Interior and modal masters in condensation methods for eigenvalue problems

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Abstract
In the dynamic analysis of structures condensation methods are often used to reduce the number of degrees of freedom to manageable size. Substructuring and choosing the master variables as the degrees of freedom on the interfaces of the substructures yields data structures which are well suited to be implemented on parallel computers. In this paper we discuss the additional use of interior masters and modal masters in substructuring. The data structure is preserved such that the condensed problem can be determined substructurewise.

1 Introduction

In the analysis of the dynamic response of a linear structure using finite element methods very often prohibitively many degrees of freedom are needed to model the behaviour of the system sufficiently accurate. In this situation static condensation methods are employed to economize the computation of a selected group of eigenvalues and eigenvectors. These methods choose from the degrees of freedom a small number of master variables which appear to be representative. The rest of the variables (termed slaves) is eliminated by use of the static equations leaving a much smaller problem for the master variables only.

Partitioning the structure under consideration into substructures and

choosing the masters as the interface degrees of freedom leads to data structures and formulae for the individual substructure which are independent of each other. Taking advantage of these properties Rothe and Voss obtained a fully parallel (improved) condensation method for eigenvalue problems (cf. [5])

The part of the spectrum which can be approximated accurately enough depends crucially on the size of the minimum slave eigenvalue which is the constrained eigenvalue problem with fixed masters. Therefore additional masters should be chosen such that the minimum slave eigenvalue is increased as much as possible without destroying the data structure that allows substructurewise determination of the condensed problem.

A method at hand is to incorporate additional master variables which are degrees of freedom at interior nodes of the substructures. In this case the data structure essentially is preserved and the parallel method of [5] carries over in a straightforward way.

A maximum increase of the minimum slave eigenvalue is obtained if we choose those modal degrees of freedom corresponding to the smallest eigenvalues of the substructures. To apply the approach of [5] directly a transformation of variables to a new orthonormal basis of the space spanned by the slave eigenvectors under consideration and its orthogonal complement has to be performed for each substructure which is very time consuming. In [2] Mackens and the author introduced a non nodal condensation method which circumvents the use of the orthonormal basis of the orthogonal complement. In this paper we apply this approach to condensed eigenvalue problems using interface masters, interior masters, and modal masters. An algorithm results which computes the the reduced problem substructurewise. A numerical example demonstrates the favourable properties of the method.

2 Nodal Condensation

Free vibration analysis of structures results in the linear eigenvalue problem

$$Kx = \lambda Mx \tag{1}$$

where the stiffness matrix $K \in \mathbb{R}^{(n,n)}$ and the mass matrix $M \in \mathbb{R}^{(n,n)}$ are real symmetric and positive definite, x is the vector of modal displacements, and λ is the square of the natural frequencies.

To reduce the number of the unknowns to manageable size the vector x is partitioned into a set of variables x_s (termed slaves) which are to be eliminated and the remaining variables x_m (termed masters) which are to be

retained. After reordering the unknowns and equations system (1) obtains the following block form:

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{Bmatrix} x_m \\ x_s \end{Bmatrix} = \lambda \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{Bmatrix} x_m \\ x_s \end{Bmatrix} \tag{2}$$

Solving the second equation of (2) for x_s one gets

$$x_s(\lambda) = -(K_{ss} - \lambda M_{ss})^{-1}(K_{sm} - \lambda M_{sm})x_m =: S(\lambda)x_m. \tag{3}$$

Hence, if $\tilde{x}_m$ is the master portion of an eigenvector $\tilde{x}$ corresponding to the eigenvalue $\tilde{\lambda}$ then $S(\tilde{\lambda})\tilde{x}_m$ is the slave part of $\tilde{x}$. Therefore, if we are interested in eigenvalues close to $\hat{\lambda}$, it is reasonable to project the eigenvalue problem (2) to the linear space

$$\left\{ \begin{bmatrix} I \\ S(\hat{\lambda}) \end{bmatrix} x_m \ : \ x_m \in \mathbb{R}^m \right\},$$

i.e. to consider the projected eigenvalue problem

$$P^t K P x_m = \lambda P^t M P x_m, \quad P := \begin{bmatrix} I \\ S(\hat{\lambda}) \end{bmatrix}. \tag{4}$$

