A model-order reduction technique for low rank rational perturbations of linear eigenproblems

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Abstract. Large and sparse rational eigenproblems where the rational term is of low rank k arise in vibrations of fluid–solid structures and of plates with elastically attached loads. Exploiting model order reduction techniques, namely the Padé approximation via block Lanczos method, problems of this type can be reduced to k–dimensional rational eigenproblems which can be solved efficiently by safeguarded iteration.

1 Introduction

In this paper we consider the rational eigenvalue problem

$$T(\lambda)x := -Kx + \lambda Mx + CD(\lambda)C^T x = 0, \tag{1}$$

where $K \in \mathbb{R}^{N\times N}$ and $M \in \mathbb{R}^{N\times N}$ are sparse symmetric and positive (semi-) definite matrices, $C \in \mathbb{R}^{N\times k}$ is a rectangular matrix of low rank k, and $D(\lambda) \in \mathbb{R}^{k\times k}$ is a real diagonal matrix depending rationally on a real parameter λ. Problems of this type arise in (finite element models of) vibrations of fluid–solid structures and of plates with elastically attached loads, e.g.

Problem (1) has a countable set of eigenvalues which can be characterized as minmax values of a Rayleigh functional [10], and its eigenpairs can be determined by iterative projection methods of Arnoldi [8] or Jacobi–Davidson type [2].

In this paper we take advantage of the fact that problem (1) can be interpreted as a rational perturbation of small rank k of a linear eigenproblem. Decomposing $x \in \mathbb{R}^N$ into its component in the range of C and its orthogonal complement, (1) can be rewritten as

$$\tilde{T}(\lambda)\tilde{x} := D(\lambda)^{-1}\tilde{x} + C^T(-K + \lambda M)^{-1}C\tilde{x} = 0, \tag{2}$$

which is a rational eigenvalue problem of much smaller dimension k.

The eigenproblem (2) retains the symmetry properties of problem (1), and hence, in principle it can be solved efficiently by safeguarded iteration. However, every step of safeguarded iteration requires the evaluation of $\tilde{T}(\lambda)$ for some λ, i.e. of $C^T(-K + \lambda M)^{-1}C$, which is too expensive because the dimension N is very large.

The term $C^T(-K+\lambda M)^{-1}C$ appears in transfer functions of time invariant linear systems, and in systems theory techniques have been developed to reduce the order of this term considerably. Taking advantage of these techniques, namely of the Padé approximation via the block Lanczos method, problem (2) is replaced by a problem of the same structure of much smaller order. Since this approximating problem inherits the symmetry properties of the original problem it can be solved efficiently by safeguarded iteration.

This paper is organized as follows. Section 2 presents the rational eigenproblems governing free vibrations of a plate with elastically attached loads, and of a fluid–solid structure. Section 3 summarizes the minmax theory for nonoverdamped nonlinear symmetric eigenvalue problems, and recalls the safeguarded iteration for determining eigenpairs of dense problems of this type in a systematic way. Section 4 discusses the reduction of problem (1) to a k dimensional rational eigenproblem and the application of the Padé–by–Lanczos method to reduce the order of the rational term. We demonstrate that the eigenvalues of the reduced problem allow a minmax characterization. Hence, it can be solved in a systematic way and efficiently by safeguarded iteration. Section 5 reports on the numerical behaviour of the model–order reduction technique for a finite element discretization of a plate problem with elastically attached loads. It demonstrates that for this type of problems the method is superior to iterative projection methods like Arnoldi's method [8]. A similar behaviour was observed in [3] for free vibrations of a fluid–solid structure.

2 Rational eigenvalue problems

In this section we briefly present two examples of rational eigenproblems of type (1).

Consider an isotropic thin plate occupying the plane domain Ω, and assume that for $j=1,\dots,p$ at points $\xi_j\in\Omega$ masses m_j are joined to the plate by elastic strings with stiffness coefficients k_j.

