Chapter 17
Evolutionary Inclusions

This chapter is devoted to the study of *evolutionary inclusions*. In contrast to evolutionary equations, we will replace the skew-selfadjoint operator A by a so-called maximal monotone relation $A \subseteq H \times H$ in the Hilbert space H. The resulting problem is then no longer an equation, but just an inclusion; that is, we consider problems of the form

$$(u, f) \in \overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}, \tag{17.1}$$

where $f \in L_{2,\nu}(\mathbb{R}; H)$ is given and $u \in L_{2,\nu}(\mathbb{R}; H)$ is to be determined. This generalisation allows the treatment of certain non-linear problems, since we will not require any linearity for the relation A. Moreover, the property that A is just a relation and not neccessarily an operator can be used to treat hysteresis phenomena, which for instance occur in the theory of elasticity and electro-magnetism.

We begin to define the notion of maximal monotone relations in the first part of this chapter. In particular, we introduce the notion of the so-called Yosida approximation of A and provide a useful perturbation result for maximal monotone relations, which will be the key argument for proving the well-posedness of (17.1). For this, we prove the celebrated Theorem of Minty, which characterises the maximal monotone relations by a range condition. The second section is devoted to the main result of this chapter, namely the well-posedness of (17.1), which generalises Picard's theorem (see Theorem 6.2.1) to a broader class of problems. In the concluding section we consider Maxwell's equations in a polarisable medium as an application.

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17.1 Maximal Monotone Relations and the Theorem of Minty

Definition Let $A \subseteq H \times H$. We call A *monotone* if

$$\forall (u, v), (x, y) \in A : \operatorname{Re} \langle u - x, v - y\rangle \geqslant 0.$$

Moreover, we call A *maximal monotone* if A is monotone and for each monotone relation $B \subseteq H \times H$ with $A \subseteq B$ it follows that $A = B$.

Remark 17.1.1 Let $A \subseteq H \times H$ be a monotone relation.

(a) It is clear that A is maximal monotone if and only if for each $x, y \in H$ with

$$\forall (u, v) \in A : \operatorname{Re} \langle u - x, v - y\rangle \geqslant 0$$

it follows that $(x, y) \in A$.

(b) From (a) it follows that A is *demiclosed*; i.e., for each sequence $((x_n, y_n))_{n\in\mathbb{N}}$ in A with $x_n \to x$ in H and $y_n \to y$ weakly or $x_n \to x$ weakly and $y_n \to y$ in H for some $x, y \in H$ as $n \to \infty$ it follows that $(x, y) \in A$ (note that in both cases we have $\langle u - x_n, v - y_n\rangle \to \langle u - x, v - y\rangle$ for each $(u, v) \in A$).

We start to present some first properties of monotone and maximal monotone relations.

Proposition 17.1.2 *Let $A \subseteq H \times H$ be monotone and $\lambda > 0$. Then the following statements hold:*

(a) *The inverse relation $(1 + \lambda A)^{-1}$ is a Lipschitz-continuous mapping, which satisfies $\left\|(1 + \lambda A)^{-1}\right\|_{\mathrm{Lip}} \leqslant 1$.*

(b) *If $1 + \lambda A$ is onto, then A is maximal monotone.*

Proof For showing (a), we assume that $(f, u), (g, x) \in (1 + \lambda A)^{-1}$ for some $f, g, u, x \in H$. Then we find $v, y \in H$ such that $(u, v), (x, y) \in A$ and $u + \lambda v = f$ as well as $x + \lambda y = g$. The monotonicity of A then yields

$$\|u - x\|^2 = \operatorname{Re} \langle f - g - \lambda(v - y), u - x\rangle \leqslant \operatorname{Re} \langle f - g, u - x\rangle \leqslant \|f - g\| \|u - x\| .$$

If now $f = g$, then $u = x$. Hence, $(1 + \lambda A)^{-1}$ is a mapping and the inequality proves its Lipschitz-continuity with $\left\|(1 + \lambda A)^{-1}\right\|_{\mathrm{Lip}} \leqslant 1$.

To prove (b), let $B \subseteq H \times H$ be monotone with $A \subseteq B$ and let $(x, y) \in B$. Since $1 + \lambda A$ is onto, we find $(u, v) \in A \subseteq B$ such that $u + \lambda v = x + \lambda y$. Since $(1 + \lambda B)^{-1}$ is a mapping by (a), we infer that

$$x = (1 + \lambda B)^{-1}(x + \lambda y) = (1 + \lambda B)^{-1}(u + \lambda v) = u$$

and hence, also $v = y$, which proves that $(x, y) \in A$ and thus, $A = B$. □

Example 17.1.3 Let $B\colon \operatorname{dom}(B) \subseteq H \to H$ be a densely defined, closed linear operator. Assume $\operatorname{Re}\langle u, Bu\rangle \geqslant 0$ and $\operatorname{Re}\langle v, B^*v\rangle \geqslant 0$ for all $u \in \operatorname{dom}(B)$ and $v \in \operatorname{dom}(B^*)$. Then B is maximal monotone. Indeed, the monotonicity follows from the linearity of B and by Proposition 6.3.1 the operator $1 + B$ is continuously invertible, hence onto. Thus, the maximal monotonicity follows by Proposition 17.1.2(b). In particular, every skew-selfadjoint operator is maximal monotone. Moreover, if $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ is a material law such that there exist $c > 0$, $\nu_0 \geqslant \mathrm{s_b}(M)$ with

$$\operatorname{Re}\langle zM(z)\phi, \phi\rangle \geqslant c\,\|\phi\|^2 \quad (\phi \in H, z \in \mathbb{C}_{\operatorname{Re}\geqslant\nu_0}),$$

then $\partial_{t,\nu}M(\partial_{t,\nu}) - c$ is maximal monotone for each $\nu \geqslant \nu_0$.

Our first goal is to show that the implication in Proposition 17.1.2(b) is actually an equivalence. This is Minty's theorem. For this, we start to introduce subgradients of convex, proper, lower semi-continuous mappings, which form the probably most prominent example of maximal monotone relations.

Definition Let $f\colon H \to (-\infty, \infty]$. We call f

(a) *convex* if for all $x, y \in H$, $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y).$$

(b) *proper* if there exists $x \in H$ with $f(x) < \infty$.
(c) *lower semi-continuous (l.s.c.)* if for each $c \in \mathbb{R}$ the sublevel set

$$[f \leqslant c] = \{x \in H\,;\ f(x) \leqslant c\}$$

is closed.
(d) *coercive* if for each $c \in \mathbb{R}$ the sublevel set $[f \leqslant c]$ is bounded.

Remark 17.1.4 If $f\colon H \to (-\infty, \infty]$ is convex, the sublevel sets $[f \leqslant c]$ are convex for each $c \in \mathbb{R}$. Hence, if f is convex, l.s.c. and coercive, the sets $[f \leqslant c]$ are weakly sequentially compact (or, by the Eberlein–Šmulian theorem [50, theorem 13.1], equivalently, weakly compact) for each $c \in \mathbb{R}$. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $[f \leqslant c]$ for some $c \in \mathbb{R}$, then it is bounded and thus, posseses a weakly convergent subsequence with weak limit $x \in H$. Since $[f \leqslant c]$ is closed and convex, Mazur's theorem [50, Corollary 2.11] yields that it is weakly closed and thus, $x \in [f \leqslant c]$ proving the claim.

Definition Let $f\colon H \to (-\infty, \infty]$ be convex. We define the *subgradient* of f by

$$\partial f \coloneqq \{(x, y) \in H \times H\,;\ \forall u \in H:\ f(u) \geqslant f(x) + \operatorname{Re}\langle y, u - x\rangle\}.$$

Remark 17.1.5 Note that $u \mapsto f(x) + \operatorname{Re}\langle y, u - x\rangle$ is an affine function touching the graph of f in x. Thus, the subgradient is the set of all pairs $(x, y) \in H$ such that there exists an affine function with slope y touching the graph of f in x. It is not hard to show that if f is differentiable in x, then $(x, y) \in \partial f$ if and only if $y = f'(x)$ (see Exercise 17.1). Thus, the subgradient of f provides a generalisation of the derivative for arbitrary convex functions.

Proposition 17.1.6 *Let $f : H \to (-\infty, \infty]$ be convex and proper. Then the following statements hold:*

(a) *If $(x, y) \in \partial f$, then $f(x) < \infty$. Moreover, the subgradient ∂f is monotone.*
(b) *If f is l.s.c. and coercive, then there exists $x \in H$ such that $f(x) = \inf_{u\in H} f(u)$.*
(c) *Let $\alpha \geqslant 0$, $x, y \in H$ and $g : H \to (-\infty, \infty]$ with $g(u) := \frac{\alpha}{2}\|u - y\|^2 + f(u)$ for $u \in H$. Then $g(x) = \inf_{u\in H} g(u)$ if and only if $(x, \alpha(y - x)) \in \partial f$.*
(d) *Let $\alpha > 0$ and $y \in H$. If f is l.s.c., then $g : H \to (-\infty, \infty]$ with $g(u) := \frac{\alpha}{2}\|u - y\|^2 + f(u)$ for $u \in H$ is convex, proper, l.s.c and coercive. In particular $1 + \alpha\partial f$ is onto and hence, ∂f is maximal monotone.*