For $\hat{\lambda} = 0$ this is the statically condensed eigenproblem

$$K_0 x_m = \lambda M_0 x_m \tag{5}$$

introduced by Guyan [1] where

$$\begin{aligned} K_0 &:= K_{mm} - K_{ms}K_{ss}^{-1}K_{sm}, \\ M_0 &:= M_{mm} - K_{ms}K_{ss}^{-1}M_{sm} - M_{ms}K_{ss}^{-1}K_{sm} + K_{ms}K_{ss}^{-1}M_{ss}K_{ss}^{-1}K_{sm}. \end{aligned}$$

For fixed $\hat{\lambda} \neq 0$ (4) is called the dynamically condensed problem, and for variable $\hat{\lambda} = \lambda$ one obtains the exactly condensed problem

$$T(\lambda)x_m = 0 \tag{6}$$

where

$$T(\lambda) := -K_{mm} + \lambda M_{mm} + (K_{ms} - \lambda M_{ms})(K_{ss} - \lambda M_{ss})^{-1}(K_{sm} - \lambda M_{ms}).$$

Usually in the literature approximations to some of the smallest eigenvalues of (1) and to the master portions x_m of the corresponding eigenvectors are obtained from the statically condensed problem (5), and the slave portions x_s are calculated by equation (3). Observe, however, that only very few eigenmodes are derived from (5) with sufficient accuracy. Several attempts have been made to enhance the quality. Most of them are very

time consuming since an iterative process is involved. A different approach which substantially improves the eigenvalue and eigenvector approximations from static condensation was introduced in [6], [4]. It takes advantage of a Rayleigh functional of the exactly condensed problem (6).

Suppose that r substructures are considered and that they connect to each other through the master variables on the interfaces only. If the slave variables are numbered appropriately, then the stiffness matrix is given by

$$K = \begin{bmatrix} K_{mm} & K_{ms1} & K_{ms2} & \dots & K_{msr} \\ K_{sm1} & K_{ss1} & O & \dots & O \\ K_{sm2} & O & K_{ss2} & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{smr} & O & O & \dots & K_{ssr} \end{bmatrix}, \tag{7}$$

and the mass matrix M has the same block form.

Taking advantage of the blockstructure of K and M the reduced matrices K_0 and M_0 can be calculated substructurewise, and hence, completely in parallel. Obviously,

$$K_0 = K_{mm} - \sum_{j=1}^{r} K_{mmj} := K_{mm} - \sum_{j=1}^{r} K_{msj} K_{ssj}^{-1} K_{smj}$$

and

$$M_0 = M_{mm} - \sum_{j=1}^{r} M_{mmj},$$

where

$$M_{mmj} := K_{msj} K_{ssj}^{-1} M_{smj} + M_{msj} K_{ssj}^{-1} K_{smj} - K_{msj} K_{ssj}^{-1} M_{ssj} K_{ssj}^{-1} K_{smj}.$$

3 Non nodal masters in condensation

We already mentioned that static condensation usually allows accurate approximations only at the lower end of the spectrum. It is known [6] that the attainable accuracy of an eigenvalue λ_k of interest increases with the distance of this value to the smallest eigenvalue $\underline{\omega}(> \lambda_k)$ of the slave eigenvalue problem, which is the constraint eigenvalue problem with the masters fixed to be zero. For the above choice of interface masters the slave problem splits into independent eigenproblems for the decoupled substructures and $\underline{\omega}$ is the minimal eigenvalue of the union of the substructures eigenvalues.

It is clear by physical intuition (or by mathematical minimax theory) that $\underline{\omega}$ will be raised (and $\underline{\omega} - \lambda_k$ be increased) by the introduction of additional constraints. In order not to destroy the parallel data structure in

substructured static condensation each additional master should use interior degrees of freedom of only one of the substructures. This obviously preserves the block structure of the mass and stiffness matrices and therefore the parallel algorithm from [5] applies.

An optimal increase of $\underline{\omega}$ with a fixed number of additional masters per substructure would be gained if the corresponding constraints would impose M-orthogonality to the substructures eigenvectors corresponding to its lowest eigenvalues.

