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On a Simple Method for Calculating Laminar Boundary Layers

by

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On a Simple Method for Calculating Laminar Boundary Layers

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Summary: A simple one parametric method, due to A. Walz and based on the momentum and energy equations, for calculating approximately laminar boundary layers is extended to cover axi-symmetric flow as well as plane flow. The necessary computing work is reduced a little.

Another known method which requires still less computing work is also extended for axi-symmetric flow and, with the amendment of a numerical constant, proves adequate for practical purposes.

1. Introduction

Since there are already several methods of calculating approximately laminar boundary layers for incompressible flow, it would seem necessary to justify the development of yet another method. The following method tries to obtain in a practical manner, results as correct as possible and with the smallest amount of computing work. Obviously a very accurate method is to represent the velocity profiles approximately by a two parametric class whereby the partial differential equation for boundary layers is replaced by two ordinary differential equations: the momentum and the energy equation[1]. But here the interpolation in the two parameters requires considerably more computing work than that with the usual one parametric methods. Therefore A. Walz[2] altered this method back again into a one parametric method using the energy equation instead of the boundary condition for the second derivative $\partial^2 u/\partial y^2$ on the wall. By this means he obtained better results than by the usual one parametric methods based on the momentum equation only. So he proved that it is more important to satisfy the energy balance on the average over the whole layer rather than the usual boundary condition for the curvature of the velocity profile at the wall. As the latter condition is too sharp for a limited variety of profiles, a wrong profile is sometimes selected by it. The method of A. Walz can be altered slightly as follows. It is extended so as to calculate boundary layers on axi-symmetric bodies (without incidence) as well as in two-dimensional plane flow. Also, the computing work is reduced.

Notation

x	the axial co-ordinate
r	the radial co-ordinate
r_0	the co-ordinate of the body of revolution

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s the distance measured along the surface of the body

n the distance normal to the surface

$\frac{r}{r_0} = 1 + \frac{r_0}{n \cos \alpha}$ with $\tan \alpha = \frac{d r_0}{dx}$

u, v the normal velocity components at any point in the layer

U the velocity at the edge of the boundary layer

U_0 the velocity of the undisturbed stream

ρ the density of the incompressible fluid

μ the viscosity

$\nu = \mu / \rho =$ the kinematic viscosity

δ the thickness of the boundary layer

$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{r}{r_0}\, dn =$ the displacement thickness

$\delta_2 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) \frac{r}{r_0}\, dn =$ the momentum thickness

$\delta_3 = \int_0^\infty \frac{u}{U}\left[1 - \left(\frac{u}{U}\right)^2\right] \frac{r}{r_0}\, dn =$ the energy thickness

$D = \int_0^\infty \left(\frac{\partial (u/U)}{\partial (n/\delta_2)}\right)^2 \frac{r}{r_0}\, d\left(\frac{n}{\delta_2}\right) =$ the dimensionless dissipation in the layer

$H = \delta_1 / \delta_2 =$ a parameter characterising the velocity profile

$H_{32} = \delta_3 / \delta_2$

$R_2 = U \delta_2 / \nu =$ the local Reynolds number of the boundary layer

$R = U_0 R' / \nu =$ the Reynolds number of the main stream

R' a characteristic length

$X = \delta_2 R_2 = U \delta_2^2 / \nu =$ a useful computing quantity

$\tau_0 = \mu \left(\frac{\partial u}{\partial n}\right)_{n=0} =$ the shear stress at the wall $=$ the skin friction

$\varepsilon = \left(\frac{\partial (u/U)}{\partial (n/\delta_2)}\right)_0 = \frac{\tau_0}{\rho U^2} R_2 =$ the dimensionless skin friction

$\lambda = -\frac{\delta_2^2}{U}\left(\frac{\partial^2 u}{\partial n^2}\right)_0 =$ another characteristic profile parameter

2. The Calculation Method

As the equations for axi-symmetric flow include those for plane flow and are only a little more complicated, the method may be derived for the boundary layer along a body of revolution, the co-ordinates of which are x (along the axis) and the radius $r_0(x)$. In the case of plane flow all the equations are to be simplified by putting $r_0(x) = \text{constant} \longrightarrow \infty$.