Proof

(a) If $(x, y) \in \partial f$ we have $f(u) \geqslant f(x) + \operatorname{Re}\langle y, u - x\rangle$ for each $u \in H$. Since f is proper, we find $u \in H$ such that $f(u) < \infty$ and hence, also $f(x) < \infty$. Let now $(u, v), (x, y) \in \partial f$. Then we have $f(u) \geqslant f(x) + \operatorname{Re}\langle y, u - x\rangle$ and $f(x) \geqslant f(u) + \operatorname{Re}\langle v, x - u\rangle = f(u) - \operatorname{Re}\langle v, u - x\rangle$. Summing up both expressions (note that $f(x), f(u) < \infty$ by what we have shown before), we infer

$$\operatorname{Re}\langle y - v, u - x\rangle \leqslant 0,$$

which shows the monotonicity.
(b) Let $(x_n)_{n\in\mathbb{N}}$ in H with $f(x_n) \to \inf_{u\in H} f(u) =: d$. Note that $d \in \mathbb{R}$, since f is proper. Without loss of generality, we can assume that $x_n \in [f \leqslant d + 1]$ for each $n \in \mathbb{N}$ and by Remark 17.1.4 we can assume that $x_n \to x$ weakly as $n \to \infty$ for some $x \in H$. Let $\varepsilon > 0$. Since $x_n \in [f \leqslant d + \varepsilon]$ for sufficiently large $n \in \mathbb{N}$, we derive $x \in [f \leqslant d + \varepsilon]$ again by Remark 17.1.4 and so, $f(x) \leqslant d + \varepsilon$ for each $\varepsilon > 0$, showing the claim.
(c) Assume that $g(x) = \inf_{u\in H} g(u)$ and let $u \in H$. Since f is proper, so is g and thus, we have $g(x) < \infty$, which in turn gives $f(x) < \infty$. Let $\lambda \in (0, 1]$ and set $w := \lambda u + (1 - \lambda)x$. Then the convexity of f yields

$$\begin{aligned}\lambda (f(u) - f(x)) \geqslant f(w) - f(x)& \\ &= g(w) - g(x) + \frac{\alpha}{2}(\|x - y\|^2 - \|w - y\|^2) \\ &\geqslant \frac{\alpha}{2}(\|x - y\|^2 - \|w - y\|^2)\end{aligned}$$

$$= \frac{\alpha}{2}\big(\|x-y\|^2 - \|\lambda(u-x)+x-y\|^2 \big)$$
$$= \frac{\alpha}{2}\big(-2\lambda \operatorname{Re}\langle u-x, x-y\rangle - \lambda^2 \|u-x\|^2 \big).$$

Dividing the latter expression by λ and taking the limit $\lambda \to 0$, we infer

$$-\alpha \operatorname{Re}\langle u-x, x-y\rangle \leqslant f(u) - f(x),$$

which proves $(x, \alpha(y-x)) \in \partial f$.
Assume now that $(x, \alpha(y-x)) \in \partial f$. For each $u \in H$ we have

$$\|x-y\|^2 - 2\operatorname{Re}\langle y-x, u-x\rangle = \|y-x-(u-x)\|^2 - \|u-x\|^2 \leqslant \|u-y\|^2$$

and thus,

$$f(u) \geqslant f(x) + \operatorname{Re}\langle \alpha(y-x), u-x\rangle \geqslant f(x) + \frac{\alpha}{2}\big(\|x-y\|^2 - \|u-y\|^2 \big),$$

which shows the claim.

(d) We first show that there exists an affine function $h\colon H \to \mathbb{R}$ with $h \leqslant f$. For this, we consider the epigraph of f given by

$$\operatorname{epi} f := \{(x,\beta) \in H \times \mathbb{R};\ f(x) \leqslant \beta\}.$$

Since f is convex and l.s.c., one easily verifies that epi f is convex and closed. Moreover, since f is proper, epi $f \neq \varnothing$. Let now $z \in H$ with $f(z) < \infty$ and $\eta < f(z)$. Then $(z, \eta) \in (H \times \mathbb{R}) \setminus \operatorname{epi} f$ and by the Hahn–Banach theorem we find $w \in H$ and $\gamma \in \mathbb{R}$ such that

$$\operatorname{Re}\langle w, z\rangle + \gamma\eta < \operatorname{Re}\langle w, x\rangle + \gamma\beta$$

for all $(x, \beta) \in \operatorname{epi} f$. In particular

$$\operatorname{Re}\langle w, z\rangle + \gamma\eta < \operatorname{Re}\langle w, x\rangle + \gamma f(x)$$

for each $x \in H$ and since this holds also for $x = z$, we infer $\gamma > 0$. Choosing $h(x) := \frac{1}{\gamma}\operatorname{Re}\langle w, z-x\rangle + \eta$ for $x \in H$, we have found the asserted affine function.

Using this, we have that

$$g(u) \geqslant \frac{\alpha}{2}\|u-y\|^2 + h(u) \quad (u \in H)$$

and since the right-hand side tends to ∞ as $\|u\| \to \infty$, we derive that g is coercive. Moreover, g is convex, proper and l.s.c. (see Exercise 17.2) and thus, there exists $x \in H$ with $g(x) = \inf_{u\in H} g(u)$ by (b). By (c), $(x, \alpha(y - x)) \in \partial f$ and thus, $(x, y) \in 1 + \alpha\partial f$. Since $y \in H$ was arbitrary, $1 + \alpha\partial f$ is onto and so, ∂f is maximal monotone by (a) and Proposition 17.1.2(b). □

We can now prove Minty's theorem.

Theorem 17.1.7 (Minty) *Let $A \subseteq H \times H$ maximal monotone. Then $1+\lambda A$ is onto for all $\lambda > 0$.*

Proof Since λA is maximal monotone for each $\lambda > 0$, it suffices to prove the statement for $\lambda = 1$. Moreover, since $A - (0, f)$ is maximal monotone for each $f \in H$, it suffices to show $0 \in \operatorname{ran}(1+A)$. For this, define $f_A \colon H \times H \to (-\infty, \infty]$ by (note that $A \neq \varnothing$ by maximal monotonicity)

$$f_A(u, v) := \sup \{\operatorname{Re}\langle u, y\rangle + \operatorname{Re}\langle v, x\rangle - \operatorname{Re}\langle x, y\rangle \ ; \ (x, y) \in A\} .$$

As a supremum of affine functions, we see that f_A is convex and l.s.c. Moreover, we have that

$$\begin{aligned} f_A(u, v) &= -\inf \{-\operatorname{Re}\langle u, y\rangle - \operatorname{Re}\langle v, x\rangle + \operatorname{Re}\langle x, y\rangle \ ; \ (x, y) \in A\} \\ &= -\inf \{\operatorname{Re}\langle x - u, y - v\rangle \ ; \ (x, y) \in A\} + \operatorname{Re}\langle u, v\rangle \end{aligned}$$

for each $u, v \in H$ and since A is maximal monotone, we get by using Remark 17.1.1

$$\begin{aligned} \inf \{\operatorname{Re}\langle x - u, y - v\rangle \ ; \ (x, y) \in A\} \geqslant 0 &\Leftrightarrow (u, v) \in A \\ &\Leftrightarrow \inf \{\operatorname{Re}\langle x - u, y - v\rangle \ ; \ (x, y) \in A\} = 0 \end{aligned}$$

and so

$$\inf \{\operatorname{Re}\langle x - u, y - v\rangle \ ; \ (x, y) \in A\} \leqslant 0 \quad (u, v \in H).$$

In particular, we get $f_A(u, v) \geqslant \operatorname{Re}\langle u, v\rangle$ for each $u, v \in H$ and $f_A(u, v) = \operatorname{Re}\langle u, v\rangle$ if and only if $(u, v) \in A$. Thus, f_A is proper since $A \neq \varnothing$. By Proposition 17.1.6(d) we obtain that $0 \in \operatorname{ran}(1 + \partial f_A)$ and thus, we find $(u_0, v_0) \in H \times H$ with $((u_0, v_0), (-u_0, -v_0)) \in \partial f_A$. Hence, by definition of ∂f_A,

$$\begin{aligned} f_A(u, v) &\geqslant f_A(u_0, v_0) + \operatorname{Re}\langle (-u_0, -v_0), (u - u_0, v - v_0)\rangle \\ &= f_A(u_0, v_0) + \|u_0\|^2 + \|v_0\|^2 - \operatorname{Re}\langle u_0, u\rangle - \operatorname{Re}\langle v_0, v\rangle \end{aligned}$$

for all $(u, v) \in H \times H$. In particular, using that $f_A(u, v) = \operatorname{Re}\langle u, v\rangle$ for $(u, v) \in A$ we get

$$0 \geqslant f_A(u_0, v_0) + \|u_0\|^2 + \|v_0\|^2 - \operatorname{Re}\langle u_0, u\rangle - \operatorname{Re}\langle v_0, v\rangle - \operatorname{Re}\langle u, v\rangle \quad ((u, v) \in A).$$

Taking the supremum over all $(u, v) \in A$, we infer

$$\begin{aligned} 0 &\geqslant f_A(u_0, v_0) + \|u_0\|^2 + \|v_0\|^2 + f_A(-u_0, -v_0), \\ &\geqslant \operatorname{Re}\langle u_0, v_0\rangle + \|u_0\|^2 + \|v_0\|^2 + \operatorname{Re}\langle -u_0, -v_0\rangle = \|u_0 + v_0\|^2 \end{aligned}$$

Thus, $u_0 + v_0 = 0$ and instead of inequalities, we actually have equalities in the expression above. Thus, $f_A(u_0, v_0) = \operatorname{Re}\langle u_0, v_0\rangle$ and so, $(u_0, v_0) \in A$. From $u_0 + v_0 = 0$ it thus follows that $0 \in \operatorname{ran}(1 + A)$. □

Next, we show how to extend maximal monotone relations on a Hilbert space H to the Bochner–Lebesgue space $L_2(\mu; H)$ for a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. The condition $(0, 0) \in A$ can be dropped if $\mu(\Omega) < \infty$.