Theoretically it is no problem to incorporate such generalized masters into the condensation process. If the columns of $Z \in \mathbb{R}^{(n,m)}$ consist of linearly independent master-vectors and the columns of $Y \in \mathbb{R}^{(n,n-m)}$ are complementary such that (Z,Y) has full rank, then every vector $x \in \mathbb{R}^n$ can be written as $x = Zx_m + Yx_s$. Going with this representation into eqn. (1) and multiplying with $(Z,Y)^t$ the following equivalent form of (1) arises:

$$\begin{bmatrix} K_{zz} & K_{zy} \\ K_{yz} & K_{yy} \end{bmatrix} \begin{Bmatrix} x_m \\ x_s \end{Bmatrix} = \lambda \begin{bmatrix} M_{zz} & M_{zy} \\ M_{yz} & M_{zz} \end{bmatrix} \begin{Bmatrix} x_m \\ x_s \end{Bmatrix} \tag{8}$$

with

$$L_{zz} := Z^t L Z,\ L_{zy} := Z^t L Y,\ L_{yz} := L_{zy}^t,\ L_{yy} := Y^t L Y,\ L \in \{K, M\}. \tag{9}$$

If (8) were accessible, the generalized degrees of freedom vector x_m could readily be used as master vector as before. We remark, that the nodal masters are still included by choosing the corresponding unit vectors as master vectors.

Practically this naive approach is not executable, however. Though the master vectors are certainly accessible, there is no chance to store the excessive number of complementary y-vectors in general.

Fortunately, it turns out, that the influence of Y in the calculation of the projection matrix P in (4) can be totally characterized by span(Y) being the orthogonal complement of span(Z) with respect to an appropriate inner product. If we let the inner product be described by the symmetric positive definite matrix V, such that $Z^T V Y = O$, one can compute the static condensation by the following result from [2], which will be the key to determining the condensed eigenvalue problem substructurewise in the presence of substructural modal masters.

Theorem 1:
With $Z \in \mathbb{R}^{(n,m)}, Y \in \mathbb{R}^{(n,n-m)}$ let $(Z,Y) \in \mathbb{R}^{(n,n)}$ be regular and $Z^t V Y = 0$ with the symmetric positive definite matrix $V \in \mathbb{R}^{(n,n)}$. Then the statically condensed eigenvalue problem corresponding to problem (8) is given by

$$P^t K P x_m = \lambda P^t M P x_m \tag{10}$$

where the matrix $P \in \mathbb{R}^{(n,m)}$ can be calculated from

$$\begin{bmatrix} K & -VZ \\ -Z^tV & O \end{bmatrix} \begin{bmatrix} P \\ S \end{bmatrix} = \begin{bmatrix} O \\ -I_m \end{bmatrix}. \tag{11}$$

Moreover the reduced stiffness matrix satisfies

$$P^tKP = S. \tag{12}$$

4 Modal condensation and substructuring

We consider the free vibrations of a structure which is decomposed into r substructures. Let the vibration problem be discretized (by finite elements or finite differences) in correspondence to the substructure decomposition, i.e. $k_{ij} = 0$ and $m_{ij} = 0$ whenever i and j denote indices of interior nodes of different substructures. We choose as masters those degrees of freedom which are located on the boundaries of the substructures. Additionally we allow generalized masters. We assume that the support of any of the generalized masters is contained in exactly one substructure. Here we have in mind nodal interior masters and modal masters, i.e. eigenvectors of the eigenvalue problem restricted to the substructure under consideration.

In this section we describe how the reduced eigenvalue problem can be computed substructurewise. For reasons of limited space we do not include how the method is deduced from Theorem 1 nor do we give details of an implementation on a parallel computer. These will be contained in a forthcoming paper [3].

We number the variables in the usual way where the coupling of the nodal masters is given by K_{mm} and M_{mm}, and the interaction of the interior degrees of freedom of the j-th substructure and the nodal masters is given by $K_{smj} = K_{msj}^t$ and $M_{smj} = M_{msj}^t$. Then the stiffness matrix K and the mass matrix M obtain the block structure given in (7).

The modal masters corresponding to the j-th substructure are collected in the matrix $Z_j \in \mathbb{R}^{(s_j,m_j)}$ where s_j denotes the number of interior degrees of freedom of the j-th substructure and m_j the number of modal masters having their support in the j-th substructure.

Finally, let

$$V = \begin{bmatrix} I_m & O & \dots & O \\ O & M_{ss1} & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & \dots & O & M_{ssr} \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} I_m & O & \dots & O \\ O & Z_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & \dots & O & Z_r \end{bmatrix}.$$

Then the condensed eigenvalue problem

$$K_0\xi := P^tKP\xi = \lambda P^tMP\xi =: \lambda M_0\xi$$

can be determined substructurewise in the following way:

(i) For $j = 1, \ldots, r$ solve the linear systems

$$\begin{bmatrix} K_{ssj} & -M_{ssj}Z_j \\ -Z_j^tM_{ssj} & O \end{bmatrix} \begin{bmatrix} P_j \\ S_j \end{bmatrix} = \begin{bmatrix} -K_{smj} \\ O \end{bmatrix}. \tag{13}$$

These are r decoupled systems of $s_j + m_j$ linear equations. Notice that most of the columns of the matrix K_{msj} are null vectors, and that only those columns of the right hand side have to be considered which correspond to the nodal master degrees of freedom on the boundary or in the interior of the j-th substructure.