Then the the flexurable vibrations are governed by the eigenvalue problem

$$Lu(\xi) = \omega^2\rho d u + \sum_{j=1}^{p} \frac{\omega^2\sigma_j}{\sigma_j-\omega^2} m_j \delta(\xi-\xi_j)u\ ,\ \xi\in\Omega \tag{3}$$

$$Bu(\xi) = 0\ ,\ \xi\in\partial\Omega \tag{4}$$

where $\rho=\rho(\xi)$ is the volume mass density, $d=d(\xi)$ is the thickness of the plate at a point $\xi\in\Omega$, and $\sigma_j=\frac{k_j}{m_j}$. B is some boundary operator depending on the support of the plate, δ denotes Dirac's delta distribution, and

$$L = \partial_{11}D(\partial_{11}+\nu\partial_{22}) + \partial_{22}D(\partial_{22}+\nu\partial_{11}) + 2\partial_{12}D(1-\nu)\partial_{12}$$

is the plate operator where $\partial_{ij}=\partial_i\partial_j$ and $\partial_i=\partial/\partial\xi_i$, $D=Ed^3/12(1-\nu^2)$ the flexurable rigidity of the plate, E is Young's modulus, and ν the Poisson ratio.

Discretizing by finite elements yields a matrix eigenvalue problem

$$-Kx + \lambda Mx + \sum_{j=1}^{p} \frac{\lambda\sigma_j}{\sigma_j - \lambda} e_{i_j} e_{i_j}^T x = 0,$$

which can be easily given the form (1). Here $\lambda = \omega^2$ and $u(\xi_j) = x_{i_j}$

Another rational eigenproblem of type (1) is governing free vibrations of a tube bundle immersed in a slightly compressible fluid under the following simplifying assumptions: The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity. Considering small vibrations of the fluid (and the tubes) around the state of rest, it can also be assumed that the fluid is irrotational.

Let $\Omega \subset \mathbb{R}^2$ denote the section of the cavity, and $\Omega_j \subset \Omega$, $j = 1, \ldots, p$, the sections of the tubes. Then the free vibrations of the fluid are governed by (cf. [4])

$$c^2 \Delta\phi + \omega^2 \phi = 0 \quad \text{in } \Omega \setminus \cup_{j=1}^{p} \Omega_j$$

$$\frac{\partial\phi}{\partial n} = \frac{\rho_0 \omega^2}{k_j - \omega^2 m_j} n \cdot \int_{\partial\Omega_j} \phi n \, ds \quad \text{on } \partial\Omega_j,\ j = 1, \ldots, p$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega$$

Here ϕ is the potential of the velocity of the fluid, c denotes the speed of sound in the fluid, ρ_0 is the specific density of the fluid, k_j represents the stiffness constant of the spring system supporting tube j, m_j is the mass per unit length of the tube j, and n is the outward unit normal on the boundary of Ω and Ω_j, respectively. Again, discretizing by finite elements yields a rational eigenproblem (1).

3 Minmax characterization for nonlinear eigenproblems

In this section we briefly summarize the variational characterization of eigenvalues for nonlinear symmetric eigenvalue problems of finite dimension.

For $\lambda \in J$ in an open real interval J let $T(\lambda) \in \mathbb{R}^{n \times n}$ be a family of symmetric matrices the elements of which are differentiable. We assume that for every $x \in \mathbb{R}^n \setminus \{0\}$ the real equation

$$f(\lambda, x) := x^T T(\lambda) x = 0 \tag{5}$$

has at most one solution $\lambda \in J$. Then equation (5) defines a functional p on some subset $D \subset \mathbb{R}^n$ which obviously generalizes the Rayleigh quotient for linear pencils $T(\lambda) = \lambda B - A$, and which we call the Rayleigh functional of the nonlinear eigenvalue problem

$$T(\lambda)x = 0. \tag{6}$$

We further assume that

$$x^T T'(p(x))x > 0 \quad \text{for every } x \in D \tag{7}$$

generalizing the definiteness requirement for linear pencils. By the implicit function theorem D is an open set, and differentiating the identity $x^T T(p(x))x = 0$ one obtains, that the eigenvectors of (6) are stationary points of p.

