European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS 2004
P. Neittaanmäki, T. Rossi, S. Korotov, E. Oñate, J. Périaux, and D. Knörzer (eds.)
Jyväskylä, 24–28 July 2004

EIGENVIBRATIONS OF A PLATE WITH ELASTICALLY ATTACHED LOADS

Heinrich Voss

Section of Mathematics
Hamburg University of Technology, D-21071 Hamburg, Germany
e-mail: voss@tu-harburg.de, web page: http://www.tu-harburg-de/mat/hp/voss/

Key words: nonlinear eigenvalue problem, Arnoldi method, free vibrations of a plate with masses.

Abstract. *In this contribution we consider the rational eigenvalue problem governing free vibrations of a plate with elastically attached masses. We discuss the numerical solution of the problem by an iterative projection method generalizing the Arnoldi method for linear eigenproblems. Taking advantage of a minmax characterization of the eigenvalues for nonoverdamped problems the projected eigenproblems are solved by safeguarded iteration. Special care is taken to determine all eigenvalues between two consecutive poles, and to inhibit the method from converging to the same eigenvalues repeatedly.*

1 INTRODUCTION

The vertical deflection $w(x,t)$ of a thin isotropic plate with elastically attached loads is governed by the equations

$$Lw(x,t) + \rho d \frac{\partial^2}{\partial t^2} w(x,t) + \sum_{j=1}^{q} m_j \frac{d^2}{dt^2} \xi_j(t) \delta(x - x_j) = 0 \quad , \quad x \in \Omega,\ t > 0 \tag{1}$$

$$Bw(x,t) = 0 \quad , \quad x \in \partial\Omega,\ t > 0 \tag{2}$$

$$m_j \frac{d^2}{dt^2} \xi_j + k_j(\xi_j(t) - w(x_j,t)) = 0 \quad , \quad t > 0,\ j = 1, \ldots, q. \tag{3}$$

Here $\Omega \subset \mathbb{R}^2$ is a domain occupied by the plate, L is the plate operator, and B a boundary operator specifying the support of the plate at its boundary $\partial\Omega$. ρ is the mass per volume density, and d the thickness of the plate. For $j = 1, \ldots, q$ at $x_j \in \Omega$ a load m_j is joined elastically to the plate with stiffness coefficient k_j, and ξ_j denotes the displacement of the mass m_j.

Using the ansatz $w(x,t) = u(x)e^{i\omega t}$ and $\xi_j(t) = c_j e^{i\omega t}$ characterizing the eigenmodes and eigenfrequencies of the vibrating plate, and eliminating c_j we obtain the rational eigenproblem

$$Lu(x) = \lambda \rho d u(x) + \sum_{j=1}^{q} \frac{\lambda \sigma_j}{\sigma_j - \lambda} m_j \delta(x - x_j) u \quad , \quad x \in \Omega \tag{4}$$

$$Bu(x) = 0 \quad , \quad x \in \partial\Omega \tag{5}$$

where $\lambda = \omega^2$ and $\sigma_j = k_j/m_j$. Discretizing by finite elements one gets a rational matrix eigenvalue problem

$$Kx = \lambda M x + \sum_{j=1}^{p} \frac{\lambda}{\sigma_j - \lambda} C_j x \tag{6}$$

where the stiffness matrix K and mass matrix M are symmetric and positive definite, and the matrices C_j are positive semidefinite with small rank. We assume that the rational terms corresponding to the same pole σ_j have been merged to the matrix C_j, and that the poles are ordered by magnitude $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_p < \sigma_{p+1} = \infty$.

It can be checked easily that for each of the intervals $J_j := (\sigma_{j-1}, \sigma_j)$, $j = 1, \ldots, p+1$ problem (6) (and the infinite dimensional problem (4), (5) as well) satisfies the conditions of the minmax theory for nonoverdamped eigenproblems [16]. Hence, in each of the intervals J_j there is a finite number of eigenvalues which can be enumerated $\lambda_\ell \le \lambda_{\ell+1} \le \ldots$ in an appropriate way (cf. Section 2).

For this type of problems we studied iterative projection methods of Jacobi–Davidson type [2] and of Arnoldi type [13], where the eigenvalues and corresponding eigenvectors can be determined from projections to subspaces of small dimension which are expanded

in the course of the algorithm. Since the projected eigenproblems inherit the symmetry properties of the original problem, they can be solved efficiently by safeguarded iteration.

In [2] and [14] we claimed that the methods while computing the ℓ-th eigenvalue usually gathers enough information about the $(\ell+1)$-th eigenvalue, such that the eigenvalues can be determined safely one after the other, and this was true for the examples from fluid–solid vibrations considered there. For the plate problem (6), however, we observed that eigenvalues were missed, and at a later stage eigenvalues were obtained repeatedly.

In this paper we modify the Arnoldi method to make sure that all eigenvalues of the rational eigenproblem (6) in a given interval are found. In particular we address the issue how to determine a suitable initial space.

The paper is organized as follows. Section 2 summarizes the minmax characterization of eigenvalues for nonoverdamped problems, and in particular introduces an appropriate enumeration of the eigenvalues. Section 3 contains the nonlinear Arnoldi method from [14], and its modification to inhibit the method to miss eigenvalues, and to determine eigenvalues which have been found previously. In Section 4 we discuss the question how many eigenvalues of problem (6) are contained in a given interval, and in particular between two consecutive poles, and Section 5 demonstrates the efficiency of the method for a numerical example. The paper closes with concluding remarks.

