Extremes
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Large components in the subcritical Norros-Reittu model

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Abstract
The Norros-Reittu model is a random (multi-)graph with n vertices and i.i.d. weights assigned to them. The number of edges between any two vertices follows an independent Poisson distribution whose parameter is increasing in the weights of the two vertices. Choosing a suitable weight distribution leads to a power-law behaviour of the degree distribution as observed in many real-world complex networks. We study this model in the subcritical regime, i.e. in the absence of a giant component. For each component, we count all its vertices to determine the component sizes and show convergence of the corresponding point process of (rescaled) component sizes to a Poisson process. More generally, one can also count only specific vertices per component, like leaves. From this one can deduce asymptotic results on the size of the largest component or the maximal number of leaves in a single component. The results also apply to the Chung-Lu model and the generalised random graph.

Keywords Norros-Reittu model · Poisson process convergence · (extremal) Counting statistics · Order statistics · Subcritical regime · Power law · Rank-1 inhomogeneous random graphs

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1 Introduction and main results

Complex networks are very large graphs with a difficult structure, which arise in a lot of different fields, ranging from biology and computer science over epidemiology to sociology. One can think of the brain, (tele-)communication networks, the internet or even root systems of trees, to name some more explicit examples. Even though

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these graphs appear in very different contexts, surprisingly they often share a couple of properties. One of them is the scale-free behaviour, which is also called a power-law. This means that the proportion of vertices having degree k is approximately proportional to $k^{-\beta-1}$ for some $\beta > 0$, resulting in the proportion of vertices having degree at least k being approximately proportional to $k^{-\beta}$, which can be weakened to $k^{-\beta}\ell(k)$ for some slowly varying function ℓ. While the first way of phrasing might be more intuitive, the latter allows for a description in terms of the tail of the typical degree distribution of the graph. See van der Hofstad (2017) for detailed information on complex networks and the survey article Voitalov et al. (2019) for the scale-free property. Due to the interest in complex networks there has been a lot of attention on finding random graph models which show a scale-free behaviour and ideally also share other properties of real-world complex networks. One of these random graphs is the Norros-Reittu model introduced in Norros and Reittu (2006), which we study in this paper. We define the model rigorously below, but essentially one takes n vertices and assigns random weights to them, which govern the probabilities of joining edges between vertices. While the Norros-Reittu model allows for multiple edges between two vertices as well as loops, i.e. edges from a vertex to itself, one can merge multiple edges into normal edges and delete loops to obtain a graph, which we call the erased Norros-Reittu model. Note that we consider a situation where the erased Norros-Reittu model is asymptotically equivalent to similar models such as the Chung-Lu model from Chung and Lu (2002a, b) or the generalised random graph from Britton et al. (2006) as shown in Example 3.6 in Janson (2010). These three models are all constructed on n vertices equipped with weights, but use slightly different connection probabilities, see (3), (4) and (5) below. This class of models is also referred to as rank-1 inhomogeneous random graphs because the connection probabilities are approximately given by a rank-1 matrix, see also Remark 1.5 in van der Hofstad (2024).

In this paper we consider the subcritical regime in which no giant component exists, i.e. the proportion of all n vertices which lie in the largest component converges in probability to zero as $n \to \infty$. In every component we count all its vertices. We study the collection of these counting statistics and prove convergence to a Poisson process. Our framework also allows to count instead of all vertices in a component only vertices of a certain type, such as leaves or vertices of a fixed degree.

The rest of this paper is organised as follows. In the remainder of this section we present and discuss our main results after introducing the required notation. All proofs are postponed to Section 2.

We consider a sequence of (multi-)graphs $(G_n)_{n\in\mathbb{N}}$ given by the Norros-Reittu model, which is defined as follows. Let $\mathcal{W} = (W_i)_{i\in\mathbb{N}}$ be a sequence of independent copies of a positive random variable W. For $n \in \mathbb{N}$, one takes $[n] = \{1, \dots, n\}$ as vertex set of G_n and assigns the weights $W_1, \dots, W_n$ to the vertices. For $x, y \in [n]$, the number of edges between x and y in G_n is given by the $\mathbb{N}_0$-valued random variable $E_n\{x, y\}$, where the random variables $(E_n\{x, y\})_{1\le x\le y\le n}$ are, conditionally on $\mathcal{W}$, independent with

$$E_n\{x, y\} \sim \text{Poisson}(W_x W_y / L_n) \quad \text{for} \quad L_n = \sum_{i=1}^{n} W_i .$$

The normalisation by L_n yields that the number of edges incident to some vertex $x \in [n]$ follows a mixed Poisson distribution whose parameter is

$$\sum_{y=1}^{n} \frac{W_x W_y}{L_n} = W_x \frac{L_n}{L_n} = W_x,$$

i.e. one counts edges according to their multiplicity and uses the convention to count each loop just once. We require the following assumptions on the weight distribution throughout this paper:

(W) The distribution of the positive random variable W has a regularly varying tail with exponent $\beta > 2$ and $\mathbf{E}[W^2] < \mathbf{E}[W]$.