The equation of continuity is

$$\frac{\partial}{\partial s}(ru) + \frac{\partial}{\partial n}(rv) = 0 \qquad (1)$$

and the boundary layer equation

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = U\frac{dU}{ds} + \nu\frac{\partial^2 u}{\partial n^2}. \qquad (2)$$

From the latter, two ordinary differential equations can be derived by multiplying by r or ur and integrating through the layer. These are the momentum equation

$$\nu r_0\left(\frac{\partial u}{\partial n}\right)_0 = \frac{d}{ds}\int_0^\infty u(U-u)\,r\,dn + \frac{dU}{ds}\int_0^\infty (U-u)\,r\,dn \qquad (3)$$

and the energy equation

$$\nu\int_0^\infty \left(\frac{\partial u}{\partial n}\right)^2 r\,dn = \frac{d}{ds}\int_0^\infty u\left(\tfrac{1}{2}U^2 - \tfrac{1}{2}u^2\right) r\,dn. \qquad (4)$$

The physical meaning of the latter equation is that the energy dissipated by the viscosity equals the loss of kinetic energy inside the whole layer.

Equations (3) and (4) can be written more shortly by introducing the quantities defined. Then equation (3) becomes

$$\frac{dX}{ds} + \left\{(3+2H)\frac{1}{U}\frac{dU}{ds} + \frac{2}{r_0}\frac{dr_0}{ds}\right\}X - 2\varepsilon = 0. \qquad (5)$$

This is the usual differential equation for X or δ_2, the momentum thickness. Another equation for the profile parameter H can be derived by combining equations (3) and (4) as follows:

$$-\frac{dH}{ds} = f\frac{1}{U}\frac{dU}{ds} + g\frac{1}{X}, \qquad (6)$$

where
$$f = -\frac{H_{32}(H-1)}{dH_{32}/dH} \quad \text{and} \quad g = \frac{\varepsilon H_{32} - 2D}{dH_{32}/dH}. \qquad (7)$$

TABLE I

H	A	B	C	D	E	F	G	ε
2	−1·11	−0·83	6·4	0·73	2·2	7·9	0·26	0·465
	4	0·25	0·8	6	0·3	0·8	9	26
2·05	−1·07	−0·58	7·2	0·79	2·5	8·7	0·35	0·439
	4	0·27	0·9	7	0·4	0·8	0·10	25
2·1	−1·03	−0·31	8·1	0·86	2·9	9·5	0·45	0·414
	5	0·30	1·0	7	0·4	0·9	0·10	24
2·15	−0·98	−0·01	9·1	0·93	3·3	10·4	0·55	0·390
	6	0·34	1·1	9	0·5	1·0	0·10	23
2·2	−0·92	+0·33	10·2	1·02	3·8	11·4	0·65	0·367
	7	0·39	1·2	9	0·5	1·1	0·11	22
2·25	−0·85	0·72	11·4	1·11	4·3	12·5	0·76	0·345
	8	0·45	1·3	0·10	0·5	1·3	0·13	21
2·3	−0·77	1·17	12·7	1·21	4·8	13·8	0·89	0·324
	9	0·51	1·4	0·11	0·6	1·5	0·13	20
2·35	−0·68	1·68	14·1	1·32	5·4	15·3	1·02	0·304
	0·11	0·59	1·6	0·12	0·7	1·6	0·14	19
2·4	−0·57	2·27	15·7	1·44	6·1	16·9	1·16	0·285
	0·13	0·66	1·8	0·14	0·7	1·8	0·15	18
2·45	−0·44	2·93	17·5	1·58	6·8	18·7	1·31	0·267
	0·15	0·74	2·0	0·15	0·8	1·9	0·17	17
2·5	−0·29	3·67	19·5	1·73	7·6	20·6	1·48	0·250
	0·17	0·82	2·2	0·16	0·9	2·1	0·18	16
2·55	−0·12	4·49	21·7	1·89	8·5	22·7	1·66	0·234
	0·18	0·90	2·4	0·18	1·0	2·4	0·19	15
2·6	+0·06	5·39	24·1	2·07	9·5	25·1	1·85	0·219
	0·19	0·99	2·7	0·20	1·0	2·6	0·22	15
2·65	0·25	6·38	26·8	2·27	10·5	27·7	2·07	0·204
	0·21	1·09	2·9	0·22	1·1	2·9	0·23	14
2·7	0·46	7·47	29·7	2·49	11·6	30·6	2·30	0·190
	0·23	1·21	3·2	0·24	1·2	3·2	0·25	13
2·75	0·69	8·68	32·9	2·73	12·8	33·8	2·55	0·177
	0·26	$1{\cdot}3_2$	3·5	0·26	1·4	3·6	0·28	12
2·8	0·95	10·0	36·4	2·99	14·2	37·4	2·83	0·165
	0·29	1·5	3·9	0·28	1·5	4·0	0·29	11
2·85	1·24	11·5	40·3	3·27	15·7	41·4	3·12	0·154
	0·32	1·7	4·3	0·30	1·7	4·4	0·31	11
2·9	1·56	13·2	44·6	3·57	17·4	45·8	3·43	0·143
	0·36	1·9	4·7	0·32	2·0	4·8	0·33	10
2·95	1·92	15·1	49·3	3·89	19·4	50·6	3·76	0·133
	0·40	2·0	5·2	0·34	2·2	5·3	0·35	10
3·0	2·32	17·1	54·5	4·23	21·6	55·9	4·11	0·123

