On extremal properties of minimal Ramsey graphs and cross intersecting families

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Summary

A graph G is said to be q-Ramsey for a graph H if for every q-colouring of the edges of H there exists a monochromatic copy of H. Moreover, G is said to be minimal q-Ramsey for H if no proper subgraph of G is q-Ramsey for H. In this thesis we study various properties of the set of minimal Ramsey graphs for many classes of graphs.

In 1976, Burr, Erdős, and Lovász initiated a systematic study of the set of minimal Ramsey graphs. In their work they introduced the parameter $s_q(H)$ which is defined as the smallest minimum degree among all minimal q-Ramsey graphs for H. Note that $q(\delta(H)-1)+1$ is a simple lower bound for $s_q(H)$. For a given q, a graph H is said to be q-Ramsey simple if $s_q(H) = q(\delta(H)-1)+1$.

In the first part of the thesis we study the Ramsey simplicity of random graphs for several ranges of p and q. We show when the graph $G_{n,p}$ is sparse enough it is q-Ramsey simple for all values of $q \geq 2$ whereas it is not Ramsey simple for any value of q whenever it is sufficiently large. We notice that the property of Ramsey simplicity is monotone in q. As a result we can meaningfully define the threshold value for q, below which a graph is Ramsey simple but ceases to be beyond it. We will establish some upper and lower bounds for this threshold in the number of colours for several ranges of p.

In the second part of the thesis we move on to the quantitative behaviour of the vertices of minimum degree in minimal Ramsey graphs. We ask the question, how many vertices of degree $s_q(H)$ can a minimal Ramsey graph have? In the same work of Burr, Erdős, and Lovász, the authors noted that for a given integer $\ell \geq 1$, there exists a minimal 2-Ramsey graph for a clique K_t which contains ℓ vertices of degree $s_2(K_t)$. We extend this observation further to show that this phenomena indeed holds for all values of q for K_t and we call this property s_q-abundance. We show that not just cliques, but also all cycles are s_q-abundant for all values of q. We will provide a general result for all 3-connected graphs which we will employ to show s_2-abundance for wheels and s_q-abundance for $G_{n,p}$ for some values of q and p in the range where $G_{n,p}$ is 3-connected.

Next, in the third part of this thesis we will consider the question of Ramsey equivalence for asymmetric pair of graphs. We say that two graphs are Ramsey equivalent if their set of minimal Ramsey graphs are the same. Recently, Fox, Grinshpun, Liebenau, Person, and Szabó asked if there exists a pair of non-isomorphic graphs that are Ramsey equivalent? In this part we analyse pairs of graphs that can and cannot be equivalent to the pair (T, K_t) where T is a tree.

In the fourth and final part of this thesis we move our attention toward families of sets. For $r, t, n \in \mathbb{N}$ we say that families $\mathcal{F}_1, \ldots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ are r-cross t-intersecting if for all $F_1 \in \mathcal{F}_1, \ldots, F_r \in \mathcal{F}_r$ we have $|\bigcap_{i\in[r]} F_i| \geq t$. We determine the maximum sum of measures of such families whenever the sets in the families are either of uniform size or are non-uniform.

Contents

1 Introduction

1.1 Overview of the thesis

Extremal problems form an integral part of research in combinatorics. They aim to find some substructures in large, seemingly disordered, structures. Trying to find structure in chaos is not unique to combinatorics but rather is an inherent trait of us humans. We are always on a quest for finding patterns to our dreams or the meaning of our lives. These may be very hard things to analyse, so here let us try our hand at something more verifiable instead.

In this work we will concern ourselves with finite graphs and sets. In 1930, in a work on formal logic, Ramsey proved a theorem that spurred a whole field of study, which later came to be known as Ramsey theory. He showed that in a sufficiently large graph, no matter however disordered it seems, there exists some given smaller structures of high degree of order. In particular [74], in every large enough graph there exists either a clique or set of independent vertices of a certain size. Even though finding the smallest graph that always contains a clique or an independent set of size three is a toy problem, this innocuous looking problem gets very hard very quickly. Then minimum size of the smallest graph that always contains either a clique or an independent set of size five is already an open problem.

Let us reformulate the above notion in a colourful setting. We say that a graph G is Ramsey for a graph H, if for every two colouring of the edges of the graph G, we will certainly find a copy of H in this such that all the edges of this copy have the same colour. Characterizing the set of Ramsey graphs has received considerable attention in the last few decades. As mentioned, determining the minimum number of vertices, which would be a prerequisite to belong to this set, is a very hard problem. One can assign various other parameters to a graph or a set through which we can continue to study the set of Ramsey graphs for a particular graph. After a moment's thought, one realizes that in some cases, it might be better to consider graphs that are in some sense *minimal*, i.e., no subgraph of this graph is Ramsey for the graph in question. After all, these graphs would function as building blocks for all other Ramsey graphs.

In 1976, Burr, Erdős, and Lovász [23] formalized this and studied certain other attributes of the set of minimal Ramsey graphs. For a graph H, let us denote the set of all minimal Ramsey graphs of H by $\mathcal{M}(H)$. Among many other parameters, they considered the minimum degree. Let us denote the minimum of the minimum degrees of all minimal Ramsey graphs of H by $s(H)$. The authors [23] were able to precisely determine this value for all cliques, which also happens to be many orders lower than the number of vertices. Moreover, they also notice that this vertex of minimum degree need not be unique in a graph. Rather one may find a minimal Ramsey graph with arbitrarily many vertices of this small degree. At this point, one may ask a question: is this phenomena observed for more graphs or is there something unique about the cliques? We will address this question in detail in Chapter 4.

A natural generalisation of this is to consider more than two colours. There have been many attempts to fully characterise the $s(K_t)$ for more colours when the graph in question is a clique. Another natural extension is to consider the asymmetric tuple of graphs. Bringing these two phenomena together, let us say that a graph G is Ramsey for a tuple $(H_1, H_2, \ldots, H_q)$ if for every q-colouring of the edges of G, there exists an $i \in [q]$ such that there is a monochromatic copy of H_i in colour i. The parameter $s(H)$ is analogously defined as $s_q(H_1, H_2, \ldots, H_q)$. Bishnoi, Boyadzhiyska, Clemens, Gupta, Lesgourgues, and Liebenau [11] considered these two generalisations together to determine the parameter s_q for the tuple consisting of cliques and cycles. There have been several other results which consider either of the two generalisations.

After the seminal work of Burr, Erdős, and Lovász [23], the parameter $s(H)$ has received much attention. Fox and Lin [45] noted that $s(H)$ is lower bounded by $2\delta(H) - 1$. This led to a notion of Ramsey simple graphs. These are the graphs whose parameter s attains this lower bound. A particular result in this direction is that of Szabó, Zumstein, and Zürcher [83] which showed that all, modulo a certain technical condition, bipartite graphs are Ramsey simple. They even conjecture that this technical condition should not be necessary and indeed all bipartite graphs must be Ramsey simple. The results on Ramsey simplicity were furthered by Grinshpun [54] in which he showed that, for the range of p in which the random graph $G(n, p)$ is asymptotically almost surely 3-connected and satisfies an extra condition, is Ramsey simple.

The properties of random graphs have been subjected to intense research. It is very natural to make them a subject for determining the parameter s_q. In Chapter 3 we will explore the Ramsey simplicity of random graphs for different ranges of p and varying values of q.

Another way to study the set of minimal Ramsey graphs is to study the set itself and not dive into characterizing the graphs within. This is to say, we would like to know if there exist two non-isomorphic graphs which have the same set of minimal Ramsey graphs. This phenomenon has been named Ramsey equivalence, and two graphs which have the same set of minimal Ramsey graphs are said to be Ramsey equivalent. There exist many instances of disconnected graphs which have the same set of minimal Ramsey graphs. A more interesting question is to determine a pair of connected graphs that are Ramsey equivalent. Fox, Grinshpun, Liebenau, Person, and Szabó [43] showed something very surprising, that no connected graph is Ramsey equivalent to a clique. In Chapter 5 we will consider the question in an asymmetric setting.

Graphs are not the only discrete structure that we study in this thesis. Consider a family of subsets of $[n]$ with the property that any two subsets in this family have a non empty intersection. It takes not long before one finds a very simple example of such a family, that is the collection of sets each with a predefined common element. Such a family is of size 2^{n-1}. Erdős, Ko, and Rado [37] showed that this is the largest possible family size . Their work was in a more general setting of k-uniform families. Such attribute of a family can be generalised to several families and also to minimum intersection size of more than one. A collection of r families of subsets of $[n]$ is said to be r-cross t-intersecting if for any selection of a set per family together intersect in at least t elements. In Chapter 6 we will determine the maximum sum of sizes of such a collection of families.

1.2 Historical background

In this section we will provide a historical perspective to questions that are addressed in the subsequent chapter of this thesis. We will begin by providing a broad perspective to Ramsey theory in Subsection 1.2.1. Thereafter we will define and see the state of the art for parameter s_q of graphs in Subsection 1.2.2. We will follow this up by a survey on Ramsey equivalence in Subsection 1.2.4 and the Ramsey simplicity of Random graphs in 1.2.3. In the last Subsection, Subsection 1.2.5 we will give an overview for the r-cross t-intersecting families.

1.2.1 Ramsey theory

A classical result of Ramsey theory states that, for every graph H, there exists an integer n such that the following property holds: For every red/blue colouring of the edges of the complete graph K_n, there exists a *monochromatic* copy of H, that is, a subgraph of K_n isomorphic to H in which all edges have the same colour. In fact, the same is true if, instead of two, we use any arbitrary number of colours. The first version of the existence of such integers was show by Ramsey [74]. Formally speaking, for any $q \in \mathbb{N}$ and a graph H, we say that a graph G is q-Ramsey for a graph H, and write $G \to_q H$, if, for any q-colouring of the edges of G, there exists a monochromatic copy of H. Then the fundamental theorem of Ramsey asserts that at least one such graph G exists for any choice of H and q. We denote the set of all such graphs for H by $\mathcal{R}_q(H)$. Through the last decades, this result has become the starting point of a field of intense studies, giving rise to a branch of combinatorics known as *Ramsey theory*. For an excellent survey on the more recent developments in the field, see [30].

One line of research is concerned with studying properties of the set of Ramsey graphs, which is the main focus of the chapters that follow. In this language, the well-known *q-colour Ramsey number* of a graph H, denoted by $r_q(H)$, can be defined as the minimum possible number of vertices in a graph that is q-Ramsey for H, i.e., $r_q(H) = \min\{v(G) : G \in \mathcal{R}_q(H)\}$. Over the years, researchers have worked hard to understand the behaviour of Ramsey numbers for various classes of graphs, which in some cases has turned out to be notoriously difficult. Perhaps the most natural example here is the clique K_t. For general t, Erdős and Szekeres [38] and Erdős [34] showed that $2^{t/2} \leq r_2(K_t) \leq 2^{2t}$, establishing that the 2-colour Ramsey number of K_t is exponential in t but leaving a large gap between the two bounds in the base of the exponent. Now, more than 70 years later, those remain essentially the best known bounds, with improvements only in the lower order terms. The current best known lower bound is due to Spencer [82]; a new upper bound was shown very recently by Sah [79], improving on the previous best known bound due to Conlon [28].

More generally, it is of interest to understand what makes a graph q-Ramsey for some chosen graph H, that is, to understand the structural properties of graphs that are q-Ramsey for H and, whenever possible, to characterize all such graphs. After considering the number of vertices, it is natural to ask about the behaviour of other graph parameters. For example, much work has been done in studying the minimum possible number of edges in a graph that is q-Ramsey for H, known as the *q-colour size-Ramsey number* of H. Formally it is written and defined as $\hat{r}_q(H) = \min\{e(G) : G \in \mathcal{R}_q(H)\}$. The parameter was introduced by Erdős, Faudree, Rousseau, Cecil, and Schelp, [39], where they noted the trivial upper bound, $\hat{r}_q(H) \leq \binom{r_q(H)}{2}$ and showed that this is indeed tight for complete graphs and two colours. It is interesting to note that this value is precisely known, of course modulo the value of Ramsey number itself. Due to [39] and Conlon, Fox, and Wigderson [32], it is known that $\hat{r}_2(K_{s,t}){=}\Theta(s^2t2^s)$ and Beck [8] had shown, $\hat{r}_2(P_n) = \Theta(n)$.

1.2.2 The parameter s_q

In the previous subsection, we considered two very natural graph parameters, namely the number of vertices and edges in the graph. Let us consider yet another parameter, the minimum degree of graphs that are q-Ramsey for a graph H. To begin with, note that asking about the smallest possible minimum degree of a graph that is q-Ramsey for H is not very interesting, as we can immediately see that the answer is zero. This is because any graph containing a q-Ramsey graph for H as a subgraph is itself q-Ramsey for H, and we can of course add an isolated vertex to obtain a graph with minimum degree zero. To avoid such trivialities, we restrict our attention to those graphs that are, in some sense, critically q-Ramsey for H. This leads to the following natural definition: We say G is *minimal q-Ramsey* for H if $G \to_q H$ and, for any proper subgraph $G' \subsetneq G$, we have $G' \not\to_q H$, that is, G loses its Ramsey property whenever we delete any vertex or edge of G. We denote the set of all minimal q-Ramsey graphs for H by $\mathcal{M}_q(H)$.

1970s saw the beginning of two prominent directions of research concerning $\mathcal{M}_q(H)$. One of the questions, first posed in [68] by Nešetřil and Rödl, was whether for a given graph H the set $\mathcal{M}_q(H)$ is finite or infinite. We call a graph H *q-Ramsey finite* (respectively *infinite*) if the set $\mathcal{M}_q(H)$ is finite (respectively infinite).

In 1978, Nešetřil and Rödl [70] showed that H is 2-Ramsey infinite in the following three cases: if H is not bipartite, if H is 2.5-connected (that is, H is 2-connected and the removal of any pair of adjacent vertices does not disconnect it), and if H is a forest containing a path of length three. In [22] the authors showed that any matching is 2-Ramsey finite. Subsequently. Burr, Erdős, Faudree, Rousseau, and Schelp [20] showed that if H is a disjoint union of non-trivial stars (i.e., all stars have at least two edges), then H is 2-Ramsey finite if and only if H is an odd star; they also showed that the disjoint union of an odd star with any number of isolated edges is 2-Ramsey-finite. Finally, the results of Rödl and Ruciński [77] imply that every graph containing a cycle is 2-Ramsey infinite. As a result, we have the following theorem.

Theorem 1.1 ([20, 22, 70, 77]). *A graph H is 2-Ramsey-finite if and only if it is the disjoint union of an odd star and any number of isolated edges.*

Around the same time, Burr, Erdős, and Lovász [23] initiated the general study of graph parameters for graphs in $\mathcal{M}_q(H)$. In their seminal paper, they considered the chromatic number, the (vertex) connectivity, and the minimum and the maximum degree of minimal

2-Ramsey graphs for the clique K_t when $t \geq 3$. In particular, they were interested in how small these parameters can be.

Surprisingly, while the 2-Ramsey number of K_t is still not known, the authors [23] could determine the mentioned values precisely. Among other results the authors showed that the minimum chromatic number amongst all Ramsey graphs for K_t is $r_2(K_t)$ and they observed that an easy corollary of this is that the minimum of the maximum degree is $r_2(K_t) - 1$. They also considered the vertex connectivity of Ramsey minimal graphs of K_t and showed that the minimum vertex connectivity is three.

Following [44], we set $s_q(H) = \min\{\delta(G) : G \in \mathcal{M}_q(H)\}$, where $\delta(G)$ denotes the minimum degree of G. When studying this parameter, there are a couple of easy general bounds one can give. For an upper bound, observe that since, by definition, $K_{r_q(H)} \to_q H$, any minimal q-Ramsey subgraph of this complete graph bears witness to the fact that $s_q(H) \leq r_q(H) - 1$. From below, as observed by Fox and Lin [45], a simple argument using the pigeonhole principle shows $s_q(H) \geq q(\delta(H) - 1) + 1$. Note that these bounds are typically very far apart: when $H = K_t$, for instance, the lower bound is linear in t while the upper bound is exponential.

One of the results that appeared in [23] establishes that $s_2(K_t) = (t-1)^2$, which is perhaps surprising, given that each graph in $\mathcal{M}_q(K_t)$ has at least exponentially many vertices. For more colours, Fox, Grinshpun, Liebenau, Person, and Szabó [44] established that $s_q(K_t) \leq 8(t-1)^6 q^3$, showing that $s_q(K_t)$ is polynomial in both t and q. Recently, this upper bound was improved by Bamberg, Bishnoi, and Lesgourgues [7] to $C(t-1)^5 q^{5/2}$. In [44] also investigated the growth of $s_q(K_t)$ as a function of q (with t being treated as a constant) and proved that $s_q(K_t) = q^2 \text{polylog}(q)$. However, a logarithmic gap remained between the lower and the upper bound. For the case of the triangle, Guo and Warnke [56] closed this gap, showing that $s_q(K_3) = \Theta(q^2 \log q)$. On the other hand, Hàn, Rödl, and Szabó [58] studied the dependence of $s_q(K_t)$ on the size of the clique with the number of colours kept constant; they showed that $s_q(K_t) = t^2 \text{polylog}(t)$.

The parameter $s_q(H)$ has also been investigated for other choices of the target graph H when $q = 2$. For instance, Szabó, Zumstein, and Zürcher [83] determined $s_2(H)$ for many interesting classes of bipartite graphs, including trees, even cycles, and biregular bipartite graphs. Later Grinshpun [54] determined $s_2(H)$ for any 3-connected bipartite graph H. A rather surprising result in this direction appeared in a paper of Fox, Grinshpun, Liebenau, Person, and Szabó [43], who studied $s_2(K_t \cdot K_2)$, where $K_t \cdot K_2$ is the graph obtained from a clique of size t by adding a new vertex and connecting it to exactly one vertex of the clique. We will call such a graph a *clique with a pendant edge*. The authors proved that $s_2(K_t \cdot K_2) = t - 1$, showing that even a single edge can significantly change the value of the parameter s_2.

The parameter $s_q(H)$ has also received, very recently, some renewed interest in the asymmetric setting. A graph G is said to be *q-Ramsey* for a q-tuple of graphs $(H_1, \ldots, H_q)$, denoted by $G \to_q (H_1, \ldots, H_q)$, if every q-edge-colouring of G contains a monochromatic copy of H_i in colour i, for some $i \in [q]$. All parameters are analogously defined. In their introductory paper, Burr, Erdős, and Lovász [23] had in fact considered the cliques in an asymmetric setting. They had shown that $s_2(K_t, K_\ell) = (t-1)(\ell-1)$. Bishnoi, Boyadzhiyska, Clemens, Gupta, Lesgourgues, and Liebenau [11] revived this area and showed, in particular, that $s_2(K_t, C_\ell) = 2(t-1)$ for all $t \geq 3$ and $\ell \geq 4$, where C_ℓ represents the cycle of length ℓ. It is a rather surprising result because the values is independent of the length of the cycle, whereas we know $r_2(K_t, C_\ell) = (t-1)(\ell-1)+1$ for $\ell = \Omega(\log t/\log\log t)$ due to Keevash, Long, and Skokan [62]. The authors [11] also considered the case for more colours.

Theorem 1.2. *For all $\ell \geq 4$, $t \geq 3$, and all $q, q_1, q_2 \geq 1$ such that $q_1 + q_2 = q$, we have*

$$s_{q_2}(K_t) + q_1 \leq s_q((\underbrace{C_\ell, \ldots, C_\ell}_{q_1 \text{ times}}, \underbrace{K_t, \ldots, K_t}_{q_2 \text{ times}})) \leq s_q(K_t). \tag{1.1}$$

1.2.3 Random graphs and Ramsey simplicity

Notice that for all the cases that have been studied the value of $s_q(H)$ is far away from the trivial upper bound. On the other hand, the lower bound of Fox and Lin [45] has been shown to be tight for many graphs. Following Grinshpun [54], we call such a graph *q-Ramsey simple.*

Definition 1.3. A graph H without isolated vertices is said to be *q-Ramsey simple* if

$$s_q(H) = q(\delta(H) - 1) + 1.$$

If H has isolated vertices, then we say that H is q-Ramsey simple if the graph obtained from H by removing all isolated vertices is q-Ramsey simple.

Observe that adding isolated vertices to a graph does not affect the structure of the corresponding Ramsey graphs significantly. Indeed, if H is a graph without isolated vertices and $H + tK_1$ is the graph obtained from H by adding $t \geq 0$ isolated vertices, it is not difficult to check that $G \in \mathcal{M}_q(H)$ if and only if $G + sK_1 \in \mathcal{M}_q(H + tK_1)$, where $s = \max\{0, t - (v(G) - v(H))\}$.

Previous work by Fox and Lin [45], Szabó, Zumstein, and Zürcher [83], and Grinshpun [54] has established the 2-Ramsey simplicity of a wide range of bipartite graphs. Further results were proven in [18], including the q-Ramsey simplicity of all cycles of length at least four, for any number of colours $q \geq 2$. Based on these results, it is believed that simplicity is a more widespread phenomenon.

Conjecture 1.4 (Szabó, Zumstein, and Zürcher [83])**.** *Every bipartite graph is 2-Ramsey simple.*

The conjecture suggests that Ramsey simplicity is quite common, but it is natural to wonder whether this extends beyond the bipartite setting, given that we know cliques are not simple. Are cliques an exceptional case, or is q-Ramsey simplicity atypical for non-bipartite graphs? In somewhat more precise terms, when can we expect the n-vertex binomial random graph $G(n, p)$, where every edge appears independently with probability p, to be q-Ramsey simple?

Random graphs have long played an important role in Ramsey Theory: Erdős' famous exponential lower bound on the Ramsey numbers of complete graphs in [34] came from analysing the clique and independence numbers of random graphs, while a key ingredient in the modern upper bounds is showing that large Ramsey graphs must be random-like. In another setting, the work of Rödl and Ruciński [76, 77] establishes, for a given graph

H and number of colours q, the range of values of p for which we have $G(n,p) \to_q H$ with high probability.

In these seminal papers, which have inspired a great deal of subsequent research, the random graph plays the role of the host graph G, while the target graph H is fixed in advance. Surprisingly, there has been considerably less work in the setting where the target graph H is itself random. When $H \sim G(n,p)$, Fox and Sudakov [46] and Conlon [29] provide some lower and upper bounds on $r_2(H)$ for different ranges of p, while Conlon, Fox, and Sudakov [31] show that $\log r_2(H)$ is well-concentrated.

In Chapter 3 we shall focus on the minimum degree of Ramsey graphs for the random graph $G(n,p)$, for various ranges of p, with the goal of determining when it is q-Ramsey simple. This line of research was initiated by Grinshpun [54], who proved that sparse random graphs are 2-Ramsey simple with high probability.

Theorem 1.5 (Corollary 2.1.4 in [54])**.** *Let $p = p(n) \in (0,1)$ and $H \sim G(n,p)$. If $\frac{\log n}{n} \ll p \ll n^{-2/3}$, then a.a.s. H is 2-Ramsey simple.*

In this range of edge probabilities the random graph is almost surely not bipartite (in fact, its chromatic number is unbounded), showing that the solution to Conjecture 1.4 would not tell the full story. Moreover, the argument in [54] can easily be extended to provide, for any fixed $q \in \mathbb{N}$, q-Ramsey simplicity for $G(n,p)$ in the above range of p. This begs two natural questions: what happens when the number of colours q grows with n, and what happens in other ranges of the edge probability p?

1.2.4 Ramsey equivalence

Recall that $s_2(K_t) = (t-1)^2$ and $s_2(K_t \cdot K_2) = (t-1)$. These results imply that there exists a 2-Ramsey graph for $K_t \cdot K_2$ that is not 2-Ramsey for K_t due to its low minimum degree.

A natural question to ask is: how does the value of s_2 change when we modify the target graph H slightly? More generally, how does the collection of Ramsey graphs change? This question motivated Szabó, Zumstein, and Zürcher [83] to define the notion of Ramsey equivalence. Two graphs H and H' are *Ramsey equivalent*, denoted by $H \sim H'$, if $\mathcal{R}_2(H) = \mathcal{R}_2(H')$.

It is not difficult to show that Ramsey equivalent pairs of graphs exist: for instance, the graph obtained by adding an isolated vertex to the clique K_t for $t \geq 3$ is Ramsey equivalent to K_t. Szabó, Zumstein, and Zürcher [83] further found that for $t \geq 4$, the graph K_t along with a disjoint edge is Ramsey equivalent to K_t. A few years later, Bloom and Liebenau [12] in fact showed that for $t \geq 4$, the disjoint union of K_t and K_{t-1} is Ramsey equivalent to K_t, which was later generalised by Reding [75] to show that this is indeed true for all $t \geq 3$ whenever $q \geq 3$. See [43] for further results in this direction.

As of now we have only seen examples of disconnected graphs being Ramsey equivalent to a clique. Szabó, Zumstein, and Zürcher [83] had indeed asked whether there exists any *connected* such graphs. Perhaps surprisingly, several years later Fox, Grinshpun, Liebenau, Person, and Szabó [43] settled this question in the negative: they showed that no connected graph is Ramsey equivalent to K_t. In light of this result, they raised the following question:

Question 1.6 ([43])**.** *Is there a pair of non-isomorphic connected graphs that are Ramsey equivalent?*

The above question is wide open. Not much is known even in the special case where the two graphs differ only by a pendent edge. It was shown by Clemens, Liebenau, and Reding [27] that no pair of 3-connected graphs can be Ramsey equivalent. A result from Grinshpun [54, Lemma 2.6.3.] allows us to show non-equivalence for further pairs consisting of a graph H and the graph H with a pendent edge. Further evidence that the answer to Question 1.6 might be negative is provided for example in [5, 80]. Axenovich, Rollin, and Ueckerdt [5] investigated the question with respect to the chromatic numbers of the two graphs in question, and eventually, Savery [80] showed that two graphs with different chromatic numbers cannot be Ramsey equivalent.

1.2.5 r-cross t-intersecting families of sets

We now move on to extremal set theory. One of the main themes here are the intersecting families. Given some $n \in \mathbb{N}$, a family $\mathcal{F} \subseteq \mathscr{P}[n]$ is said to be *intersecting* if for all $F, F' \in \mathcal{F}$ we have $F \cap F' \neq \emptyset$. Moreover, we may require that the family is k-uniform, that is all sets are of size k. We denote this by $\binom{[n]}{k}$ or equivalently $[n]^k$. The following well-known theorem by Erdős, Ko, and Rado (see e.g. [37]) is one of the earliest results in extremal set theory.

Theorem 1.7. *Let $k, n \in \mathbb{N}$ with $2k \leq n$ and let $\mathcal{F} \subseteq [n]^{(k)}$ be an intersecting family. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.*

Observe that this maximum is attained by a family which contains all the sets of size k that contain one fixed element, for instance $\mathcal{F} = \{F \in [n]^{(k)} \colon 1 \in F\}$.

There are several variations and generalisations of this result. One generalisation is to require that the intersections need to be of size at least $t \geq 1$. A family $\mathcal{F} \subseteq \mathscr{P}([n])$ is said to be t-intersecting if for all $F, F' \in \mathcal{F}$ we have $|F \cap F'| \geq t$ and again one can ask for the maximal size of a t-intersecting family. Similarly a family $\mathcal{F} \subseteq \mathscr{P}([n])$ is said to be t-intersecting k-uniform if we require that $|F| = k$ for every $F \in \mathcal{F}$. Theorem 1.7 can be generalised to this setting to show that, for $k \geq t$ and n large enough, $|\mathcal{F}| \leq \binom{n-t}{k-t}$, and this bound is attained by a family which contains all sets of size k that contain t fixed elements, for instance $\mathcal{F} = \{F \in [n]^{(k)} \colon [t] \subseteq F\}$. After some progress by several researchers, see, for instance, [35, 37, 47, 48, 86], Ahlswede and Khachatrian [2, 4] determined the maximum size of t-intersecting k-uniform family for all values of n when $n \geq k \geq t$.

A further variation are *cross intersecting* families. For $r, t, n \in \mathbb{N}$ we say that families $\mathcal{F}_1, \ldots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ are *r-cross t-intersecting* if for all $F_1 \in \mathcal{F}_1, \ldots, F_r \in \mathcal{F}_r$ we have $|\bigcap_{i \in [r]} F_i| \geq t$. We only consider non-empty r-cross t-intersecting families. If $r = 2$ or $t = 1$, we may omit the respective parameter, e.g., 2-cross 1-intersecting families $\mathcal{F}_1, \mathcal{F}_2$ are simply called cross intersecting. In this regime there are several partial results concerning the maximum sum of sizes of r-cross t-intersecting families for specific instances of r and t, starting with theorems by Hilton [60] and by Hilton and Milner [61]. In [61], the authors showed that of $\mathcal{F}_1$ and $\mathcal{F}_2$ are k-uniform intersecting families then $|\mathcal{F}_1| + |\mathcal{F}_2| \leq \binom{n}{k} - \binom{n-1}{k} + 1$ for $k \geq 2$ and $n \geq 2k$.

Another generalisation considers the use of measures (or weights) to calculate the size of families, where we sum the weights of the sets in a family instead of simply counting the number of sets in the family. Formally, consider a function $\mu : \mathscr{P}([n]) \to \mathbb{R}_{\geq 0}$, which assigns a weight to each set in $\mathscr{P}([n])$, also called a measure function. We define the measure of a family $\mathcal{F} \subseteq \mathscr{P}([n])$ as $\mu(\mathcal{F}) = \sum_{F\in\mathcal{F}} \mu(F)$. Even though we use the same variable to denote the measure of a set and a family, the notation will be clear from context.

Two commonly considered measures are the product measure ϱ_p and the uniform measure ν_k which we define following [51]. For a fixed $p \in [0,1]$, the product measure of set a set $F \in \mathscr{P}[n]$ is $\varrho_p(F) = p^{|F|}(1-p)^{n-|F|}$. Note that this can be interpreted as the probability that a specific set F is the result of a random experiment that includes each element from $[n]$ with probability p in F. The uniform measure ν_k, with $k \in [n]$, is defined as

$$\nu_k(F) = \begin{cases} 0 \text{ if } |F| \neq k \\ \frac{1}{\binom{n}{k}} \text{ if } |F| = k \end{cases} .$$

For these measures, analogues of the Erdős-Ko-Rado theorem can be considered. Indeed, we can reformulate Theorem 1.7 as follows: For $k, n \in \mathbb{N}$ with $2k \leq n$ and an intersecting family $\mathcal{F} \subseteq [n]^{(k)}$ it follows that $\nu_k(\mathcal{F}) \leq \frac{k}{n}$. For the product measure the following analogous result was first proved in [1] that for $p \leq 1/2$ and an intersecting family $\mathcal{F} \subseteq \mathscr{P}([n])$ we have $\varrho_p(\mathcal{F}) \leq p$. Despite the fact that the maximum is attained by the same family as in Theorem 1.7 the proofs for both statements are different and independent. In [51], Frankl and Tokushige ask for a general theorem that includes both theorems as special cases.

Several results for specific measures and t-intersecting families are known, in particular a product measure version of the Ahlswede-Khachatrian theorem concerning t-intersecting families, see [3, 10, 84, 33, 40, 41]. For a more thorough overview we recommend Chapter 12 in [51]. Another related result due to Borg [14] determines the maximum product of measures of two cross t-intersecting families.

1.3 Main results of this thesis

In this section we give an overview of the main results that will be presented in the subsequent chapters.

1.3.1 Ramsey simplicity of random graphs

In Chapter 3, we investigate the s_q value for random graphs for a wide range of parameters.

Let us begin by considering the range of p for which $G(n,p)$ is almost surely a forest. Szabó, Zumstein, and Zürcher[83] had shown that a certain class of bipartite graphs, which in particular include all trees are 2-Ramsey simple. Their result is easily generalisable to more colours. As a consequence we have that for $0 < p \ll n^{-1}$, the graph $G(n,p)$ is almost surely q-Ramsey simple for all $q \geq 2$.

On the other end of the spectrum we consider the range $\left(\frac{\log n}{n}\right)^{1/2} \ll p < 1$. In this range, $G(n,p)$ exhibits a very interesting property, namely that every edge is in a triangle with high probability. See Section 2.3 for more details. This in fact helps us show that for $p \gg \sqrt{\frac{\log n}{n}}$, the graph $H \sim G(n,p)$ is never q-Ramsey simple for any value of q.

This leads to the question, does a random graph always exhibit either of the two behaviours? In the following discussion we will observe that this is not the case. In fact the observed behaviour is very interesting.

We first remark that the parameter $s_q(H)$ and the notion of Ramsey simplicity are not monotone in the graph H. As we shall observe in the chapter, a Ramsey simple graph can have both subgraphs and supergraphs that are themselves not Ramsey simple, while a graph that is not Ramsey simple can have simple subgraphs and supergraphs. However, we do have monotonicity in the number of colours q, and we shall demonstrate in Lemma 3.5 that $(q+1)$-Ramsey simplicity implies q-Ramsey simplicity. Hence, we can ask for a threshold value for q, i.e., the largest number $\tilde{q}$ of colours for which a given graph is $\tilde{q}$-Ramsey simple.

Definition 3.1. $\tilde{q}(H) = \sup\{q : \ H \text{ is } q\text{-Ramsey simple}\}$.

Note that every graph is, by definition, 1-Ramsey simple, since the only minimal 1-Ramsey graph for H is H itself, and so $s_1(H) = \delta(H)$. Thus, when a graph H is not q-Ramsey simple for any number of colours $q \geq 2$, we have $\tilde{q}(H) = 1$. At the other extreme, if H is q-Ramsey simple for any number of colours q, we have $\tilde{q}(H) = \infty$. Based on our previous discussion, we can state the following theorem.

Theorem 3.2. *Let $p = p(n) \in (0,1)$ and $H \sim G(n,p)$. Then a.a.s. the following holds:*

(a) $\tilde{q}(H) = \infty$ *if* $0 < p \ll n^{-1}$.

(b) $\tilde{q}(H) = 1$ *if* $\left(\frac{\log n}{n}\right)^{1/2} \ll p < 1$.

Given the new notation, we can now state the behaviour of q-Ramsey simplicity in the intermediate range, which collects various bounds we were able to prove for the threshold $\tilde{q}(H)$ when $H \sim G(n,p)$.

Theorem 3.6. *Let $p = p(n) \in (0,1)$ and $H \sim G(n,p)$. Let $u \in V(H)$ be a vertex of minimum degree $\delta(H)$ and let $F = H[N(u)]$ be the subgraph of H induced by the neighbourhood of u. Denote by $\lambda(F)$ the order of the largest connected component in F. Then a.a.s. the following bounds hold:*

(a) $\tilde{q}(H) = \infty$ *if* $\frac{\log n}{n} \ll p \ll n^{-\frac{2}{3}}$.

(b) $\tilde{q}(H) \geq (1+o(1)) \max\left\{\frac{\delta(H)}{\lambda(F)^2}, \frac{\delta(H)}{80 \log n}\right\}$ *if* $n^{-\frac{2}{3}} \ll p \ll n^{-\frac{1}{2}}$.

(c) $\tilde{q}(H) \leq (1+o(1)) \min\left\{\frac{\delta(H)}{\Delta(F)}, \frac{\delta(H)^2}{2e(F)}\right\}$ *if* $n^{-\frac{2}{3}} \ll p \ll 1$.

As shown above, we extend Theorem 1.5 by showing that these sparse random graphs are not just q-Ramsey simple for any fixed q, but even when the number of colours q is allowed to grow with n. Most interestingly, though, the simplicity threshold for random graphs of intermediate density depends on some parameters of the random graph itself — these graphs are q-Ramsey simple for small values of q, but not when q grows too large.

Remark 1.8. As suggested by the above bounds, this dependence on q is governed by the subgraph F, and it is the appearance of edges in F that gives rise to a finite bound on $\tilde{q}(H)$. When $p \ll n^{-\frac{2}{3}}$, then F almost surely has no edges, while if $p \gg n^{-\frac{2}{3}}$, then F almost surely does, explaining the distinction between cases (a) and (c). When $p = \Theta\left(n^{-\frac{2}{3}}\right)$, then F is empty (and $\tilde{q}(H)$ thus infinite) with probability bounded away from 0 and 1.

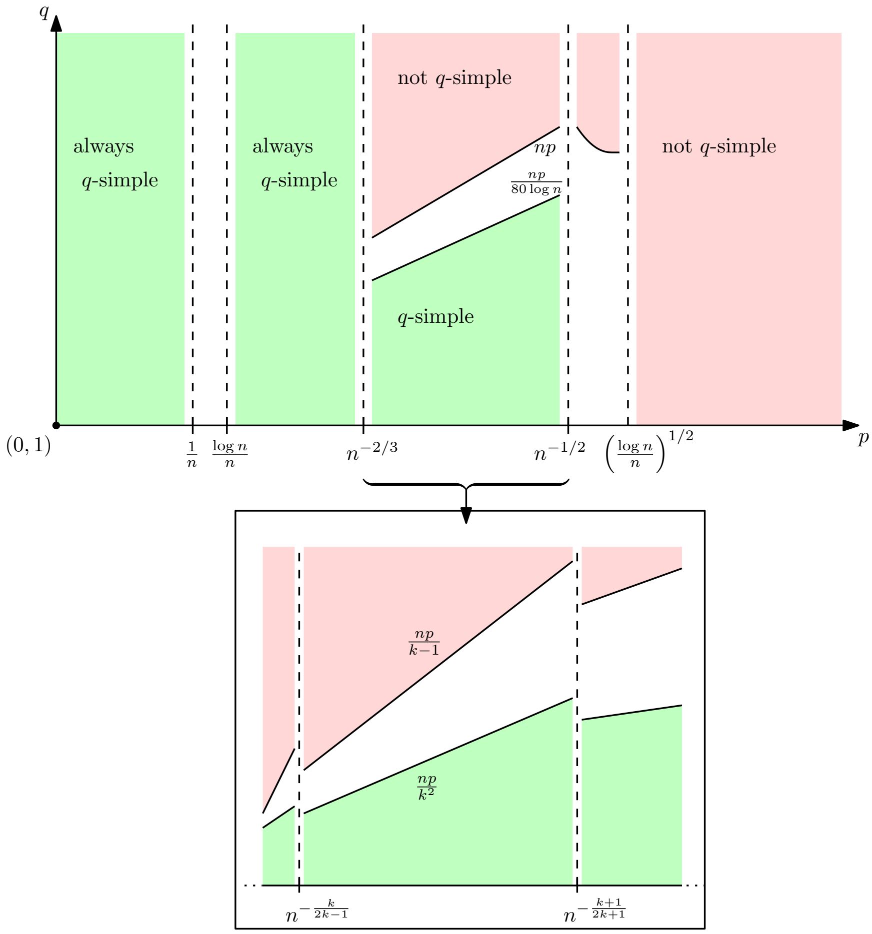

Figure 1.1: Bounds on the simplicity threshold $\tilde{q}(G(n,p))$

When $n^{-\frac{2}{3}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{2}}$, by analysing the structure of random graphs, we can give quantitative estimates for the bounds on $\tilde{q}(H)$ in this intermediate range.

In Subsection 3.3.3 we will collect the necessary random graph theoretic results to obtain the following estimates as a consequence of Theorem 3.6.

Corollary 3.16. *Let $k \geq 2$ be a fixed integer and let $f = f(n)$ satisfy $1 \ll f = n^{o(1)}$. Let $p = p(n)$ satisfy $n^{-\frac{2}{3}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{2}}$ and let $H \sim G(n,p)$. Then a.a.s. the following bounds hold:*

(a) if $n^{-\frac{k}{2k-1}} \ll p \ll n^{-\frac{k+1}{2k+1}}$, then $(1+o(1))\frac{np}{k^2} \leq \tilde{q}(H) \leq (1+o(1))\frac{np}{k-1}$.

(b) if $p = \Theta\left(n^{-\frac{k+1}{2k+1}}\right)$, then $(1+o(1))\frac{np}{(k+1)^2} \leq \tilde{q}(H) \leq (1+o(1))\frac{np}{k-1}$.

(c) if $p = n^{-\frac{1}{2}}f^{-1}$, then
$$(1+o(1))\frac{np}{\log n}\max\left\{\frac{16\log^2 f}{\log n}, \frac{1}{80}\right\} \leq \tilde{q}(H) \leq (2+o(1))\frac{np\log(f^2\log n)}{\log n}.$$

(d) if $n^{-\frac{1}{2}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{2}}$, then $1 \leq \tilde{q}(H) \leq (8+o(1))\frac{1}{p}$.

Corollary 3.16 shows that, for fixed $\varepsilon > 0$ and $n^{-\frac{2}{3}} \ll p \ll n^{-\frac{1}{2}-\varepsilon}$, we determine the threshold up to a constant factor, while for $n^{-\frac{1}{2}-o(1)}$, we know it up to a polylogarithmic factor. Most surprisingly, these bounds reveal that the threshold $\tilde{q}(H)$ evolves in a complicated fashion: while it drops from ∞ to 1 as p ranges from $\frac{\log n}{n}$ to 1, it does not do so in a monotone fashion, as it must increase in the ranges $p \in \left(n^{-\frac{k}{2k-1}}, n^{-\frac{k+1}{2k+1}}\right)$ for each fixed k. These results are illustrated in Figure 1.1.

1.3.2 Abundance

In Subsection 1.2.2 we saw that a minimal q-Ramsey graph for a given H can contain a vertex of small degree. A natural next question now is, *how many* vertices of this small degree can a minimal q-Ramsey graph for H contain? More specifically, can a minimal q-Ramsey graph have arbitrarily many vertices of the smallest possible minimum degree? This question motivates the following definition.

Definition 4.1. For a given integer $q \geq 2$, a graph H is said to be *s_q-abundant* if, for every $k \geq 1$, there exists a minimal q-Ramsey graph for H with at least k vertices of degree $s_q(H)$.

As it turns out, it is not immediate whether s_q-abundant graphs exist at all. As stated earlier, in [23], Burr et al. noted that their construction can be generalised to show that cliques are s_2-abundant. In Chapter 4, we will give several examples showing that, for all $q \geq 2$, there are infinitely many s_q-abundant graphs.

It is not hard to see that, if a graph is q-Ramsey finite, then it cannot be s_q-abundant. This immediately implies that odd stars are not s_2-abundant. On the other hand, we know that even stars *are* 2-Ramsey infinite, but as we will see below they are also not s_2-abundant. This statement follows from the following result.

Theorem 1.9 ([23])**.** *Let $m \geq 1$ be an integer. Then a connected graph G is 2-Ramsey for $K_{1,m}$ if and only if either $\Delta(G) \geq 2m-1$ or m is even and G is a $(2m-2)$-regular graph on an odd number of vertices.*

The theorem immediately implies that $\mathcal{M}_2(K_{1,m}) = \{K_{1,2m-1}\}$ whenever m is odd and $\mathcal{M}_2(K_{1,m}) = \{K_{1,2m-1}\} \cup \{G : G \text{ is connected and } (2m-2)\text{-regular and } |V(G)| \text{ is odd}\}$ if m is even. In particular, this implies that no star is s_2-abundant.