The first part of the previous assumption means that

$$\mathbf{P}(W > t) = t^{-\beta} L(t) \quad \text{for} \quad t > 0,$$

where $L : (0, \infty) \to (0, \infty)$ is a slowly varying function, i.e. for all $c > 0$,

$$\lim_{t \to \infty} \frac{L(ct)}{L(t)} = 1.$$

For more details on regular variation and slowly varying functions we refer to Chapter 2 in Resnick (2007) or to Bingham et al. (1987). Note that $\beta > 2$ in (W) implies the existence of the second moment of W.

Since the degree of any vertex follows a Poisson(W) distribution as discussed above, the regularly varying tail of W ensures that the graph is scale-free in the sense that the degree distribution has a regularly varying tail with exponent β, which can be proven along the lines of Corollary 13.1 in Bollobás et al. (2007). The condition $\mathbf{E}[W^2] < \mathbf{E}[W]$ corresponds to the subcritical regime concerning the phase transition of the component sizes. In Bollobás et al. (2007) it has been shown in Theorem 3.1, see also Section 16.4, that the graph exhibits a giant component, that is a component which contains a positive fraction of all vertices, if and only if $\mathbf{E}[W^2] > \mathbf{E}[W]$ - this is also called the supercritical regime or supercritical phase. The maximal component size in the case $\mathbf{E}[W^2] = \mathbf{E}[W]$, which is called the critical regime, has been addressed in Bhamidi et al. (2010) and Bhamidi et al. (2012). It is shown in Bhamidi et al. (2010) that if $\beta > 3$, which yields a finite third moment of W, the largest component is of order $n^{2/3}$ whereas (Bhamidi et al. 2012) provides the order $n^{(\beta-1)/\beta}$ for $\beta \in (2, 3)$, where W has an infinite third moment. The latter result requires the additional assumption that $L(t)$ converges to a constant as $t \to \infty$. For the subcritical regime where $\mathbf{E}[W^2] < \mathbf{E}[W]$ it is shown in Janson (2008) that the order of the largest component is at most $n^{1/\beta}$ and related to the maximal degree of the graph, under the assumption that $\mathbf{P}(W > t) = O(t^{-\beta})$. We complement this result by deriving the asymptotic distribution.

We define $q(n) = F^{-1}(1 - 1/n)$ for $n \in \mathbb{N}$ where F^{-1} denotes the quantile function of the distribution function F of W. For two vertices $x, y \in [n]$ we write

$x \sim y$ if x and y are connected via a path in G_n and denote the component of x by $\mathcal{C}_n(x) = \{z \in [n] \colon x \sim z\}$. To ensure that $x \in \mathcal{C}_n(x)$ for all $x \in [n]$, we use the convention that a path may consist of just a single vertex. Throughout the paper we denote by $\mathbf{P}_{\mathcal{W}}$ and $\mathbf{E}_{\mathcal{W}}$ the conditional probability and expectation when conditioning on the weights $\mathcal{W} = (W_i)_{i\in\mathbb{N}}$. For all $n \in \mathbb{N}$ we consider a function $(v, G_n) \mapsto \mathcal{X}_n(v) \subseteq [n]$ where $v \in [n]$ and assume that $\mathcal{X}_n(v) \subseteq \mathcal{C}_n(v)$. For $n \in \mathbb{N}$ and $v \in [n]$, $\mathcal{X}_n(v)$ is the class of vertices we would like to count in the component of v in G_n. We write $S_n(v)$ for the cardinality of $\mathcal{X}_n(v)$ and require the following two assumptions.

(A1) There exists some $\xi > 0$ such that

$$\frac{1}{q(n)} \sup_{v=1,\dots,n} |\mathbf{E}_{\mathcal{W}}[S_n(v)] - W_v\xi| \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty.$$

(A2) Let $n \in \mathbb{N}, v \in [n]$ and $x \in \mathcal{C}_n(v)$. Checking whether $x \in \mathcal{X}_n(v)$ only depends on all paths that start in v and contain x.

