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MATHEMATISCHE NACHRICHTEN

Series representations in spaces of vector-valued functions via Schauder decompositions

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Abstract
It is a classical result that every $\mathbb{C}$-valued holomorphic function has a local power series representation. This even remains true for holomorphic functions with values in a locally complete locally convex Hausdorff space E over $\mathbb{C}$. Motivated by this example we try to answer the following question. Let E be a locally convex Hausdorff space over a field $\mathbb{K}$, let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a set Ω and let $\mathcal{F}(\Omega, E)$ be an E-valued counterpart of $\mathcal{F}(\Omega)$ (where the term E-valued counterpart needs clarification itself). For which spaces is it possible to lift series representations of elements of $\mathcal{F}(\Omega)$ to elements of $\mathcal{F}(\Omega, E)$? We derive sufficient conditions for the answer to be affirmative using Schauder decompositions which are applicable for many classical spaces of functions $\mathcal{F}(\Omega)$ having an equicontinuous Schauder basis.

KEYWORDS
injective tensor product, Schauder basis, Schauder decomposition, series representation, vector-valued function

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1 | INTRODUCTION

The purpose of this paper is to lift series representations known from scalar-valued functions to vector-valued functions and its underlying idea was derived from the classical example of the (local) power series representation of a holomorphic function. Let $\mathbb{D}_r \subset \mathbb{C}$ be an open disc around zero with radius $r > 0$ and let $\mathcal{O}(\mathbb{D}_r)$ be the space of holomorphic functions on $\mathbb{D}_r$, i.e. the space of functions $f : \mathbb{D}_r \to \mathbb{C}$ such that the limit

$$f^{(1)}(z) := \lim_{\substack{h\to 0\\ h\in\mathbb{C}, h\neq 0}} \frac{f(z+h)-f(z)}{h}, \quad z \in \mathbb{D}_r, \tag{1.1}$$

exists in $\mathbb{C}$. It is well-known that every $f \in \mathcal{O}(\mathbb{D}_r)$ can be written as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad z \in \mathbb{D}_r,$$

where the power series on the right-hand side converges uniformly on every compact subset of $\mathbb{D}_r$ and $f^{(n)}(0)$ is the n-th complex derivative of f at 0 which is defined from (1.1) by the recursion

$$f^{(0)} := f \quad \text{and} \quad f^{(n)} := \left(f^{(n-1)}\right)^{(1)} \quad \text{for} \quad n \in \mathbb{N}.$$

By [9, 2.1 Theorem and Definition, p. 17–18] and [9, 5.2 Theorem, p. 35], this series representation remains valid if f is a holomorphic function on $\mathbb{D}_r$ with values in a locally complete locally convex Hausdorff space E over $\mathbb{C}$ where holomorphy means that the limit (1.1) exists in E and the higher complex derivatives are defined recursively as well. Analysing this example, we observe that $\mathcal{O}(\mathbb{D}_r)$, equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}_r$, is a Fréchet space, in particular barrelled, with a Schauder basis formed by the monomials $z \mapsto z^n$. Further, the formulas for the complex derivatives of a $\mathbb{C}$-valued resp. an E-valued function f on $\mathbb{D}_r$ are built up in the same way by (1.1).

Our goal is to derive a mechanism which uses these observations and transfers known series representations for other spaces of scalar-valued functions to their vector-valued counterparts. Let us describe the general setting. We recall from [11, 14.2, p. 292] that a sequence (f_n) in a locally convex Hausdorff space F over a field $\mathbb{K}$ is called a topological basis, or simply a basis, if for every $f \in F$ there is a unique sequence of coefficients $(\lambda_n^{\mathbb{K}}(f))$ in $\mathbb{K}$ such that

$$f = \sum_{n=1}^{\infty} \lambda_n^{\mathbb{K}}(f) f_n \tag{1.2}$$

where the series converges in F. Due to the uniqueness of the coefficients the map $\lambda_n^{\mathbb{K}} : f \mapsto \lambda_n^{\mathbb{K}}(f)$ is well-defined, linear and called the n-th coefficient functional associated to (f_n). Further, for each $k \in \mathbb{N}$ the map

$$P_k : F \to F, \ P_k(f) := \sum_{n=1}^{k} \lambda_n^{\mathbb{K}}(f) f_n,$$

is a linear projection whose range is span $(f_1, \ldots, f_n)$ and it is called the k-th expansion operator associated to (f_n). A basis (f_n) of F is called equicontinuous if the expansion operators P_k form an equicontinuous sequence in the linear space $L(F, F)$ of continuous linear maps from F to F (see [11, 14.3, p. 296]). A basis (f_n) of F is called a Schauder basis if the coefficient functionals are continuous, i.e. $\lambda_n^{\mathbb{K}} \in F' := L(F, \mathbb{K})$ for each $n \in \mathbb{N}$. In particular, this is already fulfilled if F is a Fréchet space by [18, Corollary 28.11, p. 351]. If F is barrelled, then a Schauder basis of F is already equicontinuous and F has the (bounded) approximation property by the uniform boundedness principle.

The starting point for our approach is Equation (1.2). Let F and E be locally convex Hausdorff spaces over a field $\mathbb{K}$ where F has an equicontinuous Schauder basis (f_n) with associated coefficient functionals $(\lambda_n^{\mathbb{K}})$. The expansion operators (P_k) form a so-called Schauder decomposition of F (see [4, p. 77]), i.e. they are continuous projections on F such that

(i) $P_k P_j = P_{\min(j,k)}$ for all $j, k \in \mathbb{N}$,
(ii) $P_k \neq P_j$ for $k \neq j$,
(iii) $(P_k f)$ converges to f for each $f \in F$.

This operator theoretic definition of a Schauder decomposition is equivalent to the usual definition in terms of closed subspaces of F given in [14, p. 377] (see [17, p. 219]). In our main Theorem 3.1 we prove that $(P_k \varepsilon \operatorname{id}_E)$ is a Schauder decomposition of Schwartz' ε-product $F\varepsilon E := L_e(F'_\kappa, E)$ and each $u \in F\varepsilon E$ has the series representation

$$u(f') = \sum_{n=1}^{\infty} u(\lambda_n^{\mathbb{K}}) f'(f_n), \quad f' \in F'.$$

Now, suppose that $F = \mathcal{F}(\Omega)$ is a space of $\mathbb{K}$-valued functions on a set Ω with a topology such that the point-evaluation functionals δ_x, $x \in \Omega$, are continuous and that there is a locally convex Hausdorff space $\mathcal{F}(\Omega, E)$ of functions from Ω to E such that the map

$$S : \mathcal{F}(\Omega)\varepsilon E \to \mathcal{F}(\Omega, E),\ u \longmapsto [x \mapsto u(\delta_x)],$$

is a (linear topological) isomorphism. Assuming that for each $n \in \mathbb{N}$ and $u \in \mathcal{F}(\Omega)\varepsilon E$ there is $\lambda_n^E(S(u)) \in E$ with

$$\lambda_n^E(S(u)) = u(\lambda_n^{\mathbb{K}}), \tag{1.3}$$

we obtain in Corollary 3.6 that $(S \circ (P_k \varepsilon \,\mathrm{id}_E) \circ S^{-1})_k$ is a Schauder decomposition of $\mathcal{F}(\Omega, E)$ and

$$f = \lim_{k\to\infty} (S \circ (P_k \varepsilon \,\mathrm{id}_E) \circ S^{-1})_k(f) = \sum_{n=1}^{\infty} \lambda_n^E(f) f_n, \quad f \in \mathcal{F}(\Omega, E),$$

which is the desired series representation in $\mathcal{F}(\Omega, E)$. Condition (1.3) might seem strange at a first glance but for example in the case of E-valued holomorphic functions on $\mathbb{D}_r$ it guarantees that the complex derivatives at 0 appear in the Schauder decomposition of $\mathcal{O}(\mathbb{D}_r, E)$ since $S(u)^{(n)}(0) = u\big(\delta_0^{(n)}\big)$ for all $u \in \mathcal{O}(\mathbb{D}_r)\varepsilon E$ and $n \in \mathbb{N}_0$ where $\delta_0^{(n)}$ is the point-evaluation of the n-th complex derivative. We apply our result to sequence spaces, spaces of continuously differentiable functions on a compact interval, the space of holomorphic functions, the Schwartz space and the space of smooth functions which are 2π-periodic in each variable.

As a byproduct of Theorem 3.1 we obtain that every element of the completion $F\,\widehat{\otimes}_\varepsilon E$ of the injective tensor product has a series representation as well if F is a complete space with an equicontinuous Schauder basis and E is complete. Concerning series representation in $F\,\widehat{\otimes}_\varepsilon E$, little seems to be known whereas for the completion $F\,\widehat{\otimes}_\pi E$ of the projective tensor product $F \otimes_\pi E$ of two metrisable locally convex spaces F and E it is well-known that every $f \in F\,\widehat{\otimes}_\pi E$ has a series representation

$$f = \sum_{n=1}^{\infty} a_n f_n \otimes e_n$$

where $(a_n) \in \ell^1$, i.e. (a_n) is absolutely summable, and (f_n) and (e_n) are null sequences in F and E, respectively (see e.g. [10, Chap. I, §2 , n°1, Théorème 1, p. 51] or [11, 15.6.4 Corollary, p. 334]). If F and E are metrisable and one of them is nuclear, then the isomorphism $F\,\widehat{\otimes}_\pi E \cong F\,\widehat{\otimes}_\varepsilon E$ holds and we trivially have a series representation of the elements of $F\,\widehat{\otimes}_\varepsilon E$ as well. Other conditions on the existence of series representations of the elements of $F\,\widehat{\otimes}_\varepsilon E$ can be found in [19, Proposition 4.25, p. 88], where F and E are Banach spaces and both of them have a Schauder basis, and in [12, Theorem 2, p. 283], where F and E are locally convex Hausdorff spaces and both of them have an equicontinuous Schauder basis.

2 | NOTATION AND PRELIMINARIES

We equip the spaces $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\mathbb{C}$ with the usual Euclidean norm $|\cdot|$. For a subset M of a topological vector space X, we write $\overline{\mathrm{acx}}(M)$ for the closure of the absolutely convex hull $\mathrm{acx}(M)$ of M in X.

By E we always denote a non-trivial locally convex Hausdorff space, in short lcHs, over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$. If $E = \mathbb{K}$, then we set $(p_\alpha)_{\alpha\in\mathfrak{A}} := \{|\cdot|\}$. We recall that for a disk $D \subset E$, i.e. a bounded, absolutely convex set, the vector space $E_D := \bigcup_{n\in\mathbb{N}} nD$ becomes a normed space if it is equipped with the gauge functional of D as a norm (see [11, p. 151]). The space E is called locally complete if E_D is a Banach space for every closed disk $D \subset E$ (see [11, 10.2.1 Proposition, p. 197]). For more details on the theory of locally convex spaces see [8, 11] or [18].

By X^Ω we denote the set of maps from a non-empty set Ω to a non-empty set X, by χ_K the characteristic function of a subset $K \subset \Omega$ and by $L(F, E)$ the space of continuous linear operators from F to E where F and E are locally convex Hausdorff spaces. If $E = \mathbb{K}$, we just write $F' := L(F, \mathbb{K})$ for the dual space. If F and E are (linearly topologically) isomorphic, we write $F \cong E$ and, if F is only isomorphic to a subspace of E, we write $F \hookrightarrow E$. We denote by $L_t(F, E)$ the space $L(F, E)$

equipped with the locally convex topology of uniform convergence on the finite subsets of F if $t = \sigma$, on the absolutely convex compact subsets of F if $t = \kappa$ and on the precompact (totally bounded) subsets of F if $t = \gamma$.

