4	4.7e-4	1.5e-4	5.8e-5
1.8e-2	4.0e-3	3.7e-3	6.2e-4	2.5e-4	1.1e-4
2.2e-2	4.3e-3	1.7e-3	6.8e-4	2.8e-4	1.1e-4
2.9e-2	9.1e-3	6.1e-3	1.2e-3	7.2e-4	3.3e-4
9.3e-2	4.3e-3	2.7e-3	1.3e-3	1.9e-4	6.0e-5
1.0e-1	3.7e-3	2.5e-3	1.5e-3	2.0e-4	1.0e-4
1.2e-1	8.4e-3	7.1e-3	2.2e-3	1.0e-3	7.5e-4
1.1e-2	1.7e-2	4.5e-3	3.0e-3	2.0e-3	1.5e-3

[9] K. Rothe and H. Voss. Improving condensation methods for eigenvalue problems via Rayleigh functional. *Comput. Meth. Appl. Mech. Engnr.*, 111:169 – 183, 1994.

[10] K. Rothe and H. Voss. A fully parallel condensation method for generalized eigenvalue problems on distributed memory computers. *Parallel Computing*, 21:907 – 921, 1995.

[11] L.E. Suarez and M.P. Singh. Dynamic condensation method for structural eigenvalue analysis. *AIAA Journal*, 30:1046 – 1054, 1992.

[12] H. Voss. An error bound for eigenvalue analysis by nodal condensation. In J. Albrecht, L. Collatz, and W. Velte, editors, *Numerical Treatment of Eigenvalue Problems, Vol. 3*, volume 69 of *International Series on Numerical Mathematics*, pages 205–214, Basel, 1984. Birkhäuser.

[13] H. Voss. Interior and modal masters in condensation methods for eigenvalue problems. In H. Power and J.J. Casares Long, editors, *Applications of High Performance Computing in Engineering*, volume V, pages 23 – 32, Southampton, 1997. Computational Mechanics Publications.