Corollary 17.1.8 *Let $A \subseteq H \times H$ maximal monotone with $(0, 0) \in A$. Moreover, let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and define*

$$A_{L_2(\mu;H)} := \{(f, g) \in L_2(\mu; H) \times L_2(\mu; H)\,;\ (f(t), g(t)) \in A \quad (t \in \Omega\ a.e.)\}\,.$$

Then $A_{L_2(\mu;H)}$ is maximal monotone.

Proof The monotonicity of $A_{L_2(\mu;H)}$ is clear. For showing the maximal monotonicity we prove that $1 + A_{L_2(\mu;H)}$ is onto (see Proposition 17.1.2(b)). For this, let $h \in L_2(\mu; H)$ and set $f(t) := (1 + A)^{-1}(h(t))$ for each $t \in \Omega$. Note that f is well-defined by Theorem 17.1.7. Since $(1 + A)^{-1}$ is continuous by Proposition 17.1.2(a) and h is Bochner-measurable, f is also Bochner-measurable. Moreover, using that $(0, 0) \in 1 + A$ and $\left\|(1 + A)^{-1}\right\|_{\mathrm{Lip}} \leqslant 1$, we compute

$$\int_\Omega \|f(t)\|^2 \,\mathrm{d}\mu(t) \leqslant \int_\Omega \|h(t)\|^2 \,\mathrm{d}\mu(t) < \infty$$

and so, $f \in L_2(\mu; H)$. Thus, $h - f \in L_2(\mu; H)$, which yields $(f, h - f) \in A_{L_2(\mu;H)}$ and so, $h \in \operatorname{ran}(1 + A_{L_2(\mu;H)})$. □

17.2 The Yosida Approximation and Perturbation Results

We now have all concepts at hand to introduce the Yosida approximation for a maximal monotone relation.

Definition Let $A \subseteq H \times H$ be maximal monotone and $\lambda > 0$. We define

$$A_\lambda := \lambda^{-1}\left(1 - (1 + \lambda A)^{-1}\right).$$

The family $(A_\lambda)_{\lambda>0}$ is called *Yosida approximation of A*.

Since for a maximal monotone relation $A \subseteq H \times H$ the resolvent $(1+\lambda A)^{-1}$ is actually a Lipschitz-continuous mapping (by Proposition 17.1.2(a)), whose domain is H (by Theorem 17.1.7), the same holds for A_λ. We collect some useful properties of the Yosida approximation.

Proposition 17.2.1 *Let $A \subseteq H \times H$ maximal monotone and $\lambda > 0$. Then the following statements hold:*

(a) *For all $x \in H$ we have $\big((1+\lambda A)^{-1}(x), A_\lambda(x)\big) \in A$.*
(b) *A_λ is monotone and $\|A_\lambda\|_{\mathrm{Lip}} \leqslant \frac{1}{\lambda}$.*

Proof

(a) For all $x \in H$ we have that $\big((1+\lambda A)^{-1}(x), x\big) \in 1+\lambda A$, and therefore, $\big((1+\lambda A)^{-1}(x), A_\lambda(x)\big) \in A$.
(b) Let $x, y \in H$. Then we compute

$$\begin{aligned}
&\lambda \operatorname{Re} \langle A_\lambda(x) - A_\lambda(y), x-y\rangle \\
&\quad = \|x-y\|^2 - \operatorname{Re}\Big\langle (1+\lambda A)^{-1}(x) - (1+\lambda A)^{-1}(y), x-y\Big\rangle \\
&\quad \geqslant \|x-y\|^2 - \Big\|(1+\lambda A)^{-1}(x) - (1+\lambda A)^{-1}(y)\Big\| \, \|x-y\| \\
&\quad \geqslant 0
\end{aligned}$$

by Proposition 17.1.2(a) and hence, A_λ is monotone. Moreover,

$$\begin{aligned}
&\operatorname{Re} \langle A_\lambda(x) - A_\lambda(y), x-y\rangle \\
&\quad = \operatorname{Re}\Big\langle A_\lambda(x) - A_\lambda(y), (1+\lambda A)^{-1}(x) - (1+\lambda A)^{-1}(y)\Big\rangle \\
&\qquad + \lambda \, \|A_\lambda(x) - A_\lambda(y)\|^2 \\
&\quad \geqslant \lambda \, \|A_\lambda(x) - A_\lambda(y)\|^2 ,
\end{aligned}$$

where we have used (a) and the monotonicity of A. The Cauchy–Schwarz inequality now yields $\|A_\lambda\|_{\mathrm{Lip}} \leqslant \frac{1}{\lambda}$. □

We state a result on the strong convergence of the resolvents of a maximal monotone relation, which we already have used in previous sections for the resolvent of $\partial_{t,\nu}$. For the projection $P_C(x)$ of $x \in H$ onto a non-empty closed convex set $C \subseteq H$, recall Exercise 4.4 and that $y = P_C(x)$ if and only if $y \in C$ and

$$\operatorname{Re} \langle x-y, u-y\rangle_H \leqslant 0 \quad (u \in C).$$

Proposition 17.2.2 *Let $A \subseteq H \times H$ be maximal monotone. Then $\overline{\operatorname{dom}(A)}$ is convex and for all $x \in H$ we have $(1+\lambda A)^{-1}(x) \to P_{\overline{\operatorname{dom}(A)}}(x)$ as $\lambda \to 0+$, where $P_{\overline{\operatorname{dom}(A)}}$ denotes the projection onto $\overline{\operatorname{dom}(A)}$.*

Proof We set $C := \overline{\operatorname{conv}\operatorname{dom}(A)}$. Then C is closed and convex. Next, we prove that $(1+\lambda A)^{-1}(x) \to P_C(x)$ as $\lambda \to 0+$ for all $x \in H$. So let $x \in H$ and set $x_\lambda := (1+\lambda A)^{-1}(x)$ for each $\lambda > 0$. Then we have $A_\lambda(x) = \frac{1}{\lambda}(x - x_\lambda)$ and hence, using Proposition 17.2.1(a) and the monotonicity of A, we infer $\operatorname{Re}\left\langle x_\lambda - u, \frac{1}{\lambda}(x - x_\lambda) - v\right\rangle \geqslant 0$ for each $(u, v) \in A$. Consequently, we obtain

$$\|x_\lambda\|^2 \leqslant \operatorname{Re}\langle x_\lambda - u, x\rangle + \operatorname{Re}\langle x_\lambda, u\rangle - \lambda \operatorname{Re}\langle x_\lambda - u, v\rangle \quad ((u, v) \in A). \tag{17.2}$$

In particular, we see that $(x_\lambda)_{\lambda>0}$ is bounded as $\lambda \to 0$ and so, for each nullsequence we find a subsequence $(\lambda_n)_n$ with $\lambda_n \to 0$ such that $x_{\lambda_n} \to z$ weakly for some $z \in H$. By (17.2) it follows that

$$\|z\|^2 \leqslant \operatorname{Re}\langle z - u, x\rangle + \operatorname{Re}\langle z, u\rangle \quad (u \in \operatorname{dom}(A)).$$

It is easy to see that this inequality carries over to each $u \in C$ and thus $\operatorname{Re}\langle z - u, z - x\rangle \leqslant 0$ for each $u \in C$ which proves $z = P_C(x)$ and hence, $x_{\lambda_n} \to P_C(x)$ weakly. Next we prove that the convergence also holds in the norm topology. From (17.2) we see that

$$\limsup_{n\to\infty} \left\| x_{\lambda_n} \right\|^2 \leqslant \operatorname{Re}\langle P_C(x) - u, x\rangle + \operatorname{Re}\langle P_C(x), u\rangle \quad (u \in \operatorname{dom}(A))$$

and again, this inequality stays true for each $u \in C$. In particular, choosing $u = P_C(x)$ we infer $\limsup_{n\to\infty} \left\| x_{\lambda_n} \right\|^2 \leqslant \|P_C(x)\|^2$, which together with the weak convergence, yields the convergence in norm (see Exercise 17.3). A subsequence argument (cf. Exercise 14.3) reveals $x_\lambda \to P_C(x)$ in H as $\lambda \to 0$.
It remains to show that $\overline{\operatorname{dom}(A)}$ is convex. By what we have shown above, we have $(1+\lambda A)^{-1}(x) \to x$ as $\lambda \to 0$ for each $x \in C$ and since $(1+\lambda A)^{-1}(x) \in \operatorname{dom}(A)$ for each $\lambda > 0$, we infer $x \in \overline{\operatorname{dom}(A)}$. Thus, $C \subseteq \overline{\operatorname{dom}(A)}$ and since the other inclusion holds trivially the proof is completed. □

We conclude this section with some perturbation results.

Lemma 17.2.3 *Let $A \subseteq H \times H$ be maximal monotone and $C \colon H \to H$ Lipschitz-continuous and monotone. Then $A + C$ is maximal monotone.*

Proof The monotonicity of $A + C$ is clear. If C is constant, then the maximality of $A + C$ is obvious. If C is non-constant we choose $0 < \lambda < \frac{1}{\|C\|_{\mathrm{Lip}}}$. Then for all $f \in H$ the mapping

$$u \mapsto (1+\lambda A)^{-1}(f - \lambda C(u))$$

defines a strict contraction (use Proposition 17.1.2(a) and $\operatorname{dom}((1+\lambda A)^{-1}) = H$ by Theorem 17.1.7) and thus, posseses a fixed point $x \in H$, which then satisfies $(x, f) \in 1+\lambda(A+C)$. Thus, $A+C$ is maximal monotone by Proposition 17.1.2(b). □

We note that the latter lemma particularily applies to $C = B_\lambda$ for a maximal monotone relation $B \subseteq H \times H$ and $\lambda > 0$ by Proposition 17.2.1(b).