The so-called ε-product of Schwartz is defined by

$$F\varepsilon E := L_e\left(F'_\kappa, E\right) \tag{2.1}$$

where $L\left(F'_\kappa, E\right)$ is equipped with the topology of uniform convergence on equicontinuous subsets of F'. This definition of the ε-product coincides with the original one by Schwartz [23, Chap. I, §1, Définition, p. 18]. It is symmetric which means that $F\varepsilon E \cong E\varepsilon F$. In the literature the definition of the ε-product is sometimes done the other way around, i.e. $E\varepsilon F$ is defined by the right-hand side of (2.1) but due to the symmetry these definitions are equivalent and for our purpose the given definition is more suitable. If we replace F'_κ by F'_γ, we obtain Grothendieck's definition of the ε-product and we remark that the two ε-products coincide if F is quasi-complete because then $F'_\gamma = F'_\kappa$. Jarchow uses a third, different definition of the ε-product (see [11, 16.1, p. 344]) which coincides with the one of Schwartz if F is complete by [11, 9.3.7 Proposition, p. 179]. However, we stick to Schwartz' definition.

For locally convex Hausdorff spaces F_i, E_i and $T_i \in L\left(F_i, E_i\right)$, $i = 1, 2$, we define the ε-product $T_1\varepsilon T_2 \in L\left(F_1\varepsilon F_2, E_1\varepsilon E_2\right)$ of the operators T_1 and T_2 by

$$\left(T_1\varepsilon T_2\right)(u) := T_2 \circ u \circ T_1^t, \quad u \in F_1\varepsilon F_2,$$

where $T_1^t : E_1' \to F_1', e' \mapsto e'\circ T_1$, is the dual map of T_1. If T_1 is an isomorphism and $F_2 = E_2$, then $T_1\varepsilon\, \mathrm{id}_{E_2}$ is also an isomorphism with inverse $T_1^{-1}\varepsilon\, \mathrm{id}_{E_2}$ by [23, Chap. I, §1, Proposition 1, p. 20] (or [11, 16.2.1 Proposition, p. 347] if the F_i are complete).

As usual we consider the tensor product $F \otimes E$ as an algebraic subspace of $F\varepsilon E$ for two locally convex Hausdorff spaces F and E by means of the linear injection

$$\Theta : F \otimes E \to F\varepsilon E,\ \sum_{n=1}^{k} f_n \otimes e_n \longmapsto \left[y \mapsto \sum_{n=1}^{k} y(f_n)e_n\right].$$

Via Θ the space $F \otimes E$ is identified with the space of operators with finite rank in $F\varepsilon E$ and a locally convex topology is induced on $F \otimes E$. We write $F \otimes_\varepsilon E$ for $F \otimes E$ equipped with this topology and $F \widehat{\otimes}_\varepsilon E$ for the completion of the injective tensor product $F \otimes_\varepsilon E$. By $\mathfrak{F}(E)$ we denote the space of linear operators from E to E with finite rank. A locally convex Hausdorff space E is said to have (Schwartz') approximation property (AP) if the identity id_E on E is contained in the closure of $\mathfrak{F}(E)$ in $L_\kappa(E,E)$. The space E has AP if and only if $E \otimes F$ is dense in $E\varepsilon F$ for every locally convex Hausdorff space (every Banach space) F by [13, Satz 10.17, p. 250]. The space E has Grothendieck's approximation property if id_E is contained in the closure of $\mathfrak{F}(E)$ in $L_\gamma(E,E)$. If E is quasi-complete, both approximation properties coincide. For more information on the theory of ε-products and tensor products see [7, 11] and [13].

A function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ to an lcHs E is called continuously partially differentiable $\left(f \text{ is } \mathcal{C}^1\right)$ if for the n-th unit vector $e_n \in \mathbb{R}^d$ the limit

$$\left(\partial^{e_n}\right)^E f(x) := \lim_{\substack{h\to 0\\ h\in\mathbb{R}, h\neq 0}} \frac{f\left(x + he_n\right) - f(x)}{h}$$

exists in E for every $x \in \Omega$ and $\left(\partial^{e_n}\right)^E f$ is continuous on Ω $\left(\left(\partial^{e_n}\right)^E f \text{ is } \mathcal{C}^0\right)$ for every $1 \le n \le d$. For $k \in \mathbb{N}$ a function f is said to be k-times continuously partially differentiable $\left(f \text{ is } \mathcal{C}^k\right)$ if f is $\mathcal{C}^1$ and all its first partial derivatives are $\mathcal{C}^{k-1}$. A function f is called infinitely continuously partially differentiable (f is $\mathcal{C}^\infty$) if f is $\mathcal{C}^k$ for every $k \in \mathbb{N}$. For $k \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ the functions $f : \Omega \to E$ which are $\mathcal{C}^k$ form a linear space which is denoted by $\mathcal{C}^k(\Omega, E)$. For $\beta \in \mathbb{N}_0^d$ with $|\beta| := \sum_{n=1}^{d} \beta_n \le k$ and a function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ to an lcHs E we set $\left(\partial^{\beta_n}\right)^E f := f$ if $\beta_n = 0$, and

$$\left(\partial^{\beta_n}\right)^E f(x) := \underbrace{\left(\partial^{e_n}\right)^E \cdots \left(\partial^{e_n}\right)^E}_{\beta_n\text{-times}} f(x)$$

if $\beta_n \neq 0$ and the right-hand side exists in E for every $x \in \Omega$. Further, we define

$$\left(\partial^{\beta}\right)^{E} f(x) =: \left(\left(\partial^{\beta_1}\right)^{E} \cdots \left(\partial^{\beta_d}\right)^{E}\right) f(x)$$

if the right-hand side exists in E for every $x \in \Omega$ and set $f^{(\beta)} := \left(\partial^{\beta}\right)^{E} f$ if $d = 1$.

3 | SCHAUDER DECOMPOSITION

Let us start with our main theorem on Schauder decompositions of ε-products.

Theorem 3.1. *Let F and E be lcHs, let $(f_n)_{n\in\mathbb{N}}$ be an equicontinuous Schauder basis of F with associated coefficient functionals $(\lambda_n)_{n\in\mathbb{N}}$ and set $Q_n : F \to F$, $Q_n(f) := \lambda_n(f) f_n$ for every $n \in \mathbb{N}$. Then the following holds.*

a) The sequence $(P_k)_{k\in\mathbb{N}}$ given by $P_k := \left(\sum_{n=1}^{k} Q_n\right) \varepsilon\, \mathrm{id}_E$ is a Schauder decomposition of $F\varepsilon E$.
b) Each $u \in F\varepsilon E$ has the series representation

$$u(f') = \sum_{n=1}^{\infty} u(\lambda_n) f'(f_n), \quad f' \in F'.$$

c) $F \otimes E$ is sequentially dense in $F\varepsilon E$.

Proof. Since (f_n) is a Schauder basis of F, the sequence $\left(\sum_{n=1}^{k} Q_n\right)$ converges to id_F in $L_\sigma(F,F)$. Thus we deduce from the equicontinuity of (f_n) that $\left(\sum_{n=1}^{k} Q_n\right)$ converges to id_F in $L_\kappa(F,F)$ by [11, 8.5.1 Theorem (b), p. 156]. For $f' \in F'$ and $f \in F$ holds

$$\begin{aligned}(Q_n^t \circ Q_m^t)(f')(f) &= Q_m^t(f')(Q_n(f)) = Q_m^t(f')(\lambda_n(f) f_n) = f'\left(\lambda_m(\lambda_n(f) f_n) f_m\right) \\ &= \lambda_m(f_n)\lambda_n(f) f'(f_m) = \begin{cases} \lambda_n(f) f'(f_n), & m = n, \\ 0, & m \neq n, \end{cases}\end{aligned}$$

due to the uniqueness of the coefficient functionals (λ_n) (see [11, 14.2.1 Proposition, p. 292]) and it follows for $k, j \in \mathbb{N}$ that

$$\left(\sum_{n=1}^{j} Q_n^t \circ \sum_{m=1}^{k} Q_m^t\right)(f')(f) = \sum_{n=1}^{\min(j,k)} \lambda_n(f) f'(f_n) = \sum_{n=1}^{\min(j,k)} Q_n^t(f')(f).$$

This implies that

$$(P_k P_j)(u) = u \circ \sum_{n=1}^{j} Q_n^t \circ \sum_{m=1}^{k} Q_m^t = u \circ \sum_{n=1}^{\min(j,k)} Q_n^t = P_{\min(j,k)}(u)$$

for all $u \in F\varepsilon E$. If $k \neq j$, w.l.o.g. $k > j$, we choose $x \in E$, $x \neq 0$, and consider $f_k \otimes x$ as an element of $F\varepsilon E$ via the map Θ. Then

$$(P_k - P_j)(f_k \otimes x) = \sum_{n=j+1}^{k} (f_k \otimes x) \circ Q_n^t = f_k \otimes x \neq 0$$

since

$$\big((f_k \otimes x)\circ Q_n^t\big)(f') = (f_k \otimes x)\big(\lambda_n(\cdot)f'(f_n)\big) = \lambda_n(f_k)f'(f_n)x = \begin{cases} (f_k \otimes x)(f'), & n = k, \\ 0, & n \neq k. \end{cases}$$

It remains to prove that for each $u \in F\varepsilon E$

$$\lim_{k\to\infty} P_k(u) = u$$

in $F\varepsilon E$. Let $(q_\beta)_{\beta\in\mathfrak{B}}$ denote the system of seminorms inducing the locally convex topology of F. Let $u \in F\varepsilon E$ and $\alpha \in \mathfrak{A}$. Due to the continuity of u there are an absolutely convex compact set $K = K(u,\alpha) \subset F$ and $C_0 = C_0(u,\alpha) > 0$ such that for each $f' \in F'$ we have

$$\begin{aligned} p_\alpha\big((P_k(u) - u)(f')\big) &= p_\alpha\left(u\left(\left(\sum_{n=1}^{k} Q_n^t - \mathrm{id}_{F'}\right)(f')\right)\right) \\ &\le C_0 \sup_{f\in K}\left|\left(\sum_{n=1}^{k} Q_n^t - \mathrm{id}_{F'}\right)(f')(f)\right| \\ &= C_0 \sup_{f\in K}\left|f'\left(\sum_{n=1}^{k} Q_n f - f\right)\right|. \end{aligned}$$

Let V be an absolutely convex 0-neighbourhood in F. As a consequence of the equicontinuity of the polar V° there are $C_1 > 0$ and $\beta \in \mathfrak{B}$ such that

$$\sup_{f'\in V^\circ} p_\alpha\big((P_k(u) - u)(f')\big) \le C_0C_1 \sup_{f\in K} q_\beta\left(\sum_{n=1}^{k} Q_n f - f\right).$$

In combination with the convergence of $\left(\sum_{n=1}^{k} Q_n\right)$ to id_F in $L_\kappa(F,F)$ this yields the convergence of $(P_k(u))$ to u in $F\varepsilon E$ and settles part a).

