

On the Decomposition of Generalized Semiautomata

Merve Nur Cakir and Karl-Heinz Zimmermann*

Department of Computer Engineering
Hamburg University of Technology
21071 Hamburg, Germany

April 21, 2020

Abstract

Semiautomata are abstractions of electronic devices that are deterministic finite-state machines having inputs but no outputs. Generalized semiautomata are obtained from stochastic semiautomata by dropping the restrictions imposed by probability. It is well-known that each stochastic semiautomaton can be decomposed into a sequential product of a dependent source and deterministic semiautomaton making partly use of the celebrated theorem of Birkhoff-von Neumann. It will be shown that each generalized semiautomaton can be partitioned into a sequential product of a generalized dependent source and a deterministic semiautomaton.

AMS Subject Classification: 68Q70, 20M35, 15A04

Keywords: Semiautomaton, stochastic automaton, monoid, Birkhoff-von Neumann.

1 Introduction

The theory of discrete stochastic systems has been initiated by the work of Shannon [14] and von Neumann [10]. While Shannon has considered memory-less communication channels and their generalization by introducing states, von Neumann has studied the synthesis of reliable systems from unreliable components. The fundamental work of Rabin and Scott [12] about deterministic finite-state automata has led to two generalizations. First, the generalization of transition functions to conditional distributions studied by Carlyle [3] and Starke [15]. This in turn yields a generalization of discrete-time Markov chains in which the chains are governed by more than one

*Email: k.zimmermann@tuhh.de

transition probability matrix. Second, the generalization of regular sets by introducing stochastic automata as described by Rabin [11].

By the work of Turakainen [16], stochastic acceptors can be viewed equivalently as generalized automata in which the "probability" is neglected. This leads to a more accessible approach to stochastic automata [5].

On the other hand, the class of nondeterministic automata [13] can be generalized to monoidal automata, where the input alphabet corresponds to an arbitrary monoid instead of a free monoid [8, 9, 17]. This leads to the class of monoidal automata whose languages are closed under a smaller set of operations when compared with regular languages.

A first step into the study of automata theory are semiautomata which are abstractions of electronic devices that are deterministic finite-state machines having inputs but no outputs [7, 9]. Generalized semiautomata are obtained from stochastic semiautomata by dropping the restrictions imposed by probability [5, 16]. It is well-known that each stochastic automaton can be decomposed into a sequential product of a dependent source and deterministic semiautomaton [2]. This result makes use in part of the celebrated theorem of Birkhoff-von Neumann that each doubly stochastic matrix can be represented as a convex combination of permutation matrices. In this paper, it will be shown that each generalized semiautomaton can be partitioned into a sequential product of a generalized dependent source and a deterministic semiautomaton.

Notation. Let X be a set. The set of all mappings on X , $T(X) = \{f \mid f : X \rightarrow X\}$, forms a monoid under function composition $(fg)(x) = g(f(x))$, $x \in X$, and the identity function $\text{id}_X : X \rightarrow X : x \mapsto x$ is the identity element. The monoid $T(X)$ is called the *full transformation monoid* of X .

2 Semiautomata

Semiautomata are abstractions of electronic devices which are deterministic finite-state machines having input but no output [7, 9].

A (*deterministic*) *semiautomaton* (SA) is a triple

$$A = (S, \Sigma, \{\delta_x \mid x \in \Sigma\})$$

where

- S is the non-empty finite set of *states*,
- Σ is the set of *input symbols*,
- $\delta_x : S \rightarrow S$ is a (partial) mapping for each $x \in \Sigma$.

Let Σ^* denote the free monoid over the alphabet Σ . By the universal property of free monoids [4, 9], the mapping $\delta : \Sigma \rightarrow T(S) : x \mapsto \delta_x$ extends

uniquely to a monoid homomorphism $\delta : \Sigma^* \rightarrow T(S) : u \mapsto \delta_u$ such that for each word $u = x_1 \dots x_k \in \Sigma^*$,

$$\delta_u = \delta_{x_1} \cdots \delta_{x_k} \quad (1)$$

and particularly $\delta_\epsilon = \text{id}_S$. The mapping δ is called the *transition function* of A . Its image $T(A) = \{\delta_u \mid u \in \Sigma^*\}$ is a submonoid of the full transformation monoid $T(S)$ generated by $\{\delta_x \mid x \in \Sigma\}$. The semiautomaton A is also denoted by $A = (S, M, \delta)$ or $A = (S^A, M^A, \delta^A)$.