These equations are still exact, of course, in general it would be impossible to compute the boundary layer only by equations (5) and (6), as these two ordinary differential equations cannot replace the partial differential equation (2) and all its boundary conditions. However, restricting oneself to an approximate calculation by assuming a one parametric class of velocity profiles, H_{32}, ε and D and consequently f and g become functions of the one parameter H alone: $f(H)$ and $g(H)$. In this case obviously equations (5) and (6) enable $X(s)$ and $H(s)$ to be computed if the body $r_0(s)$ and its pressure distribution, or $U(s)$, are given.

For axi-symmetric flow this assumption of a one parametric class implies neglecting the term r/r_0 during the integration across the layer (*see* the definitions of the characteristic quantities). This is the same approximation Mangler [3] made in his paper about the transformation of an axi-symmetric boundary layer into a corresponding plane one. Since usually the layer is very thin when compared with the radius of the body of revolution, the assumption $r/r_0=1$ is good enough as r varies only between r_0 and $r_0+\delta$. The influence of the shape of the body of revolution still remains in the term $(2/r_0)(dr_0/ds)X$ in equation (5), whereas equation (6) is approximately the same whether the flow is axi-symmetric or plane, as it is independent of r_0.

The system of the two differential equations (5) and (6) is to be solved step by step. For that the step Δs may be chosen so small that $(1/U)\,dU/ds$ can be taken as constant through the integration step. Then the local pressure gradient is characterised by

$$\gamma=\frac{U_2-U_1}{(U_1+U_2)/2}\approx\frac{1}{U}\frac{dU}{ds}\Delta s=-\frac{1}{\rho U^2}\frac{dp}{ds}\Delta s, \quad . \quad . \quad (8)$$

where the index 1 denotes the beginning and 2 the end of the interval. Practically this means that the steps Δs must be made as small as is necessary to draw the curve $U(s)$ with reasonable accuracy.

In the same way all the other functions are linearised over the interval. Hence from equation (6)

$$X_2=-\frac{2(g+\frac{1}{2}\dot{g}\Delta H)}{\Delta H+(f+\frac{1}{2}\dot{f}\Delta H)\gamma}\Delta s-X_1,\quad \text{where}\quad \Delta H=H_2-H_1$$

$$\dot{f}=\frac{df}{dH}\quad\text{and}\quad \dot{g}=\frac{dg}{dH}. \quad . \quad . \quad . \quad . \quad (9)$$

Introducing this expression for X in equation (5) a quadratic equation for ΔH is obtained. Yet for small steps Δs, ΔH will also be small so that the term with $(\Delta H)^2$ can be neglected. Then

$$\Delta H=-\frac{A(1+\omega)+B\gamma+C\dfrac{X_1}{\Delta s}\gamma}{D+E\gamma+F\dfrac{X_1}{\Delta s}\gamma+\dfrac{X_1}{\Delta s}+G\omega}, \quad . \quad . \quad (10)$$

where $$\omega=\frac{1}{r_0}\frac{dr_0}{ds}\Delta s\approx\frac{(r_0)_2-(r_0)_1}{[(r_0)_1+(r_0)_2]/2}$$

($\omega=0$ for plane flow), and A, B, ... G are the following functions of H:—

$$A=g, B=\left(\frac{3}{2}+H\right)g+\varepsilon f, C=f, D=\frac{1}{2}\dot{g}+\varepsilon, E=\frac{1}{2}\dot{B}, F=\frac{1}{2}\dot{f}, G=\frac{1}{2}\dot{g}. \qquad (11)$$

