56b | Mai 1959

SCHRIFTENREIHE SCHIFFBAU

Dr. Hellmut H.E. Baumann

Roll Resonance in a Transverse Swell

TUHH
Technische Universität Hamburg-Harburg

INSTITUT FÜR SCHIFFBAU DER UNIVERSITÖT HAMBURG

Prof. Dr.-Ing. G. Weinblum

Roll Resonance in a Transverse Swell

by

Dr. Hellmut H. E. Baumann

Hamburg, May 1959

Roll Resonance in a Transverse Swell

I n d e x

Nomenclature

a) General Symbols.

t	time
$\dot{\psi}(t) = \frac{d}{dt}\psi(t)$	a dot denotes the derivative with respect to time
$\bar{\psi}$	a bar denotes the amplitude of an oscillogram
K, D	complete elliptic integrals
k^2	their argument

b) Symbols of the Swell.

λ, H	length and height of waves
T_W	period of waves
$\mathfrak{r}(t), r$	orbital vector, orbital radius
ω	orbital angular velocity
$\mathfrak{g}$	gravitational acceleration vector
$g \approx$	981 cm sec^{-2}, its length
$\mathfrak{g}'(t) \equiv \mathfrak{g} - \omega^2 \mathfrak{r}(t)$	resultant acceleration vector
$\vartheta(t)$	angle between $\mathfrak{g}'(t)$ and $\mathfrak{g}$
$\beta(t) \equiv \frac{1}{g} \cdot \lvert \mathfrak{g}'(t) \rvert - 1$	= dimensionless acceleration normal to wave surface.
$\vartheta_a(t), \vartheta_r(t); \beta_a(t), \beta_r(t)$	active or reactive phase components with respect to ship's motion
β_0	a constant making the mean normal acceleration vanish

c) Symbols of the Ship.

$\mathfrak{m}(t)$	unit vector in the direction of ship's mast
$\varphi(t)$	ship's roll angle, measured between $\mathfrak{m}(t)$ and $\mathfrak{g}$
$\psi(t)$	ship's roll angle, measured between $\mathfrak{m}(t)$ and $\mathfrak{g}'(t)$
$k(\psi), \mathcal{U}(\psi)$	righting arm or potential energy of heel
ψ_R	end of stability range
T_S	ship's roll period
$T_0(\bar{\psi}), \nu_0(\bar{\psi})$	period or circular frequency of free roll in calm water, the plot is called "Skeleton curve

J'	ship's moment of inertia including virtual masses of water
$W(T_s)$	ship's damping coefficient
$D(T_s), D(T_w), D(\omega)$	dimensionsless damping coefficient
P	ship's weight
B	breadth of ship
$\overline{M_0 G}$	metacentric height

d) Further Symbols.

$\tau(\bar{\psi}),\ \phi(\bar{\psi})$	dimensionless period or phase integral of free roll free roll
$\eta_0(\bar{\psi}), \eta_2(\bar{\psi}), \eta_\beta(\bar{\psi})$	dimensionless mean kinetic energy
$F(\bar{\psi})$	form factor of roll oscillogram
$\alpha(t)$	phase of roll oscillogram
$w(\psi\ \bar{\psi})$	its reciprocal phase velocity of it
$g(\bar{\psi}) = w(\bar{\psi},\bar{\psi})/w(0,\bar{\psi})$	ratio of non-uniformity of phase velocity
$\gamma(t),\ \gamma_0$	phase of energy ocillogram or phase lag
$u(\psi)$	dimensionless potential heel energy
$N_1(t),\ N_2(t)$	moments neither active nor reactive to the ship's motion
C	a constant denoting the stability of roll states.

Figures

1) Explanation of terms used in the equation of motion of a ship in a transverse swell.

r = orbital radius,

ω = orbital angular velocity,

g = gravitational vector,

g' = apparent gravitational vector.

2) The amplitude function $q(\bar{\psi})$

3a) The quarter oscillograms and periods of a ship with a sinusoidal righting-arm curve rolling at various amplitudes in calm water.

3b) The quarter oscillograms and periods of a ship with an additional stability by form in calm water.

4) The planimetered phase integrals $\phi(\bar{\psi})$ of the roll oscillations of figs. 3a and 3b. The points were calculated according to (34).

5) The amplitude function $\eta_o(\bar{\psi})$ of the roll oscillations of figs. 3a and 3b. The points were calculated according to (43).

6) The form factors $F(\bar{\psi})$ of the oscillograms shown in 3a and 3b.

7) Damping moments measured for towed ship models. From Weinblum and St. Denis [9].

8) The increase of damping with roll amplitude. From Grim [10].

9) The construction of two points P_1 and P_2 of the response curves for roll states of the first kind from the skeleton curve and the damping, Construction 1 yields the blind excitation $\pm|\bar{\mathcal{F}}_r|$ from the amplitude $\bar{\psi}$ and the damping D. The corresponding frequencies are obtained from construction 2.

10a) Response curves of the roll states of the first kind for ships with righting-arm curves as shown at three constant values of the blind excitation. In the shaded region the roll states are unstable.

10b) Response curves of the roll states of the first kind for ships with righting-arm curves as shown at three constant values of the blind excitation. In the shaded regions the roll states are unstable.

11a) Response curves of the roll states of the first kind for ships with righting-arm curves as shown and damping D = 0,2 at three values of the wave steepness. In the shaded region the roll states are unstable.

11b) Response curves of the roll states of the first kind for ships with righting-arm curves as shown and damping D - 0,2 at three values of the wave steepness. In the shaded region the roll states are unstable.

12) The construction of two points P_1 and P_2 on the limiting stability curve L for roll states of the first kind from the skeleton curve and the damping.

13) Interpretation of the transformation (120).

14a) Response curves of the roll states of the second kind for ships with righting-arm curves as shown at the blind excitation $|\overline{\beta}_r| = 0,2$. The broken regions of the curve represent unstable roll states. $\overline{\psi}_{max}$ is the largest roll amplitude occuring with certainty the probability of larger amplitudes occuring is small.

14b) Response curves of the roll states of the second kind for ships with righting-arm curves as shown at the blind excitation $|\overline{\beta}_r| = 0,2$. The broken regions of the curve represent unstable roll states. $\overline{\psi}_{max}$ is the largest roll amplitude occuring with certainty, the probability of larger amplitudes occuring is small.

15) Illustrating the instable roll states of the second kind at small amplitudes.

16) Phase relation of the ship's motion in a swell of strength just exceeding the critical value for roll states of the second kind.

17) Determination of the convergence interval (125)

Footnotes

[1] Physicist, Institut für Schiffbau der Universität Hamburg, (Director Professor Dr.-Ing. G. Weinblum), Germany. - The investigations were supported by the Office of Naval Research, Washington, D.C. The treatise in German was translated by Dr. K. Hasselmann. Copies of the author's manuscript are available at the Institut für Schiffbau.

[2] Numbers in square brackets refer to the bibliography at the end of this paper.

[3] For instance, the periods of free roll can thus be approximated with an error less than 1% at amplitudes $\bar{\psi} < 0{,}7\,\psi_R$ for the following righting-armcurves:

$$h^{I}(\psi) = h_{max} \cdot \sin\frac{\pi\psi}{\psi_R}, \qquad h^{II}(\psi) = \overline{M_oG}\cdot\psi\cdot\left[1-\left(\frac{\psi}{\psi_R}\right)^2\right],$$
$$h^{III}(\psi) = \overline{M_oG}\cdot\psi\cdot\left[1-\left(\frac{\psi}{\psi_R}\right)^4\right] \text{ among others.}$$

[4] Equations Nr. 36 - 40 have been cancelled in the translated version.

[5] In case of multiple resonance (T_s/T_w = 3, 4, 5 etc.) the roll numbers satisfying condition (2) would be even greater (R = 15, 20, 25, etc.).

[6] Although the simultaneous use of the symbols $D(T_w)$ and $D(\omega)$ is not correct; there is no fear of ambiguity as the formulae will be used alternatively.

[7] $D = \frac{1}{k^2} \cdot (K - E)$, not to be confused with the damping function $D(T_s)$!

Bibliography

[1] "An Experimental Method of Ascertaining the Rolling of Ships in Waves" by G. Russo, Trans. Inst. Naval Arch., Vol. 46, 1904 pp. 30 - 52.

[2] "Die Bewegungsgleichungen des Schiffes im Seegang" by G. Weinblum, Schiffbau, Vol. 32, 1931, pp. 488 - 495, 509 - 511, and 525 - 529.

[3] "Über den Einfluss der Schiffsform auf die Bewegungen eines Schiffes im Seegang" by G. Weinblum, Werft, Reederei, Hafen Vol. 14, 1933, pp. 269 - 275 and 289 - 296.

[4] "Beitrag zur Dynamik der Schiffsbewegungen im Seegang", by W. Späth, Werft, Reederei, Hafen, Vol. 14 1933, pp. 104 - 106.

[5] "Stabilitätsgrenzen für Schiffe" by G. Kempf, Schiffbau, Vol. 41, 1940, pp. 17 - 18.

[6] "Rollschwingungen und Hebelarmkurve" by K. Wendel, Schiffbau Vol. 41, 1940, pp. 45 - 51.

[7] "Das Navipendulum als Gerät für Modell-Schlingerversuche", by H. Baumann, Werft, Reederei, Hafen, Vol. 22, 1941, pp. 103 - 104.

[8] "A Mathieu Equation for Ships Rolling among Waves I and II" by G. Vedeler, Det Kongelige Norske Videnskabers Selskabs Forhandlinger, Vol. 22, 1949, pp. 113-123

[9] "On the Motions of Ships at Sea" by G. Weinblum and M.St. Denis, Annual Meeting SNAME 1950.

[10] "Rollschwingungen, Stabilität und Sicherheit im Seegang", by O. Grim, Schiffstechnik, Vol. 1,1952 pp. 10 - 21.

[11] "Notes on Rolling in Longitudinal Waves" by F. E. Kervin Intern. Shipbuilding Progress, Vol. 2, 1955, pp. 597-614.

[12] "AComprehensive Stability Criterien for Forced Vibrations in Nonlinear Systems" by K. Klotter and E. Pinney, Journal of Applied Mechanics, March 1953, pp. 9 - 12.

[13] "Nonlinear Vibrations in Mechanical and Electrical Systems" by J. J. Stoker. Pure and Applied Mathematics, Vol. II, Interscience Publishers, New York 1950.

[14] "Nichtlineare Mechanik" von Hans Kauderer, Springer-verlag, Berlin, Göttingen, Heidelberg 1958

Roll Resonance in a Transverse Swell

By Dr. Hellmut H. E. Baumann [1)]

1. Introduction

It is known that the roll periods of dangerous ship motions differ only slightly from the corresponding natural periods at the same amplitude. In these resonance conditions the damping dissipation is balanced by the energy periodically imparted to the ship by the waves. The period of the oncoming waves need hereby not necessarily coincide with the roll period of the ship. A ship in a seaway is subjected to both vertical and horizontal alternating accelerations, and in a transverse swell we can distinguish between two types of possible roll motions corresponding to the different mechanisms of energy transfer from these two components.

The roll motion generated by the alternating horizontal acceleration component, i.e. the periodic variation of the static equilibrium position, we shall term in the following roll states of the first kind. In this case, apart from "multiple resonances", the roll period of the ship is equal to the period of the oncoming waves. The possibility of the occurence of multiple resonances, i.e. resonance between the swell and a higher harmonic of the roll oscillogram (which for larger amplitudes is no longer sinusoidal) will not be considered nearer here. For the case of equal roll and wave periods, we shall determine the response curves and their regions of stability for all roll amplitudes, assuming a damping moment on the frequency and proportional to the angular velocity. A simple method of contructing the curves graphically will also be given.