In this paper, thus also in assumption (A2), we denote by a path a sequence of *distinct* vertices $v_1 \dots v_k$ such that v_i and v_{i+1} are connected by at least one edge for $i \in [k-1]$. Consequently, assumption (A2) implies that whether $x \in \mathcal{X}_n(v)$ is satisfied cannot depend on loops or multiple edges. In contrast to (A2), (A1) also concerns the weight distribution. However, no further restrictions than in (W) are needed for our examples.

Finally, we choose from every component in G_n one vertex with the largest weight. If the largest weight is not unique, we choose the one with the smallest label. We denote the collection of these vertices by $V_n^{\max} \subseteq [n]$. We study the point processes

$$\Xi_n = \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}\}\delta_{S_n(v)q(n)^{-1}\xi^{-1}}$$

for $n \in \mathbb{N}$, where the constant $\xi > 0$ is specified in assumption (A1). Here, δ_x stands for the Dirac measure concentrated at $x \in \mathbb{R}$ and we think of point processes as random counting measures. The indicator demanding the weight of v being maximal in its component ensures that we only count each component once in Ξ_n. Note that the quantities $S_n(x)$ and $S_n(y)$ may differ for two distinct representatives $x, y \in \mathcal{C}_n(x)$, but only those of vertices belonging to $V_n^{\max}$ are relevant for us. In Theorem 3 we provide several possible choices for $\mathcal{X}_n(v)$ and thus also for $S_n(v)$, some of which depend on the choice of the representative.

Our main theorem shows point process convergence in $M_p((0,\infty])$, the space of point measures on $(0,\infty]$. We will not discuss this space in detail but instead refer to Resnick (2007), Chapter 3.

Theorem 1 *For* $n \in \mathbb{N}$, *consider the Norros-Reittu model with weights satisfying assumption* (W) *and* $(\mathcal{X}_n(v))_{v\in[n]}$ *such that assumptions* (A1) *and* (A2) *hold. Then*

$$\Xi_n = \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}\}\delta_{S_n(v)q(n)^{-1}\xi^{-1}} \xrightarrow{d} \eta_\beta \quad \textit{as} \quad n \to \infty, \tag{1}$$

where η_β is a Poisson process with intensity measure $\nu_\beta((a,b]) = a^{-\beta} - b^{-\beta}$ for all $0 < a < b \leq \infty$.

Note that the points of Ξ_n at zero are not taken into account in (1) as we work on the space of point processes on $(0,\infty]$. The key idea to prove Theorem 1 is to approximate $S_n(v)$ by $W_v\xi$ and show that Ξ_n is close to $\Theta_n = \sum_{v=1}^{n} \delta_{W_v q(n)^{-1}}$, the collection of rescaled weights, which converges to η_β as $n \to \infty$. A similar strategy was already employed in Bhattacharjee and Schulte (2022) to study large degrees of some random graphs. However, our analysis is more involved as the considered statistics of the components are less local than degrees. For example, assumption (A1) is obviously satisfied for the degrees, whereas we have to verify it by hand. Another issue of these dependencies concerns a certain variance bound, see Lemma 7, which again is immediate for the degree case. Here, we use the Poissonian nature of the Norros-Reittu model which allows us to use the Poincaré inequality for Poisson functionals, see e.g. Theorem 18.7 in Last and Penrose (2018).

From the point process convergence in Theorem 1 we deduce the following asymptotic behaviour of the maximum of $S_n(1), \ldots, S_n(n)$.

Corollary 2 *Under the same assumptions as in Theorem 1*

$$\frac{1}{q(n)\xi} \max_{v \in V_n^{\max}} S_n(v) \xrightarrow{d} Z_\beta \quad as \quad n \to \infty,$$

where Z_β is a random variable following a Fréchet distribution with parameter β.

Corollary 2 follows immediately from Theorem 1. The distribution function of the left-hand side in the corollary is given by void probabilities of the intervals $(t,\infty]$, $t \in \mathbb{R}$, of the underlying point process Ξ_n. Since these intervals are relatively compact in the usual topology on $(0,\infty]$, we can pass on to the limiting process η_β, see e.g. Theorem 3.2 in Resnick (2007), to obtain the result.

In the next theorem we provide some classes of vertices that satisfy the assumptions of Theorem 1. For two vertices $x, y \in [n]$ of G_n let $\mathrm{d}_{G_n}(x,y)$ denote the graph distance between x and y which is the number of edges of the shortest path between x and y and let $\deg_{G_n}(x)$ stand for the degree of x which is the number of neighbours of x. Since we consider a multi-graph with loops, this is not necessarily the same as the number of incident edges, see the third paragraph after the following theorem for a brief discussion on this matter.