More generally, it turns out that stars are not s_q-abundant for any $q \geq 2$: A simple argument implies that, for any $m \geq 1$ and $q \geq 2$, a minimal q-Ramsey graph for $K_{1,m}$ has either zero or $q(m-1)+1$ vertices of degree one. Indeed, if G is a minimal q-Ramsey graph for $K_{1,m}$ that is not isomorphic to $K_{1,q(m-1)+1}$, then the maximum degree of G is at most $q(m-1)$. Thus, if G contains a vertex v of degree one, then the only neighbour u of v has at most $q(m-1)-1$ other neighbours. By the minimality of G, the graph $G-v$ has a q-colouring c without a monochromatic copy of $K_{1,m}$. Since u has at most $q(m-1)-1$ neighbours in $G-v$, there is a colour that appears at most $m-2$ times on the edges incident to u. Then this colour can be used on the edge uv to extend c to a q-colouring of G without a monochromatic copy of $K_{1,m}$, leading to a contradiction. Hence, G cannot contain a vertex of degree one.

One of the goals of Chapter 4 is to systematically study s_q-abundance. First, we show that all cycles of length at least four are s_q-abundant. As a by-product, we determine $s_q(C_t)$ for all $q \geq 2$ and $t \geq 4$.

Theorem 4.10. *For any given integers $q \geq 2$, $t \geq 4$, and $k \geq 1$, there exists a minimal q-Ramsey graph for C_t that has at least k vertices of degree $q+1$. In particular, $s_q(C_t) = q+1$ and C_t is s_q-abundant.*

We will then go on to show that a clique with a pendant edge is s_2-abundant. We note that, since $s_2(K_t) = (t-1)^2$ and $s_2(K_t \cdot K_2) = t-1$ for all $t \geq 3$, Theorem 4.11 also yields that there are infinitely many graphs that are minimal 2-Ramsey for $K_t \cdot K_2$ but not minimal 2-Ramsey for K_t. One of the main building blocks used in our construction is not known to exist for $K_t \cdot K_2$ when $q > 2$, which is why we focus on the case $q = 2$.

Theorem 4.11. *For a given integer $t \geq 3$, the graph $K_t \cdot K_2$ is s_2-abundant.*

Finally, we will also show that wheel graphs are s_2-abundant. Let W_n denote the wheel graph on $n+1$ vertices. We will also determine the value of $s_2(W_n)$.

Theorem 4.15. *For a given integer $t \geq 4$, the graph W_t is s_2-abundant and $s_2(W_t) = 7$.*

We will also consider the $H \sim G(n,p)$ for certain ranges of p when H is a.a.s is 3-connected. We will show the following result.

Theorem 4.18. *Let $p = p(n) \in (0,1)$ and $H \sim G(n,p)$. Then a.a.s. H is s_q-abundant for all $q \geq 2$ whenever $\frac{\log n}{n} \ll p \ll n^{-\frac{2}{3}}$ and H is s_q-abundant for all $q \leq \tilde{q}(H)$ whenever $n^{-\frac{2}{3}} \ll p \ll n^{-\frac{1}{2}}$.*

In order to prove the statements above, we will use gadget graphs that we call *pattern gadgets*. These were proven to exist for cycles by Siggers [81] and they generalised other well-known gadgets such as signal senders, originally developed by Burr, Erdős, and Lovász [23] to study $s_2(K_t)$. Pattern gadgets help us construct minimal Ramsey graphs with many vertices of small degree.

In a nutshell, the main idea behind pattern gadgets is the following: Given some graph G and some family $\mathscr{G}$ of colourings of $E(G)$ with q colours that do not contain monochromatic copies of H, we will find some larger graph P containing G such that the colourings in $\mathscr{G}$ are exactly those colourings of G that can be extended to P without creating a monochromatic copy of H. Then, in order to prove each of the above theorems, we will choose G and $\mathscr{G}$ in such a way that we can attach k small-degree vertices to $G \subseteq P$ so that no colouring in $\mathscr{G}$ can be extended to the new edges without creating a monochromatic copy of H, but if we remove any of these new vertices, we can find a colouring in $\mathscr{G}$ that can be extended in the desired way.

The precise definition of a pattern gadget will be given in Chapter 4. We will show their existence for many target graphs H, including all 3-connected graphs in Chapter 4.

1.3.3 Ramsey equivalence for asymmetric pairs of graphs

In Chapter 5, we study Ramsey equivalence in the asymmetric setting and explore this variant of Question 1.6.

Recall that we denote the collection of all Ramsey graphs for (G,H) by $\mathcal{R}(G,H)$.

Definition 5.1. We call two pairs of graphs (G,H) and (G',H') *Ramsey equivalent*, denoted $(G,H) \sim (G',H')$, if $\mathcal{R}(G,H) = \mathcal{R}(G',H')$.

Equivalently we can define Ramsey equivalence also in terms of the corresponding Ramsey-minimal graphs, requiring that $\mathcal{M}(G,H) = \mathcal{M}(G',H')$. Our goal is to explore the notion of Ramsey equivalence for asymmetric pairs of connected graphs and in particular the asymmetric version of Question 1.6. Previously known results allow us to exclude some potential candidates. Let $\omega(G)$ denote the clique number of a graph G, defined as the largest integer n such that K_n is a subgraph of G. A famous result of Nešetřil and Rödl [69] establishes that, for every graph G, there is a Ramsey graph for G that has the same clique number as G. Hence, the disjoint union of G and H has a Ramsey graph F with clique number $\max\{\omega(G), \omega(H)\}$ and this graph F is also a Ramsey graph for (G,H). This gives the following statement which we shall use several times in our proofs.

Theorem 1.10 ([69]). *Each pair (G,H) of graphs has a Ramsey graph with clique number equal to* $\max\{\omega(G), \omega(H)\}$.

This result implies that, whenever $(G,H) \sim (G',H')$, it holds that $\max\{\omega(G), \omega(H)\} = \max\{\omega(G'), \omega(H')\}$. As a second example, Savery [80, Section 3.1] proved that $(G,H) \not\sim (G',H')$ for all graphs G, H, G', and H' with $\chi(G) + \chi(H) \neq \chi(G') + \chi(H')$, where $\chi(F)$ denotes the chromatic number of a graph F.

It turns out, however, that the asymmetric version of Question 1.6 has an affirmative answer. Let $K_{1,s}$ denote a star with s edges. Through a simple application of Petersen's Theorem [72], Burr, Erdős, Faudree, Rousseau, and Schelp [20] showed that, for any odd integers r, $s \geq 1$, the only Ramsey-minimal graph for the pair of stars $(K_{1,r}, K_{1,s})$ is the star $K_{1,r+s-1}$. Thus, any odd integers $r, s, r', s' \geq 1$ with $r + s = r' + s'$ satisfy $(K_{1,r}, K_{1,s}) \sim (K_{1,r'}, K_{1,s'})$. This example is perhaps not very satisfying, as pairs of odd stars have only a single Ramsey-minimal graph. It is then interesting to ask whether there are any Ramsey equivalent pairs of connected graphs with a larger, maybe even an infinite number of Ramsey-minimal graphs. We will later show that the answer is yes, exhibiting an infinite family of Ramsey equivalent pairs of connected graphs of the form $(T, K_t) \sim (T, K_t \cdot K_2)$, where T is a certain kind of tree.

In light of the discussion in the previous paragraph, one might ask whether there exist any other pairs of stars that are Ramsey equivalent. In the following result, we answer this question negatively. Note that $\mathcal{M}(K_{1,r}, K_{1,s})$ is infinite whenever rs is even [20].

Theorem 5.4. *Let a, b, x, y be positive integers with $\{a,b\} \neq \{x,y\}$. Then $(K_{1,a}, K_{1,b}) \sim (K_{1,x}, K_{1,y})$ if and only if $a + b = x + y$ and a, b, x, and y are odd.*

Note that each pair of stars has a Ramsey graph that is a star. Since stars cannot be Ramsey graphs for other connected graphs than (smaller) stars, pairs of stars are not Ramsey equivalent to pairs of connected graphs that are not star pairs.

We next study Ramsey equivalence for pairs of the form (T, K_t), where T is a tree and $t \geq 3$. Note that in the case where T is a single vertex or edge the collection of Ramsey graphs is trivial, as $\mathcal{M}(K_1, K_t) = \{K_1\}$ and $\mathcal{M}(K_2, K_t) = \{K_t\}$. From now on, unless otherwise specified, we will assume that T has at least two edges. It was shown by Łuczak [64] that in this case $\mathcal{M}(T, K_t)$ is infinite. Perhaps surprisingly, we find non-trivial Ramsey equivalent pairs in this setting. To describe some of those pairs, we need the following definitions. For integers $a \geq 1$, $b \geq 2$, and $t \geq 3$ with $a \leq t$, let $K_t \cdot aK_b$ denote the graph consisting of a copy of K_t and a pairwise vertex-disjoint copies of K_b, each sharing exactly one vertex with the copy of K_t (see Figure 1.2 left for an example).

Figure 1.2: The graph $K_6 \cdot 2K_3$ (left) and the largest 3-suitable caterpillar (right).

We call a tree T an *(s-)suitable caterpillar*, if T consists of a path P on three vertices and up to $3s-1$ further vertices of degree 1 such that the endpoints of P are of degree exactly $s+1$ in T and the middle vertex of P is of degree at most $s+1$ in T (see Figure 1.2 right).

Theorem 5.5. *(a) For all integers $s \geq 2$ and $t \geq 3$, we have $(K_{1,s}, K_t) \sim (K_{1,s}, K_t \cdot K_2)$.*

(b) Let $a \geq 1$ and $b \geq 2$ be integers, and let T be a star with at least two edges or a suitable caterpillar. For any sufficiently large t, we have $(T, K_t) \sim (T, K_t \cdot aK_b)$.

Observe that the first part of the above theorem holds for each $t \geq 3$, while we need a sufficiently large t to prove the second part. Our proof shows that we can take t to be quadratic in the parameters a, b and $|V(T)|$. Since we do not have a non-trivial lower bound, we make no effort to optimize this value, choosing to present a simpler proof instead.

We complement the equivalence result above by proving Ramsey non-equivalence for several other families of pairs of trees and cliques. Theorem 1.10 shows that we may restrict our attention to pairs (G, H) with $\max\{\omega(G), \omega(H)\} = t$, since otherwise $(G, H) \not\sim (T, K_t)$. Let $\mathcal{T}$ denote the class of all trees T of diameter at least three such that:

- if $\mathrm{diam}(T)$ is even, the neighbours of the central vertex of T are of degree at most two, and
- if $\mathrm{diam}(T) = 4$, the central vertex is of degree at least 3.

Note in particular that this class contains all trees of odd diameter.

Theorem 5.5 above shows that for some trees T certain modifications to the pair (T, K_t) yield a Ramsey equivalent pair. Specifically, the Ramsey graphs do not change when we attach certain disjoint pendent graphs to the second component of the pair, that is, at the vertices of K_t. The first part of the following theorem states that this behaviour does not generalise to trees from the family $\mathcal{T}$ defined above in a strong sense: for each tree $T \in \mathcal{T}$ and each $t \geq 3$, we have $(T, K_t) \not\sim (T, K_t \cdot K_2)$. The second part shows that in order to obtain a Ramsey equivalent pair (like in Theorem 5.5(b)) it is necessary that the graphs attached to different vertices of K_t do not intersect. Finally, we consider modifications to the first component of the pair, namely T. If T and T' are trees of different sizes, we have $(T, K_t) \not\sim (T', K_t)$, since the Ramsey numbers of these pairs differ, as shown by Chvátal [26]. The third part of the theorem below shows that T cannot be replaced by *any* other connected graph G if the second component of the pair stays unchanged.

Theorem 5.9. *Let T be a tree, $t \geq 3$ be an integer, and let G and H be graphs with $(G, H) \neq (T, K_t)$. Then $(T, K_t) \not\sim (G, H)$ if one of the following conditions holds:*

(a) $G = T$, $T \in \mathcal{T}$, and H is connected,

H \ G	$\omega(H) \neq t$	$H = K_t$	$H \supsetneq K_t$ and H-$E(K_t)$ has t components	$H \supsetneq K_t$ and H-$E(K_t)$ has $< t$ components
$G = T$	$\not\sim$ ([69])	$\sim$ (triv.)	$\sim$ for some H if T star or suit. cat. (5.5) $\not\sim$ if $T \in \mathcal{T}$ (5.9(a)) otherwise partial results	$\not\sim$ (5.9(b))
$G \neq T$	$\not\sim$ ([69])	$\not\sim$ (5.9(c))	open	$\not\sim$ (5.9(b))

Table 1.1: Known results about Ramsey equivalence between (T, K_t) and (G, H) where $t \geq 3$, T is a tree, G and H are connected graphs, and $\omega(G) \leq \omega(H)$. Each entry states whether the respective pairs (T, K_t) and (G, H) are Ramsey equivalent or not.

(b) H contains a copy K of K_t, and H contains a cycle with vertices from both $V(K)$ and $V(H) \setminus V(K)$,

(c) $G \neq T$, G is connected, and $H = K_t$.

Note that the Ramsey number alone is not sufficient to distinguish certain pairs (T, K_t) and (G, K_t): for example, Keevash, Long, and Skokan [62] showed that when $\ell = \Omega\left(\frac{\log t}{\log \log t}\right)$ the Ramsey numbers of (C_ℓ, K_t) and (T_ℓ, K_t) are the same, where C_ℓ and T_ℓ denote a cycle and a tree on ℓ vertices, respectively.

As we will see in Chapter 5, our construction actually allows us to prove the statement from the first part of the theorem above for a larger class of trees. Again, since our results do not lead to a complete characterization of those trees T for which $(T, K_t) \sim (T, K_t \cdot K_2)$, we choose to state the simpler, albeit somewhat weaker, result here. As a specific example, note that our results imply that for sufficiently large t and a path P we have $(P, K_t) \sim (P, K_t \cdot K_2)$ if and only if P has two or four edges.

In this thesis, we study what pairs of connected graphs (G, H) can be Ramsey equivalent to pairs of the form (T, K_t). A summary of our results is given in Table 1.1. We focus on the two cases $G = T$ and $H = K_t$.

1.3.4 Maximum sum of sizes of r-cross t-intersecting families of sets

The main result that we will show in Chapter 6 considers the three variations as has been mentioned in Subsection 1.2.5. We determine the maximum sum of measures of r-cross t-intersecting families. Given $n, a, t \in \mathbb{N}$ with $n \geq a \geq t$ consider the families

$$\mathcal{A}(n, a, t) = \{F \in \mathscr{P}([n]) \colon |F \cap [a]| \geq t\}$$
$$\mathcal{B}(n, a) = \{F \in \mathscr{P}([n]) \colon [a] \subseteq F\}.$$

Our main result essentially states that the maximum is attained by families "derived" from $\mathcal{A}(n, a, t)$ and $\mathcal{B}(n, a)$, even when we consider different kinds of measures (including ν_k and ϱ_p when $p \leq 1/2$). However, some technicalities are required to formulate it precisely in its entire generality. For this, let us define the following notation. Given a set A we write $A^{(k)}$ for the set of k-element subsets of A and similarly $A^{(\leq k)}$ for the set containing all subsets of A that are of size at most k. Further, for $\mathcal{F} \subseteq \mathscr{P}([n])$ and $k \in \mathbb{N}$, we set $\mathcal{F}^k = \{F \in \mathcal{F} : |F| = k\}$ and $\mathcal{F}^{\leq k} = \{F \in \mathcal{F} : |F| \leq k\}$.

Theorem 6.7. *Let $r \geq 2$ and $n, t \geq 1$ be integers. Further, for every $i \in [r]$ let $\mu_i : [n]_0 \to \mathbb{R}_{\geq 0}$, let $\widehat{k}_i \in [n]$ and $k_i \in [\widehat{k}_i]_0$ such that μ_i is non-increasing on $[k_i, \widehat{k}_i]$. If $\mathcal{F}_1 \subseteq [n]^{(\leq \widehat{k}_1)}, \ldots, \mathcal{F}_r \subseteq [n]^{(\leq \widehat{k}_r)}$ are non-empty r-cross t-intersecting families and $n > \max_{i\in[r]}(k_i + \min_{j\in[r]\setminus i} \widehat{k}_j) - t$, then*

$$\sum_{j\in[r]} \mu_j(\mathcal{F}_j) \leq \max\left\{\mu_\ell(\mathcal{A}(n,a,t)^{\leq \widehat{k}_\ell}) + \sum_{j\in[r]\setminus\ell} \mu_j(\mathcal{B}(n,a)^{\leq \widehat{k}_j})\colon \ell \in [r], a \in \left[t, \min_{i\in[r]\setminus\ell} \widehat{k}_i\right]\right\}.$$

Note that $\mathcal{A}(n,i,t)$ together with $r-1$ copies of $\mathcal{B}(n,i)$ are r-cross t-intersecting for every $i \geq t$. Thus, this result is sharp in the sense that there are r-cross t-intersecting families which attain the bound.

Further, we remark that for several applications, for instance, if one is just interested in the sizes of the families or their product measure, the parameters k_i and/or $\widehat{k}_i$ become trivial and so the maximum becomes significantly simpler.

We point out some particularly important special cases in the following corollaries. Firstly, if we apply Theorem 6.7 with the measure $\mu_i = \binom{n}{k}\nu_k$, and $k_i = \widehat{k}_i = k$ for every $i \in [r]$, we obtain the following result.

Corollary 6.10. *Let $r \geq 2$, and $n, t \geq 1$ be integers, $k \in [n]$, and for $i \in [r]$ let $\mathcal{F}_i \subseteq [n]^{(k)}$. If $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are non-empty r-cross t-intersecting families and $n > 2k - t$, then*

$$\sum_{j\in[r]} |\mathcal{F}_j| \leq \max_{i\in[t,k]} \left\{ \sum_{m\in[t,k]} \binom{i}{m} \cdot \binom{n-i}{k-m} + (r-1)\binom{n-i}{k-i} \right\}$$

and this bound is attained.

In fact, Theorem 6.7 also yields the more general version of this result for possibly distinct uniformities $k_1, \ldots, k_r$ when setting $\mu_i = \binom{n}{k_i}\nu_{k_i}$, and $\widehat{k}_i = k_i$ for every $i \in [r]$.

In the context of non-uniform families, one of the results of a very recent work by Frankl and Wong H.W. [52] establishes the maximum possible size of cross t-intersecting families. By taking $\mu_i \equiv 1$, $k_i = 0$, and $\widehat{k}_i = n$ for every $i \in [r]$ in Theorem 6.7, we generalise that result to r-cross t-intersecting families with $r \geq 2$.

Corollary 6.11. *Let $r \geq 2$, $n, t \geq 1$ be integers and let $\mathcal{F}_1, \ldots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be non-empty r-cross t-intersecting families. Then,*

$$\sum_{j\in[r]} |\mathcal{F}_j| \leq \max_{i\in[t,n]} \left\{2^{n-i} \sum_{m\in[t,i]} \binom{i}{m} + (r-1)2^{n-i}\right\}$$

and this bound is attained.

As a further application, note that Theorem 6.7 applied with $k_i = 0$ and $\widehat{k}_i = n$ also provides the maximum for the product measure ρ_p, if $p \leq 1/2$.

Observe that the problem, that considers only one t-intersecting family is not covered by Theorem 6.7. The following theorem gives the maximal measure of an intersecting family for several kinds of measures. As mentioned above, this solves a problem posed by Frankl and Tokushige [51] by generalising the results for the uniform measure and the product measure.

Theorem 6.8. *Let n be an integer, $\widehat{k} \in [n]$, and $k \in [\widehat{k}]_0$, let $\mathcal{F} \subseteq [n]^{(\leq \widehat{k})}$ be an t-intersecting family, and let $\mu : [n]_0 \to \mathbb{R}_{\geq 0}$ such that μ is non-increasing on $[k, \widehat{k}]$. If $n > k + \widehat{k} - t$, then*

$$\mu(\mathcal{F}) \leq \mu(\mathcal{B}(n,t)) .$$

1.4 Organisation

Let us now present an overview of the organisation of this thesis.

To begin with, in Chapter 2 we will provide a comprehensive list of the notation used throughout in this work. We will also introduce to some of the important tools and techniques in the various chapters which includes, and are not limited to, signal senders, concentration bounds for various probability distributions, and the shifting technique.

Subsequently, in Chapter 3 we will examine the phenomena of Ramsey simplicity for random graphs. We will show that for a certain range of p the graph $G(n,p)$ is Ramsey equivalent for any number of colours, for a certain range the random graph is never Ramsey simple, and what happens in between can be rather erratic. This is a joint work with Simona Boyadzhiyska, Dennis Clemens, and Shagnik Das [17].

Further, in Chapter 4, we will provide a constructive proof for the existence of pattern gadgets. We will use these pattern gadgets to show that various graphs, for example a clique with a pendant edge, are Ramsey abundant. These results extend a joint work with Simona Boyadzhiyska and Dennis Clemens [18].

In Chapter 5, we will show that there exist some non trivial pairs of pairs of graphs which are Ramsey equivalent and some which are not equivalent. In particular we will explore pairs of the form (T, K_t), where T is a tree. This is a joint work with Simona Boyadzhiyska, Dennis Clemens, and Jonathan Rollin [19].

Finally, in Chapter 6, we will investigate the maximum sum of sizes of r-cross t-intersecting families. In order to find the upper bound on the sum, we will introduce and define the notion of a necessary intersection point. We will prove our results in the more general setting of measures. This is a joint work with Yannick Mogge, Simón Piga, and Bjarne Schülke [57].

2

Notation and techniques

In this chapter we begin by defining the notation used throughout this thesis. In Section 2.2 we will then go on to define and provide some results for some well known gadget graphs like determiners, signal senders, and indicators. We will follow this up by stating some facts and establishing various properties of random graphs in Section 2.3. In the last section, Section 2.4, we will recall the shifting technique and prove a useful lemma about shifted families.

2.1 Notation

Given an integer $n \geq 1$, we define $[n] = \{1, 2, \ldots, n\}$ Given another integer $k \geq 1$, we denote by $\binom{[n]}{k}$ or $[n]^k$ the set of all subsets of $[n]$ of size k.

For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$, and we set $e(G) = |E(G)|$ and $v(G) = |V(G)|$. For any edge $\{v, w\} \in E(G)$, we write vw for short. For a graph G and vertex subsets A and B of G, we denote by $E_G(A, B)$ the edges in G with one endpoint in A and another in B. Also, $E_G(A)$ denotes the edges in G with both endpoints in A. We sometimes identify a graph G with its edge set.

For an integer $n \geq 0$, a graph K_n denotes a complete graph on n vertices, that is all pairs of vertices share an edge. C_n denotes a cycle of length n and W_n denotes a wheel graph on $n+1$ vertices, which is constructed by adding an extra vertex to C_n and connecting this to all the vertices of the cycle. $K_n \cdot K_2$ denotes what is often called a clique with a pendant edge. It is formed by adding an additional edge to some vertex of K_n. $G_{n,p}$ is a binomial random graph on n vertices, where every possible edge is added independently at random with probability p.

We let $N_G(v) = \{w \in V(G) : \ vw \in E(G)\}$ denote the neighbourhood of v in G, $d_G(v) = |N_G(v)|$ denote the degree of v in G, $\delta(G) = \min\{d_G(v) : \ v \in V(G)\}$ and $\Delta(G) = \max\{d_G(v) : \ v \in V(G)\}$ denote the minimum degree and maximum degree of G respectively. We write $\lambda(G)$ for the order of the largest connected component in G.

We say that F is a subgraph of G, denoted by $F \subseteq G$, if there is an injective map $f : V(F) \to V(G)$ such that $f(x)f(y) \in E(G)$ for all $xy \in E(F)$; further, F is a proper subgraph of G if $F \subseteq G$ and $F \neq G$. Given any subset $A \subseteq V$, the subgraph induced by A, denoted by $G[A]$, is the graph with vertex set A and edge set $E_G(A)$.

For a set $X \subseteq V(G)$ we write $G - X$ for the graph obtained from G by removing the vertices in X and all their incident edges; for a single vertex $x \in V(G)$, we write $G - x = G - \{x\}$; similarly for a subgraph F of G we let $G - F = G - V(F)$. For a set $Y \subseteq E(G)$, we write $G - Y$ for the graph obtained from G by removing the edges in Y; for a single edge $e \in E(G)$, we write $G - e = G - \{e\}$.

Let F and G be two graphs. We say that F and G are isomorphic, denoted by $F \cong G$, if there exists a bijection $f : V(F) \to V(G)$ such that $vw \in E(F)$ if and only if $f(v)f(w) \in E(G)$. In this case, we also say that F forms a copy of G. If $F \cong G[A]$ for some $A \subseteq V(G)$, then we say that F is an induced subgraph of G and write $F \subseteq_{ind} G$.

Given a graph G and any subsets A and B of the vertex set or the edge set of G, we define the *distance* between A and B, denoted by $\text{dist}_G(A, B)$, to be the number of edges in a shortest path with one endpoint in (the vertex set of) A and one endpoint in (the vertex set of) B. The girth of G, denoted by $\text{girth}(G)$, is the length of a shortest cycle in G (if G is acyclic, then $\text{girth}(G)$ is defined to be infinity). A graph G is said to be k-connected if it has more than k vertices and, for any set S of at most $k-1$ vertices, the graph $G[V(G) \setminus S]$ is connected.

In this thesis, for a colouring of some graph G we always refer to a colouring of its edge set. If G contains no monochromatic subgraph isomorphic to H under a given colouring, the colouring is said to be H-free. If a colouring uses at most q colours, we call it a q-colouring. Unless otherwise specified, we will assume in this case that our colour palette is the set $[q]$. If we are only concerned with the case $q = 2$, for the sake of convenience we will sometimes call our colours red and blue instead of colour 1 and colour 2. If c is a q-colouring of G and some subgraph F is monochromatic in some colour i, we will sometimes write $c(F) = i$. Similarly, when defining colourings, we will write for example $c(F) = i$ to indicate that we give colour i to every edge of the subgraph F.

Throughout this thesis unless otherwise specified a colouring is meant to be an edge-colouring of the given graph G. As we always call the two colours red and blue, we use red/blue-colouring and 2-colouring as synonyms of each other. Given any two graphs H_1 and H_2, we say that a 2-colouring is (H_1, H_2)-free, if there is no red copy of H_1 and no blue copy of H_2.

We define $G(n, p)$ to be the probability space of all labelled graphs on the vertex set $[n]$ where every possible edge is selected independently at random with probability p. If a graph H is sampled from $G(n, p)$ we write $H \sim G(n, p)$. Give a property $\mathcal{Q}$, we say that asymptotically almost surely (a.a.s.) $H \sim G(n, p)$ has property $\mathcal{Q}$ if $\mathbb{P}(H \text{ has } \mathcal{Q}) \to 1$ as $n \to \infty$.

2.2 Gadget graphs

In this section we will introduce three gadget graphs, namely the determiners, signal senders, and indicators. These gadget graphs will prove themselves to be really useful throughout this thesis.

2.2.1 Determiners

Determiners were introduced by Burr, Faudree, and Schelp [24] and are defined as follows.

Definition 2.1. Let $i \in [2]$, and let $G \neq H$ be graphs. A graph $D = D(G, H, \beta)$ with a distinguished edge $\beta \in E(D)$ is called a *determiner* for (G, H) if it satisfies the following two properties:

1. $D \not\to_2 (G, H)$.

2. In any (G, H)-free colouring of D with colours from $[2]$, the edge d has colour 1.

The edge β is called the *signal edge* of D. Moreover, such a determiner is called well behaved, if it has a (G, H)-free colouring in which all edges incident to β are of colour not equal to i.

Determiners are known to exist for some classes of graphs, for example whenever both target graphs are cliques $K_s \neq K_t$ [24] or more generally whenever both target graphs $H_1 \neq H_2$ are 3-connected [25].

Burr, Erdős, Faudree, Rousseau, and Schelp [21] showed that well behaved determiners exist for any pair (T, K_t) when both the tree T and the clique K_t have at least three vertices. In fact, their construction satisfies some further properties which we will use in the proof of Theorem 5.9(c). We summarize those in the following proposition.

Proposition 2.2 ([21, Proof of Theorem 8, Lemmas 9 & 10])**.** *Let $t \geq 3$ be an integer and T be a tree with at least three vertices. There exists a well-behaved (T, K_t, β)-determiner D. Moreover, the graph induced by the endpoints of β and the union of their neighbourhoods is isomorphic to K_t.*

2.2.2 Signal senders

Signal senders were introduced by Burr, Erdős, and Lovász [23] for the construction of BEL gadgets when $H = K_t$ for $t \geq 3$ and $q = 2$.

Definition 2.3. Let $q \geq 2$ and $d \geq 1$ be given integers, and let H be a graph. A *positive signal sender* $S = S^+(H, e, f, q, d)$ for H is a graph that contains two distinguished edges $e, f \in E(S)$, called the *signal edges* of S, such that the following properties hold:

$(S1)$ $S \not\to_q H$.

$(S2)$ In any H-free q-colouring of S, the edges e and f have the same colour.

$(S3)$ $\mathrm{dist}_S(e, f) \geq d$.

A *negative signal sender* $S = S^-(H, e, f, q, d)$ for H is defined similarly, except that the words "the same colour" in $(S2)$ are replaced by "different colours."

An *interior* vertex of a signal sender is a vertex that is not incident to either of the signal edges. The *interior* of a signal sender is the set of all interior vertices.

We will often use say that we *join* or *connect* two edges e_1, e_2 of a given graph by a signal sender. What we mean by that is that we create a vertex-disjoint copy of a signal sender S and identify its signal edges with e_1 and e_2, that is, the signal sender does not share any vertices or edges with the original graph except for the (vertices of the) signal edges.

Signal senders are known to exist for some important classes of graphs, as given by Theorem 2.4 below. Part (a) is due to Rödl and Siggers [78], generalising results of Burr et al. [23] and Burr et al. [25], part (b) is due to Siggers [81], and part (c) follows from a result in the PhD thesis of Grinshpun [54, Lemma 2.6.3] combined with the result of Fox et al. [43] concerning $s_2(K_t \cdot K_2)$.

Theorem 2.4.

(a) For all integers $q \geq 2$ and $d \geq 1$ and every graph H that is 3-connected or isomorphic to K_3, there exist positive and negative signal senders in which the distance between the signal edges is at least d.

(b) For all integers $q \geq 2$, $d \geq 1$, and $t \geq 4$, there exist positive and negative signal senders for C_t with girth t and distance at least d between the signal edges.

(c) For $q = 2$ and for all integers $t \geq 3$ and $d \geq 1$, there exist positive and negative signal senders for $K_t \cdot K_2$ in which the distance between the signal edges is at least d. Further, a signal sender S, positive or negative, with signal edges e and f can be chosen so that S has a $K_t \cdot K_2$-free 2-colouring in which all edges incident to e (resp. f) have a different colour from e (resp. f) and none of the vertices of e and f is contained in a monochromatic copy of K_t.

Before we continue, we make a few remarks about Theorem 2.4. First, in [54], Grinshpun does not explicitly prove that signal senders exist for $K_3 \cdot K_2$; however, his proof easily extends to this case. Further, part (c) is actually a slight strengthening of Grinshpun's result: His result is stated only in terms of negative signal senders and provides a special colouring in which neither signal edge is incident to a monochromatic copy of K_t but only one of the signal edges, say f, is required to have a colour different from all edges incident to it. We can derive the version stated above easily. Let S' be the signal sender constructed by Grinshpun. To construct a positive signal sender S^+ as in part (c), take two copies of S' and identify the two copies of e; similarly, to construct a negative signal sender as in part (c), take a copy of S^+ and a copy of S' and identify the edge e with one of the signal edges of S^+. As a final remark, in the original manuscripts where (b) and (c) appear, it is not shown explicitly that the distance between the signal edges can be arbitrarily large. However, it is easy to see that this is indeed the case. Both of the constructions do guarantee that the signal edges are not incident to each other, which means that we can increase the distance between the signal edges by stringing several signal senders together (that is, taking signal senders $S_1, \ldots, S_r$ and, for each $i \in \{2, \ldots, r-1\}$, identifying one signal edge of S_i with a signal edge of S_{i-1} and the other with a signal edge of S_{i+1}; if we take $S_1, \ldots, S_{r-1}$ to be positive signal senders, then the resulting signal sender is of the same type (positive or negative) as S_r).

2.2.3 Indicators

Indicators were introduced by Burr et al. in [24] for two colours and generalised by Clemens et al. in [27] to multiple colours. Together with signal senders, these graphs will serve as basic building blocks for our construction of pattern gadgets. For this, we need to modify slightly the definition appearing in [27], as given below. In addition, we will need both positive and negative indicators.

Definition 2.5. Let $q \geq 2$ and $d \geq 1$, and let H and F be graphs such that $H \not\subseteq F$. A *positive indicator* $I = I^+(H, F, e, q, d)$ for H is a graph such that the following properties hold:

$(I1)$ $F \subseteq_{ind} I$ and $e \in E(I)$ with $\mathrm{dist}_I(F, e) \geq d$.

$(I2)$ There exists an H-free q-colouring of I in which F is monochromatic.

$(I3)$ For every H-free q-colouring c of I in which F is monochromatic, we have $c(e) = c(F)$.

$(I4)$ For any non-constant colouring $\varphi_F : E(F) \to [q]$ and $k \in [q]$, there exists an H-free colouring $c : E(I) \to [q]$ such that $c_{|F} = \varphi_F$ and $c(e) = k$.

If I is a positive indicator with parameters H, F, e, q, and d, we call I a *positive* (H, F, e, q, d)-*indicator*. In this case, we call F the *indicator subgraph* and e the *indicator edge* of I.

A *negative indicator* $I = I^-(H, F, e, q, d)$ is the same except that in property $(I3)$ we replace "$c(e) = c(F)$" with "$c(e) \neq c(F)$."

An *interior* vertex of an indicator is a vertex that belongs to neither the indicator subgraph nor the indicator edge. The *interior* of an indicator is the set of all interior vertices.

Similar to signal senders, joining or connecting a subgraph F and an edge e by an indicator will mean that we create a vertex-disjoint copy of the indicator and identify the indicator subgraph with F and the indicator edge with e. In Chapter 4 we will introduce generalised negative indicators and will also use the same terminology in this context.

Before we move on to the existence results for indicators, let us define a very useful property of a pair of graphs, called *robustness*.

Definition 2.6. Let G be a graph and G_0 be an induced subgraph of G. We say that the pair (G, G_0) is H-robust if, in any graph obtained from G by adding any set S of new vertices and any collection of edges within $S \cup V(G_0)$, every copy of H is entirely contained either in G or in the subgraph induced by $S \cup V(G_0)$.

The construction of indicators for the case when H is 3-connected or isomorphic to K_3 was given in [24] for two colours and in [27] for more than two colours, where $(I4)$ is replaced with a similar yet slightly weaker property. Essentially the same constructions work for 3-connected graphs as well as cycles and cliques with a pendant edge with this new property $(I4)$. In our constructions, however, we need to ensure that when we put together several gadgets and later on colour each of them avoiding a monochromatic copy of our target graph H, there is still no monochromatic H in the resulting graph. We do not want to accidentally create monochromatic copies that use vertices from several different pieces of our construction. While we can get this almost immediately for 3-connected graphs, in the latter two cases we need to maintain some extra properties. Despite these additional technicalities and the slight modification in our definition of indicators, our proofs that the constructions given in [24] and [27] indeed give the required positive indicators are very similar to the proofs presented in the original papers.

Theorem 2.7. *Let $q \geq 2$ and $d \geq 1$ be integers, H be a graph, and F be a graph with $e(F) \geq 2$ such that $H \not\subseteq F$.*

(a) If H is 3-connected or $H \cong K_3$, then there exists both a positive indicator $I = I^+(H, F, e, q, d)$ and a negative indicator $I = I^-(H, F, e, q, d)$.

(b) If $H \cong C_t$ for $t \geq 4$ and girth$(F) > t$, then there exist a positive indicator $I = I^+(H, F, e, q, d)$ and a negative indicator $I = I^-(H, F, e, q, d)$, each with girth t.

(c) If $H \cong K_t \cdot K_2$ for $t \geq 3$ and $q = 2$, then there exist a positive indicator $I = I^+(H, F, e, q, d)$ and a negative indicator $I = I^-(H, F, e, q, d)$, each satisfying the following additional property: The H-free 2-colourings in $(I2)$ and $(I4)$ can be chosen so that none of the vertices of F and e is a vertex of a monochromatic copy of K_t and all edges incident to e have a different colour from e.

Further, in parts (a) and (b) the indicators can be taken so that (I, F) is H-robust and in part (c) we can ensure that (I, F) is K_t-robust.

Proof. Without loss of generality, we assume that $d > v(H)$. The existence of a negative indicator in each of the cases follows immediately from the existence of a corresponding positive indicator. We will first show this claim and then proceed to show the existence of a positive indicator in each case.

Assuming the existence of positive indicators, we can construct negative indicators easily as follows:

(i) Let $I' = I^+(H, F, e', q, d)$ be a positive indicator satisfying all required additional properties.

(ii) Let e be an edge disjoint from I'.

(iii) Connect e and e' by a negative signal sender for H.

Now, properties $(I1)$–$(I4)$ as well as the robustness property and the additional properties required in parts (b) and (c) are all easy to verify.

We now construct positive indicators. We proceed by induction on the number of edges in F. The basic construction is the same in all three cases (a)–(c); we will see that the special properties we require in the latter two cases follow almost immediately from the properties of the respective signal senders given in Theorem 2.4.

We will in fact show something stronger: Our indicators will satisfy an additional property, which, following Clemens et al. [27], we call property $\mathcal{T}$. We say that an indicator $I = I^+(H, F, e, d, q)$ satisfies *property* $\mathcal{T}$ if there is a collection of subgraphs $\{T_f \subseteq I : f \in E(F)\}$ satisfying the following properties:

$(T1)$ $V(T_f) \cap V(F) = f$ for all $f \in E(F)$.

$(T2)$ $V(I) = \bigcup_{f \in E(F)} V(T_f)$ and $E(I) = \bigcup_{f \in E(F)} E(T_f)$.

$(T3)$ for all distinct $f_1, f_2 \in E(F)$ and all $v \in V(T_{f_1}) \cap V(T_{f_2})$, we have either $v \in V(F)$ or $\text{dist}_I(v, F) \geq d$.

Property $\mathcal{T}$ will be useful for showing the required robustness properties.

We begin with the base case $e(F) = 2$. In this case, we will show that our indicators possess one additional property, as given below.

$$\text{If } F = \{f_1, f_2\} \text{ is a matching, then } \text{dist}_I(f_1, f_2) \geq d. \qquad (*)$$

We have two different constructions, one for $q = 2$ and a different one for $q > 2$. We start with the former, which is a slightly modified version of the construction given in [24].

For $q = 2$, begin with a copy H_0 of H and let $e_1, e_2 \in E(H_0)$ be arbitrary except when $H \cong K_t \cdot K_2$, in which case e_1 should not be the pendant edge. Let e be an edge disjoint from H_0 and F. Say $E(F) = \{f_1, f_2\}$. Let S^- and S^+ be a negative and a positive signal sender for H in which the distance between each pair of signal edges is at least d and which satisfy the properties guaranteed by Theorem 2.4. Let I be the graph constructed in the following way:

(i) Connect f_1 to every edge in $E(H) \setminus \{e_1, e_2\}$ by a copy of S^-.

(ii) Join f_2 and e_2 by a copy of S^-.

(iii) Join e_1 and e by a copy of S^+.

We claim that the graph I constructed in this way is a positive indicator with indicator edge e that also satisfies the required additional properties in each of the cases (a)–(c).

We first discuss where copies of H in the graph I can be located. Note that Observations 2.8 and 2.9 immediately imply the claimed robustness properties.

Observation 2.8. *Let H be 3-connected or a cycle. Let I' be a graph obtained from I by adding a new vertex set S and any collection of edges within $S \cup V(F)$. Then every copy of H in I' either lies entirely within one of the signal senders from (ii) or (iii), or is fully contained in $S \cup V(F)$, or is the starting copy H_0.*

Proof. It is not difficult to see that the claim holds when $H \cong K_3$, so assume now that $v(H) > 3$. For a contradiction, suppose there is a copy H' of H in I' forming a counterexample. Assume first that H' contains an interior vertex v of one of the signal senders from (ii) or (iii); call this signal sender S'. Since the distance between the signal edges of S' is at least $d > v(H')$, we know that H' can only contain vertices from one of the signal edges; call this signal edge f. Now, H' is a counterexample, so it needs to contain a vertex w not belonging to S'. If H is 3-connected, this is not possible, since removing the edge f disconnects the graph H' (any path from v to w in I' must contain a vertex of one of the signal edges of S'). If H is a cycle, then H' needs to contain both vertices of f, for otherwise we can disconnect H' by removing a vertex of f, contradicting the fact that H' is 2-connected. But then the vertices in $V(f) \cup \{v\}$ participate in a cycle of length strictly smaller than $v(H)$ in S', contradicting our assumption on the girth of the signal senders.

Hence, we may assume that H' is disjoint from the interior of any of the signal senders. So H' is a subgraph of the graph induced by $S \cup V(F) \cup V(H_0) \cup V(e)$, in which the sets $S \cup V(F)$, $V(H_0)$, and $V(e)$ are all disconnected from each other. Hence no copy of H can use vertices from more than one of these sets, implying the claim. □

The proof of the girth property required in part (b) is very similar to the proof of Observation 2.8.

Using a similar argument, we can show the analogous statement for $H \cong K_t \cdot K_2$, given in the observation below.

Observation 2.9. *Let $H \cong K_t \cdot K_2$. Let I' be a graph obtained from I by adding a new vertex set S and any collection of edges within $S \cup V(F)$. Then every copy of K_t in I' either lies entirely within one of the signal senders from (ii) or (iii), or is fully contained in $S \cup V(F)$, or is in the starting copy H_0.*

We now turn our attention to property (*) and property $\mathcal{T}$. If F is a matching, by the choice of the signal senders used in the construction, we indeed have $\mathrm{dist}_I(f_1, f_2) \geq d$. To verify the latter property, notice that the subgraphs T_{f_2}, consisting of the signal sender connecting f_2 and e_2, and T_{f_1}, induced by all remaining vertices together with the vertices of e_2 and the vertices in $f_1 \cap f_2$, satisfy $(T1)$–$(T3)$.

It remains to show that I satisfies properties $(I1)$–$(I4)$ as well as the additional properties required in part (c).

