Non Nodal Condensation of Eigenvalue Problems

Wolfgang Mackens and Heinrich Voss
Arbeitsbereich Mathematik, Technische Universität Hamburg-Harburg,
Kasernenstraße 12, 21073 Hamburg, Federal Republic of Germany
E-mail: {mackens, voss}@tu-harburg.d400.de,
URL: http:/www.tu-harburg.de/mat

Abstract

We generalize the Guyan condensation of large symmetric eigenvalue problems to allow general degrees of freedom to be master variables. On one hand useful information from other condensation methods (such as Component Mode Synthesis) thus can be incorporated into the method. On the other hand this opens the way to iterative refinement of eigenvector approximations. Convergence of such a procedure follows from the result, that one step of (static) condensation is equivalent to one step of inverse subspace iteration. A short outlook on several applications is included.

Zusammenfassung

Durch eine Verallgemeinerung der Guyan Kondensation großer symmetrischer Eigenwertprobleme wird die Verwendung allgemeiner Freiheitsgrade als Master-Variable ermöglicht. Dadurch können einerseits Ansatzvektoren aus anderen Kondensationsmethoden in die Guyan Kondensation eingebunden werden, und andererseits werden damit gewonnene Eigenvektorapproximationen iterativ verfeinerbar. Die Konvergenz einer solchen Verfeinerung folgt aus dem Resultat, daß die (statische) Kondensation als ein Schritt der inversen Unterraum Iteration interpretiert werden kann. Ein kurzer Ausblick auf Anwendungsgebiete schließt die Arbeit ab.

Keywords: Symmetric eigenvalue problem, Guyan condensation, generalized condensation

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1 Introduction

When discretizing continuous problems one often finds that a sufficiently accurate representation of the desired data in the discrete model requires the use of prohibitively many degrees of freedom, such that a standard treatment of the resulting large set of discrete equations is far too expensive.
For such situations several reduction techniques have been developed in different disciplines. These aim at incorporating specific parts of the (global) good approximation behaviour of the large size models into much smaller systems derived from the larger ones (cf. the survey paper [19] on reduction methods, e.g.).

In the study of structural vibrations large algebraic eigenvalue problems

$$\boldsymbol{K}\boldsymbol{x} = \lambda \boldsymbol{M}\boldsymbol{x} \tag{1}$$

arise, where the stiffness matrix $\boldsymbol{K} \in \mathbb{R}^{(n,n)}$ and the mass matrix $\boldsymbol{M} \in \mathbb{R}^{(n,n)}$ are real symmetric and positive definite, $\boldsymbol{x}$ is the vector of modal displacements and λ is the square of the natural frequencies.

Most reduction methods for eigenvalue problems consist of a **projection to a low dimensional subspace** $\mathcal{X}$. Given a basis $\tilde{\boldsymbol{x}}^1, \ldots, \tilde{\boldsymbol{x}}^m$ $(m << n)$ of $\mathcal{X}$ and putting

$$\boldsymbol{X} := \left(\tilde{\boldsymbol{x}}^1, \ldots \tilde{\boldsymbol{x}}^m\right)$$

the system (1) is replaced by the m-dimensional **projected eigenvalue problem**

$$\boldsymbol{K}_{\mathcal{X}}\boldsymbol{y} = \kappa \boldsymbol{M}_{\mathcal{X}}\boldsymbol{y}, \tag{2}$$

with the projected stiffness and mass matrices

$$\boldsymbol{K}_{\mathcal{X}} := \boldsymbol{X}^T\boldsymbol{K}\boldsymbol{X} \quad \text{and} \quad \boldsymbol{M}_{\mathcal{X}} := \boldsymbol{X}^T\boldsymbol{M}\boldsymbol{X}. \tag{3}$$

The eigenvectors $\boldsymbol{y}^i$ of (2) with their corresponding eigenvalues κ_i are afterwards lifted back to give approximate eigenpairs $(\hat{\lambda}_i, \hat{\boldsymbol{x}}^i)$ through

$$(\hat{\lambda}_i, \hat{\boldsymbol{x}}^i) := (\kappa_i, \boldsymbol{X}\boldsymbol{y}^i).$$

The elementary properties of this approach can be read up in [21] or [26], e.g.
If $m = 1$ then (2) is a scalar equation, which can be solved for the Rayleigh quotient of the onedimensional stiffness and mass matrix condensations:

$$R_{\{K,M\}}(\boldsymbol{x}) := \kappa = \frac{\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}}.$$

As is well known the Rayleigh quotient extraction of eigenvalues from eigenvector estimates produces eigenvalue approximations of increased quality: If the error of an eigenvector approximation is of first order $(O(\varepsilon))$, then the error of the corresponding Rayleigh quotient is of second order $(O(\varepsilon^2))$. The same holds for the more general multidimensional approximation (2). This is why nearly every algorithm for the approximation of eigenpairs uses projection as a basic step when it comes to compute eigenvalue approximations from eigenvector information.
Differences between algorithms are mostly found in the way in which eigenvector approximations are build or how these are updated to improve their quality.

In mathematics one likes to span the subspaces $\mathcal{X}_n$ by vectors derived from few starting vectors through the iterative application of a specific iteration matrix. Using $\mathcal{X}_n := \{\boldsymbol{z}, \boldsymbol{B}\boldsymbol{z}, \boldsymbol{B}^2\boldsymbol{z}, \ldots, \boldsymbol{B}^{n-1}\boldsymbol{z}\}$ with $\boldsymbol{B} = (\boldsymbol{K} - \sigma\boldsymbol{M})^{-1}\boldsymbol{M}$, e.g., leads to Krylov space methods, [26]. Alternatively, if one projects onto $\mathcal{X}_n := \boldsymbol{B}^{n-1}\mathcal{X}_0$, with $\mathcal{X}_0$ some starting space and $\boldsymbol{B} \in \{\boldsymbol{K}^{-1}\boldsymbol{M}, \boldsymbol{M}^{-1}\boldsymbol{K}\}$ one arrives at subspace iteration methods, [21]. In any case mathematical algorithms prefer the iterative approximation of the eigenpairs invoking little or no a priori knowledge of the eigenvectors in general. The approximation quality generally comes from a power method type enrichment of the relevant eigenvector-components within an iteratively improved set of starting vectors the initial choice of which is more or less arbitrary. What counts is the method of improvement, the starting vectors are of minor importance.

Quite contrary the engineer aims at constructing from engineering knowledge within one single step the final projection space $\mathcal{X}_m$ which gives the desired eigenpair approximations (up to a relative error of about 1 %) with no iterative refinement at all. There are several good reasons for such an approach:

- Normally, the engineer either has a lot of a priori information about the desired eigenfunctions themselves or he knows a lot about subsystems which have already been extensively analysed. It makes sense to exploit this knowledge.
- In case of eigenvalue reanalysis [6, 31] eigenpairs of very similar problems are known, since the actual eigenproblem is just a small perturbation of these. The eigenvectors of the old problems should be ideal members of a projection space to analyse the new problem.
- In Component Mode Synthesis (CMS) vibration modes of subsystems (both with fixed subsystem boundaries or with free boundaries) make up a major part of projection spaces. They are complemented by what are called attachment modes (static responses of components to applied loads) or constraint modes (static responses of components to prescribed boundary displacements) as well as coupling constraints to join them together, cf. [11, 7, 2].