We shall term the (hitherto rather neglected) motions generated by the periodic vertical acceleration, which is equivalent to a periodic variation of the ship's weight, [8][2] roll states of the second kind. Disregarding multiple resonances, the roll period in this case is twice the wave period, i.e. the roll motion is brought about by relatively short waves. We shall show that:

1) roll states of the second kind only occur if the swell amplitude exceeds a certain critical value depending on the damping;
2) above this critical value, roll motions will occur within a certain critical period interval;
3) the roll amplitude of the roll oscillations can attain dangerously large values.

The response curves and the margin of stability will be calculated for this case also.

Roll states of the second kind can also occur in a longitudinal swell from fore or aft [11]. As the waves run along the ship they cause a periodic change in the ship's stability, giving rise to the same parameter excitation in the equation of motion as a variation in the ship's weight. Our results for roll states of the second kind are thus applicable mutatis mutandis to this case also.

Our method of treating the essentially non-linear roll equations for a ship is based on physical considerations. For instance, we shall not be concerned with the exact shape of the waves but will rather characterize them generally by their amplitude and period, prescribing the wave shape otherwise in order to achieve mathematical simplification. This leads to wave profiles which for larger amplitudes are no longer sinusoidal or, say, trochoidal, for which, however, exact solutions of the equations of motion can be obtained. Our approach enables the roll states of the first and second kind to be reduced to the free, undamped oscillation of a ship in calm water. For the corresponding fundamental functions - the period and the phase integral - good approximate formulas are derived. By way of example the general theoretical calculations are applied to two ships with different righting-arm curves.

2. The Equation of Motion

The roll motion of a ship in a transverse swell is described by the differential equation

$$J' \cdot \ddot{\varphi} + W(T_s) \cdot \dot{\varphi} + [1+\beta(t)] \cdot P \cdot h(\varphi) = 0 \tag{1}$$

The following comments are necessary regarding the derivation and the notation employed. (See fig. 1): The wave length λ of the swell is assumed large in comparison to the ship's breadth B, at least.

$$\lambda > 4B. \tag{2}$$

In this case the ship participates in the orbital motion of the water and rolls with constant displacement. Apart from the gravitational force, a centrifugal force of magnitude $r\omega^2$ then acts on the ship as a result of its orbital motion (r =orbital radius, ω = orbital angular velocity). The resulting vector g' (the apparent vertical) varies periodically in magnitude and direction. The static equilibrium position of the ship thus has a variable angle relative to the horizontal. In a trochoidal wave the amplitude $\bar{\vartheta}$ of $\vartheta(t)$, the effective wave steepness, is given by

$$\sin \bar{\vartheta} = \frac{r\omega^2}{g} = \frac{2\pi r}{\lambda}. \tag{3}$$

If condition (2) is satisfied, the righting arm $h(\varphi)$ corresponding to an angle of heel φ relative to the position of static equilibrium is the same as in calm water. Because of the variation of the apparent vertical, the ship's weight has the variable value

$$P'(t) = [1 + \beta(t)] \cdot P \tag{4}$$

$\beta(t)$ is positive in the wave trough and negative on the crest.
In a sinusoidal swell

$$\dot{\vartheta}(t) = \pm\,\omega \cdot \beta(t) \tag{5a}$$

(the sign depends on the direction of propogation) and

$$\dot{\vartheta}^2(t) + \beta^2(t) = \frac{4\pi^2 r^2}{\lambda^2} \tag{5b}$$

so that for a standard wave ($2\ r/\lambda = 1/20$):

$$|\bar{\vartheta}| = |\bar{\beta}| = \frac{\pi}{20} \approx 0{,}157. \tag{6}$$

The angle of roll relative to the horizontal is

$$\varphi(t) = \psi(t) + \vartheta(t). \tag{7}$$

We shall assume that the total moment of inertia, J', of of the ship is constant. The hydrodynamic component can be taken as independent of frequency if the wave generated by the ship's motion satisfies condition (2). The damping is as yet not sufficiently well known to be described accurately by an analytical expression. It depends in a hydrodynamically complicated manner on the geometry of the ship, the position of the centre of gravity, the roll period and the momentary angle and angular velocity of roll, among other factors. In the following we shall assume it to be proportional to the angular velocity $\dot{\psi}$ relative to the water surface and take into account otherwise only the dependency on the roll period T_s.

3. The Fundamental Functions for Undamped Roll in Calm Water.

As a basis for the forced roll oscillations to be analyzed later we consider first the undamped roll oscillation in calm water. Setting

$$W = 0 \qquad \text{and} \qquad r = 0 \tag{8}$$

and writing $\psi_0(t)$ for the functions $\varphi(t)$ and $\psi(t)$ which are indentical in this case, the equation of motion becomes

$$J' \cdot \ddot{\psi}_0 + P \cdot h(\psi_0) = 0 \tag{9}$$

As the damping is zero, the sum of the kinetic and potential energies is constant:

$$\tfrac{1}{2} J' \cdot \dot{\psi}_0^2 + \mathcal{U}(\psi_0) = \mathcal{U}(\bar{\psi}) \tag{10}$$

where

$$\mathcal{U}(\psi) = P \cdot \int^{\psi} h(\psi) \cdot d\psi, \qquad \mathcal{U}(-\psi) = \mathcal{U}(+\psi), \qquad \mathcal{U}(0) = 0 \tag{11}$$

We now introduce three fundamental functions of the amplitude which will later also play an important role in the analysis of the forced oscillations in a swell:

a) the roll period,

b) the phase integral and, resulting from this,

c) the mean value of the kinetic energy.

a) From (10) the time differential of the free oscillation
$\psi_c(t)$

$$dt(\varphi_0) = \sqrt{\tfrac{1}{2}J'} \cdot \frac{d\varphi}{\sqrt{\mathcal{U}(\bar{\varphi}) - \mathcal{U}(\varphi)}} \tag{12}$$

giving the improper integral

$$T_0(\bar{\varphi}) = \sqrt{\tfrac{1}{2}J'} \cdot 4 \cdot \int_0^{\bar{\varphi}} \frac{d\varphi}{\sqrt{\mathcal{U}(\bar{\varphi}) - \mathcal{U}(\varphi)}} \tag{13}$$

for the roll period in calm water.

The integral (15) which is of fundamental importance for the dynamical behavior of the ship, can be expressed in a closed analytical form only in a few special cases. For instance, if the righting-arm curve is sinusoidal or a cubic parabols, it is well known that the amplitude function

$$\tau(\bar{\varphi}) = \frac{T_0(\bar{\varphi})}{T_0(0)} \tag{14}$$

where

$$T_0(0) = \sqrt{\frac{J'}{P \cdot \overline{M_0 G}}} \tag{15}$$

a) From (10) the time differential of the free oscillation $\psi_0(t)$ - -

$$d\,t(\psi_0) = \sqrt{\tfrac{1}{2}J'} \cdot \frac{d\psi}{\sqrt{\mathcal{U}(\bar{\psi}) - \mathcal{U}(\psi)}} \tag{12}$$

giving the improper integral

$$T_0(\bar{\psi}) = \sqrt{\tfrac{1}{2}J'} \cdot 4 \cdot \int_0^{\bar{\psi}} \frac{d\psi}{\sqrt{\mathcal{U}(\bar{\psi}) - \mathcal{U}(\bar{\psi})}} \tag{13}$$

for the roll period in calm water.

The integral (13) which is of fundamental importance for the dynamical behavior of the ship, can be expressed in a closed analytical form only in a few special cases. For instance, if the righting-arm curve is sinusoidal or a cubic parabola, it is well known that the amplitude function

$$\tau(\bar{\psi}) = \frac{T_0(\bar{\psi})}{T_0(0)} \tag{14}$$

where

$$T_0(0) = \sqrt{\frac{J'}{P \cdot \bar{M_0 G}}} \tag{15}$$

is the roll period for small amplitudes, can be expressed in terms of the complete elliptic integral

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)\cdot(1-k^2x^2)}} = \frac{\pi}{2}\cdot\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+\frac{25}{264}k^6+\dots\right). \tag{16}$$

For

$$h^I(\psi) = h_{max}\cdot \sin\frac{\pi\cdot\psi}{\psi_R} \tag{17}$$

$$\tau^I(\bar{\psi}) = \frac{2}{\pi}K, \quad k^2 = \sin^2\frac{\pi\cdot\bar{\psi}}{2\,\psi_R} \tag{18}$$

and for $$h^{II}(\psi) = \overline{M_oG}\cdot\psi\cdot\left[1-\left(\frac{\psi}{\psi_R}\right)^2\right] \tag{19}$$

$$\tau^{II}(\bar{\psi}) = \sqrt{1+k^2}\cdot\frac{2}{\pi}K, \text{ with } k^2 = \frac{\bar{\psi}^2}{2\,\psi_R^2-\psi^2} \tag{20}$$

The numerical or planimetrical evaluation of (13) generally meets with some difficulty as the integrand is singular at $\psi = \bar{\psi}$. The difficulty can be avoided by transforming to the parameter α :

$$\psi_0(t) = \bar{\psi} \cdot \sin \alpha(t) \tag{21}$$

representing the oscillation is then represented by a point moving non-uniformly but periodically round a circle at the angular velocity

$$\dot{\alpha} = + \sqrt{\frac{2}{J'} \cdot \frac{\mathcal{U}(\bar{\psi}) - \mathcal{U}(\psi_0)}{\bar{\psi}^2 - \psi_0^2}} \tag{22}$$

Equation (13) then transforms to the following equation for the amplitude function (14):

$$\tau(\bar{\psi}) = \frac{2}{\pi} \cdot \int_0^{\pi/2} \sqrt{\frac{\frac{1}{2} P \cdot \overline{M_0 G} \cdot (\bar{\psi}^2 - \psi_0^2)}{\mathcal{U}(\bar{\psi}) - \mathcal{U}(\psi_0)}} \cdot d\alpha \tag{24}$$

where expression (15) has been substituted for the roll period at small amplitudes. The integrand

$$W(\psi_0, \bar{\psi}) \equiv \sqrt{\frac{\frac{1}{2} P \cdot \overline{M_0 G} \cdot (\bar{\psi}^2 - \bar{\psi}_0^2)}{\mathcal{U}(\bar{\psi}) - \mathcal{U}(\psi_0)}} \tag{25}$$

remains finite within the entire range of integration. The values at the lower upper limits of integration are

$$w(0,\bar{\psi}) = \sqrt{\frac{\frac{1}{2}P\cdot \overline{M_0G}\cdot \bar{\psi}^2}{\mathcal{U}(\bar{\psi})}} \quad \text{and} \quad w(\bar{\psi},\bar{\psi}) = \sqrt{\frac{\overline{M_0G}\cdot \bar{\psi}}{h(\bar{\psi})}} \tag{26}$$

respectively.
As the differential coefficient quotient

$$\frac{dw}{d\alpha} = \frac{dw}{d\psi_0} \cdot \frac{d\psi_0}{d\alpha} \tag{27}$$

vanishes at the points $\psi_0 = 0$ and $\alpha = \pi/2$ the values are extrema. If the curvature of the righting-arm curve is of constant sign, the function w is monotonic in the range of integration and the values (26) are absolute extrema. The ratio

$$q(\bar{\psi}) \equiv \frac{w(\bar{\psi},\bar{\psi})}{w(0,\bar{\psi})} = \sqrt{\frac{2\,\mathcal{U}(\bar{\psi})}{\bar{\psi}\cdot P\cdot h(\bar{\psi})}} \tag{28}$$

is shown in fig. 2 for a sinusoidal righting-arm curve and a further empirical curve representing a ship with initially increasing stiffness. $q(\bar{\psi})$ is seen to be within 20% of unity for a considerable range of roll amplitudes. The integrand (25) is thus limited to a relatively small interval for moderate amplitudes, so that the integral (24) can be easily evaluated numerically or by planimetration. For righting-arm curves with curvatures of constant sign the arithmetic mean of the two extremes generally gives a satisfactory approximation for moderate amplitudes.[3)]

The roll period in calm water can thus be approximated by the algebraic expression

(29)

$$\tau(\bar{\psi}) \approx \frac{1}{2} \cdot \left(\sqrt{\frac{\frac{1}{2} P \cdot \overline{M_0 G} \cdot \bar{\psi}^2}{U(\bar{\psi})}} + \sqrt{\frac{\overline{M_0 G} \cdot \bar{\psi}}{h(\bar{\psi})}} \right)$$

i.e. by means of the ratio of the area under the initial tangent of the righting-arm curve to the area under the curve itself (both taken between the angles 0 and $\bar{\psi}$) and by the ratio of the ordinates of the tangent and the curve at the value $\bar{\psi}$.