$(I1)$ The first part is clear, since in the construction we do not add any further edges between the vertices of F. In each case, the second part of the property follows easily from the fact that $\mathrm{dist}_I(F, e)$ must be at least the distance between the signal edges in the signal senders we attach to f_1, f_2, and e.

$(I2)$ For this, consider the following colouring:

- Give colour 1 to the edges of F.
- Give colour 1 to e_1 and colour 2 to all other edges of H_0.
- Give colour 1 to e.
- Extend this colouring to each of the signal senders so that no signal sender contains a monochromatic copy of H. In case (c) choose these colourings to be $K_t \cdot K_2$-special.

Note that the extension in the last step of the colouring is possible since the colours for the signal edges are chosen so that they fit property $(S2)$. Observe also that F is monochromatic.

We claim that this colouring is H-free. Indeed, for parts (a) and (b), Observation 2.8 implies that every copy of H in I either lies entirely within some signal sender or is the starting copy H_0; by our choice of the colouring, none of these copies of H are monochromatic. For (c), again neither the starting copy H_0 nor any copy of H that is fully contained within a single signal sender is monochromatic. Any other copy of H must contain a copy of K_t that touches a signal edge and hence cannot be monochromatic by the choice of the $K_t \cdot K_2$-special 2-colouring from Theorem 2.4.

Further, we use the $K_t \cdot K_2$-special 2-colouring from Theorem 2.4 to colour each of the signal senders, so this colouring of I is also a $K_t \cdot K_2$-special 2-colouring.

$(I3)$ If F is monochromatic in, say, colour 1, then by property $(S2)$ of the signal senders, in any H-free colouring, all edges in $E(H_0) \setminus \{e_1\}$ have colour 2 and e_1 and e have the same colour. For the colouring to be H-free, the edge e_1, and hence also e, must have colour 1.

$(I4)$ To justify this property, consider the following colouring:

- Give colour $\varphi_F(f_i)$ to f_i for both $i \in [2]$.
- Give colour k to e_1, colour $\varphi_F(f_1)$ to e_2, and colour $\varphi_F(f_2)$ to all other edges of H_0.

- Give colour k to e.
- Extend this colouring to each of the signal senders so that no signal sender contains a monochromatic copy of H. In case (c), choose these colourings to be $K_t \cdot K_2$-special.

Again, the extension in the last step of the colouring is possible since the colours for the signal edges are chosen so that they fit property $(S2)$. The argument needed to check that this colouring is H-free is similar to the one used to verify property $(I2)$ above. Also, it is not hard to see that this also gives a $K_t \cdot K_2$-special 2-colouring in case (c).

We now present the construction for $q > 2$ and $e(F) = 2$, given in [27]. Say $E(F) = \{f_1, f_2\}$. Let $\{e_1, \ldots, e_{q-1}\}$ be a matching, disjoint from F. Let $H_1, \ldots, H_{q-1}$ be copies of H that are disjoint from F and $e_1, \ldots, e_{q-1}$ and that all intersect in precisely one fixed edge, which we call e. Let S^+ and S^- be a positive and a negative signal sender for H respectively in which the distance between the signal edges is at least d and which satisfy the additional properties guaranteed by Theorem 2.4. Let I be the graph constructed in the following way:

(i) Connect f_1 and e_i by a copy of S^- for all $i \in [q-2]$.

(ii) Connect f_2 and e_{q-1} by a copy of S^-.

(iii) Join each pair e_i, e_j for $1 \le i < j \le q-1$ by a copy of S^-.

(iv) For all $i \in [q-1]$, connect e_i to all edges of H_i except for e by a copy of S^+.

The verification that this indeed gives an indicator is similar to the one in the case $q = 2$. As before, every copy of H in parts (a) and (b) either is one of the starting copies $H_1, \ldots, H_{q-1}$ or is fully contained within a single signal sender (similarly for K_t in part (c)). Notice also that the robustness property, the girth property required in (b), and properties $\mathcal{T}$ and (*) are shown in a similar way as in the case $q = 2$.

Finally, we check properties $(I1)$–$(I4)$. Property $(I1)$ is straightforward.

$(I2)$ To see this property, colour f_1, f_2, and e with colour 1 and, for all $i \in [q-1]$, give e_i and all edges in $E(H_i) \setminus \{e\}$ colour $i+1$. Then extend this colouring to all of the signal senders so that each signal sender is coloured without a monochromatic copy of H, which is possible since the colours chosen above fit property $(S2)$. By the same argument as in the case $q = 2$, there in no monochromatic H anywhere in the graph.

$(I3)$ For this, suppose f_1 and f_2 have the same colour, say colour 1. Then, in any H-free colouring, each of the colours in $[q] \setminus \{1\}$ must be used on the matching $e_1, \ldots, e_{q-1}$ exactly once because of the signal senders in (i)–(iii), and each $H_i - e$ needs to be monochromatic in the colour of e_i because of the signal senders in (iv). Thus, to avoid a monochromatic copy of H, the colour of e must be 1, i.e., the same as the colour of F.

$(I4)$ Suppose that f_1 and f_2 are coloured differently, say using colours 1 and 2 respectively, and we are given any colour k. If $k \neq 1, 2$, we can colour $e_1, \ldots, e_{q-2}$ with colours $2, \ldots, k-1, k+1, \ldots, q$ respectively and e_{q-1} with colour 1; if $k = 2$, we can colour $e_1, \ldots, e_{q-2}$ with colours $3, \ldots, q$ and e_{q-1} with colour 1; finally, if $k = 1$, we colour $e_1, \ldots, e_{q-2}$ with colours $2, \ldots, q-1$ and e_{q-1} with colour q. In each case, colour k is available for e and we can still extend the colouring to all the signal senders without creating a monochromatic copy of H.

We now proceed with the induction step. Suppose there exist indicators as required in the statement of the theorem that also satisfy properties $\mathcal{T}$ and (*) when $e(F) \leq \ell$ for some $\ell \geq 2$. Assume $e(F) = \ell + 1$. Let f be any edge of F and $F' = F - f$; further, let e' and e be two edges that are disjoint from F and from each other. By the induction hypothesis, there exists a positive (H, F', e', q, d)-indicator I' satisfying all of the required properties. There also exists a positive $(H, \{e', f\}, e, q, d)$-indicator I'' in which the distance between e' and f is at least d. Now, let I be the graph obtained by joining F' and e' by I' and $\{f, e'\}$ and e by I''. We claim that I is a positive indicator satisfying all required properties.

First, as before, we discuss where copies of H can be located. For this, consider a graph obtained from I by adding a new set of vertices S and any edges within $S \cup V(F)$. We claim that in parts (a) and (b) every copy of H is contained entirely within I', I'', or the graph induced by $S \cup V(F)$, and that in part (c) every copy of K_t is contained entirely within I', I'', or the graph induced by $S \cup V(F)$. Again, this immediately implies the claimed robustness properties.

Assume first that H is either 3-connected or isomorphic to a cycle. Let H' be a copy of H in the new graph. Suppose that H' contains an interior vertex of I''. By our assumption on the distance between f and e' in I'', the graph H' can only contain vertices from one of these two edges. Again, by the condition that H is 3-connected or isomorphic to K_3 in (a) and by our assumption on the girth of each indicator and of F in (b), we conclude that $H' \subseteq I''$.

Suppose next that H' contains no such vertices but contains an interior vertex v of I'. If H' contains no vertices of F, then H' cannot contain any vertices from the set S either, and thus H' is fully contained in I'. Hence, we may assume that H' contains a vertex of F. Let $\{T_{f'} : f' \in E(F')\}$ be a collection of subgraphs of I' witnessing that I' has property $\mathcal{T}$. Then v is contained in some T_g for $g \in E(F')$; further, this g is unique, since otherwise $\mathrm{dist}_I(F, v) \geq d > v(H)$ by $(T3)$, leading to a contradiction.

Since $V(T_g) \cap V(F) = g$, the only possible copy of K_3 containing v and a vertex of F consists of v and the endpoints of g and is thus fully contained in I'. If $(S \cup V(F)) \cap V(H') \subseteq V(g)$, then H' is fully contained in I'. So H' contains a vertex from $(S \cup V(F)) \setminus V(g)$. If H is 3-connected, this is not possible, since removing the vertices of g disconnects H'. Finally, suppose H is a cycle. In this case, by the fact that H' is 2-connected, both vertices of g must be contained in H', but this means that v and the vertices of g are part of a cycle of length strictly smaller than $v(H')$, contradicting our assumption about the girth of I'.

The argument for part (c) is analogous to that for part (a). Also, a similar argument shows that the girth condition required in part (b) holds.

Now, to verify property $\mathcal{T}$, consider a collection $\{T_{f'} : f' \in E(F')\}$ of subgraphs of I' given by property $\mathcal{T}$. Adding the graph I'', we obtain a collection of subgraphs of I satisfying $(T1)$–$(T3)$.

Finally, we check the indicator properties. Property $(I1)$ is clear, since we do not add any new edges within F and $\mathrm{dist}_I(F, e) \geq \mathrm{dist}_{I''}(\{f, e'\}, e) \geq d$.

$(I2)$ The colouring c given by $c(F) = c(e') = c(e) = 1$ can be extended to I' and I'' so that neither contains a monochromatic copy of H. Furthermore, in parts (a) and (b), every copy of H lies entirely within one of I' and I'', so there is no monochromatic copy of H in I. For part (c), the same follows from the fact that every copy of K_t is contained within I' or I'' and we can choose the extensions to I' and I'' in such a way that no vertex of F, e', or e is in a monochromatic copy of K_t. The latter is possible as we can find $K_t \cdot K_2$-special 2-colourings for I' and I'' by induction. Also, using these $K_t \cdot K_2$-special 2-colourings we obtain a colouring of the whole graph I such that all edges incident to e have a different colour from e and no vertex of F or e is part of a monochromatic K_t, as required. Thus, we conclude that $(I2)$ holds and we immediately get the existence of a $K_t \cdot K_2$-special 2-colouring as required in part (c).

$(I3)$ For this, note that, if F is monochromatic in an H-free colouring, then I' forces e' to have the same colour as F, and I'' in turn makes it necessary for e to also have the same colour as F.

$(I4)$ For the last property, let φ_F be any non-constant colouring of the edges of F and $k \in [q]$. Let $f', f'' \in E(F)$ be two edges that have distinct colours, say $\varphi_F(f') = i$ and $\varphi_F(f'') = j$ for some distinct $i, j \in [q]$. First assume that $f' = f$. In this case whether $F - f$ is monochromatic or not, there exists an extension of φ_F to an H-free colouring of I' such that the colour of e' is j, which in turn means that there is an extension of this colouring also to an H-free colouring of I'' such that the colour of e is k. Otherwise, if $f' \in F - f$, then $F - f$ is not monochromatic and there is an extension of φ_F to an H-free colouring of I' in which e' has colour k. Then, whether $\{f, e'\}$ is monochromatic or not, there exists an extension of φ_F also to an H-free colouring of I'' such that e has colour k. Again, it is not difficult to check that this is an H-free colouring of I that in case (c) can be made $K_t \cdot K_2$-special. $\square$

2.3 Properties of $G(n, p)$

In this section we shall establish various properties of the random graph $G(n, p)$ needed for the proof of Theorem 3.6 and the deduction of Corollary 3.16.

2.3.1 Facts about $G(n, p)$

We start with some bounds on the degrees and edge distribution in the random graph, for which we require the following well-known concentration bounds due to Chernoff (see [66, Theorem 2.3] and [53, Theorem 22.6]).

Lemma 2.10. *Let $X \sim Bin(n, p)$ and $\mu = \mathbb{E}[X]$.*

(a) If $0 < \varepsilon < 1$, then $\mathbb{P}(X \geq (1+\varepsilon)\mu) \leq \exp\left(\frac{-\mu\varepsilon^2}{3}\right)$ and $\mathbb{P}(X \leq (1-\varepsilon)\mu) \leq \exp\left(\frac{-\mu\varepsilon^2}{2}\right)$.

(b) For all $t \geq 7\mu$ we have $\mathbb{P}(X \geq t) \leq \exp(-t)$.

With these concentration results, we can specify how many edges the random graph is likely to have. This is done in the following lemmas, which collect some folklore bounds on the degrees and number of edges in $G(n, p)$. We start by controlling the degrees.

Lemma 2.11 (Degrees in $G(n, p)$)**.** *Let $p = p(n) \in (0, 1)$, and let $H \sim G(n, p)$. Then a.a.s. the following bounds on the maximum degree hold:*

(a) for any fixed integer $k \geq 2$, we have $\Delta(H) \geq k - 1$ when $p \gg n^{-\frac{k}{k-1}}$, and

(b) *for any* $f = f(n)$ *satisfying* $1 \ll f = n^{o(1)}$, *we have* $\Delta(H) \geq \frac{\log n}{\log(f \log n)}$ *when* $p = \frac{1}{nf}$.

Moreover, if $p \gg \frac{\log n}{n}$, *then with probability at least* $1 - n^{-2}$ *we have*

(c) $d_H(v) = (1 \pm o(1))np$ *for every* $v \in V(H)$.

Proof. If $p \gg n^{-\frac{k}{k-1}}$ for any integer $k \geq 2$, then it follows from a simple second moment calculation that H a.a.s. contains a star with $k-1$ edges, and hence $\Delta(H) \geq k-1$; see Theorem 5.3 in [53] for more details.

Part (b) can be obtained from a similar application of the second moment method. For simplicity, we apply Theorem 3.1 (ii) from [13], stating that, if $n^{-3/2} \ll p \ll 1 - n^{-3/2}$ and the expected number of vertices of degree $d = d(n)$ in $H \sim G(n,p)$ tends to infinity, then with high probability H contains at least one vertex of degree d.

For $p = \frac{1}{nf}$ and $d = \frac{\log n}{\log(f \log n)}$, we can lower bound the expected number of vertices of degree d by

$$n\binom{n-1}{d}p^d(1-p)^{n-1-d} \geq n\left(\frac{(n-1)p}{d}\right)^d (1-np) \geq \tfrac{1}{2}n\left(\frac{1}{2fd}\right)^d$$
$$= \frac{1}{2}e^{\log n - d\log(2fd)} = \frac{1}{2}e^{\frac{\log n}{\log(f\log n)}\left(\log\log\sqrt{f\log n}\right)}$$

which tends to infinity. Thus we must have at least one vertex of degree d, and hence $\Delta(H) \geq d$.

Part (c) follows by applying Lemma 2.10(a) to the degree of each vertex, and then taking a union bound over all n vertices. □

We can also bound the number of edges, both globally and, provided the edge probability is not too low, in all large induced subgraphs.

Lemma 2.12 (Edge counts in $G(n,p)$)**.** *Let* $p = p(n) \in (0,1)$ *with* $p \gg n^{-2}$, *and let* $H \sim G(n,p)$. *Then a.a.s. the following statements hold:*

(a) $e(H) = (1 \pm o(1))\frac{n^2 p}{2}$, *and*

(b) *if* $p \gg \frac{\log n}{n}$, *then with probability at least* $1 - n^{-2}$, *every set* $S \subseteq V(H)$ *of size* $s \geq \frac{20 \log n}{p}$ *satisfies* $e_H(S) \geq \frac{1}{4}s^2 p$.

Proof. Part (a) follows directly from Lemma 2.10(a). For part (b), we take a union bound over all sets S of s vertices, again applying Lemma 2.10(a) to bound the probability that such a set contains too few edges. This results in a bound of

$$\sum_{s=\frac{20\log n}{p}}^{n} \binom{n}{s} e^{-\frac{1}{16}(1-o(1))s^2 p} \leq \sum_{s=\frac{20\log n}{p}}^{n} e^{s\log n - \frac{1}{16}(1-o(1))s^2 p} \leq \sum_{s=\frac{20\log n}{p}}^{n} e^{-0.2 s\log n} < n^{-2},$$

proving the lemma. □

Aside from knowing how many edges the random graph contains, we shall also need some knowledge about how they are distributed. The following result describes the structure of sparse random graphs.

Lemma 2.13. *Let $p = p(n) \in (0,1)$ with $p \ll n^{-1}$, and let $H \sim G(n,p)$. Then a.a.s. H is a forest, and moreover the order $\lambda(H)$ of its largest component satisfies the following bounds:*

(a) $\lambda(H) \leq \log n$,

(b) if $p \ll n^{-\frac{k+1}{k}}$ for some constant $k \in \mathbb{N}$, then $\lambda(H) \leq k$, and

(c) if $p = \frac{1}{nf}$ for some $f = f(n)$ satisfying $1 \ll f = n^{o(1)}$, then $\lambda(H) \leq (1+o(1))\frac{\log n}{\log f}$.

Proof. That H contains no cycles, and hence is a forest, can be shown by taking a union bound over all possible cycles; see Theorem 2.1 in [53] for the details. We now bound the orders of the trees in this forest. For part (a), we refer to Lemma 2.12(ii) in [53], which asserts that with high probability a random graph $H' \sim G(n, e^{-2}n^{-1})$ contains no trees of order larger than $\log n$. By monotonicity the same bound holds when $p \ll n^{-1}$. For the bound in (b), notice that there are only a constant number of non-isomorphic trees on $k+1$ vertices, and by a simple first moment calculation (see Theorem 5.3 in [53]) each of these trees appears in H with vanishing probability when $p \ll n^{-\frac{k+1}{k}}$. The bound in part (c) can again be obtained by running a first moment calculation, the details of which we now sketch. As there are k^{k-2} labelled trees on k vertices, the total possible number of tree components of order k is $\binom{n}{k}k^{k-2}$. For such a tree to appear as a subgraph, we need its $k-1$ edges to appear in $G(n,p)$. Hence, the probability of seeing such a tree is at most

$$\binom{n}{k}k^{k-2}p^{k-1} \leq \left(\frac{ne}{k}\right)^k \frac{(kp)^k}{k^2 p} \leq p^{-1}(nep)^k.$$

For $p = \frac{1}{nf}$ and any $\varepsilon > 0$, the sum of this expression over all $k \geq (1+\varepsilon)\frac{\log n}{\log f}$ is at most

$$p^{-1}\,(nep)^{(1+\varepsilon)\frac{\log n}{\log f}}\left(\frac{1}{1-nep}\right) \leq 2e^{\log(nf)-(1+\varepsilon)\frac{\log n}{\log f}(\log f - 1)} \leq 2e^{-\frac{\varepsilon}{2}\log n} = o(1)$$

and hence a.a.s. the largest component has order at most $(1+o(1))\frac{\log n}{\log f}$. □

Switching to a much denser range, we find that when the edge probability is sufficiently large, not only does $G(n,p)$ contain cycles, but every edge is contained in a triangle.

Lemma 2.14. *Let $p = p(n) \in (0,1)$ be such that $p \gg \sqrt{\frac{\log n}{n}}$, and let $H \sim G(n,p)$. Then a.a.s. every edge of H is contained in a triangle.*

Proof. An easy application of the union bound gives

$$\begin{aligned}
& \Pr\left(\exists e = uv \in E(H) :\ uv \text{ is not in a triangle}\right) \\
\leq\ & \Pr\left(\exists \{u,v\} \in \binom{V(H)}{2} :\ uw \notin E(H) \text{ or } vw \notin E(H) \text{ for all } w \in V(H) \setminus \{u,v\}\right) \\
\leq\ & \binom{n}{2} \cdot (1-p^2)^{n-2} < e^{2\log n - p^2(n-2)} = o(1)\,,
\end{aligned}$$

which proves the lemma. □

Finally, in our construction of minimal Ramsey graphs with vertices of low degree, we shall make use of some mild pseudorandom properties concerning the degrees, connectivity, and expansion of the target graph H. The required properties are collected in the definition below.

Definition 2.15 (Well-behaved)**.** We say an n-vertex graph H is *well-behaved* if it satisfies the following properties:

$(W1)$ H has a unique vertex u of minimum degree $\delta(H)$,

$(W2)$ every pair of vertices in H has codegree at most $\frac{1}{2}\delta(H)$,

$(W3)$ H is 3-connected, and

$(W4)$ removing $\delta(H)$ vertices from H cannot create a component of size $k \in \left[\frac{1}{2}\delta(H), \frac{1}{2}n\right]$.

As might be expected, random graphs are highly likely to be well-behaved.

Lemma 2.16. *If $\frac{\log n}{n} \ll p \ll 1$ then a.a.s. $H \sim G(n,p)$ is well-behaved.*

Proof. The property $(W1)$ is established in Theorem 3.9(i) of [13]. Moreover, due to Lemma 2.11(c) we may condition on $\delta(H) = (1 \pm o(1))np$ from now on. For property $(W2)$, observe that the distribution of the codegree of a given pair of vertices is $\mathrm{Bin}(n-2, p^2)$. We consider two cases. If $p^2 \geq \frac{10 \log n}{n}$ then by applying a Chernoff bound (Lemma 2.10(a)) we obtain $\Pr(d_H(u,v) \geq 2np^2) \leq e^{-\frac{1}{3}(n-2)p^2} < e^{-3\log n}$ for large n. Taking a union bound over all $\binom{n}{2}$ pairs of vertices, this shows that with high probability the maximum codegree is at most $2np^2 < \frac{1}{2}\delta(H)$. Otherwise $p^2 \leq \frac{10\log n}{n}$ and then Lemma 2.10(b) yields $\Pr(d_H(u,v) \geq 100 \log n) \leq e^{-100\log n}$. We can then again take a union bound over all pairs to show the maximum codegree is at most $100 \log n$, which, as $\delta(H) = (1 \pm o(1))np \gg \log n$, is again less than half the minimum degree.

Property $(W3)$ is shown to hold with high probability in Theorem 4.3 in [53].

This leaves us with property $(W4)$. Let us fix $k \in \left[\frac{1}{2}\delta(H), \frac{1}{2}n\right]$, and bound the probability that we can create a component K of size k by removing a set U of $\delta(H)$ vertices. In order for this to happen, there cannot be any edges between K and $V(H) \setminus (K \cup U)$. For given K and U, the probability of this is $(1-p)^{k(n-k-\delta(H))}$. Taking a union bound over all possible components K and cut-sets U, the probability that property $(W4)$ fails for a given k is at most

$$\binom{n}{k}\binom{n}{\delta(H)}(1-p)^{k(n-k-\delta(H))} \leq n^k n^{\delta(H)} e^{-pk(n-k-\delta(H))} \leq e^{(k+\delta(H))\log n - \frac{1}{4}pkn},$$

where the last inequality uses the bounds $k \leq \frac{1}{2}n$ and $\delta(H) = (1 \pm o(1))np \leq \frac{1}{4}n$. Now, since $p \gg \frac{\log n}{n}$, we have $k \log n \ll pkn$, and, since $k \geq \frac{1}{2}\delta(H)$, we also have $\delta(H)\log n \ll pkn$. Hence, we can bound this error probability by $e^{-\frac{1}{8}pkn} = o(n^{-1})$, using again the fact that $pkn \gg \delta(H) \log n$. Therefore, even after taking a union bound over all possible values of k, we see that property $(W4)$ holds with high probability. $\square$

2.3.2 Transference lemma

As is evident in the statement of Theorem 3.6, our bounds on the simplicity of $H \sim G(n,p)$ depend on the subgraph induced by the neighbourhood of the minimum degree vertex (which, by virtue of Lemma 2.16 and property $(W1)$, we may assume to be unique). Our next lemma allows us to transfer what we know about the random graph $G(\delta(H),p)$ to this subgraph.

Lemma 2.17. *Let $p = p(n) \in (0,1)$ be such that $p \gg \frac{\log n}{n}$. For every $s \in [0.5np, 2np]$, let $\mathcal{P}_s$ be a graph property, and assume that a random graph $G_s \sim G(s,p)$ satisfies*

$$\Pr\left(G_s \in \mathcal{P}_s\right) = 1 - o(1).$$

Then $H \sim G(n,p)$ a.a.s. has a unique minimum degree vertex u and $H[N_H(u)] \in \mathcal{P}_{d_H(u)}$.

Proof. Let us fix some $\beta_n = o(1)$ such that

$$\Pr\left(G_s \notin \mathcal{P}_s\right) = o(\beta_n) \tag{2.1}$$

for every $s \in [0.5np, 2np]$. Moreover, let X_δ denote the event that H has a unique vertex of minimum degree and $0.5np \leq \delta(H) \leq 2np$. By Lemma 2.16, specifically property $(W1)$, and Lemma 2.11 we know that X_δ holds with high probability. In particular, we can find $\delta_n = o(1)$ such that

$$\Pr(0.5np \leq \delta(H) \leq 2np) \geq \Pr(X_\delta) = 1 - \delta_n.$$

In the following we will condition on the event X_δ, and whenever we do so, we will always let u denote the unique minimum degree vertex in H. We will follow an approach similar to that used in the proof of Corollary 2.1.4 in [54]. Before we proceed with the proof, we introduce some notation and facts that we will need later on. We begin with the fact that there exists $\gamma_n = o(1)$ such that the following holds:

(1) For any $d \geq 0$, we have $\Pr(\delta(H) = d) \leq \gamma_n$.

(2) For any $d \geq 0$, if $H' \sim G(n-1,p)$, we have $\Pr(\delta(H') \geq d-1) \geq \Pr(\delta(H) \geq d) - \gamma_n$.

Part (1) follows from the proof of Theorem 3.9(i) in [13], while part (2) is shown in the proof of Corollary 2.1.4 in [54].

Next, let $\varepsilon_n = o(1)$ be chosen such that $\varepsilon_n = \omega(\max\{\beta_n, \gamma_n, \delta_n\})$. We further let t_n be the smallest integer such that $\Pr(\delta(H) \leq t_n) \geq 1 - \varepsilon_n$. Note that, by the minimality of t_n, we then have $\Pr(\delta(H) \leq t_n - 1) < 1 - \varepsilon_n$. Using (1) for $d = t_n$, we conclude

$$1 - \varepsilon_n \leq \Pr(\delta(H) \leq t_n) = \Pr(\delta(H) \leq t_n - 1) + \Pr(\delta(H) = t_n) \leq 1 - \varepsilon_n + \gamma_n. \tag{2.2}$$

Moreover, since $\varepsilon_n > \gamma_n + \delta_n$, we obtain $\Pr(\delta(H) \leq t_n) < 1 - \delta_n < \Pr(\delta(H) \leq 2np)$ and thus $t_n \leq 2np$.

Since $H \sim G(n,p)$, the subgraph $H - v$, for any fixed vertex v, has the distribution $G(n-1,p)$. However, recall that we are conditioning on the event X_δ, and that in particular there is a unique vertex u of minimum degree $d = d_H(u)$. We will be interested in the subgraph $H' = H - u$, and first need to determine how conditioning on X_δ affects its distribution.

Suppose $S \subseteq V(H')$ is the neighbourhood of u. As u is the only vertex of degree at most d in H, we must have $d_{H'}(v) \geq d+1$ for all $v \in V(H') \setminus S$, and $d_{H'}(v) \geq d$ for all $v \in S$; let C_S be the event that these lower bounds on the degrees in H' hold. Aside from C_S, however, X_δ yields no further information about the graph H', as the edges in $G(n,p)$ are independent. Thus, we have

$$\Pr_{G(n,p)}(H[S] \in \mathcal{P}_d | X_\delta \wedge \{N_H(u) = S\}) = \Pr_{G(n-1,p)}(H'[S] \in \mathcal{P}_d | C_S). \tag{2.3}$$

Now, by the Law of Total Probability,

$$\begin{aligned}
&\Pr_{G(n,p)}(H[N_H(u)] \in \mathcal{P}_{d_H(u)} | X_\delta) \\
&= \sum_{0 \leq d \leq n-1} \sum_{S \in \binom{V(H')}{d}} \Pr_{G(n,p)}(H[S] \in \mathcal{P}_d | X_\delta \wedge \{N_H(u) = S\}) \cdot \Pr_{G(n,p)}(N_H(u) = S | X_\delta) \\
&\geq \sum_{0.5np \leq d \leq t_n} \sum_{S \in \binom{V(H')}{d}} \Pr_{G(n-1,p)}(H'[S] \in \mathcal{P}_d | C_S) \cdot \Pr_{G(n,p)}(N_H(u) = S | X_\delta).
\end{aligned} \tag{2.4}$$

To estimate the first factor, we observe that

$$\begin{aligned}
\Pr_{G(n-1,p)}(C_S) &\geq \Pr(\delta(H') \geq d+1) \geq \Pr(\delta(H) \geq d+2) - \gamma_n \\
&\geq \Pr(\delta(H) \geq t_n + 2) - \gamma_n \geq \Pr(\delta(H) \geq t_n + 1) - 2\gamma_n \geq \varepsilon_n/2,
\end{aligned} \tag{2.5}$$

where the second inequality follows from (2), for the third inequality we use $d \leq t_n$, the fourth inequality follows from (1), and the last inequality comes from (2.2) and since $\varepsilon_n = \omega(\gamma_n)$. Hence we have

$$\begin{aligned}
\Pr_{G(n-1,p)}\left(H'[S] \in \mathcal{P}_d | C_S\right) &= 1 - \Pr_{G(n-1,p)}\left(H'[S] \notin \mathcal{P}_d | C_S\right) \\
&= 1 - \frac{\Pr_{G(n-1,p)}\left(\{H'[S] \notin \mathcal{P}_d\} \wedge C_S\right)}{\Pr_{G(n-1,p)}(C_S)} \\
&\geq 1 - \frac{\Pr_{G(n-1,p)}\left(H'[S] \notin \mathcal{P}_d\right)}{\Pr_{G(n-1,p)}(C_S)} \\
&\geq 1 - \frac{\Pr_{G(d,p)}\left(G_d \notin \mathcal{P}_d\right)}{\varepsilon_n/2} = 1 - o(1),
\end{aligned}$$

where for the second inequality we use (2.5) and that $H'[S] \sim G(d,p)$ and the final estimate uses (2.1) and $\beta_n = o(\varepsilon_n)$. Putting this into (2.4), we conclude that

$$\begin{aligned}
\Pr_{G(n,p)}(H[N_H(u)] \in \mathcal{P}_{d_H(u)} | X_\delta) &\geq \sum_{0.5np \leq d \leq t_n} \sum_{S \in \binom{V(H')}{d}} (1 - o(1)) \Pr_{G(n,p)}(N_H(u) = S | X_\delta) \\
&= (1 - o(1)) \Pr_{G(n,p)}(0.5np \leq \delta(H) \leq t_n | X_\delta) \\
&= (1 - o(1)) \frac{\Pr_{G(n,p)}(\{\delta(H) \leq t_n\} \wedge X_\delta)}{\Pr(X_\delta)} \\
&\geq (1 - o(1)) \frac{1 - \Pr_{G(n,p)}(\delta(H) > t_n) - \Pr_{G(n,p)}(\overline{X_\delta})}{\Pr(X_\delta)}
\end{aligned}$$

$$\geq (1 - o(1)) \frac{1 - \varepsilon_n - \delta_n}{1 - \delta_n} = 1 - o(1).$$

This proves the lemma. $\square$

2.4 Shifting technique

In this section we will introduce some well-known facts about shifting. For $F \subseteq [n]$ and $i, j \in [n]$ we set

$$\sigma_{ij}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\} \text{ if } j \in F \text{ and } i \notin F \\ F \text{ otherwise} \end{cases},$$

and note that $|\sigma_{ij}(F)| = |F|$. Moreover, for a given $\mathcal{F} \subseteq \mathscr{P}([n])$ we define the family $\sigma_{ij}(\mathcal{F}) = \{\sigma_{ij}(F) : F \in \mathcal{F}\} \cup \{F \in \mathcal{F} : \sigma_{ij}(F) \in \mathcal{F}\}$ and note that $|\sigma_{ij}(\mathcal{F})| = |\mathcal{F}|$. Further, it can easily be checked that if $\mathcal{F} \subseteq \mathscr{P}([n])$ is intersecting, then $\sigma_{ij}(\mathcal{F})$ is also intersecting. We say that $\mathcal{F} \subseteq \mathscr{P}([n])$ is *shifted* if for all $i, j \in [n]$ with $i < j$ we have $\sigma_{ij}(\mathcal{F}) = \mathcal{F}$, i.e., for all $F \in \mathcal{F}$ we have that $\sigma_{ij}(F) \in \mathcal{F}$. By shifting an intersecting family $\mathcal{F} \subseteq \mathscr{P}([n])$ repeatedly, that is, replacing $\mathcal{F}$ by $\sigma_{ij}(\mathcal{F})$ repeatedly for all $i, j \in [n]$ with $i < j$, we obtain an intersecting family $\mathcal{G}$ that is shifted and for which we have $|\mathcal{G}| = |\mathcal{F}|$ and $|\mathcal{G}^k| = |\mathcal{F}^k|$. Thus, to determine the maximum size of an intersecting family, one can restrict themselves to shifted families.

Moreover, for the sake of completeness, we prove the following Lemma.

Lemma 2.18. *Let $a, b \in [n]$. If $\mathcal{F}_1, \ldots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ are r-cross t-intersecting, then the families $\sigma_{ab}(\mathcal{F}_1), \ldots, \sigma_{ab}(\mathcal{F}_r)$ are r-cross t-intersecting.*

Proof. Assume the contrary and let $F'_1 \in \sigma_{ab}(\mathcal{F}_1), \ldots, F'_r \in \sigma_{ab}(\mathcal{F}_r)$ for every $i \in [r]$ such that $|\bigcap_{i \in [r]} F'_i| < t$. For every $i \in [r]$, let $F_i = F'_i$ if $F'_i \in \mathcal{F}$. If $F'_i \notin \mathcal{F}$, we know that $a \in F'_i$ and $b \notin F'_i$ and we set $F_i = \sigma_{ba}(F'_i) \in \mathcal{F}_i$. Since $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are r-cross t-intersecting, we have $|\bigcap_{i \in [r]} F_i| \geq t$ and so there is some $j \in [r]$ such that $F_j = \sigma_{ba}(F'_j) \neq F'_j$. But then we have $a \notin F_j$ and, thus, $a \notin \bigcap_{i \in [r]} F_i$. This yields that

$$t - 1 \leq |\bigcap_{i \in [r]} F_i \setminus \{a, b\}| = |\bigcap_{i \in [r]} F'_i \setminus \{a, b\}|. \tag{2.6}$$

Note that the assumption $|\bigcap_{i \in [r]} F'_i| < t$ tells us that in fact the left side inequality above is an equality. This in turn implies that $b \in \bigcap_{i \in [r]} F_i$.

Our assumption together with (2.6) also give some $\ell \in [r]$ such that $a \notin F'_\ell$. Then it follows by definition that $\sigma_{ab}(F_\ell) \in \mathcal{F}_\ell$ because $b \in \bigcap_{i \in [r]} F_i$. Hence, $|\sigma_{ab}(F_\ell) \cap \bigcap_{i \in [r] \setminus \ell} F_i| < t$ contradicts $\mathcal{F}_1, \ldots, \mathcal{F}_r$ being r-cross t-intersecting. $\square$

This allows us to restrict ourselves to shifted families when looking for the maximum sum of measures of r-cross t-intersecting families if the measure of a set F depends only on the size of F.

3

Ramsey simplicity of random graphs

In this chapter we will study the behaviour of parameter s_q for random graphs. For a graph H, let $\mathcal{M}_q(H)$ denote the set of all minimal Ramsey graphs for H. Then we define $s_q(H) = \min\{\delta(G) : G \in \mathcal{M}_q(H)\}$. As has been observed before, $q(\delta(H)-1)+1$ is a simple lower bound for $s_q(H)$ and H is said to be q-Ramsey simple if this bound is attained.

In the first section of this chapter, Section 3.1, we will prove Theorem 3.2. Here we will discuss the behaviour of random graphs in two extreme settings. We observe that when the random graph is sparse enough, in particular when it is almost surely a forest, it is q-Ramsey simple for all values of q whereas at the other extreme, when the graph is sufficiently dense, $G(n, p)$ is never q-Ramsey simple for any values of q.

Before we go on to analyse this behaviour further in the intermediate setting, we make an important observation in Section 3.2. We show that q-Ramsey simplicity is a monotone property and therefore one may define a threshold value for q. We define this as follows.

Definition 3.1. $\tilde{q}(H) = \sup\{q : \ H \text{ is } q\text{-Ramsey simple}\}$.

As was also noted earlier, we will define $\tilde{q}(H) = \infty$ whenever H is q-Ramsey simple for all integers $q \geq 1$ and since every graph H is trivially 1-Ramsey simple, we define $\tilde{q}(H) = 1$ whenever H is not q-Ramsey simple for any integer $q \geq 2$.

Armed with this observation and definition, in Section 3.3, we will provide some lower bound estimates on this new parameter for the intermediate ranges of p. Namely, we will prove Theorem 3.6(a) and (b). Thereafter, in Section 3.4, we will provide some upper bound estimate on $\tilde{q}(H)$. In particular we will show Theorem 3.6(c). We will conclude this chapter by stating some further question of interest in Section 3.5.

This chapter is based on joint work with Simona Boyadzhiyska, Dennis Clemens, and Shagnik Das [17].

3.1 The two extremes

We will begin by observing the behaviour of random graphs when they are either too sparse or too dense. Specifically, we will consider the ranges when the graph $H \sim G_{n,p}$ is either almost surely a forest or is such that almost surely every edge is in a triangle. We will prove the following theorem.

Theorem 3.2. *Let $p = p(n) \in (0,1)$ and $H \sim G(n,p)$. Then a.a.s. the following holds:*

(a) $\tilde{q}(H) = \infty$ *if* $0 < p \ll n^{-1}$.

(b) $\tilde{q}(H) = 1$ *if* $\left(\frac{\log n}{n}\right)^{1/2} \ll p < 1$.

To begin, we observe that we have nothing new to prove in case (a). By Lemma 2.13 we know H is a forest with high probability when $p \ll n^{-1}$. Szabó, Zumstein, and Zürcher [83] proved that all forests are 2-Ramsey simple, and their proof extends directly to show q-Ramsey simplicity for all $q \geq 3$ as well. For completeness, we provide the argument in the next lemma.

Lemma 3.3. *For every forest F without isolated vertices and every integer $q \geq 2$, we have $s_q(F) = 1$.*

Proof. Given F, fix a bipartition $V(F) = A \cup B$, where $|A| \leq |B|$ and the size of A is minimised. Set $a = |A|$, $b = |B|$, $B_1 = \{v \in B : d_F(v) = 1\}$, and $B_{\geq 2} = B \setminus B_1$. We start by showing that $|B_{\geq 2}| \leq a - 1$. Indeed, let $T_1, \dots, T_k$ be the components of F and, for each $i \in [k]$, let r_i be an arbitrary vertex in $A \cap V(T_i)$. Viewing T_i as a tree rooted at r_i, we note that each element of $B_{\geq 2} \cap V(T_i)$ must have a child in $A \cap V(T_i)$ and, since T_i is a tree, all of these children must be different. Thus, $|A \cap V(T_i)| \geq |B_{\geq 2} \cap V(T_i)| + 1$ for all $i \in [k]$, and summing up over all components yields $|A| \geq |B_{\geq 2}| + 1$.

Now, set $r = q(a-1), s = q^{r+1}v(F)$, and $t = sbq$, and let G be the graph constructed as follows:

- let $V(G) = X \dot\cup Y \dot\cup Z$, where $|X| = r$, $|Y| = s$, and $|Z| = t$,
- add a complete bipartite graph between X and Y, and
- partition Z into s subsets of size bq, indexed by the elements of Y. That is, let $Z = \bigcup_{y \in Y} Z_y$, where $|Z_y| = bq$. For each $y \in Y$, connect y to all vertices of Z_y.

Each vertex $v \in Z$ then satisfies $d_G(v) = 1$. We will now show that (i) $G - Z \not\to_q F$, and (ii) $G \to_q F$. From this it follows directly that G must contain a graph from $\mathcal{M}_q(F)$ with minimum degree one, and hence that $s_q(F) = 1$.

To see property (i), colour $E(G - Z)$ as follows: take any partition $X = X_1 \cup \ldots \cup X_q$ with $|X_i| = a - 1$ for every $i \in [q]$, and colour $E(X_i, Y)$ in colour i. Then each colour class is a bipartite graph with a partite set of size smaller than a. By the definition of a, there cannot be a monochromatic copy of F.

We prove (ii) next. Consider any q-colouring $\varphi : E(G) \to [q]$. Each vertex $y \in Y$ has bq neighbours in Z_y, and hence there must be a subset $Z'_y \subseteq Z_y$ of size b such that the edges from y to Z'_y are monochromatic. As we use only q colours, there must be a subset $Y' \subseteq Y$ of $\frac{s}{q}$ vertices $y_1, \dots, y_{s/q}$ such that, without loss of generality, the edges between y_i and Z'_{y_i} are all colour 1 for each $i \in [s/q]$. Further, set $Z' = \bigcup_{y_i \in Y'} Z'_{y_i}$. Next, let $X = \{x_1, \dots, x_r\}$. For every $y_i \in Y'$, we consider the vector $c_i := (\varphi(x_1 y_i), \dots, \varphi(x_r y_i)) \in [q]^r$, the *colour profile* of y_i. As $|Y'| = s/q = q^r v(F)$, there must be at least $v(F)$ vertices in Y' with the same colour profile c. By symmetry, we may assume that $c_1 = c_2 = \ldots = c_{v(F)} = c$. We consider two cases.

Case 1: There is a colour $i \in [q]$ that appears at least a times in c. This gives a copy of $K_{a,v(F)}$ between X and $\{y_1, \ldots, y_{v(F)}\}$ that is monochromatic in colour i. As $K_{a,v(F)}$ contains a copy of F, we are done.

Case 2: Every colour is used exactly $(a-1)$ times in c. In particular, we find a subset $X' \subseteq X$ of size $a-1$ such that the edges between X' and $\{y_1, \ldots, y_{v(F)}\}$ are monochromatic in colour 1. Using the edges between y_i and Z'_{y_i}, for $i \in [v(F)]$, we find a monochromatic copy of F: embed A into $\{y_1, \ldots, y_{v(F)}\}$ arbitrarily, embed B_1 into Z' by respecting adjacency relation, and embed $B_{\geq 2}$ into X' arbitrarily. $\square$

On the other end of the spectrum, when $H = G(n,p)$ is a dense graph, we observe that H is never simple.