Proposition 17.2.4 *Let $A, B \subseteq H \times H$ be two maximal monotone relations, $c > 0$ and $f \in H$. For $\lambda > 0$ we set*

$$x_\lambda := (c + A + B_\lambda)^{-1}(f).$$

Then $f \in \operatorname{ran}(c + A + B)$ if and only if $\sup_{\lambda>0} \|B_\lambda(x_\lambda)\| < \infty$ and in the latter case $x_\lambda \to x$ as $\lambda \to 0$ with $(x, f) \in c + A + B$, which identifies x uniquely.

Proof Note that x_λ is well-defined for $\lambda > 0$ by Lemma 17.2.3, Theorem 17.1.7 and Proposition 17.1.2.

For all $\lambda > 0$ we find $y_\lambda \in H$ such that $(x_\lambda, y_\lambda) \in A$ and $cx_\lambda + y_\lambda + B_\lambda(x_\lambda) = f$.

We first assume that there exist $x, y, z \in H$ such that $(x, y) \in A$, $(x, z) \in B$ and $cx + y + z = f$. Thus, we have

$$c(x - x_\lambda) = y_\lambda + B_\lambda(x_\lambda) - y - z,$$

which gives

$$\begin{aligned} 0 \leqslant c\,\|x_\lambda - x\|^2 &= \operatorname{Re}\langle y - y_\lambda, x_\lambda - x\rangle + \operatorname{Re}\langle z - B_\lambda(x_\lambda), x_\lambda - x\rangle \\ &\leqslant \operatorname{Re}\langle z - B_\lambda(x_\lambda), x_\lambda - x\rangle \\ &= \operatorname{Re}\Big\langle z - B_\lambda(x_\lambda), (1+\lambda B)^{-1}(x_\lambda) - x\Big\rangle + \operatorname{Re}\langle z - B_\lambda(x_\lambda), \lambda B_\lambda(x_\lambda)\rangle \\ &\leqslant \operatorname{Re}\langle z - B_\lambda(x_\lambda), \lambda B_\lambda(x_\lambda)\rangle \end{aligned}$$

where we have used the monotonicity of A in the second line and the monotonicity of B as well as Proposition 17.2.1(a) in the last line. The latter implies

$$\|B_\lambda(x_\lambda)\|^2 \leqslant \operatorname{Re}\langle z, B_\lambda(x_\lambda)\rangle\,,$$

and the claim follows by the Cauchy–Schwarz inequality.

Assume now that $K := \sup_{\lambda>0} \|B_\lambda(x_\lambda)\| < \infty$ and let $\mu, \lambda > 0$. As above, we compute

$$\begin{aligned} c\,\|x_\lambda - x_\mu\|^2 &= \operatorname{Re}\big\langle y_\mu - y_\lambda, x_\lambda - x_\mu\big\rangle + \operatorname{Re}\big\langle B_\mu(x_\mu) - B_\lambda(x_\lambda), x_\lambda - x_\mu\big\rangle \\ &\leqslant \operatorname{Re}\big\langle B_\mu(x_\mu) - B_\lambda(x_\lambda), x_\lambda - x_\mu\big\rangle \end{aligned}$$

$$\begin{aligned}
&= \operatorname{Re}\Big\langle B_\mu(x_\mu) - B_\lambda(x_\lambda), (1+\lambda B)^{-1}(x_\lambda) - (1+\mu B)^{-1}(x_\mu)\Big\rangle \\
&\quad + \operatorname{Re}\big\langle B_\mu(x_\mu) - B_\lambda(x_\lambda), \lambda B_\lambda(x_\lambda) - \mu B_\mu(x_\mu)\big\rangle \\
&\leqslant \operatorname{Re}\big\langle B_\mu(x_\mu) - B_\lambda(x_\lambda), \lambda B_\lambda(x_\lambda) - \mu B_\mu(x_\mu)\big\rangle \\
&\leqslant 2(\lambda+\mu)K^2.
\end{aligned}$$

Thus, for a nullsequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ we infer that $(x_{\lambda_n})_{n\in\mathbb{N}}$ is a Cauchy sequence whose limit we denote by x. Since $(B_{\lambda_n}(x_{\lambda_n}))_{n\in\mathbb{N}}$ is bounded, we can assume, by passing to a suitable subsequence, that $B_{\lambda_n}(x_{\lambda_n}) \to z$ weakly for some $z \in H$. Then

$$\Big\|(1+\lambda_n B)^{-1}(x_{\lambda_n}) - x\Big\| \leqslant \big\|x_{\lambda_n} - x\big\| + \big\|\lambda_n B_{\lambda_n}(x_{\lambda_n})\big\| \to 0 \quad (n\to\infty)$$

and since $((1+\lambda_n B)^{-1}(x_{\lambda_n}), B_{\lambda_n}(x_{\lambda_n})) \in B$ for each $n \in \mathbb{N}$ by Proposition 17.2.1(a), the demi-closedness of B (see Remark 17.1.1) reveals $(x,z) \in B$. Moreover,

$$y_{\lambda_n} = f - B_{\lambda_n}(x_{\lambda_n}) - cx_{\lambda_n} \to f - z - cx =: y \quad (n\to\infty)$$

weakly and hence, by the demi-closedness of A, we infer $(x,y) \in A$, which completes the proof of the asserted equivalence. By a subsequence argument (cf. Exercise 14.3) we obtain the asserted convergence (note that $x = (c+A+B)^{-1}(f)$ is uniquely determined by f). □

To treat the example in Sect. 17.4 we need another perturbation result, for which we need to introduce the notion of local boundedness of a relation.

Definition Let $A \subseteq H \times H$ and $x \in \operatorname{dom}(A)$. Then A is called *locally bounded at* x if there exists $\delta > 0$ such that

$$A[B(x,\delta)] = \{y \in H\,;\ \exists z \in B(x,\delta) : (z,y) \in A\}$$

is bounded.

Proposition 17.2.5 *Let* $A \subseteq H \times H$ *be maximal monotone such that* $\operatorname{int}\operatorname{conv}\operatorname{dom}(A) \neq \varnothing$. *Then* $\operatorname{int}\operatorname{dom}(A) = \operatorname{int}\operatorname{conv}\operatorname{dom}(A) = \operatorname{int}\overline{\operatorname{dom}(A)}$ *and* A *is locally bounded at each point* $x \in \operatorname{int}\operatorname{dom}(A)$.

In order to prove this proposition, we need the following lemma.

Lemma 17.2.6 *Let* $(D_n)_{n\in\mathbb{N}}$ *be a sequence of subsets of* H *with* $D_n \subseteq D_{n+1}$ *for each* $n \in \mathbb{N}$ *and* $D := \bigcup_{n\in\mathbb{N}} D_n$. *If* $\operatorname{int}\operatorname{conv} D \neq \varnothing$, *then* $\operatorname{int}\operatorname{conv} D = \bigcup_{n\in\mathbb{N}} \operatorname{int}\overline{\operatorname{conv} D_n}$.

Proof Set $C := \operatorname{int}\operatorname{conv} D$. By Exercise 17.4 we have $\overline{C} = \overline{\operatorname{conv} D}$. Since $(D_n)_{n\in\mathbb{N}}$ is increasing we have $\operatorname{conv} D = \bigcup_{n\in\mathbb{N}} \operatorname{conv} D_n$ and hence, $C \subseteq \bigcup_{n\in\mathbb{N}} \overline{\operatorname{conv} D_n} \subseteq \overline{C}$. Since C is a Baire space by Exercise 17.5, we find $n_0 \in \mathbb{N}$ such that $\operatorname{int}\overline{\operatorname{conv} D_{n_0}} \neq \varnothing$ and hence, $\operatorname{int}\overline{\operatorname{conv} D_n} \neq \varnothing$ for each $n \geqslant n_0$. Hence, $\overline{\operatorname{conv} D_n} = \overline{\operatorname{int}\overline{\operatorname{conv} D_n}}$ for each $n \geqslant n_0$ by Exercise 17.4. Thus,

$$\overline{C} = \overline{\bigcup_{n\in\mathbb{N}} \overline{\operatorname{conv} D_n}} = \overline{\bigcup_{n\in\mathbb{N}} \overline{\operatorname{int}\overline{\operatorname{conv} D_n}}} = \overline{\bigcup_{n\in\mathbb{N}} \operatorname{int}\overline{\operatorname{conv} D_n}}.$$

Finally, since $\bigcup_{n\in\mathbb{N}} \operatorname{int}\overline{\operatorname{conv} D_n}$ is open and convex, we infer $C = \bigcup_{n\in\mathbb{N}} \operatorname{int}\overline{\operatorname{conv} D_n}$ by Exercise 17.4. □

Proof of Proposition 17.2.5 We first show that A is locally bounded at each point in $\operatorname{int}\operatorname{conv}\operatorname{dom}(A)$. For this, we set

$$A_n := \{(x, y) \in A\,;\ \|x\|, \|y\| \leqslant n\} \quad (n \in \mathbb{N}).$$

Then $\operatorname{dom}(A) = \bigcup_{n\in\mathbb{N}} \operatorname{dom}(A_n)$ and $\operatorname{dom}(A_n) \subseteq \operatorname{dom}(A_{n+1})$ for each $n \in \mathbb{N}$. Since $\operatorname{int}\operatorname{conv}\operatorname{dom}(A) \neq \varnothing$, Lemma 17.2.6 gives $\operatorname{int}\operatorname{conv}\operatorname{dom}(A) = \bigcup_{n\in\mathbb{N}} \operatorname{int}\overline{\operatorname{conv}\operatorname{dom}(A_n)}$. Thus, it suffices to show that A is locally bounded at each $x \in \operatorname{int}\overline{\operatorname{conv}\operatorname{dom}(A_n)}$ for each $n \in \mathbb{N}$. So, let $x \in \operatorname{int}\overline{\operatorname{conv}\operatorname{dom}(A_n)}$ for some $n \in \mathbb{N}$. Then we find $\delta > 0$ such that $B[x, \delta] \subseteq \overline{\operatorname{conv}\operatorname{dom}(A_n)}$. We show that $A[B(x, \frac{\delta}{2})]$ is bounded. So, let $(u, v) \in A$ with $\|u - x\| < \frac{\delta}{2}$ and note that $u \in \overline{\operatorname{conv}\operatorname{dom}(A_n)} \subseteq B[0, n]$. Then for each $(a, b) \in A_n$ we have $\operatorname{Re}\langle u - a, v - b\rangle \geqslant 0$ and thus

$$\begin{aligned}\operatorname{Re}\langle a - u, v\rangle &= \operatorname{Re}\langle a - u, v - b\rangle + \operatorname{Re}\langle a - u, b\rangle \\ &\leqslant \operatorname{Re}\langle a - u, b\rangle \leqslant 2n^2 \quad (a \in \operatorname{dom}(A_n)).\end{aligned}$$

Clearly, this inequality carries over to each $a \in \overline{\operatorname{conv}\operatorname{dom}(A_n)}$. If $v \neq 0$ we choose $a := \frac{\delta}{2\|v\|} v + u \in B[u, \frac{\delta}{2}] \subseteq B[x, \delta] \subseteq \overline{\operatorname{conv}\operatorname{dom}(A_n)}$, and obtain

$$\|v\| \leqslant \frac{4n^2}{\delta},$$

which shows the boundedness of $A[B(x, \frac{\delta}{2})]$.