Let us turn to b) and c). Since

$$P_k(u)(f') = u\left(\sum_{n=1}^{k} Q_n^t(f')\right) = \sum_{n=1}^{k} u(\lambda_n)f'(f_n)$$

for every $f' \in F'$, we note that the range of $P_k(u)$ is contained in $\mathrm{span}\big(u(\lambda_n) \mid 1 \le n \le k\big)$ for each $u \in F\varepsilon E$ and $k \in \mathbb{N}$. Hence $P_k(u)$ has finite rank and thus belongs to $F \otimes E$ implying the sequential density of $F \otimes E$ in $F\varepsilon E$ and the desired series representation by part a). □

Remark 3.2. If F and E are complete, we have under the assumption of Theorem 3.1 that $F\widehat{\otimes}_\varepsilon E \cong F\varepsilon E$ by c) since $F\varepsilon E$ is complete by [13, Satz 10.3, p. 234] and $F\widehat{\otimes}_\varepsilon E$ is the closure of $F \otimes E$ in $F\varepsilon E$. Thus each element of $F\widehat{\otimes}_\varepsilon E$ has a series representation.

Let us apply the preceding theorem to spaces of Lebesgue integrable functions. We consider the measure space $([0,1], \mathscr{L}([0,1]), \lambda)$ of Lebesgue measurable sets and use the notation $\mathcal{L}^p[0,1]$ for the space of (equivalence classes) of Lebesgue p-integrable functions on [0,1]. The Haar system

$$h_n : [0,1] \to \mathbb{R}, \quad n \in \mathbb{N},$$

given by

$$h_1(x) := 1 \text{ for all } x \in [0,1] \quad \text{and} \quad h_{2^k+j}(x) := \begin{cases} 1, & (2j-2)/2^{k+1} \le x < (2j-1)/2^{k+1}, \\ -1, & (2j-1)/2^{k+1} \le x < 2j/2^{k+1}, \\ 0, & \text{else}, \end{cases}$$

for $k \in \mathbb{N}_0$ and $1 \le j \le 2^k$ forms a Schauder basis of $\mathcal{L}^p[0,1]$ for every $1 \le p < \infty$ and the associated coefficient functionals are

$$\lambda_n(f) := \int_{[0,1]} f(x)h_n(x)\,\mathrm{d}\lambda(x), \quad f \in \mathcal{L}^p[0,1],\ n \in \mathbb{N},$$

(see [20, Satz I, p. 317]). Because $\mathcal{L}^p[0,1]$ is Banach space and thus barrelled, its Schauder basis (h_n) is equicontinuous and we directly obtain from Theorem 3.1 the following corollary.

Corollary 3.3. *Let E be an lcHs and let $1 \le p < \infty$. Then $\left(\sum_{n=1}^{k} \lambda_n(\cdot)h_n \varepsilon \operatorname{id}_E\right)_{k\in\mathbb{N}}$ is a Schauder decomposition of $\mathcal{L}^p[0,1]\varepsilon E$ and for each $u \in \mathcal{L}^p[0,1]\varepsilon E$ there holds*

$$u(f') = \sum_{n=1}^{\infty} u(\lambda_n) f'(f_n), \quad f' \in \mathcal{L}^p[0,1]'.$$

Defining $\mathcal{L}^p([0,1],E) := \mathcal{L}^p[0,1]\varepsilon E$, we can read the corollary above as a statement on series representations in the vector-valued version of $\mathcal{L}^p[0,1]$. However, in many cases of spaces $\mathcal{F}(\Omega)$ of scalar-valued functions there is a more natural way to define the vector-valued version $\mathcal{F}(\Omega,E)$ of $\mathcal{F}(\Omega)$, see for example the space of holomorphic functions from the introduction. If $\mathcal{F}(\Omega)\varepsilon E$ and $\mathcal{F}(\Omega,E)$ are isomorphic via the map S from the introduction, we can translate Theorem 3.1 to the more natural setting $\mathcal{F}(\Omega,E)$ which motivates the following definition.

Definition 3.4 (ε-compatible). Let Ω be a set and let E be an lcHs. If $\mathcal{F}(\Omega) \subset \mathbb{K}^\Omega$ and $\mathcal{F}(\Omega,E) \subset E^\Omega$ are lcHs such that $\delta_x \in \mathcal{F}(\Omega)'$ for all $x \in \Omega$ and

$$S : \mathcal{F}(\Omega)\varepsilon E \to \mathcal{F}(\Omega,E),\ u \longmapsto [x \mapsto u(\delta_x)],$$

is an isomorphism, then we say that $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ are *ε-compatible*.

If we want to emphasise the dependence of S on $\mathcal{F}(\Omega)$, we write $S_{\mathcal{F}(\Omega)}$. Several sufficient conditions for S being an isomorphism are given in [16, Theorem 14, p. 1524].

Remark 3.5. Let Ω be a set and let E be an lcHs. If $\mathcal{F}(\Omega) \subset \mathbb{K}^\Omega$ and $\mathcal{F}(\Omega,E) \subset E^\Omega$ are lcHs such that $\delta_x \in \mathcal{F}(\Omega)'$ for all $x \in \Omega$ and

$$S : \mathcal{F}(\Omega)\varepsilon E \to \mathcal{F}(\Omega,E),\ u \longmapsto [x \mapsto u(\delta_x)],$$

is an isomorphism into, then we get by identification of isomorphic subspaces

$$\mathcal{F}(\Omega) \otimes_\varepsilon E \subset \mathcal{F}(\Omega)\varepsilon E \subset \mathcal{F}(\Omega,E)$$

and the embedding $\mathcal{F}(\Omega) \otimes E \hookrightarrow \mathcal{F}(\Omega,E)$ is given by $f \otimes e \longmapsto [x \mapsto f(x)e]$.

Proof. The inclusions obviously hold and $\mathcal{F}(\Omega)\varepsilon E$ and $\mathcal{F}(\Omega,E)$ induce the same topology on $\mathcal{F}(\Omega) \otimes E$. Further, we have

$$f \otimes e \overset{\Theta}{\longmapsto} [y \mapsto y(f)e] \overset{S}{\longmapsto} [x \longmapsto [y \mapsto y(f)e](\delta_x)] = [x \mapsto f(x)e].$$

□

Corollary 3.6. *Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-compatible spaces, let $(f_n)_{n\in\mathbb{N}}$ be an equicontinuous Schauder basis of $\mathcal{F}(\Omega)$ with associated coefficient functionals $(\lambda_n^{\mathbb{K}})_{n\in\mathbb{N}}$ and let $\lambda_n^E : \mathcal{F}(\Omega,E)\to E$ such that*

$$\lambda_n^E(S(u)) = u(\lambda_n^{\mathbb{K}}), \quad u\in\mathcal{F}(\Omega)\varepsilon E, \tag{3.1}$$

for all $n\in\mathbb{N}$. Set $Q_n^E : \mathcal{F}(\Omega,E)\to\mathcal{F}(\Omega,E)$, $Q_n^E(f) := \lambda_n^E(f)f_n$ for every $n\in\mathbb{N}$. Then the following holds.

a) The sequence $(P_k^E)_{k\in\mathbb{N}}$ given by $P_k^E := \sum_{n=1}^k Q_n^E$ is a Schauder decomposition of $\mathcal{F}(\Omega,E)$.
b) Each $f\in\mathcal{F}(\Omega,E)$ has the series representation

$$f = \sum_{n=1}^{\infty}\lambda_n^E(f)f_n.$$

c) $\mathcal{F}(\Omega)\otimes E$ is sequentially dense in $\mathcal{F}(\Omega,E)$.

Proof. For each $u\in\mathcal{F}(\Omega)\varepsilon E$ and $x\in\Omega$ we note that with P_k from Theorem 3.1 there holds

$$\begin{aligned}(S\circ P_k)(u)(x) &= u\left(\sum_{n=1}^k Q_n^t(\delta_x)\right) = u\left(\sum_{n=1}^k \lambda_n^{\mathbb{K}}(\cdot)f_n(x)\right)\\ &= \sum_{n=1}^k u(\lambda_n^{\mathbb{K}})f_n(x) = \sum_{n=1}^k \lambda_n^E(S(u))f_n(x) = (P_k^E\circ S)(u)(x)\end{aligned}$$

which means that $S\circ P_k = P_k^E\circ S$. This implies part a) and b) by Theorem 3.1 a) since S is an isomorphism. Part c) is a direct consequence of Theorem 3.1 c) and the isomorphism $\mathcal{F}(\Omega)\varepsilon E\cong\mathcal{F}(\Omega,E)$. □

In the preceding corollary we used the isomorphism S to obtain a Schauder decomposition. On the other hand, if S is an isomorphism into which is often the case (see [15, 3.9 Theorem, p. 9]), we can use a Schauder decomposition of $\mathcal{F}(\Omega,E)$ to prove the surjectivity of S.

Proposition 3.7. *Let Ω be a set and let E be an lcHs. Let $\mathcal{F}(\Omega)\subset\mathbb{K}^\Omega$ and $\mathcal{F}(\Omega,E)\subset E^\Omega$ be lcHs such that $\delta_x\in\mathcal{F}(\Omega)'$ for all $x\in\Omega$ and*

$$S : \mathcal{F}(\Omega)\varepsilon E\to\mathcal{F}(\Omega,E),\ u\longmapsto [x\mapsto u(\delta_x)],$$

is an isomorphism into. Let there be $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}(\Omega)$ and for every $f\in\mathcal{F}(\Omega,E)$ a sequence $(\lambda_n^E(f))_{n\in\mathbb{N}}$ in E such that

$$f = \sum_{n=1}^{\infty}\lambda_n^E(f)f_n, \quad f\in\mathcal{F}(\Omega,E).$$

Then the following holds.

a) $\mathcal{F}(\Omega)\otimes E$ is sequentially dense in $\mathcal{F}(\Omega,E)$.
b) If $\mathcal{F}(\Omega)$ and E are sequentially complete, then

$$\mathcal{F}(\Omega,E)\cong\mathcal{F}(\Omega)\varepsilon E.$$

c) If $\mathcal{F}(\Omega)$ and E are complete, then

$$\mathcal{F}(\Omega,E)\cong\mathcal{F}(\Omega)\varepsilon E\cong\mathcal{F}(\Omega)\widehat{\otimes}_\varepsilon E.$$

Proof. Let $f \in \mathcal{F}(\Omega, E)$ and observe that

$$P_k^E(f) := \sum_{n=1}^{k} \lambda_n^E(f) f_n = \sum_{n=1}^{k} f_n \otimes \lambda_n^E(f) \in \mathcal{F}(\Omega) \otimes E$$

for every $k \in \mathbb{N}$ by Remark 3.5. Due to our assumption we have the convergence $P_k^E(f) \to f$ in $\mathcal{F}(\Omega, E)$. Thus $\mathcal{F}(\Omega) \otimes E$ is sequentially dense in $\mathcal{F}(\Omega, E)$.

Let us turn to part b). If $\mathcal{F}(\Omega)$ and E are sequentially complete, then $\mathcal{F}(\Omega)\varepsilon E$ is sequentially complete by [13, Satz 10.3, p. 234]. Since S is an isomorphism into and

$$S\left(\Theta\left(\sum_{n=q}^{k} f_n \otimes \lambda_n^E(f)\right)\right) = \sum_{n=q}^{k} \lambda_n^E(f) f_n$$

for all $k, q \in \mathbb{N}$, $k > q$, we get that $\left(\Theta\left(\sum_{n=1}^{k} f_n \otimes \lambda_n^E(f)\right)\right)$ is a Cauchy sequence in $\mathcal{F}(\Omega)\varepsilon E$ and thus convergent. Hence we deduce that

$$S\left(\lim_{k\to\infty} \Theta\left(\sum_{n=1}^{k} f_n \otimes \lambda_n^E(f)\right)\right) = \lim_{k\to\infty} \sum_{n=1}^{k} (S \circ \Theta)\left(f_n \otimes \lambda_n^E(f)\right) = \sum_{n=1}^{\infty} \lambda_n^E(f) f_n = f$$

which proves the surjectivity of S.