This engineering approach can be very successful in the hands of a skilled engineer, since it allows him to make use of his well trained engineering intuition.
It can be very unsatisfactory, however, if the final approximations fail to have the desired precision. The case of too high a precision is of course only of minor annoyance. Only within the actual calculation work is wasted. For follow up analyses the number of basis vectors can be adequately reduced by analyzing their contributions to the actual eigenvector approximations (cf. [3]). If the precision is to low, however, one has to construct additional vectors to augment the previous basis and to rerun both the projection and the eigenvalue analysis. This is quite tedious and hence attempts have been made to reduce the effort of this task by taking into account the results of the previous condensation (cf. [23]). Computation of approximations with increased accuracy serve at the same time as a means to estimate the accuracy of the gained approximations.

Only very seldomly engineers seem to use exact computational error bounds like those of Krylov-Bogoliubov, Kahan-Krylov-Bogoliubov or the Kato-Temple-bound (cf. [21], e.g.), though these are known in the engineering literature, too (cf. [8]), and though they are easily implemented in practice [10]. Even more seldomly engineers use information from error estimates to construct additional ansatz vectors or to adapt the previously used projection space (see [4], however).
Vice versa, mathematicians do not generally design algorithms for eigenanalysis with a primary goal to allow engineering know how to be included.

The present paper is devoted to the study of Guyan condensation [9, 12]. In engineering this is a technique to construct basis vectors of projection spaces. Using the language of CMS these vectors could be called global constraint modes since they are static responses of the full system to prescribed displacements of so called master (nodal) degrees of freedom. To this end the slave (=nonmaster) components are determined as linear functions of the masters through the solution of a "slave part" of the system.
We shall generalize this approach to allow general non nodal degrees of freedom to be masters. In the course of this generalization it will become clear, that Guyan condensation can be interpreted as simultaneous inverse iteration. Consequently, Guyan condensation can be refined iteratively and we are able to hybridize engineering and mathematical approaches.

The next section reviews fundamental properties of nodal condensation. Section 3 defines non nodal condensation, and it shows that condensation can be performed without explicit access to the slave part of the system. Furthermore as a byproduct condensation is interpreted as an inverse iteration process. In Section 4 we derive a system, by which the non nodal condensation can be computed in a numerically stable way. The system is closely related to saddle point equations from constrained eigenvalue problems. We

indicate connections to the Weinstein-Stenger theory of intermediate eigenvalues (cf. [1], [32]) and to Kron's method to compute eigenvalues of coupled subsystems (cf. [28]). We close the paper in Section 5 with a short outlook on applications under investigation.

2 Nodal condensation

Condensation methods for large eigenvalue problems are subspace projection methods together with a specific Gaussian elimination flavoured approach to construct reasonable approximations of projection spaces $\mathcal{X}$.

To this end some (relatively few) components of the vector $\boldsymbol{x}$ are selected to be **masters** and to form a master part $\boldsymbol{x}_m \in \mathbb{R}^m$ of $\boldsymbol{x}$. The aim is then to construct an eigenproblem

$$\boldsymbol{K}_0 \boldsymbol{x}_m = \lambda \boldsymbol{M}_0 \boldsymbol{x}_m \tag{4}$$

for these master-vectors and the eigenparameter λ such that the eigenvectors of (4) are good approximations to the masterparts of selected eigenvectors of (1) with similarly good approximation behaviour for the accompanying eigenvalues.

To accomplish this one decomposes equation (1) into block form

$$\begin{bmatrix} \boldsymbol{K}_{mm} & \boldsymbol{K}_{ms} \\ \boldsymbol{K}_{sm} & \boldsymbol{K}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} = \lambda \begin{bmatrix} \boldsymbol{M}_{mm} & \boldsymbol{M}_{ms} \\ \boldsymbol{M}_{sm} & \boldsymbol{M}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} \tag{5}$$

where $\boldsymbol{x}_m \in \mathbb{R}^m$ containes the mastervariables, $\boldsymbol{x}_s \in \mathbb{R}^s$ collects the remaining variables, **the slaves**, and where the permutation of $\boldsymbol{x}$ leading to the new order $\boldsymbol{x}_m, \boldsymbol{x}_s$ of the variables has been applied likewise to the rows as to the columns of $\boldsymbol{K}$ and $\boldsymbol{M}$. Then these matrices are still symmetric and positive definite in their permuted form.

Now we see that if the master part $\tilde{\boldsymbol{x}}_m$ of an eigenvector $\tilde{\boldsymbol{x}}$ is given together with the corresponding eigenvalue $\tilde{\lambda}$ then the slavepart $\tilde{\boldsymbol{x}}_s$ can be computed from the second row of (5) through the **master-slave-extension**

$$\tilde{\boldsymbol{x}}_s = \boldsymbol{S}(\tilde{\lambda})\tilde{\boldsymbol{x}}_m := -(\boldsymbol{K}_{ss} - \tilde{\lambda}\boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \tilde{\lambda}\boldsymbol{M}_{sm})\tilde{\boldsymbol{x}}_m \tag{6}$$

as long as the matrix

$$(\boldsymbol{K}_{ss} - \tilde{\lambda}\boldsymbol{M}_{ss}) \text{ is regular.} \tag{7}$$

The latter condition is usually expressed as $\tilde{\lambda}$ not being an eigenvalue of the **slave-eigenvalue problem**

$$\boldsymbol{K}_{ss}\boldsymbol{x}_s = \lambda \boldsymbol{M}_{ss}\boldsymbol{x}_s, \tag{8}$$

which (cf. Section 4) can be seen to be the the eigenvalue problem corresponding to the vibration of the slave-portion of the system with the master degrees of freedom restricted to be zero.
With the master-slave extension a prolongation

$$\mathcal{P}(\tilde{\lambda})\boldsymbol{x}_m := \begin{pmatrix} \boldsymbol{x}_m \\ \boldsymbol{S}(\tilde{\lambda})\boldsymbol{x}_m \end{pmatrix} \tag{9}$$

of $\mathbb{R}^m$-vectors $\boldsymbol{x}_m$ to full space vectors is defined which would reproduce an eigenvector $\tilde{\boldsymbol{x}}$ from its master part $\tilde{\boldsymbol{x}}_m$ (if the latter is different from zero).