To complete the proof we need to show that $\operatorname{int}\operatorname{dom}(A) = \operatorname{int}\operatorname{conv}\operatorname{dom}(A) = \operatorname{int}\overline{\operatorname{dom}(A)}$. First we note that $\overline{\operatorname{dom}(A)}$ is convex by Proposition 17.2.2 and hence, $\overline{\operatorname{conv}\operatorname{dom}(A)} = \overline{\operatorname{dom}(A)}$. Now Exercise 17.4(b) gives

$$\operatorname{int}\overline{\operatorname{dom}(A)} = \operatorname{int}\overline{\operatorname{conv}\operatorname{dom}(A)} = \operatorname{int}\operatorname{conv}\operatorname{dom}(A).$$

To show the missing equality it suffices to prove that $\operatorname{int}\operatorname{conv}\operatorname{dom}(A) \subseteq \operatorname{dom}(A)$. So, let $x \in \operatorname{int}\operatorname{conv}\operatorname{dom}(A)$. Then $x \in \overline{\operatorname{dom}(A)}$ and hence, we find a sequence

$((x_n, y_n))_{n\in\mathbb{N}}$ in A with $x_n \to x$. Since A is locally bounded at x, the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded and hence, we can assume without loss of generality that $y_n \to y$ weakly for some $y \in H$. The demi-closedness of A (see Remark 17.1.1) yields $(x, y) \in A$ and thus, $x \in \operatorname{dom}(A)$. □

Now we can prove the following perturbation result.

Theorem 17.2.7 *Let $A, B \subseteq H \times H$ be maximal monotone, $\big(\operatorname{int}\operatorname{dom}(A)\big) \cap \operatorname{dom}(B) \neq \varnothing$. Then $A + B$ is maximal monotone.*

Proof By shifting A and B, we can assume without loss of generality that $(0, 0) \in A\cap B$ and $0 \in (\operatorname{int}\operatorname{dom}(A))\cap\operatorname{dom}(B)$. We need to prove that $\operatorname{ran}(1+A+B) = H$. So, let $y \in H$ and set

$$x_\lambda := (1 + A + B_\lambda)^{-1}(y) \quad (\lambda > 0).$$

Since $(0, 0) \in A \cap B_\lambda$ and $\left\|(1 + A + B_\lambda)^{-1}\right\|_{\mathrm{Lip}} \leqslant 1$, we infer that $\|x_\lambda\| \leqslant \|y\|$ for each $\lambda > 0$. For showing $y \in \operatorname{ran}(1 + A + B)$ we need to prove that $\sup_{\lambda>0} \|B_\lambda(x_\lambda)\| < \infty$ by Proposition 17.2.4. By definition we find $y_\lambda \in H$ such that $(x_\lambda, y_\lambda) \in A$ and $y = x_\lambda + y_\lambda + B_\lambda(x_\lambda)$ for each $\lambda > 0$. Since A is locally bounded at $0 \in \operatorname{int}\operatorname{dom}(A)$ by Proposition 17.2.5 we find $R, \delta > 0$ with $B(0, \delta) \subseteq \operatorname{dom}(A)$ and

$$\forall (u, v) \in A : \|u\| < \delta \Rightarrow \|v\| \leqslant R.$$

For $\lambda > 0$ we define $u_\lambda := \frac{\delta}{2\|y_\lambda\|} y_\lambda$ if $y_\lambda \neq 0$ and $u_\lambda := 0$ if $y_\lambda = 0$. Then $\|u_\lambda\| \leqslant \frac{\delta}{2} < \delta$ and thus, $u_\lambda \in \operatorname{dom}(A)$. Hence, there exist $v_\lambda \in H$ with $(u_\lambda, v_\lambda) \in A$ and $\|v_\lambda\| \leqslant R$ for each $\lambda > 0$. The monotonicity of A then yields

$$\begin{aligned}
0 &\leqslant \operatorname{Re}\langle y_\lambda - v_\lambda, x_\lambda - u_\lambda\rangle \\
&= \operatorname{Re}\langle y_\lambda, x_\lambda\rangle - \operatorname{Re}\langle v_\lambda, x_\lambda\rangle - \operatorname{Re}\langle y_\lambda, u_\lambda\rangle + \operatorname{Re}\langle v_\lambda, u_\lambda\rangle \\
&\leqslant \operatorname{Re}\langle y - x_\lambda - B_\lambda(x_\lambda), x_\lambda\rangle - \operatorname{Re}\langle y_\lambda, u_\lambda\rangle + R\,\|y\| + \frac{\delta}{2}R \\
&\leqslant \operatorname{Re}\langle y, x_\lambda\rangle - \operatorname{Re}\langle y_\lambda, u_\lambda\rangle + R\,\|y\| + \frac{\delta}{2}R \\
&\leqslant \|y\|^2 - \operatorname{Re}\langle y_\lambda, u_\lambda\rangle + R\,\|y\| + \frac{\delta}{2}R,
\end{aligned}$$

where we have used the monotonicity of B_λ and $B_\lambda(0) = 0$ in the fourth line. Hence, we obtain

$$\frac{\delta}{2}\,\|y_\lambda\| = \operatorname{Re}\langle y_\lambda, u_\lambda\rangle \leqslant \|y\|^2 + R\,\|y\| + \frac{\delta}{2}R,$$

which shows that $(y_\lambda)_{\lambda>0}$ is bounded and thus, also $\sup_{\lambda>0} \|B_\lambda(x_\lambda)\| < \infty$. □

17.3 A Solution Theory for Evolutionary Inclusions

In this section we provide a solution theory for evolutionary inclusions by generalising Picard's theorem (see Theorem 6.2.1) to the following situation.

Throughout, we assume that $A \subseteq H \times H$ is a maximal monotone relation with $(0,0) \in A$. Moreover, let $M \colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law satisfying the usual positive definiteness constraint

$$\exists \nu_0 \geqslant \mathrm{s_b}\left(M\right), c > 0 \,\forall z \in \mathbb{C}_{\mathrm{Re} \geqslant \nu_0}, \phi \in H : \; \mathrm{Re}\left\langle \phi, zM(z)\phi\right\rangle \geqslant c \left\|\phi\right\|^2.$$

Then for $\nu \geqslant \max\{\nu_0, 0\}$, $\nu \neq 0$, we consider *evolutionary inclusions* of the form

$$(u, f) \in \overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A_{L_{2,\nu}(\mathbb{R};H)}}, \tag{17.3}$$

where $A_{L_{2,\nu}(\mathbb{R};H)}$ is defined as in Corollary 17.1.8. The solution theory for this kind of problems is as follows.

Theorem 17.3.1 *Let* $\nu \geqslant \max\{\nu_0, 0\}$*,* $\nu \neq 0$*. Then the inverse relation* $S_\nu := \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A_{L_{2,\nu}(\mathbb{R};H)}}\right)^{-1}$ *is a Lipschitz-continuous mapping,* $\operatorname{dom}(S_\nu) = L_{2,\nu}(\mathbb{R}; H)$ *and* $\|S_\nu\|_{\mathrm{Lip}} \leqslant \frac{1}{c}$*. Moreover, the solution mapping* S_ν *is causal and independent of* ν *in the sense that* $S_\nu(f) = S_\mu(f)$ *for each* $f \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\mu}(\mathbb{R}; H)$ *and* $\mu \geqslant \nu \geqslant \max\{\nu_0, 0\}$*,* $\nu \neq 0$*.*

In order to prove this theorem, we need some prerequisites. We start with an estimate, which will give us the uniqueness of the solution as well as the causality of the solution mapping S_ν.