In equation (10) the argument H in these functions is to be taken at the beginning of the step: $A(H_1)$, $B(H_1)$, ... $G(H_1)$. Once ΔH is computed ΔX follows from an equation derived from equation (5):

$$\Delta X=\frac{\varepsilon_1+\varepsilon_2-[(H_1+H_2+3)\gamma+2\omega]\,X_1/\Delta s}{1+\frac{1}{2}[(H_1+H_2+3)\gamma+2\omega]}\,\Delta s. \qquad (12)$$

According to its definition X gives the momentum thickness

$$\delta_2{}^2=X\nu/U \quad \text{or} \quad \frac{\delta_2}{R'}\sqrt{R}=\sqrt{\left(\frac{X/R'}{U/U_0}\right)}, \qquad (13)$$

where $R=U_0R'/\nu$ is the Reynolds number of the undisturbed stream and R' a characteristic length. With δ_2 giving the size and H the shape of the velocity profile the boundary layer is known once $X(s)$ and $H(s)$ are calculated.

To establish the functions $A, B, \ldots G$ a certain class of velocity profiles has to be chosen. Walz[2] has shown that the results do not depend very much on the particular class if the energy equation is used together with that for the momentum. This independence obviously is fundamental for the usefulness of the whole method. So the Hartree-profiles which have been proved useful before may be taken again. Hartree[4] tabulated the velocity profiles for the case $U\sim s^m$ where the boundary layer equation (2) can be tackled analytically. For each of his profiles (for various m) the integrations for δ_1, δ_2 and δ_3 were carried out numerically and the functions $A, B, \ldots G$ calculated (*see* Table I). It was not necessary to integrate the dissipation D to find $g(H)$ after equation (7), as for these profiles

$$g=-\lambda f \quad \text{with} \quad \lambda=-\frac{\delta_2{}^2}{U}\left(\frac{\partial^2 u}{\partial n^2}\right)_{n=0}. \qquad (14)$$

For, in this case, $U\sim s^m$, the velocity profile does not change its shape at all along the distance from the stagnation point s as long as m is constant. Hence $dH/ds=0$ and, according to equation (6), $g=-(1/U)(dU/ds)Xf$. Further, as these are exact solutions the general boundary condition for equation (2)

$$\lambda=\frac{\delta_2{}^2}{\nu}\frac{dU}{ds}=\frac{X}{U}\frac{dU}{ds}\left(\text{and by definition } \lambda=-\frac{\delta_2{}^2}{U}\left(\frac{\partial^2 u}{\partial n^2}\right)_0\right). \qquad (15)$$

is satisfied, and so simply $g=-\lambda f$. Yet it is just this boundary condition (15) which has been dropped in favour of the energy equation in the approximate method already described. Therefore, it will be fulfilled only approximately when the relation $U\sim s^m$ no longer holds.

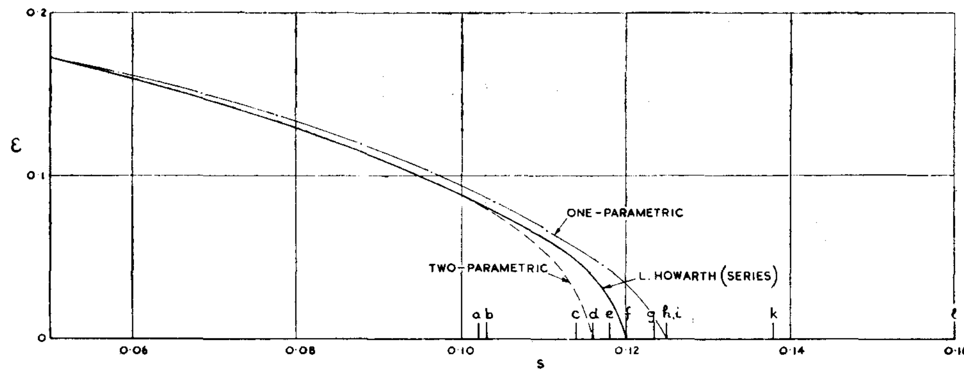

Fig. 1.

Howarth's flow: $U=1-s$. Comparison of the dimensionless skin friction ε calculated by various methods.