The free-roll oscillograms at various amplitudes are shown in fig. 3a for the ship with a sinusoidal righting-arm curve and in fig. 3b for the ship with initially increasing stiffness. The dependence of the natural period on the roll amplitude is given by the curve passing through the maxima.

b) The normed phase integral

(30)

$$\phi(\bar{\psi}) \equiv \frac{1}{\pi \bar{\psi}} \cdot \sqrt{\frac{J'}{2 U(\bar{\psi})}} \cdot \oint \dot{\psi}_0 \cdot d\psi_0$$

which is equal to unity for a linear righting-arm curve, can easily be evaluated by planimetration. The $\phi(\bar{\psi})$-curves corresponding to the ship types of fig. 3a and 3b are shown in fig. 4. It is also possible to approximate the phase integral by an algebraic expression. With the substitution (21), integral (30) transforms to

$$\phi(\bar{\psi}) = \sqrt{\frac{J'}{2\,\mathcal{U}(\bar{\psi})}} \cdot \frac{\bar{\psi}}{\pi} \cdot \int_0^{2\pi} \cos^2\alpha \cdot \dot{\alpha} \cdot d\alpha \qquad (31)$$

where α is given by (22).
The extrema of $\dot{\alpha}(\alpha)$ at the values $\alpha = 0, \pi/2, \pi$ etc. are thus

$$\left.\begin{aligned} \dot{\alpha}(0) = \dot{\alpha}(\pi) &= \dot{\alpha}(2\pi) = \sqrt{\frac{2\,\mathcal{U}(\bar{\psi})}{J' \cdot \bar{\psi}^2}} \\ \dot{\alpha}\left(\frac{\pi}{2}\right) = \dot{\alpha}\left(\frac{3\pi}{2}\right) &= \frac{1}{q(\bar{\psi})} \cdot \sqrt{\frac{2\,\mathcal{U}(\bar{\psi})}{J' \cdot \bar{\psi}^2}} \end{aligned}\right\} \qquad (32)$$

with $q(\bar{\psi})$ as in (28).
For righting-arm curves with curvatures of constant sign, $\dot{\alpha}(\alpha)$ is monotonic and can be approximated by

$$\dot{\alpha}(\alpha) \approx \sqrt{\frac{2\,\mathcal{U}(\psi)}{J' \cdot \bar{\psi}^2}} \cdot \frac{1}{2}\left[\left(1 + \frac{1}{q(\bar{\psi})}\right) + \left(1 - \frac{1}{q(\bar{\psi})}\right) \cdot \cos 2\alpha\right] \qquad (33)$$

From (31) we thus obtain the algebraic expression

$$\begin{aligned} \phi(\bar{\psi}) &\approx \frac{1}{2}\cdot\left(1+\frac{1}{q(\bar{\psi})}\right)+\frac{1}{4}\cdot\left(1-\frac{1}{q(\bar{\psi})}\right) \\ &\approx \frac{3}{4}+\frac{1}{4q(\bar{\psi})} \end{aligned} \tag{34}$$

which is plotted in fig. 4.

c) We consider finally the normed mean value of the kinetic energy

$$\eta_0(\bar{\psi}) \equiv \frac{2}{T_0(\bar{\psi})}\cdot\int_0^{T_0(\bar{\psi})}\left(1-\frac{U(\psi_0)}{U(\bar{\psi})}\right)\cdot dt \tag{35}$$

which will be used frequently in the following. It depends in general on the amplitude and the righting-arm curve. The expression is normed to equal unity for restoring moments proportional to the roll angle. For ships with increasing stiffness, $\eta_0(\bar{\psi})$ is greater than one, as lower velocities occur for shorter times. For decreasing stiffness, $\eta_0(\bar{\psi})$ is smaller than one and approaches zero as the amplitude approaches the stability limit. The upper limit for $\eta_0(\bar{\psi})$ is two, corresponding to a body which can move freely within a certain interval and is abruptly reflected at the ends. This case is, of course, not realizable in practice for ships, and we can take as the upper limit approximately $\eta_{max} = 1{,}5$.

Thus

$$0 \leq \eta_0(\bar{\psi}) \leq \eta_{max} = \begin{cases} 1{,}0 \text{ for decreasing stiffness} \\ 1{,}5 \text{ for increasing stiffness} \end{cases} \qquad (41)^{4)}$$

The $\eta_0(\bar{\psi})$-curves for the two ships described above are shown in fig. 5. The curves were obtained from the planimetered values of $\phi(\bar{\psi})$ and $\tau(\bar{\psi})$ as in fig. 3a and 3b using the identy

$$\tau(\bar{\psi}) \cdot \eta_0(\bar{\psi}) \equiv \phi(\bar{\psi}) \cdot \sqrt{\frac{\frac{1}{2} P \cdot \overline{M_0G} \cdot \bar{\psi}^2}{\mathcal{U}(\bar{\psi})}} \qquad (42)$$

which can easily be deduced from the energy equation (10). For ships with monotonically increasing or decreasing stiffness we can use the approximations (34) and (29) for $\phi(\bar{\psi})$ and $\tau(\bar{\psi})$ respectively obtaining

$$\eta_0(\bar{\psi}) \approx \frac{1}{1+q(\bar{\psi})} + \frac{1}{2q(\bar{\psi})} \,. \qquad (43)$$

The approximate values calculated from this expression are plotted in fig. 5.

4. The Steady Roll States in a Transverse Swell.

We confine our considerations to the steady roll states of period T_s, for which

$$\psi(t + T_s) \equiv \psi(t) \tag{44}$$

Multiplying the equation of motion (1) with $\dot{\psi}(t)$ and $\psi(t)$ respectively and integrating over the roll period we obtain, using (7)

$$\int_0^{T_s} J' \cdot {}_{2}\ddot{\varphi} \cdot \dot{\psi} \cdot dt + \int_0^{T_s} W(T_s) \cdot \dot{\psi}^2 \cdot dt + \int_0^{T_s} \beta \cdot P \cdot h(\psi) \cdot \dot{\psi} \cdot dt = 0 \tag{45}$$

$$\int_0^{T_s} J' \cdot {}_{2}\ddot{\varphi} \cdot \psi \cdot dt + \int_0^{T_s} \left[J' \cdot \ddot{\psi} + P \cdot h(\psi) \right] \cdot \psi \cdot dt + \int_0^{T_s} \beta \cdot P \cdot h(\psi) \cdot \psi \cdot dt = 0 \tag{46}$$

From these equations certain general time-relations between the excitation and the oscillation can be deduced.

a) Relations between wave and ship periods.

The second integral in equation (45) represents the energy dissipated by the damping; it is always positive. The remaining integrals in the equation must therefore be negative, singly or in the sum, to yield the necessary energy supply. The second integral in equation (46) vanishes if $\varphi(t)$ satisfies equation (9) describing the free undamped oscillation in calm water or, in other words, if the ship rolls with its natural period. This integral is thus a measure of the detuning brought about by the remaining two integrals, which cause the ship to roll with a different period T_s. If we now let T_w be the period of the swell and $\frac{1}{n} \cdot T_s$ the period of a higher harmonic of the (generally non-sinusoidal) oscillogram, the first integrals in equation (45) and (46) will then be unequal to zero only if

$$\frac{1}{n} \cdot T_s = T_W, \quad n \text{ integer.} \tag{47}$$

On the other hand, the third integrals in (45) and (46) will be different from zero only if

$$\frac{1}{2n'} \cdot T_s = T_W, \quad n' \text{ integer.} \tag{48}$$

The conditions (47) and (48) can be satisfied by an infinite number of values n and n'. In the following, however, we shall confine our attention to the two cases n - 1 or n' = 1, disregarding the possibility of multiple resonances. This gives the mutually exclusive conditions

$$T_s = T_W \quad \text{or} \quad T_s = 2T_W. \qquad (49) \quad (50)$$

b) <u>Phase decomposition.</u>

From equations (45) and (46) we can also determine the relative phases of the ship and wave motions. The multipliers $\dot{\psi}(t)$ and $\psi(t)$ are 90° out of phase, as

$$\int_0^{T_s} \psi \cdot \dot{\psi} \cdot dt = 0.$$

The exciting moment can similarly be decomposed into two terms which are 90° out of phase:

$$\vartheta(t) = \vartheta_a(t) + \vartheta_r(t) ; \qquad \beta(t) = \beta_a(t) + \beta_r(t) \tag{51}$$

We define as the active excitation the components contributing solely to the energy supply integrals in (45). For these components then

$$\int_0^{T_s} \ddot{\vartheta}_a \cdot \psi \cdot dt = 0 ; \qquad \int_0^{T_s} \beta_a \cdot \psi \cdot dt = 0. \tag{52}$$

Conversely, the components contributing solely to the detuning integrals in (46) will be termed the reactive excitation. For these

$$\int_0^{T_s} \ddot{\vartheta}_r \cdot \dot{\psi} \cdot dt = 0 ; \qquad \int_0^{T_s} \beta_r \cdot \dot{\psi} \cdot dt = 0. \tag{53}$$

With condition (49) and (46) simplify to

$$\int_0^{T_s} \left[J' \cdot \ddot{\vartheta}_a + W(T_s) \cdot \dot{\psi}_1 \right] \cdot \dot{\psi}_1 \cdot dt = 0, \tag{54}$$

$$\int_0^{T_s} \left[J' \cdot \ddot{\vartheta}_r + J' \cdot \ddot{\psi}_1 + P \cdot h(\psi_1) \right] \cdot \psi_1 \, dt = 0. \tag{55}$$

The condition (50), on the other hand, leads to

$$\int_0^{T_s} \left[W(T_s) \cdot \dot{\psi}_2 + \beta_a \cdot P \cdot h(\psi_2) \right] \cdot \dot{\psi}_2 \cdot dt = 0, \tag{56}$$

$$\int_0^{T_s} \left[J' \cdot \ddot{\psi}_2 + (1+\beta_r) \cdot P \cdot h(\psi_2) \right] \cdot \psi_2 \cdot dt = 0. \tag{57}$$

c) Two kinds of roll states.

In a transverse swell we can thus distinguish between two kinds of roll motions, $\varphi_1(t)$ and $\varphi_2(t)$.

The roll states of the first kind are generated by the periodic horizontal acceleration $b_h(t) = \vartheta(t) \cdot g$ accompanying the swell; they have the same period as the swell. - The roll states of the second kind are generated by the periodic vertical acceleration $b_v(t) = \beta(t) \cdot g$ due to the swell and have a period equal to twice the swell period. In the first case the condition (2) is well satisfied. According to Kempf [5] the roll number

$$R = T_s \cdot \sqrt{\frac{g}{B}} \tag{58}$$

in general
is not less than eight, so that for roll states of the first kind

$$\lambda > 10B. \tag{59}$$

For roll states of the second kind, condition (2) requires

$$R > 10. \tag{60}$$

We shall therefore have to exclude very stiff ships from our considerations of this case.[5)]

5. The Roll States of the First Kind, $T_S = T_W$.

For roll states of the first kind, the periodic vertical acceleration $\beta(t)\cdot g$ yields no contribution to equations (45) and (46) and is thus only of secondary physical importance. We shall therefore set $\beta(t) \equiv 0$ for the first and investigate the influence of the vertical acceleration afterwards. The equation of motion now becomes

$$J'\cdot\left(\ddot{\vartheta}+\ddot{\psi}_1\right)+W(T_s)\cdot\dot{\psi}_1+P\cdot h(\psi_1)=0 \tag{61}$$

which by phase decomposition according to (51) reduces the two equations

$$J'\cdot\ddot{\vartheta}_a + W(T_s)\cdot\dot{\psi}_1 = +N_1(t) \tag{62}$$

$$J'\cdot\left(\ddot{\vartheta}_r+\ddot{\psi}_1\right)+P\cdot h(\psi_1) = -N_1(t) \tag{63}$$

each of which contains only components of the same phase. The moment $N_1(t)$ contributes to neither the excitation nor the detuning of the ship, as from (54) and (55)

$$\int_0^{T_s} N_1\cdot\dot{\psi}_1\cdot dt = 0 \quad \text{and} \quad \int_0^{T_s} N_1\cdot\psi_1\cdot dt = 0; \qquad (64) \quad (65)$$

$N_1(t)$ is a combination of higher harmonics which can still be disposed of within certain limits.