Theorem 3 *Under assumption* (W)*, the following classes of vertices satisfy* (1) *with ξ as stated below.*

1. ***All vertices:*** $\mathcal{X}_n(v) = \mathcal{C}_n(v)$ *with* $\xi = \frac{\mathbf{E}[W]}{\mathbf{E}[W]-\mathbf{E}[W^2]}$.
2. ***Vertices in a fixed distance*** $m \in \mathbb{N}$ ***to*** v*:* $\mathcal{X}_n(v) = \{x \in \mathcal{C}_n(v) \colon \mathrm{d}_{G_n}(x,v) = m\}$ *with* $\xi = \left(\frac{\mathbf{E}[W^2]}{\mathbf{E}[W]}\right)^{m-1}$.
3. ***Vertices with fixed degree*** $m \in \mathbb{N}$*:* $\mathcal{X}_n(v) = \{x \in \mathcal{C}_n(v) \colon \deg_{G_n}(x) = m\}$ *with* $\xi = \frac{1}{(m-1)!} \frac{\mathbf{E}[W^m e^{-W}]}{\mathbf{E}[W]-\mathbf{E}[W^2]}$.

4. ***Terminal trees:*** *Let $m \in \mathbb{N}$ and consider a rooted tree T with vertex set $V(T) = [m]$ and root 1. For $x \in [n]$ we say that $x \in \mathcal{X}_n(v)$ if and only if there is exactly one path from v to x and the subgraph generated by x and its descendants with x as root is isomorphic to T, ignoring loops and multiple edges. We have*

$$\xi = \frac{1}{c(T)} \frac{\mathbf{E}[W^{\deg_T(1)+1} e^{-W}]}{\mathbf{E}[W] - \mathbf{E}[W^2]} \prod_{i=2}^{m} \frac{\mathbf{E}[W^{\deg_T(i)} e^{-W}]}{\mathbf{E}[W]},$$

where $c(T)$ is the order of the automorphism group of T that preserves the root and $\deg_T$ denotes the degree of a vertex in the tree T.

We wish to add some remarks on the classes of vertices in Theorem 3, starting with related work. The first class of vertices in Theorem 3 results in $S_n(v)$ being the component size of $v \in [n]$. Therefore, Corollary 2 provides asymptotic results on the size of the largest component. In Janson (2008) the author shows in a similar setting that the k-th largest component size satisfies $|\mathcal{C}_{(k)}| = \xi d_{(k)} + o_p(q(n))$ where $d_{(k)}$ denotes the k-th largest degree of the graph. The idea here is to do a breadth-first exploration of the component, starting in the vertex having the largest degree, which is similar in spirit to our approach. The proof is based on finding lower and upper bounds for $|\mathcal{C}_{(k)}|$ by a coupling to the configuration model. It might be possible to use a similar approach in our setting when considering, say, vertices of fixed degree. In this case, the main obstacle resides in finding suitable lower and upper bounds because not every vertex uncovered in the exploration process contributes to the vertex count.

In the second class of vertices we are interested in the number of vertices in a certain distance from a fixed vertex. Counting the vertices in distance $m = 1$ of some vertex provides its degree so that one studies the point process of degrees of vertices having maximal weight in their component in Theorem 3. In that case, our result yields the same limiting process as obtained in Bhattacharjee and Schulte (2022), where the point process of all degrees was investigated. If one puts together the results from Janson (2008) mentioned in the previous paragraph and Bhattacharjee and Schulte (2022), one can also conclude the first claim of Theorem 3.

Contrary to our convention of counting neighbours, the degree in Bhattacharjee and Schulte (2022) is the number of incident edges, including loops and multiplicities, which we here refer to by $\deg'_{G_n}$. This leads to the question whether we can count the incident edges for vertices with maximal weights in their components to study this type of degree. Unfortunately, the usage of $\deg'_{G_n}$ does not fit into our framework as we are studying quantities $S_n(v) = |\mathcal{X}_n(v)|$ for $v \in [n]$ and $\mathcal{X}_n(v) \subseteq \mathcal{C}_n(v)$. Therefore, every vertex $x \in \mathcal{C}_n(v)$ can contribute at most one to the counting statistic $S_n(v)$. For $\deg'_{G_n}(v)$ instead, every vertex has to contribute according to the number of edges connecting it to v. Moreover, assumption (A2) does not provide information on multiple edges and loops. Thus, the third class of vertices in Theorem 3 also cannot use $\deg'_{G_n}$ instead of $\deg_{G_n}$. However, under (W) one can show that there will asymptotically be neither loops nor multiple edges in components that contain a vertex of large weight. Therefore, assumption (A2) denying the usage of information regarding multiple edges and loops is no real restriction. Moreover, one observes the same asymptotic behaviour for both degree notions even though the second one does not fit directly into our framework.