3. Examples

To check the accuracy of the calculation method the plane boundary layer has been computed for the case $U=1-s$, which Howarth[5] has treated analytically by series. The calculation begins at $s=0{\cdot}05$ where $\delta_2=0{\cdot}162$, $X=0{\cdot}162^2\ (1-0{\cdot}05)=0{\cdot}0249$ and $H=2{\cdot}77$, $\varepsilon=0{\cdot}172$. The result as given in Fig. 1 (the dimensionless skin friction over the distance s) deviates very little from the exact solution. The separation point is only found by extrapolating as the range of Table I is not great enough. Namely, between $H=3{\cdot}480$ and $4{\cdot}031$ (for the separation profile) no profiles are tabulated by Hartree so that an extension of Table I is impossible without computing more velocity profiles afresh. However, in practice the laminar boundary layer will turn into a turbulent one at some distance before separation. Hence it would not pay to strive after greater accuracy near to the separation point itself.

For comparison Fig. 1 gives the separation point as found by various methods when starting at $s=0{\cdot}05$:

(*a*) Th. v. Kármán-C. B. Millikan[6] $s_{sep}=0{\cdot}102$

(*b*) A. Walz (Hartree profiles; based on the momentum equation only)[7] $=0{\cdot}103$

(*c*) A. Walz (Hartree-profiles)[2] $=0{\cdot}114$

(*d*) Two parametric class of velocity profiles[1] $=0{\cdot}116$

(*e*) H. Görtler (relaxation method)[8] $=0{\cdot}118$

(*f*) L. Howarth (series)[5] $=0{\cdot}120$

(*g*) D. Meksyn (series)[9] $=0{\cdot}1235$

(*h*) Present method $=0{\cdot}125$

(*i*) A. Walz (profile: polynomial of the fourth degree)[2] $=0{\cdot}125$

(*k*) H. Schlichting (profile: polynomial of the sixth degree)[10] $=0{\cdot}138$

(*l*) K. Pohlhausen (profile: polynomial of the fourth degree)[11] $=0{\cdot}160$,

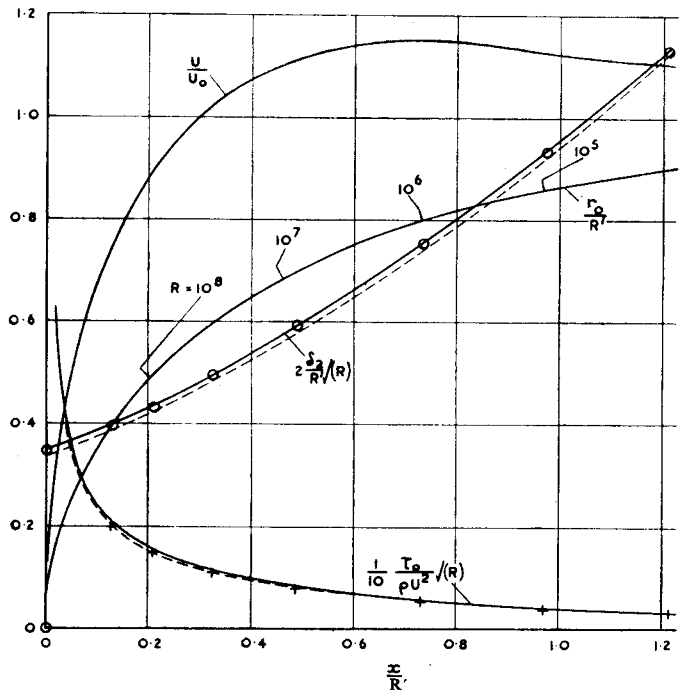

Fig. 2.

Half-body of revolution (produced by a single source): Velocity outside the boundary layer, momentum thickness, skin friction and neutral stability point for various Reynolds numbers.

Further, the boundary layer has been computed for three types of head of an infinite long body of revolution:

(*a*) half-body
(*b*) hemisphere and cylinder
(*c*) $\frac{1}{4}$-calibre rounded head and cylinder.

The pressure distribution in (*b*) and (*c*) was that calculated recently by Vandrey[12] by a new method (applicable to bodies with discontinuous curvature of the meridian). As these pressure distributions are calculated for ideal fluid the results as given in Figs. 2, 3 and 4 hold only for high Reynolds numbers.

The strokes on the surface of the bodies indicate the position of the neutral stability point at the respective Reynolds number, *i.e.* the point after which the laminar layer might become unstable. The actual transition to the turbulent boundary layer will occur between this point and the separation point. The estimation of the neutral stability point is given in the Appendix.

In (*b*), $U(s)$ was given only at a few points s, as indicated by Fig. 3, so that the actual curve is rather uncertain near the end of the head. But obviously as the separation is there in any case it does not matter very much.