2 MINMAX CHARACTERIZATION FOR NONLINEAR PROBLEMS

For λ in an open real interval J let $T(\lambda) \in \mathbb{R}^{n\times n}$ be a family of symmetric matrices the elements of which are differentiable. We consider the nonlinear eigenvalue problem to identify parameters $\lambda \in J$ such that the linear system of equations

$$T(\lambda)x = 0 \tag{7}$$

has a nontrivial solution $x \neq 0$. As in the linear case we call λ with this property an eigenvalue and x a corresponding eigenvector.

We assume that for every $x \in \mathbb{R}^n \setminus \{0\}$ the real equation

$$f(\lambda, x) := x^T T(\lambda)x = 0 \tag{8}$$

has at most one solution $\lambda \in J$. Then equation (8) defines a functional P on some subset $D \subset \mathbb{R}^n$ which obviously generalizes the Rayleigh quotient for a linear pencil $T(\lambda) = \lambda B - A$, and which we call the Rayleigh functional of the nonlinear eigenvalue problem (7). We further assume that

$$x^T T'(P(x))x > 0 \quad \text{for every } x \in D \tag{9}$$

generalizing the definiteness requirement for linear pencils. By the implicit function theorem D is an open set, and differentiating the identity $x^T T(P(x))x = 0$ one obtains, that the eigenvectors of (7) are stationary points of P.

Obviously these conditions are satisfied for the plate problem (6) in each of the intervals J_j, $j = 1, \ldots, p+1$, if we define

$$T(\lambda) := -K + \lambda M + \sum_{j=1}^{p} \frac{\lambda}{\sigma_j - \lambda} C_j. \tag{10}$$

Under the general conditions above we proved in [16] a minmax principle for the nonlinear eigenproblem (7) (which applies to the infinite dimensional problem (4), (5), too) if the eigenvalues are enumerated appropriately.

$\lambda \in J$ is an eigenvalue of (7) if and only if $\mu = 0$ is an eigenvalue of the matrix $T(\lambda)$, and by Poincaré's maxmin principle there exists $k \in \mathbb{N}$ such that

$$0 = \max_{\dim V = k} \min_{x \in V,\ x \neq 0} \frac{x^T T(\lambda) x}{\|x\|^2}. \tag{11}$$

Then we assign this k to λ as its number and call λ an k-th eigenvalue of problem (7). Note that k is not uniquely determined for multiple eigenvalues.

With this enumeration it holds (cf. [16]) that for every $k \in \{1, \ldots, n\}$ problem (7) has at most one k-th eigenvalue in J, which can be characterized by

$$\lambda_k = \min_{\dim V = k, D \cap V \neq \emptyset} \sup_{v \in D \cap V} P(v), \tag{12}$$

and conversely, if

$$\lambda_k := \inf_{\dim V = k, D \cap V \neq \emptyset} \sup_{v \in D \cap V} P(v) \in J, \tag{13}$$

then λ_k is a k-th eigenvalue of (7), and the characterization (12) holds. The minimum is attained by the invariant subspace of $T(\lambda_k)$ corresponding to its k largest eigenvalues, and the supremum is attained by any eigenvector of $T(\lambda_k)$ corresponding to $\mu = 0$. From (13) it follows immediately, if (7) has a k_1-th eigenvalue $\lambda_{k_1} \in J$ and a k_2-th eigenvalue $\lambda_{k_2} \in J$ with $k_1 < k_2$ then there exists an ℓ-th eigenvalue λ_ℓ for every $\ell \in \{k_1, k_1 + 1, \ldots, k_2\}$, and $\lambda_{k_1} \leq \lambda_{k_1+1} \leq \cdots \leq \lambda_{k_2}$.

The enumeration of eigenvalues and the fact that the eigenvectors of (7) are the stationary vectors of the Rayleigh functional suggests the method in Algorithm 1 called safeguarded iteration for computing the k–th eigenvalue. Since in each iteration step a linear eigenproblems has to be solved this method is not appropriate for the plate problem (6) which will be large and sparse. However, in iterative projection methods the problem under consideration is projected to eigenproblems of small dimension, and solving these ones by safeguarded iteration the occurring linear eigenproblems of small dimension can be treated with standard software.

Algorithm 1 Safeguarded iteration

1: Start with an approximation σ_1 to the k-th eigenvalue of (7)
2: **for** $\ell = 1, 2, \ldots$ until convergence **do**
3: determine an eigenvector x_ℓ corresponding to the k-largest eigenvalue of $T(\sigma_\ell)$
4: solve $x_\ell^T T(\sigma_{\ell+1}) x_\ell = 0$ for $\sigma_{\ell+1} = P(x_\ell)$
5: **end for**

Safeguarded iteration has the following approximation properties [15].

THEOREM 1

(i) If $\lambda_1 := \inf_{x \in D} P(x) \in J$ and $x_1 \in D$ then the safeguarded iteration converges globally to λ_1.

(ii) If $\lambda_k \in J$ is a k-th eigenvalue of (7) which is simple then the safeguarded iteration converges locally and quadratically to λ_k.

(iii) Let $T(\lambda)$ be twice continuously differentiable, and assume that $T'(\lambda)$ is positive definite for $\lambda \in J$. If x_ℓ in step 3. of Algorithm 1 is chosen to be an eigenvector corresponding to the k largest eigenvalue of the generalized eigenproblem $T(\sigma_\ell)x = \mu T'(\sigma_\ell)x$ then the convergence is even cubic.