The fourth class of vertices in Theorem 3 is motivated by the fact that the large components are tree-like as $n \to \infty$, which follows from the local convergence we discuss in the next paragraph. We are able to count the number of subgraphs which are isomorphic to a given rooted tree and are "terminal" in the sense that they only spread away from the vertex $v \in V_n^{\max}$ (in the graph distance sense) and have no further edges attached to them. See also Fig. 1 for a visualisation where the tree T is a wedge with the vertex of degree 2 as root. In the picture one can see a tree with hub v in which there are two vertices that give birth to a terminal wedge, v_3 and v_9. The vertex v itself does not qualify although v, v_1, v_2 form a wedge, because there are further edges. Similarly, the vertex v_4 is a root of the wedge v_4, v_8, v_9, which is not terminal since v_9 has more neighbours than just v_4. The subgraph induced by v, v_2, v_5 with v_2 as root forms also a wedge, but is not terminal since the wedge spreads from its root v_2 towards v.

We would like to convey some heuristic explanations for the values of ξ in Theorem 3. By Theorem 3.18 in van der Hofstad (2024), the neighbourhoods of vertices in the Norros-Reittu model look like unimodular Galton-Watson trees as $n \to \infty$, see Subsection 1.5 in van der Hofstad (2024) for a definition of this object and Section 2 therein for a rigorous treatment of the underlying convergence. Intuitively, one has the following: When one fixes a vertex v of the Norros-Reittu model and explores its neighbourhood, it looks asymptotically as if one attaches roughly Poisson(W_v) many neighbours to it and then starts independent Galton-Watson trees in each of the neighbours whose offspring distribution satisfies

$$p_k = \mathbf{P}(\text{number of offsprings} = k) = \frac{\mathbf{E}[W^{k+1}e^{-W}]}{k!\mathbf{E}[W]} \tag{2}$$

for $k \in \mathbb{N}_0$. Note that the offspring distribution has the same distribution as $D^s - 1$, where D^s follows the size-biased degree distribution, and has the mean $\mu = \mathbf{E}[W^2]/\mathbf{E}[W]$. Since the total progeny of a (subcritical) Galton-Watson tree equals $1/(1-\mu)$, see e.g. Theorem 3.5 in van der Hofstad (2017), one obtains for the component size of v roughly

$$\mathbf{E}\big[|\mathcal{C}(v)|\big|W_v\big] \approx W_v \frac{1}{1-\mu} = W_v \frac{\mathbf{E}[W]}{\mathbf{E}[W] - \mathbf{E}[W^2]} = W_v \xi$$

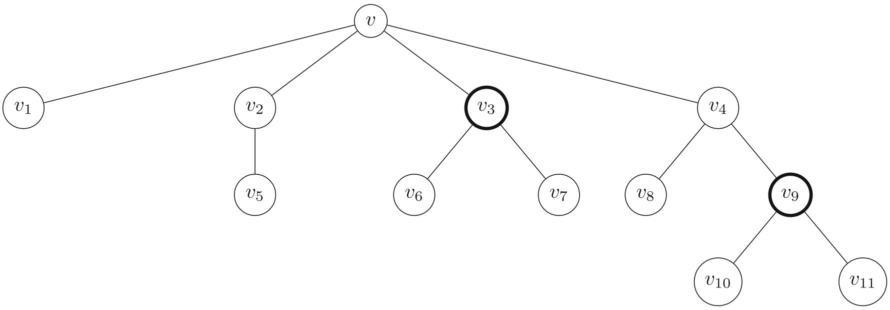

Fig. 1 Counting terminal wedges in $\mathcal{C}_n(v)$, here with v_3 and v_9 as roots

with ξ as in the first claim. Similarly, as the expected number of individuals in the k-th generation of a Galton-Watson tree with mean offspring μ equals μ^k, see e.g. Theorem 3.3 in van der Hofstad (2017), one obtains our choice for ξ in the second claim. Note here that v is, so to say, in generation -1 so that vertices having distance m to v are in generation $m-1$. If one is interested in vertices with a fixed degree $m \in \mathbb{N}$ as in the third claim, this means that an individual needs to have exactly $m-1$ descendants. If one approximates the number of vertices having degree m by multiplying the expected component size with the probability that an individual has $m-1$ descendants, one obtains with the formulae above