Theorem 3.4. *Let $p \gg \sqrt{\frac{\log n}{n}}$ and $H \sim G(n,p)$. Then $\tilde{q}(H) = 1$.*

Proof of Theorem 3.6(b). Let $q \geq 2$ be some integer. For a contradiction suppose that H is q-Ramsey simple. Let G be a minimal q-Ramsey graph for H such that G has a vertex w with $d_G(w) = q\delta(H) - (q-1)$. By the minimality of G, we can find an H-free q-colouring c of the graph $G - w$. Now fix an arbitrary vertex $v \in N_G(w)$ and observe that, by the pigeonhole principle, there must be a set $W \subseteq N_G(w) \setminus \{v\}$ of size $\delta(H) - 1$ such that all edges between v and any of its neighbours in W have the same colour; without loss of generality, let this be colour 1 and set $U = W \cup \{v\}$. We can extend the colouring c by giving each colour $i \in [q-1]$ to all exactly $\delta - 1$ edges from w to $N_G(w) \setminus U$ and giving colour q to all edges from w to vertices in U. With this colouring we cannot create a monochromatic copy of H in any colour $[q-1]$, as w is only incident to $\delta(H) - 1$ edges of any colour in $[q-1]$. On the other hand, w is incident to exactly $\delta(H)$ edges of colour q, which all lie between w and U. Hence, if there were a monochromatic copy of H in colour q, the edge wv would need to be part of it. However, since all edges in U involving v are of colour 1, that means the edge wv is not contained in any triangle of colour q, which is a contradiction to Lemma 2.14. Hence there cannot be in a monochromatic copy of H. $\square$

3.2 Monotonicity in q

In this section we will prove that the property of being q-Ramsey simple is monotone decreasing in the number of colours; that is, we will show that if a graph is not q-Ramsey simple for some q, then it cannot be q'-Ramsey simple for any $q' \geq q$.

Note that q-Ramsey simplicity does not observe any monotonicity with respect to the graph H. Indeed, we know that any tree on t vertices is 2-Ramsey simple, whereas the clique K_t is not. Similarly, there exist graphs that are q-Ramsey simple but contain subgraphs that are not. For instance, Theorem 2.1.3 in [54] shows that any 3-connected graph H containing a vertex v of minimum degree such that $N(v)$ is contained in an independent set of size $2\delta(H) - 1$ is 2-Ramsey simple. Hence, while K_δ for $\delta \geq 3$ is not 2-Ramsey simple, the following supergraph of it is: add $2\delta - 1$ new vertices to K_δ with a complete bipartite graph connecting them to the clique, and then add another vertex v connected to exactly δ of the $2\delta - 1$ new vertices.

Lemma 3.5. *If H is not q-Ramsey simple, then H is not $(q+1)$-Ramsey simple.*

Proof. Assume H is not q-Ramsey simple, that is, $s_q(H) > q(\delta(H) - 1) + 1$. Suppose for a contradiction that there exists a graph $G \in \mathcal{M}_{q+1}(H)$ such that G contains a vertex v of degree $(q+1)(\delta(H) - 1) + 1$. Let e be an arbitrary edge incident to v.

By the minimality of G, we know that the graph $G - e$ has an H-free $(q+1)$-colouring c. Now, if there are at most $\delta(H) - 2$ edges that are incident to v and have colour $q+1$ under c, then we can give e colour $q+1$ to obtain an H-free $(q+1)$-colouring of G, contradicting $G \in \mathcal{M}_{q+1}(H)$. Hence we may assume that there are at least $\delta(H) - 1$ edges incident to v that have colour $q+1$. Let G_0 be the subgraph of G containing all edges that have colours in $[q]$ under c together with the edge e, i.e., $G_0 = G - c^{-1}(q+1)$. We then know that $d_{G_0}(v) \leq (q+1)(\delta(H) - 1) + 1 - (\delta(H) - 1) = q(\delta(H) - 1) + 1 < s_q(H)$. If G_0 is not q-Ramsey for H, then G_0 has an H-free q-colouring c', and extending c' to the graph G by colouring the edges in $E(G) \setminus E(G_0)$ with colour $q+1$ gives an H-free $(q+1)$-colouring of G, a contradiction. Therefore, $G_0 \to_q H$.

But $d_{G_0}(v) < s_q(H)$, so G_0 cannot be minimal q-Ramsey for H, and in particular, the vertex v cannot be part of a minimal q-Ramsey subgraph of G_0. Thus $G_0 - v \to_q H$. But the restriction of c to $G_0 - v$ is H-free by our choice of c, which again leads to a contradiction.

Hence, H cannot be $(q+1)$-Ramsey simple. □

3.3 Simplicity for $G(n, p)$

In this section we prove the lower bounds on $\tilde{q}(G(n,p))$ from Theorem 3.6. These are the positive results, showing that with high probability $H \sim G(n,p)$ is q-Ramsey simple for the appropriate values of q.

Theorem 3.6. *Let $p = p(n) \in (0, 1)$ and $H \sim G(n,p)$. Let $u \in V(H)$ be a vertex of minimum degree $\delta(H)$ and let $F = H[N(u)]$ be the subgraph of H induced by the neighbourhood of u. Denote by $\lambda(F)$ the order of the largest connected component in F. Then a.a.s. the following bounds hold:*

(a) $\tilde{q}(H) = \infty$ *if* $\frac{\log n}{n} \ll p \ll n^{-\frac{2}{3}}$.

(b) $\tilde{q}(H) \geq (1 + o(1)) \max\left\{ \frac{\delta(H)}{\lambda(F)^2}, \frac{\delta(H)}{80 \log n} \right\}$ *if* $n^{-\frac{2}{3}} \ll p \ll n^{-\frac{1}{2}}$.

(c) $\tilde{q}(H) \leq (1 + o(1)) \min\left\{ \frac{\delta(H)}{\Delta(F)}, \frac{\delta(H)^2}{2e(F)} \right\}$ *if* $n^{-\frac{2}{3}} \ll p \ll 1$.

For the remaining cases, we will show that H is typically such that one can construct a minimal q-Ramsey graph G for H with $\delta(G) = q(\delta(H) - 1) + 1$, provided, in case (b), that q is not too large. We first establish a general sufficient condition for the existence of such a graph G in Section 3.3.1, and then show in Section 3.3.2 that it is satisfied with high probability by the random graph H. In Section 4.2.5 we shall extend these results by showing H admits minimal Ramsey graphs with arbitrarily many vertices of degree $q(\delta(H) - 1) + 1$.

Before we start, we introduce a piece of notation we shall use throughout this section. Given a graph Γ with a q-colouring $f : E(\Gamma) \to [q]$ and any colour $i \in [q]$, the colour-i subgraph Γ_i of Γ is the graph $\Gamma_i = (V(\Gamma), f^{-1}(i))$ consisting of all edges of Γ with the colour i.

3.3.1 Reducing to the smallest neighbourhood

In this subsection we shall show that when establishing the q-Ramsey simplicity of a well-behaved graph H (recall Definition 2.15), we can focus our attention on the neighbourhood of the minimum degree vertex.

Proposition 3.7. *Let $q \geq 2$, let H be a well-behaved graph, and let $F = H[N(u)]$ be the subgraph induced by the neighbourhood of the unique minimum degree vertex u. Suppose there exists a q-edge-coloured graph Γ on $q(\delta(H) - 1) + 1$ vertices such that:*

(i) for every set $U \subseteq V(\Gamma)$ of $\delta(H)$ vertices and for every colour $i \in [q]$, there exists a copy $F_{U,i}$ of F in $\Gamma[U]$ whose edges are all of colour i, and

(ii) for each $i \in [q]$, the colour-i subgraph Γ_i of Γ has maximum degree at most $\delta(H) - 1$.

Then H is q-Ramsey simple.

This proposition provides a sufficient condition: to establish the q-Ramsey simplicity of a well-behaved graph, one need only construct the coloured graph Γ. Before proceeding with its proof, we remark that the condition is very close to being necessary as well.

Remark 3.8. Let H be q-Ramsey simple with a unique vertex u of minimum degree, and let G be a minimal q-Ramsey graph for H with a vertex w of degree $q(\delta(H) - 1) + 1$. Let $\Gamma = G[N(w)]$ be the subgraph of G induced by the neighbourhood of w. By minimality, there is a q-colouring c of $G - w$, and in particular of Γ, without any monochromatic copies of H.

Since G itself is q-Ramsey for H, no matter how we extend the colouring c to the edges incident to w, we must create a monochromatic copy of H. Given any subset U of $\delta(H)$ vertices in Γ and any colour $i \in [q]$, colour the edges from w to U with colour i, and colour the remaining edges incident to w evenly with the other colours, so that each is used $\delta(H) - 1$ times. Any monochromatic copy of H must involve at least $\delta(H)$ edges incident to w, and hence must be of colour i and contain all the vertices in U. As w has degree $\delta(H)$ in this monochromatic subgraph, it must play the role of u in H, and therefore we must find a colour-i copy of F in $\Gamma[U]$.

Thus, if H is q-Ramsey simple, there must exist a q-coloured graph Γ on $q(\delta(H)-1)+1$ vertices satisfying property (i) of Proposition 3.7. While the well-behavedness of H and property (ii) may not be necessary, they shall enable us to maintain control over potential copies of H when constructing the minimal q-Ramsey graph G.

Given the graph Γ, when we build from it a q-Ramsey graph G we shall, as is common practice in the field, make extensive use of signal senders, which are gadgets that allow us to prescribe colour patterns on the edges of a graph. These have already been introduced in Section 2.2.2, and see Definition 2.3 therein.

Fortunately for us, signal senders exist for all 3-connected graphs, as shown by Rödl and Siggers [78], building on earlier work of Burr, Erdős, and Lovász [23] and Burr, Nešetřil, and Rödl [25]. In Theorem 2.4 we have listed several useful existence results for signal senders. Nonetheless, we extract Theorem 2.4(a) here again for the sake of readability.

Theorem 3.9 ([78])**.** *If H is 3-connected, then for any $q \geq 2$ and $d \geq 1$, there are positive and negative signal senders $S^+(H, q, d, e, f)$ and $S^-(H, q, d, e, f)$.*

The utility of signal senders lies in the ability to force pairs of edges in an H-free colouring of a graph G to have the same (or different, in the negative case) colours. This is achieved through the process of *attachment*; given a graph G and a pair of distinct edges $h_1, h_2 \in E(G)$, we attach to G a signal sender $S^+(H,q,d,e,f)$ (or $S^-(H,q,d,e,f)$), defined on a disjoint set of vertices, between h_1 and h_2 by identifying the signal edges e and $f \in E(S)$ with the edges h_1 and $h_2 \in E(G)$. In this next result, we show that attachment cannot create unexpected copies of our target graph H, provided that the signal edges are sufficiently far apart.

Lemma 3.10. *Let $q \geq 2$, let H be any 3-connected graph, and let $d \geq v(H)$. Let $S = S^+(H,q,d,e,f)$ or $S = S^-(H,q,d,e,f)$ be a signal sender and let G be any graph on a disjoint set of vertices. If the graph G' is formed by attaching S to any two distinct edges of G, then, for any copy H_0 of H in G', we have either $V(H_0) \subseteq V(G)$ or $V(H_0) \subseteq V(S)$.*

Proof. Let H_0 be a copy of H in G and suppose for the sake of contradiction that H_0 is fully contained neither in G nor in S. We can then find vertices $x \in V(H_0) \cap (V(S) \setminus V(G))$ and $y \in V(H_0) \cap (V(G) \setminus V(S))$. Now, by 3-connectivity, H_0 contains three internally-vertex-disjoint paths between x and y.

Since $V(S) \cap V(G) = e \cup f$, each of these paths must pass through a distinct endpoint of one of the signal edges e and f. There must be one path meeting e and another meeting f, and the portions of these paths that lie within the signal sender contain a path from e to f within $V(H_0) \cap V(S)$. However, this contradicts e and f being at distance $d \geq v(H)$. $\square$

Armed with these preliminaries, we can now prove Proposition 3.7.

Proof of Proposition 3.7. We shall take a slightly indirect route to certifying the q-Ramsey simplicity of H. Rather than constructing a minimal q-Ramsey graph with minimum degree $q(\delta(H)-1)+1$, we will instead build a graph G such that:

(a) $G \to_q H$,

(b) G has a vertex w of degree $q(\delta(H)-1)+1$, and

(c) $G - w \not\to_q H$.

Since G is q-Ramsey for H, it must contain a minimal q-Ramsey subgraph $G' \subseteq G$. By virtue of (c), we have $w \in V(G')$, and hence $\delta(G') \leq d_{G'}(w) \leq d_G(w) = q(\delta(H)-1)+1$. In light of the general lower bound, we must in fact have equality, and hence G' bears witness to the q-Ramsey simplicity of H.

To construct this q-Ramsey graph G, we start with the graph Γ. Recall that, for each set U of $\delta(H)$ vertices of Γ and for each colour $i \in [q]$, there is a colour-i copy $F_{U,i}$ of F in $\Gamma[U]$. We will wish to complete these to potential monochromatic copies of H. To this end, let $R = H - (\{u\} \cup N(u))$ be the remainder of H after we remove the minimum degree vertex u and its neighbourhood. Then, for every U and i, we include a copy $R_{U,i}$ of R on a disjoint set of vertices, adding the necessary edges so that $R_{U,i} \cup F_{U,i}$ forms a copy of $H - u$. We call the resulting graph Γ^+.

Now recall that the graph Γ comes with an edge-colouring, which we extend by colouring the edges in $R_{U,i}$ and between $R_{U,i}$ and $F_{U,i}$ with the colour i. Denote by c the resulting colouring of Γ^+. To force the correct colouring, we shall use signal senders. Note that, since

H is well-behaved, property $(W3)$ ensures H is 3-connected, and hence by Theorem 3.9 positive and negative signal senders exist.

We introduce a matching $e_1, e_2, \ldots, e_q$ of q edges, again on a set of new vertices. For every pair $i < j$, we attach a negative signal sender $S_{i,j} = S^-(H, q, v(H), e_i, e_j)$ between e_i and e_j. As we shall see later, this will ensure that these edges all receive distinct colours in an H-free colouring. Now, for every edge f in Γ^+, we attach a positive signal sender $S_f = S^+(H, q, v(H), e_{c(f)}, f)$ between $e_{c(f)}$ and f. Finally, we introduce a new vertex w and make it adjacent to every vertex in Γ. This completes our construction of the graph G, which is depicted in Figure 3.1.

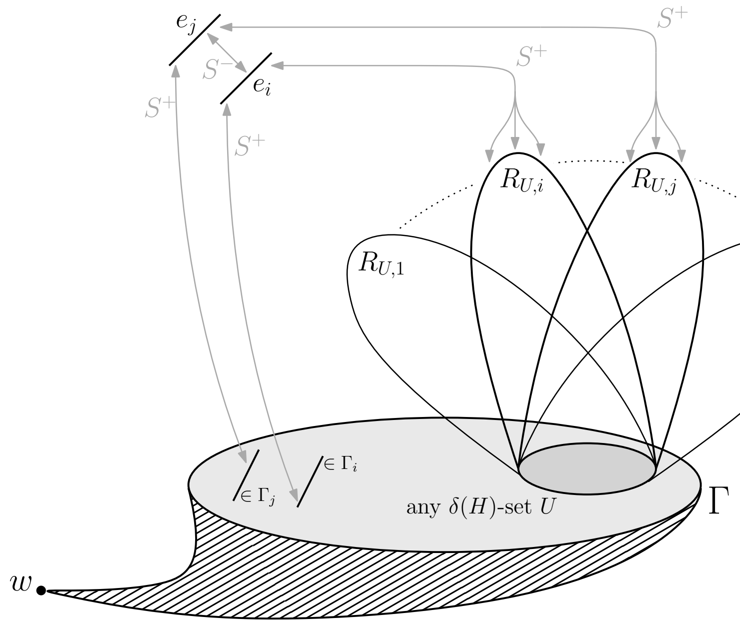

Figure 3.1: Construction of G

Observe that $d_G(w) = v(\Gamma) = q(\delta(H) - 1) + 1$, and so condition (b) is already satisfied. We shall now verify conditions (a) and (c) in the following claims.

Claim 3.11. *The graph G is q-Ramsey for H.*

Proof. Suppose for a contradiction that we have an H-free q-colouring of G. First observe that, by Definition 2.3$(S2)$, if the signal sender $S_{i,j}$ is H-free, then the edges e_i and e_j must receive different colours. As this is true for each pair $i < j$, we may, relabelling colours if necessary, assume that each edge e_i receives colour i.

Next, for each edge f in Γ^+, consider the signal sender S_f. If this does not contain a monochromatic copy of H, then $e_{c(f)}$ and f must have the same colour, and thus f receives the colour $c(f)$. Hence we have forced the desired colouring on Γ^+.

This brings us to the vertex w. Since it has degree $q(\delta(H) - 1) + 1$, there must be some colour i and a set $U \subseteq V(\Gamma)$ of size $\delta(H)$ such that the edges between w and U are all of colour i. However, appealing to condition (i) of Proposition 3.7, we find a colour-i copy $F_{U,i}$ of F in $\Gamma[U]$, which we can complete to a copy of H by attaching w and $R_{U,i}$, contradicting our supposition. $\square$

Claim 3.12. *The graph $G - w$ is not q-Ramsey for H.*

Proof. We provide an H-free q-colouring of $G - w$. To start, we give Γ^+ the colouring c, and, for each $i \in [q]$, colour the edge e_i of the matching with the colour i. Observe that, under this colouring, the signal edges of each positive signal sender S_f in G have the same colour, while those of negative signal senders $S_{i,j}$ receive different colours. By Definition 2.3 we can find an H-free colouring of each signal sender that agrees with the colouring of the signal edges. We use these to extend our colouring to the signal senders as well, thereby obtaining a q-colouring of $G - w$.

Now suppose for a contradiction that this colouring gives rise to a colour-i copy H_0 of H for some $i \in [q]$. First, observe that it follows from Lemma 3.10 that H_0 either is fully contained in a signal sender or is contained in $\Gamma^+ \cup \{e_i : i \in [q]\}$. Since the signal senders were coloured without monochromatic copies of H, and the edges $\{e_i : i \in [q]\}$ are isolated in the latter graph, we need only show that we cannot have $H_0 \subseteq \Gamma^+$.

We next claim that H_0 can only meet at most one subgraph $R_{U,i}$. Indeed, suppose instead that there are two sets U and U' such that $V(H_0) \cap V(R_{U,i})$ and $V(H_0) \cap V(R_{U',i})$ are both non-empty. As the sets $V(R_{U,i})$ and $V(R_{U',i})$ are disjoint, we may assume without loss of generality that $|V(H_0) \cap V(R_{U,i})| \leq \frac{1}{2}n$.

Since $R_{U,i}$ is only attached to Γ through the vertices in U, the set U must be a cut-set for the subgraph H_0. Let $x \in V(H_0) \cap V(R_{U,i})$ be an arbitrary vertex, and let K be the component of x in $H_0 - U$. We clearly have $|K| \leq |V(H_0) \cap V(R_{U,i})| \leq \frac{1}{2}n$.

On the other hand, observe that x is also in the copy $H_{U,i}$ of H supported on $\{w\} \cup V(F_{U,i}) \cup V(R_{U,i})$. In $H_{U,i}$, the set U is the neighbourhood of w, and, since H is well-behaved, condition $(W2)$ implies x has at most $\frac{1}{2}\delta(H)$ neighbours in U. As $d_{H_0}(x) \geq \delta(H)$, this means x must have at least $\frac{1}{2}\delta(H)$ neighbours in $H_0 - U$. Hence, we also have $|K| \geq \frac{1}{2}\delta(H)$. However, this contradicts condition $(W4)$, as the removal of the vertices in $U \cap V(H_0)$ cannot create a component in H_0 of size between $\frac{1}{2}\delta(H)$ and $\frac{1}{2}n$.

Thus, H_0 meets at most one subgraph $R_{U,i}$. Now, by property (ii) of the colouring c of Γ, we have that any vertex is incident to fewer than $\delta(H)$ edges of colour i in Γ. Thus, in order to be part of H_0, a vertex from Γ must have neighbours in $R_{U,i}$ as well. However, the only such vertices are those in U, and since $|U \cup V(R_{U,i})| = n - 1$, this does not leave us with enough vertices for a copy of H.

Our colouring is therefore indeed H-free, thereby proving the claim. □

This shows that the graph G satisfies conditions (a), (b), and (c), completing the proof. □

3.3.2 Constructing coloured neighbourhoods

The path to proving the lower bounds of Theorem 3.6 is now clearly signposted. By Lemma 2.16, we know that when $\frac{\log n}{n} \ll p \ll 1$, the random graph $H \sim G(n,p)$ is well-behaved with high probability, and hence we are in position to apply Proposition 3.7. We shall then use the results of Section 2.3 to describe the subgraph F induced by the minimum degree vertex in H. This subgraph evolves as the edge probability p increases, and in each range we will construct an appropriate coloured graph Γ that satisfies the conditions of the proposition.

We start with the sparse range, where $p \ll n^{-\frac{2}{3}}$.

Proof of Theorem 3.6(a). Let $q \geq 2$, let p satisfy $\frac{\log n}{n} \ll p \ll n^{-\frac{2}{3}}$, and let $H \sim G(n,p)$. By Lemma 2.16 and Corollary 3.13, we have with high probability that H is well-behaved and the subgraph $F = H[N(u)]$ induced by the neighbourhood of the minimum degree vertex u is empty. In this case, we can simply take Γ to be an empty graph on $q(\delta(H)-1)+1$ vertices. Properties (i) and (ii) of Proposition 3.7 are then trivially satisfied, and so it follows that H is q-Ramsey simple. □

When $p \gg n^{-\frac{2}{3}}$, we will begin to see edges in the neighbourhood of the minimum degree vertex. Provided $p \ll n^{-\frac{1}{2}}$, though, the neighbourhood remains simple in structure, and we can get reasonably sharp bounds on the number of colours for which the random graph is Ramsey simple.

Proof of Theorem 3.6(b), first bound. Let $n^{-2/3} \ll p \ll n^{-1/2}$ and $H \sim G(n,p)$. By Lemma 2.16, we know that with high probability H is well-behaved. Let $\lambda(F)$ be the order of the largest component of the subgraph $F = H[N_H(u)]$ induced by the neighbourhood of the minimum degree vertex u. Given any $\varepsilon > 0$, we shall show that, as n tends to infinity, H is with high probability q-Ramsey simple for every $q \leq (1-5\varepsilon)\frac{\delta(H)}{\lambda(F)^2}$.

By Corollaries 3.14 and 3.15 the graph F is with high probability a very sparse forest. More precisely, if we denote by $T_1, T_2, \ldots, T_t$ the components of F that contain at least one edge, then each T_j is a tree spanning at most $\lambda(F)$ vertices and $\sum_j v(T_j) \leq \varepsilon\delta(H)$.

To prove simplicity, we provide a geometric construction of an edge-coloured graph Γ on $q(\delta(H)-1)+1$ vertices. Let s be the largest prime number that is at most $(1-\varepsilon)\frac{\delta(H)}{\lambda(F)}$. By the upper bound of Baker, Harman, and Pintz [6] on prime gaps, we have $s \geq (1-2\varepsilon)\frac{\delta(H)}{\lambda(F)}$. Now consider the finite affine plane $\mathbb{F}_s^2$, which has s^2 points. Each line in the plane consists of s points, and the set of lines can be partitioned into $s+1$ parallel classes $C_1, C_2, \ldots, C_{s+1}$ of s lines each.

To form the graph Γ, we take as vertices an arbitrary set of $q(\delta(H)-1)+1$ points from $\mathbb{F}_s^2$. Note that our choices of q and s ensure that $q(\delta(H)-1)+1 \leq s^2$ and $q \leq s \leq \delta(H)$. Then, given $x, y \in V(\Gamma)$, we add the edge $\{x,y\}$ if and only if the line they span lies in one of the first q parallel classes. We colour the edges by the parallel classes; that is, if the corresponding line lies in C_i, for some $i \in [q]$, we give the edge $\{x,y\}$ the colour i.

We shall now show that Γ satisfies properties (i) and (ii) of Proposition 3.7, which will show that H is q-Ramsey simple. We start with the latter property. The colour-i subgraph Γ_i of Γ consists of pairs of points in lines in the parallel class C_i. Each such line gives rise to a clique in Γ, and since the lines are parallel, these cliques are vertex-disjoint. Finally, since each line has at most s points in Γ, it follows that $\Delta(\Gamma_i) \leq s-1 \leq \delta(H)-1$, and hence property (ii) holds.

For property (i), we need to show that for any $\delta(H)$-set $U \subseteq V(\Gamma)$ and any colour $i \in [q]$, we can find a copy of F in $\Gamma_i[U]$. We shall embed the trees T_j one at a time. Suppose, for some $j \geq 1$, we have already embedded $T_1, T_2, \ldots, T_{j-1}$, and let $U' \subseteq U$ be the set of vertices we have not yet used. Since F has at most $\varepsilon\delta(H)$ non-isolated vertices, it follows that $|U'| \geq (1-\varepsilon)\delta(H)$.

As observed when showing property (ii), the colour-i subgraph Γ_i is a disjoint union of at most s cliques. Hence, by the pigeonhole principle, U' meets one of these cliques in at least $\frac{|U'|}{s}$ vertices. By our choice of s, this is at least $\lambda(F)$, and so $\Gamma_i[U']$ contains a clique on $\lambda(F)$ vertices, in which we can freely embed T_j.

Repeating this process, we can embed all the trees, thereby obtaining a copy of F in $\Gamma_i[U]$. Hence property (i) is satisfied as well, and thus H is indeed q-Ramsey simple. $\square$

The above construction allows us to obtain lower bounds on $\tilde{q}(H)$ whenever $n^{-2/3} \ll p \ll n^{-1/2}$. However, when $p = n^{-\frac{1}{2}-o(1)}$ and $\lambda(F)$ gets larger, a probabilistic construction yields a better bound.

Proof of Theorem 3.6(b), second bound. Let $p \ll n^{-\frac{1}{2}}$, and let $H \sim G(n, p)$. Our goal is to show that if $q \leq \frac{\delta(H)}{80 \log n}$, then with high probability H is q-Ramsey simple. We again start by collecting some information about the random graph H, before constructing an appropriate graph Γ for Proposition 3.7.

By Lemma 2.11(c) and Lemma 2.16, we may assume that H is well-behaved with $\delta(H) = (1 \pm o(1))np$. Furthermore, applying Corollaries 3.14 and 3.15, we know that with high probability, the subgraph $F = H[N(u)]$ induced by the neighbourhood of the minimum degree vertex is a forest with $o(\delta(H))$ edges containing no tree on more than $\log n$ vertices. We label the components of F as $T_1, T_2, \ldots, T_t$.

We now define the q-coloured graph Γ on $N = q(\delta(H) - 1) + 1$ vertices. We take $\Gamma \sim G(N, \frac{1}{2})$ to be a random graph with edge probability $\frac{1}{2}$. Once we have sampled the graph, we also equip it with a random colouring, colouring each edge independently and uniformly at random from the q colours.

Observe that for each colour $i \in [q]$, the colour-i subgraph $\Gamma_i \subseteq \Gamma$ has the distribution $G(N, \frac{1}{2q})$. Hence, it follows from Lemma 2.11(c), combined with a union bound over the number of colours q, that with high probability $\Delta(\Gamma_i) \leq (1 + o(1))\frac{N}{2q} < \delta(H)$ for every $i \in [q]$. This establishes property (ii) of Proposition 3.7.

We now need to show that property (i) also holds with high probability. That is, we need to ensure that, for every colour $i \in [q]$ and every set $U \subseteq V(\Gamma)$ of $\delta(H)$ vertices, we can find a copy of F in $\Gamma_i[U]$. We shall once again do this by proving the stronger fact that, taking $\varepsilon \geq 0$, for any set U' of $(1 - \varepsilon)\delta(H) \geq \frac{1}{2}np$ vertices, and any tree T on at most $\log n$ vertices, we can embed a copy of T in $\Gamma_i[U']$. We can then greedily embed the components of F one at a time; as F only has $o(\delta(H))$ edges, we will always have at least $(1 - \varepsilon)\delta(H)$ vertices remaining when embedding one of its components.

Applying Lemma 2.12(b) combined with a union bound over the colours $i \in [q]$, we know that with high probability the monochromatic subgraphs Γ_i have the property that the number of edges spanned by any set of $\frac{1}{2}np$ vertices is at least $\frac{1}{4}\left(\frac{1}{2}np\right)^2 \frac{1}{2q} > 2np \log n$.

Since the set U' spans at least $2np \log n$ edges, the average degree in any such subgraph is at least $2 \log n$. By repeatedly removing low-degree vertices, we obtain a subgraph with minimum degree at least $\log n$. It is then trivial to embed a tree on at most $\log n$ vertices in this subgraph, as at each vertex, we will always have enough unused neighbours to embed its children. Thus, we can find disjoint copies of the trees $T_1, T_2, \ldots, T_t$, thereby constructing a copy of F in $\Gamma_i[U]$. This proves property (i), and so by Proposition 3.7 it follows that H is q-Ramsey simple. $\square$

3.3.3 The smallest neighbourhood and quantitative simplicity

We can now combine the results from Section 2.3.1 with Lemma 2.17 to obtain a sequence of corollaries describing the subgraph F induced by the neighbourhood of the minimum degree vertex, which we shall later apply when proving Theorem 3.6. We will also use these to derive Corollary 3.16 from Theorem 3.6.

To start with, for the proof of the Ramsey simplicity of H in case (a) of Theorem 3.6, it will be important that F is an empty graph. This is guaranteed by the following corollary.

Corollary 3.13. *Let $p = p(n) \in (0,1)$ be such that $\frac{\log n}{n} \ll p \ll n^{-\frac{2}{3}}$, and let $H \sim G(n,p)$. Then a.a.s. H has a unique minimum degree vertex u, and $e(N_H(u)) = 0$.*

Proof. By Lemma 2.17 it is enough to prove that, for every $s \in [0.5np, 2np]$, with high probability $G_s \sim G(s,p)$ has no edges. This holds, since by the assumptions on s and p we obtain $\mathbb{E}[e(G_s)] < s^2 p \leq 4n^2p^3 = o(1)$. □

For larger values of p, we can control the number of edges appearing in F, which we will require for the proofs of both simplicity and non-simplicity.

Corollary 3.14. *Let $p = p(n) \in (0,1)$ be such that $n^{-\frac{2}{3}} \ll p \ll 1$, and let $H \sim G(n,p)$. Then a.a.s. H has a unique minimum degree vertex u, and the graph $F = H[N(u)]$ satisfies $\frac{1}{16}n^2p^3 \leq e(F) \leq 4n^2p^3$.*

Proof. For every $s \in [0.5np, 2np]$, Lemma 2.12(a) guarantees that $G_s \sim G(s,p)$ almost surely has $(1+o(1))\frac{s^2p}{2} \in [\frac{1}{16}n^2p^3, 4n^2p^3]$ edges. The above statement now follows by an application of Lemma 2.17. □

Finally, in the range $n^{-\frac{2}{3}} \ll p \ll n^{-\frac{1}{2}}$, when determining the q-Ramsey simplicity of H, we will make use of the fact that F is typically a forest with small components, while also appealing to the fact that its maximum degree cannot be too small.

Corollary 3.15. *Let $p = p(n) \in (0,1)$ be such that $n^{-\frac{2}{3}} \ll p \ll n^{-\frac{1}{2}}$, and let $H \sim G(n,p)$. Then a.a.s. H has a unique minimum degree vertex u, the graph $F = H[N(u)]$ induces a forest, and the order $\lambda(F)$ of the largest component in F satisfies the following bounds:*

(a) $\lambda(F) \leq \frac{1}{2}\log n$,

(b) if $p \ll n^{-\frac{k+1}{2k+1}}$ for some fixed integer $k \geq 2$, then $\lambda(F) \leq k$, and

(c) if $p = n^{-\frac{1}{2}}f^{-1}$ for some $f = f(n)$ satisfying $1 \ll f = n^{o(1)}$, then $\lambda(F) \leq \left(\frac{1}{4} + o(1)\right)\frac{\log n}{\log f}$.

Moreover, the maximum degree $\Delta(F)$ of F a.a.s. satisfies the following:

(d) if $p \gg n^{-\frac{k}{2k-1}}$ for some fixed integer $k \geq 2$, then $\Delta(F) \geq k-1$, and

(e) if $p = n^{-1/2}f^{-1}$ for some $1 \ll f = f(n) = n^{o(1)}$, then $\Delta(F) \geq \left(\frac{1}{2} - o(1)\right)\frac{\log n}{\log(f^2 \log n)}$.

Proof. By Lemma 2.17, it suffices to verify that the corresponding bounds on $\lambda(G_s)$ and $\Delta(G_s)$ for $G_s \sim G(s,p)$ hold with high probability when $s \in [0.5np, 2np]$. These bounds are obtained as follows: for property (a) observe that $p \ll n^{-\frac{1}{2}}$ implies $p \ll s^{-1}$ and $s \ll n^{\frac{1}{2}}$, in which case Lemma 2.13(a) gives that $\lambda(G_s) \leq \log s \leq \frac{1}{2}\log n$ holds a.a.s.. For property (b) we use that $p \ll n^{-\frac{k+1}{2k+1}}$ implies $p \ll s^{-\frac{k+1}{k}}$, and hence

$\lambda(G_s) \le k$ holds a.a.s. by Lemma 2.13(b). For properties (c) and (e) observe that $p = n^{-\frac{1}{2}} f^{-1}$ implies $\frac{0.5}{sf^2} \le p \le \frac{2}{sf^2}$ and $s = n^{\frac{1}{2}-o(1)}$, which a.a.s. leads to $\lambda(G_s) \le (1+o(1))\frac{\log s}{\log f^2} \le \left(\frac{1}{4}+o(1)\right)\frac{\log n}{\log f}$ by Lemma 2.13(c), and to $\Delta(G_s) \ge (1-o(1))\frac{\log s}{\log(f^2 \log s)} \ge \left(\frac{1}{2}-o(1)\right)\frac{\log n}{\log(f^2\log n)}$ by Lemma 2.11(b). Finally, for property (d) we note that $p \gg n^{-\frac{k}{2k-1}}$ implies $p \gg s^{-\frac{k}{k-1}}$, and hence Lemma 2.11(a) ensures that $\Delta(G_s) \ge k-1$ a.a.s. $\square$

With these bounds on the parameters of the subgraph F induced by the neighbourhood of the minimum degree vertex, we are now in position to deduce Corollary 3.16, giving quantitative estimates on the value of $\tilde{q}(H)$ in the intermediate range.

Corollary 3.16. *Let $k \ge 2$ be a fixed integer and let $f = f(n)$ satisfy $1 \ll f = n^{o(1)}$. Let $p = p(n)$ satisfy $n^{-\frac{2}{3}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{2}}$ and let $H \sim G(n,p)$. Then a.a.s. the following bounds hold:*

(a) if $n^{-\frac{k}{2k-1}} \ll p \ll n^{-\frac{k+1}{2k+1}}$, then $(1+o(1))\frac{np}{k^2} \le \tilde{q}(H) \le (1+o(1))\frac{np}{k-1}$.

(b) if $p = \Theta\left(n^{-\frac{k+1}{2k+1}}\right)$, then $(1+o(1))\frac{np}{(k+1)^2} \le \tilde{q}(H) \le (1+o(1))\frac{np}{k-1}$.

(c) if $p = n^{-\frac{1}{2}} f^{-1}$, then
$$(1+o(1))\frac{np}{\log n}\max\left\{\frac{16\log^2 f}{\log n}, \frac{1}{80}\right\} \le \tilde{q}(H) \le (2+o(1))\frac{np\log(f^2\log n)}{\log n}.$$

(d) if $n^{-\frac{1}{2}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{2}}$, then $1 \le \tilde{q}(H) \le (8+o(1))\frac{1}{p}$.

Proof. We start by appealing to Lemma 2.11(c) to observe that a.a.s. $\delta(H) = (1+o(1))np$.

Let us begin by establishing the lower bounds on $\tilde{q}(H)$. By Theorem 3.6 we have $\tilde{q}(H) \ge (1+o(1))\max\left\{\frac{\delta(H)}{\lambda(F)^2}, \frac{\delta(H)}{80\log n}\right\}$, and we can bound $\lambda(F)$ using Corollary 3.15.

When $p \ll n^{-\frac{k+1}{2k+1}}$ for some fixed integer $k \ge 2$, then, by Corollary 3.15(b), we a.a.s. have $\lambda(F) \le k$. Thus, in this range, we have $\tilde{q}(H) \ge (1+o(1))\frac{np}{k^2}$ a.a.s., which yields the lower bounds for parts (a) and (b) of Corollary 3.16 (note that when $p = \Theta\left(n^{-\frac{k+1}{2k+1}}\right)$, we have $p \ll n^{-\frac{(k+1)+1}{2(k+1)+1}}$). The lower bound in part (c) follows by substituting the bound on $\lambda(F)$ from Corollary 3.15(c), while the lower bound in part (d) is trivial.

For the upper bounds, Theorem 3.6 gives $\tilde{q}(H) \le \min\left\{\frac{\delta(H)}{\Delta(F)}, \frac{\delta(H)^2}{2e(F)}\right\}$. The upper bounds in parts (a), (b), and (c) come from substituting the appropriate lower bounds on $\Delta(F)$ given by Corollary 3.15. When $p \gg n^{-\frac{k}{2k-1}}$ for some fixed k, Corollary 3.15(d) yields $\Delta(F) \ge k-1$ a.a.s., which provides the upper bounds in parts (a) and (b) of Corollary 3.16. The upper bound in part (c) follows similarly, using the lower bound on $\Delta(F)$ from Corollary 3.15(e). Finally, for the upper bound in part (d) of Corollary 3.16, we use Corollary 3.14, which asserts that a.a.s. $e(F) \ge \frac{1}{16}n^2p^3$. Thus $\frac{\delta(H)^2}{2e(F)} \le \frac{8+o(1)}{p}$, as required. $\square$

3.4 Non-simplicity for $G(n,p)$

In this section we prove the upper bounds on $\tilde{q}(H)$ from Theorem 3.6. These are the negative results, showing that with high probability $H \sim G(n,p)$ is not q-Ramsey simple for large values of q.

For the proofs, the centre of attention will again be the neighbourhood of the minimum degree vertex of H. We will prove upper bounds for case (c) of our theorem. In the proof below, we first establish that, if the neighbourhood of the minimum degree vertex exhibits a high maximum degree, then H cannot be q-Ramsey simple for a large enough q. For this, we will need the following result from [63].

Theorem 3.17 ([63])**.** *Let G be an n-vertex graph of average degree d and let $k \in \mathbb{N}$. Then there is a set U of at least $(k+1)n/(d+k+1)$ vertices such that $\Delta(G[U]) \leq k$.*

Proof of Theorem 3.6(c). Let $n^{-2/3} \ll p \ll 1$ and $H \sim G(n,p)$. By Lemma 2.16, we know that H has a unique vertex u of minimum degree. As before, we set $F = H[N_H(u)]$.

Suppose that H is q-Ramsey simple and G is a minimal q-Ramsey graph for H with minimum degree $N = q(\delta(H)-1)+1$. Let w be a vertex of minimum degree in G, and $\Gamma = G[N_G(w)]$. It follows from Remark 3.8 that there is an edge-colouring of Γ such that the induced graph on every $\delta(H)$-set of vertices contains, in each colour, a copy of F.

We are now ready to prove that $\tilde{q}(H) \leq (1+o(1))\frac{\delta(H)}{\Delta(F)}$. The above observation implies that the induced graph on each $\delta(H)$-set has, in each colour, a vertex of degree at least $\Delta(F)$. However, the average degree of the sparsest colour class in Γ is at most $\frac{2\binom{N}{2}}{qN} = \frac{N-1}{q} = \delta - 1$. Thus, by Theorem 3.17, Γ has a set of $\frac{\Delta(F)N}{\delta(H)+\Delta(F)-1}$ vertices that induce a graph with maximum degree less than $\Delta(F)$ in this colour. Hence, we must have

$$\frac{\Delta(F)N}{\delta(H)+\Delta(F)-1} \leq \delta(H)-1,$$

which rearranges to give $q \leq \frac{\delta(H)+\Delta(F)-1}{\Delta(F)} - \frac{1}{\delta(H)-1}$, from which the conclusion follows.

We turn our attention to the second bound, namely $\tilde{q}(H) \leq (1+o(1))\frac{\delta(H)^2}{2e(F)}$. For any subset $U \subseteq V(\Gamma)$ of size $\delta(H)$, there must be a colour $i \in [q]$ such that there are at most $\frac{1}{q}\binom{\delta(H)}{2}$ edges of colour i inside U. Using once again our observation above, we know that $\Gamma[U]$ contains a copy of F in colour i, and therefore we must have $\frac{1}{q}\binom{\delta(H)}{2} \geq e(F)$, which yields the claimed bound. $\square$

We remark that the proofs of both upper bounds in Theorem 3.6(c) do not use the fact that H is a random graph, and are valid for any graph that has a unique vertex of minimum degree whose neighbourhood is not an independent set.

3.5 Concluding remarks

In this chapter we built upon the work of Grinshpun [54] and studied the q-Ramsey simplicity of $H \sim G(n,p)$ for a wide range of values of p and q. We encountered three different types of behaviour: for very sparse ranges, i.e., when $p \ll \frac{1}{n}$ or $\frac{\log n}{n} \ll p \ll n^{-\frac{2}{3}}$, we showed that a.a.s. H is q-Ramsey simple for every possible number of colours q; for much denser ranges, i.e., when $p \gg \left(\frac{\log n}{n}\right)^{\frac{1}{2}}$, a.a.s. we do not have Ramsey simplicity even when $q = 2$; in between these ranges, when $n^{-\frac{2}{3}} \ll p \ll n^{-\frac{1}{2}}$, there exists a finite threshold value $\tilde{q}(H) \geq 2$ on the number of colours q such that H is q-Ramsey simple if and only if $q \leq \tilde{q}(H)$. We determined this threshold up to a constant or, when $p = n^{-\frac{1}{2}-o(1)}$, logarithmic factor. Several natural questions remain open.

First, our main result does not provide any information on the Ramsey simplicity of $G(n,p)$ when p is between $\frac{1}{n}$ and $\frac{\log n}{n}$.

Question 3.18. *What can be said about $\tilde{q}(H)$ when $H \sim G(n,p)$ for $p = \Omega\left(\frac{1}{n}\right)$ and $p = O\left(\frac{\log n}{n}\right)$? In particular, is H a.a.s. 2-Ramsey simple in this case?*

In the range $p \gg \frac{\log n}{n}$ our simplicity proofs rely heavily on the fact that a.a.s. $H \sim G(n,p)$ is 3-connected, implying the existence of signal senders for H, which in turn allow us to deduce a fairly general recipe for constructing suitable Ramsey graphs. When $p \ll \frac{1}{n}$, we know that $H \sim G(n,p)$ is a.a.s. a forest, and simplicity follows from the construction of Szabó, Zumstein, and Zürcher [83], which works for certain bipartite graphs. When $\frac{1}{n} \ll p \ll \frac{\log n}{n}$, however, the random graph $G(n,p)$ becomes more complex (in particular, it is non-bipartite) but it is not yet connected. As a result, resolving the aforementioned question will likely require new ideas.

