Modal and Interior Nodal Masters in Parallel Condensation Methods for Generalized Eigenvalue Problems

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ABSTRACT

For large–scale eigenvalue problems the authors introduced in [4] a coarse grained parallel algorithm for distributed memory computers based on substructuring and static condensation. The approach can be generalized to non–nodal masters if the support of each of the generalized masters is contained in the interior of one substructure. In this note we demonstrate that modal masters are superior to interior nodal masters.

INTRODUCTION

The finite element analysis of the dynamic response of a complex structure usually results in a large matrix eigenvalue problem. To reduce the number of unknowns to manageable size static condensation is often employed. The method is based on eliminating most of the variables (called slaves), and retaining a small number of unknown coordinates (called masters).

Partitioning the structure under consideration into substructures which connect to each other only through master variables leads to data structures and formulae which are well suited to being implemented on distributed memory MIMD parallel computers. Taking advantage of these properties in [4] we obtained a fully parallel condensation method.

The part of the spectrum which can be approximated accurately by the condensation method depends crucially on the size of the minimum slave eigenvalue. Therefore additional masters should be chosen such that the minimum slave eigenvalue is increased as much as possible without destroying the data structure that allows substructurewise determination of the condensed problem.

Obviously the data structure is preserved if we choose degrees of freedom at interior nodes of the substructures as additional masters. In [2] it is shown that the condensed problem can also be determined substructurewise and therefore completely in parallel if we allow generalized masters as long as the support of each of them is contained in the interior of one substructure.

By the minmax theorem the minimum slave eigenvalue is maximized within this class if we choose modal masters, i.e. eigenvectors of the eigenvalue problem restricted to the substructures which correspond to the minimum eigenvalues. In this note we compare the use of modal masters to interior nodal masters for plate problems. It turns out that modal masters have much better approximation properties than interior nodal masters.

We notice that the eigenvalue approximations can be enhanced by orders of magnitude using the Rayleigh functional of the exactly condensed problems which can be evaluated substructurewise, too. For reasons of limited space we do not include this improvement in this note. Details are given in [2].

NON–NODAL CONDENSATION AND SUBSTRUCTURING

Free vibration analysis of structures results in the linear eigenvalue problem

$$Kx = \lambda Mx \tag{1}$$

where the stiffness matrix $K \in \mathbb{R}^{(n,n)}$ and the mass matrix $M \in \mathbb{R}^{(n,n)}$ are real symmetric and positive definite.

The static condensation method using nodal masters involves partitioning x, K and M as

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{Bmatrix} x_m \\ x_s \end{Bmatrix} = \lambda \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{Bmatrix} x_m \\ x_s \end{Bmatrix} \tag{2}$$

where the subscripts m and s refer to master and slave, respectively. In order to eliminate the slaves the inertia terms in the second equation of (2) are neglected. Hence, $x_s = K_{ss}^{-1} K_{sm} x_m$, and we obtain from the first equation the statically condensed problem.

In this note we allow non–nodal masters. Let $\{z_1, \ldots, z_m\}$ be a set of linearly independent master-vectors. The nodal masters correspond to unit vectors with one component 1 and all others zero. Let $V \in \mathbb{R}^{(n,n)}$ be a positive definite matrix, and let $y_{m+1}, \ldots, y_n$ be an orthogonal basis of $\{z_1, \ldots, z_m\}^\perp := \{y \ : \ y^t V z_j = 0, \ j = 1, \ldots, m\}$.

With $Z := (z_1, \ldots, z_m) \in \mathbb{R}^{(n,m)}$ and $Y := (y_{m+1}, \ldots, y_n) \in \mathbb{R}^{(n,n-m)}$ every vector $x \in \mathbb{R}^n$ can be written as $x = Zx_m + Yx_s$, $x_m \in \mathbb{R}^m$, $x_s \in \mathbb{R}^{n-m}$. Using this representation, and multiplying equation (1) with the regular matrix $(Z, Y)^t$ one obtains the eigenvalue problem

$$\begin{bmatrix} K_{zz} & K_{zy} \\ K_{yz} & K_{yy} \end{bmatrix} \begin{Bmatrix} x_m \\ x_s \end{Bmatrix} = \lambda \begin{bmatrix} M_{zz} & M_{zy} \\ M_{yz} & M_{yy} \end{bmatrix} \begin{Bmatrix} x_m \\ x_s \end{Bmatrix} \tag{3}$$

where

$$K_{zz} := Z^t K Z,\ M_{zz} := Z^t M Z,\ K_{zy} := Z^t K Y,\ M_{zy} := Z^t M Y,\ \dots. \tag{4}$$