If $\mathcal{F}(\Omega)$ and E are complete, then $\mathcal{F}(\Omega)\widehat{\otimes}_\varepsilon E$ is the closure of $\mathcal{F}(\Omega) \otimes_\varepsilon E$ in the complete space $\mathcal{F}(\Omega)\varepsilon E$ by [13, Satz 10.3, p. 234]. As $\lim_{k\to\infty} \Theta\left(\sum_{n=1}^{k} f_n \otimes \lambda_n^E(f)\right)$ is an element of the closure, we obtain part c). □

4 | APPLICATIONS

4.1 | Sequence spaces

For our first application we recall the definition of some sequence spaces. A matrix $A := \left(a_{k,j}\right)_{k,j\in\mathbb{N}}$ of nonnegative numbers is called Köthe matrix if it fulfils:

(1) $\forall\, k \in \mathbb{N}\ \exists\, j \in \mathbb{N}:\ a_{k,j} > 0$,
(2) $\forall\, k, j \in \mathbb{N}:\ a_{k,j} \leq a_{k,j+1}$.

For an lcHs E we define the Köthe space

$$\lambda^\infty(A, E) := \left\{ x = \left(x_k\right) \in E^{\mathbb{N}} \mid \forall\, j \in \mathbb{N},\ \alpha \in \mathfrak{A}:\ |x|_{j,\alpha} := \sup_{k\in\mathbb{N}} p_\alpha\left(x_k\right) a_{k,j} < \infty \right\}$$

and the topological subspace

$$c_0(A, E) := \left\{ x = \left(x_k\right) \in E^{\mathbb{N}} \mid \forall\, j \in \mathbb{N}:\ \lim_{k\to\infty} x_k a_{k,j} = 0 \right\}.$$

In particular, the space $c_0(\mathbb{N}, E)$ of null-sequences in E is obtained as $c_0(\mathbb{N}, E) = c_0(A, E)$ with $a_{k,j} := 1$ for all $k, j \in \mathbb{N}$. The space of convergent sequences in E is defined by

$$c(\mathbb{N}, E) := \left\{ x \in E^{\mathbb{N}} \mid x = \left(x_k\right) \text{ converges in } E \right\}$$

and equipped with the system of seminorms

$$|x|_{\alpha} := \sup_{k\in\mathbb{N}} p_{\alpha}\big(x_k\big), \quad x \in c(\mathbb{N}, E),$$

for $\alpha \in \mathfrak{A}$. We define the spaces of E-valued rapidely decreasing sequences which we need for our subsection on Fourier expansion by

$$s(\Omega, E) := \left\{ x = \big(x_k\big) \in E^{\Omega} \mid \forall\, j \in \mathbb{N},\, \alpha \in \mathfrak{A} : \ |x|_{j,\alpha} := \sup_{k\in\Omega} p_{\alpha}\big(x_k\big)\big(1+|k|^2\big)^{j/2} < \infty \right\}$$

with $\Omega = \mathbb{N}^d, \mathbb{N}_0^d, \mathbb{Z}^d$. Furthermore, we equip the space $E^{\mathbb{N}}$ with the system of seminorms given by

$$\|x\|_{l,\alpha} := \sup_{k\in\mathbb{N}} p_{\alpha}\big(x_k\big)\chi_{\{1,\dots,l\}}(k), \quad x = \big(x_k\big) \in E^{\mathbb{N}},$$

for $l \in \mathbb{N}$ and $\alpha \in \mathfrak{A}$. For a non-empty set Ω we define for $n \in \Omega$ the n-th unit function by

$$\varphi_{n,\Omega} : \Omega \to \mathbb{K}, \quad \varphi_{n,\Omega}(k) := \begin{cases} 1, & k = n, \\ 0, & \text{else}, \end{cases}$$

and we simply write φ_n instead of $\varphi_{n,\Omega}$ if no confusion seems to be likely. Further, we set

$$\varphi_{\infty} : \mathbb{N} \to \mathbb{K}, \quad \varphi_{\infty}(k) := 1, \quad \text{and} \quad x_{\infty} := \delta_{\infty}(x) := \lim_{k\to\infty} x_k \quad \text{for} \quad x \in c(\mathbb{N}, E).$$

For series representations of the elements in these sequence spaces we do not need Corollary 3.6 due to the subsequent proposition but we can use the representation to obtain the surjectivity of S for sequentially complete E.

Proposition 4.1. *Let E be an lcHs and let $\ell\mathcal{V}(\Omega, E)$ be one of the spaces $c_0(A, E)$, $E^{\mathbb{N}}$, $s(\mathbb{N}^d, E)$, $s(\mathbb{N}_0^d, E)$ or $s(\mathbb{Z}^d, E)$.*

a) Then $\big(\sum_{n\in\Omega, |n|\le k} \delta_n\varphi_n\big)_{k\in\mathbb{N}}$ is a Schauder decomposition of $\ell\mathcal{V}(\Omega, E)$ and

$$x = \sum_{n\in\Omega} x_n\varphi_n, \quad x \in \ell\mathcal{V}(\Omega, E).$$

b) Then $\big(\delta_{\infty}\varphi_{\infty} + \sum_{n=1}^{k}(\delta_n - \delta_{\infty})\varphi_n\big)_{k\in\mathbb{N}}$ is a Schauder decomposition of $c(\mathbb{N}, E)$ and

$$x = x_{\infty}\varphi_{\infty} + \sum_{n=1}^{\infty} \big(x_n - x_{\infty}\big)\varphi_n, \quad x \in c(\mathbb{N}, E).$$

Proof. Let us begin with a). For $x = \big(x_n\big) \in \ell\mathcal{V}(\Omega, E)$ let $\big(P_k^E\big)$ be the sequence in $\ell\mathcal{V}(\Omega, E)$ given by

$$P_k^E(x) := \sum_{|n|\le k} x_n\varphi_n.$$

It is easy to see that P_k^E is a continuous projection on $\ell\mathcal{V}(\Omega, E)$, $P_k^E P_j^E = P_{\min(k,j)}^E$ for all $k, j \in \mathbb{N}$ and $P_k^E \neq P_j^E$ for $k \neq j$. Let $\varepsilon > 0$, $\alpha \in \mathfrak{A}$ and $j \in \mathbb{N}$. For $x \in c_0(A, E)$ there is $N_0 \in \mathbb{N}$ such that $p_{\alpha}\big(x_n a_{n,j}\big) < \varepsilon$ for all $n \ge N_0$. Hence we have for $x \in c_0(A, E)$

$$\big|x - P_k^E(x)\big|_{j,\alpha} = \sup_{n>k} p_{\alpha}\big(x_n\big)a_{n,j} \le \sup_{n\ge N_0} p_{\alpha}\big(x_n\big)a_{n,j} \le \varepsilon$$

for all $k \geq N_0$. For $x \in E^{\mathbb{N}}$ and $l \in \mathbb{N}$ we have

$$\left\|x - P_k^E(x)\right\|_{l,\alpha} = 0 < \varepsilon$$

for all $k \geq l$. For $x \in s(\Omega, E)$, $\Omega = \mathbb{N}^d, \mathbb{N}_0^d, \mathbb{Z}^d$, we notice that there is $N_1 \in \mathbb{N}$ such that for all $n \in \Omega$ with $|n| \geq N_1$ we have

$$\frac{\left(1+|n|^2\right)^{j/2}}{\left(1+|n|^2\right)^{j}} = \left(1+|n|^2\right)^{-j/2} < \varepsilon.$$

Thus we deduce for $|n| \geq N_1$

$$p_\alpha\left(x_n\right)\left(1+|n|^2\right)^{j/2} < \varepsilon p_\alpha\left(x_n\right)\left(1+|n|^2\right)^{j} \leq \varepsilon |x|_{2j,\alpha}$$

and hence

$$\left|x - P_k^E(x)\right|_{j,\alpha} = \sup_{|n|>k} p_\alpha\left(x_n\right) a_{n,j} \leq \sup_{|n|\geq N_1} p_\alpha\left(x_n\right)\left(1+|n|^2\right)^{j/2} \leq \varepsilon |x|_{2j,\alpha}$$

for all $k \geq N_1$. Therefore $\left(P_k^E(x)\right)$ converges to x in $\ell\mathcal{V}(\Omega, E)$ and

$$x = \lim_{k\to\infty} P_k^E(x) = \sum_{n\in\Omega} x_n \varphi_n.$$

Now, we turn to b). For $x = \left(x_n\right) \in c(\mathbb{N}, E)$ let $\left(\widetilde{P}_k^E(x)\right)$ be the sequence in $c(\mathbb{N}, E)$ given by

$$\widetilde{P}_k^E(x) := x_\infty \varphi_\infty + \sum_{n=1}^{k} (x_n - x_\infty)\varphi_n.$$

Again, it is easy to see that $\widetilde{P}_k^E$ is a continuous projection on $c(\mathbb{N}, E)$, $\widetilde{P}_k^E \widetilde{P}_j^E = \widetilde{P}_{\min(k,j)}^E$ for all $k, j \in \mathbb{N}$ and $\widetilde{P}_k^E \neq \widetilde{P}_j^E$ for $k \neq j$. Let $\varepsilon > 0$ and $\alpha \in \mathfrak{A}$. Then there is $N_2 \in \mathbb{N}$ such that $p_\alpha\left(x_n - x_\infty\right) < \varepsilon$ for all $n \geq N_2$. Thus we obtain

$$\left|x - \widetilde{P}_k^E(x)\right|_\alpha = \sup_{n>k} p_\alpha\left(x_n - x_\infty\right) \leq \sup_{n\geq N_2} p_\alpha\left(x_n - x_\infty\right) \leq \varepsilon$$

for all $k \geq N_2$ implying that $\left(\widetilde{P}_k^E(x)\right)$ converges to x in $c(\mathbb{N}, E)$ and

$$x = \lim_{k\to\infty} \widetilde{P}_k^E(x) = x_\infty \varphi_\infty + \sum_{n=1}^{\infty} \left(x_n - x_\infty\right)\varphi_n.$$

□

Theorem 4.2. *Let E be a sequentially complete lcHs and $\ell\mathcal{V}(\Omega, E)$ one of the spaces $c_0(A, E)$, $E^{\mathbb{N}}$, $s\left(\mathbb{N}^d, E\right)$, $s\left(\mathbb{N}_0^d, E\right)$ or $s\left(\mathbb{Z}^d, E\right)$. Then*

(i) $\ell\mathcal{V}(\Omega, E) \cong \ell\mathcal{V}(\Omega)\varepsilon E$,
(ii) $c(\mathbb{N}, E) \cong c(\mathbb{N})\varepsilon E$.