We now consider the swell to be characterized by its period T_w and steepness $\bar{\alpha}$. We shall not take the exact shape of the wave profile into account but rather prescribe the wave-slope function $\alpha(t)$ in such a manner that $N_1(t)$ vanishes, thus enabling easy integration of equations (62) and (63).

a) The roll amplitudes.
If we integrade (62) twice under the side condition

$$\int_0^{T_w} \alpha_a(t) \cdot dt = 0 \tag{66}$$

we obtain the active excitation

$$\alpha_a(t) = -\frac{W(T_s)}{J'} \cdot \int^t \psi_1(t) \cdot dt. \tag{67}$$

Introducing the form factor

$$F(\bar{\psi}) = \frac{2\pi}{\bar{\psi} \cdot T_s} \cdot \int_0^{\frac{1}{4}T_s} \psi_1 \cdot dt \tag{68}$$

[$\psi_1(0) = 0$; $\psi_1(\frac{1}{4}T_s) = \bar{\psi}$] (F (Fis plotted in fig. 6 over the amplitude for the oscillograms given in figs. 3a and 3b)

The amplitudes of the active excitation becomes

$$\nu_a = F(\bar{\psi}) \cdot D(T_s) \cdot \bar{\psi} \tag{70}$$

where

$$D(T_s) = \frac{T_s \cdot W(T_s)}{2\pi\, J'} . \tag{71}$$

As can be seen from fig. 6, F can practically be taken equal to one - at least until more precise information is available on the damping D. Examples of D are shown in figs. 7 and 8.

To determine the reactive excitation $\nu_r(t)$ we now make the assumption that its oscillogram is similar in shape to the roll oscillogram $\psi_1(t)$:

$$\ddot{\nu}_r(t) = \frac{\bar{\nu}_r}{\bar{\psi}} \cdot \ddot{\psi}_1(t) \tag{72}$$

Equation (63) then yields the differential equation

$$J' \cdot \left(1 + \frac{\bar{\nu}_r}{\bar{\psi}}\right) \cdot \ddot{\psi}_1 + P \cdot h(\psi_1) = 0 \tag{73}$$

which is the same as (9) except for the change in the moment of inertia. The detuning factor relative to the natural roll period in calm water is thus

$$\frac{T_s}{T_0(\bar{\psi})} = \sqrt{1 + \frac{\bar{\nu}_r}{\bar{\psi}}} \tag{74}$$

The factor can be greater or smaller than one, depending on the phase angle between $\vartheta_r^{\varphi}(t)$ and $\psi_1(t)$. For the amplitude of the reactive excitation we find $T_s = T_w$

$$\bar{\vartheta}_r^{\varphi} = \left(\frac{T_w^2}{T_o^2(\bar{\psi})} - 1 \right) \cdot \bar{\psi} \tag{75}$$

so that

$$\vartheta_r^{\varphi}(t) = \left(\frac{T_w^2}{T_o^2(\bar{\psi})} - 1 \right) \cdot \psi_1(t) . \tag{76}$$

b) <u>The wave profile</u>.

From equation (67) and (76) we can now determine the wave profile corresponding to the exact solutions found above. The oscillogram of the reactive excitation component is the same as the undamped roll oscillogram in calm water, apart from the scale factor (74). From (46) it can be shown that this component dominates outside the resonance interval. The amplitude here, however, is small, so that there is only a relatively small contribution from the higher harmonics. The active excitation component, on the other hand, is the integral of $\psi_o(t)$ so that here the n^{th} harmonic is reduced by a factor 1/n. However, the active excitation becomes important near the resonance point where the larger roll amplitudes entail a larger proportion of higher harmonics (figs. 3a and 3b).

The wave profile itself is finally obtained by integrating the wave slope $\vartheta(t)$, which further reduces the higher harmonic contribution. Outside the resonance interval the profile is approximately equal to the integral of the basic oscillogram $\psi_o(t)$ for small amplitudes, whereas near the resonance peak it is approximately equal to the second integral of the basic oscillogram for large amplitudes. The difference between a sinusoidal profile and the profile obtained in our case is thus acceptable, in view of the inherent difficulty of defining the profile of an actual swell at sea.

c) The response curves.

Having determined the fundamental function $T_o(\bar{\psi}) = 2\pi/\nu_o(\bar{\psi})$ it is now a simple matter to evaluate graphically the response curves $\bar{\psi}(\omega^2, \bar{\vartheta})$, i.e. the roll amplitude $\bar{\psi}$ as a function of the square of the exciting frequency $\omega = 2\pi/T_W$, with the effective wave steepness $\bar{\vartheta}$ as a parameter. If the skeleton curve $\bar{\psi}(\nu_o^2)$ is plotted in the $(\omega^2, \bar{\psi})$-plane, the value ω^2 corresponding to a pair of values $(\bar{\psi}, \bar{\vartheta}_r)$ can be obtained by projection, using

$$\nu_o^2(\bar{\psi}) : \omega^2 = (\bar{\psi} + \bar{\vartheta}_r) : \bar{\psi} \tag{77}$$

(which is identical with (75)).

If the amplitude of the reactive excitation is known we can thus construct as many points $(\omega^2, \bar{\psi})$ on the response curve as required. The method is illustrated in region 2 of fig. 9.

In the idealized case of a ship without damping, $\bar{\vartheta}_r$ equals $\pm\bar{\vartheta}$. The signs correspond to the two possible phase angles between $\psi_1(t)$ and $\vartheta_r(t)$ and must both be considered. In this case the response curves can be constructed immediately and are plotted in figs. 10 a and 10 b for the fundamental functions shown in figs. 3a and 3b.

In the general case of finite damping the reactive excitation has first to be determined. If the oscillograms $\vartheta_a^{\mathfrak{R}}(t)$ and $\vartheta_r^{\mathfrak{R}}(t)$ are sinusoidal,

$$\overline{\vartheta_r^{\mathfrak{R}}} = \pm \sqrt{\overline{\vartheta^{\mathfrak{R}}}^2 - \overline{\vartheta_a^{\mathfrak{R}}}^2} \tag{78}$$

In applying this relation to the general case of non-sinusoidal oscillograms, an appreciable error arises only if both amplitudes are of comparable magnitude. This is neither the case xlose to the resonance peak, where the active excitation dominates, nor further away, from it where the reverse is the case. We shall accept this small error in the following in order to facilitate the representation of the response curves.

The curves calculated using (78) are shown in figs. 11a and 11b with $D(T_s) = 0{,}2$ and $F(\bar{\varphi}) = 1$. The reactive excitation amplitudes were determined as shown in region 1 of fig. 9. For constant D and F = 1, equation (70) becomes

$$\overline{\vartheta_a^{\mathfrak{R}}} = D \cdot \bar{\varphi}_{m} \tag{79}$$

In the resonance case $\overline{\vartheta_r^{\mathfrak{R}}} = 0$ therefore

$$\overline{\vartheta^{\mathfrak{R}}} = D \cdot \bar{\varphi}_{res} \tag{80}$$

so that (78) becomes

$$\vartheta_r = \pm D \cdot \sqrt{\bar{\varphi}_{res}^{\,2} - \bar{\varphi}^{2}} \tag{81}$$

The length ϑ_r/D for a given roll amplitude $\bar{\varphi} < \bar{\varphi}_{res}$ is obtained from the quarter circle of radius $\bar{\varphi}_{res} = \bar{\vartheta}/D$ (fig.9 - the radius can also be larger than φ_R) by Pythagoras (shaded triangle). The reactive excitation amplitudes themselves can then be obtained from the two straight lines of slope $\pm$ D : 1.

The response curves for the roll states of the first kind can be obtained analytically from (70), (75) and (78):

$$\bar{\vartheta}^{2} = \left[\left(\frac{T_W^{\,2}}{T_o^{\,2}(\bar{\varphi})} - 1\right)^{2} + F^{2}(\bar{\varphi}) \cdot D^{2}(T_W)\right] \cdot \bar{\varphi}^{2} \tag{82a}$$

or, in terms of the angular frequencies [6])

$$\bar{\vartheta}^{2} = \left[\left(\frac{\nu_o^{2}(\bar{\varphi})}{\omega^{2}} - 1\right)^{2} + F^{2}(\bar{\varphi}) \cdot D^{2}(\omega)\right] \cdot \bar{\varphi}^{2} \tag{82b}$$

d) Stable roll states.
The roll states derived above all satisfy the equation of motion (1). It remains to be investigated, however, if they can also be realized physically. This question is of particular interest in view of the existance of multiple-valued regions of the response curves.

If we confine our attention to steady roll states, the question of general dynamic stability cannot be immediately answered. We shall therefore formulate a stability criterion which is applicable to our response curves derived for the steady roll states as follows:

We shall regard a roll state as stable if for a given period an increment in the magnitude of the exciting amplitude leads to an increment of the same sign in the magnitude of the oscillation amplitude. Analytically,

$$\mathcal{J} \equiv \frac{\partial_\nu \overline{\vartheta}^2}{\partial \overline{\psi}^2} > 0 \qquad (83)$$

If we now consider the response curve tangent for constant D,

$$\frac{d\overline{\psi}}{dT_w^2} = -\frac{\dfrac{\partial_\nu \overline{\vartheta}^2}{\partial T_w^2}}{2\psi \cdot \dfrac{\partial_\nu \overline{\vartheta}^2}{\partial \overline{\psi}^2}} = -\frac{\overline{\psi}}{C \cdot T_o^2(\overline{\psi})} \cdot \left(\frac{T_w^2}{T_o^2(\overline{\psi})} - 1\right) \qquad (84a)$$

$$\frac{d\psi}{d\omega^2} = -\frac{\dfrac{\partial_\nu \overline{\vartheta}^2}{\partial \omega^2}}{2\psi \cdot \dfrac{\partial_\nu \overline{\vartheta}^2}{\partial \overline{\psi}^2}} = -\frac{\overline{\psi} \cdot \nu_o(\overline{\psi})}{C \cdot \omega^4} \cdot \left(\frac{\nu_o^2(\overline{\psi})}{\omega^2} - 1\right) \qquad (84b)$$

we find that in both cases for stable roll states ($C > o$) the tangent is positive to the left of the skeleton curve and negative to the right. The tangents become infinite on the boundary curve separating the stable and unstable regions for which C = 0. From the amplitude relation (82b) and with F = 1 the limiting curve of the stability domain can be obtained in form of the equation

$$C \equiv \left(\frac{\nu_o^2(\bar{\psi})}{\omega^2} - 1\right)^2 + D^2(\omega) + \left(\frac{\nu_o^2(\bar{\psi})}{\omega^2} - 1\right) \cdot \frac{\bar{\psi}}{\omega^2} \cdot \frac{d\nu_o^2(\bar{\psi})}{d\bar{\psi}} = 0 \tag{85}$$

$$\equiv \left(\frac{\nu_o^2(\bar{\psi})}{\omega^2} - 1\right) \cdot \left(\frac{\nu_o^2(\bar{\psi})}{\omega^2} + \frac{\bar{\psi}}{\omega^2} \cdot \frac{d\nu_o^2(\psi)}{d\bar{\psi}} - 1\right) + D^2(\omega) = 0. \tag{85}$$