The masterparts of relevant eigenvectors are most easily caught by varying $\boldsymbol{x}_m$ over all of $\mathbb{R}^m$. The linear space of all full space vectors of the form $\mathcal{P}(\tilde{\lambda})\boldsymbol{x}_m, \quad \boldsymbol{x}_m \in \mathbb{R}^m$ is the span of the columns of

$$\mathcal{P}(\tilde{\lambda}) := \begin{bmatrix} \boldsymbol{I}_m \\ \boldsymbol{S}(\tilde{\lambda}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_m \\ -(\boldsymbol{K}_{ss} - \tilde{\lambda}\boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \tilde{\lambda}\boldsymbol{M}_{sm}) \end{bmatrix},$$

which is the matrix representation of the operator $\mathcal{P}(\tilde{\lambda})$. Subspace projection of the original problem onto the column space of this matrix leads to a small problem for the masterparts $\boldsymbol{x}_m$ which is called **the condensation to the chosen master-variables at** $\tilde{\lambda}$,

$$(\underbrace{\mathcal{P}(\tilde{\lambda})^T \boldsymbol{K} \mathcal{P}(\tilde{\lambda})}_{\boldsymbol{K}_0} - \lambda \underbrace{\mathcal{P}(\tilde{\lambda})^T \boldsymbol{M} \mathcal{P}(\tilde{\lambda})}_{\boldsymbol{M}_0}) \boldsymbol{x}_m = 0. \tag{10}$$

This eigenproblem certainly would reproduce $\tilde{\lambda}$ since we assumed $\tilde{\lambda}$ to be an eigenvalue of the original problem. Furthermore the corresponding eigenvectors would be prolongated to eigenvectors of the original system. Thus this reduction would retain the approximation quality of the large system with respect to the eigeninformation connected with the eigenvalue $\tilde{\lambda}$.

Of course an exact eigenvalue $\tilde{\lambda}$ of the original system is not available. Hence one has to use suitable substitutes for the eigenvalue. Depending on the specific choice of $\tilde{\lambda}$ specific names for the condensation processes are common use:

- $\tilde{\lambda} = 0$ is known as **static condensation** [9, 12],
- $\tilde{\lambda} = \hat{\lambda}$ with a given fixed $\hat{\lambda}$ is called **dynamic condensation** [13, 14] and
- $\tilde{\lambda} = \lambda$ with variable λ leads to **exact condensation** [14, 29, 22], which produces a nonlinear condensed eigenvalue problem.

Explanations of the names and some further discussion of the pros and cons of these methods may, e.g., be found in [10]. It should be remarked that an extension of dynamic condensation presented there gives particularly good results. From eigenpairs of the statically ($\tilde{\lambda} = 0$) condensed equation (10), one derives vectors of a new projection space by prolongating the small space eigenvector-approximations $\boldsymbol{y}^i$ with their corresponding eigenvalue approximations: $P(\kappa_i)\boldsymbol{y}^i$. This method independently has been found in [20].

All the cited condensation methods are **nodal condensation methods** in that the masters and slaves are always chosen from the nodal degrees of freedom in which the problem has been set up. Usually, these degrees are dictated by the method of discretization and are not adapted to the specific eigenproblem.
Intuition tells us that the condensation will produce the better results the better the true eigenvectors (to be approximated) can be represented by master components. If e.g. the first unit vector would be an eigenvector, then chosing the first component as a master would approximate this eigenvector and the corresponding eigenvalue exactly and independently of the choice of $\tilde{\lambda}$.
This situation can be induced by a basis transformation with an exact eigenvector as first new basis vector. With an approximate eigenvector instead such a proceeding should still give similar results. We shall see that condensation to general degrees of freedom can be performed without having to carry out a corresponding basis transformation.

3 Non nodal condensation, fundamental results

Our aim here is to generalize the condensation techniques to allow for general degrees of freedom to be chosen as masters. To this end we assume that a set of linearly independent **master-vectors**

$$\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m$$

is given, with a specific (not yet specified) potential to help with the modeling of the problem. Within the nodal condensation methods these vectors are unit vectors (with all components equal to zero except for exactly one unit component) with the totality of nonzero components (of all $\boldsymbol{z}$-vectors) specifying the master components. As we shall see it makes sense to use as $\boldsymbol{z}$-vectors all of the above cited ansatz vectors from engineering approaches to projection methods.

Together with the $\boldsymbol{z}$-vectors we assume an inner product on $\mathbb{R}^n$ to be given by a symmetric positive definite matrix $\boldsymbol{V}$ via

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^T \boldsymbol{V} \boldsymbol{y}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

If we then let

$$\boldsymbol{y}_{m+1}, \ldots, \boldsymbol{y}_n$$

denote a complementary orthonormal basis of

$$\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m\}^\perp := \{\boldsymbol{y} \ : \ \boldsymbol{y}^T \boldsymbol{V} \boldsymbol{z}_j = 0, \ j = 1, \ldots, m\}$$

and if we put

$$\begin{aligned} \boldsymbol{Z} &:= (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m) \in \mathbb{R}^{(n,m)}, \\ \boldsymbol{Y} &:= (\boldsymbol{y}_{m+1}, \ldots, \boldsymbol{y}_n) \in \mathbb{R}^{(n,n-m)}, \end{aligned} \tag{11}$$

such that

$$(\boldsymbol{Z}, \boldsymbol{Y}) \in \mathbb{R}^{(n,n)} \ \text{ is regular and } \ \boldsymbol{Z}^T \boldsymbol{V} \boldsymbol{Y} = \boldsymbol{O}, \tag{12}$$

then the master-slave-splitting of a vector $\boldsymbol{x} \in \mathbb{R}^n$ into its master- and slave-parts with master- and slave-component-vectors $\boldsymbol{x}_m \in \mathbb{R}^m$ and $\boldsymbol{x}_s \in \mathbb{R}^{n-m}$, respectively, is given by

$$\boldsymbol{x} = \boldsymbol{Z} \boldsymbol{x}_m + \boldsymbol{Y} \boldsymbol{x}_s \,.$$

Going with this representation into (1) and multiplying with the regular matrix $(\boldsymbol{Z}, \boldsymbol{Y})^T$ one arrives at

$$\begin{bmatrix} \boldsymbol{K}_{zz} & \boldsymbol{K}_{zy} \\ \boldsymbol{K}_{yz} & \boldsymbol{K}_{yy} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} = \lambda \begin{bmatrix} \boldsymbol{M}_{zz} & \boldsymbol{M}_{zy} \\ \boldsymbol{M}_{yz} & \boldsymbol{M}_{yy} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} \tag{13}$$

with

$$\begin{aligned} \boldsymbol{L}_{zz} &:= \boldsymbol{Z}^T \boldsymbol{L} \boldsymbol{Z}, \quad \boldsymbol{L}_{zy} := \boldsymbol{Z}^T \boldsymbol{L} \boldsymbol{Y} \\ \boldsymbol{L}_{yz} &:= \boldsymbol{Y}^T \boldsymbol{L} \boldsymbol{Z}, \quad \boldsymbol{L}_{yy} := \boldsymbol{Y}^T \boldsymbol{L} \boldsymbol{Y} \end{aligned} \ , \quad \boldsymbol{L} \in \{\boldsymbol{K}, \boldsymbol{M}\} \,. \tag{14}$$

In the special case that $\boldsymbol{Z}$ and $\boldsymbol{Y}$ consist of unit vectors only and $\boldsymbol{V} = \boldsymbol{I}$ this splitting reproduces the splitting (5).