Proof. The map $S_{\ell\mathcal{V}(\Omega)}$ is an isomorphism into by [15, 3.9 Theorem, p. 9] and [15, 4.13 Proposition, p. 23]. Considering $c(\mathbb{N}, E)$, we observe that for $x \in c(\mathbb{N})$

$$\delta_n(x) = x_n \to x_\infty = \delta_\infty(x)$$

which implies the concergence $\delta_n \to \delta_\infty$ in $c(\mathbb{N})'_\gamma$ by the Banach–Steinhaus theorem since $c(\mathbb{N})$ is a Banach space. Hence we get

$$u(\delta_\infty) = \lim_{n\to\infty} u(\delta_n) = \lim_{n\to\infty} S(u)(n) = \delta_\infty(S(u))$$

for every $u \in c(\mathbb{N})\varepsilon E$ which implies that $S_{c(\mathbb{N})}$ is an isomorphism into by [15, 3.9 Theorem, p. 9]. From Proposition 4.1 and Proposition 3.7 we deduce our statement. □

4.2 | Continuous and differentiable functions on a closed interval

We start with continuous functions on compact sets. We recall the following definition from [25, p. 259]. A locally convex Hausdorff space is said to have the metric convex compactness property (metric ccp) if the closure of the absolutely convex hull of every metrisable compact set is compact. In particular, every sequentially complete space has metric ccp and this implication is strict (see [16, pp. 1512–1513] and the references therein). Let E be an lcHs, let $\Omega \subset \mathbb{R}^d$ compact and denote by $\mathcal{C}(\Omega, E) := \mathcal{C}^0(\Omega, E)$ the space of continuous functions from Ω to E. We equip $\mathcal{C}(\Omega, E)$ with the system of seminorms given by

$$|f|_{0,\alpha} := \sup_{x\in\Omega} p_\alpha(f(x)), \quad f \in \mathcal{C}(\Omega, E),$$

for $\alpha \in \mathfrak{A}$. We want to apply our preceding results to intervals. Let $-\infty < a < b < \infty$ and $T := (t_j)_{0\leq j\leq n}$ be a partition of the interval $[a, b]$, i.e. $a = t_0 < t_1 < \cdots < t_n = b$. The hat functions $h^T_{t_j} : [a, b] \to \mathbb{R}$ for the partition T are given by

$$h^T_{t_j}(x) := \begin{cases} \dfrac{x - t_j}{t_j - t_{j-1}}, & t_{j-1} \leq x \leq t_j, \\ \dfrac{t_{j+1} - x}{t_{j+1} - t_j}, & t_j < x \leq t_{j+1}, \\ 0, & \text{else}, \end{cases}$$

for $2 \leq j \leq n - 1$ and

$$h^T_a(x) := \begin{cases} \dfrac{t_1 - x}{t_1 - a}, & a \leq x \leq t_1, \\ 0, & \text{else}, \end{cases} \qquad h^T_b(x) := \begin{cases} \dfrac{x - t_{n-1}}{b - t_{n-1}}, & t_{n-1} \leq x \leq b, \\ 0, & \text{else}. \end{cases}$$

Let $\mathcal{T} := (t_n)_{n\in\mathbb{N}_0}$ be a dense sequence in $[a, b]$ with $t_0 = a$, $t_1 = b$ and $t_n \neq t_m$ for $n \neq m$. For $T^n := \{t_0, \dots, t_n\}$ there is a (unique) enumeration $\sigma : \{0, \dots, n\} \to T^n$ such that $T_n := (t_{\sigma(j)})_{0\leq j\leq n}$ is a partition of $[a, b]$. The functions

$$\varphi^{\mathcal{T}}_0 := h^{T_1}_{t_0}, \quad \varphi^{\mathcal{T}}_1 := h^{T_1}_{t_1} \quad \text{and} \quad \varphi^{\mathcal{T}}_n := h^{T_n}_{t_n} \quad \text{for } n \geq 2$$

are called Schauder hat functions for the sequence $\mathcal{T}$ and form a Schauder basis of $\mathcal{C}([a, b])$ with associated coefficient functionals given by

$$\lambda^{\mathbb{K}}_0(f) := f(t_0), \quad \lambda^{\mathbb{K}}_1(f) := f(t_1) \quad \text{and} \quad \lambda^{\mathbb{K}}_{n+1}(f) := f(t_{n+1}) - \sum_{k=0}^{n} \lambda^{\mathbb{K}}_k(f)\varphi^{\mathcal{T}}_k(t_{n+1}), \quad f \in \mathcal{C}([a, b]),\ n \geq 1,$$

by [24, 2.3.5 Proposition, p. 29]. Looking at the coefficient functionals, we see that the right-hand sides even make sense if $f \in C([a,b],E)$ and thus we define λ_n^E on $C([a,b],E)$ for $n \in \mathbb{N}_0$ accordingly.

Theorem 4.3. *Let E be an lcHs with metric ccp and let $\mathcal{T} := (t_n)_{n\in\mathbb{N}_0}$ be a dense sequence in $[a,b]$ with $t_0 = a$, $t_1 = b$ and $t_n \neq t_m$ for $n \neq m$. Then $\left(\sum_{n=0}^{k} \lambda_k^E \varphi_n^{\mathcal{T}}\right)_{k\in\mathbb{N}_0}$ is a Schauder decomposition of $C([a,b],E)$ and*

$$f = \sum_{n=0}^{\infty} \lambda_n^E(f)\varphi_n^{\mathcal{T}}, \quad f \in C([a,b],E).$$

Proof. The spaces $C([a,b])$ and $C([a,b],E)$ are ε-compatible by [15, 5.4 Example, p. 25] if E has metric ccp. $C([a,b])$ is a Banach space and thus barrelled implying that its Schauder basis $\left(\varphi_n^{\mathcal{T}}\right)$ is equicontinuous. We note that for all $u \in C([a,b])\varepsilon E$ and $x \in [a,b]$

$$\lambda_n^E(S(u))(x) = u\left(\delta_{t_n}\right) = u\left(\lambda_n^{\mathbb{K}}\right), \quad n \in \{0,1\},$$

and by induction

$$\begin{aligned}
\lambda_{n+1}^E(S(u))(x) &= u\left(\delta_{t_{n+1}}\right) - \sum_{k=0}^{n} \lambda_k^E(S(u))\varphi_k^{\mathcal{T}}\left(t_{n+1}\right) \\
&= u\left(\delta_{t_{n+1}}\right) - \sum_{k=0}^{n} u\left(\lambda_k^{\mathbb{K}}\right)\varphi_k^{\mathcal{T}}\left(t_{n+1}\right) \\
&= u\left(\lambda_{n+1}^{\mathbb{K}}\right), \quad n \geq 1.
\end{aligned}$$

Thus (3.1) is fulfilled proving our claim by Corollary 3.6. □

If $a = 0$, $b = 1$ and $\mathcal{T}$ is the sequence of dyadic numbers given in [24, 2.1.1 Definitions, p. 21], then $\left(\varphi_n^{\mathcal{T}}\right)$ is the so-called Faber–Schauder system. Using the Schauder basis and coefficient functionals of the space $C_0(\mathbb{R})$ of continuous functions vanishing at infinity given in [24, 2.7.1, pp. 41–42] and [24, 2.7.4 Corollary, p. 43] a corresponding, weaker result for the E-valued counterpart $C_0(\mathbb{R},E)$ holds as well by a similar argumentation. Another corresponding result holds for the space $C_{0,0}^{[\gamma]}([0,1],E)$, $0 < \gamma < 1$, of γ-Hölder continuous functions on [0,1] with values in E that vanish at zero and at infinity if one uses the Schauder basis and coefficient functionals of $C_{0,0}^{[\gamma]}([0,1])$ from [6, Theorem 2, p. 220] and [5, Theorem 3, p. 230]. The results are a bit weaker in both cases since [1, 2.4 Theorem (2), pp. 138–139] and [15, 5.9 Example b), p. 28] only guarantee that $S_{C_0(\mathbb{R})}$ and $S_{C_{0,0}^{[\gamma]}([0,1])}$ are isomorphisms if E is quasi-complete.

Now, we turn to spaces of continuously differentiable functions on an interval (a,b) such that all derivatives can be continuously extended to the boundary. For an lcHs E and $k \in \mathbb{N}$ the space $C^k([a,b],E)$ is given by

$$C^k([a,b],E) := \left\{ f \in C^k((a,b),E) \mid \left(\partial^{\beta}\right)^E f \text{ continuously extendable on } [a,b] \text{ for all } \beta \in \mathbb{N}_0,\ \beta \leq k \right\}$$

and equipped with the system of seminorms given by

$$|f|_{\alpha} := \sup_{\substack{x\in(a,b)\\ \beta\in\mathbb{N}_0,\beta\leq k}} p_{\alpha}\left(\left(\partial^{\beta}\right)^E f(x)\right), \quad f \in C^k([a,b],E),$$

for $\alpha \in \mathfrak{A}$. From the Schauder hat functions $\left(\varphi_n^{\mathcal{T}}\right)$ for a dense sequence $\mathcal{T} := \left(t_n\right)_{n\in\mathbb{N}_0}$ in $[a,b]$ with $t_0 = a$, $t_1 = b$ and $t_n \neq t_m$ for $n \neq m$ and the associated coefficient functionals $\lambda_n^{\mathbb{K}}$ we can easily get a Schauder basis for the space $C^k([a,b])$, $k \in \mathbb{N}$, by applying $\int_a^{(\cdot)}$ k-times to the series representation

$$f^{(k)} = \sum_{n=0}^{\infty} \lambda_n^{\mathbb{K}}(f^{(k)})\varphi_n^{\mathcal{T}}, \quad f \in C^k([a,b]),$$

where we identified $f^{(k)}$ with its continuous extension on $[a,b]$. The resulting Schauder basis $f_n^{\mathcal{T}} : [a,b] \to \mathbb{R}$ and associated coefficient functionals $\mu_n^{\mathbb{K}} : C^k([a,b]) \to \mathbb{K}$, $n \in \mathbb{N}_0$, are

$$f_n^{\mathcal{T}}(x) = \frac{1}{n!}(x-a)^n, \qquad \mu_n^{\mathbb{K}}(f) = f^{(n)}(a), \qquad 0 \le n \le k-1,$$

$$f_n^{\mathcal{T}}(x) = \int_a^x \int_a^{t_{k-1}} \cdots \int_a^{t_2} \int_a^{t_1} \varphi_{n-k}^{\mathcal{T}} \, dt \, dt_1 \dots dt_{k-1}, \qquad \mu_n^{\mathbb{K}}(f) = \lambda_{n-k}^{\mathbb{K}}(f^{(k)}), \qquad n \ge k,$$

for $x \in [a,b]$ and $f \in C^k([a,b])$ (see e.g. [21, pp. 586–587], [24, 2.3.7, p. 29]). Again, the mapping rule for the coefficient functionals still makes sense if $f \in C^k([a,b],E)$ and so we define μ_n^E on $C^k([a,b],E)$ for $n \in \mathbb{N}_0$ accordingly.