Each case needed only a couple of hours' computing work.

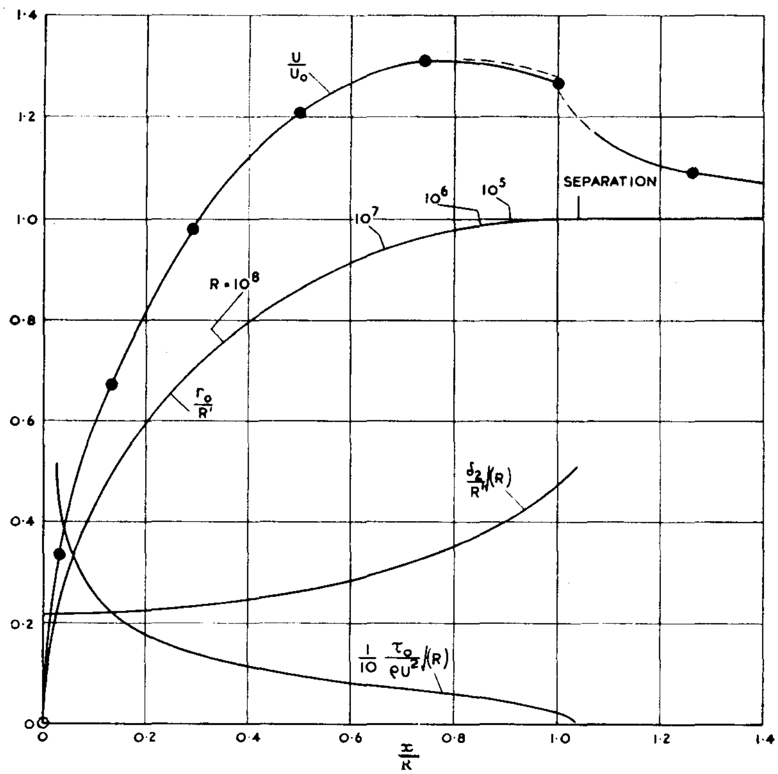

Fig. 3.

Hemispherical head: Velocity outside the boundary layer, momentum thickness, skin friction and neutral stability point for various Reynolds numbers.

4. Comparison with Another Very Simple Method

For plane boundary layers a somewhat simpler calculation method has been proposed by H. Holstein and T. Bohlen[14] and A. Walz[15] in Germany and by Young and Winterbottom[16] and, in detail, by Thwaites[17] in this country. It is based on the momentum equation (5) alone which might be written

$$\frac{dX}{ds} = -(3+2H)\frac{X}{U}\frac{dU}{ds} + 2\varepsilon. \qquad (5a)$$

Supposing that the velocity profiles can be represented by a one parametric class, the right side of (*5a*) is a function of the parameter H alone. For ε this is obvious. Yet it holds also for $(X/U)\,dU/ds$ which equals another parameter