For overdamped problems, i.e. if the Rayleigh functional p is defined on $\mathbb{R}^n \setminus \{0\}$, Rogers [7] generalized the minmax characterization of Poincaré for symmetric eigenproblems to nonlinear ones. In this case problem (6) has exactly n eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ in J, and it holds

$$\lambda_j = \min_{\dim V = j} \max_{v \in V,\ v \neq 0} p(v).$$

For the general symmetric nonlinear case this characterization does not hold. This is easily seen considering a linear family $T(\lambda) = \lambda B - A$ on an interval J which does not contain the smallest eigenvalue of $Ax = \lambda Bx$.

The key idea introduced in [11] is to enumerate the eigenvalues appropriately. $\lambda \in J$ is an eigenvalue of problem (6) if and only if $\mu = 0$ is an eigenvalue of the matrix $T(\lambda)$, and by Poincaré's maxmin principle there exists $m \in \mathbb{N}$ such that

$$0 = \max_{\dim V = m} \min_{x \in V \setminus \{0\}} \frac{x^T T(\lambda) x}{\|x\|^2}.$$

Then we assign this m to λ as its number and call λ an m-th eigenvalue of problem (6).

With this enumeration the following minmax characterization holds (cf. [11]):

Theorem 1. *Assume that for every $x \neq 0$ equation* (5) *has at most one solution $p(x)$ in the open real interval J, and that condition* (7) *holds.*

Then for every $m \in \{1, \ldots, n\}$ the nonlinear eigenproblem (6) *has at most one m-th eigenvalue λ_m in J, which can be characterized by*

$$\lambda_m = \min_{\substack{\dim V = m \\ D \cap V \neq \emptyset}} \sup_{v \in D \cap V} p(v). \tag{8}$$

Conversely, if

$$\lambda_m := \inf_{\substack{\dim V = m \\ D \cap V \neq \emptyset}} \sup_{v \in D \cap V} p(v) \in J, \tag{9}$$

then λ_m is an m-th eigenvalue of (6)*, and the characterization* (8) *holds.*

The minimum is attained by the invariant subspace of the matrix $T(\lambda_m)$ corresponding to its m largest eigenvalues, and the supremum is attained by any eigenvector of $T(\lambda_m)$ corresponding to $\mu = 0$.

To prove this characterization we took advantage of the following relation

$$\lambda \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \lambda_m \quad \Leftrightarrow \quad \mu_m(\lambda) := \max_{\dim V = m} \quad \min_{x\in V,\ x\neq 0} \quad \frac{x^T T(\lambda) x}{\|x\|^2} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0. \quad (10)$$

The enumeration of eigenvalues and the fact that the eigenvectors of (6) are the stationary vectors of the Rayleigh functional suggests the method in Algorithm 1 called safeguarded iteration for computing the m–th eigenvalue.

Algorithm 1 Safeguarded iteration

1: Start with an approximation σ_1 to the m-th eigenvalue of (6)
2: **for** $k = 1, 2, \dots$ until convergence **do**
3: determine an eigenvector x_k corresponding to the m-largest eigenvalue of $T(\sigma_k)$
4: evaluate $\sigma_{k+1} = p(x_k)$, i.e. solve $x_k^T T(\sigma_{k+1}) x_k = 0$ for σ_{k+1}
5: **end for**

The following theorem contains the convergence properties of the safeguarded iteration (cf. [12], [9]).

Theorem 2. *(i) If $\lambda_1 := \inf_{x\in D} p(x) \in J$ and $\sigma_1 \in p(D)$ then the safeguarded iteration converges globally to λ_1.*
(ii) If $\lambda_m \in J$ is an m-th eigenvalue of (6) *which is simple then the safeguarded iteration converges locally and quadratically to λ_m.*
(iii) Let $T(\lambda)$ be twice continuously differentiable, and assume that $T'(\lambda)$ is positive definite for $\lambda \in J$. If x_k in step 3. of Algorithm 1 is chosen to be an eigenvector corresponding to the m largest eigenvalue of the generalized eigenproblem $T(\sigma_k)x = \mu T'(\sigma_k)x$ then the convergence is even cubic.