(ii) For $j = 1, \ldots, r$ solve the linear systems

$$\begin{bmatrix} K_{ssj} & -M_{ssj}Z_j \\ -Z_j^tM_{ssj} & O \end{bmatrix} \begin{bmatrix} Q_j \\ R_j \end{bmatrix} = \begin{bmatrix} O \\ -I_{m_j} \end{bmatrix}. \tag{14}$$

Systems (14) and (13) share the same coefficient matrix. Hence, the LDL^T-decomposition of problem (13) can be reused in the solution process of (14).

(iii) Compute

$$\tilde{S} := K_{mm} + \sum_{j=1}^{r} K_{msj}P_j.$$

(iv) From (12) it follows that the reduced stiffness matrix is given by

$$K_0 = \begin{bmatrix} \tilde{S} & S_1^t & \ldots & S_r^t \\ S_1 & R_1 & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ S_r & O & \ldots & R_r \end{bmatrix}.$$

(v) The reduced mass matrix $M_0 = P^tMP$ can be determined in the following way: For $j = 1, \ldots, r$ compute

$$U_j := M_{msj}P_j,\ V_j := M_{msj}Q_j,\ X_j := M_{ssj}P_j,\ Y_j := M_{ssj}Q_j.$$

Then

$$M_0 = \begin{bmatrix} M_{mm} + \sum\limits_{j=1}^{r}(U_j + U_j^t + P_j^tX_j) & V_1 + P_1^tY_1 & \ldots & V_r + P_r^tY_r \\ V_1^t + Y_1^tP_1 & Q_1^tY_1 & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ V_r^t + Y_r^tP_r & O & \ldots & Q_r^tY_r \end{bmatrix}$$

Notice that most of the computations (namely, solving the linear systems and determining the matrix products) can be done substructurewise and therefore completely in parallel. Only in the compilation of the reduced matrices K_0 and M_0 communication between the processes is needed.

5 A numerical example

We consider the L-shaped membrane problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{15}$$

where $\Omega := ((-1,1)\times(0,1))\cup((0,1)\times(-1,0])$ denotes an L-shaped region in $\mathbb{R}^2$. We discretized problem (15) on a square grid with mesh size $h = 1/24$ using the ordinary 5-point difference approximation of the Laplace operator which yields a linear eigenvalue problem of dimension 1633.

If we decompose Ω into 12 identical substructures each being a square with side length 0.5, and if we choose the masters and slaves as the boundary and interior grid points of the substructures, respectively, then we arrive at a reduced problem of dimension 181. The approximation properties of this condensed problem are really poor. The smallest eigenvalue of the condensed problem approximates the smallest eigenvalue of the original problem with a relative error of 8.23%. The approximation properties are enhanced substantially if one evaluates the Rayleigh functional of the exactly condensed problems at the eigenvectors of the statically condensed problem (cf. [6]).

The following table contains the ten smallest eigenvalues of the discrete version of the L-shaped membrane, the relative errors of its approximations from the statically condensed problem, and the values of the Rayleigh functional at the eigenvectors of the condensed problem. We were able to approximate 10 eigenvalues at the lower end of the spectrum with a relative error of less than 1%.

#	eigenvalue	condensation	Rayleigh fct.
1	$9.662291e+00$	$8.23e-02$	$3.42e-05$
2	$1.517498e+01$	$1.24e-01$	$1.55e-04$
3	$1.971104e+01$	$1.59e-01$	$3.25e-04$
4	$2.944159e+01$	$2.19e-01$	$2.86e-03$
5	$3.189298e+01$	$2.54e-01$	$5.82e-03$
6	$4.133373e+01$	$4.95e-01$	$-5.34e-03$
7	$4.470593e+01$	$5.93e-01$	$-2.42e-03$
8	$4.910897e+01$	$6.10e+01$	0
9	$4.910897e+01$	$6.10e+01$	0
10	$5.651630e+01$	$5.53e+01$	$-8.91e-03$

In each substructure we added 1 interior master in the center and 5 interior master (one in the center and the other ones at the points $(0.25\ell, 0.25\ell)$, $(0.25\ell, 0.75\ell)$, $(0.75\ell, 0.25\ell)$, $(0.75\ell, 0.75\ell)$ where ℓ denotes the sidelength of the substructure). The condensed eigenvalue problems of dimensions 193 and 241, respectively, improved the eigenvalue approximations a little. The relative errors are contained in the following table. We were able to approximate 11 and 13 eigenvalues, respectively, at the lower end of the spectrum with a relative error less than 1% with 1 and 5 additional interior masters, respectively.