From this equation one could eliminate the variables x_s in the usual way and determine the condensed problem. However, in practice the matrix Y is not available, and one has access only to the small number of vectors $z_1, \dots, z_m$. In [1] a general method to perform condensation using only the matrix Z without having access to Y is presented:

Theorem 1: *Let $z_1, \dots, z_m \in \mathbb{R}^n$ be linearly independent. Then the statically condensed eigenvalue problem corresponding to problem (3) is given by*

$$P^t K P x_m = \lambda P^t M P x_m \tag{5}$$

where the matrix $P \in \mathbb{R}^{(n,m)}$ can be calculated from

$$\begin{bmatrix} K & -VZ \\ -Z^t V & O \end{bmatrix} \begin{bmatrix} P \\ S \end{bmatrix} = \begin{bmatrix} O \\ -I_m \end{bmatrix}. \tag{6}$$

Moreover

$$P^t K P = S. \tag{7}$$

We now assume that the structure under consideration is decomposed into r substructures and that m_0 nodal masters are chosen such that the substructures are connected to each other through these masters only. Then the stiffness matrix can be written as

$$K = \begin{bmatrix} K_{mm} & K_{ms1} & K_{ms2} & \dots & K_{msr} \\ K_{sm1} & K_{ss1} & O & \dots & O \\ K_{sm2} & O & K_{ss2} & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{smr} & O & O & \dots & K_{ssr} \end{bmatrix},$$

where $K_{mm} \in \mathbb{R}^{(m_0,m_0)}$, $K_{msj} \in \mathbb{R}^{(m_0,s_j)}$ and $K_{ssj} \in \mathbb{R}^{(s_j,s_j)}$, and the mass matrix M has the same block form.

We assume that the support of each of the additional masters is contained in the interior of one substructure. We collect the masters corresponding to the j–th substructure in the matrix $Z_j \in \mathbb{R}^{(s_j,m_j)}$ where s_j denotes the number of interior degrees of freedom of the j-th substructure and m_j is the number of these additional interior nodal or modal masters having their support in the j-th substructure.

Finally, let

$$V = \operatorname{diag}\{I_{m_0}, M_{ss1}, \dots, M_{ssr}\}.$$

Then the condensed eigenvalue problem

$$K_0\xi := P^t K P \xi = \lambda P^t M P \xi =: \lambda M_0 \xi, \tag{8}$$

can be determined in the following way (cf. [2])

(i) For $j = 1, \dots, r$ solve the linear systems

$$\begin{bmatrix} K_{ssj} & -M_{ssj}Z_j \\ -Z_j^t M_{ssj} & O \end{bmatrix} \begin{bmatrix} P_j & Q_j \\ S_j & R_j \end{bmatrix} = \begin{bmatrix} -K_{smj} & O \\ O & -I_{m_j} \end{bmatrix} \tag{9}$$

with $P_j \in \mathbb{R}^{(s_j,m_0)}$, $Q_j \in \mathbb{R}^{(s_j,m_j)}$, $S_j \in \mathbb{R}^{(m_j,m_0)}$ and $R_j \in \mathbb{R}^{(m_j,m_j)}$.

These are r decoupled systems of $s_j + m_j$ linear equations. Notice that most of the columns of the matrix K_{smj} are null vectors, and that only those columns of K_{smj} have to be considered which correspond to the nodal master degrees of freedom on the boundary of the j-th substructure.

(ii) Compute

$$\tilde{S} := K_{mm} + \sum_{j=1}^{r} K_{msj} P_j.$$

(iii) From (7) it follows that the reduced stiffness matrix is given by

$$K_0 = \begin{bmatrix} \tilde{S} & S_1^t & \dots & S_r^t \\ S_1 & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ S_r & O & \dots & R_r \end{bmatrix}.$$

(iv) The reduced mass matrix $M_0 = P^t M P$ can be determined in the following way: For $j = 1, \dots, r$ compute

$$U_j := M_{msj} P_j,\ V_j := M_{msj} Q_j,\ X_j := M_{ssj} P_j,\ Y_j := M_{ssj} Q_j.$$

Then

$$M_0 = \begin{bmatrix} M_{mm} + \sum_{j=1}^{r} (U_j + U_j^t + P_j^t X_j) & V_1 + P_1^t Y_1 & \dots & V_r + P_r^t Y_r \\ V_1^t + Y_1^t P_1 & Q_1^t Y_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ V_r^t + Y_r^t P_r & O & \dots & Q_r^t Y_r \end{bmatrix}$$

Observe that most of the computations can be done substructurewise and therefore completely in parallel. Only the compilation of the reduced matrices K_0 and M_0 requires communication between the processes.