3 ITERATIVE PROJECTION METHODS

For sparse linear eigenvalue problems $Ax = \lambda x$ iterative projection methods are very efficient. Here the dimension of the eigenproblem is reduced by projection to a subspace of much smaller dimension, and the reduced problem is handled by a fast technique for dense problems. The subspaces are expanded in the course of the algorithm in an iterative way with the aim that some of the eigenvalues of the reduced matrix become good approximations to some of the wanted eigenvalues of the given large matrix.

Essentially two types of methods are in use: methods which project the problem to a sequence of Krylov spaces like the Lanczos or the Arnoldi method [1], [10] and methods which aim at a specific eigenpair expanding a search space by a direction which has a high approximation potential for the eigenvector under consideration like the Jacobi–Davidson method [1], [11], [12]. The Krylov subspace approaches take advantage of the linear structure of the underlying problem and construct an approximate incomplete Schur factorization (or incomplete spectral decomposition in the Hermitean case) from which they derive approximations to some of the extreme eigenvalues and corresponding eigenvectors, whereas the second type aims at the wanted eigenvalues one after the other using the Schur decomposition only to prevent the method from converging to eigenpairs which have been obtained already in a previous step. Since for general nonlinear eigenproblems a normal form like the Schur factorization does not exist generalizations of iterative projection methods have to be of the second type.

Iterative projection methods for nonlinear eigenproblems are discussed in [2], [3], [4], [5], [8], [9], [13], [14]. A typical example is the nonlinear symmetric Arnoldi method [13], [14] where the current search space is expanded by the increment of the residual inverse iteration ([7]) which has a high local approximation potential. If it is applied to a linear problem then the preconditioned Arnoldi method results, which motivates its name although no Arnoldi recursion and no Krylov spaces appear.

A crucial point in iterative projection methods for general nonlinear eigenvalue problems when approximating more than one eigenvalue is to inhibit the method from converging to eigenvalues which have been found previously.

If $T(\lambda)$ is a family of symmetric matrices allowing a minmax characterization of its eigenvalues then the projected problems inherit this property suggesting to compute the eigenvalues one after the other by safeguarded iteration. Algorithm 2 contains a template of the Arnoldi method for the symmetric problem (7) for computing all eigenvalues in a given interval J where we assume that λ_{m_1} is the smallest eigenvalue in J and λ_{m_2} is the largest one.

Algorithm 2 Nonlinear Arnoldi method for symmetric problems

start with an initial pole σ and an initial basis V, $V^HV = I$; $m = m_1$
determine preconditioner $M \approx T(\sigma)^{-1}$, σ close to first wanted eigenvalue
while $m \leq m_2$ **do**
  compute k-th eigenvalue μ and corresponding eigenvector y of projected problem $V^HT(\mu)Vy = 0$ by safeguarded iteration.
  determine Ritz vector $u = Vy$, $\|u\| = 1$, and residual $r = T(\mu)u$
  if $\|r\| < \epsilon$ **then**
    accept $\lambda_k = \mu$, $x_k = u$,
    choose new pole σ and determine $M \approx T(\sigma)^{-1}$ if indicated
    restart if necessary
    choose approximations μ and u to next eigenvalue and eigenvector and determine residual $r = T(\mu)u$
    $m = m + 1$
  end if
  $v = Mr$
  $v = v - VV^Hv$,$\tilde{v} = v/\|v\|$, $V = [V, \tilde{v}]$
  reorthogonalize if necessary
end while

In [14] we commented on the individual steps of the algorithm, namely when and how to renew the preconditioner, how to restart if the increasing storage and/or the computational cost for solving the projected eigenvalue problems make it necessary to purge some of the basis vectors, and how to choose an approximation to the eigenvalues targeted at next. Most important are the questions how to start the algorithm, i.e. what

the number m_1 of the smallest eigenvalue in J is and how to find a suitable initial search space V, and how many eigenvalues exist in J. For rational eigenproblems like the plate problem with attached masses this was discussed in [6] and [15], and will be summarized in the next section.

Usually, while computing the m_1-th eigenvalue λ_{m_1} the algorithm gathers enough information in the search space V about the next eigenvector to compute λ_{m_1+1} safely, and the eigenvalues in J can be computed one after the other without determining the same eigenvalue repeatedly.

This procedure worked fine in all of our examples from vibrations of fluid–solid structures (cf. [14]). For plate problems with attached loads, however, we observed that the algorithm missed eigenvalues, and at a later stage determined eigenvalues that were found already previously.

After having found the k-th eigenpair (λ_k, x_k) the method misses the $(k+1)$-th eigenvalue λ_{k+1}, if the orthogonal projection of the corresponding eigenvector x_{k+1} to the space V is small compared to the projection of some x_ℓ with $k+1 < \ell$ where V denotes the search space for which the termination condition $\|T(\lambda_k)Vu\| < \varepsilon$ for λ_k was satisfied. In this case it may happen that the enumeration of the eigenvalues of the projected problem does not coincide with the enumeration for the original problem, and an approximation to a higher eigenpair is returned in the next step. Since the residual inverse iteration defining the expansion of V converges locally to the eigenvalue closest to the current approximation the component of x_{k+1} in the current search space will not be amplified sufficiently, and the method converges to a higher eigenvalue. As the method proceeds the component of x_{k+1} in the search spaces will grow, and the enumeration of the eigenvalues of the projected problem will be corrected. In this situation the method when converging will replicate an eigenvalue that was found already previously, usually the last but one. Of course we have to check whether this multiply detected eigenvalue is a multiple eigenvalue. If this is not the case we reduce the number of the eigenvalue we are searching for.