A semiautomaton $A = (S, \Sigma, \delta)$ serves as a skeleton of a deterministic finite-state machine that is exactly in one state at a time. If the semiautomaton A is in state s and reads the word $u \in \Sigma^*$, it transits into the state $s' = \delta_u(s)$.

Example 1. Consider the semiautomaton $A = (S, \Sigma, \delta)$ with state set $S = \{1, 2, 3\}$, input alphabet $\Sigma = \{x, y\}$, and transition function δ given by the automaton graph in Fig. 1. The associated transformation monoid is generated by the transformations

$$\delta_x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \delta_y = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}.$$

We have

$$\begin{aligned} \delta_{xx} &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, & \delta_{xy} &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \\ \delta_{yx} &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, & \delta_{yy} &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}. \end{aligned}$$

Hence, the transformation monoid $T(A)$ is given by $\{\text{id}_S, \delta_x, \delta_y, \delta_{xy}\}$. \diamond

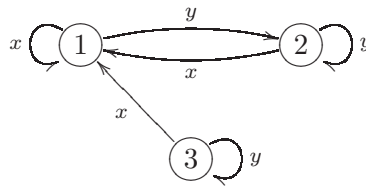


Figure 1: Semiautomaton.

3 Generalized Semiautomata

Stochastic automata are a generalization of non-deterministic finite state automata [5]. Generalized automata can be obtained from stochastic automata by dropping the restrictions imposed by probability [5, 16, 17].

A *generalized semiautomaton* (GSA) is a triple

$$A = (S, \Sigma, \{Q_x \mid x \in \Sigma\}),$$

where

- S is the non-empty finite set of *states*,
- Σ is the *input alphabet*, and
- Q is a collection of $n \times n$ nonnegative matrices Q_x , $x \in \Sigma$, where n is the number of states.

In view of the universal property of free monoids [4, 9], the mapping $Q : \Sigma \rightarrow \mathbb{R}^{n \times n} : x \mapsto Q_x$ extends uniquely to a monoid homomorphism $Q : \Sigma^* \rightarrow \mathbb{R}^{n \times n}$ such that for each word $u = x_1 \dots x_k \in \Sigma^*$,

$$Q_u = Q_{x_1} \cdots Q_{x_k} \quad (2)$$

and particularly $Q_\epsilon = I_n$ is the $n \times n$ identity matrix. The mapping Q is called the *transition function* of A . Its image $T(A) = \{Q_u \mid u \in \Sigma^*\}$ is a submonoid of the full transformation monoid $T(S)$ generated by $\{Q_x \mid x \in \Sigma\}$. The generalized semiautomaton A is also denoted by $A = (S, \Sigma, Q)$ or $A = (S^A, \Sigma^A, Q^A)$.

The state set $S = \{s_1, \dots, s_n\}$ can be viewed as the standard basis for the Euclidean vector space \mathbb{R}^n , where s_i is the basis vector whose i th coordinate is 1 and all others are 0. In this way, the (i, j) th entry of the matrix $Q_u = (s_{ij}^{(u)})$ is given by $s_{ij}^{(u)} = s_i^T Q_u s_j$.

