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The Cauchy-Riemann operator on smooth Fréchet-valued functions with exponential growth on rotated strips

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We study the Cauchy-Riemann operator $\overline{\partial}$ on spaces $\mathcal{E}^{\gamma}(\Omega_{(\theta)}, E)$ of smooth functions on an open set $\Omega_{(\theta)} \subset \mathbb{R}^2$ with values in a complex Fréchet space E which grow exponentially ($\gamma = 1$) on strips rotated by an angle θ. We state sufficient conditions for the surjectivity of $\overline{\partial}: \mathcal{E}^{\gamma}(\Omega_{(\theta)}, E) \to \mathcal{E}^{\gamma}(\Omega_{(\theta)}, E)$.

1 Introduction

It is a classical result that the Cauchy-Riemann operator $\overline{\partial}: \mathcal{C}^{\infty}(\Omega) \to \mathcal{C}^{\infty}(\Omega)$ is surjective on the space $\mathcal{C}^{\infty}(\Omega)$ of smooth $\mathbb{C}$-valued functions on an open set $\Omega \subset \mathbb{R}^2$ (see e.g. [1, Theorem 1.4.4, p. 12]). The interest in the surjectivity of the Cauchy-Riemann operator $\overline{\partial}: \mathcal{C}^{\infty}(\Omega, E) \to \mathcal{C}^{\infty}(\Omega, E)$ on the space $\mathcal{C}^{\infty}(\Omega, E)$ of smooth functions on Ω with values in a complex Fréchet space E comes from a parameter dependence problem.

Given a family of right-hand sides $(f_\lambda)_{\lambda\in U}$ in $\mathcal{C}^{\infty}(\Omega)$ depending holomorphically on the parameter λ is there a family of solutions $(u_\lambda)_{\lambda\in U}$ in $\mathcal{C}^{\infty}(\Omega)$ with $\overline{\partial}u_\lambda = f_\lambda$ for all $\lambda \in U$ which depends holomorphically on the parameter λ as well? Here, holomorphic parameter dependence of $(f_\lambda)_{\lambda\in U}$ means that the map $\lambda \mapsto f_\lambda(x)$ is holomorphic on the open set $U \subset \mathbb{C}$ for all $x \in \Omega$. The answer to this question is affirmative since the Cauchy-Riemann operator $\overline{\partial}: \mathcal{C}^{\infty}(\Omega, \mathcal{O}(U)) \to \mathcal{C}^{\infty}(\Omega, \mathcal{O}(U))$ is surjective by [2, Theorem 10.10, p. 240] where $\mathcal{O}(U)$ is the Fréchet space of holomorphic functions on U equipped with the compact-open topology.

In this short note we consider the problem whether for any given $f \in \mathcal{C}^{\infty}(\Omega, E)$ there is a solution u of $\overline{\partial}u = f$ not only in $\mathcal{C}^{\infty}(\Omega, E)$ but conserving the growth of f. We solve this problem in the affirmative for functions having exponential growth on rotated strips using a corresponding result on horizontal strips given in [3].

2 Notation and Preliminaries

We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^2$ and identify $\mathbb{R}^2$ and $\mathbb{C}$ as normed linear spaces. In the following E stands for a non-trivial locally convex Hausdorff space over the field $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$. If $E = \mathbb{C}$, then we set $(p_\alpha)_{\alpha\in\mathfrak{A}} := \{|\cdot|\}$. We recall the following well-known definitions regarding continuous partial differentiability of E-valued functions (c.f. [3, p. 3-4]). A function $f: \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^2$ to E is called continuously partially differentiable (f is $\mathcal{C}^1$) if for the n-th unit vector $e_n \in \mathbb{R}^2$ the limit

$$\partial^{e_n} f(x) := \partial_n f(x) := \lim_{\substack{h\to 0\\ h\in\mathbb{R}, h\neq 0}} \frac{f(x + he_n) - f(x)}{h}$$