From this equation the boundary curve can easily be constructed. Neglecting the damping for the first, we obtain the two solutions (the bracket-terms in (85) being zero):

$$\omega^2 = \nu_o^2(\bar{\psi}) \qquad \text{and} \qquad \omega^2 = \nu_*^2(\bar{\psi}). \tag{86}$$

The first solution is the skeleton curve; the second is related to this curve by the equation

$$\nu_*^2(\bar{\psi}) = \nu_o^2(\bar{\psi}) + \bar{\psi} \cdot \frac{d\nu_o^2(\bar{\psi})}{d\bar{\psi}} \tag{87}$$

In the construction of the limiting curve $\nu_o^2(\bar{\varphi})$, as shown in fig. 12, each point of the skeleton curve has been simply translated laterally by the subtangent length $s = \left| \bar{\varphi} \cdot \frac{d\nu_o^2(\bar{\varphi})}{d\bar{\varphi}} \right|$. Introducing $\nu_*^2(\bar{\varphi})$ from (87) equation (85) becomes

$$C \equiv \left(\frac{\nu_o^2(\bar{\varphi})}{\omega^2} - 1\right) \cdot \left(\frac{\nu_*^2(\bar{\varphi})}{\omega^2} - 1\right) + D^2(\omega) = 0 \qquad (88)$$

For D = 0, C is negative in the region between the two limiting curves found above, so that this region is unstable. For finite damping, the unstable region becomes smaller, as can be seen from (88). To construct the limiting curve L in the case of finite damping we rewrite (88) in the form

$$\left(\nu_o^2(\bar{\varphi}) - \omega^2\right) \cdot \left(\nu_*^2(\bar{\varphi}) - \omega^2\right) + \left(\omega^2 \cdot D(\omega)\right)^2 = 0 \qquad (89)$$

and apply the second law. The damping function $D(\omega)$ can be arbitrary. If $\omega^2 \cdot D(\omega)$ is drawn below the ω^2-axis, the abscissae ω_1^2, ω_2^2 of the limiting curve for a given $\bar{\varphi}$ are obtained as the intercepts of the curve with the semicircle drawn below the ω^2-axis through the points $\nu_o^2(\bar{\varphi}), \nu_*^2(\bar{\varphi})$ on the axis. The diameter of the semicircle ist intersected normally in the points ω_1^2 and ω_2^2 by a secant of half length $\omega^2 \cdot D(\omega)$, giving equation (89). The method is illustrated in fig. 12 for constant D = 0,2. The shaded regions of instability in figs. 10a - 11b were determined in this way.

e) Probable roll states.

Having determined which regions of the response curves correspond to physically realizable states, the question arises which of the different possible states for a given period will actually occur in practice. This is a statistical problem, depending on the initial condition of the ship and the random moments which could bring about a transition from one state to another. These effects lie beyond the scope of our considerations, which are confined only to steady roll states. However, the following two comments may serve for a general orientation: Firstly, we shall expect roll states of larger amplitude to be less probable than those of smaller amplitude, as their higher energy content has to be imparted to the ship initially. Secondly, in experiments with a navipendulum corresponding to a sinusoidal righting-arm curve, the author always found that the higher energy roll states could be realized only with great care, very small disturbance sufficing to cause transition to the lower-energy state. The phenomenon became more marked the higher the energy of the state. Energy considerations and experiment thus suggest a greater probability for the roll states of smaller amplitude. The random disturbances which in practice are always superimposed on the swell will seldom give the ship just the impulse required for transition to the higher energy state, though they will generally be strong enough to disturb this state on account of its smaller stability.

f) The influence of the vertical acceleration.

The roll states of the first kind as derived above will be slightly modified by the periodical vertical acceleration which we have so far neglected. To obtain an estimate of this influence we substitute the variable factor $1 + \beta(t)$ in the restoring moment (see equation (1)) by a suitable mean value $1 \pm \varepsilon \cdot |\bar{\beta}|$ during the appropiate semicycle ($0 < \varepsilon < 1$, - the sign is positive in the wave trough and negative on the crest). It is then seen immediately that the time required for a semicycle is reduced in the wave trough and increased on the wave crest by the vertical acceleration. The total period, however, remains the same, as for roll states of the first kind

$$\int_0^{T_s} \beta \cdot h(\psi) \cdot \psi \cdot dt = 0 \tag{90}$$

i.e. the detuning equation (46) remains unchanged.

A further result due to the vertical acceleration is an unsymmetry of the oscillation. This is particularly marked in the resonance case $\bar{\vartheta}_r = 0$. The roll oscillogram $\psi_1(t)$ here lags a quarter of a period behind the wave slope $\vartheta(t) = \vartheta_a^m(t)$ (equation (67)). On a wave crest the ship therefore has a maximum angle of roll, the direction being towards the latest wave slope, i.e. to lee; in a trough the ship has its maximum windward roll angle. Outside of the resonance intervals the maximum roll angles occur earlier or later. As the restoring moment is smaller on the crest than in the trough, the lee roll angle will be greater than the windward roll angle. This results in a mean heel to lee ψ_- of the roll oscillogram

$$\psi_1(t) = \psi_- + \psi_\sim(t). \tag{91}$$

To estimate the magnitude of ψ_- we equate the potential energies of the two extreme positions:

$$\left[1-\varepsilon\cdot|\bar{\beta}|\right]\cdot\mathcal{U}(\psi_-+\bar{\varphi})=\left[1+\varepsilon\cdot|\bar{\beta}|\right]\cdot\mathcal{U}(\psi_--\bar{\varphi}). \tag{92}$$

Expanding U in powers of ψ_- and making use of the symmetry of U we obtain in the first approximation:

$$\mathcal{U}(\psi_-+\bar{\varphi})\pm\mathcal{U}(\psi_--\bar{\varphi})\approx\begin{cases}2\,\mathcal{U}(\bar{\varphi})\\ 2\,\psi_-\cdot P\cdot h(\bar{\varphi})\end{cases} \tag{93}$$

and thus finally

$$\frac{|\psi_-|}{|\bar{\varphi}|}\approx\frac{\varepsilon\cdot\mathcal{U}(\bar{\varphi})\cdot|\bar{\beta}|}{\bar{\varphi}\cdot P\cdot h(\bar{\varphi})}<\tfrac{1}{2}q^2(\bar{\varphi})\cdot|\bar{\beta}| \tag{94}$$

where $q(\bar{\varphi})$ is the function defined in (28) and plotted in fig. 2. The finite mean heel angle does not invalidate our original approach, as ψ_- has no influence on the angular acceleration. However, we must emphasize subsequently that the amplituden $\bar{\varphi}$ should be taken as the amplitude of $\psi_\sim(t)$.

6. The Roll States of the Second Kind, $T_s = 2\ T_w$.

In the roll states of the second kind the periodic variation of the apparent vertical, $\vartheta(t)$ yields no contribution to equations (45) and (46). We shall therefore set

$$\varphi(t) \equiv \psi(t) = \psi_2(t) \tag{95}$$

for the first and consider the influence of the variation in the apparent vertical later. The equation of motion then becomes

$$J' \cdot \ddot{\psi}_2 + W(T_S) \cdot \dot{\psi}_2 + \left[1 + \beta(t)\right] \cdot P \cdot h(\psi_2) = 0 \tag{96}$$

which can be decomposed again into the two equations

$$W(T_S) \cdot \dot{\psi}_2 + \beta_a \cdot P \cdot h(\psi_2) = + N_2(t) \tag{97}$$

$$J' \cdot \ddot{\psi}_2 + \left[1 + \beta_r\right] \cdot P \cdot h(\psi_2) = - N_2(t) \tag{98}$$

containing only components of the same phase. The moment $N_2(t)$ again has no influence on the excitation or detuning of the ship as from (56) and (57)

$$\int_0^{T_S} N_2 \cdot \dot{\psi}_2 \cdot dt = 0 \quad \text{and} \quad \int_0^{T_S} N_2 \cdot \psi_2 \cdot dt = 0. \qquad (99) \quad (100)$$

$N_2(t)$ is a combination of higher harmonics which can be disposed of within certain limits. It cannot vanish identically as the terms on the left of (97) have different periods. It contains essentially a component of period $\frac{4}{3} T_S$. We shall postpone the discussion of this term until later, however.

If we again disregard the precise shape of the wave profile we can prescribe the periodical vertical acceleration in a manner enabling simplification of the analysis. We shall thus relate the active and reactive exciting moments (which must satisfy the conditions (52) and (53) as simply as possible to the potential energy (11):

$$\beta_a(t) = -\overline{\beta}_a \cdot 2 \sqrt{\mathcal{U}(\varphi_2) \cdot \left(1 - \mathcal{U}(\varphi_2)\right)} \; \operatorname{sg} \varphi_2 \cdot \dot{\varphi}_2 \tag{101}$$

$$\beta_r(t) = \overline{\beta}_r \cdot \left(1 - 2\mathcal{U}(\varphi_2)\right) + \beta_0 \tag{102}$$

where

$$0 \leq \mathcal{U}(\varphi) \equiv \frac{U(\varphi)}{U(\overline{\varphi})} \leq 1, \quad \varphi_2(t). \tag{103}$$

The active excitation amplitude $\overline{\beta_a}$ is alsways positive. whereas the reactive excitation amplitude $\overline{\beta_r}$ can also become negative depending on the phase angle. β_0 is a constant chosen to make the mean acceleration vanish:

$$\int_0^{T_W} \beta(t) \cdot dt = 0 \tag{104}$$

The equations can also be written in the form

$$\beta(t) = -\bar{\beta} \cdot \sin\left(2\gamma(t) - \gamma_0\right) + \beta_0 \tag{105}$$

$$= \bar{\beta}_r \cdot \cos 2\gamma(t) - \bar{\beta}_a \cdot \sin 2\gamma(t) + \beta_0$$

where

$$\bar{\beta}_r = \bar{\beta} \cdot \sin \gamma_0 , \qquad \bar{\beta}_a = \bar{\beta} \cdot \cos \gamma_0 \tag{106}$$

Here γ_0 is a phase constant and $\gamma(t)$ the (non-uniformly) varying phase angle. The amplitude of the total excitation is given by the exact expression

$$\bar{\beta}^2 = \bar{\beta}_r^2 + \bar{\beta}_a^2 \tag{107}$$

As the active excitation component (101) is symmetrical it yields no contribution to the integral (104) so that β_0 depends only on the oscillogram of the reactive excitation component. From (102) then

$$\frac{\beta_0}{\beta_r} = \frac{2}{T_W} \cdot \int_0^{T_W} \mathcal{U}(\psi_2) \cdot dt - 1 \tag{108}$$

$$= 1 - \eta_2(\bar{\psi})$$

where T_w can also be replaced by a multiple, equ. by $T_s = 2\,T_w$. The function $\eta_2(\bar{\psi})$ differs from the fundamental function $\eta_0(\bar{\psi})$ given in equation (35) as the oscillogram $\psi_2(t)$ of the forced oscillation is not the same as the oscillogram $\psi_0(t)$ of the free oscillation. It will be shown later that to a first approximation

$$\eta_2(\bar{\psi}) = \eta_0(\bar{\psi}) - \frac{1}{8}\bar{\beta}_r \cdot \left(2\eta_0(\bar{\psi}) - \eta_0^2(\bar{\psi}) + \delta\right) \tag{109}$$

where

$$|\delta| \leq \eta_0(\bar{\psi}) \cdot \left(2 - \eta_0(\bar{\psi})\right) \leq 1.$$

a) <u>The roll amplitude</u>.