$$\lambda(H) = -\frac{\partial^2 u/U}{\partial(n/\delta_2)^2}$$

(compare equation (15)) according to the general boundary condition which has

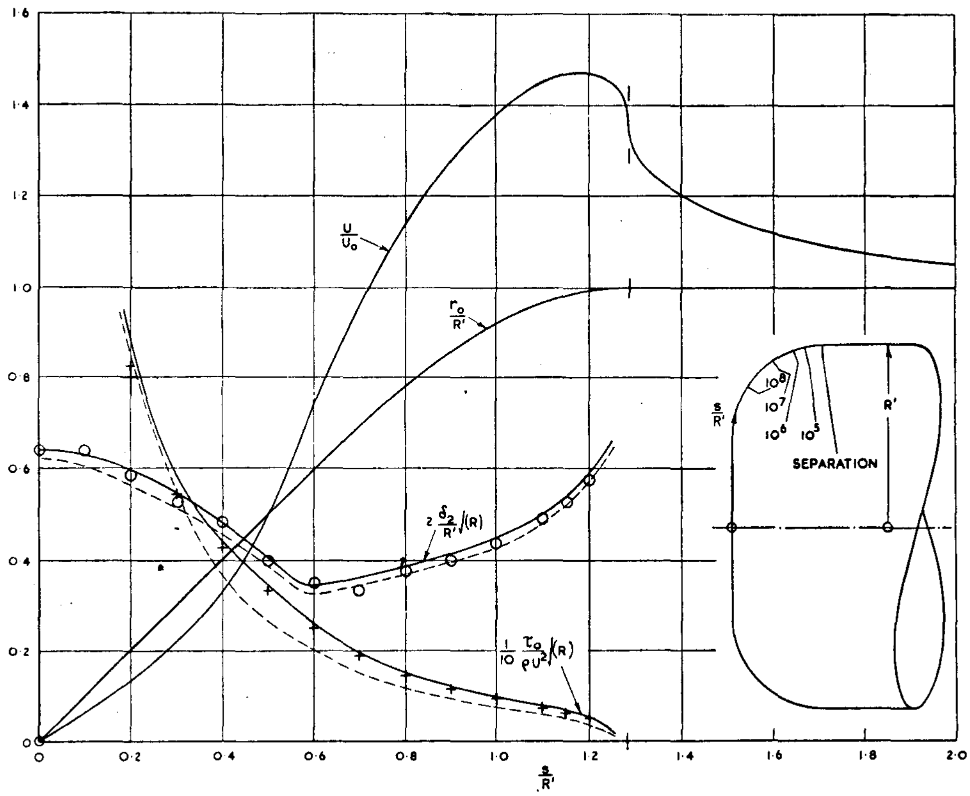

Fig. 4.

1/4-calibre head: Velocity outside the boundary layer, momentum thickness, skin friction and neutral stability point for various Reynolds numbers.

been dropped in the method already described. Hence, the left side of equation (5*a*), dX/ds, is also a function of H alone.

Using Thwaites's notation for $\lambda = -m$, $\varepsilon = l(m)$ and $L(m) = (2H+4)m + 2l$, the momentum equation becomes

$$\frac{dX}{ds} = L(m) - m. \qquad (16)$$

For various known solutions of plane boundary layer problems the relation $L(m)$ is almost the same, which proves the main assumption of a one parametric class of velocity profiles. Furthermore, $L(m)$ can be approximated by a linear function such as

$$L(m) = a + bm. \qquad (17)$$

Then equation (16) can easily be integrated to give an explicit formula

$$X = U\frac{\delta_2^2}{\nu} = \frac{a}{U^{b-1}}\int_0^s U^{b-1}\,ds, \qquad . \quad . \quad . \quad (18)$$

and the parameter is to be found by

$$m = -\frac{X}{U}\frac{dU}{ds}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

For axi-symmetric flow Mangler's transformation[3] alters this equation (18) into

$$X = \frac{a}{U^{b-1}}\frac{1}{r_0^2}\int_0^s U^{b-1}\,r_0^2\,ds \qquad . \quad . \quad . \quad . \quad (20)$$

with $r_0\,(s)$=local radius of the body of revolution. This might also be written

$$\frac{\delta_2^2}{R'^2}R = \frac{a}{(U/U_0)^b}\,\frac{1}{r_0/R'}\int_0^{s/R'}(U/U_0)^{b-1}\,(r_0/R')^2\,ds/R', \qquad . \quad . \quad (21)$$

$$m = -\frac{dU/U_0}{ds/R'}\left(\frac{\delta_2}{R'}\sqrt{R}\right)^2 \qquad . \quad (22)$$

and

$$\frac{\tau_0}{\rho U^2}\sqrt{R} = \frac{1}{U/U_0}\,\frac{R'}{\delta_2\sqrt{R}}\,l\,(m), \qquad . \quad . \quad . \quad . \quad (23)$$

with $l\,(m)$ after Table I in (17). The results of using this formula are given for two examples in Figs. 2 and 4 (half-body and ¼-calibre rounded head). The broken lines correspond to the constants as proposed by Thwaites[17], a=0·45 and b=6.

As he was interested mainly in (plane) cases with increasing pressure—whereas in these examples the region with decreasing pressure is of importance—better results (single marked points in Figs. 2 and 4) are reached by a=0·45 and b=5·5*.

Thus it is shown that this simple formula, (18) or (20), can be used for axi-symmetric flow as well as for plane flow with an accuracy sufficient for most practical purposes. But it seems advisable to make b=5·5 as long as $dU/ds > 0$ and afterwards b=6 for $dU/ds < 0$, with a=0·45 in both cases.

APPENDIX

For practical use some known formulae are given, (a) for the flow near the stagnation point and, (b) to estimate the indifference point.

Note added in proof: Meantime E. TRUCKENBRODT, *Ing.-Arch.* 20 (1952), p. 211, proposed the same method with a=0·47 and b=6·0.