A NUMERICAL EXAMPLE

Consider the free vibration analysis of a uniform thin clamped plate covering the rectangular region $\Omega := [0,4] \times [0,3]$ which is governed by the eigenvalue problem

$$\Delta^2 u = \lambda u \quad \text{in } \Omega \qquad \text{and} \qquad u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{10}$$

We discretized problem (10) by rectangular Bogner-Fox-Schmidt elements with node variables u, u_x, u_y, u_{xy}. Figure 1 shows the mesh under consideration. The finite element model contains 144 elements, 121 nodes, and $n = 484$ degrees of freedom.

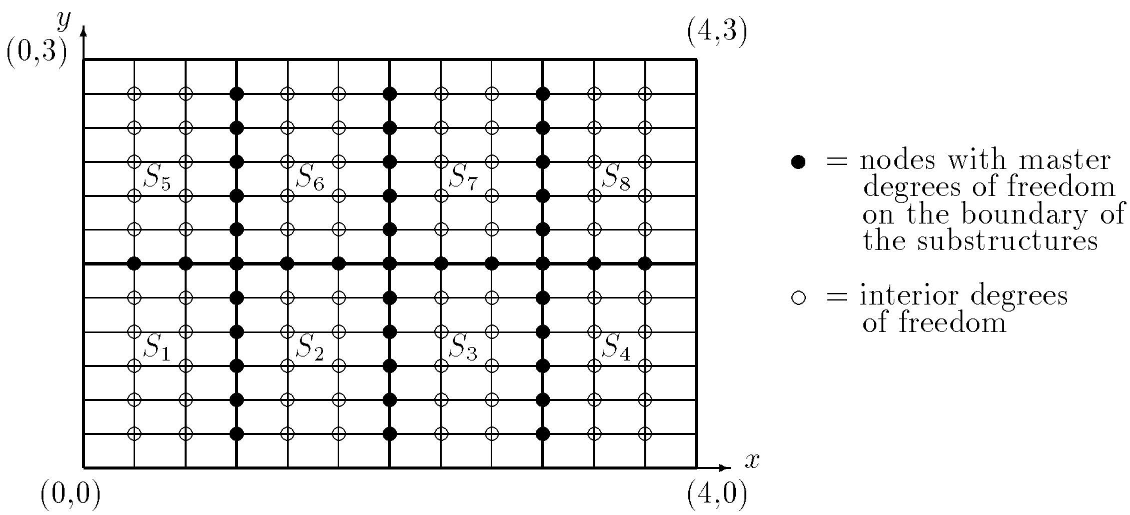


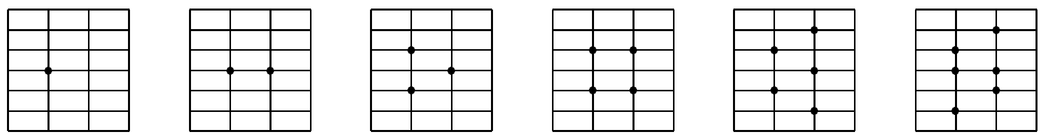
Figure 1. Clamped plate (divided into 8 substructures)

We divided Ω into $r = 8$ identical rectangles of edge length 1 in x-direction and 3/2 in y-direction each of which has $s_j = 40$, $j = 1, \cdots, 8$ interior degrees of freedom. Choosing the unknowns on the boundaries of the substructures as masters we obtained a condensed eigenvalue problem of dimension $m_0 = 164$.

Next we considered an identical number m_j of additional modal masters in each substructure, i.e. the generalized masters defined by the matrix Z_j in (9) which contains in its columns the eigenvectors of the discretized rectangular clamped plate covering the region $[0,1] \times [0,1.5]$ corresponding to the m_j smallest eigenvalues.

In order to demonstrate the merits of modal masters we compared them to interior nodal masters. In each substructure we chose an identical number m_j of displacements at interior nodes of the substructures as additional masters (slopes at interior nodes yielded inferior results). The resulting condensed problem (8) can be determined substructurewise in the same way as in [4] or as in the previous section if the matrix Z_j in (9) consists of unit vectors with exactly one unit component.

Figure 2 shows the optimal positioning of the interior master displacements for $m_j = 1, \cdots, 6$ which were obtained experimentally.