Proposition 3.1. *Each deterministic semiautomaton is a generalized automaton.*

Proof. Let $A = (S, \Sigma, \delta)$ be a deterministic semiautomaton and let $S = \{s_1, \dots, s_n\}$. Define the generalized semiautomaton $B = (S, \Sigma, Q)$, where for each $x \in \Sigma$, the (i, j) th entry of Q_x is 1 if $\delta_x(s_i) = s_j$ and otherwise 0. Then the mapping $T(A) \rightarrow T(B) : \delta_u \mapsto Q_u$ is a monoid isomorphism. \square

A generalized semiautomaton $A = (S, \Sigma, P)$ is called *stochastic* if the matrices P_x , $x \in \Sigma$, are stochastic, i.e., P_x is a matrix of nonnegative real numbers such that each row sum is equal to 1. The product of stochastic matrices is again a stochastic matrix and so the transition monoid $T(A)$ consists of the stochastic matrices P_u , $u \in \Sigma^*$. In particular, the (i, j) th element $p(s_j \mid u, s_i)$ of the matrix P_u is the transition probability that the automaton enters state s_j when started in state s_i and reading the word u .

Example 2. Let $m \geq 2$ be an integer. Put $\Sigma = \{0, \dots, m-1\}$. The stochastic semiautomaton $\mathcal{A} = (\{s_1, s_2\}, \Sigma, P)$ given by

$$P_x = \frac{1}{m} \begin{pmatrix} m-x & x \\ m-x-1 & x+1 \end{pmatrix}, \quad x \in \Sigma,$$

is called *m-adic semiautomaton*. For each word $u = x_1 \dots x_k \in \Sigma^*$,

$$P_u = \frac{1}{m^k} \begin{pmatrix} m^k - w_k & w_k \\ m^k - w_k - 1 & w_k + 1 \end{pmatrix},$$

where $w_k = x_k m^{k-1} + \dots + x_2 m + x_1$ and the entry $\frac{1}{m^k} w_k$ corresponds in the *m*-adic representation to $0.x_k \dots x_1$. \diamond

A generalized semiautomaton $A = (S, \Sigma, D)$ is called *doubly stochastic* if the matrices D_x , $x \in \Sigma$, are doubly stochastic, i.e., D_x is a matrix of nonnegative real numbers such that each row and column sum is equal to 1. The product of doubly stochastic matrices is again a doubly stochastic matrix and so the transition monoid $T(A)$ consists of the doubly stochastic matrices D_u , $u \in \Sigma^*$.

4 Decomposition of Generalized Semiautomata

The objective is to decompose each generalized semiautomata into a sequential product of a generalized dependent source and a deterministic semiautomaton. The corresponding result for stochastic semiautomata has been proved by Bukharaev [2].

A *generalized dependent source* is a triple

$$\Gamma = (\Sigma, \Xi, \{\gamma(z \mid x) \mid x \in \Sigma, z \in \Xi\}),$$

where Σ and Ξ are alphabets and $\gamma : \Sigma \times \Xi \rightarrow \mathbb{R}_{\geq 0} : (x, z) \rightarrow \gamma(z \mid x)$ is a mapping which is extended recursively to $\Sigma^* \times \Xi^*$ as follows:

- $\gamma(\epsilon \mid \epsilon) = 1$,
- $\gamma(v \mid u) = 0$ for all $u \in \Sigma^*$ and $v \in \Xi^*$ with $|u| \neq |v|$, and
- $\gamma(zv \mid xu) = \gamma(x \mid z)\gamma(u \mid v)$ for all $x \in \Sigma$, $u \in \Sigma^*$, $z \in \Xi$ and $v \in \Xi^*$.

A generalized dependent source Γ is also denoted by $\Gamma = (\Sigma, \Xi, \gamma)$.

In particular, a *dependent source* is a generalized dependent source $\Gamma = (\Sigma, \Xi, \gamma)$, where Σ and Ξ are alphabets and for each $x \in \Sigma$, $\gamma(\cdot \mid x)$ defines a (conditional) probability measure on Ξ . This measure can be extended for each $u \in \Sigma^*$ to a (conditional) probability measure $\gamma(\cdot \mid u)$ on Ξ^* along the same lines as above. Note that a dependent source can be viewed as a stochastic input-output automaton with a single state [2, 5].