Theoretically, (13) could serve as a basis for all three sorts of condensation approaches for general masters just as (5) was the basis in the case of nodal masters.
There is, however, a very strong practical objection to such a naive approach. In the non nodal case we can in general not exclude the case that *though the vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m$ are well accessible the complementary vectors $\boldsymbol{y}_{m+1}, \ldots, \boldsymbol{y}_n$ are definitely not!*
While it is always possible to compute and handle a small number of m master-vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m$ it is impossible for large systems in general to store the data needed for a representation of the $\boldsymbol{y}$-vectors.
Concerning the system (13) this means (in contrast to system (5)) that with the exception of $\boldsymbol{K}_{zz}$ and $\boldsymbol{M}_{zz}$ all other submatrices are not accessible. Hence condensation — as a naive adaptation of the nodal case — will no longer work.

In the present paper we focus on showing how the condensation method can be performed using only the basis $\boldsymbol{Z} = (\boldsymbol{z}^1, \ldots, \boldsymbol{z}^m)$ and without having access to a basis $\boldsymbol{Y}$ of the orthogonal complement $\boldsymbol{Z}^\perp$. To do so let us first derive an expression for the

prolongation $\mathcal{P}$ from (9) correspondig to (13). Written with the coordinates $\boldsymbol{x}_m$ and $\boldsymbol{x}_s$ the master-slave-extension (6) of a master vector $\tilde{\boldsymbol{x}}_m$ reads:

$$\tilde{\boldsymbol{x}}_s = -(\boldsymbol{K}_{yy} - \tilde{\lambda}\boldsymbol{M}_{yy})^{-1}(\boldsymbol{K}_{yz} - \tilde{\lambda}\boldsymbol{M}_{yz})\tilde{\boldsymbol{x}}_m. \tag{15}$$

The vector of coordinates $(\tilde{\boldsymbol{x}}_m, \tilde{\boldsymbol{x}}_s)$ with respect to the basis $(\boldsymbol{Z}, \boldsymbol{Y})$ corresponds to the vector

$$\tilde{\boldsymbol{x}} = \boldsymbol{Z}\tilde{\boldsymbol{x}}_m + \boldsymbol{Y}\tilde{\boldsymbol{x}}_s. \tag{16}$$

Let us restrict our attention to static condensation first (i.e. let $\tilde{\lambda} = 0$). Inserting $\tilde{\boldsymbol{x}}_m$ and $\tilde{\boldsymbol{x}}_s$ from (15) into (16) and using the expressions (14) for the projected matrices $\boldsymbol{K}_{yy}, \boldsymbol{K}_{yz}$ results in

$$\tilde{\boldsymbol{x}} = \mathcal{P}\tilde{\boldsymbol{x}}_m := \left(\boldsymbol{I} - \boldsymbol{Y}\left[\boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{Y}\right]^{-1}\boldsymbol{Y}^T\boldsymbol{K}\right)\boldsymbol{Z}\tilde{\boldsymbol{x}}_m. \tag{17}$$

Using the prolongation operator $\mathcal{P}$ the statically condensed matrices $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$ read

$$\boldsymbol{K}_0 := \mathcal{P}^T\boldsymbol{K}\mathcal{P} = \boldsymbol{Z}^T\boldsymbol{K}\boldsymbol{Z} - \boldsymbol{Z}^T\boldsymbol{K}\boldsymbol{Y}\left(\boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{Y}\right)^{-1}\boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{Z} \tag{18}$$

and

$$\boldsymbol{M}_0 := \mathcal{P}^T\boldsymbol{M}\mathcal{P}. \tag{19}$$

Thus the condensation data seem to rely heavily on the large matrix $\boldsymbol{Y}$. The next theorem gives a first expression for $\mathcal{P}$ which does not involve $\boldsymbol{Y}$ at all.

Theorem 1:

Let $\boldsymbol{Z} \in \mathbb{R}^{(n,m)}$ and $\boldsymbol{Y} \in \mathbb{R}^{(n,n-m)}$ such that (12) holds with the symmetric positive definite metric matrix $\boldsymbol{V}$. Then with $\boldsymbol{X} := \boldsymbol{V}\boldsymbol{Z}$ one has

$$\boldsymbol{Y}\left(\boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{Y}\right)^{-1}\boldsymbol{Y}^T = \boldsymbol{K}^{-1} - \boldsymbol{K}^{-1}\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{K}^{-1}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{K}^{-1}, \tag{20}$$

such that

$$\mathcal{P} = \boldsymbol{K}^{-1}\boldsymbol{X}\left[\boldsymbol{X}^T\boldsymbol{K}^{-1}\boldsymbol{X}\right]^{-1}\boldsymbol{X}^T\boldsymbol{Z}. \tag{21}$$

Proof:

Let $\boldsymbol{v} := \boldsymbol{Y}(\boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{Y})^{-1}\boldsymbol{Y}^T\boldsymbol{w}$.

From the assumptions on $\boldsymbol{Z}$ and $\boldsymbol{Y}$ it follows that $(\boldsymbol{V}\boldsymbol{Z}, \boldsymbol{K}\boldsymbol{Y})$ is regular since

$$\boldsymbol{V}\boldsymbol{Z}\boldsymbol{\alpha} = \boldsymbol{K}\boldsymbol{Y}\boldsymbol{\beta} \Rightarrow \boldsymbol{0} = \boldsymbol{Y}^T\boldsymbol{V}\boldsymbol{Z}\boldsymbol{\alpha} = \underbrace{\boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{Y}}_{\text{SPD}}\boldsymbol{\beta} = \boldsymbol{0} \Rightarrow \boldsymbol{\beta} = \boldsymbol{0} \Rightarrow \boldsymbol{\alpha} = \boldsymbol{0}.$$

With the regularity of $(\boldsymbol{V}\boldsymbol{Z}, \boldsymbol{K}\boldsymbol{Y})$ one concludes now that

$$\begin{array}{rcl}
\boldsymbol{v} & := & \boldsymbol{Y}(\boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{Y})^{-1}\boldsymbol{Y}^T\boldsymbol{w} \\
 & \Longleftrightarrow & \\
\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{v} = \boldsymbol{0} & \text{and} & \boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{v} = \boldsymbol{Y}^T\boldsymbol{w}, \\
 & \Longleftrightarrow & \\
\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{v} = \boldsymbol{0} & \text{and} & \boldsymbol{Y}^T(\boldsymbol{K}\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{0}, \\
 & \Longleftrightarrow & \\
\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{v} = \boldsymbol{0} & \text{and} & (\boldsymbol{K}\boldsymbol{v} - \boldsymbol{w}) \in \operatorname{span}(\boldsymbol{V}\boldsymbol{Z}), \\
 & \Longleftrightarrow & \\
\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{v} = \boldsymbol{0} & \text{and} & \boldsymbol{v} = \boldsymbol{K}^{-1}\boldsymbol{w} + \boldsymbol{K}^{-1}\boldsymbol{V}\boldsymbol{Z}\boldsymbol{\alpha} \text{ for some } \boldsymbol{\alpha} \in \mathbb{R}^m.
\end{array}$$