$m_j = 1$ $m_j = 2$ $m_j = 3$ $m_j = 4$ $m_j = 5$ $m_j = 6$

Figure 2. Positioning of interior nodal masters in each substructure

Table 1 compares the application of interior modal masters to that of interior nodal ones in the following way: It shows the number of eigenvalues at the lower end of the spectrum that are approximated with relative error less than one percent. Particularly, if we employ 6 additional masters in each substructure then we increase the dimension of the condensed problem from $m_0 = 164$ to $m = 212$, and using modal masters we are able to approximate the first 27 harmonics with relative error of less than 1% whereas with nodal masters we obtain only the the first 16 harmonics with this accuracy.

$j = 1, \cdots, 8, \quad m_j =$	0	1	2	3	4	5	6
dimension	164	172	180	188	196	204	212
modal	1	5	6	10	17	22	27
nodal	1	2	5	6	8	12	16

Table 1. Number of eigenvalues approximated with less than one percent

This demonstrates the superiority of the interior modal masters to the interior nodal ones.

For the parallel computation we used a distributed memory transputer system by PARSYTEC with four processors and multiprocessing capability. In our example the calculation is divided into 12 processes, which are connected according to Figure 3. The mapping of the processes to the processors of the transputer system can be seen in Figure 4.

The parallel computation is done in two steps. In the first step the condensed eigenvalue problem (8) is determined. To this end nine processes 'Ma'and 'S_1',$\cdots$, 'S_8' are created where the process S_j is associated to the

j-substructure (cf. Figure 1), and its task is to compute the data which are required from the j-substructure. This computation can be done fully in parallel without any communication between the processes.

The process 'Ma' compiles the condensed matrices K_0 and M_0. Therefore 'Ma' computes K_{mm} and M_{mm}, and at the end of step one it collects the information from the slaves. The processes are connected by a tree topology (cf. Figure 3), because an intermediate summation of the slave information in 'S_1' and 'S_5' is necessary to avoid a bottleneck at 'Ma'.

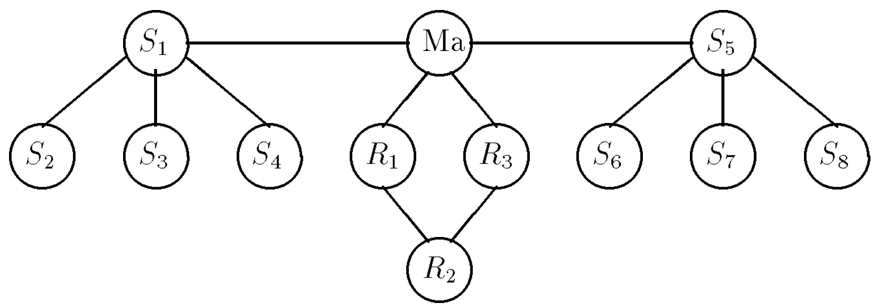


Figure 3. Process topology for 8 substructures

In the second step the condensed eigenvalue problem (8) is solved by a parallel solver for dense matrix eigenvalue problems, which is described in [4] and in detail in [3]. This parallel solver is based on a ring topology which in our example is realized by the processes 'Ma', 'R_1', 'R_2' and 'R_3' (cf. Figure 3).

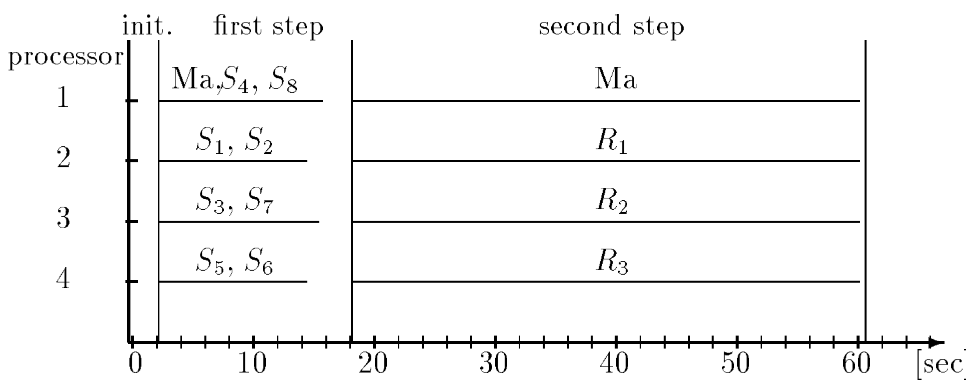


Figure 4. Loading diagram for 4 processors

Figure 4 shows the computational load of the processors for $m_j = 5$ as well as the mapping of the processes to the processors. It demonstrates that the parallel computation is well balanced. It is also efficient because of its coarse granularity, especially the parallel eigensolver has an efficiency of about 90 per cent (cf. [3]).

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