Theorem 4.4. *Let E be an lcHs with metric ccp, let $k \in \mathbb{N}$, let $\mathcal{T} := (t_n)_{n\in\mathbb{N}_0}$ be a dense sequence in $[a,b]$ with $t_0 = a$, $t_1 = b$ and $t_n \neq t_m$ for $n \neq m$. Then $\left(\sum_{n=0}^{l} \mu_n^E f_n^{\mathcal{T}}\right)_{l\in\mathbb{N}_0}$ is a Schauder decomposition of $C^k([a,b],E)$ and*

$$f = \sum_{n=0}^{\infty} \mu_n^E(f) f_n^{\mathcal{T}}, \quad f \in C^k([a,b],E).$$

Proof. The spaces $C^k([a,b])$ and $C^k([a,b],E)$ are ε-compatible by [16, Example 20, p. 1529] if E has metric ccp. The Banach space $C^k([a,b])$ is barrelled giving the equicontinuity of its Schauder basis. Due to [16, Proposition 10 b), p. 1520] we have for all $u \in C^k([a,b])\varepsilon E$, $\beta \in \mathbb{N}_0$, $\beta \le k$, and $x \in (a,b)$

$$(\partial^\beta)^E S(u)(x) = u\left(\delta_x \circ (\partial^\beta)^{\mathbb{K}}\right).$$

Further, for every sequence (x_n) in (a,b) converging to $t \in \{a,b\}$ we obtain by [16, Proposition 19, p. 1529] applied to $T := (\partial^\beta)^{\mathbb{K}}$

$$\lim_{n\to\infty} (\partial^\beta)^E S(u)(x_n) = u\left(\lim_{n\to\infty} \delta_{x_n} \circ (\partial^\beta)^{\mathbb{K}}\right).$$

From these observations we deduce that $\mu_n^E(S(u)) = u(\mu_n^{\mathbb{K}})$ for all $n \in \mathbb{N}_0$, i.e. (3.1) holds. Therefore our statement is a consequence of Corollary 3.6. □

4.3 | Holomorphic functions

In this short subsection we show how to get the result on power series expansion of holomorphic functions from the introduction. Let E be an lcHs over $\mathbb{C}$, $z_0 \in \mathbb{C}$, $r \in (0,\infty]$ and $\Omega := \mathbb{D}_r(z_0)$ where $\mathbb{D}_r(z_0) \subset \mathbb{C}$ is an open disc around z_0 with radius $r > 0$. We equip $\mathcal{O}(\Omega,E)$ with the system of seminorms given by

$$|f|_{K,\alpha} := \sup_{z\in K} p_\alpha(f(z)), \quad f \in \mathcal{O}(\Omega,E),$$

for $K \subset \Omega$ compact and $\alpha \in \mathfrak{A}$. We note that

$$\mathcal{O}(\Omega,E) = \left\{ f \in C^\infty(\Omega,E) \mid \overline{\partial}^E f = 0 \right\}$$

if E is locally complete where

$$\overline{\partial}^E f(z) := \frac{1}{2}\Big(\big(\partial^{e_1}\big)^E + i\big(\partial^{e_2}\big)^E\Big)f(z), \quad f \in C^1(\Omega, E),\ z \in \Omega,$$

is the Cauchy–Riemann operator. Moreover, we set $\big(\partial_{\mathbb{C}}^0\big)^E f := f$,

$$\big(\partial_{\mathbb{C}}^1\big)^E f(z) := \lim_{\substack{h\to 0\\ h\in\mathbb{C},h\neq 0}} \frac{f(z+h)-f(z)}{h}, \quad z \in \Omega,$$

and $\big(\partial_{\mathbb{C}}^{n+1}\big)^E f := \big(\partial_{\mathbb{C}}^1\big)^E\Big(\big(\partial_{\mathbb{C}}^n\big)^E f\Big)$ for $n \in \mathbb{N}_0$ and $f \in \mathcal{O}(\Omega, E)$ (see (1.1)). We observe that real and complex derivatives are related by

$$\big(\partial^{\beta}\big)^E f(z) = i^{\beta_2}\Big(\partial_{\mathbb{C}}^{|\beta|}\Big)^E f(z), \quad z \in \Omega, \tag{4.1}$$

for every $f \in \mathcal{O}(\Omega, E)$ and $\beta = \big(\beta_1, \beta_2\big) \in \mathbb{N}_0^2$.

Theorem 4.5. *Let E be a locally complete lcHs over $\mathbb{C}$, $z_0 \in \mathbb{C}$ and $r \in (0, \infty]$. Then*

$$\left(\sum_{n=0}^{k} \frac{\delta_{z_0} \circ \big(\partial_{\mathbb{C}}^n\big)^E}{n!}\big(\cdot - z_0\big)^n\right)_{k\in\mathbb{N}_0}$$

is a Schauder decomposition of $\mathcal{O}\big(\mathbb{D}_r\big(z_0\big), E\big)$ and

$$f = \sum_{n=0}^{\infty} \frac{\big(\partial_{\mathbb{C}}^n\big)^E f\big(z_0\big)}{n!}\big(\cdot - z_0\big)^n, \quad f \in \mathcal{O}\big(\mathbb{D}_r\big(z_0\big), E\big).$$

Proof. The spaces $\mathcal{O}\big(\mathbb{D}_r\big(z_0\big)\big)$ and $\mathcal{O}\big(\mathbb{D}_r\big(z_0\big), E\big)$ are ε-compatible by [2, Theorem 9, p. 232] if E is locally complete. Further, the Schauder basis $\Big(\big(\cdot - z_0\big)^n\Big)$ of $\mathcal{O}\big(\mathbb{D}_r\big(z_0\big)\big)$ is equicontinuous since the Fréchet space $\mathcal{O}\big(\mathbb{D}_r\big(z_0\big)\big)$ is barrelled. Due to [15, 4.12 Proposition, p. 22] and (4.1) we have for all $u \in \mathcal{O}\big(\mathbb{D}_r\big(z_0\big)\big)\varepsilon E$ and $z \in \mathbb{D}_r\big(z_0\big)$

$$\big(\partial_{\mathbb{C}}^n\big)^E S(u)(z) = u\Big(\delta_z \circ \big(\partial_{\mathbb{C}}^n\big)^{\mathbb{C}}\Big)$$

which means that (3.1) is satisfied. Corollary 3.6 implies our statement. □

4.4 | Fourier expansions

In this subsection we turn our attention to Fourier expansions in the space of vector-valued rapidly decreasing functions and in the space of vector-valued smooth, 2π-periodic functions. We start with the definition of the Pettis-integral which we need to define the Fourier coefficients for vector-valued functions. For a measure space (X, Σ, μ) and $1 \leq p < \infty$ let

$$\mathfrak{L}^p(X, \mu) := \left\{ f : X \to \mathbb{K} \text{ measurable} \mid q_p(f) := \int_X |f(x)|^p \,\mathrm{d}\mu(x) < \infty \right\}$$

and define the quotient space of p-integrable functions by $\mathcal{L}^p(X, \mu) := \mathfrak{L}^p(X, \mu)/\big\{f \in \mathfrak{L}^p(X, \mu) \mid q_p(f) = 0\big\}$ which becomes a Banach space if it is equipped with the norm $\|f\|_p := q_p(F)^{1/p}$, $f = [F] \in \mathcal{L}^p(X, \mu)$. From now on we do not distinguish between equivalence classes and their representants anymore.

For a measure space (X,Σ,μ) and $f : X \to \mathbb{K}$ we say that f is integrable on $\Lambda \in \Sigma$ and write $f \in \mathcal{L}^1(\Lambda,\mu)$ if $\chi_\Lambda f \in \mathcal{L}^1(X,\mu)$. Then we set

$$\int_\Lambda f(x)\,\mathrm{d}\mu(x) := \int_X \chi_\Lambda(x) f(x)\,\mathrm{d}\mu(x).$$

Definition 4.6 (Pettis-integral)**.** Let (X,Σ,μ) be a measure space and let E be an lcHs. A function $f : X \to E$ is called *weakly (scalarly) measurable* if the function $e'\circ f : X \to \mathbb{K}$, $(e'\circ f)(x) := \langle e', f(x)\rangle := e'(f(x))$, is measurable for all $e' \in E'$. A weakly measurable function is said to be *weakly (scalarly) integrable* if $e'\circ f \in \mathcal{L}^1(X,\mu)$. A function $f : X \to E$ is called *Pettis-integrable* on $\Lambda \in \Sigma$ if it is weakly integrable on Λ and

$$\exists\, e_\Lambda \in E\ \forall e' \in E' : \langle e', e_\Lambda\rangle = \int_\Lambda \langle e', f(x)\rangle\,\mathrm{d}\mu(x).$$

In this case e_Λ is unique due to E being Hausdorff and we set

$$\int_\Lambda f(x)\,\mathrm{d}\mu(x) := e_\Lambda.$$

If we consider the measure space $(X,\mathscr{L}(X),\lambda)$ of Lebesgue measurable sets for $X \subset \mathbb{R}^d$, we just write $\mathrm{d}x := \mathrm{d}\lambda(x)$.

Lemma 4.7. *Let E be a locally complete lcHs, let $\Omega \subset \mathbb{R}^d$ open and let $f : \Omega \to E$. If f is weakly C^1, i.e. $e'\circ f \in C^1(\Omega)$ for every $e' \in E'$, then f is Pettis-integrable on every compact subset of $K \subset \Omega$ with respect to any locally finite measure μ on Ω and*

$$p_\alpha\left(\int_K f(x)\,\mathrm{d}\mu(x)\right) \leq \mu(K)\sup_{x\in K} p_\alpha(f(x)), \quad \alpha \in \mathfrak{A}.$$

Proof. Let $K \subset \Omega$ be compact and let (Ω,Σ,μ) be a measure space with locally finite measure μ, i.e. Σ contains the Borel σ-algebra $\mathcal{B}(\Omega)$ on Ω and for every $x \in \Omega$ there is a neighbourhood $U_x \subset \Omega$ of x such that $\mu(U_x) < \infty$. Since the map $e'\circ f$ is differentiable for every $e' \in E'$, thus Borel-measurable, and $\mathcal{B}(\Omega) \subset \Sigma$, it is measurable. We deduce that $e'\circ f \in \mathcal{L}^1(K,\mu)$ for every $e' \in E'$ because locally finite measures are finite on compact sets. Hence the map

$$I : E' \to \mathbb{K},\ I(e') := \int_K \langle e', f(x)\rangle\,\mathrm{d}\mu(x)$$

is well-defined and linear. We estimate

$$|I(e')| \leq \mu(K)\sup_{x\in f(K)} |e'(x)| \leq \mu(K)\sup_{x\in\overline{\mathrm{acx}}(f(K))} |e'(x)|, \quad e' \in E'.$$

Due to f being weakly C^1 and [3, Proposition 2, p. 354] the absolutely convex set $\overline{\mathrm{acx}}(f(K))$ is compact yielding $I \in \left(E'_\kappa\right)' \cong E$ by the theorem of Mackey–Arens which means that there is $e_K \in E$ such that

$$\langle e', e_K\rangle = I(e') = \int_K \langle e', f(x)\rangle\,\mathrm{d}\mu(x), \quad e' \in E'.$$

Therefore f is Pettis-integrable on K with respect to μ. For $\alpha \in \mathfrak{A}$ we set $B_\alpha := \left\{x \in E \mid p_\alpha(x) < 1\right\}$ and observe that

$$
\begin{aligned}
p_\alpha\left(\int_K f(x)\,\mathrm{d}\mu(x)\right) &= \sup_{e' \in B_\alpha^\circ} \left|\left\langle e', \int_K f(x)\,\mathrm{d}\mu(x)\right\rangle\right| \\
&= \sup_{e' \in B_\alpha^\circ} \left|\int_K e'(f(x))\,\mathrm{d}\mu(x)\right| \\
&\le \mu(K) \sup_{e' \in B_\alpha^\circ} \sup_{x \in K} |e'(f(x))| \\
&= \mu(K) \sup_{x \in K} p_\alpha(f(x))
\end{aligned}
$$

where we used [18, Proposition 22.14, p. 256] in the first and last equation to get from p_α to $\sup_{e' \in B_\alpha^\circ}$ and back. □

For an lcHs E we define the Schwartz space of E-valued rapidly decreasing functions by

$$
\mathcal{S}(\mathbb{R}^d, E) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d, E) \mid \forall\, l \in \mathbb{N},\, \alpha \in \mathfrak{A} :\ |f|_{l,\alpha} < \infty \right\}
$$

where

$$
|f|_{l,\alpha} := \sup_{\substack{x \in \mathbb{R}^d \\ \beta \in \mathbb{N}_0^d, |\beta| \le l}} p_\alpha\left(\left(\partial^\beta\right)^E f(x)\right)\left(1 + |x|^2\right)^{l/2}.
$$