The *sequential product* of generalized dependent source $\Gamma = (\Sigma, \Xi, \gamma)$ and generalized semiautomaton $B = (S, \Xi, Q^B)$ defines a generalized semiautomaton $A = (S, \Sigma, Q^A)$ such that for all $x \in \Sigma$,

$$Q_x^A = \sum_{z \in \Xi} \gamma(z \mid x) \cdot Q_z^B. \quad (3)$$

By induction, for all $u \in \Sigma^*$,

$$Q_u^A = \sum_{v \in \Xi^*} \gamma(v \mid u) \cdot Q_v^B. \quad (4)$$

A permutation matrix P is a square binary matrix which has exactly one entry of 1 in each row and each column and 0's elsewhere. By the Birkhoff-von Neumann theorem [6], for each $n \times n$ doubly stochastic matrix P there exist real numbers $\alpha_1, \dots, \alpha_N \geq 0$ with $\sum_{i=1}^N \alpha_i = 1$ and permutation matrices P_1, \dots, P_N such that

$$P = \alpha_1 P_1 + \dots + \alpha_N P_N. \quad (5)$$

This representation is also known as Birkhoff-von Neumann decomposition. Such a representation of a doubly stochastic matrix as a convex combination of permutation matrices may not be unique. By the Marcus-Ree Theorem [1], $N \leq n^2 - 2n + 2$ for dense matrices.

A square matrix P is called *deterministic* if it has exactly one entry of 1 in each row and 0's elsewhere. In particular, each permutation matrix is deterministic. For each $n \times n$ stochastic matrix P there exist real numbers $\alpha_1, \dots, \alpha_N \geq 0$ with $\sum_{i=1}^N \alpha_i = 1$ and deterministic matrices P_1, \dots, P_N such that

$$P = \alpha_1 P_1 + \dots + \alpha_N P_N. \quad (6)$$

Such a representation of a stochastic matrix as a convex combination of deterministic matrices may not be unique.

A square matrix P is called *semideterministic* if in each nonzero row there is exactly one entry of 1 and 0's elsewhere. In particular, each deterministic matrix is semideterministic.

Proposition 4.1. *For each nonnegative square matrix A , there exist real numbers $\alpha_1, \dots, \alpha_N \geq 0$ and semideterministic matrices P_1, \dots, P_N such that*

$$A = \alpha_1 P_1 + \dots + \alpha_N P_N. \quad (7)$$

Proof. For each nonnegative square matrix $P = (p_{ij})$ let $p_{i,\pi(i)}$ be a minimal nonzero entry in row i . Consider the semideterministic matrix $D = (d_{ij})$ with $d_{i,\pi(i)} = 1$ for each i and $d_{ij} = 0$ otherwise. Moreover, put $m(P) = \min\{p_{ij} \mid p_{ij} \neq 0\}$. Then $P - m(P)D$ is a nonnegative matrix with at least one more zero entry than P . Iterating this step a finite number N of times gives a sequence $(P_k)_{1 \leq k \leq N}$ of nonnegative matrices and a sequence $(D_k)_{1 \leq k \leq N}$ of semideterministic matrices such that $P_1 = A$, $P_{k+1} = P_k - m(P_k)D_k$ for $1 \leq k \leq N$, and $P_{N+1} = 0$. This yields the decomposition of A as a linear combination of semideterministic matrices $A = \sum_{k=1}^N m(P_k)D_k$. \square

For doubly stochastic and stochastic matrices, the proof is similar.

Example 3. Consider the nonnegative matrix

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 2 & 2 & 8 \\ 3 & 3 & 6 \end{pmatrix}.$$

A sequence of reductions showing the selected entries at each step is

$$\begin{pmatrix} \underline{2} & 4 & 6 \\ \underline{2} & 2 & 8 \\ \underline{3} & 3 & 6 \end{pmatrix}, \begin{pmatrix} 0 & \underline{4} & 6 \\ 0 & \underline{2} & 8 \\ 0 & \underline{3} & 6 \end{pmatrix}, \begin{pmatrix} 0 & \underline{3} & 6 \\ 0 & \underline{1} & 8 \\ 0 & \underline{3} & 6 \end{pmatrix}, \begin{pmatrix} 0 & \underline{2} & 6 \\ 0 & 0 & \underline{8} \\ 0 & \underline{2} & 6 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \underline{6} \\ 0 & 0 & \underline{6} \\ 0 & 0 & \underline{6} \end{pmatrix},$$

yields the decomposition

$$\begin{aligned} A &= 2 \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &\quad + 2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 6 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