The first of these two equations uniquely determines $\boldsymbol{\alpha}$ from the second one to be

$$\boldsymbol{\alpha} = -\left[\boldsymbol{X}^T\boldsymbol{K}^{-1}\boldsymbol{X}\right]^{-1}\boldsymbol{X}^T\boldsymbol{K}^{-1}\boldsymbol{w}.$$

Putting this into

$$\boldsymbol{v} = \boldsymbol{K}^{-1}\boldsymbol{w} + \boldsymbol{K}^{-1}\boldsymbol{V}\boldsymbol{Z}\boldsymbol{\alpha},$$

gives the desired formula (20). Inserting (20) into (17) we obtain (21). ∎

In the next corollary we collect the consequences of Theorem 1 for the condensed stiffness and mass matrices, $\boldsymbol{K}_0 = \mathcal{P}^T\boldsymbol{K}\mathcal{P}$ and $\boldsymbol{M}_0 = \mathcal{P}^T\boldsymbol{M}\mathcal{P}$. To give the result a comfortable form, we assume that the column vectors of $\boldsymbol{Z}$ form a $\boldsymbol{V}$-orthonormal basis of the projection space. Then the matrix $\boldsymbol{X}^T\boldsymbol{Z}$ at the end of equation (21) can be dropped. Since the number of $\boldsymbol{z}$-vectors is small, this is numerically not too restrictive an assumption.

Corollary 2:

With $\boldsymbol{Z}$ and $\boldsymbol{Y}$ as in Theorem 1 and under the additional assumption of the $\boldsymbol{V}$-orthonormality

$$\boldsymbol{X}^T\boldsymbol{Z} = \boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{Z} = \boldsymbol{I}_m$$

of the $\boldsymbol{z}$-vectors, one has

$$\begin{aligned} \boldsymbol{K}_0 &= \left[\boldsymbol{X}^T\boldsymbol{K}^{-1}\boldsymbol{X}\right]^{-1}, \\ \mathcal{P} &= \boldsymbol{K}^{-1}\boldsymbol{X}\boldsymbol{K}_0, \\ \boldsymbol{M}_0 &= \boldsymbol{K}_0\boldsymbol{X}^T\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{X}\boldsymbol{K}_0. \end{aligned} \tag{22}$$

Remarks:

1. Observe that the inverse of $\boldsymbol{K}_0$ is the projection of $\boldsymbol{K}^{-1}$ to the space span$\{\boldsymbol{X}\}$. For the case of nodal condensation this means that ${\boldsymbol{K}_0}^{-1}$ is obtained from $\boldsymbol{K}^{-1}$ by deleting all rows and columns corresponding to the slave variables.
   Notice, however, that the expressions for $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$ from (22) are mainly of theoretical interest. For the practical computation of these see Theorem 5 below.

2. Condensation is projection of the eigenvalue problem to the column space of the prolongation matrix $\mathcal{P}$. Looking at the above formula for $\mathcal{P}$

   $$\mathcal{P} = \boldsymbol{K}^{-1}\boldsymbol{X}\boldsymbol{K}_0$$

   one sees that the column space of $\mathcal{P}$ is not influenced by the matrix $\boldsymbol{K}_0 = \left[\boldsymbol{X}^T\boldsymbol{K}^{-1}\boldsymbol{X}\right]^{-1}$ on its right. Hence one could drop it without changing the results and the condensation data would be simplified to

   $$\begin{aligned} \tilde{\mathcal{P}} &= \boldsymbol{K}^{-1}\boldsymbol{X}, \\ \tilde{\boldsymbol{K}}_0 &= \tilde{\mathcal{P}}^T\boldsymbol{K}\tilde{\mathcal{P}} = \boldsymbol{X}^T\boldsymbol{K}^{-1}\boldsymbol{X}, \\ \tilde{\boldsymbol{M}}_0 &= \tilde{\mathcal{P}}^T\boldsymbol{M}\tilde{\mathcal{P}} = \boldsymbol{X}^T\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{X}. \end{aligned} \tag{23}$$

   From the simplification of the condensation data it is tempting to try to use these new formulae. It should be noted, however, that often sparsity structures of $\boldsymbol{K}$ or $\boldsymbol{M}$ are exploited when calculating $\mathcal{P}$ from the blocked system (13). Using the new version (23) will often destroy these structures.
   We shall present an alternate system for the calculation of $\mathcal{P}$ in the next section, anyhow, which is perfectly suited for the exploitation of sparseness and substructuring.
   The last set of data (23) has proven useful, however, in prototype investigations of new sets of potential masters.

3. Though the last set of data may not be advisable for practical large scale implementation, it is clearly seen from it that condensation is nothing else but one step of inverse subspace iteration (iteration matrix is $\boldsymbol{K}^{-1}\boldsymbol{M}$) with starting subspace chosen as the span of the columns of the matrix $\boldsymbol{M}^{-1}\boldsymbol{V}\boldsymbol{Z}$.
   In the case of modal masters, where the $\boldsymbol{z}$-vectors approximate eigenvectors, the natural metric matrix would be $\boldsymbol{V} := \boldsymbol{M}$. Then condensation would result in simultaneous inverse iteration with start space span($\boldsymbol{Z}$).

Results on non nodal dynamic condensation at $\hat{\lambda}$ can be read from the results on static condensation by replacing the matrix $\boldsymbol{K}$ within the master-slave prolongation by $(\boldsymbol{K} - \hat{\lambda}\boldsymbol{M})$.