The amplitude of the active excitation component can be obtained by substituting (101) in (97) and evaluating the integral (99). With (11) we find

$$\bar{\beta}_a \cdot 2\mathcal{U}(\bar{\psi}) \cdot \oint_{T_s = 2T_W} \sqrt{\mathcal{U}(\psi_2)\cdot\left(1 - \mathcal{U}(\psi_2)\right)} \cdot d\,\mathcal{U}(\psi_2) = W(T_s) \cdot \int_0^{T_s} \dot{\psi}_2^2 \cdot dt, \tag{110}$$

where the root has the same sign as $\psi_2 \cdot \dot{\psi}_2$.

The plot of the integrand on the left side over $\mathcal{U}(\psi)$ is a circle of radius 1/2. As $\mathcal{U}(\psi_2)$ completes two oscillations in the time T_s the left side of (110) is equal to $\bar{\beta}_a \cdot \mathcal{U}(\bar{\psi}) \cdot \pi$ so that the equation becomes

$$\bar{\beta}_a = \frac{T_s \cdot W(T_s)}{\pi \cdot J'} \cdot \frac{2}{T_s} \cdot \int_0^{T_s} \frac{\frac{1}{2} \cdot J' \cdot \dot{\psi}_2^2}{\mathcal{U}(\bar{\psi})} \cdot dt \tag{111}$$

The first factor is twice the dimensionless damping coefficient of equation (71). If we allow for the modification of the $\psi_2(t)$ -oscillogram by the reactive excitation $\beta_r(t)$ the integral can be reduced to $\eta_o(\bar{\psi})$ defined in (35). It will be shown later that to a first approximation

$$\frac{2}{T_s} \cdot \int_0^{T_s} \frac{\frac{1}{2} J' \cdot \dot{\psi}_2^2}{\mathcal{U}(\bar{\psi})} \cdot dt = \eta_o(\bar{\psi}) + \frac{1}{8} \bar{\beta}_r \cdot \left(2\eta_o(\bar{\psi}) - 5\eta_o^2(\bar{\psi}) + \delta\right) \equiv \eta_\beta(\bar{\psi}) \qquad (112)$$

where $\qquad |\delta| \leq \eta_o(\bar{\psi}) \cdot \left(2 - \eta_o(\bar{\psi})\right) \leq 1.$

The amplitude of the active excitation can thus be expressed as the product

$$\bar{\beta}_a = 2 D(T_s) \cdot \eta_\beta(\bar{\psi}). \qquad (113)$$

As we have neglected the dependence of the damping on the roll amplitude we can with the same consequence also neglect the dependence of the active excitation on the amplitude. This can be further justified by the fact that experience indicates an increase of D with the amplitude (fig. 8), whereas the mean value $\eta_o(\bar{\psi})$ decreases (fig. 5), the two effects thus tending to cancel. Within the limits of these uncertainties we shall thus use the simplified formula

$$\bar{\beta}_a \approx 2 D(T_s) \approx \text{const.} \qquad (114)$$

In contrast to the roll states of the first kind, where the active excitation was essentially proportinal to the roll amplitude the active excitation in this case is practically independent of the roll amplitude.

This means that roll oscillations of the second kind can be generated only if the swell exceeds the critical value

$$H : \lambda \geq 2 D(T_s) : \pi . \tag{1115}$$

The amplitude of the reactive excitation is then determined by (107), and is also practically constant for a given swell strength $\overline{\beta}$.

If we substitute the reactive excitation (102) in (98) and neglect the higher harmonic combination $N_2(t)$ for the first, we obtain the equation of motion for a ship rolling in calm water with a modified restoring moment given by

$$h_\beta(\psi) = \left[1 + \beta_r(\psi)\right] \cdot h(\psi) \tag{116}$$

$$= \left[1 + \beta_0 + \overline{\beta}_r \left(1 - 2u(\psi)\right)\right] \cdot h(\psi) .$$

We can then derive the following expressions analogous to equations (11) - (13); for the potential heel energy

$$\mathcal{U}_\beta(\psi) = P \cdot \int h_\beta(\psi) \cdot d\psi$$
$$= \mathcal{U}(\bar{\psi}) \cdot \left[(1+\beta_0)\,\mathcal{U}(\psi) + \beta_r \cdot \mathcal{U}(\psi) \cdot (1-\mathcal{U}(\psi)) \right], \qquad (117)$$

for the total energy

$$\frac{1}{2} J' \cdot \dot{\psi}^2 + \mathcal{U}_\beta(\psi) = \mathcal{U}_\beta(\bar{\psi}) \qquad (118)$$

and for the time differential of the forced oscillation $\psi_2(t)$

$$dt(\psi_2) = \sqrt{\tfrac{1}{2} J'} \cdot \frac{d\psi}{\sqrt{\mathcal{U}_\beta(\bar{\psi}) - \mathcal{U}_\beta(\psi)}}$$
$$= \sqrt{\tfrac{1}{2} J'} \cdot \frac{d\psi}{\sqrt{\mathcal{U}(\bar{\psi}) - \mathcal{U}(\bar{\psi})} \cdot \sqrt{1+\beta_0 - \bar{\beta}_r \cdot \mathcal{U}(\psi)}} \qquad (119)$$

Introducing the time differential for the free oscillation $\psi_0(t)$ given by (12), equation (119) becomes (fig. (13))

$$dt(\psi_2) = \frac{dt(\psi_0)}{\sqrt{1+\beta_0 - \bar{\beta}_r \cdot \mathcal{U}(\psi_0)}} \qquad (120)$$

and the period of the forced oscillation

$$T_S = \int_0^{T_0(\bar{\psi})} \frac{dt}{\sqrt{1+\beta_0 - \bar{\beta}_r \cdot \mathcal{U}(\psi_0)}}, \quad \psi_0(t). \tag{121}$$

The detuning factor

$$\frac{T_S}{T_0(\bar{\psi})} = \frac{1}{T_0(\bar{\psi})} \cdot \int_0^{T_0(\bar{\psi})} \left[1+\beta_0 - \bar{\beta}_r \cdot \mathcal{U}(\bar{\psi}_0)\right]^{-1/2} \cdot dt \tag{122}$$

resulting from the periodic vertical acceleration is thus the mean value of the expression

$$\left[1+\beta_0 - \bar{\beta}_r \cdot \mathcal{U}(\psi_0)\right]^{-1/2}$$

taken over the natural period.

The exact evaluation of this integral is generally very difficult. We shall therefore limit ourselves to an approximation of the first order in $\bar{\beta}_r$ which is valid for all $\bar{\psi}$ (see also (108)):

$$\left[1+\beta_0 - \bar{\beta}_r \cdot \mathcal{U}(\psi_0)\right]^{-1/2} = 1 - \frac{1}{2}\bar{\beta}_r \cdot \left(1 - \eta_2(\bar{\psi}) - \mathcal{U}(\psi_0)\right) \pm \cdots \tag{123}$$

The condition for absolute convergence is

$$\left| \overline{\beta}_r \cdot \left(1 - \eta_2(\overline{\varphi}) - \mathcal{U}(\varphi_0)\right) \right| < 1 \tag{124}$$

As will be shown later, this leads to the conditions

$$-0{,}83 < \overline{\beta}_r < +0{,}83 \quad \text{for decreasing stiffness}$$

$$-0{,}60 < \overline{\beta}_r < +0{,}66 \quad \text{for increasing stiffness} \tag{125}$$

With

$$\eta_2(\overline{\varphi}) \approx \eta_0(\overline{\varphi})$$

and - see (35) -

$$\frac{1}{T_0(\overline{\varphi})} \cdot \int_0^{T_0(\overline{\varphi})} \mathcal{U}(\varphi_0) \cdot dt = 1 - \tfrac{1}{2}\eta_0(\overline{\varphi})$$

the detuning factor then becomes approximately

$$\frac{T_s}{T_0(\overline{\varphi})} \approx 1 - \tfrac{1}{2}\overline{\beta}_r \cdot \left(1 - \eta_0(\overline{\varphi}) - 1 + \tfrac{1}{2}\eta_0(\overline{\varphi})\right) \tag{126}$$

$$\approx 1 + \tfrac{1}{4}\overline{\beta}_r \cdot \eta_0(\overline{\varphi}) .$$

As $T_s = 2\,T_W$ we have thus derived an implicite representation of the response curves of the second kind. A transformation to the explicit form $\bar{\varphi}(T_W, \bar{\beta}_r)$ cannot be carried out, as the functions $T_o(\bar{\varphi})$ and $\eta_o(\bar{\varphi})$ are transcendent. However, the two branches of the response curve corresponding to a given reactive excitation amplitude $\pm|\bar{\beta}_r|$ can easily be gained from the given skeleton curve $T_o(\bar{\varphi})$ using equation (126). The results are plotted in fig. 14a and 14b for the righting-arm curves shown and $|\bar{\beta}_r| = 0{,}2$.

We found that the roll period of a ship is only slightly detuned by the periodical vertical acceleration. Roll states of the second kind are thus not possible for all swell periods. As the response curve rises very stepply, however, the amplitude may become dangerously large.

b) <u>Stable roll states.</u>
From (107), (114) and (126):

$$\bar{\beta}^2 = \left(\frac{T_s}{T_o(\bar{\varphi})} - 1\right)^2 \cdot \frac{16}{\eta_o^2(\bar{\varphi})} + 4D^2(T_s). \tag{127}$$

To determine the stable regions of the response curves we apply our stability criterion of section 5d. For stable roll motions we then have

$$C = \frac{\partial \bar{\beta}^2}{\partial \bar{\varphi}^2} > 0. \tag{128}$$

The derivatives of the response curves $\bar{\varphi}(T_s, \bar{\beta})$ for a given wave strength $\bar{B}$ = const, are

$$\frac{d\bar{\varphi}}{dT_s'} = -\frac{\partial\bar{\beta}^2/\partial T_s'}{\partial\bar{\beta}^2/\partial\bar{\varphi}} = -\frac{\partial\bar{\beta}^2/\partial T_s'}{2\bar{\varphi}\cdot C}. \tag{129}$$

The response curves thus have infinite derivatives on the limiting curve of the stability region, C = 0, and also on the axis $\bar{\varphi} = 0$. From (127) we find further for constant D

$$\frac{d\bar{\varphi}}{dT_s'} = \frac{16\cdot(T_o(\bar{\varphi})-T_s')}{C\cdot\bar{\varphi}\cdot T_o'^2(\bar{\varphi})\cdot \eta_o^2(\bar{\varphi})}. \tag{130}$$

This yields the following rule for the response curves of the second kind (plotted over either the period or the frequency): For stable roll motions the curve tangents are positive on the left of the skeleton curve and negative on the right. This determines the sign of $\bar{\beta}_r$:

$$\begin{aligned} \bar{\beta}_r &> 0 \qquad \text{for increasing stiffness,} \\ \bar{\beta}_r &< 0 \qquad \text{for decreasing stiffness.} \end{aligned} \tag{131}$$

The unstable parts of the response curves are shown dotted in fig. 14a and 14b. In the following it will be shown that the dotted interval on the T_s-axis, corresponding to the trivial solution $\varphi_2(t) \equiv 0$ of (96), for which our stability criterion is not applicable, is also unstable.

c) <u>The critical period interval.</u>

In the limiting case of small amplitudes

$$h(\psi) = \overline{M_0G} \cdot \psi , \qquad \mathcal{U}(\psi) = \tfrac{1}{2} P \cdot \overline{M_0G} \cdot \psi^2 \tag{132}$$

and

$$T_0(\bar{\psi}) = T_0(0) = 2\pi \cdot \sqrt{\frac{J'}{P \cdot \overline{M_0G}}} . \tag{133}$$

This case is particularly useful for studying certain characteristic properties of the roll states of the second kind. Neglecting $N_2(t)$, (98) becomes a linear (Hill) differential equation

$$J' \cdot \ddot{\psi}_2 + \left[1 + \beta_r(t)\right] \cdot P \cdot \overline{M_0G} \cdot \psi_2 = 0 \tag{134}$$

With

$$\mathcal{U}(\psi) = \frac{\psi^2}{\bar{\psi}^2} \tag{135}$$

equation (102) then leads to the non-linear differential equation

$$J' \cdot \ddot{\psi}_2 + \left[1 + \beta_0 + \bar{\beta}_r \left(1 - 2 \cdot \frac{\psi_2^2}{\bar{\psi}^2}\right)\right] \cdot P \cdot \overline{M_0G} \cdot \psi_2 = 0 \tag{136}$$

for the steady roll states. This is the free roll equation for a ship in calm water whose righting-arm curve is a cubic parabola:

$$h_{\beta}(\psi) = \left[1+\beta_0+\bar{\beta}_r - 2\bar{\beta}_r \cdot \frac{\psi^2}{\bar{\Psi}^2}\right] \cdot \overline{M_0G} \cdot \psi . \qquad (137)$$

The roll oscillogram is therefore not sinusoidal, the difference increasing with the amplitude of the reactive excitation. Nevertheless, the period of oscillation is independent of the roll amplitude, as the latter is proportinal to the stability limit of the righting-arm curve (137), (i.e. to the angle $\psi = \Psi_{(R)}$ for which the expression in rectangular brackets vanishes) - the propertionality factor depending only on the reactive excitation amplitude -

$$\frac{\bar{\Psi}^2}{\Psi^2_{(R)}} = \frac{2\bar{\beta}_r}{1+\beta_0+\bar{\beta}_r} . \qquad (138)$$

The response curves therefore run parallel to the ordinate axis, so that $\bar{\Psi}$ is variable independently of $\bar{\mathrm{B}}$. The resonance states of the second kind are thus indifferent for small amplitudes, as the expression (128) vanishes.