4 Order reduction for rational eigenproblems

Let $K \in \mathbb{R}^{N\times N}$ and $M \in \mathbb{R}^{N\times N}$ be sparse symmetric matrices where M is positive definite and K is positive semidefinite, let $C \in \mathbb{R}^{N\times k}$ be a rectangular matrix of low rank $k \ll N$, and let $D(\lambda) := \mathrm{diag}\{\frac{\lambda}{\kappa_j - \lambda m_j}\} \in \mathbb{R}^{k\times k}$ be a real diagonal matrix depending rationally on a real parameter λ.

We consider the rational eigenvalue problem

$$T(\lambda)x := -Kx + \lambda Mx + CD(\lambda)C^T x = 0. \quad (11)$$

Decomposing $x = Cy + z$ with $y \in \mathbb{R}^k$ and $z \in \mathrm{range}\{C\}^\perp$, and multiplying equation (11) by $C^T(-K + \lambda M)^{-1}$ one obtains

$$C^T(Cy + z) + C^T(-K + \lambda M)^{-1} CD(\lambda)C^T(Cy + z) = 0$$

which is equivalent to

$$\tilde{T}(\lambda)\tilde{x} := -D(\lambda)^{-1}\tilde{x} + C^T(K - \lambda M)^{-1}C\tilde{x} = 0, \quad \tilde{x} := D(\lambda)C^T Cy. \quad (12)$$

This eigenproblem is of much smaller dimension than problem (11), and it retains the symmetry properties of (11). It is easily seen that $\tilde{T}(\lambda)$ satisfies the conditions of the minmax characterization in each interval $J := (\pi_j, \pi_{j+1})$ where π_j denotes the eigenvalues of the generalized problem $Kx = \pi Mx$ in ascending order. Hence, (12) could be solved by safeguarded iteration. However, since the dimension N of the original problem is usually very large, it is very costly to evaluate $C^T(K - \lambda M)^{-1}C$ and therefore $\tilde{T}(\lambda)$ for some given λ.

The term $H(\lambda) := C^T(K - \lambda M)^{-1}C$ appears in transfer functions of time invariant linear systems, and in systems theory techniques have been developed to reduce the order of this term considerably. One way to define such a reduction is by means of Padé approximation of $H(\lambda)$, which is a rational matrix function of the same type with a much smaller order than N.

Let $\lambda_0 \in \mathbb{C}$ be a shift which is not a pole of H. Then H has a Taylor series about λ_0

$$H(\lambda) = \sum_{j=0}^{\infty} \mu_j (\lambda - \lambda_0)^j \tag{13}$$

where the moments μ_j are $k \times k$ matrices. A reduced-order model of state-space dimension n is called an n-th Padé model of system (13), if the Taylor expansions of the transfer function H of the original problem and H_n of the reduced model agree in as many leading terms as possible, i.e.

$$H(\lambda) = H_n(\lambda) + \mathcal{O}\big((\lambda - \lambda_0)^{q(n)}\big), \tag{14}$$

where $q(n)$ is as large as possible, and which was proved by Freund [6] to satisfy

$$q(n) \geq 2\lfloor \frac{n}{k} \rfloor.$$

Although the Padé approximation is determined via a local property (14) it usually has excellent approximation properties in large domains which may even contain poles. As introduced by Feldmann and Freund [5] the Padé approximation H_n can be evaluated via the Lanczos process.