exists in E for every $x \in \Omega$ and $\partial^{e_n} f$ is continuous on Ω ($\partial^{e_n} f$ is $\mathcal{C}^0$) for $n = 1, 2$. For $k \in \mathbb{N}$ a function f is said to be k-times continuously partially differentiable (f is $\mathcal{C}^k$) if f is $\mathcal{C}^1$ and all its first partial derivatives are $\mathcal{C}^{k-1}$. A function f is called infinitely continuously partially differentiable (f is $\mathcal{C}^\infty$) if f is $\mathcal{C}^k$ for every $k \in \mathbb{N}$. The linear space of all functions $f: \Omega \to E$ which are $\mathcal{C}^\infty$ is denoted by $\mathcal{C}^{\infty}(\Omega, E)$. For $f \in \mathcal{C}^{\infty}(\Omega, E)$ and a multi-index $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$ we set $\partial^{\beta_n} f := f$ if $\beta_n = 0$, and

$$\partial^{\beta_n} f := \underbrace{(\partial^{e_n})\cdots(\partial^{e_n})}_{\beta_n\text{-times}} f$$

if $\beta_n \neq 0$ as well as

$$\partial^{\beta} f := \partial^{\beta_1}\partial^{\beta_2} f.$$

Due to the vector-valued version of Schwarz' theorem $\partial^\beta f$ is independent of the order of the partial derivatives on the right-hand side and we call $|\beta| := \beta_1 + \beta_2$ the order of differentiation.

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3 Cauchy-Riemann operator

Let $\Omega \subset \mathbb{R}^2$ be an open non-empty set and $(\Omega_n)_{n\in\mathbb{N}}$ a family of open sets given by

$$\Omega_n := \{x = (x_1, x_2) \in \Omega \mid |x_2| < n \text{ and } \inf_{y\in\partial\Omega} |y - x| > 1/n\}$$

where we use the convention $\inf_{y\in\varnothing} |y - x| := \infty$. In particular, we obtain for $\Omega = \mathbb{R}^2$ the horizontal strips

$$\mathbb{R}^2_n = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_2| < n\}, \quad n \in \mathbb{N}.$$

Next, we rotate the sets Ω and Ω_n. For an angle $\theta \in \mathbb{R}$ we define the rotated sets $\Omega_{n,(\theta)} := e^{i\theta}\Omega_n$, $n \in \mathbb{N}$, and $\Omega_{(\theta)} := e^{i\theta}\Omega$ where we use the identification $\mathbb{C} = \mathbb{R}^2$ to set

$$e^{i\theta}x := (\cos(\theta)x_1 - \sin(\theta)x_2, \sin(\theta)x_1 + \cos(\theta)x_2), \quad x = (x_1, x_2) \in \Omega.$$

Then $\Omega_{(0)} = \Omega$, $\Omega_{n,(0)} = \Omega_n$ for all $n \in \mathbb{N}$ and $\Omega_{(\theta)} = \bigcup_{n\in\mathbb{N}} \Omega_{n,(\theta)}$. Now, we introduce the spaces we want to consider.

Definition 3.1 Let $\Omega \subset \mathbb{R}^2$ be an open non-empty set, $\theta \in [0, 2\pi)$, the sequence $(a_n)_{n\in\mathbb{N}}$ strictly increasing, $\gamma > 0$ and $(E, p_\alpha)_{\alpha\in\mathfrak{A}}$ a locally convex Hausdorff space. We set

$$\mathcal{E}^\gamma(\Omega_{(\theta)}, E) := \{f \in \mathcal{C}^\infty(\Omega_{(\theta)}, E) \mid \forall\, n \in \mathbb{N},\, m \in \mathbb{N}_0,\, \alpha \in \mathfrak{A}: \, |f|_{n,m,\alpha,\theta} := \sup_{\substack{x\in\Omega_{n,(\theta)} \\ \beta\in\mathbb{N}_0^2,\, |\beta|\le m}} p_\alpha\big(\partial^\beta f(x)\big)e^{a_n|x|^\gamma} < \infty\}.$$

For $\theta = 0$ we have the following result concerning the surjectivity of the Cauchy-Riemann operator for smooth functions with values in a Fréchet space, i.e. in a complete metrisable locally convex space.

Theorem 3.2 ([3, 4.10 Example a), p. 22]) *Let $\Omega \subset \mathbb{R}^2$ be an open non-empty set and $(a_n)_{n\in\mathbb{N}}$ strictly increasing. If $a_n \le 0$ for all $n \in \mathbb{N}$, $0 < \gamma \le 1$ and $(E, p_\alpha)_{\alpha\in\mathfrak{A}}$ is a Fréchet space over $\mathbb{C}$, then the Cauchy-Riemann operator*

$$\overline{\partial} := \frac{1}{2}(\partial_1 + i\partial_2): \mathcal{E}^\gamma(\Omega, E) \to \mathcal{E}^\gamma(\Omega, E)$$

is surjective.