Corollary 3:

With $\boldsymbol{Z}$ and $\boldsymbol{Y}$ as in Theorem 1 and with $\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{Z} = \boldsymbol{I}_m$ assume that $\tilde{\lambda}$ is not a slave eigenvalue, i.e. assume that

$$\boldsymbol{Y}^T(\boldsymbol{K} - \tilde{\lambda}\boldsymbol{M})\boldsymbol{Y} \text{ is regular.}$$

Assume additionally that $\tilde{\lambda}$ is not an eigenvalue of the original problem. Then the prolongation operator

$$\mathcal{P}(\tilde{\lambda}) = (\boldsymbol{I} - \boldsymbol{Y}[\boldsymbol{Y}^T(\boldsymbol{K} - \tilde{\lambda}\boldsymbol{M})\boldsymbol{Y}]^{-1}\boldsymbol{Y}^T(\boldsymbol{K} - \tilde{\lambda}\boldsymbol{M}))\boldsymbol{Z} \tag{24}$$

can be written without $\boldsymbol{Y}$-dependence as

$$\mathcal{P}(\tilde{\lambda}) = (\boldsymbol{K} - \tilde{\lambda}\boldsymbol{M})^{-1}\boldsymbol{X}\left(\boldsymbol{X}^T(\boldsymbol{K} - \tilde{\lambda}\boldsymbol{M})^{-1}\boldsymbol{X}\right)^{-1}. \tag{25}$$

Proof: Taking into account the introductory remark of the corollary, the only thing that has to be assured for the proof is the regularity of the matrix $\left(\boldsymbol{X}^T(\boldsymbol{K} - \tilde{\lambda}\boldsymbol{M})^{-1}\boldsymbol{X}\right)$ from (25). With $\boldsymbol{A} := (\boldsymbol{K} - \tilde{\lambda}\boldsymbol{M})$ it follows from

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{X}^T\boldsymbol{A}^{-1} & \boldsymbol{I} \end{pmatrix}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{X} \\ \boldsymbol{0} & -\boldsymbol{X}^T\boldsymbol{A}^{-1}\boldsymbol{X} \end{pmatrix}$$

that $\left(\boldsymbol{X}^T(\boldsymbol{K} - \tilde{\lambda}\boldsymbol{M})^{-1}\boldsymbol{X}\right)$ is regular if and only if $\begin{pmatrix} \boldsymbol{A} & \boldsymbol{X} \\ \boldsymbol{X}^T & 0 \end{pmatrix}$ is. But from

$$\begin{pmatrix} (\boldsymbol{X},\boldsymbol{Y})^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} (\boldsymbol{X},\boldsymbol{Y}) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}^T\boldsymbol{A}\boldsymbol{X} & \boldsymbol{X}^T\boldsymbol{A}\boldsymbol{Y} & \boldsymbol{X}^T\boldsymbol{X} \\ \boldsymbol{Y}^T\boldsymbol{A}\boldsymbol{X} & \boldsymbol{Y}^T\boldsymbol{A}\boldsymbol{Y} & \boldsymbol{0} \\ \boldsymbol{X}^T\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$$

it is clear that this is equivalent to $\boldsymbol{Y}^T(\boldsymbol{K} - \tilde{\lambda}\boldsymbol{M})\boldsymbol{Y}$ being regular. ∎

It would cause no problems to derive in an additional corollary the formal expression

$$(\boldsymbol{X}^T(\boldsymbol{K} - \lambda\boldsymbol{M})^{-1}\boldsymbol{X})^{-1}\boldsymbol{x}_m = \boldsymbol{0}$$

for the exact non nodal condensation, where the parameter $\tilde{\lambda}$ is chosen to be the unknown eigenvalue itself. If $(\boldsymbol{K} - \lambda\boldsymbol{M})$ is expressed in its diagonalized form, relations to the Weinstein formulae from the theory of intermediate eigenvalue problems (see [32, 1], e.g.) can be seen. However, we will not invest any further thought into this formula, since it shares with the last corollary the unpleasant feature, that the value $\tilde{\lambda}(= \lambda$ here) has to

be different from an eigenvalue of the original problem. This is particularly nasty because condensing dynamically with an eigenvalue $\tilde{\lambda}$ would reproduce this eigenvalue (if $\tilde{\lambda}$ is not a slave eigenvalue as well and if there exists an eigenvector corresponding to $\tilde{\lambda}$ with a nonvanishing master part).
Normally, one applies dynamic (and exact) condensation for $\tilde{\lambda}$ values in the interval $(0, \underline{\omega})$ with $\underline{\omega}$ denoting the smallest slave-eigenvalue. Dynamic condensation (and exact condensation) works for all values of $\tilde{\lambda}$ in this region without any exception, and this seems to be one of the reasons why this interval is often called the region "where condensation is valid" (see [27] for an additional reason). Actually, dynamic condensation formally works whenever $\tilde{\lambda}$ is different from one of the $n-m$ slave eigenvectors, but since in most cases the lower part of the spectrum is of predominant interest, the interval $(0, \underline{\omega})$ is in fact the most interesting region.
It is hence utterly annoying, that our condensation formulae do not work for the most interesting points of that interval.
From (24) one infers, that $\mathcal{P}(\tilde{\lambda})$ depends continuously on $\tilde{\lambda}$ in the complement of the slave spectrum. Thus one could try to define the condensations by analytic continuation of the derived formulae. However, this is at least numerically not to be recommended since there is a method to compute the prolongations in a numerically stable way. The latter is the subject of the next section.

4 Non nodal condensation and constrained eigenproblems

We start with a result, that has already been implicit in the proof of Corollary 3.

Lemma 4:

Under the assumptions of Corollary 2 one has:

$$\det(\boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{Y} - \mu\boldsymbol{Y}^T\boldsymbol{M}\boldsymbol{Y}) = (-1)^m \det(\boldsymbol{Y}\ \boldsymbol{Z})^2 \cdot \det \begin{pmatrix} \boldsymbol{K}-\mu\boldsymbol{M} & \boldsymbol{V}\boldsymbol{Z} \\ \boldsymbol{Z}^T\boldsymbol{V} & \boldsymbol{O} \end{pmatrix} \tag{26}$$

such that the system $(\boldsymbol{Y}^T\boldsymbol{K}\boldsymbol{Y} - \mu\boldsymbol{Y}^T\boldsymbol{M}\boldsymbol{Y})$ *is singular if and only if*

$$\begin{pmatrix} \boldsymbol{K}-\mu\boldsymbol{M} & \boldsymbol{V}\boldsymbol{Z} \\ \boldsymbol{Z}^T\boldsymbol{V} & \boldsymbol{O} \end{pmatrix} \tag{27}$$

is.

Proof:

Letting $\boldsymbol{A}(\mu) := \boldsymbol{K} - \mu\boldsymbol{M}$,

$$\boldsymbol{L} = \begin{pmatrix} \boldsymbol{I}_{n-m} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{0} & \boldsymbol{I}_m & \boldsymbol{O} \\ -\boldsymbol{Z}^T\boldsymbol{A}(\mu)\boldsymbol{Y} & -\frac{1}{2}\boldsymbol{Z}^T\boldsymbol{A}(\mu)\boldsymbol{Z} & \boldsymbol{I}_m \end{pmatrix}$$

$$\boldsymbol{W} = \begin{pmatrix} \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{Z} \end{pmatrix} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_m \end{pmatrix}$$

one has

$$\boldsymbol{L}^T\boldsymbol{W}^T \begin{pmatrix} \boldsymbol{A}(\mu) & \boldsymbol{V}\boldsymbol{Z} \\ \boldsymbol{Z}^T\boldsymbol{V} & \boldsymbol{O} \end{pmatrix} \boldsymbol{W}\boldsymbol{L} = \begin{pmatrix} \boldsymbol{Y}^T\boldsymbol{A}(\mu)\boldsymbol{Y} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I}_m \\ \boldsymbol{O} & \boldsymbol{I}_m & \boldsymbol{O} \end{pmatrix},$$

wherefrom the given formula follows. ∎

The second main result of this paper is the following formula for the computation of the dynamic condensation.