Proposition 17.3.2 *Let* $\nu \geqslant \max\{\nu_0, 0\}$*,* $\nu \neq 0$*, and*

$$(u, f), (x, g) \in \overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A_{L_{2,\nu}(\mathbb{R};H)}}.$$

Then for all $a \in \mathbb{R}$

$$\left\|\mathbb{1}_{(-\infty,a]}(u - x)\right\|_{L_{2,\nu}} \leqslant \frac{1}{c} \left\|\mathbb{1}_{(-\infty,a]}(f - g)\right\|_{L_{2,\nu}}.$$

Proof By definition, we find sequences $((u_n, f_n))_{n\in\mathbb{N}}$ and $((x_n, g_n))_{n\in\mathbb{N}}$ in $\partial_{t,\nu} M(\partial_{t,\nu}) + A_{L_{2,\nu}(\mathbb{R};H)}$ such that $u_n \to u, x_n \to x$, $f_n \to f$ and $g_n \to g$ as $n \to \infty$. In particular, for each $n \in \mathbb{N}$ we find $v_n, y_n \in L_{2,\nu}(\mathbb{R}; H)$ such that $(u_n, v_n), (x_n, y_n) \in A_{L_{2,\nu}(\mathbb{R};H)}$ and

$$\begin{aligned} \partial_{t,\nu} M(\partial_{t,\nu}) u_n + v_n &= f_n, \\ \partial_{t,\nu} M(\partial_{t,\nu}) x_n + y_n &= g_n. \end{aligned}$$

Since $(0,0) \in A$, we infer $(\mathbb{1}_{(-\infty,a]}u_n, \mathbb{1}_{(-\infty,a]}v_n), (\mathbb{1}_{(-\infty,a]}x_n, \mathbb{1}_{(-\infty,a]}y_n) \in A_{L_{2,\nu}(\mathbb{R};H)}$ and hence, we may estimate

$$\begin{aligned}
&\operatorname{Re}\left\langle \mathbb{1}_{(-\infty,a]}(f_n - g_n), u_n - x_n \right\rangle \\
&\quad = \operatorname{Re}\left\langle \mathbb{1}_{(-\infty,a]}\partial_{t,\nu} M(\partial_{t,\nu})(u_n - x_n), u_n - x_n \right\rangle \\
&\qquad + \operatorname{Re}\left\langle \mathbb{1}_{(-\infty,a]}v_n - \mathbb{1}_{(-\infty,a]}y_n, \mathbb{1}_{(-\infty,a]}u_n - \mathbb{1}_{(-\infty,a]}x_n \right\rangle \\
&\quad \geqslant \operatorname{Re}\left\langle \mathbb{1}_{(-\infty,a]}\partial_{t,\nu} M(\partial_{t,\nu})(u_n - x_n), u_n - x_n \right\rangle,
\end{aligned}$$

where we used Corollary 17.1.8. Moreover, since $z \mapsto (zM(z))^{-1}$ is a material law, $(\partial_{t,\nu} M(\partial_{t,\nu}))^{-1}$ is causal. By Proposition 16.2.3, for $\phi \in \operatorname{dom}(\partial_{t,\nu} M(\partial_{t,\nu}))$ we have $\operatorname{Re}\left\langle \mathbb{1}_{(-\infty,a]}\partial_{t,\nu} M(\partial_{t,\nu})\phi, \phi \right\rangle \geqslant c \left\| \mathbb{1}_{(-\infty,a]}\phi \right\|^2$. Thus, we end up with

$$\operatorname{Re}\left\langle \mathbb{1}_{(-\infty,a]}(f_n - g_n), u_n - x_n \right\rangle \geqslant c \left\| \mathbb{1}_{(-\infty,a]}(u_n - x_n) \right\|^2,$$

which yields

$$\left\| \mathbb{1}_{(-\infty,a]}(u_n - x_n) \right\| \leqslant \frac{1}{c} \left\| \mathbb{1}_{(-\infty,a]}(f_n - g_n) \right\|.$$

Letting $n \to \infty$, we derive the assertion. □

Next, we address the existence of a solution for (17.3) for suitable right-hand sides f. For this, we provide another useful characterisation for the weak differentiability of a function in $L_{2,\nu}(\mathbb{R}; H)$.

Lemma 17.3.3 *Let $\nu \in \mathbb{R}$, $u \in L_{2,\nu}(\mathbb{R}; H)$. Then $u \in \operatorname{dom}(\partial_{t,\nu})$ if and only if $\sup_{0<h\leqslant h_0} \frac{1}{h} \|\tau_h u - u\| < \infty$ for some $h_0 > 0$. In either case*

$$\frac{1}{h}(\tau_h u - u) \to \partial_{t,\nu} u \quad (h \to 0)$$

in $L_{2,\nu}(\mathbb{R}; H)$.

Proof For $h > 0$ we consider the operator $D_h \colon L_{2,\nu}(\mathbb{R}; H) \to L_{2,\nu}(\mathbb{R}; H)$ given by $D_h v = \frac{1}{h}(\tau_h v - v)$. If $v \in C_c^1(\mathbb{R}; H)$ we estimate

$$\begin{aligned}
\|D_h v\|^2 &= \int_{\mathbb{R}} \frac{1}{h^2} \|v(t+h) - v(t)\|^2 \mathrm{e}^{-2\nu t}\, \mathrm{d}t = \int_{\mathbb{R}} \frac{1}{h^2} \left\| \int_0^h v'(t+s)\, \mathrm{d}s \right\|^2 \mathrm{e}^{-2\nu t}\, \mathrm{d}t \\
&\leqslant \int_{\mathbb{R}} \frac{1}{h} \int_0^h \left\| v'(t+s) \right\|^2 \mathrm{d}s\, \mathrm{e}^{-2\nu t}\, \mathrm{d}t = \frac{1}{h} \int_0^h \int_{\mathbb{R}} \left\| v'(t+s) \right\|^2 \mathrm{e}^{-2\nu t}\, \mathrm{d}t\, \mathrm{d}s \\
&\leqslant \mathrm{e}^{2\nu h} \left\| v' \right\|^2.
\end{aligned}$$

By density of $C_c^1(\mathbb{R}; H)$ in $H_\nu^1(\mathbb{R}; H)$ we infer that

$$\sup_{0\leqslant h\leqslant 1} \|D_h\|_{L(H_\nu^1(\mathbb{R};H), L_{2,\nu}(\mathbb{R};H))} \leqslant \mathrm{e}^{\nu}.$$

Moreover, for $v \in C_c^1(\mathbb{R}; H)$ it is clear that $D_h v \to v'$ in $L_{2,\nu}(\mathbb{R}; H)$ as $h \to 0$ by dominated convergence. Since $(D_h)_{0\leqslant h\leqslant 1}$ is uniformly bounded, the convergence carries over to elements in $H_\nu^1(\mathbb{R}; H)$, which proves the first asserted implication and the convergence statement.

Assume now that $\sup_{0<h\leqslant h_0} \frac{1}{h} \|\tau_h u - u\| < \infty$ for some $h_0 > 0$. Choosing a suitable sequence $(h_n)_{n\in\mathbb{N}}$ in $(0, h_0]$ with $h_n \to 0$ as $n \to \infty$, we can assume that $\frac{1}{h_n}(\tau_{h_n} u - u) \to v$ weakly for some $v \in L_{2,\nu}(\mathbb{R}; H)$. Then we compute for each $\phi \in C_c^\infty(\mathbb{R}; H)$

$$
\begin{aligned}
\langle v, \phi\rangle &= \lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{h_n} \langle u(t+h_n) - u(t), \phi(t)\rangle\, \mathrm{e}^{-2\nu t}\, \mathrm{d}t \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{h_n} \Big\langle u(t), \phi(t-h_n)\mathrm{e}^{2\nu h_n} - \phi(t)\Big\rangle\, \mathrm{e}^{-2\nu t}\, \mathrm{d}t \\
&= \int_{\mathbb{R}} \big\langle u(t), -\phi'(t) + 2\nu\phi(t)\big\rangle\, \mathrm{e}^{-2\nu t}\, \mathrm{d}t = \big\langle u, \partial_{t,\nu}^* \phi\big\rangle,
\end{aligned}
$$

which—as $C_c^\infty(\mathbb{R}; H)$ is a core for $\partial_{t,\nu}^*$ (see Proposition 3.2.4 and Corollary 3.2.6)—shows $u \in \operatorname{dom}(\partial_{t,\nu}^{**}) = \operatorname{dom}(\partial_{t,\nu})$. □

Proposition 17.3.4 *Let $\nu \geqslant \nu_0$ and $f \in \operatorname{dom}(\partial_{t,\nu})$. Then there exists $u \in \operatorname{dom}(\partial_{t,\nu})$ such that*

$$
(u, f) \in \partial_{t,\nu} M(\partial_{t,\nu}) + A_{L_{2,\nu}(\mathbb{R}; H)}.
$$

Proof We recall that $B := \partial_{t,\nu} M(\partial_{t,\nu}) - c$ is maximal monotone by Example 17.1.3. Let $\lambda > 0$ and set

$$
u_\lambda := \Big(c + B + \big(A_{L_{2,\nu}(\mathbb{R}; H)}\big)_\lambda\Big)^{-1}(f) = \Big(\partial_{t,\nu} M(\partial_{t,\nu}) + \big(A_{L_{2,\nu}(\mathbb{R}; H)}\big)_\lambda\Big)^{-1}(f).
$$

We remark that $\big(A_{L_{2,\nu}(\mathbb{R}; H)}\big)_\lambda = \big(A_\lambda\big)_{L_{2,\nu}(\mathbb{R}; H)}$ (see Exercise 17.6). Hence, we have $\tau_h\big(A_{L_{2,\nu}(\mathbb{R}; H)}\big)_\lambda = \big(A_{L_{2,\nu}(\mathbb{R}; H)}\big)_\lambda \tau_h$ for each $h > 0$. Thus, we obtain

$$
\tau_h u_\lambda = \Big(\partial_{t,\nu} M(\partial_{t,\nu}) + \big(A_{L_{2,\nu}(\mathbb{R}; H)}\big)_\lambda\Big)^{-1}(\tau_h f)
$$

and so, due to the monotonicity of B and $\big(A_{L_{2,\nu}(\mathbb{R}; H)}\big)_\lambda$,

$$
\|\tau_h u_\lambda - u_\lambda\| \leqslant \frac{1}{c} \|\tau_h f - f\|.
$$

Dividing both sides by h and using Lemma 17.3.3, we infer that $u_\lambda \in \operatorname{dom}(\partial_{t,\nu})$ and

$$
\big\|\partial_{t,\nu} u_\lambda\big\| = \lim_{h\to 0} \frac{1}{h} \|\tau_h u_\lambda - u_\lambda\| \leqslant \frac{1}{c} \sup_{0<h\leqslant 1} \frac{1}{h} \|\tau_h f - f\| =: K
$$

and hence,

$$\sup_{\lambda>0}\left\|\left(A_{L_{2,\nu}(\mathbb{R};H)}\right)_\lambda(u_\lambda)\right\| = \sup_{\lambda>0}\left\|f-\partial_{t,\nu}M(\partial_{t,\nu})u_\lambda\right\| \leqslant \|f\|+K\left\|M(\partial_{t,\nu})\right\|.$$