Using (133), (135) and the substitution $x = \Psi/\bar{\Psi}$ the roll period can be obtained from (119) as an elliptic integral of the first kind:

$$T_s = \frac{T_0(0)}{\sqrt{1+\beta_0}} \cdot \frac{2}{\pi} \cdot \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{T_0(0)}{\sqrt{1+\beta_0}} \cdot \frac{2}{\pi} K \qquad (139)$$

with

$$k^2 = \frac{\bar{\beta}_r}{1+\beta_0} . \tag{140}$$

The excitation can also be expressed as a function of the parameter k^2 by means of elliptic integrals. From (140) and (108) we obtain two simultaneous equations for the amplitude of the reactive excitation

$$\left.\begin{aligned} \bar{\beta}_r - k^2 \cdot \beta_0 &= k^2 \\ \left[1-\eta_2(0)\right] \cdot \bar{\beta}_r - \beta_0 &= 0 \end{aligned}\right\} \tag{141}$$

with the solutions

$$\bar{\beta}_r = \frac{k^2}{1-k^2\left[1-\eta_2(0)\right]} , \quad \beta_0 = \frac{k^2\left[1-\eta_2(0)\right]}{1-k^2\left[1-\eta_2(0)\right]} \tag{142}$$

Using equations (119) and (135) and the substitution $x = \Psi_2/\bar{\Psi}$ the mean value

$$\eta_2(0) = \frac{2}{T_S} \cdot \int_0^{T_S} \left[1-\mathcal{U}(\psi_2)\right] \cdot dt \tag{143}$$

which depends on the oscillogram shape but not on its amplitude, can be written

$$\eta_2(0) = 2 - \frac{2}{T_S} \cdot \sqrt{\frac{J'}{P \cdot M_0 G \cdot (1+\beta_0)}} \cdot 4 \cdot \int_0^1 \frac{x^2 \cdot dx}{\sqrt{(1-x^2)(1-k^2x^2)}} , \tag{144}$$

so that from (133) and (139)

$$\eta_2(0) = 2 \cdot \left(1 - \frac{D}{K}\right) , \tag{146}$$

where D is the complete elliptic integral

$$D = \int_0^1 \frac{x^2 \cdot dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \, . \quad {}^{7)} \tag{145}$$

With the aid of (142) we can thus determine both the reactive excitation amplitude $\overline{\beta_r}$ and the roll period (139) as functions of k^2.

To determine the active excitation we evaluate the mean value of $\eta_\beta(\overline{\psi})$ defined by equations (111) and (113). For small amplitudes $\overline{\psi} \to 0$ we find from (118), (117), (132)and (135) with $x = \psi_2/\overline{\psi}$:

$$\eta_\beta(0) = \frac{2}{T_S} \cdot \sqrt{\frac{J' \cdot (1+\beta_0)}{P \cdot M_0 G}} \cdot 4 \cdot \int_0^1 \sqrt{(1-x^2) \cdot (1-k^2x^2)} \cdot dx \tag{147}$$

This expression can also be reduced to the complete elliptic integrals K and D , since

$$\int_0^1 \sqrt{(1-x^2)(1-k^2x^2)} \cdot dx = \frac{3}{2}K - \frac{1}{3}(1+k^2)\cdot D \tag{148}$$

Using (139) and (133), the expression (147) then becomes

$$\eta_\beta(0) = \frac{1+\beta_0}{3} \cdot \left(4 - (1+k^2) \cdot \frac{2D}{K}\right). \tag{149}$$

In fig. 15 the active exciting amplitude $\overline{\beta_a} - 2D(T_s)\cdot\eta_\beta(0)$ and the detuning factor $T_s/T_o(0)$ are plotted over the reactive excitation amplitude for constant D = 0,2. From the figure we can read off the magnitude $\overline{\mathrm{B}}$ and phase γ_0 of the excitation (or the equivalent components $\overline{\beta_a}$ and $\overline{\beta_r}$) required to sustain a roll state of small amplitude within a finite interval around the natural period $T_o(o)$.

The figure is also useful for explaining qualitatively the nascent state of the roll motions of the second kind and the way in which the final steady state is attained. If twice the value of the exciting period, 2 T_w, lies within the interval determined by the arc of radius $\overline{\mathrm{B}}$ about the origin the ship becomes unstable as soon as the periodic vertical acceleration exceeds the critical value $\overline{\mathrm{B}}$ crit (which is practically equal to 2 D (T_o) as the curve is only slightly inclined to the $\overline{\beta_r}$-axis). The active excitation amplitude $\bar{\beta} \cdot \cos \gamma_0$ then exceeds the damping value $2D(T_s) \cdot \eta_\beta(0)$ and after an initial random impulse the ship begins to roll with the period $T_S = 2\ T_W$. As the inital roll amplitude can be assumed arbitrarily small for this argumentation, it is seen that within this critical period interval the trivial solution $\psi_2(t) \equiv 0$ of equation (96) to which our criterion (128) could not be applied, represents an unstable roll state of the second kind. Using (126), the critical period interval determined by the arc in fig. 15 can be written for small $|\bar{\beta}_r|$ in the approximate form

$$\left| \frac{T_s}{T_o(0)} - 1 \right| < \frac{1}{4} \left| \bar{\beta}_r \right| . \tag{150}$$

Under the said conditions the roll amplitude increases after an initial random impulse until a point is reached where the reactive excitation corresponding to the changed natural period $T_o(\bar{\psi})$ reduces the active excitation amplitude $\bar{\beta}_a = +\sqrt{\bar{\beta}^2 - \bar{\beta}_r^2}$ to the value $2D(T_s) \cdot \eta_\beta(\bar{\psi})$ just balancing the damping. As the active excitation (113) required to balance the damping scarcely varies with the amplitude, the final steady state is attained primarily by detuning, (see figs. 14a and 14b). It can readily be seen that the state is stable as a larger roll amplitude would lead to a further detuning. This would necessate a larger reactive excitation, which could only by yielded by a larger total excitation - provided the damping does not decrease excessively with increasing amplitude. Out stability criterion (128) can thus also be understood from this view-point.

We have seen that roll states of the second kind will always occur in the critical interval (150). Outside this interval the trivial solution $\psi_2(t) \equiv 0$ is stable, so that a specific impulse is necessary to bring the ship into a a non-trivial roll state of the second kind. Although all kinds of impulses are to be expected in a seaway, the probability for the impulse required is relatively small, as not only must it impart to the ship a large amount of energy, but it must also have just the right magnitude and phase.

The maximum roll amplitude $\bar{\psi}_{max}$ which will occur with certainty is marked in figs. 14a and 14b. As the point representing this roll state lies above the limiting point of the critical period interval which is also the limiting point of the (dotted) instability curve, we obtain the following equation for $\bar{\psi}_{max}$ using (126):

$$\left(1+\frac{1}{4}\bar{\beta}_r\cdot\eta_0(\bar{\psi}_{max})\right)\cdot T_0(\bar{\psi}_{max})=\left(1-\frac{1}{4}\bar{\beta}_r\right)\cdot T_0(0). \qquad (151)$$

In solving the equation the sign rule (131) has to be observed. For normal values of $|\bar{\beta}_r|$ the steady state will be attained by a relatively small degree of detuning. The quotient $\tau(\bar{\psi}_{max})=T_0(\bar{\psi}_{max})/T_0(0)$ is then only slightly different from one so that it can be well approximated by (29). If we further set (see figs 2 and 5)

$$g(\bar{\psi}_{max})\approx 1 \quad \text{and} \quad \eta_0(\bar{\psi}_{max})\approx 1 \qquad (152)$$

The two expressions under the roots in (29) become approximately equal, yielding the (somewhat crude) approximate formula

$$\tau(\bar{\psi}_{max})\approx\sqrt{\frac{M_0G\cdot\bar{\psi}_{max}}{h(\bar{\psi}_{max})}}\,. \qquad (153)$$

From (153) we obtain the following simple formula for the maximum amplitude occuring with certainty:

$$h(\overline{\psi}_{max}) \approx \left(1+\overline{\beta}_r\right) \cdot \overline{M_0G} \cdot \overline{\psi}_{max} . \tag{154}$$

This expression can be easily evaluated by intersecting the righting-arm curve with the straight line passing through the origin whose slope is $(1+\overline{\beta}_r)$ -times the slope of the initial tangent $\overline{M_0G} \cdot \psi$ of the righting-arm curve. The algebraic solution of (154) for $h(\psi) = \overline{M_0G} \cdot \psi \cdot (1+a\,\psi^2)$ is

$$\overline{\psi}_{max} \approx \sqrt{\frac{\overline{\beta}_r}{a}} \tag{155}$$

From (131) is follows that the solution is real.

d) Appendices.

I The Influence of the Wave Slope $\vartheta(t)$ and the Higher Harmonic Combination $N_2(t)$.

We consider finally the influences of the periodically varying apparent vertical direction $\vartheta(t)$ and the higher harmonic combination $N_2(t)$ which is given by (97), (101) and the oscillogram $\psi_2(t)$. We can use hereby the result found above that for the maximum amplitude occuring with certainty in roll states of the second kind the natural period $T_0(\bar{\psi}_{max})$ is $(1-\frac{1}{2}\bar{\beta}_r)$-times the natural period at small amplitudes - i.e. the influence of finite amplitude is in practice only of the order of a few percent. In the critical period interval the ship can thus be gegarded as a quasi-linear system. Its motion can then be represented as a superposition of the three solutions $\psi_1(t), \psi_2(t), \psi_3(t)$ of the differential equation (1) corresponding to the excitation by a)

a) the periodic variation of the apparent vertical direction, $\vartheta(t)$,
b) the periodic variation of the magnitude of the apparent vertical vector $\beta(t)\cdot g$,
c) the moment $N_2(t)$ representing the difference between the exciting and damping moments.

The sum of the three solutions gained from the differential equations

$$J'\cdot\left(\ddot{\psi}_1+\ddot{\vartheta}(t)\right)+W(T_s)\cdot\dot{\psi}_1+P\cdot h(\psi_1)=0 \qquad (156)$$

$$J'\cdot\ddot{\psi}_2 \quad +\left[1+\beta_r(t)\right]\cdot P\cdot h(\psi_2)=0 \qquad (157)$$

$$J'\cdot\ddot{\psi}_3 \quad +W(T_s)\cdot\dot{\psi}_3+P\cdot h(\psi_3)=-N_2(t) \qquad (158)$$

must be sufficiently small.