Second, when p lies in the range $p = \Omega\left(n^{-\frac{1}{2}}\right)$ and $p = O\left(\left(\frac{\log n}{n}\right)^{\frac{1}{2}}\right)$, we proved that $\tilde{q}(H) = O(p^{-1})$, which shows that the threshold value here is of smaller order than when $p = n^{-\frac{1}{2}-o(1)}$, as demonstrated in Corollary 3.16. However, we did not provide any non-trivial lower bounds, and we wonder if that might not be possible.

Question 3.19. *Is it true that H is a.a.s. not 2-Ramsey simple when $H \sim G(n,p)$ with $p = \Omega\left(n^{-\frac{1}{2}}\right)$ and $p = O\left(\left(\frac{\log n}{n}\right)^{\frac{1}{2}}\right)$?*

In this case, signal senders for H do exist, but the neighbourhood of the minimum degree vertex becomes more complex than just a forest, making it difficult to construct a graph as described in Remark 3.8. On the other hand, the presence of isolated vertices makes it likely that a more delicate argument than the one used in part (b) would be needed to show non-simplicity for smaller q. Nevertheless, we tend to believe that a.a.s. $\tilde{q}(H) = 1$ for all $p \gg n^{-\frac{1}{2}}$.

The bounds on $\tilde{q}(H)$ presented in cases (a) and (c) are already quite close, but it would be interesting to close the remaining gaps.

Question 3.20. *Let $H \sim G(n,p)$ with $n^{-2/3} \ll p \ll n^{-1/2}$. What are the asymptotics of $\tilde{q}(H)$?*

In this range, as we have seen in Section 3.3, the question about q-Ramsey simplicity is tightly linked to the problem of finding a q-coloured graph Γ on $q(\delta(H)-1)+1$ vertices such that the following holds: For every set $U \subseteq V(\Gamma)$ of $\delta(H)$ vertices and for every colour $i \in [q]$, there exists a copy $F_{U,i}$ of $F = H[N(u)]$ in $\Gamma[U]$ whose edges are all of colour i. The proofs of our lower bounds in Section 3.3 are obtained by finding such Γ (with additional properties as given in Proposition 3.7) through explicit constructions or probabilistic arguments. In order to prove that a.a.s. H is not q-Ramsey simple, it would suffice to prove that such Γ does not exist, that is, every q-coloured graph Γ on $q(\delta(H)-1)+1$ contains *at least* one subset $U \subseteq V(\Gamma)$ of size $\delta(H)$ such that $\Gamma[U]$ is missing a copy of F in at least one colour. Note that in the proof of our second bound in case (c) of Theorem 3.6 we obtain such a result by a simple counting argument which guarantees that we cannot pack q copies of F into any graph on $\delta(H)$ vertices. Related to this argument, it seems challenging to determine how many copies of a given random graph can be packed into a complete graph, leading us to suggest the following question.

Question 3.21. *Let $H \sim G(n,p)$ with $0 < p < 1$. How many copies of H can be packed into K_n?*

In the densest range, that is, when $p \gg \left(\frac{\log n}{n}\right)^{\frac{1}{2}}$, we know that $H \sim G(n,p)$ is a.a.s. not q-Ramsey simple for any $q \geq 2$. We wonder, however, what the behaviour of $s_q(H)$ in this case is; in particular, it would be interesting to determine whether $s_q(H)$ is still typically close to the easy lower bound $q(\delta(H)-1)+1$. Note that the answer is no if $p=1$ and $q \geq 2$, since $s_2(K_n) = (n-1)^2$. However, when $\left(\frac{\log n}{n}\right)^{\frac{1}{2}} \ll p \ll 1$, we do not know of any bounds other than the general ones mentioned in the introduction. In particular, we propose the following problem, similar to one posed by Grinshpun, Raina, and Sengupta [55].

Question 3.22. *How large is $s_2(H)$ for $H \sim G(n, \frac{1}{2})$ a.a.s.?*

Related to the above discussion, we also note that our methods can be applied to the 2-colour asymmetric Ramsey setting, in which a graph G is said to be 2-Ramsey for a pair of graphs (H_1, H_2) if every red-/blue-colouring of its edges leads to a red copy of H_1 or a blue copy of H_2. In this setting, we define minimal Ramsey graphs and the smallest minimum degree $s_2(H_1, H_2)$ in the obvious way; the general lower bound is replaced by $s_2(H_1, H_2) \geq \delta(H_1) + \delta(H_2) - 1$ and again we call a pair (H_1, H_2) 2-Ramsey simple if this lower bound is attained. Our constructions can be modified to show that for $H_1 \sim G(n, p_1)$ and $H_2 \sim G(n, p_2)$ the pair (H_1, H_2) is a.a.s. 2-Ramsey simple if $\frac{\log n}{n} \ll p_1 \leq p_2 \ll n^{-1/2}$. When $p_1, p_2 \ll n^{-1}$, then again a modification of the argument of Szabó, Zumstein, and Zürcher [83] can be used to show that we a.a.s. have 2-Ramsey simplicity. Still, the following questions remain.

Question 3.23. *Let $H_1 \sim G(n, p_1)$ and $H_2 \sim G(n, p_2)$ with $p_1 \ll n^{-1}$ and $\frac{\log n}{n} \ll p_2 \ll n^{-1/2}$. Is the pair (H_1, H_2) a.a.s. 2-Ramsey simple? What happens if one of the graphs comes from the dense range?*

We also remark that our ideas from Section 4.2.5 can be used to resolve a special case of a conjecture due to Grinshpun [54], stating that all triangle-free graphs are 2-Ramsey simple. In [55], Grinshpun, Raina, and Sengupta use a construction similar to ours to show that the conjecture is true for all regular 3-connected triangle-free graphs satisfying one extra technical condition. Our approach allows us to prove that every well-behaved triangle-free graph is q-Ramsey simple for any $q \geq 2$.

Finally, let us emphasise that there has been little study of (minimal) Ramsey graphs for $G(n,p)$. The only results we are aware of concern the Ramsey number of $G(n,p)$, as mentioned in Section 1.2.3. Hence, as a more general direction for future research, it would be interesting to explore other aspects of the Ramsey behaviour of $G(n,p)$ as the target graph.

4 Abundance of graphs

In this chapter we will investigate the property of abundance for various graph classes. Let us begin by recalling the definition of abundance for graphs.

Definition 4.1. For a given integer $q \geq 2$, a graph H is said to be s_q*-abundant* if, for every $k \geq 1$, there exists a minimal q-Ramsey graph for H with at least k vertices of degree $s_q(H)$.

In this formulation, Burr, Erdős, and Lovász [23] had shown that for all $t \geq 3$, the graph K_t is s_2 abundant. We will further this study and develop tools that will allow us to show that K_t is in fact s_q-abundant for all values of $q \geq 2$. These tools, that we will call *pattern gadgets* are a far reaching generalisation of the gadgets which were first developed in [23].

In Section 4.1, we will first define a pattern gadget. Before we provide an explicit construction for these graphs, we will require a generalisation of indicators. We will define these intermediate gadgets and provide a construction for them, see Subsection 2.2.3 for an overview on indicators. Hereafter, we will be ready to show the existence of patter gadgets for many target graphs H, including all 3-connected graphs.

Thereafter, in Section 4.2, we will go on to use the gadgets to show that large classes of graphs are s_q-abundant. To begin with we will see that for $t \geq 4$ all cycles C_t are s_q-abundant for $q \geq 2$. We will then show that cliques with a pendant edge $K_t \cdot K_2$, for $t \geq 3$, are s_2-abundant.

In Subsection 4.2.3 we will go on to apply the pattern gadgets to prove a general result on abundance for 3-connected graphs. We will use this result to show that the wheel graph W_t is s_2-abundant for $t \geq 4$. We will then go on to show that for certain ranges of p and q the graph $G(n, p)$ is s_q-abundant.

As has been mentioned earlier, this chapter is an extension of joint work with Simona Boyadzhiyska, and Dennis Clemens [18] and Subsection 4.2.5 is based on a joint work with Simona Boyadzhiyska, Dennis Clemens, and Shagnik Das [17].

4.1 Construction of pattern gadgets

As has been discussed, most of our constructions of minimal Ramsey graphs will rely on the existence of certain gadget graphs; these graphs will have the property that, in every colouring not containing a monochromatic copy of our target graph H, some fixed *colour patterns* need to appear on certain sets of edges. Such an approach has already been used in the paper of Burr et al. [23] when proving that $s_2(K_t) = (t-1)^2$. In their paper, the authors introduced gadget graphs that are now known as BEL gadgets and are defined as follows: Let H and G be fixed graphs such that $G \not\to_q H$, and let φ be an H-free q-colouring of G; a *BEL gadget* for H with respect to the pair (G, φ) is a graph B containing G as an induced subgraph such that B is not q-Ramsey for H but in every H-free q-colouring of the edges of B, the subgraph G has the colouring given by φ (up to a permutation of colours). Burr et al. showed the existence of BEL gadgets for all cliques on at least three vertices when $q = 2$ (for any appropriate choice of G and φ). Later results imply that BEL gadgets exist for more general graphs and for more colours; an overview of those results can be found in Section 2.2.3.

Suppose we want to construct a minimal q-Ramsey graph for H that contains a vertex of degree at most d. Provided that a BEL gadget with certain properties exists, it suffices to find a graph G that contains a vertex v of degree d and a q-colouring φ of $G - v$ that contains no monochromatic copy of H but cannot be extended to an H-free colouring of G. Indeed, we can construct $\widetilde{G}$ by taking a copy G' of $G - v$ and a BEL gadget for H with respect to (G', φ) and adding the vertex v along with d edges so that $V(G') \cup \{v\}$ induces a copy of G. Now it is not difficult to check that $\widetilde{G} \to_q H$, and if H satisfies certain conditions, then we can also ensure that $\widetilde{G} - v \not\to_q H$. This means that any minimal q-Ramsey subgraph of $\widetilde{G}$ needs to contain v, that is, v is *important* for $\widetilde{G}$ to be a q-Ramsey graph, and $s_q(H) \leq d_{\widetilde{G}}(v)$.

For our main theorems, we will aim to find graphs $\widetilde{G}$ with many vertices of small degree, each of which is important for $\widetilde{G}$ to be a Ramsey graph for H. In order to do so, we will use a gadget that allows for more flexibility than a BEL gadget. This gadget again comes with a subgraph G on which fixed colour patterns are forced in any H-free q-colouring. However, while for a BEL gadget we fix only a single permissible pattern (up to a permutation of the colour classes), our gadget graph allows us to fix a family of permissible colour patterns for G such that each of these patterns, and no other, can be extended to an H-free colouring of the whole graph.

To make this more precise, let us first define colour patterns and an isomorphism relation between them.

Definition 4.2. Let $q \geq 2$ be a given integer and H and G be graphs. A *q-colour pattern* for G is a partition $g = \{G_1, G_2, \ldots, G_q\}$ of the edges of G. If $H \not\subseteq G_i$ for every $i \in [q]$, we say that g is *H-free.* Given any subset $A \subseteq V(G)$, we call the partition $g[A] = \{G_1[A], G_2[A], \ldots, G_q[A]\}$ the *induced q-colour pattern* on A.

Let G' be a copy of G, and let $g' = \{G'_1, \ldots, G'_q\}$ be a q-colour pattern for G'. Then we say that g and g' are *isomorphic*, denoted by $g \cong g'$, if there exists a permutation π of $[q]$ such that $G_i \cong G'_{\pi(i)}$ for every $i \in [q]$.

Using the above terminology, we can now give a precise definition of the gadget graphs that we are interested in.

Definition 4.3. Let $q \geq 2$ be a given integer and H and G be graphs such that $G \not\to_q H$. Also let $\mathscr{G}$ be a family of H-free q-colour patterns for G. Then we call a graph $P = P(H, G, \mathscr{G}, q)$ a *pattern gadget* if the following properties hold:

$(P1)$ $G \subseteq_{ind} P$.

$(P2)$ If $c : E(P) \to [q]$ is an H-free colouring of P, then $\{c_{|G}^{-1}(1), \ldots, c_{|G}^{-1}(q)\} \in \mathscr{G}$.

$(P3)$ For every pattern $\{G_1, \ldots, G_q\} \in \mathscr{G}$, there exists an H-free colouring $c : E(P) \to [q]$ such that $\{c_{|G}^{-1}(1), \ldots, c_{|G}^{-1}(q)\} = \{G_1, \ldots, G_q\}$.

A variant of these gadgets was defined by Siggers [81], who showed its existence for cycles. The rest of this section is mainly devoted to our proof that pattern gadgets exist for certain choices of the graph H. In the proof, we will combine various intermediate gadgets and for that to work we will often require them to satisfy an additional property of robustness which has already been defined in Section 2.2.3. We recall the definition here.

Definition 2.6. Let G be a graph and G_0 be an induced subgraph of G. We say that the pair (G, G_0) is H-robust if, in any graph obtained from G by adding any set S of new vertices and any collection of edges within $S \cup V(G_0)$, every copy of H is entirely contained either in G or in the subgraph induced by $S \cup V(G_0)$.

The main theorem of this section states that, if H is 3-connected or isomorphic to a cycle or $K_t \cdot K_2$, then pattern gadgets that satisfy certain robustness properties exist for H.

Theorem 4.4. *Let $q \geq 2$ be a given integer, and let H and G be graphs with $G \not\to_q H$. Further, let $\mathscr{G}$ be a family of H-free q-colour patterns for G.*

(a) If H is 3-connected or a triangle, then a pattern gadget $P = P(H, G, \mathscr{G}, q)$ exists.

(b) If H is a cycle of length at least four, then a pattern gadget $P = P(H, G, \mathscr{G}, q)$ exists.

(c) If $H \cong K_t \cdot K_2$ and $q = 2$ and G does not contain a copy of H, then a pattern gadget $P = P(H, G, \mathscr{G}, q)$ exists. Further, we can ensure that in the 2-colourings in $(P3)$ every monochromatic copy of K_t using a vertex from G is fully contained in G.

Further, in parts (a) and (b), the pattern gadget can be taken so that (P, G) is H-robust, and in part (c), it can be taken so that (P, G) is K_t-robust.

Before we give the proof of Theorem 4.4 in Section 4.1.2, we need to construct a generalisation of the *indicators*, which were introduced in Section 2.2.3. We will call these the *generalised negative indicators* and will define and construct them shortly in the next section, Section 4.1.1.

4.1.1 Generalised negative indicators

Before we can prove the existence of pattern gadgets as stated in Theorem 4.4, we will first need to construct slightly weaker gadget graphs, which we call generalised negative indicators.

Recall that a negative indicator $I = I^-(H, F, e, q, d)$ comes with an indicator subgraph F and an indicator edge e and has the following property: In any H-free q-colouring of I that colours F monochromatically, e needs to get a colour different from that of F; but once F is not monochromatic, we can extend the q-colouring to an H-free q-colouring of I, independently of which colour is chosen for e. That is, in short, when F is monochromatic we get some information on the colour given to e, while otherwise we do not.

The gadgets I^* described in the following will generalise this concept by replacing e with another graph G. Now, whenever the indicator subgraph F is monochromatic in an H-free q-colouring of I^*, we again want to get some information on the colouring given to G, namely that a certain colour pattern is forced on G. Otherwise, when F is not monochromatic, we do not get any information on G in the sense that we can still colour this subgraph by any H-free q-colouring and then find an H-free extension to I^*. We give a precise definition below.

Definition 4.5. Let $q \geq 2$ and $d \geq 1$ be integers, and let H, F, and G be graphs with $H \not\subseteq F$. Further, let $G = G_1 \cup G_2 \cup \ldots \cup G_{q-1}$ be a partition with $H \not\subseteq G_k$ for every $k \in [q-1]$. We call a graph $I^* = I^*(H, F, \{G_k\}_{k \in [q-1]}, q, d)$ a *generalised negative indicator* if the following properties hold:

$(GI1)$ $F, G \subseteq_{ind} I^*$ and $\mathrm{dist}_{I^*}(F, G) \geq d$.

$(GI2)$ There exists an H-free q-colouring of I^* such that F is monochromatic.

$(GI3)$ In any H-free colouring $c : E(I^*) \to [q]$ in which F is monochromatic, each of the graphs G_i needs to be monochromatic so that $\{c(F), c(G_1), \ldots, c(G_{q-1})\} = [q]$.

$(GI4)$ Let $\varphi_F : E(F) \to [q]$ be any non-constant colouring, and let $\varphi_G : E(G) \to [q]$ be any H-free colouring. Then there exists an H-free colouring $c : E(I^*) \to [q]$ such that $c_{|F} = \varphi_F$ and $c_{|G} = \varphi_G$.

If I^* is a generalised negative indicator with parameters $H, F, \{G_k\}_{k \in [q-1]}, q$, and d, we call I^* a *generalised negative* $(H, F, \{G_k\}_{k \in [q-1]}, q, d)$*-indicator*. In this case, we call F and G the *indicator subgraphs* of I^*.

An *interior* vertex of a generalised negative indicator is a vertex that belongs to neither of the indicator subgraphs. The *interior* of a generalised negative indicator is the set of all interior vertices.

The following lemma states that, if H is 3-connected or isomorphic to a cycle or $K_t \cdot K_2$, then generalised negative indicators that satisfy additional robustness properties exist for H.

Lemma 4.6. *Let $q \geq 2$ and $d \geq 1$ be integers, and H, F, and G be graphs with $H \not\subseteq F$. Further, let $G = G_1 \cup \ldots \cup G_{q-1}$ be a partition such that $H \not\subseteq G_k$ for every $k \in [q-1]$.*

(a) If H is 3-connected or $H \cong K_3$, then a generalised negative indicator $I^* = I^*(H, F, \{G_k\}_{k \in [q-1]}, q, d)$ exists.

(b) If $H \cong C_t$ for $t \geq 4$ and $girth(F) > t$, then a generalised negative indicator $I^* = I^*(H, F, \{G_k\}_{k \in [q-1]}, q, d)$ exists.

(c) If $H \cong K_t \cdot K_2$ for $t \geq 3$ and $q = 2$, then a generalised negative indicator $I^* = I^*(H, F, \{G_k\}_{k \in [q-1]}, q, d)$ with the following additional property exists: The H-free 2-colourings in $(GI2)$ and $(GI4)$ can be chosen so that every monochromatic copy of K_t using a vertex from $F \cup G$ is contained fully in $F \cup G$.

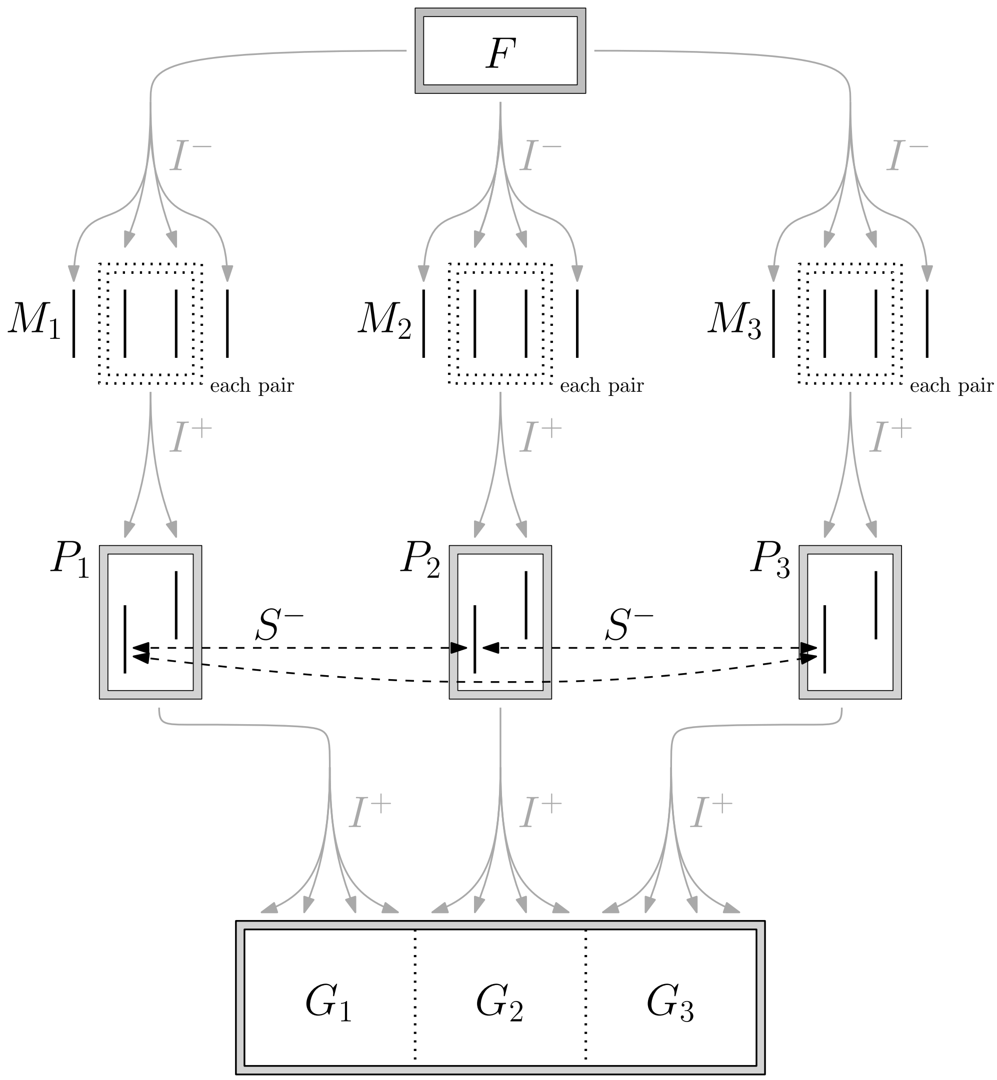

Figure 4.1: Generalised negative indicator for $q = 4$.

Further, in parts (a) and (b), the generalised negative indicator can be taken so that (I^*, F) and (I^*, G) are H-robust. In part (c), we can ensure that (I^*, F) and (I^*, G) are K_t-robust.

Proof. Let q, d, H, F, and G be as given, and without loss of generality assume that $d \geq v(H) + 1$. Let $M_1, \ldots, M_{q-1}$ be matchings of size q, let $P_1, \ldots, P_{q-1}$ be matchings of size two, and let e_k be a fixed edge of P_k for each $k \in [q-1]$.

In order to construct I^*, we take the vertex-disjoint union of F, G and all of the above matchings and we join them with signal senders and indicators in the following way:

(i) For every $k \in [q-1]$ and every edge $m \in M_k$, join F and m by a negative (H, F, m, q, d)-indicator.

(ii) For every $k \in [q-1]$, every submatching $S \subseteq M_k$ of size two, and every edge $p \in P_k$, join S and p by a positive (H, S, p, q, d)-indicator.

(iii) For every $1 \leq k_1 < k_2 \leq q-1$, join the distinguished edges $e_{k_1} \in P_{k_1}$ and $e_{k_2} \in P_{k_2}$ by a negative signal sender $S^- = S^-(H, e_{k_1}, e_{k_2}, q, d)$.

(iv) For every $k \in [q-1]$ and every edge $g \in E(G_k)$, join P_k and g by a positive (H, P_k, g, q, d)-indicator.

Moreover, let all the indicators satisfy the robustness property promised by Theorem 2.7 respectively. When H is a cycle of length $t \geq 4$, choose the gadgets in (i)–(iv) so that their girth equals t. When $H \cong K_t \cdot K_2$ for some $t \geq 3$ and $q = 2$, choose these gadgets so that they have a $K_t \cdot K_2$-special 2-colouring. Note that the existence of all these gadgets and colourings is given by Theorem 2.4 and Theorem 2.7. An illustration of the construction for the case $q = 4$ can be found in Figure 4.1.

Let $M_k = \{m_1^k, \ldots, m_q^k\}$ for every $k \in [q-1]$. Before showing that I^* satisfies $(GI1)$–$(GI4)$, we first discuss where copies of H can be located in the graph I^*. Note that from the following two observations we immediately obtain the desired robustness properties as stated in Lemma 4.6.

Observation 4.7. *Let H be 3-connected or a cycle. Let I' be a graph obtained from I^* by adding two new vertex sets S_F and S_G and any collection of edges within $S_F \cup V(F)$ and within $S_G \cup V(G)$. Then every copy of H in I' is fully contained in one of the indicators from (i), (ii), or (iv), in one of the signal senders from (iii), or in one of the subgraphs induced by $S_F \cup V(F)$ or $S_G \cup V(G)$.*

Proof. For a contradiction, assume that some copy H' of H in I' forms a counterexample. Consider first the case when H' uses a vertex $v \in S_G \cup V(G)$. Since H' is a counterexample, we have $V(H') \not\subseteq S_G \cup V(G)$. Hence, H' needs to use an interior vertex of one of the indicators in (iv); without loss of generality, assume it is an indicator $I^+_{P_1}$ joining P_1 with an edge of G_1. We then have $\mathrm{dist}_{I^*}(P_1, G) \geq d > v(H')$ by property $(I1)$ of the indicators in (iv), and thus, since H' is 3-connected or a triangle or a cycle with $v(H') = \mathrm{girth}(I^+_{P_1})$, it follows that $H' \subseteq I^+_{P_1}$, a contradiction. We may therefore assume that H' is vertex-disjoint from $S_G \cup V(G)$.

Consider next the case when H' uses a vertex $v \in S_F \cup V(F)$. As before, we have $V(H') \not\subseteq S_F \cup V(F)$. Hence, H' needs to use an interior vertex of an indicator in (i);

without loss of generality, assume it is an indicator I_1 between F and an edge $m \in M_1$. But then, since $\mathrm{dist}_{I_1}(m, F) \geq d > v(H')$ by property $(I1)$ and since (I_1, F) is H-robust by Theorem 2.7, we conclude that $H' \subseteq I_1$ must hold, contradicting our assumption. Hence, we may also assume that H' is vertex-disjoint from $S_F \cup V(F)$.

Now, if H' uses an interior vertex of one of the signal senders S^- in (iii), say between the edges e_{k_1} and e_{k_2}, then again, using that $\mathrm{dist}_S(e_{k_1}, e_{k_2}) \geq d$ by property $(S3)$ and that H' is 3-connected or isomorphic to a triangle or H' is a cycle with $v(H') = \mathrm{girth}(S)$, we deduce that H' must be fully contained in that signal sender.

Next, if H' uses an interior vertex of one of the indicators in (i), (ii) or (iv), using the same argument and the robustness properties of our indicators, guaranteed by Theorem 2.7 for positive and negative indicators respectively, we again conclude that H' must be fully contained in that indicator.

Hence, we are left with the case when H' uses neither vertices from $S_F \cup V(F)$, nor vertices from $S_G \cup V(G)$, nor interior vertices from one of the gadgets in (i)–(iv). But then $H' \subseteq \bigcup_{k \in [q-1]} (M_k \cup P_k)$, which contradicts the fact that H' contains at least one cycle. ✓

Observation 4.8. *Let $H \cong K_t \cdot K_2$. Let I' be a graph obtained from I^* by adding two new vertex sets S_F and S_G and any collection of edges within $S_F \cup V(F)$ and within $S_G \cup V(G)$. Then every copy of K_t in I' is fully contained in one of the indicators from (i), (ii), or (iv), in one of the signal senders from (iii), or in one of the subgraphs induced by $S_F \cup V(F)$ or $S_G \cup V(G)$.*

Proof. The proof is analogous to the previous proof, except that we use the robustness properties of all gadget graphs with respect to K_t, guaranteed by Theorem 2.7 for the indicators in (i), (ii), and (iv). ✓

It remains to show that I^* satisfies $(GI1)$–$(GI4)$ and to verify the additional property required in case (c) regarding the existence of $K_t \cdot K_2$-special 2-colourings for $(GI2)$ and $(GI4)$.

$(GI1)$ The graph F is an induced subgraph of I^*, as it is an induced subgraph of each of the negative indicators in (i) by property $(I1)$. Also G is an induced subgraph of I^*, since in the construction of I^* we attach gadget graphs to single edges of G without adding any further edges inside $V(G)$. Moreover, we have $\mathrm{dist}_{I^*}(F, G_k) \geq d$, since, for every $k \in [q-1]$ and every $m \in M_k$, the joining (H, F, m, q, d)-indicator I_F^- from (i) satisfies $\mathrm{dist}_{I_F^-}(F, m) \geq d$ by property $(I1)$.

$(GI2)$ We define a colouring $c : E(I^*) \to [q]$ as follows:

- Give colour 1 to the edges of F.
- For every $k \in [q-1]$, give colour $k+1$ to the edges m_1^k and m_2^k.
- For every $k \in [q-1]$, colour the edges of $M_k \setminus \{m_1^k, m_2^k\}$ such that each colour from $[q] \setminus \{1, k+1\}$ is used exactly once.
- For every $k \in [q-1]$, give colour $k+1$ to the edges of P_k and G_k.
- Finally, extend this colouring to each of the indicators and signal senders in (i)–(iv) so that none of these contains a monochromatic copy of H. In case (c), choose these colourings to be $K_t \cdot K_2$-special.

The extension in the last step of the colouring is possible for the following reason: For the indicators in (i), we can find such an extension by properties $(I2)$ and $(I3)$ for negative indicators and since $c(F) = 1 \neq c(m)$ for every $k \in [q-1]$ and $m \in M_k$. For the indicators in (ii), consider two cases. If $S = \{m_1^k, m_2^k\}$, then we have $c(S) = c(P_k) = k+1$, and hence we can colour as desired by properties $(I2)$ and $(I3)$. Otherwise, if $S \in \binom{M_k}{2}$ is different from $\{m_1^k, m_2^k\}$, the colouring on S is not constant and hence we can extend as desired by property $(I4)$. For the signal senders in (iii), the described extension is possible by properties $(S1)$ and $(S2)$ for negative signal senders and since $c(e_{k_1}) \neq c(e_{k_2})$ for every distinct $k_1, k_2 \in [q-1]$. For the indicators in (iv), we again use properties $(I2)$ and $(I3)$ plus the fact that $c(P_k) = c(G_k)$ for every $k \in [q-1]$.

It remains to check that the resulting colouring c on I^* is H-free. Consider first the case when H is a cycle or 3-connected. By Observation 4.7, we know that each copy of H must be fully contained in one of the gadgets in (i)–(iv) or in the graph G. By the choice of the colouring, we know that each of the gadgets is coloured without a monochromatic copy of H. Moreover, the colouring c splits the graph G into colour classes given by the subgraphs $G_1, \ldots, G_{q-1}$, none of which contains a copy of H by the assumption of the lemma. Hence, c is H-free in this case.

Next, consider the case when $H \cong K_t \cdot K_2$. Assume that there is a monochromatic copy H' of H, and let K' denote the copy of K_t in H'. According to Observation 4.8, K' needs to be fully contained in one of the gadget graphs or in one of the subgraphs F or G. If H' is fully contained in one of these parts, then H' cannot be monochromatic by the same argument as above. Hence, we may assume that K' uses a vertex of one of the signal edges, indicator edges, or indicator subgraphs. If K' is contained in one of the gadget graphs, then by the choice of the $K_t \cdot K_2$-special colouring for this gadget graph, K' cannot be monochromatic, a contradiction.

So assume next that $K' \subseteq G = G_1$. We need to check that no edge adjacent to K' can be of the same colour. Indeed, since $H \nsubseteq G_1$ by the assumption of the lemma, every edge incident to K' must belong to one of the indicators from (iv) and must be incident to the corresponding indicator edge which is part of K'. But the 2-colouring of each indicator was chosen to be $K_t \cdot K_2$-special, so any such edge has the opposite colour, and hence H' cannot be monochromatic, a contradiction. We are left with the case $K' \subseteq F$. As we have $H \nsubseteq F$ by the assumption of the theorem, we know that any edge adjacent to K' must be part of one of the indicators from (i). But then H' is fully contained in such an indicator and hence cannot be monochromatic, as the colouring on every gadget is H-free, a contradiction.

Note that the last argument also shows half of the additional property in case (c), i.e., that the H-free 2-colourings in $(GI2)$ can be chosen so that every monochromatic copy of K_t using a vertex from $F \cup G$ is contained fully in $F \cup G$.

$(GI3)$ Let c be any H-free colouring of I^* such that F is monochromatic, say $c(F) = 1$. By properties $(I2)$ and $(I3)$ for negative indicators, the indicators in (i) make sure that all edges in the matchings M_k need to get a colour different from 1. Then, by the pigeonhole principle, in each matching M_k there needs to be at least one colour from $[q] \setminus \{1\}$ that appears at least twice. For each matching M_k, fix one such colour and denote it by c_k. By symmetry, we assume without loss of generality that $c(m_1^k) = c(m_2^k) = c_k$. By property $(I3)$ for the indicators in (ii), we conclude that $c(P_k) = c(e_k) = c_k$. Similarly, using property $(S2)$ for the signal senders in (iii), we obtain that all edges in $\{e_1, \ldots, e_{q-1}\}$

need to have distinct colours. Since colour 1 is excluded, we may assume by symmetry that $c_k = c(e_k) = k + 1$ and thus $c(P_k) = k + 1$. Then, applying property $(I3)$ for the positive indicators in (iv) yields that $c(G_k) = k + 1$ and hence $\{c(F), c(G_1), \ldots, c(G_k)\} = [q]$.

$(GI4)$ Let φ_F and φ_G satisfy the assumption in property $(GI4)$. We define a colouring $c : E(I^*) \to [q]$ as follows:

- colour F according to φ_F.
- colour G according to φ_G.
- For every $k \in [q-1]$ and $\ell \in [q]$, give colour ℓ to m_ℓ^k.
- For every $k \in [q-1]$, give colour k to e_k and give colour $k+1$ to the edge in $P_k - e_k$.
- Finally, extend this colouring to each of the indicators and signal senders in (i)–(iv) so that none of these contains a monochromatic copy of H. In case (c), choose these colourings to be $K_t \cdot K_2$-special.

The extension in the last step of the colouring is possible for the following reason: For the indicators in (i), we can find such an extension by property $(I4)$ for negative indicators and since φ_F is not constant by assumption. For the indicators in (ii), such an extension exists by property $(I4)$ and since no subgraph $S \subseteq M_k$ of size two is coloured monochromatically. For the signal senders in (iii), this extension is possible by properties $(S1)$ and $(S2)$ and since $c(e_{k_1}) \neq c(e_{k_2})$ for every distinct $k_1, k_2 \in [q-1]$. For the indicators in (iv), we again use property $(I4)$ plus the fact that P_k is not monochromatic.

Finally, as in the discussion of $(GI2)$, it follows that c must be H-free. Moreover, if $H \cong K_t \cdot K_2$ and $q = 2$ then, taking a $K_t \cdot K_2$-special 2-colouring for each of the gadget graphs, we deduce that every monochromatic copy of K_t that uses a vertex from $F \cup G$ is fully contained in $F \cup G$. That is, we obtain the second half of the additional property required in case (c). □

4.1.2 Existence of pattern gadgets

We are now ready to prove Theorem 4.4.

Set $t = |\mathscr{G}|$. For every $g = \{G_1, \ldots, G_q\} \in \mathscr{G}$, fix an *ordered colour pattern* $\vec{g} = (G_1, \ldots, G_q)$ with an arbitrary ordering of the subgraphs $G_i \in g$, and denote the jth component of $\vec{g}$ by $\vec{g}_j$. Further, let $\vec{\mathscr{G}} = \{\vec{g} : g \in \mathscr{G}\}$. Choose $r \in \mathbb{Z}_{\geq 1}$ such that

$$\binom{(r-1)q+1}{r} \geq t \ .$$

Fix a matching M of size $(r-1)q+1$ and a surjection $s : \binom{M}{r} \to \vec{\mathscr{G}}$, which exists by the choice of r. We construct a pattern gadget $P = P(H, G, \mathscr{G}, q)$ as follows. Take G together with the given family $\mathscr{G}$ of H-free q-colour patterns for G. Further, take the matching M to be vertex-disjoint from G and join submatchings of M and edges of G by generalised negative indicators and positive indicators as described below. For this, choose an integer d such that $d > v(H)$.

(i) For every $A \in \binom{M}{r}$ and every edge $e \in E(s(A)_q)$, join the submatching A and the edge e by a positive (H, A, e, q, d)-indicator.

(ii) For every $A \in \binom{M}{r}$, join the submatching A and the graph $G - (s(A))_q$ by a generalised negative $(H, A, \{s(A)_k\}_{k \in [q-1]}, q, d)$-indicator.

The existence of the indicators needed in (i) and (ii) is given by Theorem 2.7 and Lemma 4.6.

In the case when $H \cong K_t \cdot K_2$ and $q = 2$, we additionally choose all gadgets so that they have $K_t \cdot K_2$-special 2-colourings as described in Theorem 2.7 and Lemma 4.6(c) respectively. Moreover, we choose all the indicators so that they satisfy the robustness properties described in Theorem 2.7 and Lemma 4.6. Then, analogously to Observation 4.7 and Observation 4.8, we can prove the following.

Observation 4.9. *Let P' be a graph obtained from P by adding a vertex set S and any collection of edges within $S \cup V(G)$. If H is 3-connected or a cycle, then every copy of H in P' is fully contained in one of the indicators from (i) or (ii) or in the subgraph induced by $S \cup V(G)$. If $H \cong K_t \cdot K_2$, then every copy of K_t in P' is fully contained in one of the indicators from (i) or (ii) or in the subgraph induced by $S \cup V(G)$.*

Given this observation, it follows immediately that (P, G) is H-robust if H is 3-connected or a cycle and that (P, G) is K_t-robust if $H \cong K_t \cdot K_2$. Hence, it remains to verify that P satisfies $(P1)$–$(P3)$, and that in the case when $H \cong K_t \cdot K_2$ and $q = 2$ we can find 2-colourings for $(P3)$ as described in part (c) of Theorem 4.4.

$(P1)$ Since P is constructed by attaching different gadgets to G without adding edges inside $V(G)$, we have $G \subseteq_{ind} P$.

$(P2)$ Let $c : E(P) \to [q]$ be any H-free colouring of P. By the pigeonhole principle, at least one colour is used at least r times on the matching M. Without loss of generality, say $c(A) = q$ for some $A \in \binom{M}{r}$. Consider the pattern $g = \{s(A)_k\}_{k \in [q]}$. By property $(I3)$ of the indicators in (i), we deduce that every edge in $E(s(A)_q)$ also needs to have colour q. Moreover, by property $(GI3)$ of the generalised negative indicators in (ii), each of the subgraphs $s(A)_k$ with $k \neq q$ is forced to be monochromatic, and all colours except for $c(A) = q$ get used among these subgraphs. Hence, $\{c_{|G}^{-1}(1), \ldots, c_{|G}^{-1}(q)\} = g \in \mathscr{G}$.

$(P3)$ Let $g = \{G_1, \ldots, G_q\} \in \mathscr{G}$ be given. Fix an arbitrary set $A_0 \in \binom{M}{r}$ such that $s(A_0) = \vec{g}$. Without loss of generality, assume that $s(A_0)_k = G_k$ for every $k \in [q]$; otherwise relabel the subgraphs in g. We define a colouring $c : E(P) \to [q]$ as follows:

- Give colour q to each edge in A_0.
- colour $M \setminus A_0$ so that each colour from $[q-1]$ appears exactly $r - 1$ times.
- For every $k \in [q]$, give colour k to the edges of G_k.
- Finally, extend this colouring to each of the gadgets in (i) and (ii) so that none of these contains a monochromatic copy of H. In case (c), choose these colourings to be $K_t \cdot K_2$-special.

We claim that the extension in the last step of the colouring is indeed possible. Recall that each gadget from (i) and (ii) is associated to a submatching $A \in \binom{M}{r}$. Suppose first that $A = A_0$. Then we have $c(A) = q = c(s(A)_q)$, and by properties $(I2)$ and $(I3)$, we find an extension as desired for the corresponding positive indicators in (i). Moreover, we have $c(s(A)_k) = k \neq q = c(A)$ for every colour $k \in [q-1]$. Hence, by properties $(GI2)$ and $(GI3)$, we find extensions as desired for the corresponding generalised negative indicators in (ii). Consider next the case when $A \neq A_0$. Then A is not monochromatic, since A_0 is the only monochromatic subset of M of size r. Now, let I be any positive indicator between A and any edge $e \in E(s(A)_q)$ as described in (i). Then, by property $(I4)$, we find an extension for I as desired. Finally, let I be the generalised negative indicator from (ii) for the set A. Then, using property $(GI4)$, we conclude analogously that an extension for I can be found.

Finally, we have $\{G_1, \ldots, G_q\} = \{c_{|G}^{-1}(1), \ldots, c_{|G}^{-1}(q)\}$. Since $g = \{G_1, \ldots, G_q\}$ is an H-free q-colour pattern by the assumption of the theorem, we know that $c_{|G}$ is H-free. Now, if H is 3-connected or a cycle, then every copy of H in P that is not contained in G must be a subgraph of some indicator from (i) or (ii), according to Observation 4.9. But we already know that the colouring c is H-free on every indicator, and hence it is H-free on the whole graph P.

It remains to consider the case when $H \cong K_t \cdot K_2$ and $q = 2$. Assume that there is a monochromatic copy H' of H, and let K' denote its copy of K_t. As above, if H' is fully contained in one of the indicators, then it cannot be monochromatic. Hence, we may assume that K' intersects the vertex set of an indicator edge or an indicator subgraph. Then, by the $K_t \cdot K_2$-special 2-colourings for the indicators, we know that K' needs to be a subgraph of G. Without loss of generality, let $K' \subseteq G_1$. Since G does not contain a copy of $K_t \cdot K_2$ by assumption, we know that $E_G(V(K'), V(G_2)) = \varnothing$. Hence, the pendant edge f of H' needs to belong either to a positive indicator between some $A \in \binom{M}{r}$ and some $e \in E(K')$, or to a generalised negative indicator between some $A \in \binom{M}{r}$ and the graph $G_1 \supseteq K'$. In the former case, the edge f needs to be incident to the indicator edge e and hence $c(e) \neq c(f)$ by the $K_t \cdot K_2$-special 2-colouring of the corresponding positive indicator. In the latter case, we have $c(f) \neq c(K')$ as the colouring of the generalised negative indicator was chosen to be H-free. Hence, in both cases H' cannot be monochromatic, a contradiction. □

4.2 Applications of pattern gadgets

In this section, we present several applications of the pattern gadgets constructed in the previous section. We first prove Theorem 4.10 and Theorem 4.11 directly. The proof of Theorem 4.15 is given later in the section as a consequence of a more general result about 3-connected graphs (Theorem 4.12).

4.2.1 Cycles

Theorem 4.10. *For any given integers $q \geq 2$, $t \geq 4$, and $k \geq 1$, there exists a minimal q-Ramsey graph for C_t that has at least k vertices of degree $q+1$. In particular, $s_q(C_t) = q+1$ and C_t is s_q-abundant.*

Proof. Let $H \cong C_t$ and $t \geq 4$ and $q \geq 2$ be fixed. We first note that $s_q(H) \geq q+1$. Indeed, suppose there is a minimal q-Ramsey graph G for C_t with a vertex v of degree at most q; by the minimality of G, there exists a C_t-free q-colouring of $G - v$. Now, colouring the edges incident to v so that no two of them share a colour gives a q-colouring of G with no monochromatic C_t, a contradiction.