Proposition 17.2.4 implies $u_\lambda \to u$ as $\lambda \to 0$ and $(u, f) \in \partial_{t,\nu}M(\partial_{t,\nu})+A_{L_{2,\nu}(\mathbb{R};H)}$. Moreover, since $(\partial_{t,\nu}u_\lambda)_{\lambda>0}$ is uniformly bounded, we can choose a suitable nullsequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ such that $\partial_{t,\nu}u_{\lambda_n} \to v$ weakly for some $v \in L_{2,\nu}(\mathbb{R}; H)$. Since $\partial_{t,\nu}$ is closed and hence, weakly closed (either use $\partial_{t,\nu}^{**} = \partial_{t,\nu}$ or Mazur's theorem [50, Corollary 2.11]) again), we infer that $u \in \operatorname{dom}(\partial_{t,\nu})$. □

We are now in the position to prove Theorem 17.3.1.

Proof of Theorem 17.3.1 Let $\nu \geqslant \nu_0$. Since $\partial_{t,\nu}M(\partial_{t,\nu}) - c$ is monotone (Example 17.1.3), the relation $\partial_{t,\nu}M(\partial_{t,\nu}) + A_{L_{2,\nu}(\mathbb{R};H)} - c$ is monotone and thus, $(\partial_{t,\nu}M(\partial_{t,\nu}) + A_{L_{2,\nu}(\mathbb{R};H)})^{-1}$ defines a Lipschitz-continuous mapping with smallest Lipschitz-constant less than or equal to $\frac{1}{c}$. Since this mapping is densely defined by Proposition 17.3.4, it follows that $S_\nu = \left(\overline{\partial_{t,\nu}M(\partial_{t,\nu}) + A_{L_{2,\nu}(\mathbb{R};H)}}\right)^{-1}$ is Lipschitz-continuous with $\|S_\nu\|_{\mathrm{Lip}} \leqslant \frac{1}{c}$ and $\operatorname{dom}(S_\nu) = L_{2,\nu}(\mathbb{R}; H)$. Moreover, S_ν is causal, since for $f, g \in L_{2,\nu}(\mathbb{R}; H)$ with $\mathbb{1}_{(-\infty,a]}f = \mathbb{1}_{(-\infty,a]}g$ for some $a \in \mathbb{R}$ it follows that $\mathbb{1}_{(-\infty,a]}S_\nu(f) = \mathbb{1}_{(-\infty,a]}S_\nu(g)$ by Proposition 17.3.2. Thus, the only thing left to be shown is the independence of the parameter ν. So, let $f \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\mu}(\mathbb{R}; H)$ for some $\nu_0 \leqslant \nu \leqslant \mu$. Then we find a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $C_c^1(\mathbb{R}; H)$ with $\phi_n \to f$ in both $L_{2,\nu}(\mathbb{R}; H)$ and $L_{2,\mu}(\mathbb{R}; H)$. We set $u_n := S_\nu(\phi_n) \in L_{2,\nu}(\mathbb{R}; H)$ and since $0 = S_\nu(0)$, we derive that $\inf\operatorname{spt} u_n \geqslant \inf\operatorname{spt}\phi_n > -\infty$ by Proposition 17.3.2. Thus, $u_n \in L_{2,\mu}(\mathbb{R}; H)$ and since $u_n \in \operatorname{dom}(\partial_{t,\nu})$ by Proposition 17.3.4 and $\operatorname{spt}\partial_{t,\nu}u_n \subseteq \operatorname{spt} u_n$, we infer that also $\partial_{t,\nu}u_n \in L_{2,\mu}(\mathbb{R}; H)$, which shows $u_n \in \operatorname{dom}(\partial_{t,\mu})$ and $\partial_{t,\mu}u_n = \partial_{t,\nu}u_n$ by Exercise 11.1. By Theorem 5.3.6 it follows that

$$\begin{aligned}\partial_{t,\nu}M(\partial_{t,\nu})u_n &= M(\partial_{t,\nu})\partial_{t,\nu}u_n = M(\partial_{t,\nu})\partial_{t,\mu}u_n\\ &= M(\partial_{t,\mu})\partial_{t,\mu}u_n = \partial_{t,\mu}M(\partial_{t,\mu})u_n.\end{aligned}$$

Since we have $(u_n, \phi_n - \partial_{t,\nu}M(\partial_{t,\nu})u_n) \in A_{L_{2,\nu}(\mathbb{R};H)}$ it follows that $(u_n, \phi_n - \partial_{t,\mu}M(\partial_{t,\mu})u_n) \in A_{L_{2,\mu}(\mathbb{R};H)}$ by the definition of $A_{L_{2,\mu}(\mathbb{R};H)}$ and thus, $u_n = S_\mu(\phi_n)$. Letting $n \to \infty$, we finally derive $S_\mu(f) = S_\nu(f)$. □

17.4 Maxwell's Equations in Polarisable Media

We recall Maxwell's equations from Chap. 6. Let $\Omega \subseteq \mathbb{R}^3$ open. Then the electric field E and the magnetic induction B are linked via Faraday's law

$$\partial_{t,\nu}B + \operatorname{curl}_0 E = 0,$$

where we assume the electric boundary condition for E. Moreover, the electric displacement D, the current j_c and the magnetic field H are linked via Ampère's law

$$\partial_{t,\nu} D + j_c - \operatorname{curl} H = j_0,$$

where j_0 is a given external current. Classically, D and E as well as B and H are linked by the constitutive relations

$$D = \varepsilon E, \text{ and } B = \mu H,$$

where $\varepsilon, \mu \in L(L_2(\Omega)^3)$ model the dielectricity and magnetic permeability, respectively. In a non-polarisable medium, we would additionally assume Ohm's law that links j_c and E by $j_c = \sigma E$ with $\sigma \in L(L_2(\Omega)^3)$. In polarisable media however, this relation is replaced as follows

$$\begin{aligned} \|E\| < E_0 &\Rightarrow j_c = \sigma E \\ \|E\| = E_0 &\Rightarrow \exists \lambda \geqslant 0 : \ j_c = (\sigma + \lambda) E, \end{aligned} \tag{17.4}$$

where $E_0 > 0$ is the called the threshold of ionisation of the underlying medium. The above relation is used to model the following phenomenon: Assume that the medium is not or weakly electrically conductive (i.e., σ is very small) but if the electric field is strong enough (i.e., reaching the threshold E_0), the medium polarises and allows for a current flow proportional to the electric field. Such phenomena occur for instance in certain gases between two capacitor plates, where the gas becomes a conductor if the electric field is strong enough.

Our first goal is to formulate (17.4) in terms of a binary relation. For this, we set

$$B := \left\{ (u, v) \in L_2(\Omega)^3 \times L_2(\Omega)^3 \, ; \ \|u\| \leqslant E_0, \ \operatorname{Re} \langle u, v\rangle = E_0 \|v\| \right\}.$$

Lemma 17.4.1 *Let $u, v \in L_2(\Omega)^3$. Then $(u, v) \in B$ if and only if*

$$(\|u\| \leqslant E_0) \textit{ and } (\|u\| < E_0 \Rightarrow v = 0) \textit{ and } (\|u\| = E_0 \Rightarrow \exists \lambda \geqslant 0 : \ v = \lambda u).$$

Proof Assume first that $(u, v) \in B$. Then $\|u\| \leqslant E_0$ by definition. Moreover,

$$E_0 \|v\| = \operatorname{Re} \langle u, v\rangle \leqslant \|u\| \|v\|$$

and hence, if $\|u\| < E_0$ it follows that $v = 0$. Moreover, if $\|u\| = E_0$ we have equality and thus, u and v are linearly dependent; that is, we find $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \lambda_2 \neq 0$ such that $\lambda_1 u + \lambda_2 v = 0$. Note that $\lambda_2 \neq 0$ since $u \neq 0$ and hence, we get

$v = \lambda u$ with $\lambda := -\frac{\lambda_1}{\lambda_2}$. We then have

$$0 \leqslant |\lambda| E_0^2 = \|v\|\, E_0 = \operatorname{Re}\langle u, v\rangle = \operatorname{Re}\lambda\, \|u\|^2 = \operatorname{Re}\lambda\, E_0^2,$$

which shows $0 \leqslant \operatorname{Re}\lambda = |\lambda|$ and thus, $\lambda \geqslant 0$. The other implication is trivial. □

The latter lemma shows that (E, j_c) satisfies (17.4) if and only if $(E, j_c - \sigma E) \in B$, or equivalently $(E, j_c) \in \sigma + B$. Thus, we may reformulate Maxwell's equations in a polarisable medium Ω as follows

$$\left(\begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} j_0 \\ 0 \end{pmatrix}\right) \in \partial_{t,\nu} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}.$$

To apply our solution theory in Theorem 17.3.1, we need to ensure that

$$A := \begin{pmatrix} B & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix} \tag{17.5}$$

defines a maximal monotone relation on $L_2(\Omega)^6 \times L_2(\Omega)^6$. This will be done by the perturbation result presented in Theorem 17.2.7. We start by showing the maximal monotonicity of B.