	1 interior master		5 interior masters	
#	condensation	Rayleigh fct.	condensation	Rayleigh fct.
1	$5.95e-02$	$2.80e-05$	$3.33e-02$	$1.49e-05$
2	$9.11e-02$	$1.30e-04$	$5.45e-02$	$6.85e-05$
3	$1.18e-01$	$3.10e-04$	$7.05e-02$	$1.63e-04$
4	$1.69e-01$	$1.89e-03$	$1.03e-01$	$9.02e-04$
5	$1.94e-01$	$3.25e-03$	$1.18e-01$	$1.46e-03$
6	$3.33e-01$	$-2.51e-04$	$1.93e-01$	$2.82e-04$
7	$3.89e-01$	$1.31e-03$	$2.24e-01$	$8.44e-04$
8	$4.21e-01$	$3.79e-03$	$2.44e-01$	$1.95e-03$
9	$4.21e-01$	$3.79e-03$	$2.44e-01$	$1.95e-03$
10	$4.28e-01$	$7.55e-03$	$2.58e-01$	$4.66e-03$

Finally, we incorporated 1, 3 and 9 modal masters of each substructure, respectively, in the condensation process yielding reduced problems of dimensions 193, 217 and 289, respectively. The relative errors for the ten smallest eigenvalues are contained in the following tables. We were able to approximate 10, 26 and 66 eigenvalues at the lower end of the spectrum with a relative error less than 1%.

	1 modal master		3 modal masters	
#	condensation	Rayleigh fct.	condensation	Rayleigh fct.
1	$7.18e-03$	$6.77e-06$	$3.37e-03$	$1.11e-06$
2	$1.48e-02$	$4.01e-05$	$4.88e-03$	$3.96e-06$
3	$2.14e-02$	$9.17e-05$	$6.21e-03$	$9.94e-06$
4	$3.85e-02$	$7.56e-04$	$8.74e-03$	$3.93e-05$
5	$3.62e-02$	$1.21e-03$	$9.08e-03$	$5.90e-05$
6	$2.54e-02$	$-3.40e-04$	$9.07e-03$	$8.00e-05$
7	$1.98e-02$	$7.51e-05$	$8.71e-03$	$8.13e-05$
8	$2.93e-02$	$1.05e-03$	$7.65e-03$	$7.24e-05$
9	$2.93e-02$	$1.05e-03$	$7.65e-03$	$7.24e-05$
10	$5.53e-02$	$4.79e-03$	$7.29e-03$	$1.23e-05$

#	eigenvalue	9 modal masters condensation	Rayleigh fct.
1	$9.662291e+00$	$7.53e-04$	$6.21e-08$
2	$1.517498e+01$	$1.38e-03$	$3.25e-07$
3	$1.971104e+01$	$1.96e-03$	$8.17e-07$
4	$2.944159e+01$	$3.22e-03$	$4.81e-06$
5	$3.189298e+01$	$3.16e-03$	$7.67e-06$
6	$4.133373e+01$	$2.40e-03$	$5.68e-06$
7	$4.470593e+01$	$2.02e-03$	$2.84e-06$
8	$4.910897e+01$	$2.73e-03$	$9.27e-06$
9	$4.910897e+01$	$2.73e-03$	$9.27e-06$
10	$5.651630e+01$	$4.47e-03$	$3.00e-05$
20	$1.007678e+02$	$9.42e-03$	$3.59e-05$
30	$1.598435e+02$	$1.11e-02$	$1.64e-03$
40	$1.983320e+02$	$1.55e-02$	$3.62e-04$
50	$2.427713e+02$	$2.60e-02$	$2.50e-04$
60	$2.817866e+02$	$2.22e-02$	$3.02e-04$
70	$3.257477e+02$	$3.91e-02$	$1.62e-03$

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