$$\mathbf{E}\big[S_n(v)\big|W_v\big] \approx \mathbf{E}\big[|\mathcal{C}_n(v)|\big|W_v\big]p_{m-1} \approx W_v \frac{\mathbf{E}[W]}{\mathbf{E}[W]-\mathbf{E}[W^2]} \frac{\mathbf{E}[W^m e^{-W}]}{(m-1)!\mathbf{E}[W]},$$

which yields the choice of ξ in the third claim. The ξ in the fourth claim is recovered by a similar argument, where one needs to fix the number of descendants for all individuals involved in the tree.

Finally, it would be possible to study vertices of a certain degree or the terminal trees from the fourth example in a fixed distance $k \in \mathbb{N}$ to v, as in the second example. This of course affects the choice of ξ. Following the intuition on the values of ξ above, one needs to multiply the values of ξ in the respective settings with

$$\left(\frac{\mathbf{E}[W^2]}{\mathbf{E}[W]}\right)^{k-1} \frac{\mathbf{E}[W]-\mathbf{E}[W^2]}{\mathbf{E}[W]},$$

which corresponds to the ratio of the values for ξ in the second and first claim of Theorem 3, as we do not consider all vertices, but only those in distance k.

After borrowing results from local convergence for intuition, the question arises whether one could use local convergence to rigorously prove statements in our context. First of all, one could try to verify assumption (A1) with the heuristics provided above, but this requires the conditional expectation with respect to all weights, not just that of v, and it is unclear how to treat the supremum. One could also wonder if Theorem 1 can be proven directly via local convergence. The key point in our proof is to control the probabilities of particular events for given vertices. Due to some rescalings, these events depend on n and thus do not seem to fit into the framework of local convergence.

We conclude this section with a transfer of the results to related models. Write $p_{ij} = W_i W_j / L_n$ for $i, j \in [n]$ and $n \in \mathbb{N}$. For the following discussion it is helpful to think of p_{ij} as desired connection probability for the vertices i and j. However, p_{ij} is not necessarily bounded by one and the different models use different ways to cope with this problem. In the Norros-Reittu model denoted by $\mathrm{NR}(n)$ two distinct vertices $i, j \in [n]$ are connected by (at least) one edge with probability

$$\mathbf{P}_{\mathcal{W}}(i \leftrightarrow j) = 1 - \exp(-p_{ij}), \tag{3}$$

conditionally on the weights $\mathcal{W}$. We write $\mathrm{ENR}(n)$ for the corresponding erased Norros-Reittu model without loops or multiple edges. There are a couple of similar models, for example the Chung-Lu model presented by Chung and Lu (2002a, b) denoted by $\mathrm{CL}(n)$, where

$$\mathbf{P}_{\mathcal{W}}(i \leftrightarrow j) = \min(1, p_{ij}), \tag{4}$$

or the generalised random graph model considered by Britton et al. (2006) we call $\mathrm{GRG}(n)$ with

$$\mathbf{P}_{\mathcal{W}}(i \leftrightarrow j) = \frac{p_{ij}}{p_{ij}+1} \tag{5}$$

for distinct vertices $i, j \in [n]$. Note that we consider both $\mathrm{CL}(n)$ and $\mathrm{GRG}(n)$ with slight modifications from their original form as in van den Esker et al. (2008) (see also Chapter 6 in van der Hofstad (2017)). We come back to the idea of p_{ij} being the desired connection probability, which is typically small. Then it is clear that the right-hand sides of (4) and (5) are close to p_{ij}, while a first order Taylor expansion shows the same for (3). Therefore, one can expect the three random graph models $\mathrm{ENR}(n)$, $\mathrm{CL}(n)$ and $\mathrm{GRG}(n)$ to behave similarly. These heuristics are made precise in Example 3.6 from Janson (2010) and we will come back to it in Subsection 2.3.

From the strong law of large numbers we obtain that $n^{-1}L_n \to \mathbf{E}[W]$ almost surely as $n \to \infty$. This paves the way for another slight variation of $\mathrm{ENR}(n)$, $\mathrm{CL}(n)$ and $\mathrm{GRG}(n)$ one can find in the literature, namely when one replaces p_{ij} in (3), (4) and (5) by $p'_{ij} = W_i W_j/(n\mathbf{E}[W])$. We denote the corresponding models by $\mathrm{ENR}'(n)$, $\mathrm{CL}'(n)$ and $\mathrm{GRG}'(n)$. For the same reason as in the previous paragraph, $\mathrm{ENR}'(n)$, $\mathrm{CL}'(n)$ and $\mathrm{GRG}'(n)$ can be expected to behave similarly and Example 3.1 in Janson (2010) makes this precise. Note that some authors also use $\tilde{p}_{ij} = W_i W_j/n$ instead of p'_{ij}, but this amounts to simply rescaling the weights.