To cope with the problem of missing eigenvalues and getting some repeatedly we introduced the following if clause into the nonlinear Arnoldi method after step 6:

```
6a: if (λ_m − λ_{m−1})/λ_m < ε_1 & |⟨u, x_k⟩| > 1 − ε_2 then
6b:     m=m-1
6c: else
```

and closed the else clause after statement 11. Here ε_1 and ε_2 are small positive constants, and $\langle u, x_k\rangle$ denotes the scalar product of u and x_k.

4 LOCATING EIGENVALUES OF RATIONAL EIGENPROBLEMS

To determine the number of eigenvalues between two consecutive poles σ_k and σ_{k+1} of problem (6) we consider for $\mu \in (\sigma_k, \sigma_{k+1})$ the parameter dependent linear eigenvalue

problem

$$\Big(K+\sum_{j=1}^{k}\frac{\mu}{\mu-\sigma_j}C_j\Big)x=\lambda\Big(M+\sum_{j=k+1}^{p}\frac{1}{\sigma_j-\mu}C_j\Big)x. \tag{14}$$

We denote by $\lambda_m(\mu)$ the m-smallest eigenvalue. Then $\hat\lambda$ is an eigenvalue of the rational eigenproblem (6) if and only if $\hat\lambda$ is a fixed point of $\lambda_m : (\sigma_k,\sigma_{k+1})\to\mathbb{R}$, and it is easily checked that it is an m-th eigenvalue.

For the Rayleigh quotient $R_\mu(x)$ of problem (14) it holds $R_{\mu_1}(x)\geq R_{\mu_2}(x)$ for $\mu_1\leq\mu_2$ and every $x\neq 0$, and therefore each of the functions $\lambda_m(\cdot)$ is monotonely nonincreasing. Hence, if $N(\mu)$ for $\mu\in(\sigma_k,\sigma_{k+1})$ denotes the number of eigenvalues of problem (14) which are less than μ, then for $\sigma_k<\alpha<\beta<\sigma_{k+1}$ the interval $[\alpha,\beta)$ contains $N(\beta)-N(\alpha)$ eigenvalues of the rational problem (6), and they are enumerated by $N(\alpha)+1, N(\alpha)+2,\dots,N(\beta)$.

To determine the number of eigenvalues between the poles σ_k and σ_{k+1} we have to study the limit behaviour of the function $\lambda_m(\mu)$ for μ tending to the boundaries of the interval. In [6] we obtained the following results which were even proven for the infinite dimensional case.

LEMMA 1

$$\kappa_m:=\lim_{\mu\to\sigma_k+}\lambda_m(\mu) \tag{15}$$

is the m-th eigenvalue of the reduced problem

Find $\lambda\in\mathbb{R}$ and $x\in H_k:=\{x\in H : C_kx=0\}$, $x\neq 0$ such that

$$\Big(K+\sum_{j=1}^{k-1}\frac{\sigma_k}{\sigma_k-\sigma_j}C_j\Big)x=\lambda\Big(M+\sum_{j=k+1}^{p}\frac{1}{\sigma_j-\sigma_k}C_j\Big)x, \tag{16}$$

LEMMA 2

Let $r_{k+1}=\operatorname{rank}(C_{k+1})$. Then

$$\lim_{\mu\to\sigma_{k+1}-}\lambda_j(\mu)=0\quad\text{for } j=1,\dots,r_{k+1}. \tag{17}$$

For $m>r_{k+1}$

$$\lim_{\mu\to\sigma_{k+1}-}\lambda_m(\mu)=:\tilde\kappa_m=\tilde\lambda_{m-r}, \tag{18}$$

where $\tilde\lambda_{m-r}$ is the $m-r$ smallest eigenvalue of the reduced problem

Find $\lambda\in\mathbb{R}$ and $x\in H_{k+1}:=\{x\in H : C_{k+1}x=0\}$, $x\neq 0$ such that

$$\Big(K+\sum_{j=1}^{k}\frac{\sigma_{k+1}}{\sigma_{k+1}-\sigma_j}C_j\Big)x=\lambda\Big(M+\sum_{j=k+2}^{p}\frac{1}{\sigma_j-\sigma_{k+1}}C_j\Big)x, \tag{19}$$