We recall the definition of the Hermite functions. For $n \in \mathbb{N}_0$ we set

$$
h_n : \mathbb{R} \to \mathbb{R},\ h_n(x) := \left(2^n n! \sqrt{\pi}\right)^{-1/2} \left(x - \frac{d}{dx}\right)^n e^{-x^2/2} = \left(2^n n! \sqrt{\pi}\right)^{-1/2} H_n(x) e^{-x^2/2},
$$

with the Hermite polynomials H_n of degree n which can be computed recursively by

$$
H_0(x) = 1 \quad \text{and} \quad H_{n+1}(x) = 2x H_n(x) - H_n'(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}_0.
$$

For $n = (n_k) \in \mathbb{N}_0^d$ we define the n-th Hermite function by

$$
h_n : \mathbb{R}^d \to \mathbb{R},\ h_n(x) := \prod_{k=1}^d h_{n_k}(x_k), \quad \text{and} \quad H_n : \mathbb{R}^d \to \mathbb{R},\ H_n(x) := \prod_{k=1}^d H_{n_k}(x_k).
$$

Proposition 4.8. *Let E be a sequentially complete lcHs, let $f \in \mathcal{S}(\mathbb{R}^d, E)$ and let $n \in \mathbb{N}_0^d$. Then $f h_n$ is Pettis-integrable on $\mathbb{R}^d$.*

Proof. For $k \in \mathbb{N}$ we define the Pettis-integral

$$
e_k := \int_{[-k,k]^d} f(x) h_n(x)\,\mathrm{d}x
$$

which is a well-defined element of E by Lemma 4.7. We claim that (e_k) is a Cauchy sequence in E. First, we notice that there are $j \in \mathbb{N}$ and $C > 0$ such that $|H_n(x)| \le C\left(1 + |x|^2\right)^{j/2}$ for all $x \in \mathbb{R}^d$ as H_n is a product of polynomials in one variable. Now, let $\alpha \in \mathfrak{A}$, $k, m \in \mathbb{N}$, $k > m$, and set $C_n := \left(\prod_{i=1}^d 2^{n_i} n_i! \sqrt{\pi}\right)^{-1/2}$ as well as $Q_{k,m} := [-k,k]^d \setminus [-m,m]^d$. We observe that $\left|\prod_{i=1}^d x_i\right| \ge 1$ for every $x \in Q_{k,m}$ and

$$
\begin{aligned}
p_\alpha(e_k-e_m) &= \sup_{e'\in B_\alpha^\circ} |e'(e_k-e_m)| \\
&= \sup_{e'\in B_\alpha^\circ} \left| \int_{Q_{k,m}} \langle e', f(x)h_n(x)\rangle \,\mathrm{d}x \right| \\
&\leq C_n \int_{Q_{k,m}} e^{-|x|^2/2}\,\mathrm{d}x \sup_{e'\in B_\alpha^\circ}\sup_{x\in\mathbb{R}^d} |e'(f(x)H_n(x))| \\
&= C_n \int_{Q_{k,m}} e^{-|x|^2/2}\,\mathrm{d}x \sup_{x\in\mathbb{R}^d} p_\alpha(f(x))|H_n(x)| \\
&\leq C_n C|f|_{j,\alpha} \int_{Q_{k,m}} \left|\prod_{i=1}^d x_i\right| e^{-|x|^2/2}\,\mathrm{d}x \\
&= C_n C|f|_{j,\alpha}\left(\left(2\int_{[0,k]} xe^{-x^2/2}\,\mathrm{d}x\right)^d - \left(2\int_{[0,m]} xe^{-x^2/2}\,\mathrm{d}x\right)^d \right) \\
&= 2^d C_n C|f|_{j,\alpha}\left(\left(1-e^{-k^2/2}\right)^d - \left(1-e^{-m^2/2}\right)^d \right)
\end{aligned}
$$

proving our claim. Since E is sequentially complete, the limit $y := \lim_{k\to\infty} e_k$ exists in E. From $e'\circ f \in \mathcal{S}(\mathbb{R}^d)$ for every $e' \in E'$ and the dominated convergence theorem we deduce

$$
\langle e', y\rangle = \lim_{k\to\infty}\langle e', e_k\rangle = \lim_{k\to\infty}\int_{[-k,k]^d} \langle e', f(x)\rangle h_n(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \langle e', f(x)h_n(x)\rangle\,\mathrm{d}x, \quad e'\in E',
$$

which yields the Pettis-integrability of fh_n on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} f(x)h_n(x)\,\mathrm{d}x = y$. □

Due to the previous proposition we can define the n-th Fourier coefficient of $f\in\mathcal{S}(\mathbb{R}^d,E)$ by

$$
\widehat{f}(n) := \int_{\mathbb{R}^d} f(x)\overline{h_n(x)}\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)h_n(x)\,\mathrm{d}x, \quad n\in\mathbb{N}_0^d,
$$

if E is sequentially complete. We know that the map

$$
\mathscr{F}^{\mathbb{K}}: \mathcal{S}(\mathbb{R}^d)\to s(\mathbb{N}_0^d),\ \mathscr{F}^{\mathbb{K}}(f) := (\widehat{f}(n))_{n\in\mathbb{N}_0^d},
$$

is an isomorphism and its inverse is given by

$$
(\mathscr{F}^{\mathbb{K}})^{-1}: s(\mathbb{N}_0^d)\to\mathcal{S}(\mathbb{R}^d),\ (\mathscr{F}^{\mathbb{K}})^{-1}(x) := \sum_{n\in\mathbb{N}_0^d} x_n h_n,
$$

(see e.g. [13, Satz 3.7, p. 66]). It is already known that $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d,E)$ are ε-compatible if E is quasi-complete by [22, Proposition 9, p. 108, Théorème 1, p. 111] (cf. [16, Example 17, p. 1526]). We improve this to sequentially complete E and derive a Schauder decomposition of $\mathcal{S}(\mathbb{R}^d,E)$ as well using our observations on the sequence space $s(\mathbb{N}_0^d,E)$.

Theorem 4.9. *Let E be a sequentially complete lcHs. Then the following holds.*

a) The map

$$
\mathscr{F}^{E}: \mathcal{S}(\mathbb{R}^d,E)\to s(\mathbb{N}_0^d,E),\ \mathscr{F}^{E}(f) := (\widehat{f}(n))_{n\in\mathbb{N}_0^d},
$$

is an isomorphism, $\mathcal{S}(\mathbb{R}^d,E)\cong\mathcal{S}(\mathbb{R}^d)\varepsilon E$ and

$$
\mathscr{F}^{E} = S_{s(\mathbb{N}_0^d)}\circ(\mathscr{F}^{\mathbb{K}}\varepsilon\,\mathrm{id}_E)\circ S^{-1}_{\mathcal{S}(\mathbb{R}^d)}.
$$

b) $\left(\sum_{n\in\mathbb{N}_0^d,|n|\leq k}\mathscr{F}_n^E h_n\right)_{k\in\mathbb{N}}$ *is a Schauder decomposition of* $\mathcal{S}(\mathbb{R}^d,E)$ *and*

$$f=\sum_{n\in\mathbb{N}_0^d}\widehat{f}(n)h_n,\quad f\in\mathcal{S}(\mathbb{R}^d,E).$$

Proof. First, we show that the map $\mathscr{F}^E$ is well-defined. Let $f\in\mathcal{S}(\mathbb{R}^d,E)$. Then $e'\circ f\in\mathcal{S}(\mathbb{R}^d)$ and

$$\langle e',\mathscr{F}^E(f)_n\rangle=\langle e',\widehat{f}(n)\rangle=\widehat{e'\circ f}(n)=\mathscr{F}^{\mathbb{K}}(e'\circ f)_n$$

for every $n\in\mathbb{N}_0^d$ and $e'\in E'$. Thus we have $\mathscr{F}^{\mathbb{K}}(e'\circ f)\in s(\mathbb{N}_0^d)$ for every $e'\in E'$ which implies by [18, Mackey's theorem 23.15, p. 268] that $\mathscr{F}^E(f)\in s(\mathbb{N}_0^d,E)$ and that $\mathscr{F}^E$ is well-defined.

We notice that

$$s(\mathbb{N}_0^d,E)\cong s(\mathbb{N}_0^d)\varepsilon E\cong\mathcal{S}(\mathbb{R}^d)\varepsilon E\hookrightarrow\mathcal{S}(\mathbb{R}^d,E)$$

where the first isomorphism is $S^{-1}_{s(\mathbb{N}_0^d)}$ by Theorem 4.2 (i), the second is the map $\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\varepsilon\,\mathrm{id}_E$ and the third isomorphism into is the map $S_{\mathcal{S}(\mathbb{R}^d)}$ by [15, 3.9 Theorem, p. 9] and [16, Proposition 10 b), p. 1520] since $\mathcal{S}(\mathbb{R}^d)$ is barrelled. Next, we show that $S_{\mathcal{S}(\mathbb{R}^d)}\circ\left(\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\varepsilon\,\mathrm{id}_E\right)\circ S^{-1}_{s(\mathbb{N}_0^d)}$ is surjective and the inverse of $\mathscr{F}^E$. We can explicitly compute the composition of these maps. By the proof of Proposition 3.7 b) we get that the inverse of $S_{s(\mathbb{N}_0^d)}$ is given by

$$S^{-1}_{s(\mathbb{N}_0^d)}:s(\mathbb{N}_0^d,E)\to s(\mathbb{N}_0^d)\varepsilon E,\ w\mapsto\lim_{k\to\infty}\sum_{n\in\mathbb{N}_0^d,|n|\leq k}\Theta(\varphi_n\otimes w_n).$$

Let $w\in s(\mathbb{N}_0^d,E)$. Then for every $x\in\mathbb{R}^d$ and $e'\in E'$

$$\begin{aligned}S^{-1}_{s(\mathbb{N}_0^d)}\left(\left(\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\right)^t(\delta_x)\right)&=\sum_{n\in\mathbb{N}_0^d}\Theta(\varphi_n\otimes w_n)\left(\left(\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\right)^t(\delta_x)\right)\\&=\sum_{n\in\mathbb{N}_0^d}\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}(\varphi_n)(x)w_n\\&=\sum_{n\in\mathbb{N}_0^d}w_nh_n(x)\end{aligned}$$

which gives

$$\left(S_{\mathcal{S}(\mathbb{R}^d)}\circ\left(\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\varepsilon\,\mathrm{id}_E\right)\circ S^{-1}_{s(\mathbb{N}_0^d)}\right)(w)(x)=\sum_{n\in\mathbb{N}_0^d}w_nh_n(x).$$

Let $f\in\mathcal{S}(\mathbb{R}^d,E)$. Then $w:=\mathscr{F}^E(f)\in s(\mathbb{N}_0^d,E)$ and

$$e'\left(\sum_{n\in\mathbb{N}_0^d}w_nh_n\right)=\sum_{n\in\mathbb{N}_0^d}e'\left(\widehat{f}(n)\right)h_n=\sum_{n\in\mathbb{N}_0^d}\widehat{e'\circ f}(n)h_n=\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\left(\left(\widehat{e'\circ f}(n)\right)_{n\in\mathbb{N}_0^d}\right)=e'\circ f$$

for all $e' \in E'$ resulting in

$$f = \sum_{n\in\mathbb{N}_0^d} w_n h_n = \sum_{n\in\mathbb{N}_0^d} \mathscr{F}^E(f)_n h_n$$

and

$$\left(S_{\mathcal{S}(\mathbb{R}^d)}\circ\left(\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\varepsilon\,\mathrm{id}_E\right)\circ S^{-1}_{s\left(\mathbb{N}_0^d\right)}\right)(w)(x) = f(x)$$

for every $x \in \mathbb{R}^d$. Thus we conclude

$$\left[\left(S_{\mathcal{S}(\mathbb{R}^d)}\circ\left(\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\varepsilon\,\mathrm{id}_E\right)\circ S^{-1}_{s\left(\mathbb{N}_0^d\right)}\right)\circ\mathscr{F}^E\right](f) = f$$

yielding the surjectivity of the composition. Therefore $S_{\mathcal{S}(\mathbb{R}^d)}\circ\left(\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\varepsilon\,\mathrm{id}_E\right)\circ S^{-1}_{s\left(\mathbb{N}_0^d\right)}$ is an isomorphism with right inverse $\mathscr{F}^E$ which implies that $\mathscr{F}^E$ is its inverse. In addition, the bijectivity of $S_{\mathcal{S}(\mathbb{R}^d)}\circ\left(\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\varepsilon\,\mathrm{id}_E\right)\circ S^{-1}_{s\left(\mathbb{N}_0^d\right)}$ and $S^{-1}_{s\left(\mathbb{N}_0^d\right)}$ implies that $S_{\mathcal{S}(\mathbb{R}^d)}\circ\left(\left(\mathscr{F}^{\mathbb{K}}\right)^{-1}\varepsilon\,\mathrm{id}_E\right)$ is bijective and thus $S_{\mathcal{S}(\mathbb{R}^d)}$ is surjective. Hence $S_{\mathcal{S}(\mathbb{R}^d)}$ is also an isomorphism completing the proof of part a).