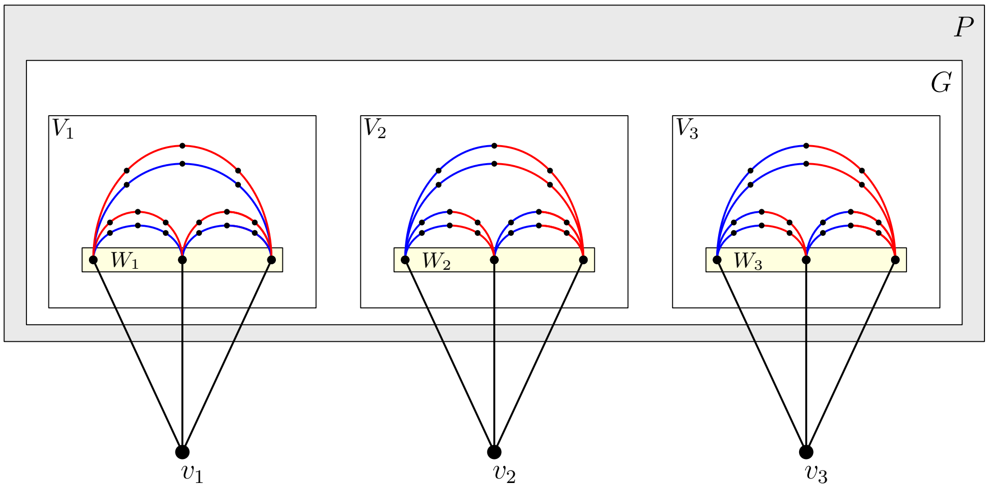

Figure 4.2: Graph $\widetilde{G}$ for $q = 2$, $t = 6$ and $k = 3$.

We now turn our attention to showing that there can be arbitrarily many vertices of degree $q+1$, also implying that $s_q(C_t) = q+1$. Let $k \geq 1$. We now construct a minimal q-Ramsey graph for H with at least k vertices of degree $q+1$. Our graph will be constructed in several steps. We refer the reader to Figure 4.2 for an illustration of our construction in the case when $q = 2$, $t = 6$, and $k = 3$.

To begin with, let W be a set of $q+1$ vertices. For every $u, w \in W$ and $u \neq w$, add q internally vertex-disjoint paths of length $t-2$ with u and w as endpoints. Call the resulting graph F. Let $c_1 : E(F) \to [q]$ be a colouring of the edges of F such that, for every distinct $u, w \in W$, every path between u and w is monochromatic but no two such paths are monochromatic in the same colour. Let $c_2 : E(F) \to [q]$ be another colouring of the edges of F such that, for every distinct $u, w \in W$, no path between u and w is monochromatic. We define f_1 and f_2, two q-colour patterns for F, by setting $f_1 = \{c_1^{-1}(i)\}_{i\in[q]}$ and $f_2 = \{c_2^{-1}(i)\}_{i\in[q]}$. Note that f_1 and f_2 are H-free.

We now take k vertex-disjoint copies $F_1, \ldots, F_k$ of F, where $F_i = (V_i, E_i)$ for all $1 \leq i \leq k$, and denote by W_i the subset of V_i corresponding to W in $V(F)$. Call this graph G, and define $V = \bigcup_{i=1}^{k} V_i$. Note that $G \not\to_q H$, since $F \not\to_q H$. Let $\mathscr{G}$ be a family of q-colour patterns for G such that $g \in \mathscr{G}$ if and only if there exists an $i \in [k]$ such that $g[V_i] \cong f_1$ and $g[V_j] \cong f_2$ for all $j \neq i$. Note that $\mathscr{G}$ is a family of H-free q-colour patterns for G.

By Theorem 4.4, we know that there exists a pattern gadget $P = P(H, G, \mathscr{G}, q)$. Moreover, we can choose the pattern gadget P in such a way that the pair (P, G) is H-robust. We add k additional vertices $v_1, \ldots, v_k$ to P, and for all $i \in [k]$, we add edges from v_i to all vertices in W_i. We call the resulting graph $\widetilde{G}$.

We now show that $\widetilde{G} \to_q H$ and that each of the new vertices v_i is important for $\widetilde{G}$ to have this property, that is, $\widetilde{G} - v_i \not\to_q H$ for every $i \in [k]$. This then implies the existence of a minimal q-Ramsey graph for H with the desired properties. Indeed, consider any minimal q-Ramsey graph $\widetilde{G}' \subseteq \widetilde{G}$. Since $\widetilde{G} - v_i \not\to_q H$, we know that $v_i \in V(\widetilde{G}')$ for every $i \in [k]$. Also $q+1 \leq s_q(C_t) \leq d_{\widetilde{G}'}(v_i) \leq q+1$, which means that $d_{\widetilde{G}'}(v_i) = s_q(C_t) = q+1$.

First, we show that $\widetilde{G} \to_q H$. Let $c : E(\widetilde{G}) \to [q]$ be a q-colouring of the edges of $\widetilde{G}$, and assume that c is H-free. For each $i \in [q]$, define $c_i = c^{-1}(i)$ to be the ith colour class with respect to c. By property $(P2)$ of the pattern gadget P, we know that $g = \{c_1[V], \ldots, c_q[V]\} \in \mathscr{G}$; by the definition of $\mathscr{G}$, there exists an $i \in [k]$ such that $\{c_1[V_i], \ldots, c_q[V_i]\} \cong f_1$. Without loss of generality, we may assume that $i = 1$. Consider the edges from v_1 to the vertices of W_1. There are $q+1$ such edges and they are coloured in q colours, so by the pigeonhole principle there are two vertices in W_1, say u and w, such that $c(v_1u) = c(v_1w)$. Again without loss of generality, we may assume that $c(v_1u) = 1$. By our choice of f_1, we know that there is a monochromatic path of length $t-2$ in colour 1 between the vertices u and w. This monochromatic path along with the edges v_1u and v_1w gives a monochromatic cycle of length t, contradicting our assumption.

Next, we show that $\widetilde{G} - v_i \not\to_q H$ for every $i \in [k]$. By symmetry, it is enough to show this for $i = 1$. Partition the vertices in V in the following way: For every $\ell \in [k]$, write $G[V_\ell] = G_{\ell,1} \cup \cdots \cup G_{\ell,q}$ so that $\{G_{1,j}\}_{j \le q} \cong f_1$ and $\{G_{\ell,j}\}_{j \le q} \cong f_2$ for $\ell \neq 1$. We define a colouring $c : E(G) \to [q]$ by setting $c(G_{\ell,j}) = j$ for every $\ell \in [k]$ and $j \in [q]$. The q-colour pattern on V defined by c, namely $\{c^{-1}_{|G}(1), \ldots, c^{-1}_{|G}(q)\}$, is in $\mathscr{G}$, and by property $(P3)$, we can extend c to an H-free colouring of P. We then colour the remaining edges in $\widetilde{G} - v_1$ arbitrarily, and denote the resulting q-colouring of $\widetilde{G} - v_1$ by $\tilde{c}$. Since $\tilde{c}_{|P}$ is H-free, any monochromatic copy of H in $\widetilde{G} - v_1$ needs to contain a vertex v_ℓ for some $\ell \ge 2$. Now, due to the C_t-robustness of the pair (P, G), any possible monochromatic copy of H must be contained in some $V_\ell \cup \{v_\ell\}$. Such a copy then needs to contain two vertices of W_ℓ and a path of length $t-2$ between them. But we know that $\{c^{-1}_{|G[V_\ell]}(j)\}_{j \in [q]} \cong f_2$, and by the definition of f_2, no such path is monochromatic. Hence, no monochromatic copy of H exists. $\square$

4.2.2 Cliques with a pendant edge

Theorem 4.11. *For a given integer $t \ge 3$, the graph $K_t \cdot K_2$ is s_2-abundant.*

Proof. It was shown by Fox et al. [43] that $s_2(K_t \cdot K_2) = t - 1$ for every $t \ge 3$. We now show that a minimal 2-Ramsey graph for $K_t \cdot K_2$ can contain arbitrarily many vertices of this minimum degree.

Let $H \cong K_t \cdot K_2$ for some $t \ge 3$, and let $k \ge 1$ be fixed. Our construction of a minimal 2-Ramsey graph for H containing at least k vertices of degree $t-1$ will combine ideas similar to those in the proof of Theorem 4.10 with ideas from the construction given by Fox et al. [43]. We again refer the reader to Figure 4.3 for an illustration of the case $t = 4$ and $k = 3$.

We begin by defining F to be the vertex disjoint union of $t-1$ copies of K_t. For every copy of K_t, we fix an arbitrary vertex and call the set of all these vertices W. Let $c_1 : E(F) \to \{red, blue\}$ be a 2-colouring that colours every edge of F red. Let $c_2 : E(F) \to \{red, blue\}$ be another 2-colouring of the edges of F such that no copy of K_t is monochromatic (in either colour). We define two colour patterns f_1 and f_2 for F by setting $f_1 = \{c_1^{-1}(red), c_1^{-1}(blue)\}$ and $f_2 = \{c_2^{-1}(red), c_2^{-1}(blue)\}$. Note that f_1 and f_2 are H-free.

Now take k vertex-disjoint copies $F_1, \ldots, F_k$ of F, where $F_i = (V_i, E_i)$ for $1 \le i \le k$, and let W_i be the subset of V_i corresponding to the set W in $V(F)$. Call this graph G, and define $V = \bigcup_{i=1}^{k} V_i$. Note that G does not contain any copies of H. Let $\mathscr{G}$ be a family of 2-colour patterns for G such that $g \in \mathscr{G}$ if and only if there exists an $i \in [k]$ such that $g[V_i] \cong f_1$ and $g[V_j] \cong f_2$ for all $j \neq i$. Note that $\mathscr{G}$ is a family of H-free 2-colour patterns for G.

By Theorem 4.4, we deduce that there exists a pattern gadget $P = P(H, G, \mathscr{G}, 2)$. Moreover, we can choose the pattern gadget P in such a way that the pair (P, G) is K_t-robust and that for property $(P3)$ there is always an H-free 2-colouring such that, if a monochromatic copy of K_t uses a vertex from G, then it lies entirely in G. We add k additional vertices $v_1, \ldots, v_k$ to P with edges from v_i to all vertices of W_i for all $i \in [k]$; also, for all $i \in [k]$, we add an edge between each pair of distinct vertices in W_i. Lastly, we choose an arbitrary vertex in W_i and add a pendant edge e_i incident to that vertex. We call the resulting graph $\widetilde{G}$.

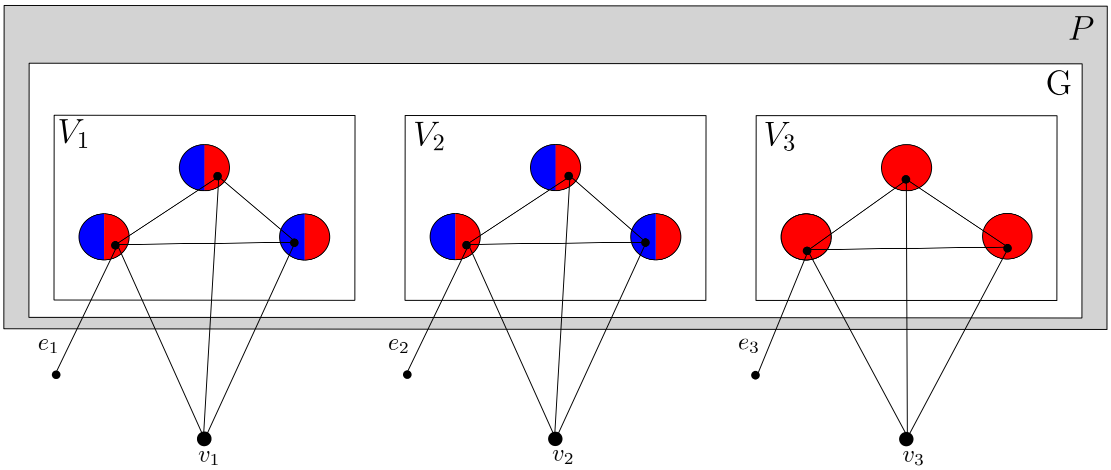

Figure 4.3: Graph $\widetilde{G}$ for $t = 4$ and $k = 3$.

We now show that $\widetilde{G} \to_2 H$ and that $\widetilde{G} - v_i \not\to_2 H$ for every $i \in [k]$. This, as argued in the proof of Theorem 4.10, implies the existence of a minimal 2-Ramsey graph with the desired properties.

First we show that $\widetilde{G} \to_2 H$. Let $c : E(\widetilde{G}) \to \{red, blue\}$ be a 2-colouring of the edges of $\widetilde{G}$; assume that c is H-free. Define $c_{red} = c^{-1}(red)$ and $c_{blue} = c^{-1}(blue)$ to be the two colour classes with respect to c. By property $(P2)$ of the pattern gadget P, we know that $g = \{c_{red}[V], c_{blue}[V]\} \in \mathscr{G}$, and by the definition of $\mathscr{G}$, there exists an $i \in [k]$ such that $\{c_{red}[V_i], c_{blue}[V_i]\} \cong f_1$. Without loss of generality, we may assume that $i = 1$ and every edge inside V_i is red. Consider the edges with endpoints in the set $W' = W_1 \cup \{v_1\}$. Since c is an H-free colouring of $\widetilde{G}$ and each such edge e has at least one endpoint in W_1 (and is hence incident to an all-red copy of K_t), we obtain that $c(e) = blue$. As a result, the graph induced by W' is a monochromatic blue copy of K_t. Now, the pendant edge e_1 is incident to monochromatic copies of K_t in both colours and thus creates a monochromatic copy of H irrespective of its colour. This contradicts our assumption.

Next, we show that, for every $i \in [k]$, we have $\widetilde{G} - v_i \not\to_2 H$. By symmetry, it suffices to show this for $i = 1$. For every $\ell \in [k]$, take a partition $G[V_\ell] = G_{\ell,red} \cup G_{\ell,blue}$ such that $\{G_{1,red}, G_{1,blue}\} \cong f_1$ and $\{G_{\ell,red}, G_{\ell,blue}\} \cong f_2$ for $\ell \neq 1$. We define a colouring $c : E(\widetilde{G}) \to \{red, blue\}$ by first setting $c(G_{\ell,j}) = j$ for every $\ell \in [k]$ and $j \in \{red, blue\}$. The colour pattern defined on G by c is in $\mathscr{G}$, and by property $(P3)$ of P, we can extend this to all of P so that the colouring $c_{|P}$ is H-free and has the following additional property:

(P) If a monochromatic copy of K_t in the colouring $c_{|P}$ uses a vertex from G, then it lies entirely in G.

Now, for every $\ell \geq 2$, colour one edge between v_ℓ and W_ℓ red and colour the remaining edges in $E_{\widetilde{G}}(W_\ell \cup \{v_\ell\}) \cup \{e_\ell\}$ blue. Further, colour all edges in $E_{\widetilde{G}}(W_1) \cup \{e_1\}$ with the colour not used on $G[V_1]$ (recall that $G[V_1]$ was coloured monochromatically as $\{G_{1,red}, G_{1,blue}\} \cong f_1$).

We claim that this colouring is H-free. For a contradiction, assume that there is a monochromatic copy H' of H produced by the colouring c. Since $c_{|P}$ is H-free, H' needs to use at least one edge e_0 from $E_{\widetilde{G}}(W_1) \cup \{e_1\}$ or from $E_{\widetilde{G}}(W_\ell \cup \{v_\ell\}) \cup \{e_\ell\}$ for some $\ell \geq 2$.

Consider first the case when $e_0 \in E_{\widetilde{G}}(W_1) \cup \{e_1\}$. We know that $G[V_1]$ is monochromatic and that e_0 has the opposite colour, say $G[V_1]$ is red and e_0 is blue. Then, by property (P) and the fact that $|W_1| = t - 1$, there can be no blue copy of K_t in the subgraph induced by the set $V_1 \supseteq W_1$. But this means that e_0 cannot be part of a blue copy of $K_t \cdot K_2$, a contradiction.

Consider now the case when $e_0 \in E_{\widetilde{G}}(W_\ell \cup \{v_\ell\}) \cup \{e_\ell\}$ for some $\ell \geq 2$, and assume without loss of generality that $\ell = 2$. By the K_t-robustness of the pair (P, G), the copy H' of H must be contained within $E_{\widetilde{G}}(V_2 \cup \{v_2\}) \cup \{e_2\}$. Since $c_{|G[V_2]} \cong f_2$, i.e., the copies of K_t in F_2 are not monochromatic, and c satisfies property (P), we obtain that $G[V_2]$ does not contain a monochromatic copy of K_t. From this and the fact that e_2 is a pendant edge it follows that the vertices of the copy of K_t in H' must be contained entirely in $W_2 \cup \{v_2\}$. But this set contains precisely t vertices that do not form a monochromatic copy of K_t, again giving a contradiction. □

4.2.3 3-connected graphs

Before turning to the proof of Theorem 4.15 and Theorem 4.18, we state and prove a more general statement concerning 3-connected graphs. Roughly speaking, it reduces the problem of showing s_q-abundance to that of finding a suitable minimal q-Ramsey graph containing at least one vertex of the desired small degree. In fact, we can even relax the condition that the q-Ramsey graph be minimal and that the desired small degree be precisely $s_q(H)$ for the given graph H.

Theorem 4.12. *Let H be 3-connected or a triangle and assume there exists a graph F together with a vertex $v \in V(F)$ and an edge $e \in E(F)$ satisfying the following properties:*

$(F1)$ $F \to_q H$.

$(F2)$ *v and e do not share a copy of H in F.*

$(F3)$ $F - e \not\to_q H$.

$(F4)$ *$F - g \not\to_q H$ for every $g \in E(F)$ which is incident to v.*

Then, for any $k \in \mathbb{Z}_{\geq 1}$, there exists a minimal q-Ramsey graph for H that has k vertices of degree $d_F(v)$.

Proof. Given a graph F with the required properties, denote the edges incident to v in F by $g_1, \dots, g_{d_F(v)}$. Let $F' = F - v - e$. In order to define q-colour patterns for an application of Theorem 4.4, we first observe the existence of two types of H-free q-colourings on F'.

Claim 4.13. *For every $j \in [d_F(v)]$, there exists an H-free q-colouring $c_{1,j}$ of F' such that*

- *$c_{1,j}$ can be extended to an H-free q-colouring of $F - \{e, g_j\}$, and*
- *$c_{1,j}$ cannot be extended to an H-free q-colouring of $F - e$.*

Proof. By property $(F4)$, there exists an H-free q-colouring φ of $F - g_j$. We set $c_{1,j} := \varphi_{|F'}$. One observes easily that this is an H-free q-colouring of F' and that $\varphi_{|F-\{e,g_j\}}$ is an extension to $F - \{e, g_j\}$ that is H-free. Hence, it remains to check that there is no H-free extension to the graph $F - e$.

For a contradiction, assume that there exists some H-free colouring $\psi : E(F - e) \to [q]$ extending $c_{1,j}$. The q-colouring $\widetilde{\psi} : E(F) \to [q]$ defined by

$$\widetilde{\psi}(f) = \begin{cases} \psi(f) & \text{if } f \neq e \\ \varphi(e) & \text{if } f = e \end{cases}$$

cannot be H-free by property $(F1)$. Thus, there must be a copy H' of H that is monochromatic under $\widetilde{\psi}$; moreover, H' needs to use the edge e as $\widetilde{\psi}_{|F-e} = \psi$ is H-free. By property $(F2)$, we have $v \notin V(H')$, that is, H' lies entirely in the graph $F - v$. However, $\widetilde{\psi}_{|F-v} = \varphi_{|F-v}$, since $\widetilde{\psi}_{|F'} = \psi_{|F'} = c_{1,j} = \varphi_{|F'}$ and $\widetilde{\psi}(e) = \varphi(e)$. Hence, since φ is H-free, H' cannot be monochromatic, a contradiction. ✓

Claim 4.14. *There exists an H-free q-colouring c_2 of F' that can be extended to an H-free q-colouring of $F - e$.*

Proof. By property $(F3)$ there exists an H-free q colouring φ of $F - e$. We set $c_2 := \varphi_{|F'}$. ✓

Given the colourings of our previous claims, we next define H-free q-colour patterns $f_{1,j}$, with $j \in [d_F(v)]$, and f_2 for F' by partitioning F' into its colour classes with respect to $c_{1,j}$ and c_2, respectively. More precisely, we set

$$f_{1,j} = \{c_{1,j}^{-1}(i)\}_{i \in [q]} \quad \text{and} \quad f_2 = \{c_2^{-1}(i)\}_{i \in [q]}.$$

Now let $k \geq 1$ be an integer. We proceed similarly as in the proof of Theorem 4.10 and construct a graph $\widetilde{G}$ that will be a q-Ramsey graph for H with the additional property that there are at least k vertices of degree $d_F(v)$, each of which is important for G to be q-Ramsey for H.

First, let $F_1, \dots, F_q$ be k vertex-disjoint copies of $F - e$. For each $i \in [k]$, let $v_i \in V(F_i)$ represent the vertex $v \in V(F - e)$ and let $g_1^i, \dots, g_{d_F(v)}^i \in E(F_i)$ be the edges representing $g_1, \dots, g_{d_F(v)}$. Moreover, for every $i \in [k]$, let $F_i' = F_i - v_i$ and $W_i := N_{F_i}(v_i)$.

We fix $G = (V, E)$ to be the vertex-disjoint union of the graphs $F_i' = (V_i', E_i')$, i.e., we set $V = \cup_{i=1}^k V_i'$ and $E = \cup_{i=1}^k E_i'$. Then we fix a family $\mathscr{G}$ of q-colour patterns for G such that $g \in \mathscr{G}$ if and only if there exist $i \in [k]$ and $j \in [d_F(v)]$ such that $g[V_i'] \cong f_{1,j}$ and such that $g[V_\ell'] \cong f_2$ for all $\ell \neq i$.

By the definition of the patterns $f_{1,j}$ and f_2, and since the vertex sets V_i' for $i \in [k]$ are pairwise disjoint, we know that $\mathscr{G}$ is a family of H-free q-colour patterns for G. Hence, applying Theorem 4.4, we can find a pattern gadget $P = P(H, G, \mathscr{G}, q)$ such that (P, G) is H-robust. Finally, we obtain $\widetilde{G}$ from P by adding the vertices $v_1, \ldots, v_k$ and by connecting v_i to all vertices in W_i via the edges $g_1^i, \ldots, g_{d_F(v)}^i$ for all $i \in [k]$.

Analogously to the proof of Theorem 4.10, we now show that $\widetilde{G} \to_q H$ and that each of the edges g_j^i, for $i \in [k]$ and $j \in [d_F(v)]$, is important for $\widetilde{G}$ to be Ramsey in the sense that $\widetilde{G} - g_j^i \not\to_q H$. This then implies the existence of a minimal q-Ramsey graph as claimed by the theorem. Indeed, assuming these properties, let $\widetilde{G}' \subseteq \widetilde{G}$ be minimal q-Ramsey for H. Since $\widetilde{G} - g_j^i \not\to_q H$, we can conclude that $g_j^i \in E(\widetilde{G}')$ for every $i \in [k]$ and $j \in [d_F(v)]$. This then implies that $d_{\widetilde{G}'}(v_i) = d_F(v)$. Hence, $\widetilde{G}'$ is a minimal q-Ramsey graph for H with at least k vertices of degree $d_F(v)$.

Let us show first that $\widetilde{G} \to_q H$. For a contradiction, suppose we can find an H-free q-colouring $c : E(\widetilde{G}) \to [q]$. For each $i \in [q]$, define $c_i = c^{-1}(i)$ to be the ith colour class with respect to c. By property $(P2)$ of the pattern gadget P, we know that $g := \{c_{|G}^{-1}(1), \ldots, c_{|G}^{-1}(q)\} \in \mathscr{G}$. Hence, by the definition of $\mathscr{G}$, there exist $i \in [k]$ and $j \in [d_F(v)]$ such that $g[V_i'] \cong f_{1,j}$. But then, by the choice of $f_{1,j}$ and the properties of $c_{1,j}$, we deduce that $c_{|\widetilde{G}[V_i']}$ cannot be extended to an H-free q-colouring of $\widetilde{G}[V_i' \cup \{v_i\}]$. This a contradiction, since $c_{|\widetilde{G}[V_i' \cup \{v_i\}]}$ is already such an H-free extension by the assumption on c.

Next, we show that $\widetilde{G} - g_j^i \not\to_q H$ for every $i \in [k]$ and $j \in [d_F(v)]$. By symmetry, we may only consider the case when $i = j = 1$. We first partition G in the following way: For every $\ell \in [k]$, we fix a partition $G[V_\ell] = G_{\ell,1} \cup \cdots \cup G_{\ell,q}$ such that $\{G_{1,r}\}_{r \leq q} \cong f_{1,1}$ and $\{G_{\ell,r}\}_{r \leq q} \cong f_2$ for $\ell \neq 1$. By the choice of $f_{1,1}$ and f_2, we know that the colouring $c : E(G) \to [q]$ defined by $c(G_{\ell,r}) = r$, for every $\ell \in [k]$ and $r \in [q]$, is H-free. Moreover, $\{c^{-1}(1), \ldots, c^{-1}(q)\} \in \mathscr{G}$ and therefore, by property $(P3)$, we can extend c to an H-free q-colouring φ_P of P. By the definition of $f_{1,1}$ and the properties of $c_{1,1}$, we know that the colouring $c|_{G[V_1]}$ can be extended to an H-free q-colouring φ_1 of $\widetilde{G}[V_1 \cup \{v_1\}] - g_1^1$. By the definition of f_2 and the properties of c_2 we know that, for each $\ell \neq 1$, the colouring $c|_{G[V_\ell]}$ can be extended to an H-free q-colouring φ_ℓ of $\widetilde{G}[V_\ell \cup \{v_\ell\}]$. We now put all these colourings together to form the colouring $\varphi : E(\widetilde{G} - g_1^1) \to [q]$ given by

$$\varphi(f) := \begin{cases} \varphi_P(f) & \text{if } f \in E(P), \\ \varphi_\ell(f) & \text{if } v_\ell \in f \text{ for some } \ell \in [k]. \end{cases}$$

We claim that this colouring is H-free.

Assume for a contradiction that there is a monochromatic copy H' of H in the colouring φ. Then, since (P, G) is H-robust, we know that $H' \subseteq P$ or $H' \subseteq \widetilde{G}[V \cup \{v_\ell\}_{\ell \in [k]}] - g_1^1$. Since the colouring φ_P on P is H-free, we can assume that $H' \subseteq \widetilde{G}[V \cup \{v_\ell\}_{\ell \in [k]}] - g_1^1$. But then, since H' is connected, we have $H' \subseteq \widetilde{G}[V_\ell \cup \{v_\ell\}]$ for some $\ell \neq 1$ or $H' \subseteq \widetilde{G}[V_1 \cup \{v_1\}] - g_1^1$. In both cases we know that H' cannot be monochromatic, since the colourings $\varphi_1, \ldots, \varphi_k$ are H-free. This is a contradiction. $\square$

4.2.4 Wheels

Finally, we illustrate how to apply Theorem 4.12 by deriving Theorem 4.15 as a consequence of it. Note that for $t \geq 4$, the wheel graph W_t is 3-connected.

Theorem 4.15. *For a given integer $t \geq 4$, the graph W_t is s_2-abundant and $s_2(W_t) = 7$.*

Proof. Let $H \cong W_t$ and $t \geq 4$ be fixed. We first note that $s_2(H) \geq 7$. Indeed, suppose there is a minimal 2-Ramsey graph G for W_t with a vertex v of degree at most 7; by the minimality of G, there exists a W_t-free 2-colouring of $G - v$, say c. Without any loss to our argument, we may assume that neighbourhood of v induces a complete graph K_6. It is an easy exercise to see that K_6 is Ramsey for C_4. Let us assume that the copy of C_4 in $N(v)$ is blue. We now extend the colouring c to edges incident to v. We colour the edges incident to v and the copy of C_4 red and the remaining edges blue. Since the degree of v in colour blue is two and $\delta(H) = 3$ we know that we did not create a copy of W_t in blue. Moreover, we did not create a red copy of W_t. Indeed, for the first case, say that in the extended colouring there exists a red copy of H and the vertex v plays the role of the central vertex. Observe that, in H, any two non adjacent vertices of the cycle are connected by a path of length at least three, but in the neighbourhood of v there are at most two red matching edges. In the second case assume that v plays the role of a cycle vertex in the red copy of H. This means, that there must be a vertex in the neighbourhood of v which is connected to v and at least two other neighbouring verteices in red, which again does not exist. Therefore $s_2(W_t) \geq 7$.

We now turn our attention to showing that there can be arbitrarily many vertices of degree 7, also implying that $s_2(W_t) = 7$. Let $k \geq 1$. We now construct a minimal 2-Ramsey graph for H with at least k vertices of degree 7.

Let A and B be two disjoint set of vertices with sizes four and three respectively and W_t' be the copy of the wheel on $t+1$ vertices with one cycle vertex removed along with the three edges incident to it and the two edges induced by the vertices in its neighbourhood. Denote the neighbourhood of the deleted vertex by x_1, x_2, and x_3 with x_2 being the central vertex of the wheel. Let G_1 be a graph constructed from a complete bipartite graph between A and B as follows: for any three vertices, henceforth called a *triple*, in $A \cup B$, such that they are not all in the same part, attach to these vertices a copy of W_t' by identifying x_1, x_2, and x_3 with the triple, in such a way that the vertex which does not share its part with the other two vertices of the triple is a copy of x_2. Let G_2 be the graph obtained by adding a copy of C_4 on the vertices of A, a copy of P_3 on the vertices of B, and for any triple of vertices entirely from A or from B we add a copy of W_t' by identifying x_2 with the middle vertex of the path induced on the vertices of the triplet. Notice that G_1 and G_2 are W_t-free. We define $G = G_1 \cup G_2$.

Fix a graph G as described above. We take the given graph G, an isolated vertex v, and a matching $M = \{e_1, e_2\}$ that is vertex-disjoint from G and v; next, we take a negative signal sender $S^- := S^-(W_t, e, f, q, d)$ and a positive signal sender $S^+ := S^+(W_t, e, f, q, d)$ with $d > 3$, the existence of which is guaranteed by Theorem 2.4. We then obtain $\widetilde{G}$ as follows:

(i) Join e_1 and e_2 by a copy of S^-.

(ii) For every $i \in [2]$ and every $f \in E(G_i)$, join e_i and f by a copy of S^+.

(iii) Connect v to all vertices in $A \cup B$ by an edge.

We will see in the following that $\widetilde{G} \to_2 W_t$, $\widetilde{G} - v \not\to_2 W_t$, and $\widetilde{G} - M \not\to_2 W_t$. From this, we can then conclude the existence of a graph $F \in \mathcal{M}_2(W_t)$ satisfying the hypothesis of Theorem 4.12. Indeed, consider any minimal 2-Ramsey graph F for W_t contained in $\widetilde{G}$. Since $\widetilde{G} - v \not\to_2 W_t$, we conclude that F must contain the vertex v; moreover, we have $s_2(W_t) \leq d_F(v) \leq d_{\widetilde{G}}(v) = 7$, so $d_F(v) = s_2(W_t)$. Further, using that $\widetilde{G} - M \not\to_2 W_t$, we also deduce that $\widetilde{G}$ must contain at least one edge $e \in M$. Since $\mathrm{dist}_F(v, e) \geq \mathrm{dist}_{\widetilde{G}}(v, e) \geq 4$, v and e cannot share a copy of W_t, implying that property $(F2)$ holds. By the minimality of F, properties $(F1)$, $(F3)$, and $(F4)$ are immediate. We split the remainder of the proof into two claims.

Claim 4.16. *We have $\widetilde{G} \to_2 W_t$ and $\widetilde{G} - v \not\to_2 W_t$.*

Proof. We begin by showing that $\widetilde{G} \to_2 W_t$. For a contradiction, assume that there exists a W_t-free colouring $c : E(\widetilde{G}) \to [2]$. The signal senders in (i) then ensure that the edges of M must receive distinct colours, say without loss of generality that $c(e_i) = i$ for every $i \in [2]$. The signal senders in (ii) ensure that $c(G_i) = c(e_i) = i$ for every $i \in [2]$. Now, consider the edges incident to v. A simple case disctinction shows that, irrespective of the colouring of these edges, there exists three monochromatic edges at v such that the graph induced by this neighbourhood of v is a path of the same colour. Hence, v along with the aforementioned edges and the copy of W_t' on the desired neighbourhood of v forms a copy of W_t. This is a contradiction.

Next, let us show that $\widetilde{G} - v \not\to_2 W_t$. In order to do so, we define a 2-colouring c of $\widetilde{G} - v$. We first set $c(G_i) = c(e_i) = i$ for every $i \in [2]$; afterwards we extend the colouring c to $\widetilde{G} - v$ in such a way that c is W_t-free on each signal sender from (i) and (ii). Note that the latter is possible by property $(S1)$ and $(S2)$. Analogously to previous proofs, each copy of W_t is fully contained either in a signal sender or in the graph G. Since the colouring restricted to any signal sender is W_t-free and since $\{G_1, G_2\}$ is a W_t-free 2-colour pattern, it follows that c is W_t-free. ✓

Claim 4.17. *$\widetilde{G} - M \not\to_2 W_t$.*

Proof. In order to see this claim, we define a 2-colouring c of $\widetilde{G} - M$ as follows: We first fix a W_t-free 2-colouring of the graph G as follows: For each edge e that is contained entirely in $A \cup B$ we assign $c(e) = i$ if $e \in G_i$, for each edge e in a copy of W_t' we assign $c(e) = i$ if $e \notin G_i$, and for every edge e incident to v we assign $c(e) = 1$. Indeed it is easy to verify that this is a W_t-free 2-colouring of G. Afterwards, we extend the colouring to every signal sender so that it is K_t-free. The latter is possible since every signal sender is missing at least one signal edge in the graph $\widetilde{G} - M$ (and hence we can always pretend that the missing signal edge has a colour that fits property $(S2)$). Now, each copy of W_t is fully contained either in a signal sender or in the graph G, and hence, the resulting colouring of $\widetilde{G} - M$ is W_t-free. ✓

Putting Claims 4.16 and 4.17 together, we obtain the theorem.

□

4.2.5 Random graphs

In this section we will provide yet another application of Theorem 4.12. We will consider certain rages of p for which $G(n,p)$ is almost surely 3-connected.

In Chapter 3, via Proposition 3.7 we showed that, when establishing the q-Ramsey simplicity of a graph H, it suffices to consider the neighbourhood of a minimum degree vertex w. In the construction of the Ramsey host graph G, the vertex w will have the desired degree $\delta(G) = s_q(H)$, but we can expect all other vertices to have much higher degree. Indeed, they are all contained in signal senders, which tend to be large and complicated structures. It is then natural to ask if this must be the case, or if we can apply the results of this chapter to find minimal q-Ramsey graphs for H with arbitrarily many vertices of the lowest possible degree. To that end we will show the following theorem.

Theorem 4.18. *Let $p = p(n) \in (0,1)$ and $H \sim G(n,p)$. Then a.a.s. H is s_q-abundant for all $q \geq 2$ whenever $\frac{\log n}{n} \ll p \ll n^{-\frac{2}{3}}$ and H is s_q-abundant for all $q \leq \tilde{q}(H)$ whenever $n^{-\frac{2}{3}} \ll p \ll n^{-\frac{1}{2}}$.*

Here $\tilde{q}(H)$ is as defined in Definition 3.1. We show that in the mentioned ranges of p and q, the graph $G(n,p)$ is almost surely not just q-Ramsey simple but also s_q-abundant.

Proposition 4.19. *Let $q \geq 2$ and let H be a well-behaved n-vertex graph. If there is a q-edge-coloured graph Γ on $q(\delta(H)-1)+1$ vertices satisfying the conditions of Proposition 3.7, and if either $e(\Gamma) = 0$ or $n > q(\delta(H)-1)+2$, then not only is H q-Ramsey simple, but it is also s_q-abundant.*

As we have shown in the previous chapter in Section 3.3.2, for the ranges of parameters covered by Theorem 4.18, H is well-behaved and admits the construction of a suitable q-coloured graph Γ. Moreover, when $p \ll n^{-2/3}$, we have $e(\Gamma) = 0$, while when $n^{-2/3} \ll p \ll n^{-1/2}$, we have $\delta(H) = (1+o(1))np$ and $q \leq np$, and so $q(\delta(H)-1)+2 \leq (1+o(1))(np)^2 \ll n$. Hence, once we prove Proposition 4.19, we will have shown that in these cases $G(n,p)$ is also s_q-abundant. To do so, we shall apply Theorem 4.12.

To prove Proposition 4.19, we shall show that the q-Ramsey graph G we built in the proof of Proposition 3.7 admits a subgraph $G' \subseteq G$ satisfying conditions of Theorem 4.12 when we take v_0 to be the minimum degree vertex $w \in V(G)$, implying that H is s_q-abundant.

Proof of Proposition 4.19. Consider the graph G constructed in the proof of Proposition 3.7, and recall that it in particular contained a vertex w of degree $q(\delta(H)-1)+1$, and a matching $M = \{e_1, e_2, \ldots, e_q\}$ of edges that were attached to the rest of the graph by signal senders.

By Claim 3.11, we know $G \to_q H$. Let $G' \subseteq G$ be a minimal subgraph that is still q-Ramsey for H. Claim 3.12 shows that we must have $w \in V(G')$, and in our application of Theorem 4.12, we shall take $v_0 = w$. The following claim, which we shall prove later, shows that G' must contain at least one edge from the matching M.

Claim 4.20. *The graph $G - M$ is not q-Ramsey for H.*

We thus have $e_i \in E(G')$ for some $i \in [q]$, and we take $e = e_i$ in Theorem 4.12. Given this preparation, it is simple to verify the conditions of the theorem. Indeed, we took G'

to be a minimal q-Ramsey graph which ensures that Properties $(F1)$, $(F3)$, and $(F4)$ are trivially true. Moreover, recall that the neighbourhood of w in G is the vertex set of Γ. As e_i is only connected to Γ via signal senders, in which the distance between the signal edges is at least $v(H)$, it follows that there cannot be any copy of H containing both e_i and w, thereby satisfying Property $(F2)$. We can therefore apply Theorem 4.12 to deduce the existence of minimal q-Ramsey graphs for H with arbitrarily many vertices of degree $d_{G'}(w) \leq d_G(w) = s_q(H)$, showing that H is s_q-abundant. □

All that remains, then, is to prove Claim 4.20, a task we now complete.

Proof of Claim 4.20. We need to exhibit an H-free colouring of $G - M$. This graph consists of three types of edges:

1. those incident to w or in the graph Γ,
2. those in the subgraphs $R_{U,i}$ and between $R_{U,i}$ and Γ, for $U \in \binom{V(\Gamma)}{\delta(H)}$ and $i \in [q]$, and
3. the edges within the signal senders.

We colour all edges of (1) with the colour 1, and all edges of (2) with the colour 2. We finish by extending this colouring to an H-free q-colouring of each of the signal senders; note that this is possible, as each signal sender is missing at least one of its signal edges from M.

From Lemma 3.10, we know that any copy of H is either within a signal sender or outside it, and as we coloured the signal senders in an H-free fashion, it is only the colour-1 edges of (1) or the colour-2 edges of (2) that could give rise to a monochromatic copy of H.

We can rule out the former immediately. Either $e(\Gamma) = 0$, in which case the edges of (1) are simply a star around the vertex w, which cannot contain a copy of the well-behaved (and therefore 3-connected) graph H, or $n > q(\delta(H) - 1) + 2 = v(\Gamma) + 1$, and so $\Gamma + \{w\}$ does not have enough vertices to support a copy of H.

To handle the latter case, observe that the argument in Claim 3.12 shows that no copy of H can intersect two different subgraphs $R_{U,i}$ and $R_{U',i'}$, for $i, i' \in [q]$ and $U, U' \in \binom{V(\Gamma)}{\delta(H)}$. Hence, any copy of H among the edges of (1) and (2) must use vertices of $R_{U,i}$ and U for some $U \in \binom{V(\Gamma)}{\delta(H)}$ along with some vertex in $\Gamma - U$ or the vertex w. In either case these involve edges from (1) and therefore have the colour 1, and hence we cannot have a colour-2 copy of H.

This completes the proof of Claim 4.20 and, with it, the proof of Proposition 4.19. □

4.3 Concluding remarks

In this chapter, we developed a new tool for studying (minimal) Ramsey graphs and showed some applications to questions concerning minimum degrees. In particular, we used pattern gadgets to find examples of graphs H such that a minimal q-Ramsey graph for H can contain arbitrarily many vertices of degree $s_q(H)$, that is, s_q-abundant graphs. In a joint work with Boyadzhiyska and Clemens [18], we in fact further used Theorem 4.12 for the case of general q to show the following result for cliques.

Theorem 4.21. *For any given integers $q \geq 2$ and $t \geq 3$, the clique K_t is s_q-abundant.*

Interestingly, Theorem 4.21 illustrates that we can sometimes show s_q-abundance without knowing the precise value of s_q. Further, we also established a sufficient condition for a given 3-connected graph to be s_q-abundant in Theorem 4.12. Given the tools developed in this paper, we believe that all 3-connected graphs should be s_q-abundant and propose Conjecture 4.22 below.

Conjecture 4.22. *Every 3-connected graph H is s_q-abundant for any integer $q \geq 2$.*

It is easy to show that the path P_4 with three edges is s_2-abundant. Indeed, let $k \geq 3$ be an odd integer and G be the graph obtained from the cycle C_k by adding a distinct pendant edge to each vertex of the cycle. Using the fact that in every 2-colouring of C_k there must be two consecutive edges of the same colour, it is not difficult to check that G is a minimal 2-Ramsey graph for P_4. Further, G has k vertices of degree one, establishing the claim.

Thus, we have seen that stars are not s_2-abundant but P_4 is. For all other trees T, the question of whether T is s_2-abundant (or, more generally, s_q-abundant) remains open. This leads us to propose the following problem.

Question 4.23. *Let $q \geq 2$ be an integer. Is every tree that is not a star s_q-abundant?*

As explained above, a positive answer to this question would be rather surprising.

More generally, we would like to understand better which graphs H are s_q-abundant. In particular, besides stars, we do not have any examples of graphs that are not s_q-abundant; we propose the following question.