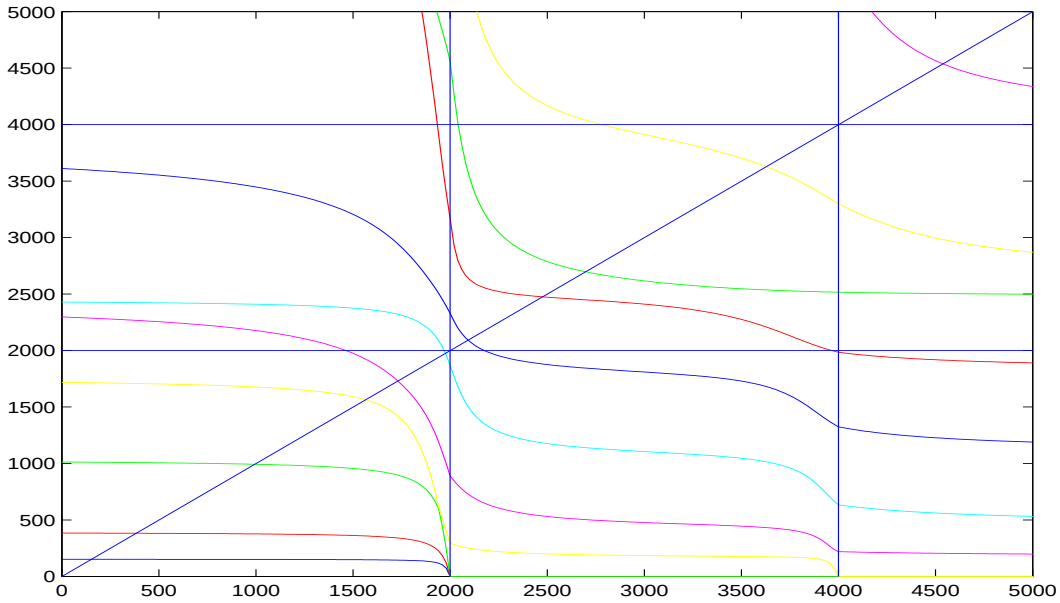

Figure 1: Eigencurves of equation (14) for a plate with 4 masses

These results demonstrate that it is reasonable to call a pole σ_k an m-th eigenvalue of the rational eigenproblem (6) if and only if it is the m-smallest eigenvalue of the restricted problem (16).

If we connect the $m + r_{k+1}$-th eigencurve in (σ_k, σ_{k+1}) to the m-th eigencurve in $(\sigma_{k+1}, \sigma_{k+2})$ then we obtain a joint curve which is continuous at σ_{k+1}. Figure 1 contains the eigencurves for a plate to which 3 identical masses and one further mass are attached.

Concerning the number of eigenvalues of (6) between two consecutive poles we obtain from Lemmas 1 and 2.

THEOREM 2
Let m_k be the number of eigenvalues λ_j of the reduced problem (16) satisfying $\lambda_j \le \sigma_k$, and let r_k be the rank of C_k.

Then the rational eigenproblem

$$Kx = \lambda\Big(M + \sum_{j=1}^{p} \frac{1}{\sigma_j - \lambda} C_j\Big)x, \tag{20}$$

has $n_{k+1} + r_{k+1} - n_k$ eigenvalues in $(\sigma_k, \sigma_{k+1}]$ enumerated by $n_k + 1, n_k + 2, \ldots, n_{k+1} + r_{k+1}$.

Theorem 2 answers the question how to initialize the Arnoldi method for the rational eigenproblem (6). For the interval $J_1 := (0, \sigma_1)$ the infimum of the Rayleigh functional

is contained in J_1, and due to the global convergence of the safeguarded iteration we can start with any one dimensional space V such that $P(V) \in J_1$, and can compute the eigenvalues in J_1 one after the other until the method leaves the interval J_1. Let $\tilde{N}_1$ be the number of the largest eigenvalue $\lambda_{\tilde{N}_1}$ of (1) found in J_1.

To start the method for $J_2 := (\sigma_1, \sigma_2)$ we choose $\hat{\mu} = \sigma_1 + \varepsilon$, $\varepsilon > 0$ small, and determine the eigenvalues of the linear problem (14) for $\mu = \hat{\mu}$ which are less than $\hat{\mu}$. We assume that these are $\tilde{n}_1$. If $\tilde{n}_1 + r_1 = N_1$ then all eigenvalues in J_1 have been found, no eigenvalue exist in $(\sigma_1, \hat{\mu})$, and we can start the Arnoldi method for J_2 with an orthonormal basis of the invariant subspace of problem (14) corresponding to the $\tilde{n}_1 + 1$ smallest eigenvalues. Otherwise we have to explore the intervals $(\lambda_{\tilde{N}_1}, \sigma_1)$ and $(\sigma_1, \hat{\mu})$ for further eigenvalues. For the subsequent intervals we can proceed in an analogous way.

5 NUMERICAL EXAMPLE

Consider the clamped plate occupying the domain $\Omega = (0, 4) \times (0, 3)$ with constant coefficients $\rho = d = 1$. We assume that 6 masses are attached to the plate at $x_1 = (1, 1)$, $x_2 = (2, 1)$, $x_3 = (3, 1)$, $x_4 = (1, 2)$, $x_5 = (2, 2)$ and $x_6 = (3, 2)$, where $\sigma_1 = \sigma_2 = \sigma_3 = 1000$ $\sigma_4 = \sigma_6 = 2000$ and $\sigma_5 = 3000$, and $m_1 = 1$, $m_2 = 1/2$ and $m_3 = 1/3$.

We discretized the eigenproblem by Bogner-Fox-Schmit elements on a quadratic mesh with stepsize $h = 0.05$ which yielded a matrix eigenvalue problem

$$Kx = \lambda Mx + \frac{1000\lambda}{1000 - \lambda} C_1 x + \frac{1000\lambda}{2000 - \lambda} C_2 x + \frac{1000\lambda}{3000 - \lambda} C_3 x \tag{21}$$

of dimension 18644. Here C_j for $j = 1, 2, 3$ is a diagonal matrix of rank 3, 2 and 1, respectively, corresponding to the loads $\{m_1, m_2, m_3\}$, $\{m_4, m_6\}$ and m_5.

By Theorem 2 problem (21) has 24 eigenvalues smaller than the smallest pole σ_1 enumerated $\lambda_1, \ldots, \lambda_{24}$. The Arnoldi method found all eigenvalues. Figure 2 contains the convergence history of the method, where we show graphically the norm of the residual vector as a function of the iteration number. Every time when the norm is less than 10^{-3} (marked by $\times$) an eigenvalue has been found. It is flagged by $\circ$ if it is assigned the correct number.