The following theorem concerns the applicability of our results when considering one of the related models above instead of the multigraph $\mathrm{NR}(n)$. To this end, we define the set $\mathcal{G} = \{\mathrm{ENR}, \mathrm{CL}, \mathrm{GRG}\}$.

Theorem 4 *Under assumption* (W) *the following hold.*

1. *Let* $(\mathcal{X}_n(v))_{v\in[n],n\in\mathbb{N}}$ *be identical for the Norros-Reittu model* $\mathrm{NR}(n)$ *and its erased version* $\mathrm{ENR}(n)$*, i.e.* $\mathcal{X}_n(v)$ *does not take loops or multiple edges into account. If* (1) *is valid for the Norros-Reittu model* $\mathrm{NR}(n)$*, then it also holds for the random graph models* $G(n)$ *with* $G \in \mathcal{G}$.
2. *Suppose that the assumptions (A1) and (A2) hold true for* $\mathrm{NR}(n)$*. Then, the convergence results from Theorem 1 and Corollary 2 remain true when considering* $G(n)$ *and* $G'(n)$ *for* $G \in \mathcal{G}$ *instead of* $\mathrm{NR}(n)$.
3. *For all* $G \in \mathcal{G}$*, Theorem 3 remains true when considering* $G(n)$ *and* $G'(n)$ *instead of* $\mathrm{NR}(n)$.

Note that the first part of Theorem 4 does not require assumptions (A1) and (A2) to hold. However, as soon as assumption (A2) is satisfied, $(\mathcal{X}_n(v))_{v\in[n],n\in\mathbb{N}}$ coincide for the Norros-Reittu model and its erased version since neither multiple edges nor loops are taken into account.

While we use techniques from Janson (2010) to prove Theorem 4 by transferring the result for the Norros-Reittu model to the related random graphs, one might also be able to adapt our proof strategy for the Norros-Reittu model to these models.

2 Proofs

2.1 Proof of Theorem 1

Before going to the graph-related proofs we need a fact about the asymptotic behaviour of the maximum of the first n weights $W_1, \dots, W_n$ under assumption (W). It is part of a well-known result from extreme value theory called Fisher-Tippet-Gnedenko theorem, see e.g. Theorem 1.2.1 and Corollary 1.2.4 in de Haan and Ferreira (2006). For $n \in \mathbb{N}$ write $W_{(n)} = \max_{i=1,\dots,n} W_i$. It holds that

$$\frac{W_{(n)}}{q(n)} \xrightarrow{d} Z_\beta \quad \text{as} \quad n \to \infty, \tag{6}$$

where Z_β is a random variable following a Fréchet distribution with parameter β. For the scaling factor $q(n)$ we obtain from Lemma 3.3 in Bhattacharjee and Schulte (2022) that

$$q(n) = F^{-1}(1 - 1/n) = n^{1/\beta}\ell(n) \tag{7}$$

for $n \in \mathbb{N}$ with some slowly varying function ℓ. In particular it follows that $n^{-1/2} W_{(n)}$ converges in probability to zero as $n \to \infty$, due to $\beta > 2$ from assumption (W) and the fact that the slowly varying function $\ell(n)$ grows slower than any positive power of n, see Proposition 1.3.6 (v) in Bingham et al. (1987). Combined with the weak law of large numbers we obtain

$$\frac{W_{(n)}^2}{L_n} = \frac{W_{(n)}^2}{n} \frac{1}{n^{-1} L_n} \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty. \tag{8}$$

Next, we introduce notation and discuss convergence of often occurring series. For a set A we denote by $A^k_{\neq}$ the set of all k-tuples of pairwise distinct elements of A. Recall that $\mathbf{E}_{\mathcal{W}}$ denotes the conditional expectation with respect to the weights. Naturally, all equalities and inequalities only hold $\mathbf{P}$-almost surely which we will not write explicitly in the following to keep the notation simple. We typically approximate the number of vertices in the set $\mathcal{X}_n(v_0)$ for $v_0 \in [n]$ by counting paths originating in v_0 and leading to vertices in $\mathcal{X}_n(v_0)$, resulting in a sum of the following kind,