Question 4.24. *Let $q \geq 2$ be an integer. Does there exist a graph H that is not s_q-abundant, but has infinitely many Ramsey-minimal graphs of minimum degree $s_q(H)$?*

5

Ramsey equivalence for asymmetric pairs

In this chapter we study the question: When are two pair of graphs Ramsey equivalent? In [43], the authors raised a question: Does there exist a pair a graphs such that they are Ramsey equivalent? In this chapter we will study a variant of this problem in the asymmetric setting. Let us first recall what it means for two graphs to be equivalent.

Definition 5.1. We call two pairs of graphs (G, H) and (G', H') *Ramsey equivalent*, denoted $(G, H) \sim (G', H')$, if $\mathcal{R}(G, H) = \mathcal{R}(G', H')$.

We will begin this study with a consideration for two pair of graphs which only consists of stars. In Section 5.1, we will provide a necessary and sufficient condition for a pair of pair of star graphs to be equivalent, that is we will prove Theorem 5.4.

Section 5.2 contains the proof of our main equivalence result, namely Theorem 5.5. We consider a pair of graphs which consist of a tree and a clique. We will show that such a pair is equivalent to another pair which consist of the same tree and a clique with other structures hanging off of it.

In Section 5.3 we prove Theorem 5.9. Here we again consider a pair of graphs which consists of a tree and a clique, but in this section we exhibit a collection which is non equivalent to the pair in question.

To conclude, in Section 5.4 we state some open problems and provide some easy observations for the properties of the pair (G, H) such that $(G, H) \sim (K_t, K_t)$.

This chapter is based on a joint work with Simona Boyadzhiyska, Dennis Clemens, and Jonathan Rollin [19].

5.1 Star pairs

In this section, we prove Theorem 5.4. We note that this theorem can be deduced from Theorem 1 in [71]. However, the calculations are tedious and for completeness we present explicit constructions here when we want to show that there exist graphs that are Ramsey for certain pairs of stars and not Ramsey for other star pairs.

Observe that, given positive integers a and b, an $(a+b-2)$-regular graph F is a Ramsey graph for a pair $(K_{1,a}, K_{1,b})$ if and only if $E(F)$ cannot be decomposed into an $(a-1)$-regular subgraph and a $(b-1)$-regular subgraph. A k-regular spanning subgraph is also called a *k-factor*. Our results rely on the rich theory on factors. Specifically we need the following fact.

Lemma 5.2. *Let p, q and r be integers, with p and q being odd. Further, assume that $p < q \leq r$ if r is odd, and that $p < q \leq r/2$ if r is even. There is an r-regular graph that has a q-factor and no p-factor.*

In order to prove the above lemma, we apply a theorem due to Belck [9] (a special case of the well-known f-factor theorem of Tutte [85]), which provides a necessary and sufficient condition for the existence of k-factors in regular graphs. For a graph G and a set $D \subseteq V(G)$, we call a component C of $G - D$ an *odd component* with respect to D if $|V(C)|$ is odd, and we let $q_G(D)$ denote the number of such components. We use the following corollary of Theorem IV from [9].

Theorem 5.3 ([9]). *Let G be a graph and let $p > 0$ be an odd integer. If there exists a set $D \subseteq V(G)$ such that $p|D| < q_G(D)$, then G has no p-factor.*

Proof of Lemma 5.2. Given p, q and r as described in the statement, we aim to construct an r-regular graph F that has a q-factor and no p-factor. The graph F will be constructed in three steps.

In the first step, we find an r-regular graph G with an even number of vertices that has a q-factor G_q and a matching M_G with $\lfloor (r-1)/2 \rfloor$ edges which contains exactly $(q-1)/2$ edges of G_q. To this end, define G to be the graph obtained by taking $2r(r-q+1)$ copies $Q_{i,j}$ of K_{q+1}, with $i \in [r-q+1]$ and $j \in [2r]$, and adding a perfect matching between any two copies $Q_{i_1,j}$ and $Q_{i_2,j}$ with $i_1 \neq i_2$ and $j \in [2r]$. Then G has an even number of vertices and is r-regular. Moreover, the subgraph G_q that consists of all $Q_{i,j}$ is a q-factor. The matching M_G can be found by taking $\frac{q-1}{2}$ independent edges from $Q_{1,1}$ and one edge from every matching between $Q_{1,j}$ and $Q_{2,j}$ with $2 \leq j \leq \lfloor \frac{r-q+2}{2} \rfloor$. Set $M_q := M_G \cap E(G_q)$.

For the second step, let H and H_q denote the graphs obtained from G, respectively G_q, by adding a new vertex u and replacing every edge $vw \in M_G$, respectively $vw \in M_q$, by the edges uv and uw. Then H has an odd number of vertices, u is of degree $2\lfloor (r-1)/2 \rfloor$ in H, and all other vertices are of degree r. Moreover, H_q is a spanning subgraph of H in which u is of degree $q-1$ and all other vertices are of degree q.

For the third step, we consider two cases depending on the parity of r.

Case 1: r is odd. In this case u has degree $r-1$ in the graph H. Let $t = r - q + 1$, and let $F = F(q,r)$ denote the graph obtained from a copy of K_t with vertex set $D = \{d_j : j \in [t]\}$ and qt vertex disjoint copies $H^1, \ldots, H^{qt}$ of the graph H as follows: For each $i \in [qt]$, let u^i denote the copy of u in H^i. We partition the set $\{u^i : i \in [qt]\}$ into t sets $U_1, U_2, \ldots, U_t$ each of size q and, for each $j \in [t]$, add an edge between d_j and each vertex in U_j. An illustration of the construction is given in Figure 5.1 (left). Then F is r-regular. Moreover, F has a q-factor, given by the subgraph consisting of all copies of H_q (coming from the H^i with $i \in [qt]$) and all edges between D and the copies of u.

It thus remains to show that F does not admit a p-factor. This follows from Theorem 5.3. Indeed, the odd components of $F - D$ are exactly the $q(r-q+1)$ copies of H, and therefore

$$p|D| - q_G(D) = p|D| - q(r-q+1) = (p-q)(r-q+1) < 0\,,$$

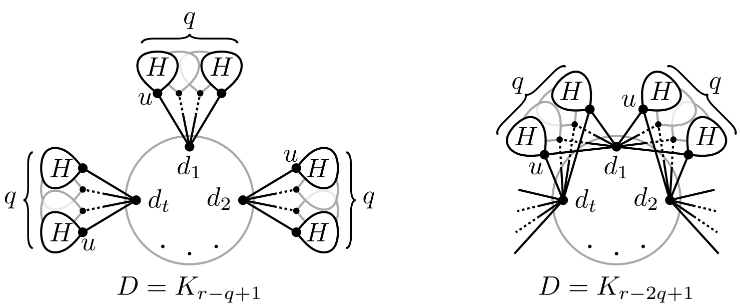

Figure 5.1: A construction of an r-regular graph with a q-factor and no p-factor for odd r (left) and even r (right).

since $p < q \leq r$ by assumption.

Case 2: r is even. In this case u has degree $r-2$ in the graph H. Moreover, by assumption we have $q \leq r/2$. Let $t = r - 2q + 1$, and let $F = F(q,r)$ denote the graph obtained from a copy of K_t with vertex set $D = \{d_1, d_2, \dots, d_t\}$ and qt vertex disjoint copies $H^1, \dots, H^{qt}$ of the graph H as follows: For each $i \in [qt]$, let u^i denote the copy of u in H^i. We partition the set $\{u^i : i \in [qt]\}$ into t sets $U_1, U_2, \dots, U_t$ each of size q and, for each $j \in [t]$, add an edge between d_j and each vertex in $U_j \cup U_{j+1}$, where $U_{t+1} := U_1$. This is illustrated in Figure 5.1 (right).

Then the graph F is r-regular. Moreover, it contains a q-factor consisting of all copies of H_q (coming from the H^i with $i \in [qt]$) and all edges between d_j and U_j for every $j \in [t]$. Furthermore, by Theorem 5.3, F does not have a p-factor. Indeed, the odd components of $F - D$ are exactly the qt copies of H, and therefore

$$p|D| - q_G(D) = p|D| - qt = (p-q)t < 0\,,$$

since $p < q$ and $q \leq r/2$ and hence $t \geq 1$ by assumption. □

Theorem 5.4. *Let a, b, x, y be positive integers with $\{a,b\} \neq \{x,y\}$. Then $(K_{1,a}, K_{1,b}) \sim (K_{1,x}, K_{1,y})$ if and only if $a+b = x+y$ and a, b, x, and y are odd.*

Proof. First observe that $K_{1,a+b-1}$ is a Ramsey graph for $(K_{1,x}, K_{1,y})$ if and only if $x+y \leq a+b$. This shows that $(K_{1,a}, K_{1,b}) \not\sim (K_{1,x}, K_{1,y})$ when $a+b \neq x+y$. For the remainder of the proof assume that $a+b = x+y$.

As discussed in the introduction, if a and b are both odd, then $K_{1,a+b-1}$ is the unique minimal Ramsey graph for $(K_{1,a}, K_{1,b})$ [20]. So $(K_{1,a}, K_{1,b}) \sim (K_{1,x}, K_{1,y})$ if a, b, x, y are all odd. It remains to consider the case where at least one of a, b, x, and y is even and find a distinguishing graph, that is, a graph that is Ramsey for one of the pairs of stars and not Ramsey for the other pair. Without loss of generality, assume that a is the largest even number in $\{a,b,x,y\}$. Let $r = a+b-2 = x+y-2$. Recall that an r-regular graph is a Ramsey for $(K_{1,a}, K_{1,b})$ if and only if it has no $(a-1)$-factor. We consider several cases.

Case 1: xy is odd. Then each Ramsey graph for $(K_{1,x}, K_{1,y})$ contains $K_{1,a+b-1}$, as remarked above, and hence no graph of maximum degree at most r is a Ramsey graph for $(K_{1,x}, K_{1,y})$. Consider an r-regular graph on an odd number of vertices, which exists since r is even. Since $(a-1)$ is odd, this graph does not have an $(a-1)$-factor and is therefore Ramsey for $(K_{1,a}, K_{1,b})$ but not Ramsey for $(K_{1,x}, K_{1,y})$. Thus $(K_{1,a}, K_{1,b}) \not\sim (K_{1,x}, K_{1,y})$.

Case 2: xy is even. We may assume that x is the larger even number in $\{x, y\}$. Then $a > x$, since a is the largest even number in $\{a, b, x, y\}$, and since $\{a, b\} \neq \{x, y\}$. We again distinguish two cases.

Case 2.1: b is odd. Then r is odd. Setting $q = a - 1$ and $p = x - 1$, we know that p and q are odd, and $p < q \leq r$. Hence, using Lemma 5.2, we find an r-regular graph F that has an $(a-1)$-factor and no $(x-1)$-factor. Thus $F \not\rightarrow (K_{1,a}, K_{1,b})$ and $F \rightarrow (K_{1,x}, K_{1,y})$. Hence $(K_{1,a}, K_{1,b}) \not\sim (K_{1,x}, K_{1,y})$.

Case 2.1: b is even. Then r and y are even. As we have $a > x \geq y$ and $a + b - 2 = r = x + y - 2$, we obtain $b < y \leq \frac{r+2}{2}$. Setting $q = y - 1$ and $p = b - 1$, we know that p and q are odd, and $p < q \leq \frac{r}{2}$. Hence, using Lemma 5.2, we find an r-regular graph F that has a $(y-1)$-factor and no $(b-1)$-factor. Thus $F \not\rightarrow (K_{1,x}, K_{1,y})$ and $F \rightarrow (K_{1,a}, K_{1,b})$. Hence $(K_{1,a}, K_{1,b}) \not\sim (K_{1,x}, K_{1,y})$. □

5.2 Equivalence results for trees and cliques

Let us now move onto a pair which is more interesting than the one that only contains stars. In this section we will investigate the pair that consists of a tree and a clique and exhibit some non trivial pairs of graphs which are equivalent to it.

Theorem 5.5.

(a) *For all integers $s \geq 2$ and $t \geq 3$, we have $(K_{1,s}, K_t) \sim (K_{1,s}, K_t \cdot K_2)$.*

(b) *Let $a \geq 1$ and $b \geq 2$ be integers, and let T be a star with at least two edges or a suitable caterpillar. For any sufficiently large t, we have $(T, K_t) \sim (T, K_t \cdot aK_b)$.*

Proof of Theorem 5.5(a). If $F \rightarrow (K_{1,s}, K_t \cdot K_2)$, then also $F \rightarrow (K_{1,s}, K_t)$. It suffices to show that, if $F \not\rightarrow (K_{1,s}, K_t \cdot K_2)$, then also $F \not\rightarrow (K_{1,s}, K_t)$.

Let F denote a graph that is not Ramsey for $(K_{1,s}, K_t \cdot K_2)$ and let c be a $(K_{1,s}, K_t \cdot K_2)$-free colouring of F that minimizes the number of blue copies of K_t among all such colourings. We claim that c has no blue copies of K_t and hence $F \not\rightarrow (K_{1,s}, K_t)$.

For a contradiction, assume that there exists a blue copy of K_t under c. Note that the blue copies of K_t must be pairwise disjoint and that there are no blue edges leaving any of these copies, that is, the copies of K_t form isolated components in the blue subgraph of F under c. A walk in F is a subgraph of F formed by a sequence $u_1, \ldots, u_\ell$ of (not necessarily distinct) vertices of F with edges $u_i u_{i+1}$ for all $i \in [\ell - 1]$. We call the vertices u_1 and u_ℓ the endpoints of W, also when $u_1 = u_\ell$. If W is a walk and K is a blue copy of K_t in F under c, we say that K is visited (by W) if W contains an edge of K; otherwise K is unvisited (by W). A walk W is *feasible* if it satisfies the following properties:

i) Each edge of F occurs at most once in W.

ii) W contains at least one blue edge.

iii) The edges of W are alternately coloured red and blue, that is, $c(u_i u_{i+1}) \neq c(u_{i+1} u_{i+2})$ for $i \in [\ell - 2]$.

iv) Each blue edge in W is contained in a blue copy of K_t, and each blue copy of K_t has at most one edge in W.

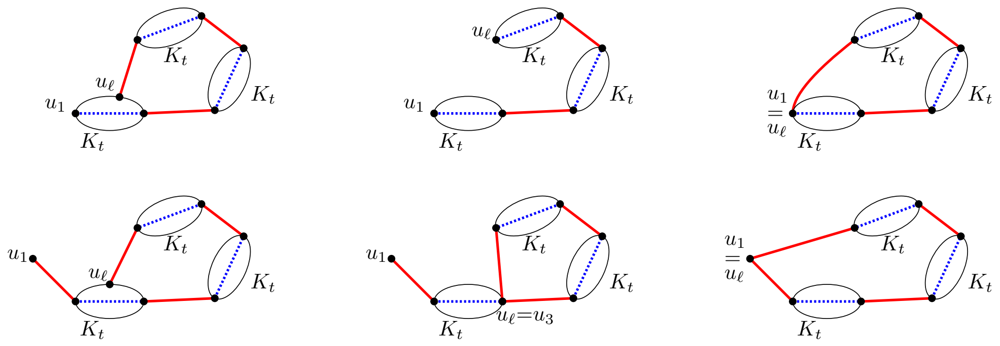

Figure 5.2: Several examples of feasible walks (with solid lines representing red edges and dotted lines representing blue edges).

v) No vertex of W is contained in an unvisited blue copy of K_t in F.

vi) An endpoint of W that is not incident to any red edge in W is not incident to any red edge in F.

Refer to Figure 5.2 for an illustration.

We first observe that feasible walks exist and can be found with the following greedy procedure: Start with an arbitrary edge e belonging to a blue copy of K_t. This satisfies the first five properties above but not necessarily the last. For each endpoint of e that is incident to some red edge in F, the walk then follows one such red edge. Now the first four properties are still satisfied, but property v) might become invalid (for the current endpoints), while property vi) becomes valid. If (in either direction) the walk has reached a so far unvisited blue copy of K_t, the walk follows an arbitrary blue edge in this copy. In this way, the procedure continues in both directions, extending W so that the first four properties are satisfied in each step, until the latter two conditions are satisfied as well. Observe that the procedure is guaranteed to terminate, since each edge of F occurs at most once in W.

We next make some observations about the structure of W. If W repeats a vertex, this vertex must be u_1 or u_ℓ. Indeed, if $u_i = u_j$ for some $1 < i < j < \ell$, then u_i has degree at least four in W by property i) and, since W is alternating by property iii), u_i is incident to at least two blue edges in W. But this is not possible since the blue copies of K_t are disjoint and W traverses at most one edge from each such copy by iv). In other words, W must be a red/blue-alternating path, except possibly the edges u_1u_2 and $u_{\ell-1}u_\ell$. Therefore, each vertex of W that is not an endpoint is incident to exactly one red and one blue edge; each of the endpoints may be incident to multiple red edges but again to at most one blue edge.

We now choose a feasible walk W with the smallest number of red edges. We obtain a new colouring $\tilde{c}$ by switching the colours of the edges in W. We claim that the colouring $\tilde{c}$ contains

1. no red copy of $K_{1,s}$,

2. fewer blue copies of K_t than c, and

3. no blue copy of $K_t \cdot K_2$.

By the definition of c, this leads to the desired contradiction.

To prove (1), first note that the switch does not change the number of red edges incident to vertices not in W. Now consider a vertex u of W. By our earlier observation, u is incident to at most one blue edge in W. If u is also incident to a red edge in W, then the switch does not increase the total number of red edges incident to u. If there is no red edge incident to u in W, then by property vi), we know that there are no red edges incident to u in F under c; hence u is only incident to one red edge under $\tilde{c}$. Therefore, there is no red copy of $K_{1,s}$ under $\tilde{c}$ since $s \geq 2$.

We prove (2) now. By properties ii) and iv), W contains at least one edge belonging to a copy of K_t which is blue under c. So after switching colours, this copy of K_t contains a red edge. Therefore, the only way for (2) to fail is that we create a new blue copy of K_t by switching the colours along W. So, consider any edge uv in W whose colour switched from red to blue and such that uv is contained in a copy K of K_t. We aim to show that K is not monochromatic blue under $\tilde{c}$. To do so, we choose an arbitrary vertex $x \in V(K) \setminus \{u, v\}$, which exists as $t \geq 3$. What we will see is that either ux or vx is red under $\tilde{c}$, which will prove the claim. Assume that the statement is false. We distinguish three cases depending on the colours of ux and vx under c.

Case 1: Assume ux and vx are both red under c. Then, by assumption, both of these edges and uv must have switched colours and hence belong to W. This however contradicts the above observation that at most two vertices in W are incident to two red edges of W under c.

Case 2: Assume ux and vx are both blue under c. Since W is alternating and contains at least one blue edge, at least one of u and v, say u, must be contained in a blue copy K' of K_t under c. But then K' together with the edge ux (if $x \notin V(K')$) or the edge vx (if $x \in V(K')$) forms a blue copy of $K_t \cdot K_2$ under c, a contradiction to the choice of c.

Case 3: Assume ux is red and vx is blue under c (the case where ux is blue and vx is red is similar). Then vx did not switch colours, i.e., $vx \notin W$, and both ux and uv switched colours, i.e., $ux, uv \in W$. Therefore, u is incident to two red edges of W under the colouring c, and hence must be an endpoint of W as observed above. So, one of v and x, say x, is not an endpoint of W and is therefore contained in a blue copy K' of K_t under c. Then $v \in V(K')$, since vx is blue and c does not contain a blue copy of $K_t \cdot K_2$. Since W is alternating, since x is not an endpoint of W, and since the blue copies of K_t are disjoint, the walk W contains an edge xy from K'. Since $vx \notin W$, we have $y \neq v$. By property iv) of W and again since blue copies of K_t under c are disjoint, it follows that v is not incident to any blue edge in W and must therefore be an endpoint. Removing v from W, and hence the red edge uv, yields a feasible walk W with fewer red edges, a contradiction to the choice of W.

It remains to check property (3). Assume that there is a blue copy of $K_t \cdot K_2$ under $\tilde{c}$, with blue copy K of K_t and pendent blue edge f. As we have already seen, the switching of colours does not create new blue copies of K_t. Hence, K is blue under c and all edges intersecting K in exactly one vertex are red under c, since c does not contain a blue copy of $K_t \cdot K_2$. This means that f is red under c and hence $f \in W$. By property v) and K is disjoint from all other blue copies of K_t under c, the walk W contains an edge from K. The colour of this edge is then switched from blue to red, a contradiction. Altogether we see that $\tilde{c}$ is $(K_{1,s}, K_t)$-free and hence $F \not\to (K_{1,s}, K_t)$.

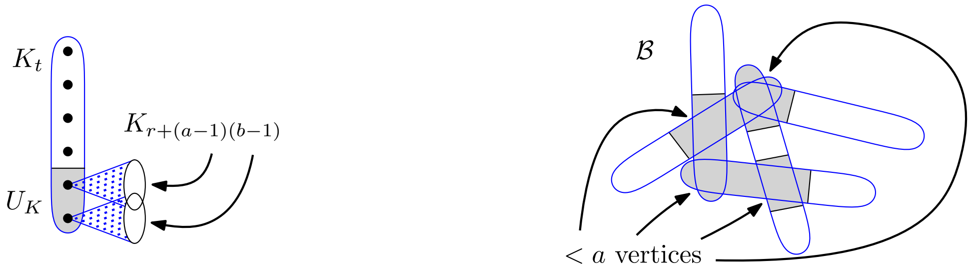

Figure 5.3: Left: The set U_K (grey background) in a copy K of K_t. Right: A set $\mathcal{B}$ of blue copies of K_t under φ_1 with pairwise intersection of size less than a. The intersections are contained in the respective sets U_K by Claim 5.7.

□

To prove Theorem 5.5(b) we make use of the following definition. We call a graph G *k-woven* if, for each graph F that contains an edge uv which is contained in all copies of G in F, there is a set $Y_{uv} \subseteq E(F) \setminus \{uv\}$ such that the following holds: Y_{uv} consists of at most k edges incident to u and at most k edges incident to v, and each copy of G in F contains an edge from Y_{uv}. In other words, Y_{uv} is a set of edges of size at most $2k$ whose removal yields a graph with no copies of G that still contains the edge uv. As a simple example, it is not difficult to check that stars with at least two edges are 1-woven. Indeed, if a graph F has an edge uv that is contained in each copy of some star $K_{1,s}$ in F, then u and v are of degree at most s in F (and all other vertices are of degree at most $s-1$). So any set Y_{uv} consisting of one edge incident to u and one edge incident to v in $G - uv$ satisfies the condition stated above, that is, each copy of $K_{1,s}$ in F contains an edge from Y_{uv}. We begin by demonstrating the utility of k-woven graphs in Proposition 5.6 below. We will then prove Theorem 5.5(b) by showing that suitable caterpillars are k-woven for appropriately chosen k. For any pair of graphs (G, H), we write $r(G, H)$ for the Ramsey number of (G, H), i.e., the smallest integer n such that $K_n \to (G, H)$.

Proposition 5.6. *Let G be a k-woven graph, let $a \geq 1$ and $b \geq 2$ be integers, and let $r = r(G, K_{b-1})$. If $t \geq 4k + 2(r + (a-1)(b-1)) + (a-1)$, then $(G, K_t) \sim (G, K_t \cdot aK_b)$.*

Proof. Clearly, each Ramsey graph for $(G, K_t \cdot aK_b)$ is also a Ramsey graph for (G, K_t). So consider a graph F with $F \not\to (G, K_t \cdot aK_b)$. We shall show that $F \not\to (G, K_t)$. Let φ_1 denote a $(G, K_t \cdot aK_b)$-free colouring of $E(F)$. For each blue copy K of K_t in F, let $U_K \subseteq V(K)$ denote the set of vertices u in K such that there are at least $r + (a-1)(b-1)$ blue edges between u and $F - K$ whose endpoints induce a complete graph in $F - K$. See Figure 5.3 (left).

Let $\mathcal{B}$ denote a maximal set of blue copies of K_t in F such that any two copies of K_t in $\mathcal{B}$ intersect in fewer than a vertices. See Figure 5.3 (right). We first make several general observations.

Claim 5.7. *For any two copies K, $K' \in \mathcal{B}$, we have $V(K) \cap V(K') \subseteq U_K \cap U_{K'}$ and hence $(V(K) \setminus U_K) \cap (V(K') \setminus U_{K'}) = \emptyset$.*

Proof. For each vertex $u \in V(K) \cap V(K')$ the number of blue edges between u and $K - K'$, as well as between u and $K' - K$, is at least $t - (a-1) \geq 4k + 2(r + (a-1)(b-1)) \geq$

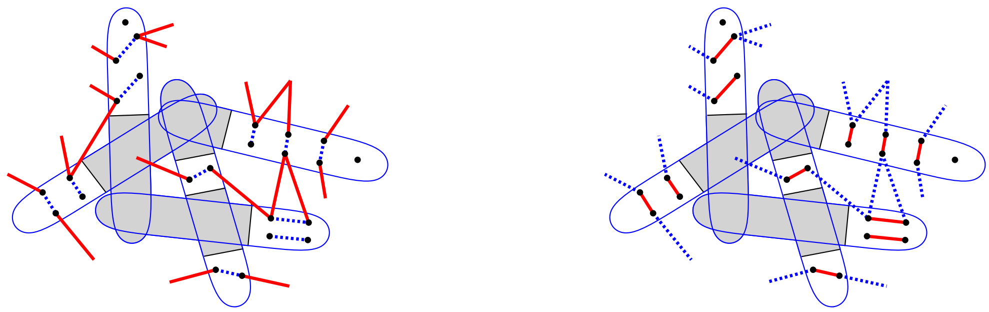

Figure 5.4: Left: A largest blue (dotted) matching M in $\cup_{K\in\mathcal{B}}(V(K)\setminus U_K)$ with some pendent red (solid) edges under ϕ_1. Right: The final colouring φ_3 is obtained by switching colours in M (from blue to red) and at most k further incident edges at each vertex in M (from red to blue).

$r+(a-1)(b-1)$. Since these blue neighbourhoods induce a complete graph, we have $u\in U_K$ and $u\in U_{K'}$. □

Claim 5.8. *For every $K\in\mathcal{B}$, we have $|U_K|\le a-1$ and hence $|V(K)\setminus U_K|\ge 2$.*

Proof. We first argue that $|U_K|\le a-1$. This holds since, under φ_1, each vertex in U_K has a blue neighbourhood of size at least $r+(a-1)(b-1)$ in $F-K$ that induces a complete graph. As there are no red copies of G under φ_1, by the definition of Ramsey number, we iteratively find $|U_K|$ vertex-disjoint blue copies of K_{b-1} in the blue neighbourhood of U_K, one for up to $\min\{a,|U_K|\}$ vertices in U_K. So $|U_K|\le a-1$, as there is no blue copy of $K_t\cdot aK_b$ under φ_1. This shows that $|V(K)\setminus U_K|\ge t-a+1\ge 2$, as required. □

We shall now recolour some edges contained in or incident to cliques in $\mathcal{B}$ to obtain an (G,K_t)-free colouring of $E(F)$. Let M denote a largest matching in the graph induced by $\cup_{K\in\mathcal{B}}(V(K)\setminus U_K)$. That is, M consists of a largest matching from each of the cliques $K-U_K$, which are vertex-disjoint by Claim 5.7 (see Figure 5.4). Let φ_2 denote the colouring obtained from φ_1 by switching the colour of each edge in M from blue to red. Each red copy of G under φ_2 contains an edge from M. Let $u_1v_1,\dots,u_{|M|}v_{|M|}$ denote the edges of M in an arbitrary order. We shall use the fact that G is k-woven to find sets $Y_1,\dots,Y_{|M|}\subseteq E(F)\setminus M$ such that each set Y_i consists of at most k red edges incident to u_i and at most k red edges incident to v_i and such that each red copy of G in F under φ_2 contains an edge from Y_i for some $i\in[|M|]$. To do so consider the subgraph F_1 of F formed by all red copies of G under φ_2 containing u_1v_1 and not containing u_jv_j for $j>1$. Then $F_1-u_1v_1$ contains no copy of G, and hence, since G is k-woven, there is a desired set $Y_1\subseteq E(F_1)\setminus\{u_1v_1\}$. For $i>1$ we proceed iteratively. Having chosen $Y_1,\dots,Y_{i-1}$, let F_i denote the subgraph of F formed by all red copies of G under φ_2 containing u_iv_i, not containing any edge from Y_j for $j<i$, and not containing u_jv_j for $j>i$. Then $F_i-u_iv_i$ contains no copy of G. Hence, since G is k-woven, there is a desired set $Y_i\subseteq E(F_i)\setminus\{u_iv_i\}$.

Now let φ_3 denote the colouring obtained from φ_2 by switching the colour of each edge in $\cup_{1\le i\le|M|}Y_i$ from red to blue (see Figure 5.4). Then there are no red copies of G under φ_3. Indeed, each red copy G' of G under φ_2 contains an edge from Y_i for some i. We shall prove that there are no blue copies of K_t under φ_3. Let K' denote a copy of K_t in F.

First suppose that $K' \in \mathcal{B}$. By Claim 5.8, we have $|V(K') \setminus U_{K'}| \geq 2$, and hence K' contains a red edge under φ_3 from $E(K') \cap E(M)$.

We may assume then that $K' \notin \mathcal{B}$. If for each $K \in \mathcal{B}$ we have $V(K) \cap V(K') \subseteq U_K$, then $|V(K) \cap V(K')| < a$ by Claim 5.8. By the maximality of $\mathcal{B}$, K' contains a red edge under φ_1. This edge is red under φ_3, since only edges incident to M switched colours from red to blue and the edges in K' are not incident to M (as $V(K) \cap V(K') \subseteq U_K$ for each $K \in \mathcal{B}$ here). So K' is not blue in this case.

If there is $K \in \mathcal{B}$ with $|V(K) \cap V(K')| > \lceil \frac{t-|U_K|}{2} \rceil + |U_K|$, then K' contains an edge from $M \cap K$. This edge is red under φ_3 and so K' is not blue.

If neither of the two previous cases holds, then let $V = \cup_{K \in \mathcal{B}} (V(K') \cap (V(K) \setminus U_K))$. By assumption $|V| \geq 1$ (since we are not in the first case). Each vertex $v \in V$ is contained in exactly one $K \in \mathcal{B}$ and, since we are not in the second case, the number of edges between v and $K' - K$ is at least $t - \lceil \frac{t-|U_K|}{2} \rceil - |U_K| = \lfloor \frac{t-|U_K|}{2} \rfloor$. Since $v \notin U_K$, fewer than $r + (a-1)(b-1)$ of those edges are coloured blue under φ_1 (as their endpoints induce a complete subgraph of K'). Together, this means that the number of red edges under φ_1 incident to v in K' is at least $\lfloor \frac{t-|U_K|}{2} \rfloor - r - (a-1)(b-1) + 1 \geq 2k+1$, using the fact that $|U_K| \leq a-1$ by Claim 5.8. In total there are at least $(2k+1)|V|/2 > k|V|$ red edges in K' under φ_1. To obtain φ_3, for each $v \in V$, at most k incident edges were chosen to switch colours from red to blue (in total more edges incident to v than those k might have switched colours due to other vertices in V). So at most $k|V|$ edges in K' switched their colour from red to blue. This shows that at least one edge in K' is red under φ_3.

Altogether there are no red copies of G and no blue copies of K_t under φ_3 and hence $F \not\to (G, K_t)$. $\square$

Proof of Theorem 5.5(b). By Proposition 5.6, it suffices to show that stars and suitable caterpillars are k-woven for some k. As mentioned above, it is not difficult to check that stars with at least two edges are 1-woven. We now focus on caterpillars, and claim that every s-suitable caterpillar is $2(s+1)^2$-woven.

Let T be an s-suitable caterpillar, that is, T consists of a path abc and s leaves adjacent to a, s leaves adjacent to c, and $s' < s$ leaves adjacent to b. We shall prove that T is k-woven for $k = 2(s+1)^2$. Let F be a graph with an edge uv that is contained in all copies of T in F, and let $F' = F - uv$. We need to find a set $Y_{uv} \subseteq E(F')$ consisting of at most k edges incident to u and at most k edges incident to v such that Y_{uv} contains an edge from each copy of T in F.

First suppose that there is a copy T_0 of T in F in which u is a leaf. The neighbour of u in T_0 is v, since T_0 contains uv by assumption. Then v is of degree at most $|V(T)| - 2 = 1 + 2s + s' \leq k$ in F', since otherwise uv can be replaced in T_0 by some edge vw in F' to form a copy of T entirely in F', which does not exist by assumption. Let Y_v consist of all edges in F' incident to v. If each copy of T in F contains an edge from Y_v, then we can choose $Y_{uv} = Y_v$ as our desired edge set. Otherwise, there is a copy of T in F containing no edges from Y_v. In such a copy of T the vertex v is a leaf, since uv is the only edge incident to v not in Y_v. Similarly as above, u is of degree at most k in F', and hence we can choose Y_{uv} to consist of all edges incident to u and all edges incident to v in F'.

It remains to consider the case where neither u nor v is a leaf in any copy of T in F. By the symmetry of T, we may assume that in each copy of T in F the edge ab corresponds to uv, where a corresponds to either u or v. Let N_u and N_v denote the set of neighbours

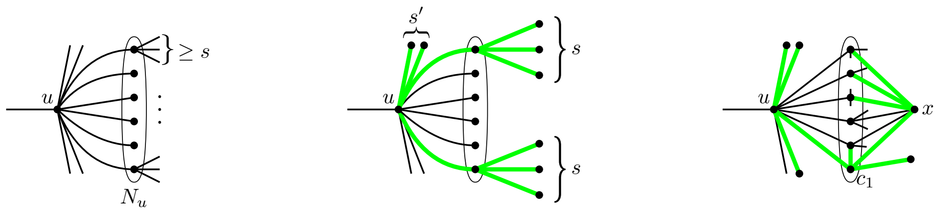

Figure 5.5: Left: The vertex u and the set N_u of at least $k+1$ neighbours of u of degree at least $s+1$ each. Middle: If two vertices in N_u have only u as a common neighbour, then there is copy of T (green/thick grey). Right: Otherwise, there is a vertex x that is a leaf in at least $2s+1$ copies of $K_{1,s}$ centred in N_u and there is a copy of T (here a 3-suitable caterpillar) as well (green/thick grey).

of u and v in F', respectively, that are of degree at least $s+1$ in F' (see Figure 5.5 left). In each copy of T in F the edge bc corresponds to an edge uw with $w \in N_u$ or an edge vw with $w \in N_v$. In particular choosing $Y_{uv} = \{uw\colon w \in N_u\} \cup \{vw\colon w \in N_v\}$ yields the desired edge set, provided that $|N_u|$, $|N_v| \leq k$. In the following, we prove $|N_u| \leq k$. By symmetry the same bound holds for $|N_v|$.

For a contradiction, assume that $|N_u| \geq k+1$, which in particular implies that u is of degree at least $k+1$ in F'. We shall prove that there is a copy of T in F' under this assumption. For each $w \in N_u$, choose a star in F' with centre vertex w and exactly s leaves not containing u, and let $\mathcal{S}$ denote the set of all chosen stars. For any two such stars S, $S' \in \mathcal{S}$ there are at least $k+1-2(s+1) \geq s > s'$ neighbours of u in F' not contained in $V(S) \cup V(S')$. Since F' does not contain a copy of T, the stars S and S' must intersect in some vertex, which could be the centre of one of the stars but not of both (see Figure 5.5 middle).

Now, consider some fixed star $S \in \mathcal{S}$. By the pigeonhole principle, there is a vertex x in $V(S)$ that is contained in at least $|\mathcal{S} \setminus \{S\}|/|V(S)| = (|N_u|-1)/(s+1) \geq \frac{k}{s+1} = 2(s+1)$ of the stars in $\mathcal{S}$. It may happen that x is the centre vertex of one such star, but in any case there is a family of $2s+1$ stars in $\mathcal{S}$ that have x as a leaf. Let $X = \{c_1, c_2, \ldots, c_{2s+1}\} \subseteq N_u$ denote the set of their centres. Then we find a copy of T in F' as follows: Let X_1 denote a set of s' neighbours of c_1 distinct from u and x in F', which exists since $s' < s$ and $c_1 \in X$ is of degree $s+1$ in F'. Let X_2 denote a set of s neighbours of x in X disjoint from $X_1 \cup \{c_1\}$, which exists since $|X| = 2s+1 \geq s+s'+2$. Finally, let X_3 denote a set of s neighbours of u in F' disjoint from $X_1 \cup X_2 \cup \{x, c_1\}$, which exists since the degree of u in F' is at least $k+1 = 2(s+1)^2+1 \geq 2s+s'+2$. Then the path uc_1x together with the vertices in X_1, X_2, and X_3 induces a copy of T in F', a contradiction (see Figure 5.5 right). This shows that $|N_u| \leq k$ and, by symmetry, $|N_v| \leq k$. Hence, $Y_{uv} = \{uw\colon w \in N_u\} \cup \{vw\colon w \in N_v\}$ is the desired edge set. □

5.3 Non-equivalence results for trees and cliques

In this section, we prove each part of Theorem 5.9 in turn. When constructing appropriate distinguishing graphs in our proofs, we will often combine several smaller graphs, which we call building blocks, by identifying some of their vertices or edges. We will assume that, except for the specified intersections, all of these building blocks are disjoint from one another. Before we begin, let us restate the theorem.

Theorem 5.9. *Let T be a tree, $t \geq 3$ be an integer, and let G and H be graphs with $(G,H) \neq (T,K_t)$. Then $(T,K_t) \not\sim (G,H)$ if one of the following conditions holds:*

(a) $G = T$, $T \in \mathcal{T}$, and H is connected,

(b) H contains a copy K of K_t, and H contains a cycle with vertices from both $V(K)$ and $V(H) \setminus V(K)$,

(c) $G \neq T$, G is connected, and $H = K_t$.

Proof of Theorem 5.9(a)

Recall that the diameter of a tree T, denoted $\mathrm{diam}(T)$, is the length of its longest path. Our construction gives $(T,K_t) \not\sim (T,H)$ for each tree T from the following slightly larger class $\mathcal{T}'$ consisting of all trees T that either have odd diameter, or have even diameter and additionally satisfy the following:

- The central vertex of T has at most one neighbour of degree at least three that is contained in a longest path in T.
- If T is of diameter four, the central vertex is of degree at least three.

See Figure 5.6 (left) for an illustration.

Theorem 5.9(a) is clearly a direct consequence of Theorem 5.10 below.

Theorem 5.10. *Let $t \geq 3$, let H be a connected graph, and let $T \in \mathcal{T}'$. Then $(T,K_t) \not\sim (T,H)$.*

Proof. Consider a tree $T \in \mathcal{T}'$. If $\omega(H) \neq t$, then $(T,K_t) \not\sim (T,H)$ by Theorem 1.10. So we assume $K_t \subsetneq H$ for the remainder of the proof. We will construct a Ramsey graph for (T,K_t) that is not Ramsey for $(T,K_t \cdot K_2)$ and hence not Ramsey for (T,H). The construction differs slightly depending on the parity of the diameter of T. We begin by introducing a useful gadget graph.

Throughout the proof, we let $U_{k,i}$ denote the rooted tree in which every leaf is at distance i from the root and every vertex that is not a leaf has exactly k children. Here, the distance between two vertices x and y is the length of a shortest path that has x and y as its endpoints. Note that $U_{k,i}$ contains every tree of diameter at most $2i$ and maximum degree at most k.

Let d denote the maximum degree of T. Let Γ be a Ramsey graph for (T,K_{t-1}) that does not contain a copy of K_t, which exists by Theorem 1.10. Write $k = d|V(\Gamma)|$. For a positive integer i, let $\Lambda_i = \Lambda_i(T,\Gamma)$ denote the graph obtained from a copy of $U_{k,i}$ by adding edges so that, for each non-leaf vertex of $U_{k,i}$, its set of children induces d vertex-disjoint copies of Γ. We refer to the root of $U_{k,i}$ as the root of Λ_i. Let $\Phi_i = \Phi_i(\Lambda_i)$ be the red/blue-colouring that assigns red to all edges in $U_{k,i}$ and blue to all the other edges, see Figure 5.6 (middle) for an illustration. Observe that, if $i < \mathrm{diam}(T)$, then Φ_i is a (T,K_t)-free colouring of $E(\Lambda_i)$. We have the following Ramsey property of Λ_i.

Claim 5.11. *Every red/blue-colouring of $E(\Lambda_i)$ yields a red copy of T, a blue copy of K_t, or a red copy of $U_{d,i}$ whose root is the root of Λ_i.*

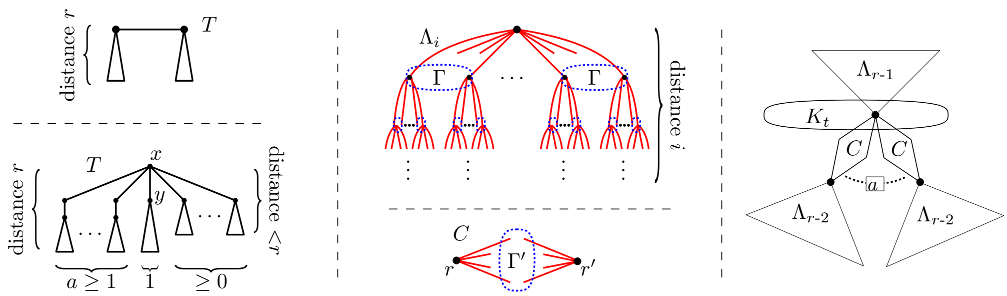

Figure 5.6: Left: An odd diameter tree and an even diameter tree from the class $\mathcal{T}'$. Middle: Graphs Λ_i and C with the respective $(T, K_t \cdot K_2)$-free colourings. Right: A graph F with $F \to (T, K_t)$ and $F \not\to (T, K_t \cdot K_2)$ in case T is of even diameter.

Proof. To see why this is true, consider an arbitrary 2-colouring of $E(\Lambda_i)$ with no red copy of T. Then each copy of Γ contains a blue copy of K_{t-1}. If some non-leaf vertex in $U_{k,i}$ has only blue edges to one of the copies of Γ formed by its children, then there is a blue copy of K_t. Otherwise, every such vertex has a red edge to each of the d copies of Γ formed by its children, yielding a copy of $U_{d,i}$ as required. □

First consider the case where T is of diameter $2r+1$ for some integer r. We construct a graph F as follows: Start with a copy K of K_t. For each vertex u of K, add a copy of Λ_r rooted at u so that the copies of Λ_r are pairwise disjoint. We claim that $F \to (T, K_t)$ and $F \not\to (T, K_t \cdot K_2)$.

To prove the first claim, we consider an arbitrary 2-colouring of $E(F)$ with no red copy of T. By Claim 5.11, some copy of Λ_r contains a blue copy of K_t or each vertex of K is the root of a red copy of $U_{d,r}$. If we find a blue copy of K_t, we are done, and hence we may assume that the latter happens for every vertex of K. If there is a red edge in K, then this edge and the red copies of $U_{d,r}$ rooted at its endpoints form a graph which contains a red copy of T. Otherwise, all edges of K are coloured blue, yielding a blue copy of K_t. This shows $F \to (T, K_t)$.