In accordance with Theorem 1 the method converges to the smallest eigenvalue λ_1 first, but then it detects λ_3, λ_4, λ_6, and λ_6 again as second, third, forth and fifth eigenvalue, respectively. Since λ_6 is found to be not a double eigenvalue the method returns to computing a forth eigenvalue and finds λ_4 a second time. Reducing the number of the desired eigenvalue again, λ_3 is obtained again, and after a further reduction the method finally converges to the second eigenvalue λ_2. Thereafter the eigenvalues $\lambda_3, \ldots, \lambda_6$ are found correctly, each of them requiring an expansion of the search space V by one vector only. For λ_3, λ_4 and λ_6 this is no surprise since these eigenvalues were found already previously, and therefore the search space already contained a good approximation of the corresponding eigenvector. After determining λ_8 twice and returning to searching for the seventh eigenvalue the method finds all remaining eigenvalues one after the other in the

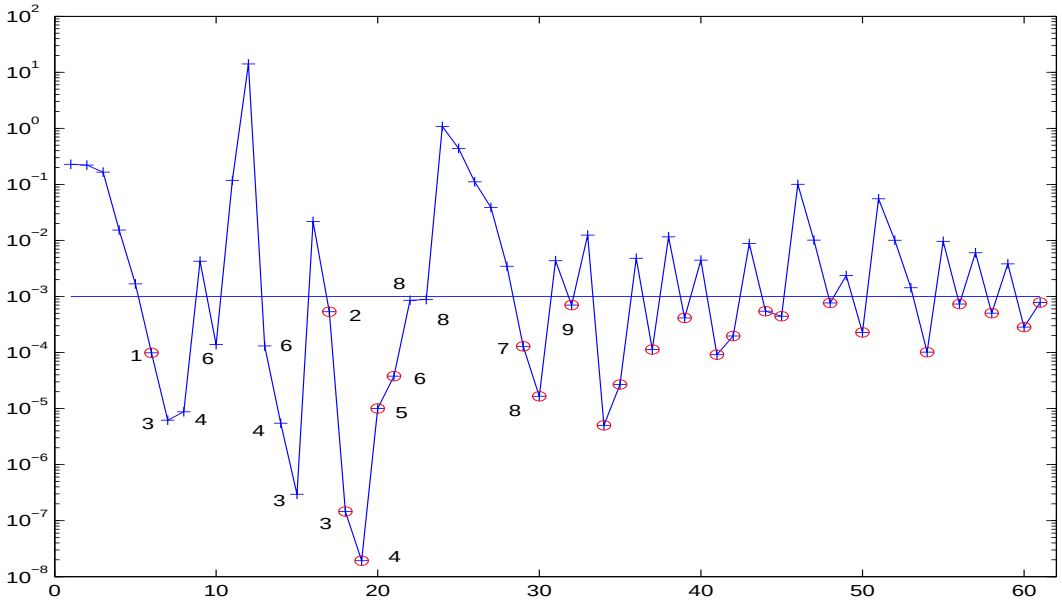

Figure 2: Convergence history; interval $(0, 1000)$

correct order. In Figure 2 we assigned the numbers to the converged eigenvalues up to λ_9.

Although in the whole run the algorithm reduced the number of the desired eigenvalue four times the method performs efficiently. The dimension of the search space containing good approximations to all 24 wanted eigenvectors grows only to 61, hence, an average of only 2.5 iteration steps is needed to find an eigenvalue.

On an Intel Centrino M processor with 1.7 GHz and 1 GB RAM under MATLAB 6.5 the nonlinear Arnoldi method required 138.6 seconds CPU time, where we terminated the iteration if the residual norm was less than 10^{-3}. 0.54 seconds were needed to solve all 61 projected eigenproblems by safeguarded iteration, and 36.3 seconds to determine the preconditioners (namely, two LU factorizations of $T(\sigma)$).

The interval $J_2 = (\sigma_1, \sigma_2)$ contains 8 eigenvalues enumerated $\tilde{\lambda}_{22}, \ldots, \tilde{\lambda}_{29}$ (notice that there are 24 eigenvalues $\lambda_1, \ldots, \lambda_{24} \in (0, \sigma_1)$ and $\operatorname{rank}(C_1) = 3$; thus, by Theorem 2 the smallest eigenvalue in J_2 must be a 22nd eigenvalue). According to Section 4 an appropriate initial search space is the invariant subspace corresponding to the 22 largest eigenvalues of the linear problem

$$Kx + \frac{1000(1000+\varepsilon)}{\varepsilon} C_1 x = \lambda M x + \frac{1000\lambda}{1000-\varepsilon} C_2 x + \frac{1000\lambda}{2000-\varepsilon} C_3 x \tag{22}$$

Figure 3 shows the convergence history of the nonlinear Arnoldi method for the interval J_2 where we tagged the correct number to the converged eigenvalues and marked the correctly found eigenvalues by $\circ$. The method required 32 iteration steps, and 86.5 seconds CPU time, 0.45 of which were needed to solve the projected eigenproblems and 36.7 seconds to obtain the preconditioners.