Lemma 17.4.2 *We define the function $I \colon L_2(\Omega)^3 \to (-\infty, \infty]$ by*

$$I(u) = \begin{cases} 0 & \text{if } \|u\| \leqslant E_0 \\ \infty & \text{otherwise.} \end{cases}$$

Then I is convex, proper and l.s.c. Moreover, $B = \partial I$. In particular, B is maximal monotone.

Proof This is part of Exercise 17.7. □

Proposition 17.4.3 *The relation A given by (17.5) is maximal monotone with* $(0, 0) \in A$.

Proof Since B is maximal monotone by Lemma 17.4.2, it is easy to see that $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ is maximal monotone, too. Moreover, by definition we see that $0 \in \operatorname{int}\operatorname{dom}(B)$ and thus, $0 \in \operatorname{int}\operatorname{dom}\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{int}\operatorname{dom}(B) \times L_2(\Omega)^3$. Since clearly $0 \in \operatorname{dom}\begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}$ is maximal monotone (see Example 17.1.3), the assertion follows from Theorem 17.2.7. □

Theorem 17.4.4 *Let $\varepsilon, \mu, \sigma \in L(L_2(\Omega)^3)$ with ε, μ selfadjoint. Moreover, assume there exist $\nu_0, c > 0$ such that*

$$\nu\varepsilon + \operatorname{Re}\sigma \geqslant c \text{ and } \mu \geqslant c \quad (\nu \geqslant \nu_0).$$

Then for each $\nu \geqslant \nu_0$ we have that

$$S_\nu := \left(\overline{\partial_{t,\nu}\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}_{L_{2,\nu}(\mathbb{R};L_2(\Omega)^6)}}\right)^{-1}$$

is a Lipschitz-continuous mapping with $\operatorname{dom}(S_\nu) = L_{2,\nu}(\mathbb{R}; L_2(\Omega)^6)$ and $\|S_\nu\|_{\mathrm{Lip}} \leqslant \frac{1}{c}$. Moreover, S_ν is causal and independent of ν in the sense that $S_\nu(f) = S_\eta(f)$ whenever $\nu, \eta \geqslant \nu_0$ and $f \in L_{2,\nu}(\mathbb{R}; L_2(\Omega)^6) \cap L_{2,\eta}(\mathbb{R}; L_2(\Omega)^6)$.

Proof This follows from Theorem 17.3.1 applied to $M(z) := \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + z^{-1}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$ and A as in (17.5). □

17.5 Comments

The concept of maximal monotone relations in Hilbert spaces was first introduced by Minty in 1960 for the study of networks [66] and became a well-studied subject also with generalisations to the Banach space case. For this topic we refer to the monographs [16] and [49, Chapter 3]. The concept of subgradients is older and it was found out by Rockafellar [99] that subgradients are maximal monotone. Indeed, one can show that subgradients are precisely the cyclically maximal monotone relations (see e.g. [16, Theoreme 2.5]).

The Theorem of Minty was proved in 1962, [65] and generalised to the case of reflexive Banach spaces by Rockafellar in 1970 [100]. The proof presented here follows [106] and was kindly communicated by Ralph Chill and Hendrik Vogt.

The classical way to approach differential inclusions of the form $(u, f) \in \partial_t + A$ where A is maximal monotone uses the theory of nonlinear semigroups of contractions, introduced by Komura in the Hilbert space case, [56] and generalised to the Banach space case by Crandall and Pazy, [24]. The results on evolutionary inclusions presented in this chapter are based on [117, 118] and were further generalised to non-autonomous problems in [122, 126].

The model for Maxwell's equations in polarisable media can be found in [36, Chapter VII]. We note that in this reference, condition (17.4) is replaced by

$$|E| < E_0 \Rightarrow j_c = \sigma E$$
$$|E| = E_0 \Rightarrow \exists \lambda \geqslant 0 : \; j_c = (\sigma + \lambda)E,$$

which should hold almost everywhere. To solve this problem, one cannot apply Theorem 17.2.7, since 0 is not an interior point of the domain of the corresponding relation and thus, a weaker notion of solution is needed to tackle this problem, see [36, Theorem 8.1].

Exercises

Exercise 17.1 Let $f: H \to (-\infty, \infty]$ be convex, proper and l.s.c. Moreover, assume that f is differentiable in $x \in H$ (in particular, $f < \infty$ in a neighbourhood of x). Show that $(x, y) \in \partial f$ if and only if $y = f'(x)$.

Exercise 17.2 Let $f, g: H \to (-\infty, \infty]$. Prove that

(a) $f + g$ is convex if f and g are convex.
(b) $f + g$ is l.s.c. if f and g are l.s.c.

Exercise 17.3 Let H be a Hilbert space, $(x_n)_{n\in\mathbb{N}}$ in H and $x \in H$. Show, that $x_n \to x$ if and only if $x_n \to x$ weakly and $\limsup_{n\to\infty} \|x_n\| \leqslant \|x\|$.

Exercise 17.4 Let X be a normed space (or, more generally, a topological vector space) and $C \subseteq X$ convex. Prove the following statements:

(a) If $x \in \operatorname{int} C$ and $y \in \overline{C}$, then $(1-t)x + ty \in \operatorname{int} C$ for each $t \in [0, 1)$.
(b) If $\operatorname{int} C \neq \varnothing$, then $\overline{C} = \overline{\operatorname{int} C}$ and $\operatorname{int} \overline{C} = \operatorname{int} C$.
(c) If C is open and $K \subseteq X$ is open with $\overline{K} \subseteq \overline{C}$. Then $K \subseteq C$.

Hint: For (a) take an open set $U \subseteq X$ with $0 \in U$ such that $x + U - U \subseteq C$ and show $(1-t)x + ty + (1-t)U \subseteq C$.

Exercise 17.5 Let X be a topological space and $U \subseteq X$ open. We equip U with the trace topology. Prove the following statements:

(a) For $A \subseteq U$ we have $\overline{A}^U = \overline{A}^X \cap U$ and $\operatorname{int}_U A = \operatorname{int}_X A$.
(b) If $A \subseteq U$ is closed in U and $\operatorname{int}_U A = \varnothing$, then $\operatorname{int}_X \overline{A}^X = \varnothing$.
(c) If X is a Baire space, then U is a Baire space.

Recall, that a topological space X is a *Baire space* if for each sequence $(A_n)_{n\in\mathbb{N}}$ of closed sets with $\operatorname{int} A_n = \varnothing$ it follows that $\operatorname{int} \bigcup_{n\in\mathbb{N}} A_n = \varnothing$ or, equivalently, if for each sequence $(U_n)_{n\in\mathbb{N}}$ of open and dense sets it follows that $\bigcap_{n\in\mathbb{N}} U_n$ is dense.

Exercise 17.6 Let $A \subseteq H \times H$ be maximal monotone.

(a) Let $\mu, \lambda > 0$. Show that $(A_\lambda)_\mu = A_{\lambda+\mu}$.
(b) Let $(0, 0) \in A$ and $(\Omega, \mathcal{A}, \mu)$ a σ-finite measure space. Prove that $(A_\lambda)_{L_2(\mu)} = (A_{L_2(\mu)})_\lambda$ for each $\lambda > 0$.

Exercise 17.7 Let H be a Hilbert space and $C \subseteq H$ non-empty, convex and closed. Moreover, define $I_C : H \to (-\infty, \infty]$ by

$$I_C(x) := \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{otherwise.} \end{cases}$$

Show that I_C is convex, proper and l.s.c. and show

$$(x, y) \in \partial I_C \Leftrightarrow x \in C, \forall u \in C : \operatorname{Re} \langle y, u - x \rangle \leqslant 0.$$

Moreover, prove Lemma 17.4.2.

References

16. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, vol. 5 (Elsevier, Amsterdam, 1973)
24. M.G. Crandall, A. Pazy, Semi-groups of nonlinear contractions and dissipative sets. J. Funct. Anal. **3**, 376–418 (1969). English.
36. G. Duvaut, J.L. Lions, *Inequalities in Mechanics and Physics. Translated from the French by C.W. John*, vol. 219 (Springer, Berlin, 1976)
49. S. Hu, N.S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I*, Vol. 419. Mathematics and Its Applications. Theory (Kluwer Academic Publishers, Dordrecht, 1997)
50. J. Voigt, *A Course on Topological Vector Spaces* (Birkhäuser-Verlag, Cham, Switzerland, 2020)
56. Y. Komura, Nonlinear semigroups in Hilbert space. J. Math. Soc. Jpn. **19**, 493–507 (1967)
65. G.J. Minty, Monotone (nonlinear) operators in Hilbert space. Duke Math. J. **29**, 341–346 (1962)
66. G.J. Minty, Monotone networks. Proc. R. Soc. Lond. Ser. A **257**, 194–212 (1960)
99. R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings. Pac. J. Math. **33**, 209–216 (1970)
100. R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators. Trans. Am. Math. Soc. **149**, 75–88 (1970)
106. S. Simons, C. Zalinescu, A new proof for Rockafellar's characterization of maximal monotone operators. Proc. Am. Math. Soc. **132**(10), 2969–2972 (2004)
117. S. Trostorff, An alternative approach to well-posedness of a class of differential inclusions in Hilbert spaces. Nonlinear Anal. **75**(15), 5851–5865 (2012)
118. S. Trostorff, Autonomous evolutionary inclusions with applications to problems with nonlinear boundary conditions. Int. J. Pure Appl. Math. **85**(2), 303–338 (2013)
122. S. Trostorff, Well-posedness for a general class of differential inclusions. J. Differ. Equ. **268**, 6489–6516 (2020)
126. S. Trostorff, M. Wehowski, Well-posedness of non-autonomous evolutionary inclusions. Nonlinear Anal. Theory Methods Appl. Ser. A Theory Methods **101**, 47–65 (2014)