The rest of part b) follows from the isomorphism $\mathcal{S}(\mathbb{R}^d, E) \cong s(\mathbb{N}_0^d, E)$ via $\mathscr{F}^E$ and Proposition 4.1 a). □

Our last example of this subsection is devoted to Fourier expansions of vector-valued 2π-periodic smooth functions. We equip the space $\mathcal{C}^\infty(\mathbb{R}^d, E)$ for locally convex Hausdorff E with the system of seminorms generated by

$$|f|_{K,l,\alpha} := \sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d,|\beta|\le l}} p_\alpha\left(\left(\partial^\beta\right)^E f(x)\right)\chi_K(x) = \sup_{\substack{x\in K\\ \beta\in\mathbb{N}_0^d,|\beta|\le l}} p_\alpha\left(\left(\partial^\beta\right)^E f(x)\right), \quad f \in \mathcal{C}^\infty(\mathbb{R}^d, E),$$

for $K \subset \mathbb{R}^d$ compact, $l \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$. By $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E)$ we denote the topological subspace of $\mathcal{C}^\infty(\mathbb{R}^d, E)$ consisting of the functions which are 2π-periodic in each variable. Due to Lemma 4.7 we are able to define the n-th Fourier coefficient of $f \in \mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E)$ by

$$\hat{f}(n) := \mathfrak{F}_n^E(f) := (2\pi)^{-d}\int_{[-\pi,\pi]^d} f(x)e^{-i\langle n,x\rangle}\,\mathrm{d}x, \quad n \in \mathbb{Z}^d,$$

where $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbb{R}^d$, if E is locally complete.

Theorem 4.10. *Let E be a locally complete lcHs over $\mathbb{C}$. Then a Schauder decomposition of $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E)$ is given by $\left(\sum_{n\in\mathbb{Z}^d,|n|\le k}\mathfrak{F}_n^E e^{i\langle n,\cdot\rangle}\right)_{k\in\mathbb{N}}$ and*

$$f = \sum_{n\in\mathbb{Z}^d} \hat{f}(n)e^{i\langle n,\cdot\rangle}, \quad f \in \mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E).$$

Proof. First, we prove that $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)$ and $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E)$ are ε-compatible. It follows from [2, p. 228] (cf. [16, Example 18 a), p. 1528]) that $S_{\mathcal{C}^\infty(\mathbb{R}^d)} : \mathcal{C}^\infty(\mathbb{R}^d)\varepsilon E \to \mathcal{C}^\infty(\mathbb{R}^d, E)$ is an isomorphism. Furthermore, for each $x \in \mathbb{R}^d$ and $1 \le n \le d$ we have $\delta_x = \delta_{x+2\pi e_n}$ in $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)'$ and thus

$$S_{\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)}(u)(x) - S_{\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)}(u)(x + 2\pi e_n) = u(\delta_x - \delta_{x+2\pi e_n}) = 0, \quad u \in \mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)\varepsilon E,$$

implying that $S_{\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)}(u)$ is 2π-periodic in each variable. In addition, we observe that $e'\circ f$ is 2π-periodic in each variable for all $e' \in E'$ and $f \in \mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E)$. An application of [15, 3.26 Proposition (i), p. 17] yields that

$$S_{\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)} : \mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)\varepsilon E \to \mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E)$$

is an isomorphism, i.e. $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)$ and $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E)$ are ε-compatible.

The space $C^{\infty}_{2\pi}(\mathbb{R}^d)$ is barrelled since it is a Fréchet space and thus its Schauder basis $\left(e^{i\langle n,\cdot\rangle}\right)$ is equicontinuous. By [16, Theorem 14, p. 1524] the inverse of $S_{C^{\infty}_{2\pi}(\mathbb{R}^d)}$ is given by

$$R^t : C^{\infty}_{2\pi}(\mathbb{R}^d, E) \to C^{\infty}_{2\pi}(\mathbb{R}^d)\varepsilon E, \quad f \mapsto \mathcal{J}^{-1} \circ R^t_f,$$

where $\mathcal{J} : E \to E'^{\star}$ is the canonical injection in the algebraic dual $E'^{\star}$ of E' and

$$R^t_f : C^{\infty}_{2\pi}(\mathbb{R}^d)' \to E'^{\star},\ y \longmapsto [e' \mapsto y(e' \circ f)],$$

for $f \in C^{\infty}_{2\pi}(\mathbb{R}^d, E)$. Let $u \in C^{\infty}_{2\pi}(\mathbb{R}^d)\varepsilon E$. Then $f := S_{C^{\infty}_{2\pi}(\mathbb{R}^d)}(u) \in C^{\infty}_{2\pi}(\mathbb{R}^d, E)$ and from the Pettis-integrability of $fe^{-i\langle n,\cdot\rangle}$ we obtain

$$\begin{aligned} R^t_f(\mathfrak{F}^{\mathbb{C}}_n)(e') = \mathfrak{F}^{\mathbb{C}}_n(e' \circ f) &= (2\pi)^{-d} \int_{[-\pi,\pi]^d} \left\langle e', f(x)e^{-i\langle n,x\rangle} \right\rangle \mathrm{d}x \\ &= \left\langle e', (2\pi)^{-d} \int_{[-\pi,\pi]^d} f(x)e^{-i\langle n,x\rangle}\,\mathrm{d}x \right\rangle \\ &= \left\langle e', \mathfrak{F}^E_n(f) \right\rangle, \quad e' \in E', \end{aligned}$$

for all $n \in \mathbb{Z}^d$ which results in

$$u(\mathfrak{F}^{\mathbb{C}}_n) = S^{-1}_{C^{\infty}_{2\pi}(\mathbb{R}^d)}(f)(\mathfrak{F}^{\mathbb{C}}_n) = \mathcal{J}^{-1}\left(R^t_f(\mathfrak{F}^{\mathbb{C}}_n)\right) = \mathfrak{F}^E_n(f) = \mathfrak{F}^E_n\left(S_{C^{\infty}_{2\pi}(\mathbb{R}^d)}(u)\right)$$

and thus shows the validity of (3.1). Now, our statement follows from Corollary 3.6. □

Considering the coefficients in the series expansion above, we know that the map

$$\mathfrak{F}^{\mathbb{C}} : C^{\infty}_{2\pi}(\mathbb{R}^d) \to s(\mathbb{Z}^d), \quad \mathfrak{F}^{\mathbb{C}}(f) := \left(\hat{f}(n)\right)_{n\in\mathbb{Z}^d},$$

is an isomorphism (see e.g. [13, Satz 1.7, p. 18]). Thus we have the following relation if E is a locally complete Hausdorff space over $\mathbb{C}$

$$C^{\infty}_{2\pi}(\mathbb{R}^d, E) \cong C^{\infty}_{2\pi}(\mathbb{R}^d)\varepsilon E \cong s(\mathbb{Z}^d)\varepsilon E \hookrightarrow s(\mathbb{Z}^d, E)$$

where the first isomorphism is $S^{-1}_{C^{\infty}_{2\pi}(\mathbb{R}^d)}$, the second is the map $\mathfrak{F}^{\mathbb{C}}\varepsilon\, \mathrm{id}_E$ and the third isomorphism into is the map $S_{s(\mathbb{Z}^d)}$ by [15, 3.9 Theorem, p. 9]. We can explicitely compute the composition of these maps. With the notation from the proof above we have for every $f \in C^{\infty}_{2\pi}(\mathbb{R}^d, E)$ and $n \in \mathbb{Z}^d$

$$\begin{aligned} \left(S_{s(\mathbb{Z}^d)} \circ (\mathfrak{F}^{\mathbb{C}}\varepsilon\, \mathrm{id}_E) \circ S^{-1}_{C^{\infty}_{2\pi}(\mathbb{R}^d)}\right)(f)(n) &= S_{s(\mathbb{Z}^d)}\left(\left(\mathfrak{F}^{\mathbb{C}}\varepsilon\, \mathrm{id}_E\right)\left(S^{-1}_{C^{\infty}_{2\pi}(\mathbb{R}^d)}(f)\right)\right)(n) \\ &= S_{s(\mathbb{Z}^d)}\left(S^{-1}_{C^{\infty}_{2\pi}(\mathbb{R}^d)}(f) \circ (\mathfrak{F}^{\mathbb{C}})^t\right)(n) \\ &= S^{-1}_{C^{\infty}_{2\pi}(\mathbb{R}^d)}(f)\left((\mathfrak{F}^{\mathbb{C}})^t(\delta_n)\right) \\ &= S^{-1}_{C^{\infty}_{2\pi}(\mathbb{R}^d)}(f)(\mathfrak{F}^{\mathbb{C}}_n) \\ &= \mathfrak{F}^E_n(f) \\ &= \hat{f}(n). \end{aligned}$$

Thus the map

$$\mathfrak{F}^E : C_{2\pi}^\infty(\mathbb{R}^d, E) \to s(\mathbb{Z}^d, E),\ \mathfrak{F}^E(f) := (\hat{f}(n))_{n\in\mathbb{Z}^d},$$

is well-defined and an isomorphism into if E is locally complete. If E is sequentially complete, it is even an isomorphism to the whole space $s(\mathbb{Z}^d, E)$ because $S_{s(\mathbb{Z}^d)}$ is surjective to the whole space then by Theorem 4.2 (i). Hence we have:

Theorem 4.11. *If E is a sequentially complete lcHs over $\mathbb{C}$, then*

$$\mathfrak{F}^E : C_{2\pi}^\infty(\mathbb{R}^d, E) \to s(\mathbb{Z}^d, E),\ \mathfrak{F}^E(f) := (\hat{f}(n))_{n\in\mathbb{Z}^d},$$

is an isomorphism and

$$\mathfrak{F}^E = S_{s(\mathbb{Z}^d)} \circ (\mathfrak{F}^{\mathbb{C}} \varepsilon \operatorname{id}_E) \circ S^{-1}_{C_{2\pi}^\infty(\mathbb{R}^d)}.$$

For quasi-complete E Theorem 4.10 and Theorem 4.11 are already known by [13, Satz 10.8, p. 239].

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