Theorem 5:

Under the standard assumptions from Corollary 2 the following conclusions hold:

a. *If $\tilde{\lambda}$ is not a slave eigenvalue the prolongation operator $\mathcal{P}(\tilde{\lambda})$ can be calculated from*

$$\begin{pmatrix} \boldsymbol{K}-\tilde{\lambda}\boldsymbol{M} & \boldsymbol{V}\boldsymbol{Z} \\ \boldsymbol{Z}^T\boldsymbol{V} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \mathcal{P}(\tilde{\lambda}) \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{I}_m \end{pmatrix}. \tag{28}$$

b. *In the case of static condensation ($\tilde{\lambda}=0$), the negative of $\boldsymbol{\alpha}$ from the last equation is already the condensed stiffness matrix:*

$$-\boldsymbol{\alpha} = \boldsymbol{K}_0 = \mathcal{P}^T\boldsymbol{K}\mathcal{P}. \tag{29}$$

Proof:

From Lemma 4 it follows that the matrix from (28) is regular if and only if $\det\left(\boldsymbol{Y}^T(\boldsymbol{K}-\tilde{\lambda}\boldsymbol{M})\boldsymbol{Y}\right) \neq 0$, i.e. if $\tilde{\lambda}$ is not a slave eigenvector. If $\boldsymbol{K}-\tilde{\lambda}\boldsymbol{M}$ is regular, too, block elimination in (28) yields

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} (\boldsymbol{K}-\tilde{\lambda}\boldsymbol{M})^{-1}\boldsymbol{X}\left(\boldsymbol{X}^T\left[\boldsymbol{K}-\tilde{\lambda}\boldsymbol{M}\right]^{-1}\boldsymbol{X}\right)^{-1} \\ -\left(\boldsymbol{X}^T\left[\boldsymbol{K}-\tilde{\lambda}\boldsymbol{M}\right]^{-1}\boldsymbol{X}\right)^{-1} \end{pmatrix}.$$

Under the above conditions the first part of the solution agrees with $\mathcal{P}(\tilde{\lambda})$ from (25). Since one infers from formula (24) that $\mathcal{P}(\tilde{\lambda})$ depends continuously on $\tilde{\lambda}$ for $\tilde{\lambda}$ not a slave eigenvalue the proof of part a. is complete.
Part b. follows by inspection from Corollary 2. ∎

Remark:

The assumptions of the last two results (Lemma 4 and Theorem 5) included the $\boldsymbol{V}$-orthonormality $\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{Z}=\boldsymbol{I}_m$ of the z-vectors. This assumption has been made more or less for the sake of convenience. It could have been dropped without altering the results essentially.
If $\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{Z} \neq \boldsymbol{I}_m$ then in formula (26) a factor $\det(\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{Z})^{-2}$ has to be added on the right hand side. Hence the conclusion following that formula remains as it is.
In Theorem 5 the given formulae can still be used for condensation if $\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{Z} \neq \boldsymbol{I}_m$. Notice, however, that under these circumstances the calculated projection matrix differs from the usual one by a right hand side factor of $(\boldsymbol{Z}^T\boldsymbol{V}\boldsymbol{Z})^{-1}$. Since this corresponds to a regular basis transformation of the span of the column vectors of $\mathcal{P}$ the condensation results do not change. Since the negative of the $\boldsymbol{\alpha}$ matrix from the second part of the theorem corresponds to the projection of the stiffness matrix with the projector from part a. it can be used within the condensation process as usual.

Normally, the matrix from (27) occurs in the constrained eigenvalue problem

$$\begin{pmatrix} \boldsymbol{K}-\lambda\boldsymbol{M} & \boldsymbol{X} \\ \boldsymbol{X}^T & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\alpha} \end{pmatrix} = \boldsymbol{0}, \tag{30}$$

where the solution $\boldsymbol{x}$ of a normal (unconstrained) eigenvalue problem

$$(\boldsymbol{K}-\lambda\boldsymbol{M})x = 0$$

is subjected to linear constraints $\boldsymbol{X}^T\boldsymbol{x} = 0$ which are enforced by the Lagrange parameters $\boldsymbol{\alpha}$.
Physically the constraints can be interpreted as generalized bearings of the investigated vibrating entity, the Lagrange parameters $\boldsymbol{\alpha}$ in $\boldsymbol{X}\boldsymbol{\alpha}$ from the first equation of (30) can be seen as the strengths of generalized bearing forces (cf. [28], e.g.). An explanation of (30) within a mathematical context can be found in [1].
For the applications aimed at her it may be of interest to formulate the equivalence of (30) and the slave-eigenproblem

$$\boldsymbol{Y}^T(\boldsymbol{K}-\lambda\boldsymbol{M})\boldsymbol{Y}\boldsymbol{y} = 0 \tag{31}$$

in the following lemma, the easy verification of which is left to the reader.

Lemma 6:

Let the standard assumptions on the symmetric positive definite matrix $\boldsymbol{V}$ and the matrices $\boldsymbol{Z}, \boldsymbol{X} := \boldsymbol{V}\boldsymbol{Z} \in \mathbb{R}^{(n,m)}$ and $\boldsymbol{Y} \in \mathbb{R}^{(n,n-m)}$ hold, such that

$$\mathit{rank}(\boldsymbol{Z},\boldsymbol{Y}) = n,\ \boldsymbol{X}^T\boldsymbol{Z} = \boldsymbol{I}_m \ \ \mathit{and}\ \ \boldsymbol{X}^T\boldsymbol{Y} = \boldsymbol{0}.$$

Then

$(\lambda, \boldsymbol{x})$ *is an eigenpair of the constrained eigenvalue problem (30)*

(with corresponding Lagrange coefficient $\boldsymbol{\alpha} = -\boldsymbol{Z}^T(\boldsymbol{K}-\lambda\boldsymbol{M})\boldsymbol{x}$) if and only if

$\boldsymbol{x} = \boldsymbol{Y}\boldsymbol{y}$ *and* $(\lambda, \boldsymbol{y})$ *is an eigenpair of the slave problem (31).*

5 Applications, a short outlook

Iterative improvement of condensation

In [10] we presented a procedure to derive realistic but safe error bounds for eigenvalue approximations gained by condensation methods. The methods works well except for the usual disadvantage of condensation approaches that the found approximation quality is not easily increased if it turns out at the end that it is not good enough. There are attempts to refine approximations (cf. [23] or [24]) but these again offer only one refinement step. After this there is no chance to further improve the results without a total restart of the whole procedure with new condensation data.
The concept of non nodal condensation allows to improve all or selected eigenpair approximations up to a desired approximation quality. Several variants are under investigation. We give some short indications of possible procedures.