To apply the Padé–by–Lanczos process to the rational eigenproblem we transform $\tilde{T}$ further to a more convenient form. Choosing a shift λ_0 close to the eigenvalues we are interested in problem (12) can be rewritten as

$$\tilde{T}(\lambda)\tilde{x} = -\frac{1}{\lambda} D_1 \tilde{x} + D_2 \tilde{x} + H_{\lambda_0} \tilde{x} + (\lambda - \lambda_0) B^T (I - (\lambda - \lambda_0)A)^{-1} B \tilde{x} = 0 \tag{15}$$

where $M = EE^T$ is the Cholesky factorization of M, $H_{\lambda_0} := C^T(K - \lambda_0 M)^{-1}C$, $B := E^T(K - \lambda_0 M)^{-1}C$, $A := E^T(K - \lambda_0 M)^{-1}E$, and D_1 and D_2 are diagonal matrices with positive entries κ_j and m_j, respectively.

If no deflation is necessary the order of $B^T(I - (\lambda - \lambda_0)A)B$ can be reduced by block Lanczos method, and the following theorem holds. A more general version taking into account deflation is proved in [6], a different approach based on a coupled recurrence is derived in [1]. Note that we will consider only real shifts and therefore all appearing matrices can be assumed to be real.

Theorem 3. *Let $V_m \in \mathbb{R}^{N\times mk}$ be an orthonormal basis of the block Krylov space $\mathcal{K}_m(A,B) := span\{B, AB, \dots, A^{m-1}B\}$ generated by the block Lanczos process such that the following recursion holds*

$$AV_m = V_m A_m + [O, \dots, O, \hat{V}_{m+1}\beta_{m+1}] \tag{16}$$

where $\hat{V}_{m+1} \in \mathbb{R}^{N\times k}$, $\beta_{m+1} \in \mathbb{R}^{k\times k}$, and $A_m \in \mathbb{R}^{mk\times mk}$.

Then with $B = V_1\Phi$, and $B_m := [I_k, O, \dots, O]^T\Phi$ the moments are given by

$$BA^iB = B_m^T A_m^i B_m, \quad i = 0, 1, \dots, 2m-1, \tag{17}$$

and it holds

$$B^T(I - sA)^{-1}B = B_m^T(I - sA_m)^{-1}B_m + \mathcal{O}(|s|^{2m}). \tag{18}$$

Replacing $B^T(I-(\lambda-\lambda_0)A)^{-1}B$ by $B_m^T(I-(\lambda-\lambda_0)A_m)^{-1}B_m$ one obtains the reduced eigenvalue problem

$$S(\lambda)\tilde{x} := -\frac{1}{\lambda}D_1\tilde{x} + D_2\tilde{x} + H_{\lambda_0}\tilde{x} + (\lambda-\lambda_0)B_m^T(I-(\lambda-\lambda_0)A_m)^{-1}B_m\tilde{x} = 0 \tag{19}$$

which again is a rational eigenproblem with poles $\tilde{\pi}_0 = 0$ and $\tilde{\pi}_j = \lambda_0 + 1/\alpha_j$ where α_j, $j = 1, \dots, km$ denote the eigenvalues of A_m.

Let $\tilde{\pi}_\nu < \tilde{\pi}_{\nu+1}$ denote two consecutive poles of S, and let $J_\nu = (\tilde{\pi}_\nu, \tilde{\pi}_{nu+1})$. Then for $\lambda \in J_\nu$ it holds

$$\tilde{x}^T S'(\lambda)\tilde{x} = \frac{1}{\lambda^2}\tilde{x}^T D_1\tilde{x} + \sum_{j=1}^{km}\frac{\beta_j^2}{(1-\alpha_j(\lambda-\lambda_0))^2} > 0, \; B_m\tilde{x} = (\beta_j)_{j=1,\dots,km},$$

and hence the conditions of the minmax characterization are satisfied for the reduced eigenproblem in every interval J_ν, and therefore its eigenvalues can be determined by safeguarded iteration.