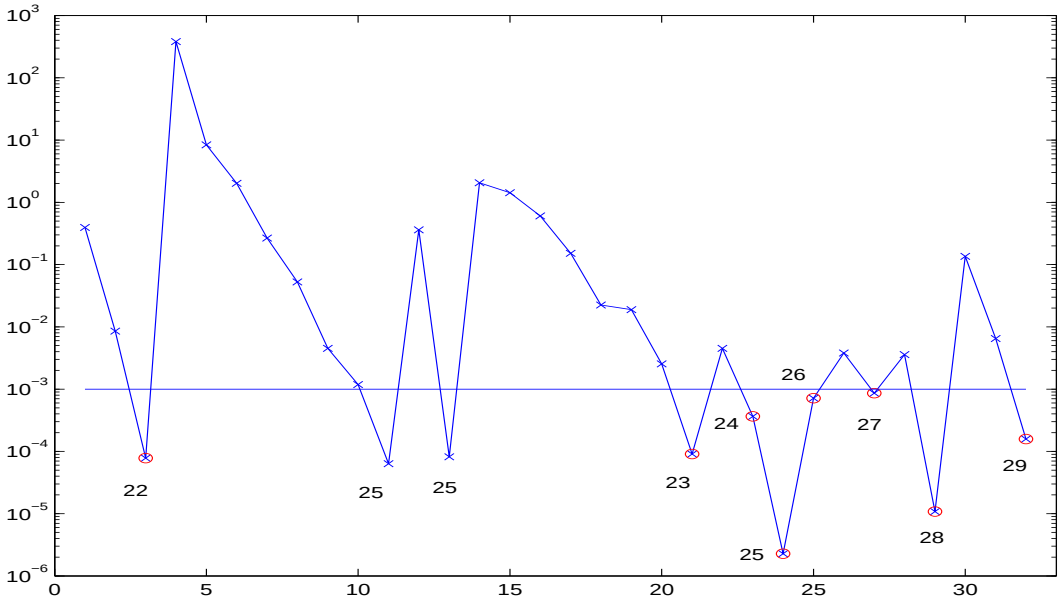

Figure 3: Convergence history; interval $(1000, 2000)$

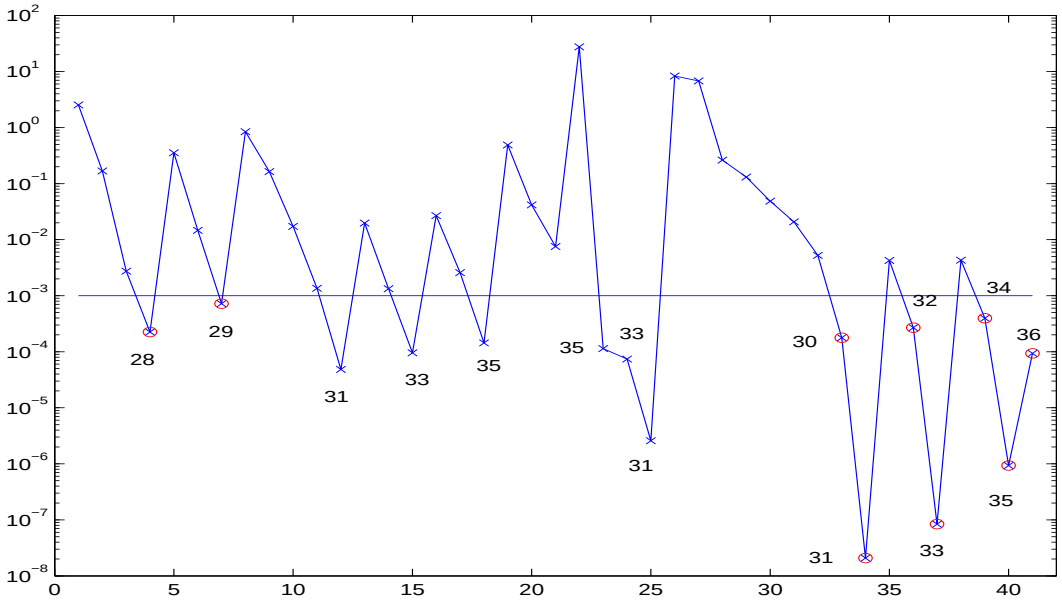

Figure 4: Convergence history; interval $(2000, 3000)$

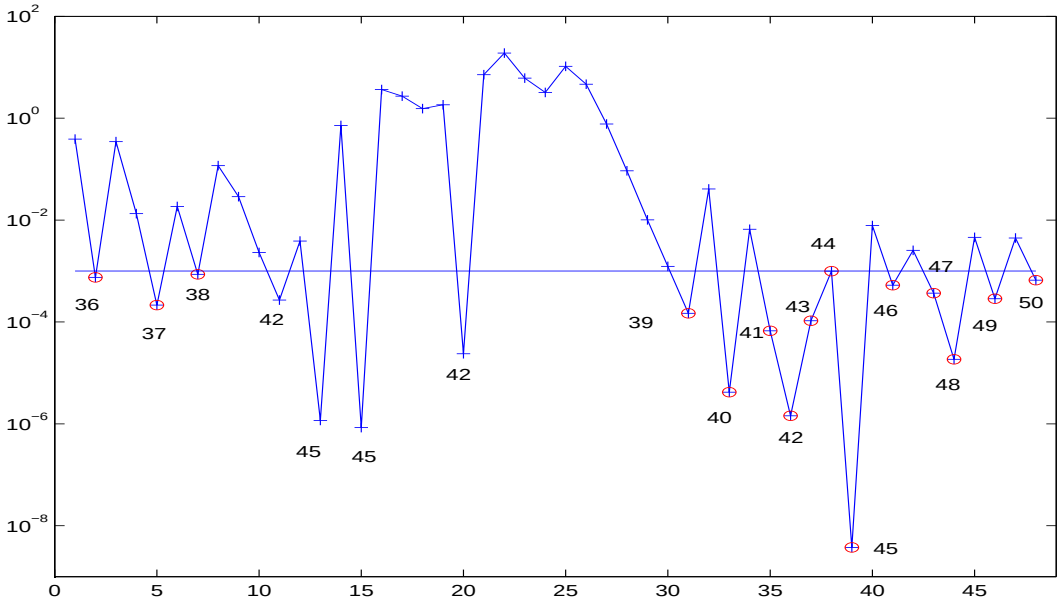

Figure 5: Convergence history; interval $(3000, 5000)$

For the interval J_3 and the eigenvalues which are larger than the largest pole the method behaved similar. 9 eigenvalues are contained in J_3 enumerated $\hat{\lambda}_{29}, \ldots, \hat{\lambda}_{36}$ which were found requiring 41 iteration steps and 84.4 seconds CPU time, were 0.90 second were needed to solve the projected eigenproblems and 18.1 seconds to determine the preconditioner. The convergence history is contained in Figure 4.

Finally, there are 18609 eigenvalues greater than σ_3 the smallest of them being a 36th one. The interval $J_4 = (3000, 5000)$ contains 15 eigenvalues. The Arnoldi method needed 48 iterations and 149.9 seconds CPU time. 1.42 seconds were required to solve the projected eigenproblems and 73.4 seconds to determine the preconditioners (4 LU factorizations). Figure 5 shows the convergence history for this interval.

6 CONCLUSIONS

For the rational eigenvalue problem governing the free vibrations of plate with attached loads we considered a nonlinear iterative projection methods of Arnoldi type. Taking advantage of symmetry properties and the minmax characterization of its eigenvalues the projected eigenproblems of small dimension can be solved efficiently by safeguarded iteration. Particular care is taken that the method does not miss eigenvalues in a specified interval, and the method is prohibited from converging to eigenvalues which have been found already in a preceding step.

REFERENCES

[1] Z. Bai, J. Demmel, J. Dongarra, A. Ruhe, and H.A. van der Vorst, editors. *Templates for the Solution of Algebraic Eigenvalue Problems: A Practical Guide*. SIAM, Philadelphia, 2000.

[2] T. Betcke and H. Voss. A Jacobi–Davidson–type projection method for nonlinear eigenvalue problems. Technical Report 47, Section of Mathematics, Hamburg University of Technology, 2002. To appear in Future Generation Computer Systems.

[3] P. Hager. *Eigenfrequency Analysis. FE-Adaptivity and a Nonlinear Eigenvalue Problem*. PhD thesis, Chalmers University of Technology, Göteborg, 2001.