According to (5b) the exciting functions $\beta(t)$ and $\vartheta(t)$ have the same amplitude $\frac{2\pi r}{\lambda}$ and period T_W but are 90^o out of phase. Their phase lag relative to the roll motion of the second kind $\psi_2(t)$ of period $T_s = 2\ T_W$ depends on the damping and can be determined from (101). The situation is illustrated qualitatively in fig. 16 for the case $\bar{\beta}_r \approx 0$ where the vertical acceleration $\beta(t) \cdot g$ of the wave just exceeds the critical damping value $2D(T_s) \cdot g$. As $\beta_a(t)$ (here $\approx \beta(t)$) and $\psi_2 \cdot \dot{\psi}_2$ are of opposite sign the ship will be in its position either of static equilibrium or of maximum roll on the maximum wave slope for which $\beta(t) = 0$. In the wave trough $\beta(t) > 0$ the ship rolls towards its static equilibrium position and on the wave crest $\beta(t) < 0$ in the reverse direction. In fig. 16 the successive position of the ship are to be read from left to right. The wave is approaching from the right and the maximum roll angle $\pm|\bar{\psi}_2|$ occurs on the rear slope of the wave.

On this roll motion we now superimpose a motion of the first kind $\psi_1(t)$ of period T_W. In the linear approximation the differential equation (156) can be written

$$J' \cdot (\ddot{\psi}_1 + \ddot{\vartheta}) + W(T_s) \cdot \dot{\psi}_1 + J' \cdot \nu_1^2 \cdot \psi_1 = 0, \tag{159}$$

$\nu_1 \approx 2\pi/T_s$ being substituted for the ship's natural frequency. As the exciting frequency $\omega_1 = 2\pi/T_W \approx 2\nu_1$ is well away from the resonance point we can neglect the damping, obtaining the approximate steady solution

$$\psi_1(t) \approx -\frac{4}{3}\,\vartheta(t). \tag{160}$$

The angles

The angles of roll on the rear wave slope are thus

$$\overline{\psi}_{\substack{I\\II}} = \overline{\psi}_1 \pm |\overline{\psi}_2| = -\frac{4}{3}\overline{\vartheta} \pm |\overline{\psi}_2| . \tag{161}$$

The maximum roll angle $\overline{\psi}_{II}$ is directed towards the wave crest i.e. to lee. From (114) the wave steppness $|\overline{\vartheta}| = |\overline{\beta}|$ is equal to $2D(T_s)$ at the critical damping value. From (159) we can thus estimate the largest roll angle for simultaneous roll oscillations of the first and second kind

$$\overline{\psi}_{II} \approx \frac{8}{3} D(T_s) + \sqrt{\frac{\overline{\beta}_r}{a}} \tag{162}$$

In the higher harmonic combination $N_2(t)$ the dominating term is the third harmonic

$$\omega_3 = 3\,\frac{2\pi}{T_s} , \tag{163}$$

For the limiting case of a sinusoidal oscillogram this is, in fact, the only term. $N_2(t)$ has its maximum value $\overline{N}$ when the angular velocity $\dot{\psi}_2$ is a maximum. Using (118) the amplitude is approximately

$$\overline{N} \approx W(T_s) \cdot \sqrt{\frac{2\,\mathcal{U}(\overline{\psi}_2)}{\mathcal{J}'}} \tag{164}$$

for small $|\overline{\beta}_r|$.

If we now substitute

$$\nu_3 \approx \frac{2\pi}{T_s} \approx \sqrt{\frac{2\,\mathcal{U}(\bar{\varphi}_2)}{J' \cdot \bar{\varphi}_2^{\,2}}} \tag{165}$$

for the ship's natural frequency (see equ. (32)), the aaproximate steady solution of (158) in which the damping can be neglected as the exciting frequency (163) is well away from the resonance value, becomes

$$\varphi_3(t) \approx \frac{N_2(t)}{8J' \cdot \nu_3^{\,2}} \tag{166}$$

The amplitude is

$$\begin{aligned} \bar{\varphi}_3 &\approx \frac{W(T_s) \cdot \nu_2}{8J' \cdot \nu_3^{\,2}} \cdot \bar{\varphi}_2 \\ &\approx \tfrac{1}{8}\, D(T_s) \cdot \bar{\varphi}_2 \end{aligned} \tag{167}$$

As $D(T_s)$ seldom exceeds 0,2 for ships the amplitude $\bar{\varphi}_3$ of the higher harmonic generated by the difference moment $N_2(t)$ is only a few percent of the total roll amplitude.

II. The mean values $\eta_2(\bar{\psi})$ and $\eta_\beta(\bar{\psi})$.

We next investigate the influence of the reactive excitation $\beta_r(t)$ on the mean values $\eta_2(\psi)$ and $\eta_\beta(\bar{\psi})$ defined in (108) or (111) and (113) respectively. We consider first the integrals

$$\eta_2(\bar{\psi}) = \frac{2}{T_s} \cdot \int_0^{T_s} \left[1 - \mathcal{U}(\psi_2)\right] \cdot dt \tag{168}$$

and

$$\eta_\beta(\bar{\psi}) = \frac{2}{T_s} \int_0^{T_s} \frac{\frac{1}{2} J' \cdot \dot{\psi}_2^2}{\mathcal{U}(\bar{\psi})} \cdot dt \tag{169}$$

Transforming to the time differential of the free roll oscillation $\psi_0(t)$ according to (120) we obtain

$$\eta_2(\bar{\psi}) = \frac{2}{T_s} \cdot \int_0^{T_0(\bar{\psi})} \left[1 - \mathcal{U}(\psi_0)\right] \left[1 + \beta_0 - \bar{\beta}_r \cdot \mathcal{U}(\psi_0)\right]^{-1/2} \cdot dt \tag{170}$$

and with (118) and (117)

$$\eta_\beta(\bar{\psi}) = \frac{2}{T_s} \cdot \int_0^{T_0(\bar{\psi})} \left[1 - \mathcal{U}(\psi_0)\right] \cdot \left[1 + \beta_0 - \bar{\beta}_r \cdot \mathcal{U}(\psi_0)\right]^{+1/2} \cdot dt \tag{171}$$

Expanding the roots and using (108) and (126) we obtain to the first order in $\bar{\beta}_r$

$$\eta_2(\bar{\psi}) \approx \frac{2}{T_0(\bar{\psi})} \cdot \int_0^{T_0(\bar{\psi})} \left[1-\mathcal{U}(\psi_0)\right] \cdot \left[1+\tfrac{1}{4}\bar{\beta}_r\left(\eta_0(\bar{\psi})-2\left[1-\mathcal{U}(\psi_0)\right]\right)\right] \cdot dt \qquad (172)$$

and

$$\eta_\beta(\bar{\psi}) \approx \frac{2}{T_0(\bar{\psi})} \cdot \int_0^{T_0(\bar{\psi})} \left[1-\mathcal{U}(\psi_0)\right] \cdot \left[1-\tfrac{1}{4}\bar{\beta}_r \cdot \left(3\eta_0(\bar{\psi})-2\left[1-\mathcal{U}(\psi_0)\right]\right)\right] \cdot dt \qquad (173)$$

The expression involve the mean values of the function $\left[1-\mathcal{U}(\psi_0)\right]$ and its square.

Using equation (35) and introducing the symbol

$$\Delta = \frac{1}{T_0(\bar{\psi})} \cdot \int_0^{T_0(\bar{\psi})} \left[1-\mathcal{U}(\psi_0)\right]^2 \cdot dt \qquad (174)$$

we can write

$$\eta_2(\bar{\psi}) \approx \eta_0(\bar{\psi}) + \tfrac{1}{4}\bar{\beta}_r \cdot \eta_0^2(\bar{\psi}) - \bar{\beta}_r \cdot \Delta \qquad (175)$$

and

$$\eta_\beta(\bar{\psi}) \approx \eta_0(\bar{\psi}) - \tfrac{3}{4}\bar{\beta}_r \cdot \eta_0^2(\bar{\psi}) + \bar{\beta}_r \cdot \Delta \qquad (176)$$

Now the mean square of a value is always greater than the square of the mean value and further

$$0 \leq \left[1-u(\varphi_0)\right]^2 \leq \left[1-u(\varphi_0)\right] \leq 1 \tag{177}$$

so that the following bounds can be given for Δ :

$$\frac{1}{4}\eta_0^2(\bar{\varphi}) \leq \Delta \leq \frac{1}{2}\eta_0(\bar{\varphi}). \tag{178}$$

We can just as well write

$$\Delta = \frac{1}{2}\cdot\left(\frac{1}{4}\eta_0^2(\bar{\varphi})+\frac{1}{2}\eta_0(\bar{\varphi})\right)+\frac{1}{8}\delta \tag{179}$$

where

$$|\delta| \leq \eta_0(\bar{\varphi})\cdot\left(2-\eta_0(\bar{\varphi})\right) \leq 1. \tag{180}$$

The influence of the reactive excitation on $\eta_2(\bar{\varphi})$ and $\eta_\beta(\bar{\varphi})$ can thus be estimated by the formulae

$$\eta_2(\bar{\varphi}) \approx \eta_0(\bar{\varphi}) - \frac{1}{8}\sqrt{\beta_r}\cdot\left[\eta_0(\bar{\varphi})\cdot\left(2-\eta_0(\bar{\varphi})\right)+\delta\right] \tag{181}$$

and

$$\eta_\beta(\bar{\varphi}) \approx \eta_0(\bar{\varphi}) + \frac{1}{8}\sqrt{\beta_r}\cdot\left[\eta_0(\bar{\varphi})\left(2-5\eta_0(\bar{\varphi})\right)+\delta\right] \tag{182}$$

III. The convergence interval of the expansion (123).

To determine the convergence interval of the expansion (123) we write (181) in the form

$$\eta_2(\bar{\varphi}) = \eta_0(\bar{\varphi}) - A \cdot \bar{\beta}_r \tag{183}$$

with

$$A = \left[\frac{1}{8}\,\eta_0(\bar{\varphi}) \cdot (2 - \eta_0(\bar{\varphi})) + \delta\right] \tag{184}$$

The convergence condition (124) then becomes

$$\left| A \cdot \bar{\beta}_r^2 + m \cdot \bar{\beta}_r \right| < 1 \tag{185}$$

with

$$m = 1 - \eta_0(\bar{\varphi}) - u(\varphi_0) . \tag{186}$$

The convergence interval

$$\beta_I < \bar{\beta}_r < \beta_{II} \tag{187}$$

can thus be determined from the intersects of the family of parabolas

$$\left.\begin{array}{l} y_1 = A \cdot \bar{\beta}_r^{\,2} \\ 0 \le A \le \frac{1}{4} \end{array}\right\} \tag{188}$$

with the two families of straight lines

$$\left.\begin{aligned} y_{\frac{2}{3}} &= \pm 1 - m \cdot \overline{\beta}_r \\ -\eta_{max} &\leq m \leq 1. \end{aligned}\right\} \qquad (189)$$

The parameter intervals of A and m follow from (41), (103), and (180). The curve families are shown in fig. 17. The absissae of the two points on opposite sides of the y-axis which lie nearest to the axis are the limiting values $\beta_I < 0$ and $\beta_{II} > 0$ of the convergence interval, as all $\overline{\beta}_r$ lying between these limits satisfy condition (185). The intercepts and the corresponding convergence intervals are shown in the figure for $\eta_{max} = 1$ and $\eta_{max} = 1,5$. Having determined the limiting values of the parameters A and m from the intercepts nearest to the y-axis the convergence intervals were evuated numerically:

$-0,83 < \overline{\beta}_r < +0,83$ for $= 0,0$ (decreasing stiffness) (190)

(190)

$-0,60 < \overline{\beta}_r < +0,66$ for $= 1,5$ (increasing stiffness)