Moreover, for every $\tilde{x}$ the Rayleigh quotient of $S(\lambda)$

$$R(\tilde{x};\lambda) := \frac{\tilde{x}^T S(\lambda)\tilde{x}}{\|\tilde{x}\|_2^2}$$

is monotonely increasing in J_ν with respect to λ. Hence, if $\mu_j(\lambda)$ denote the eigenvalues of $S(\lambda)$ ordered in decreasing order, then every $\mu_j(\lambda)$ is monotonely increasing, and it follows immediately from (10)

Theorem 4. *For two consecutive poles $\tilde{\pi}_\nu < \tilde{\pi}_{\nu+1}$ of $S(\cdot)$, and $\tilde{\pi}_\nu < \xi < \zeta < \tilde{\pi}_{nu+1}$ let*

$$\mu_{\ell_1}(\xi) \le 0 < \mu_{\ell_1-1} \quad \textit{and} \quad \mu_{\ell_2}(\zeta) < 0 \le \mu_{\ell_2-1}(\zeta).$$

Then the reduced eigenvalue problem (19) has $\ell_2 - \ell_1$ eigenvalues

$$\xi \le \lambda_{\ell_1} \le \lambda_{\ell_1+1} \le \dots \le \lambda_{\ell_2-1} \le \zeta$$

which can be determined by (the cubically convergent version of) safeguarded iteration.

5 Numerical results

We consider a clamped plate occupying the region $\Omega = (0,4) \times (0,3)$, and we assume that 6 identical masses are attached at the points (i,j), $i = 1,2,3$, $j = 1,2$.

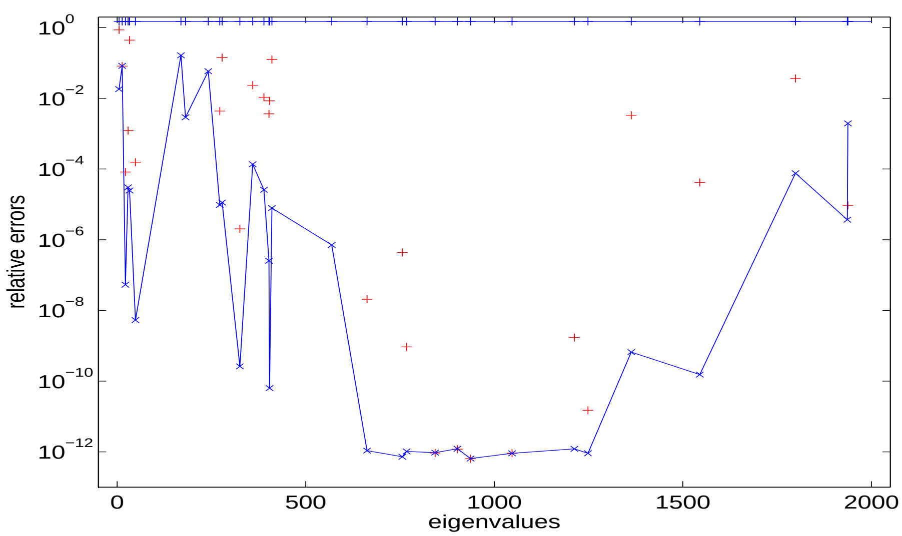
Discretizing with Bogner–Fox–Schmid elements with stepsize $h = 0.05$ one gets a rational eigenvalue problem

$$T(\lambda)x := -Kx + \lambda Mx + \frac{1000\lambda}{1000-\lambda} C^T Cx = 0$$

of dimension $N = 18644$ and $k = 6$ governing free vibrations of the plate which has 32 eigenvalues in the interval $J = (0, 2000)$.

For $m = 12$ with shift $\lambda_0 = 1000$ all 32 eigenvalues are found requiring 103.5 seconds on an Intel Centrino M processor with 1.7 GHz and 1 GB RAM under MATLAB 6.5. For $m = 6$ only 27 eigenvalues are found in 50.8 sec. For comparison, the nonlinear Arnoldi in [8] requires 227.1 seconds

Figure 1 demonstrates the approximation properties of the reduced problem. The eigenvalues are marked as vertical bars at the top of the picture, crosses indicate the relative errors of the eigenvalue approximations obtained for $m = 12$, and plus signs the errors for 27 eigenvalue approximations obtained for $m = 6$.



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