[4] P. Hager and N.E. Wiberg. The rational Krylov algorithm for nonlinear eigenvalue problems. In B.H.V. Topping, editor, *Computational Mechanics for the Twenty-First Century*, pages 379 – 402. Saxe–Coburg Publications, Edinburgh, 2000.

[5] E. Jarlebring and H. Voss. Rational Krylov for nonlinear eigenproblems, an iterative projection method. Technical Report 69, Section of Mathematics, Hamburg University of Technology, 2003. Submitted to Appl.Math.

[6] L. Mazurenko and H. Voss. On the number of eigenvalues of a rational eigenproblem. Technical Report 61, Section of Mathematics, Hamburg University of Technology, 2003. Submitted to SIAM J.Numer.Anal.

[7] A. Neumaier. Residual inverse iteration for the nonlinear eigenvalue problem. *SIAM J. Numer. Anal.*, 22:914 – 923, 1985.

[8] A. Ruhe. Computing nonlinear eigenvalues with spectral transformation Arnoldi. *ZAMM*, 76:S2: 17 – 20, 1996.

[9] A. Ruhe. A rational Krylov algorithm for nonlinear matrix eigenvalue problems. *Zapiski Nauchnyh Seminarov POMI*, 268:176 – 180, 2000.

[10] Y. Saad. *Numerical Methods for Large Eigenvalue Problems*. Manchester University Press, John Wiley & Sons, New York, Brisbane, Toronto, 1992.

[11] G.L. Sleijpen, G.L. Booten, D.R. Fokkema, and H.A. van der Vorst. Jacobi-Davidson type methods for generalized eigenproblems and polynomial eigenproblems. *BIT*, 36:595 – 633, 1996.

[12] G.L. Sleijpen and H.A. van der Vorst. A Jacobi-Davidson iteration method for linear eigenvalue problems. *SIAM J.Matr.Anal.Appl.*, 17:401 – 425, 1996.

[13] H. Voss. An Arnoldi method for nonlinear eigenvalue problems. Technical Report 56, Section of Mathematics, Hamburg University of Technology, 2002. To appear in BIT Numerical Mathematics.

[14] H. Voss. An Arnoldi method for nonlinear symmetric eigenvalue problems. In *Online Proceedings of the SIAM Conference on Applied Linear Algebra, Williamsburg*, http://www.siam.org/meetings/laa03/, 2003.

[15] H. Voss. Initializing iterative projection methods for rational symmetric eigenproblems. In *Online Proceedings of the Dagstuhl Seminar Theoretical and Computational Aspects of Matrix Algorithms, Schloss Dagstuhl 2003*, ftp://ftp.dagstuhl.de/pub/Proceedings/03/03421/03421.VoszHeinrich.Other.pdf, 2003.

[16] H. Voss and B. Werner. A minimax principle for nonlinear eigenvalue problems with applications to nonoverdamped systems. *Math.Meth.Appl.Sci.*, 4:415–424, 1982.