◇

Theorem 4.2. *Each generalized semiautomaton $A = (S, \Sigma, Q)$ can be represented as a sequential product of a generalized dependent source $\Gamma = (\Sigma, \Xi, \gamma)$ and a semideterministic semiautomaton $B = (S, \Xi, \delta)$.*

In particular, each stochastic (or strongly stochastic) semiautomaton $A = (S, \Sigma, P)$ can be represented as a sequential product of a dependent source $\Gamma = (\Sigma, \Xi, \gamma)$ and a deterministic (or permutation) semiautomaton $B = (S, \Xi, \delta)$.

Proof. Let $\{D_1, \dots, D_N\}$ denote the collection of $n \times n$ semideterministic matrices. Put $\Xi = \{1, \dots, N\}$ and for each $x \in \Sigma$, write Q_x as a conical combination of semideterministic matrices

$$Q_x = \sum_{z \in \Xi} \alpha(z, x) D_z.$$

This defines the generalized dependent source $\Gamma = (\Sigma, \Xi, \gamma)$, where for each $x \in \Sigma$ and $z \in \Xi$,

$$\gamma(z \mid x) = \alpha(z, x),$$

and the deterministic automaton $B = (S, \Xi, \delta)$, where for each $z \in \Xi$, the transition $\delta_z : S \rightarrow S$ is given by the matrix D_z as in the proof of Prop. 3.1. Then we obtain for each $x \in \Sigma$,

$$Q_x^A = \sum_{z \in \Xi} \gamma(z \mid x) Q_z^B.$$

The second part is clear from the above remarks. □

Example 4. Consider the generalized semiautomaton

$$A = (\{s_1, s_2\}, \{x_1, x_2\}, \{Q_{x_1}, Q_{x_2}\}),$$

where

$$Q_{x_1} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_{x_2} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

Then

$$Q_{x_1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$Q_{x_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Put $\Xi = \{z_1, \dots, z_5\}$ and

$$\begin{aligned} D_{z_1} &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, & D_{z_2} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & D_{z_3} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ D_{z_4} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & D_{z_5} &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then

$$Q_{x_1} = D_{z_1} + D_{z_2} + 3D_{z_3} \quad \text{and} \quad Q_{x_2} = D_{z_4} + 2D_{z_5}.$$

This gives the state transition table of the deterministic semiautomaton $B = (S, \Xi, \delta)$, where

δ^B	z_1	z_2	z_3	z_4	z_5
s_1	s_1	s_1	s_2	s_1	s_2
s_2	s_1	—	—	s_2	s_2

and the transitions of the generalized dependent source $\Gamma = (\Sigma, \Xi, \gamma)$, where

γ	z_1	z_2	z_3	z_4	z_5
x_1	1	1	3	0	0
x_1	0	0	0	1	2

◇

Example 5. Reconsider the m -adic semiautomaton $\mathcal{A} = (\{s_1, s_2\}, \Sigma, P)$. For each $x \in \Sigma$,

$$P_x = \frac{m-x-1}{m} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{x}{m} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Put $\Xi = \{z_1, z_2, z_3\}$ and

$$D_{z_1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_{z_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_{z_3} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then for each $x \in \Sigma$,

$$P_x = \frac{m-x-1}{m}D_{z_1} + \frac{1}{m}D_{z_2} + \frac{x}{m}D_{z_3}.$$

This provides the state transition table of the deterministic semiautomaton $B = (S, \Xi, \delta)$, where

δ^B	z_1	z_2	z_3
s_1	s_1	s_1	s_2
s_2	s_1	s_2	s_2

and the transitions of the dependent source $\Gamma = (\Sigma, \Xi, \gamma)$, where for each $x \in \Sigma$,

γ	z_1	z_2	z_3
x	$\frac{m-x-1}{m}$	$\frac{1}{m}$	$\frac{x}{m}$

◇

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