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^2$ be an open non-empty set, $\theta \in [0, 2\pi)$ and $(a_n)_{n\in\mathbb{N}}$ strictly increasing. If $a_n \le 0$ for all $n \in \mathbb{N}$, $0 < \gamma \le 1$ and $(E, p_\alpha)_{\alpha\in\mathfrak{A}}$ is a Fréchet space over $\mathbb{C}$, then the Cauchy-Riemann operator*

$$\overline{\partial}: \mathcal{E}^\gamma(\Omega_{(\theta)}, E) \to \mathcal{E}^\gamma(\Omega_{(\theta)}, E)$$

is surjective.

Proof. Let $f \in \mathcal{E}^\gamma(\Omega_{(\theta)}, E)$. Then we set $\widetilde{f}: \Omega \to E$, $\widetilde{f}(x) := f(e^{i\theta}x)$, which is a well-defined map, $\widetilde{f} \in \mathcal{C}^\infty(\Omega_{(\theta)}, E)$ and by induction over the order of differentiation we obtain

$$\partial^\beta \widetilde{f}(x) = e^{i\theta|\beta|}(\partial^\beta f)(e^{i\theta}x), \quad x \in \Omega,\ \beta \in \mathbb{N}_0^2.$$

Furthermore, for each $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$ we have

$$\begin{aligned} |\widetilde{f}|_{n,m,\alpha,0} &= \sup_{\substack{x\in\Omega_n \\ \beta\in\mathbb{N}_0^2,\, |\beta|\le m}} p_\alpha\big(\partial^\beta \widetilde{f}(x)\big)e^{a_n|x|^\gamma} = \sup_{\substack{x\in\Omega_n \\ \beta\in\mathbb{N}_0^2,\, |\beta|\le m}} p_\alpha\big(e^{i\theta|\beta|}(\partial^\beta f)(e^{i\theta}x)\big)e^{a_n|e^{-i\theta}e^{i\theta}x|^\gamma} \\ &= \sup_{\substack{y\in\Omega_{n,(\theta)} \\ \beta\in\mathbb{N}_0^2,\, |\beta|\le m}} |e^{i\theta|\beta|}|p_\alpha\big((\partial^\beta f)(y)\big)e^{a_n|e^{-i\theta}|^\gamma|y|^\gamma} = \sup_{\substack{y\in\Omega_{n,(\theta)} \\ \beta\in\mathbb{N}_0^2,\, |\beta|\le m}} p_\alpha\big((\partial^\beta f)(y)\big)e^{a_n|y|^\gamma} = |f|_{n,m,\alpha,\theta} < \infty \end{aligned}$$

implying $\widetilde{f} \in \mathcal{E}^\gamma(\Omega, E)$. Hence it follows from Theorem 3.2 that there is $\widetilde{u} \in \mathcal{E}^\gamma(\Omega, E)$ such that $\overline{\partial}\widetilde{u} = \widetilde{f}$. We set $u: \Omega_{(\theta)} \to E$, $u(x) := e^{-i\theta}\widetilde{u}(e^{-i\theta}x)$, which is a well-defined map, $u \in \mathcal{C}^\infty(\Omega_{(\theta)}, E)$ and like before we derive that $|u|_{n,m,\alpha,\theta} = |\widetilde{u}|_{n,m,\alpha,0} < \infty$ for all $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$ which yields $u \in \mathcal{E}^\gamma(\Omega_{(\theta)}, E)$. Thus we get from the chain rule for the Cauchy-Riemann operator

$$\overline{\partial}u(x) = e^{-i\theta}(\overline{\partial}\widetilde{u})(e^{-i\theta}x) \cdot \overline{\partial}(x \mapsto e^{i\theta}(x_1, -x_2)) = e^{-i\theta}(\overline{\partial}\widetilde{u})(e^{-i\theta}x)e^{i\theta} = (\overline{\partial}\widetilde{u})(e^{-i\theta}x) = \widetilde{f}(e^{-i\theta}x) = f(x)$$

for all $x \in \Omega$. □

References

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