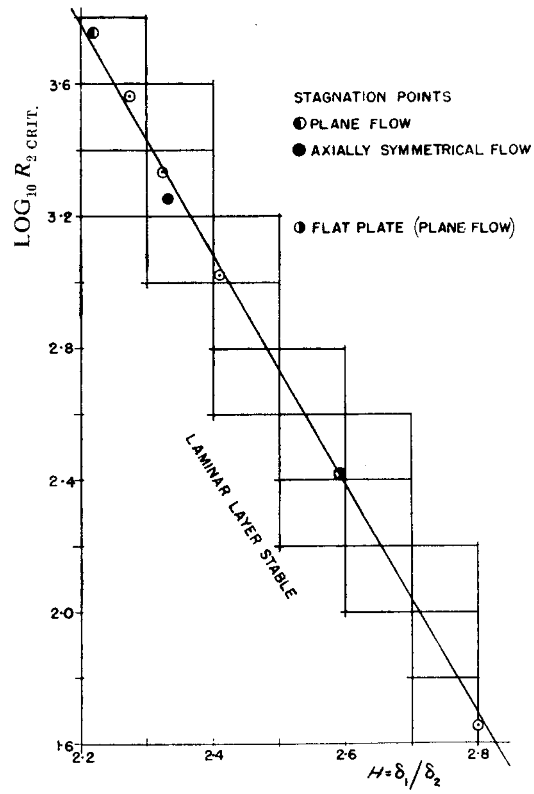

Fig. 5.

Diagram for estimating the neutral stability point from the profile parameter H and the local Reynolds number of the boundary layer (for Hartree profiles).

(*a*) *Flow near the stagnation point*

(α) Two-dimensional:

$$U/U_0=u_1\,(s/R'),\quad H=2{\cdot}21_7,\quad \varepsilon=0{\cdot}36_0,\quad X=0{\cdot}0845\,s.$$

$$\frac{\tau_0}{\rho U^2}\sqrt{R}=\frac{1{\cdot}233}{\sqrt{u_1}}\,\frac{1}{s/R'},\quad \frac{\delta_2}{R'}\sqrt{R}=\frac{0{\cdot}292}{\sqrt{u_1}}.$$

(β) Axi-symmetric:

$$U/U_0=u_1\,(s/R'),\quad H=2{\cdot}29_7,\quad \varepsilon=0{\cdot}32_5,\quad X=0{\cdot}0613\,s,$$

$$\frac{\tau_0}{\rho U^2}\sqrt{R}=\frac{1{\cdot}312}{\sqrt{u_1}}\,\frac{1}{s/R'},\quad \frac{\delta_2}{R'}\sqrt{R}=\frac{0{\cdot}248}{\sqrt{u_1}}.$$

(*b*) *Neutral stability point*

Schlichting[10] and Pretsch[13] have shown that it is possible to attach to each boundary layer profile approximately a critical Reynolds number:

$$R_{2\mathrm{crit}} = (U\,\delta_2/\nu)_{\mathrm{crit}}.$$

If the actual R_2 is smaller, the laminar layer is stable in any case; if $R_2 > R_{2\mathrm{crit}}$ the layer is unstable to certain disturbances so that transition can occur. The real transition to the turbulent layer will take place somewhere between this neutral stability point and the theoretical separation point of the laminar boundary layer.

Usually $R_{2\mathrm{crit}}$ is given as a function of the parameter λ. Yet as λ is determined by H for one parametric classes, Fig. 5 gives directly $R_{2\mathrm{crit}}$—actually the logarithm of it—depending on H.

This diagram holds for axially symmetrical flow as well as for plane flow. It is to be seen that a good approximation is given by the straight line corresponding to

$$(R_2)_{\mathrm{crit}} = e^{26\cdot 3 - 8H} \qquad (24)$$

When $X(s)$ and $H(s)$ have been calculated the critical Reynolds number of the main flow $R_{\mathrm{crit}} = (U_0\,R'/\nu)_{\mathrm{crit}}$ can be estimated by (24) for any distance s of the neutral stability point from the stagnation point, respectively the position of the neutral stability point s for a given Reynolds number R. For this it is practical to write (24) as follows

$$\log_{10} R_{\mathrm{crit}} = 7\cdot 55 - 6\cdot 95\,(H - 2\cdot 2) - \log_{10}\left(\frac{X}{R'}\,\frac{U}{U_0}\right). \qquad (25)$$

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