To see that $F \not\to (T, K_t \cdot K_2)$, colour all edges of K blue and give all copies of Λ_r the colouring Φ_r. Then K is the only blue copy of K_t and it cannot be extended to a copy of $K_t \cdot K_2$, as all edges leaving K are coloured red and all the other blue edges form vertex-disjoint copies of Γ, which was chosen such that $K_t \nsubseteq \Gamma$. The red edges form vertex-disjoint trees of diameter $2r < \mathrm{diam}(T)$. Hence, there is no red copy of T and no blue copy of $K_t \cdot K_2$ and so $F \not\to (T, K_t \cdot K_2)$.

Now consider the case where T is of diameter $2r$ for some integer r. In this case the assumptions on $\mathcal{T}'$ imply that at most one neighbour of the central vertex is of degree at least three and is contained in a longest path. Further, if the diameter is exactly four, the central vertex of T is of degree at least three. Let x denote the central vertex of T, let y denote a neighbour of x in T that is of largest degree among all neighbours of x contained in a longest path in T, and let a denote the number of all other neighbours of x contained in a longest path in T (see Figure 5.6 (left) for an illustration). By the assumption on the structure of T, all neighbours of x counted by a are of degree exactly two in T. As in the previous case, we will use the graphs Λ_i as building blocks. We now define the second type of building block that we will use in the construction.

Let J denote a graph containing no copy of K_t such that, for any 2-colouring of the *vertices* of J, there is a vertex-monochromatic copy of K_{t-1}. Such a graph exists by [42]. Let Γ' be a Ramsey graph for (T, J) not containing a copy of K_t, which exists by Theorem 1.10, and let $k' = |V(\Gamma')|$. Let C denote the graph obtained from a copy of Γ' by adding two non-adjacent vertices r and r' and a complete bipartite graph between these two vertices and the vertices of the copy of Γ'. For convenience we call r the root of C. See Figure 5.6 (middle) for an illustration.

Claim 5.12. *In every red/blue-colouring of $E(C)$, there exists a red copy of T, a blue copy of K_t, or a red path rvr' for some $v \in V(\Gamma')$.*

Proof. To see why this is true, consider a red/blue-colouring of $E(C)$ with no red copy of T. Then the copy of Γ' in C contains a blue copy J' of J. In particular, each copy of K_{t-1} in J' is blue. Moreover, either each copy of K_{t-1} has a red edge going to each of r and r', or there is a blue copy of K_t. In the latter case, we are done, so assume the former. Consider an auxiliary vertex colouring of J' obtained by colouring each vertex v in J' with the colour of the edge rv. Since there cannot be a vertex-monochromatic blue copy of K_{t-1}, there is a vertex-monochromatic red copy of K_{t-1}. We assume that there is a red edge between this copy of K_{t-1} and r', and hence we find a red path rvr', where $v \in V(J')$. $\square$

We now construct a graph F' as follows: Start with a copy K' of K_t. For each vertex u of K', add a copies of C and a copy of Λ_{r-1} all rooted at u. Further, for each copy of C, add a copy of Λ_{r-2} rooted at the copy of r'. See Figure 5.6 (right) for an illustration. We claim that $F' \to (T, K_t)$ and $F' \not\to (T, K_t \cdot K_2)$.

To prove the first claim we consider an arbitrary 2-colouring of $E(F')$ with no red copy of T. By Claim 5.11, either there is a blue copy of K_t in some copy of Λ_{r-1} or Λ_{r-2} or the root of each copy of Λ_{r-1} or Λ_{r-2} is the root of a red copy of $U_{d,r-1}$ or $U_{d,r-2}$, respectively. Assume the latter is true, since otherwise we are done. By Claim 5.12, either there exists a blue copy of K_t in some copy of C or each copy of C in F contains a red path of length two connecting the copies of r and r'. Again, we may assume that we are in the latter case. For each copy of C, the copy of r' is the root of a copy of Λ_{r-2}. Now, every vertex of K' is the root of a copies of C and a copy of Λ_{r-1}. If there is a red edge e in K, then there is a red copy of T formed by e and subtrees of the red trees rooted at its endpoints, with e playing the role of the edge xy in T. Otherwise all edges of K are coloured blue, proving the claim.

To show that $F' \not\to (T, K_t \cdot K_2)$ we colour the edges of F' as follows: All edges of K' are coloured blue and all mentioned copies of Λ_i $(i \in \{r-1, r-2\})$ are coloured according to the colouring Φ_i, as defined earlier. For each mentioned copy of C, all edges in the copy of Γ' are blue and all other edges red. Then K' is the only blue copy of K_t, as all other blue edges form vertex-disjoint copies of Γ or Γ' and these graphs do not contain copies of K_t. Moreover, K' has only red incident edges. So there is no blue copy of $K_t \cdot K_2$. Now consider the red subgraph of F', and recall that the central vertex of T has $a+1$ neighbours contained in paths of length $2r$. If $r > 2$, then each longest red path in F' has $2r$ edges and the middle vertex of each such path is in K'. In particular, the central vertex of each red tree of diameter $2r$ is in K'. But for each vertex $u \in V(K')$ there are at most a red paths of length $2r$ that meet at u and are otherwise pairwise vertex-disjoint, so there is no red copy of T. If $r = 2$, then for the same reason there is no red copy of T rooted in K'. In this case there are also red paths of length $2r = 4$ whose central vertex is in the neighbourhood of K'. However, these vertices are of degree at most two in the red subgraph and, by assumption, the root of T is of degree at least three in this case. This shows that $F \not\rightarrow (T, K_t \cdot K_2)$. $\square$

Proof of Theorem 5.9(b)

We first introduce some definitions. A *cycle of length s* in a hypergraph $\mathcal{H}$ is a sequence $e_1, v_1, e_2, v_2 \ldots, e_s, v_s$ of distinct hyperedges and vertices such that $v_i \in e_i \cap e_{i+1}$ for all $1 \leq i \leq s$ where $e_{s+1} = e_1$. The *girth* of a hypergraph $\mathcal{H}$ is the length of a shortest cycle in $\mathcal{H}$ (if no cycle exists, then we say that the girth of $\mathcal{H}$ is infinity). The *chromatic number* of a hypergraph $\mathcal{H}$ is the minimum number r for which there exists an r-colouring of the vertex set of $\mathcal{H}$ with no monochromatic edges. A hypergraph is *d-degenerate* if every subhypergraph contains a vertex of degree at most d, and we define its *degeneracy* to be the smallest d for which this property holds.

Proof of Theorem 5.9(b). Suppose that H contains a copy K of K_t, and H contains a cycle with vertices from both $V(K)$ and $V(H) \setminus V(K)$. Let $g \geq 3$ denote the length of a shortest such cycle in H and let $k = |E(T)|$.

We shall use a hypergraph of high girth and high minimum degree. The existence of such a hypergraph follows from a well-known result of Erdős and Hajnal [36]; we sketch the argument here for the sake of completeness. Erdős and Hajnal [36] showed that there exists a t-uniform hypergraph $\mathcal{H}'$ with girth at least $g + 1$ and chromatic number at least $kt+1$. It is not difficult to show, using a greedy algorithm, that a d-degenerate hypergraph has chromatic number at most $d+1$. Hence, the degeneracy of $\mathcal{H}'$ must be at least kt, and thus $\mathcal{H}'$ must contain a t-uniform subhypergraph $\mathcal{H}$ with girth at least $g+1$ and minimum degree at least kt.

We construct a graph F with vertex set $V(\mathcal{H})$ by embedding a copy of K_t into each edge of $\mathcal{H}$. First observe that F does not contain a copy of H, since $\mathcal{H}$ has girth larger than g and hence each cycle of length at most g is fully contained in one of the copies of K_t, that is, in one of the hyperedges, and no two copies of K_t in F share an edge. In particular $F \not\rightarrow (T, H)$. Next we shall prove that $F \rightarrow (G, K_t)$. Consider a 2-colouring of $E(F)$ without blue copies of K_t. Then each copy of K_t in F (each hyperedge of $\mathcal{H}$) contains a red edge. Since $\mathcal{H}$ has minimum degree kt and is t-uniform, there are at least $v(\mathcal{H})k$ red edges, that is, the average red degree of F is at least $2k$. It follows from a standard greedy argument that the red subgraph of F contains a subgraph of minimum degree at least k. By greedily embedding the vertices of T in this subgraph, we find a red copy of T. Hence $F \rightarrow (T, K_t)$ and $(T, K_t) \not\sim (G, H)$. $\square$

Proof of Theorem 5.9(c)

Proof of Theorem 5.9(c). We may assume that $G \not\subseteq T$, since otherwise G is a tree and we can switch the graphs G and T in the statement. Suppose that the pairs (T, K_t) and (G, K_t) are Ramsey equivalent. In order to reach a contradiction, we will construct a graph F which is Ramsey for (T, K_t) but not Ramsey for (G, K_t). To do so, first fix a

(T, K_t, β)-determiner D, as given by Proposition 2.2. To create F, we start with a copy T_0 of T, and for each edge e of T_0 we take a copy D_e of D on a new set of vertices and identify e with the copy of β in D_e.

We first observe that F is a Ramsey graph for (T, K_t). Indeed, if we assume that F has a (T, K_t)-free colouring, then this induces a (T, K_t)-free colouring on each copy of D, so each copy of β needs to be red by the definition of a determiner. But then T_0 becomes a red copy of T, a contradiction.

It remains to prove that F is not Ramsey for (G, K_t), i.e., to find a (G, K_t)-free colouring of $E(F)$. For this, fix any edge $e_0 \in E(T_0)$. We first observe that the graph $F - e_0$ is not Ramsey for (T, K_t) by considering the following colouring: give each copy of D a (T, K_t)-free colouring such that its copy of β is red (or not coloured if $\beta = e_0$) and all edges incident to β in D are blue. The existence of such a colouring is guaranteed by the fact that F is well-behaved.

By our assumption that (T, K_t) and (G, K_t) are Ramsey equivalent, we conclude that $F - e_0$ is not a Ramsey graph for (G, K_t). Therefore, we can find a (G, K_t)-free colouring c of $F - e_0$. We now extend this colouring to F by assigning the colour blue to e_0. If this does not create a blue copy of K_t, we have already found the required colouring. So we may assume that this extension leads to a blue copy K of K_t. Notice that every copy of K_t in F is fully contained in a copy of the determiner D. Then by Proposition 2.2 this blue copy of K_t is the graph induced by the endpoints of e_0 and the union of their neighbourhood in D_{e_0}, i.e., it must be contained in the copy D_{e_0} of D and is unique. We now use this information to recolour all other copies of $D - \beta$ in F using the colouring of $E(D_{e_0} - e_0)$; we further colour T_0 fully red. In this new colouring of $E(F)$, there cannot be a blue copy of K_t as there were none in $D_{e_0} - e_0$. Moreover, there cannot be a red copy of G, since every copy of $D - \beta$ has a (G, K_t)-free colouring, every edge incident to T_0 is blue, and $G \not\subseteq T$. This is a contradiction to the assumption $F \to (G, K_t)$ and hence $(T, K_t) \not\sim (G, K_t)$. $\square$

5.4 Concluding remarks

In this chapter we identify a non-trivial infinite family of Ramsey equivalent pairs of connected graphs of the form $(T, K_t) \sim (T, K_t \cdot K_2)$, where T is a non-trivial star or a so-called suitable caterpillar. We also prove that $(T, K_t) \not\sim (T, K_t \cdot K_2)$ for a large class of other trees T including all trees of odd diameter. It remains open whether for the remaining trees the respective pairs are Ramsey equivalent or not. Our proof actually shows $(G, K_t) \sim (G, K_t \cdot K_2)$ for all so-called woven graphs G and sufficiently large t. This leads to the following two questions: Are there any woven graphs other than the trees mentioned in Theorem 5.5(b)? Are there non-woven graphs G and integers t with $(G, K_t) \sim (G, K_t \cdot K_2)$?

One of the questions that drove the study of Ramsey equivalence is: What graphs H are Ramsey equivalent to the clique K_t? This question was addressed in [12, 43, 83]. In particular, it follows from the results of Folkman [42] and Nešetřil and Rödl [69] and Fox, Grinshpun, Liebenau, Person, and Szabó [43] that there is no connected graph $H \neq K_t$ such that $H \sim K_t$. It is then natural to ask: what about an asymmetric pair of connected graphs?

Question 5.13. *Are there connected graphs G and H and an integer t such that, for $(G, H) \neq (K_t, K_t)$ it holds $(G, H) \sim (K_t, K_t)$?*

Some known results allow us to easily exclude many possible pairs (G, H). For example, the results of Folkman [42] and Nešetřil and Rödl [69], as stated in Theorem 1.10 above, show that, if $\max\{\omega(G), \omega(H)\} \neq t$, then $(G, H) \not\sim (K_t, K_t)$, while the work of Fox, Grinshpun, Liebenau, Person, and Szabó [43] shows that we cannot have $\omega(G) = \omega(H) = t$. Thus, we can restrict our attention to pairs (G, H) with $\omega(G) < t$ and $\omega(H) = t$. Combining several results concerning Ramsey properties of the random graph $G(n, p)$ [16, 59, 67, 76, 77], we can restrict (G, H) even further: namely, we can show that $m_2(G) = m_2(H) = m_2(K_t)$. Using the ideas developed by Savery in [80], we can also prove that the chromatic numbers of the graphs G and H must satisfy either $\chi(G) = t-1$ and $\chi(H) = t+1$, or $\chi(G) = t$ and $H = K_t$. In addition, the theory of determiners developed in [25] for 3-connected graphs allows us to conclude that G and H cannot both be 3-connected. It would be very interesting to provide a complete answer to Question 5.13.

Our study focuses on pairs of connected graphs. Disconnected graphs have also received some attention in the symmetric setting; the central question here asks which graphs are Ramsey equivalent to a complete graph [12, 43, 83]. Similar questions arise in the asymmetric setting, for instance for which graphs G and integers t we have $(G, K_t) \sim (G, K_t + K_{t-1})$, where $K_t + K_{t-1}$ is the disjoint union of K_t and K_{t-1} (this holds in case $G = K_t$ by [12]).

6

On r-cross t-intersecting families

In this chapter we determine the maximum of sum of measures of r-cross t-intersecting families. Before we go any further, let us recall that for $r, t, n \in \mathbb{N}$ we say that families $\mathcal{F}_1, \ldots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ are *r-cross t-intersecting* if for all $F_1 \in \mathcal{F}_1, \ldots, F_r \in \mathcal{F}_r$ we have $|\bigcap_{i\in[r]} F_i| \geq t$. We will prove Theorems 6.7 and 6.8.

Before we provide a formal proof, we will provide a short idea of our methods in Section 6.1.

In Section 6.2 we will define a *necessary intersection point* which is a central tool for our work. We will then go on the provide a proof of the theorems which directly follow from a more common result as stated in Proposition 6.9.

To conclude, in Section 6.3 we will discuss some related open problems.

This chapter is based on joint work with Yannick Mogge, Simón Piga, and Bjarne Schülke [57].

6.1 Idea of the proof

Our proof is based on what we call *necessary intersection points* (see Definition 6.1). Roughly speaking we say that an element $a \in [n]$ is a necessary intersection point for r-cross t-intersecting families $\mathcal{F}_1, \ldots, \mathcal{F}_r$ if there are sets in the families which "depend" on this element to fulfil their intersection property. For example, if we consider the 2-cross 1-intersecting families $\mathcal{A}(n, 2, 1)$ and $\mathcal{B}(n, 2)$, the element 2 is a necessary intersection point because there are pairs of sets that intersect only in 2. In this case, 1 and 2 are the only necessary intersection points of these families. The idea is to "decrease" the maximal necessary intersection point as long as possible, i.e., replace the presently considered r-cross t-intersecting families by r-cross t-intersecting families whose sum of measures is not smaller but which have a smaller maximal necessary intersection point.

Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be some r-cross t-intersecting families and let $a \in [n]$ be their maximal necessary intersection point. To construct the new families we first remove all sets that "depend" on a in one family, say $\mathcal{F}_r$; we call the family of these sets $\mathcal{F}_r(a)$. Then a will no longer be a necessary intersection point. Potentially, there are some subsets of $[n]$ which could not be in any of the other families because they would not intersect "correctly" with some set in $\mathcal{F}_r(a)$. However, after removing $\mathcal{F}_r(a)$ from $\mathcal{F}_r$ and depending on how such a set relates with $\mathcal{F}_r \setminus \mathcal{F}_r(a)$, it may be added to one of the other families without breaking the intersection property.

There are some structural properties that follow from a being the maximal necessary intersection point and the fact that the families are shifted. These will help us to analyse which new sets can actually be added to the families $\mathcal{F}_1, \ldots, \mathcal{F}_{r-1}$ and to prove that in fact the measure of the newly added sets is at least as large as the measure of the removed sets. Moreover, this analysis guarantees that the new maximal necessary intersection point is at most $a-1$.

We can iterate this construction and decrease the maximal necessary intersection point in every step. This process has to stop at a certain point, and we show that then the resulting families are contained in families with the desired structure (namely $\mathcal{A}(n,a,t)$ and $\mathcal{B}(n,a)$).

6.2 Proof of main results

In this section we will prove Theorems 6.7 and 6.8. We begin by introducing necessary intersection points which are central to our proofs.

Definition 6.1. Let $\mathcal{F}_1 \subseteq \mathscr{P}([n]), \ldots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be r-cross t-intersecting families. We say $a \in [n]$ is a *necessary intersection point* of $\mathcal{F}_1, \ldots, \mathcal{F}_r$ if for all $j \in [r]$ there is an $F_j \in \mathcal{F}_j$ such that

$$|[a] \cap \bigcap_{j\in[r]} F_j| = t \text{ and } a \in \bigcap_{j\in[r]} F_j\,. \tag{6.1}$$

The following easy lemma is one of the useful properties of necessary intersection points used together with shifting.

Lemma 6.2. *Let $\mathcal{F}_1 \subseteq \mathscr{P}([n]), \ldots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be shifted r-cross t-intersecting families and let a be their maximal necessary intersection point. If $i \in [r]$, $F \in \mathcal{F}_i$, and $F_j \in \mathcal{F}_j$ for $j \in [r] \setminus i$ are such that $|[a-1] \cap F \cap \bigcap_{j\in[r]\setminus i} F_j| < t$, then $[a-1] \subseteq F \cup \bigcap_{j\in[r]\setminus i} F_j$.*

Proof. We will assume that there is a $b \in [a-1] \setminus (F \cup \bigcap_{j\in[r]\setminus i} F_j)$ and derive a contradiction. Suppose $a \notin F$. Then $|[a] \cap F \cap \bigcap_{j\in[r]\setminus i} F_j| < t$. Thus, since $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are r-cross t-intersecting, there is a necessary intersection point larger than a. This contradicts the assumption that a is the maximal necessary intersection point of $\mathcal{F}_1, \ldots, \mathcal{F}_r$. We conclude that $a \in F$.

Further, we know that $\sigma_{ba}(F) \in \mathcal{F}_i$ since $\mathcal{F}_i$ is shifted and $b < a$. But then we have $|[a] \cap \sigma_{ba}(F) \cap \bigcap_{j\in[r]\setminus i} F_j| < t$, which again contradicts $\mathcal{F}_1, \ldots, \mathcal{F}_r$ being r-cross t-intersecting with maximal necessary intersection point a. $\square$

Roughly speaking, the proof proceeds by iteratively decreasing the maximal necessary intersection point, i.e., replacing the currently considered families by families with a smaller maximal necessary intersection point. In this "updating" process we need to be careful with those sets which need a fulfil the intersection property. To make this more precise, we introduce the following notation.

Let $\mathcal{F}_1 \subseteq \mathscr{P}([n]), \ldots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be r-cross t-intersecting families and let a be their maximal necessary intersection point. For every $j \in [r]$ define $\mathcal{F}_j(a)$ to be the set of all $F \in \mathcal{F}_j$ for which there exist $F_i \in \mathcal{F}_i$ for every $i \in [r] \setminus j$ such that (6.1) holds. We also refer to the sets in $\mathcal{F}_j(a)$ as the sets in $\mathcal{F}_j$ *depending on a*. Further, for $A \subseteq [a-1]$ set $\mathcal{F}_j(A,a) = \{F \in \mathcal{F}_j(a) : F \cap [a-1] = A\}$.

The following lemma is the key of our proof. It will allow us to "push down" the maximal necessary intersection point of the families considered in case that we are not already done. Since we will prove Theorem 6.7 and Theorem 6.8 simultaneously (by proving Proposition 6.9), we phrase this lemma in a general setting. The families with indices in $[r_1]$ are families as in Theorem 6.7 and the remaining families are as in Theorem 6.8.

Lemma 6.3. *Let $r, t, n \in \mathbb{N}$, $r_1 \in \mathbb{N}_0$ with $r \geq r_1$, $r \geq 2$, and $r_1 \neq 1$, and let $a \in [n]$. If $r_1 \geq 2$, suppose that $k_1, \ldots, k_{r_1} \in [n]$ are such that $n \geq 2 \max_{i \in [r_1]} k_i + \operatorname{secmin}_{i \in [r_1]} k_i - t$, and let $\mu_1, \ldots, \mu_{r_1} : [n]_0 \to \mathbb{R}_{\geq 0}$. For $i \in [r_1+1, r]$, set $k_i = n$ and let $\mu_i : [n]_0 \to \mathbb{R}_{\geq 0}$ be non-increasing. For $i \in [r]$, let $\mathcal{F}_i \subseteq [n]^{(\leq k_i)}$. If $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are shifted r-cross t-intersecting families with maximal necessary intersection point $a \geq t+1$ such that for all $i \in [r]$, the family $\mathcal{F}_i \setminus \mathcal{F}_i(a)$ is non-empty, then there are non-empty families $\mathcal{H}_1, \ldots, \mathcal{H}_r$ such that*

(a) for $i \in [r]$ we have $\mathcal{H}_i \subseteq [n]^{(\leq k_i)}$,

(b) $\mathcal{H}_1, \ldots, \mathcal{H}_r$ are r-cross t-intersecting with maximal necessary intersection point at most $a-1$, and

(c) $\sum_{j \in [r]} \mu_j(\mathcal{H}_j) \geq \sum_{j \in [r]} \mu_j(\mathcal{F}_j)$.

Proof. Roughly speaking, the families $\mathcal{H}_1, \ldots, \mathcal{H}_r$ will be obtained from $\mathcal{F}_1, \ldots, \mathcal{F}_r$ by deleting $\mathcal{F}_i(a)$ from some of them and adding new sets to the others. More precisely, define for every $i \in [r_1]$ the family

$$\mathcal{F}_i^{\text{add}} = \bigcup_{k \in [k_i]} \bigcup_{\substack{A \subseteq [a-1]: \\ \mathcal{F}_i(A,a)^k \neq \emptyset}} \{A \cup T : T \in [a+1,n]^{(k-|A|)}\}, \tag{6.2}$$

and for $i \in [r_1+1, r]$ define the family $\mathcal{F}_i^{\text{add}} = \{F \setminus a : F \in \mathcal{F}_i(a)\}$. Next, for $i \in [r]$ we set $\mathcal{F}_i^- = \mathcal{F}_i \setminus \mathcal{F}_i(a)$ and $\mathcal{F}_i^+ = \mathcal{F}_i \cup \mathcal{F}_i^{\text{add}}$. Note that for all $i \in [r]$ we have $\mathcal{F}_i^+, \mathcal{F}_i^- \subseteq [n]^{(\leq k_i)}$ and, hence, they satisfy (a).

We aim to show that considering $\mathcal{F}_i^-$ for some indices and $\mathcal{F}_j^+$ for the other indices will yield families as desired. To this end let us now observe the following claim, ensuring that such a collection will fulfil (b).

Claim 6.4. *Let $i \in [r]$.*

1. *The families $\mathcal{F}_1^-, \ldots, \mathcal{F}_{i-1}^-, \mathcal{F}_i^+, \mathcal{F}_{i+1}^-, \ldots, \mathcal{F}_r^-$ are r-cross t-intersecting with maximal necessary intersection point at most $a-1$.*

2. *The families $\mathcal{F}_1^+, \ldots, \mathcal{F}_{i-1}^+, \mathcal{F}_i^-, \mathcal{F}_{i+1}^+, \ldots, \mathcal{F}_r^+$ are r-cross t-intersecting with maximal necessary intersection point at most $a-1$.*

Proof. (1): Assume the contrary and let $F_j \in \mathcal{F}_j^-$ for $j \in [r] \setminus i$ and $F_i \in \mathcal{F}_i^+$ such that $|[a-1] \cap \bigcap_{j \in [r]} F_j| < t$. Since $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are r-cross t-intersecting, this means that there is some $F' \in \mathcal{F}_i(a)$ (potentially $F' = F_i$) with $F_i \cap [a-1] = F' \cap [a-1]$. But then $|[a-1] \cap F' \cap \bigcap_{j\in[r]\setminus i} F_j| < t$, which is a contradiction because $F_j \in \mathcal{F}_j^- = \mathcal{F}_j \setminus \mathcal{F}_j(a)$.

(2): Assume the contrary and let $F_j \in \mathcal{F}_j^+$ for $j \in [r]\setminus i$ and $F_i \in \mathcal{F}_i^-$ such that $|[a-1] \cap \bigcap_{j\in[r]} F_j| < t$. Since $\mathcal{F}_1, \dots, \mathcal{F}_r$ are r-cross t-intersecting, this means that for all $j \in [r] \setminus i$ there is an $F_j' \in \mathcal{F}_j(a)$ with $F_j \cap [a-1] = F_j' \cap [a-1]$. But then $|[a-1] \cap F_i \cap \bigcap_{j\in[r]\setminus i} F_j'| < t$, which is a contradiction because $F_i \in \mathcal{F}_i^- = \mathcal{F}_i \setminus \mathcal{F}_i(a)$. □

Now, let us show that the updated families will still have maximum measure, that is, that (c) holds. This essentially follows from the next two claims.

Claim 6.5. *For $i \in [r]$ we have $\mu_i(\mathcal{F}_i^{add}) \geq \mu_i(\mathcal{F}_i(a))$.*

Proof. If $i \in [r_1+1, r]$, note that the definition of $\mathcal{F}_i^{\text{add}}$ implies an injection $\varphi : \mathcal{F}_i(a) \to \mathcal{F}_i^{\text{add}}$ with $|\varphi(F)| = |F| - 1$. Thus, recalling that μ_i is non-increasing, the claim is proved.

If $i \in [r_1]$, we need to work a bit more. If $r_1 = 0$, there is nothing else to show, so assume that $r_1 \geq 2$. First, we want to get an upper bound on a. Let s be the minimal integer such that there is some $m_* \in [r_1]$ and $A_* \in [a-1]^{(s)}$ such that $\mathcal{F}_{m_*}(A_*, a) \neq \emptyset$. By definition we know that for $F \in \mathcal{F}_{m_*}(A_*, a)$ there are $F_j \in \mathcal{F}_j$ for all $j \in [r] \setminus m_*$ such that $|[a-1] \cap F \cap \bigcap_{j\in[r]\setminus m_*} F_j| < t$. Thus, Lemma 6.2 yields that $[a-1] \subseteq F \cup \bigcap_{j\in[r]\setminus m_*} F_j$. Since $|F \cap [a-1]| = |A_*| = s$ and $r_1 \geq 2$, this entails $a \leq s + 1 + \min_{j\in[r_1]\setminus m_*} k_j - t$.

To show $\mu_i(\mathcal{F}_i^{\text{add}}) \geq \mu_i(\mathcal{F}_i(a))$ it is enough to show that for all $k \in [k_i]$ we have $|(\mathcal{F}_i^{\text{add}})^k| \geq |(\mathcal{F}_i(a))^k|$.

Further, it is easy to see that for all $k \in [k_i]$,

$$(\mathcal{F}_i(a))^k \subseteq \bigcup_{\substack{A\subseteq[a-1]:\\ \mathcal{F}_i(A,a)^k\neq\emptyset}} \{A \cup a \cup T : T \in [a+1, n]^{(k-1-|A|)}\}.$$

Hence, in view of (6.2), to show $|(\mathcal{F}_i^{\text{add}})^k| \geq |(\mathcal{F}_i(a))^k|$ it is enough to show that for every $A \subseteq [a-1]$ with $\mathcal{F}_i(A,a)^k \neq \emptyset$ we have $\binom{n-a}{k-1-|A|} \leq \binom{n-a}{k-|A|}$, which in turn holds if $\frac{n-a}{2} > k - 1 - |A|$. And indeed, the bounds on a and n entail

$$\frac{n-a}{2} \geq \frac{n - s - 1 - \min_{j\in[r_1]\setminus m_*} k_j + t}{2} \geq \frac{2\max_{i\in[r_1]} k_i - s - 1}{2} > k - 1 - |A|.$$

□

Further let us observe the following.

Claim 6.6. *For $i \in [r]$ we have $\mathcal{F}_i \cap \mathcal{F}_i^{add} = \emptyset$.*

Proof. Assume there is some $F \in \mathcal{F}_i \cap \mathcal{F}_i^{\text{add}}$. Then, because $F \in \mathcal{F}_i^{\text{add}}$, there is some $F' \in \mathcal{F}_i(a)$ with $[a-1] \cap F = [a-1] \cap F'$. For F' on the other hand, there are $F_j \in \mathcal{F}_j$ for all $j \in [r] \setminus i$ such that $|[a] \cap F' \cap \bigcap_{j\in[r]\setminus i} F_j| = t$ and $a \in F' \cap \bigcap_{j\in[r]\setminus i} F_j$. But since $F \in \mathcal{F}_i^{\text{add}}$, we know that $a \notin F$ and thus we have $|[a] \cap F \cap \bigcap_{j\in[r]\setminus i} F_j| < t$. This gives us a contradiction since $F \in \mathcal{F}_i$ and $\mathcal{F}_1, \dots, \mathcal{F}_r$ are r-cross t-intersecting with maximal necessary intersection point a. □

Finally, we can "update" the collection of families. If $\mu_r(\mathcal{F}_r(a)) \leq \sum_{i\in[r-1]} \mu_i(\mathcal{F}_i(a))$, we consider the families $\mathcal{H}_i = \mathcal{F}_i^+$ for $i \in [r-1]$ and $\mathcal{H}_r = \mathcal{F}_r^-$. Recall that we have $\mathcal{H}_i \subseteq [n]^{(\leq k_i)}$ for $i \in [r]$ and that they are non-empty by the condition that $\mathcal{F}_i \setminus \mathcal{F}_i(a) \neq \emptyset$ for all $i \in [r]$. By Claim 6.4 these families are r-cross t-intersecting with their maximal necessary intersection point at most $a-1$ and by Claim 6.5, Claim 6.6, and $\mu_r(\mathcal{F}_i(a)) \leq \sum_{i\in[r-1]} \mu_i(\mathcal{F}_i(a))$ we have $\sum_{i\in[r]} \mu_i(\mathcal{F}_i) \leq \sum_{i\in[r-1]} \mu_i(\mathcal{F}_i^+) + \mu_r(\mathcal{F}_r^-)$. Together, this yields (a)-(c) in the conclusion of the lemma.

If $\mu_r(\mathcal{F}_r(a)) \geq \sum_{i\in[r-1]} \mu_i(\mathcal{F}_i(a))$, we consider the families $\mathcal{F}_1^-, \ldots, \mathcal{F}_{r-1}^-, \mathcal{F}_r^+$. Similarly as before, it follows that these will satisfy (a)-(c). □

We are now ready to prove our main theorems.

Theorem 6.7. *Let $r \geq 2$ and $n, t \geq 1$ be integers. Further, for every $i \in [r]$ let $\mu_i : [n]_0 \to \mathbb{R}_{\geq 0}$, let $\widehat{k}_i \in [n]$ and $k_i \in [\widehat{k}_i]_0$ such that μ_i is non-increasing on $[k_i, \widehat{k}_i]$. If $\mathcal{F}_1 \subseteq [n]^{(\leq \widehat{k}_1)}, \ldots, \mathcal{F}_r \subseteq [n]^{(\leq \widehat{k}_r)}$ are non-empty r-cross t-intersecting families and n is at least $\max_{i\in[r]}(k_i + \min_{j\in[r]\setminus i} \widehat{k}_j) - t + 1$, then*

$$\sum_{j\in[r]} \mu_j(\mathcal{F}_j) \leq \max\Big\{\mu_\ell(\mathcal{A}(n,a,t)^{\leq \widehat{k}_\ell}) + \sum_{j\in[r]\setminus\ell} \mu_j(\mathcal{B}(n,a)^{\leq \widehat{k}_j}) \colon \ell \in [r], a \in \big[t, \min_{i\in[r]\setminus\ell} \widehat{k}_i\big]\Big\}.$$

Theorem 6.8. *Let n be an integer, $\widehat{k} \in [n]$, and $k \in [\widehat{k}]_0$, let $\mathcal{F} \subseteq [n]^{(\leq \widehat{k})}$ be an t-intersecting family, and let $\mu : [n]_0 \to \mathbb{R}_{\geq 0}$ such that μ is non-increasing on $[k, \widehat{k}]$. If n is at least $k + \widehat{k} - t + 1$, then*

$$\mu(\mathcal{F}) \leq \mu(\mathcal{B}(n,t)).$$

Both Theorem 6.7 and Theorem 6.8 can be obtained from the following more general result by setting $r_1 = r$ and $r_1 = 0$ respectively. Moreover, this result also provides the maximum of $\sum_{i\in[r]} \mu_i(\mathcal{F}_i)$ in the case when some of the families and measures satisfy the conditions in Theorem 6.7 and the others satisfy those in Theorem 6.8.

Proposition 6.9. *Let $r, t, n \in \mathbb{N}$, $r_1 \in \mathbb{N}_0$ with $r \geq r_1$, $r \geq 2$, and $r_1 \neq 1$, and let $a \in [n]$. If $r_1 \geq 2$, for $i \in [r_1]$ let $k_i \in [n]$ be such that $n \geq 2\max_{i\in[r_1]} k_i + \operatorname{secmin}_{i\in[r_1]} k_i - t$, and let $\mu_i : [n]_0 \to \mathbb{R}_{\geq 0}$. For $i \in [r_1+1, r]$ set $k_i = n$ and let $\mu_i : [n]_0 \to \mathbb{R}_{\geq 0}$ be non-increasing. For $i \in [r]$ let $\mathcal{F}_i \subseteq [n]^{(\leq k_i)}$. If $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are non-empty r-cross t-intersecting families with maximal necessary intersection point at most a, then*

$$\sum_{j\in[r]} \mu_j(\mathcal{F}_j) \leq \max\Big\{\mu_\ell(\mathcal{A}(n,a_*,t)^{\leq k_\ell}) + \sum_{j\in[r]\setminus\ell} \mu_j(\mathcal{B}(n,a_*)^{\leq k_j})\Big\}, \tag{6.3}$$

where the maximum is taken over $\ell \in [r]$ and $a_* \in \Big[t, \min\big\{a, \min_{i\in[r]\setminus\ell} k_i\big\}\Big]$.

Proof. We perform an induction on r. The beginning is the same for the induction start and the induction step. Let all the parameters and μ_i be given as in the statement of the theorem and note that without restriction $t \leq \min_{i\in[r]} k_i$. Further, let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be such that

1. for $i \in [r]$ we have $\mathcal{F}_i \subseteq [n]^{(\leq k_i)}$,

2. they are r-cross t-intersecting with maximal necessary intersection point at most a,
3. they maximise $\sum_{j\in[r]} \mu_j(\mathcal{F}_j)$ among all families satisfying (1) and (2),
4. their maximal necessary intersection point is minimal among those families that fulfil (1), (2), and (3).

Since the properties (1), (2), (3), and (4) are preserved when shifting, we may assume that $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are shifted. Denote the maximal necessary intersection point of $\mathcal{F}_1, \ldots, \mathcal{F}_r$ by a_* and observe that if $a_* = t$, we are done. So we assume that $a_* \geq t+1$.

First, consider the case that for all $i \in [r]$ we have that $\mathcal{F}_i^- \neq \emptyset$. Then Lemma 6.3 yields families $\mathcal{H}_1, \ldots, \mathcal{H}_r$ satisfying (1)-(3) with a maximal necessary intersection point smaller than a_*. This is a contradiction to the choice of the families (see (4)) and thereby completes the proof of both the induction start and the induction step.

Second, consider the case that for some $j \in [r]$, without loss of generality r, it holds that $\mathcal{F}_r \setminus \mathcal{F}_r(a_*) = \emptyset$. That is to say, all sets in $\mathcal{F}_r$ depend on a_*.

Assume that there is a $b \in [a_* - 1]$ and $F \in \mathcal{F}_r$ such that $b \notin F$. As $\mathcal{F}_r$ is shifted, we have that $\sigma_{ba_*}(F) \in \mathcal{F}_r$, but this set does not depend on a_*. Hence, for every $F \in \mathcal{F}_r$ we have $[a_*] \subseteq F$, in other words $\mathcal{F}_r \subseteq \mathcal{B}(n, a_*)^{\leq k_r}$.

For $r = 2$ notice that since a_* is the maximal necessary intersection point, every $F_1 \in \mathcal{F}_1$ has at least t elements in $[a_*]$. This yields $\mathcal{F}_1 \subseteq \mathcal{A}(n, a_*, t)^{\leq k_1}$ and hence

$$\mu_1(\mathcal{F}_1) + \mu_2(\mathcal{F}_2) \leq \mu_1(\mathcal{A}(n, a_*, t)^{\leq k_1}) + \mu_2(\mathcal{B}(n, a_*)^{\leq k_2}),$$

which finishes the proof of the induction start.

For $r \geq 3$ observe that the families $\mathcal{F}_1, \ldots, \mathcal{F}_{r-1}$ are $(r-1)$-cross t-intersecting families with maximal necessary intersection point at most a_* which maximise $\sum_{j\in[r-1]} \mu_j(\mathcal{F}_j)$ (among all $(r-1)$-cross t-intersecting families $\mathcal{G}_i \subseteq [n]^{(\leq k_i)}$ with maximal necessary intersection point at most a_*). Thus, the induction hypothesis implies that there is an $\ell \in [r-1]$ and an $a_{**} \in [a_*]$ such that

$$\sum_{j\in[r-1]} \mu_j(\mathcal{F}_j) \leq \mu_\ell(\mathcal{A}(n, a_{**}, t)^{\leq k_\ell}) + \sum_{j\in[r-1]\setminus\ell} \mu_j(\mathcal{B}(n, a_{**})^{\leq k_j}).$$

Since $\mathcal{F}_r \subseteq \mathcal{B}(n, a_*)^{\leq k_r} \subseteq \mathcal{B}(n, a_{**})^{\leq k_r}$, this entails

$$\sum_{j\in[r]} \mu_j(\mathcal{F}_j) \leq \mu_\ell(\mathcal{A}(n, a_{**}, t)^{\leq k_\ell}) + \sum_{j\in[r]\setminus\ell} \mu_j(\mathcal{B}(n, a_{**})^{\leq k_j}),$$

which finishes the induction step. □

6.3 Concluding remarks

We point out some particularly important special cases in the following corollaries. Firstly, if we apply Theorem 6.7 with the measure $\mu_i = \binom{n}{k}\nu_k$, and $k_i = \widehat{k}_i = k$ for every $i \in [r]$, we obtain the following result.

Corollary 6.10. *Let $r \geq 2$, and $n, t \geq 1$ be integers, $k \in [n]$, and for $i \in [r]$ let $\mathcal{F}_i \subseteq [n]^{(k)}$. If $\mathcal{F}_1, \dots, \mathcal{F}_r$ are non-empty r-cross t-intersecting families and $n > 2k - t$, then*

$$\sum_{j \in [r]} |\mathcal{F}_j| \leq \max_{i \in [t,k]} \left\{ \sum_{m \in [t,k]} \binom{i}{m} \cdot \binom{n-i}{k-m} + (r-1)\binom{n-i}{k-i} \right\}$$

and this bound is attained.

In the context of non-uniform families, one of the results of a very recent work by Frankl and Wong H.W. [52] establishes the maximum possible size of cross t-intersecting families. By taking $\mu_i \equiv 1$, $k_i = 0$, and $\widehat{k}_i = n$ for every $i \in [r]$ in Theorem 6.7, we generalise that result to r-cross t-intersecting families with $r \geq 2$.

Corollary 6.11. *Let $r \geq 2$, $n, t \geq 1$ be integers and let $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathscr{P}([n])$ be non-empty r-cross t-intersecting families. Then,*

$$\sum_{j \in [r]} |\mathcal{F}_j| \leq \max_{i \in [t,n]} \left\{ 2^{n-i} \sum_{m \in [t,i]} \binom{i}{m} + (r-1)2^{n-i} \right\}$$

and this bound is attained.

Observe that the maxima in our results are attained for different i (and ℓ), depending on the measures and r, t, and n. However, we remark the following.

Remark 6.12. For given $t, n, k \in \mathbb{N}$ and a measure μ there is an r_0 such that if $r \geq r_0$, the maximum in Theorem 6.7 and Theorem 6.8 is always attained for $i = t$ if $\mu = \mu_j$ (and $k_j = k$) for all $j \in [r]$.

One can also ask for the maximum of the product of sizes or, more generally, the product of measures of r-cross t-intersecting families, instead of the sum. More precisely, for given measures $\mu_1, \dots, \mu_r$ find the maximum possible value of

$$\prod_{i \in [r]} \mu_i(\mathcal{F}_i) \tag{6.4}$$

for $\mathcal{F}_1, \dots, \mathcal{F}_r$ being r-cross t-intersecting families.

There are some partial results concerning this problem ([49, 14, 73, 65]). Frankl and Tokushige [50] determined the maximum product of the sizes of r-cross 1-intersecting families. In [14], Borg determined the maximum of (6.4) for r-cross 1-intersecting families and measures with certain properties (which include the product measure, the uniform measure, and the constant measure) (see also [15] for a general result). It is well known that for $a_1, \dots, a_r \in \mathbb{R}_{\geq 0}$ with $\sum_{i \in [r]} a_i \leq a$ the product $\prod_{i \in [r]} a_i$ is maximised if $a_i = \frac{a}{r}$ for all $i \in [r]$. Therefore, considering Remark 6.12, given n, measures $\mu_i = \mu$ (and $k_i = k$) with μ (and k and n) satisfying the conditions in Theorem 6.7 or Theorem 6.8, there is an r_0 such that for $r \geq r_0$ these theorems actually also yield that the maximum of (6.4) is $(\mu(\mathcal{B}(n,t)^{\leq k}))^r$. This particularly includes the product measure, the uniform measure, and the constant measure, and solves a few instances of the Problems 12.10 and 12.11, and of the Conjectures 12.12 and 12.13 posed by Frankl and Tokushige in [51].

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