Static improvement of selected eigenapproximations

If the approximation $(\hat{\lambda}_i, \boldsymbol{w}_i)$ to a specific eigenpair $(\lambda_i, \boldsymbol{x}_i)$ is chosen to be improved, the projection matrix $\mathcal{P}$ of the condensation will be complemented by a suitable prolongation $\boldsymbol{p}_+$ of $\boldsymbol{w}_i$. According to formula (23) this could be the inverse iteration prolongation

$$\boldsymbol{p}_+ := \boldsymbol{K}^{-1}\boldsymbol{V}\boldsymbol{w}_i \tag{32}$$

where $\boldsymbol{V} := \boldsymbol{M}$ corresponds directly to an inverse iteration for the generalized eigenvalue problem.
Alternatively, formula (28) would suggest to compute $\boldsymbol{p}_+$ from

$$\begin{pmatrix} \boldsymbol{K} & \boldsymbol{V}\boldsymbol{W} \\ \boldsymbol{W}^T\boldsymbol{V} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_+ \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{e}^1 \end{pmatrix} \tag{33}$$

where $\boldsymbol{W}$ contains $k \leq m$ of the already gained eigenvector approximations including $\boldsymbol{w}_i$ in the first column, $\boldsymbol{e}^1$ is the first unit vector in $\mathbb{R}^k$ and $\boldsymbol{V} := \boldsymbol{M}$ is a sensible choice for $\boldsymbol{V}$.
With $\mathcal{P}_+ := (\mathcal{P}, \boldsymbol{p}_+)$ the eigenproblem is now approximated by projection to the column space of $\mathcal{P}_+$.
If care is taken with this latter projection, the projected eigenvalue problem can be solved with considerably less effort than the previous problem projected onto span $\mathcal{P}$ only.
Clearly the procedure can be repeated and it can be executed for the simultaneous improvement of several eigenpair approximations.
The development of adequately implementable versions of the algorithms including version for parallel computers is under investigation [17].

Dynamic improvement of selected eigenapproximations

In dynamic condensation the condensation data is gained with the use of a shifted operator $\boldsymbol{K} - \hat{\lambda}_i\boldsymbol{M}$ instead of $\boldsymbol{K}$. In the light of (23) and (25) the static improvements just reported on can equally well be performed with $\boldsymbol{p}_+$-vectors which are computed with a shifted inverse iteration (either using the shifted versions of (32) or (33)). In the same way as Rayleigh-quotient shifts lead to considerable improvement of inverse iteration, the shifts here improve the iteration a great deal, too. It is found, however, that simultaneous (different) shifts for several simultaneous eigenpair improvements safely lead to the expected results only if the corresponding eigenapproximations are of good quality already. The same has been found for the related projection-condensation iteration in [10]. The need for good starting approximations makes sense as well by observing that shifted simultaneous inverse iterations with projective approximation of the eigenvalues can be interpreted as a modified Newton-Iteration for the computation of invariant subspaces (cf. [15]). For Newton type iterations only locally fast convergence is expected.
It appears from numerical results, however, that the iteration is well behaved in the sense, that it converges in most cases to eigendata. This would conform with the known global convergence properties of Rayleigh-quotient iteration (cf. [21]).

Substructuring and non nodal condensation

Joint application of substructuring and condensation is particularly well suited for implementation on parallel computers (cf. e.g. [25]). Partitioning the structure under consideration into substructures and choosing nodal masters as the interface degrees of freedom leads to data structures and formulae for the individual substructures which are independent of each other. A way to improve the approximation quality is to incorporate additional nodal masters from the interior of the substructures. In this cases the data structure essentially is preserved and the parallel method carries over in a straightforward way [25].

The part of the spectrum which can be approximated accurately enough depends crucially on the size of the minimal slave eigenvalue. In the substructuring case the slave eigenvalues consist of the eigenvalues of the independently vibrating substructures. The eigenvalues of these are of course optimally increased if their eigenvectors are constrained to be orthogonal to the substructures first eigenmodes. On the one hand this falls into the scope of non nodal condensation (the additional Z-vectors consists of eigenvectors of the substructures filled up with zeros at the complementary positions). On the other hand such vectors again allow the substructurewise computation of condensation, if the metric matrix V has block diagonal structure with the blocks defined by the nodal interface degrees of freedom and the collections of interior points of the individual substructures. This approach has been defined and investigated in [30]. It has been found there that the use of substructure modes leads to considerably better results than the use of a comparable number of interior nodal masters. Details of the parallel implementation of the methods will be given in [16]. It will as well be there, where the substructured version of the statical and dynamical improvement of approximations of specific eigenpairs will be dealt with. In principle this is done by representing an eigenvector approximation from nodal substructured condensation through its parts from the interior of the different substructures and using these parts in the same way as the substructure modes in [16].

Eigenreanalysis

In a typical situation in applied eigenanalysis the interesting part of the eigendata (eigenvectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m$ with corresponding eigenvalues $\hat{\lambda}_1, \ldots, \hat{\lambda}_m$) of a very similar problem is perfectly known. The actual problem is a slight perturbation of the latter one and since the eigendata ars most probably but slightly perturbed, too (cf. [6, 31]), it makes sense, to use the known data to approximate the unknown. Approximations of the new data can be obtained by projection on the space of old eigenvectors. Better results will normally be produced by using the old vectors as (part of the) master vectors within a condensation approximation, possibly with a static or dynamic improvement of the desired data.

Truncated iterative improvement

Within some of the above computational schemes the system

$$\begin{pmatrix} \boldsymbol{K} - \tilde{\lambda}\boldsymbol{M} & \boldsymbol{V}\boldsymbol{Z} \\ \boldsymbol{Z}^T\boldsymbol{V} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \mathcal{P} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{I}_m \end{pmatrix}$$

has to be solved. If this system is large, it makes sense to solve it only approximately. Due to the problems structure block versions of CG or MINRES (the system is symmetric, regular but always indefinite, cf. the proof of Lemma 4) seem to be especially suited. Sensible steerings of the number of iterations are under investigation.

Algorithms for the choice of masters

A central question in nodal condensation has been how to choose the condensation masters optimally (cf. [5, 18, 27], e.g.). The aim of most approaches to the automatical determination of masters has been to maximize $\underline{\omega}$, the minimal slave eigenvalue. Due to a result from [29] this makes sense, since an a priori bound on the approximation error of static condensation from that paper decreases monotonically with increasing $\underline{\omega}$.
With non nodal condensation the situation is seen to be more complicated. At least the quality of a set of masters is not caught by $\underline{\omega}$ only. There are easy examples where quite different masters with very different approximation behaviour in condensation lead to the same $\underline{\omega}$. The question how to mathematically characterize the quality of non nodal masters is open. We shall investigate it in the near future.
At the moment — and most probably for ever — engineering know how will be of valuable help in constructing starting approximations in non nodal condensation.

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