$$T_n(v_0) = \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in ([n] \setminus \{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\} \mathbf{1}\{v_k \in \mathcal{X}_n(v_0)\}, \tag{9}$$

where $x \leftrightarrow y$ means that the vertices x and y are connected by at least one edge. We first sum over the length of the path starting in v_0 and then over its vertices, which we demand to be distinct. Writing $L_n^{(2)} = \sum_{v=1}^n W_v^2$ and using the upper bound one for the last indicator in (9), conditional independence of the remaining indicators and $\mathbf{1}\{x \leftrightarrow y\} \le E_n\{x, y\}$ for $x, y \in [n]$ leads to

$$\mathbf{E}_{\mathcal{W}}[T_n(v_0)] \le \sum_{k=1}^n \sum_{(v_1,\dots,v_k)\in[n]^k_{\neq}} \prod_{i=1}^k \mathbf{E}_{\mathcal{W}}[E_n\{v_i, v_{i-1}\}] = \sum_{k=1}^n \sum_{(v_1,\dots,v_k)\in[n]^k_{\neq}} \prod_{i=1}^k \frac{W_{v_i} W_{v_{i-1}}}{L_n}$$

$$\le W_{v_0} \sum_{k=1}^n \sum_{v_1,\dots,v_k=1}^n \prod_{i=1}^{k-1} \frac{W_{v_i}^2}{L_n} \frac{W_{v_k}}{L_n} \le W_{v_0} \sum_{k=0}^n \left(\frac{L_n^{(2)}}{L_n}\right)^k =: W_{v_0} S_{n,\mathcal{W}}.$$

From the strong law of large numbers it follows that

$$\frac{L_n^{(2)}}{L_n} = \frac{n^{-1} L_n^{(2)}}{n^{-1} L_n} \xrightarrow{a.s.} \frac{\mathbf{E}[W^2]}{\mathbf{E}[W]} \quad \text{as} \quad n \to \infty. \tag{10}$$

The geometric sum formula together with $\mathbf{E}[W^2] < \mathbf{E}[W]$ by assumption (W) yields

$$S_{n,\mathcal{W}} = \frac{1 - \left(L_n^{(2)}/L_n\right)^{n+1}}{1 - L_n^{(2)}/L_n} \xrightarrow{a.s.} \frac{\mathbf{E}[W]}{\mathbf{E}[W] - \mathbf{E}[W^2]} \quad \text{as} \quad n \to \infty. \tag{11}$$

This argument also provides almost sure convergence for related expressions of the form

$$\sum_{k=0}^n p(k) \left(\frac{L_n^{(2)}}{L_n}\right)^k, \tag{12}$$

where p is an arbitrary polynomial of finite degree. We would like to point out that the limit in (10) equals the mean of the offspring distribution of the local limit in (2). In fact, it is the mean of the size-biased version of the weights which govern the degrees.

For $n \in \mathbb{N}$, $x, y \in [n]$ with $x \neq y$, $A \subseteq [n]$ and $k \ge 2$ we write

$$\mathcal{P}_k^{(n)}(x, y, A) = \{(v_1, \dots, v_k) \in [n]^k_{\neq} : v_1 = x, v_k = y, v_2, \dots, v_{k-1} \notin A, v_1 \leftrightarrow \dots \leftrightarrow v_k\},$$
$$\mathcal{P}_k^{(n)}(x, A) = \{(v_1, \dots, v_k) \in [n]^k_{\neq} : v_1 = x, v_2, \dots, v_k \notin A, v_1 \leftrightarrow \dots \leftrightarrow v_k\}.$$

The first set consists of all k-tuples of vertices of G_n that form a path with endpoints x and y whose inner vertices do not lie in A. Similarly, the second set consists of all k-tuples which form a path whose starting vertex is x and all other vertices do not belong to A. We are often interested in bounding the expected number of such paths, conditionally on the weights, where we can use the following lemma.

Lemma 5 *Let $x, y \in [n]$ with $x \neq y$, $A \subseteq [n]$ and $k \ge 2$. Then*

$$\mathbf{E}_{\mathcal{W}}[|\mathcal{P}_k^{(n)}(x, y, A)|] \le \frac{W_x W_y}{L_n} \left(\frac{L_n^{(2)}}{L_n}\right)^{k-2} \quad \textit{and} \quad \mathbf{E}_{\mathcal{W}}[|\mathcal{P}_k^{(n)}(x, A)|] \le W_x \left(\frac{L_n^{(2)}}{L_n}\right)^{k-2}.$$

Proof We only prove the first assertion, the second one follows immediately from the first by summing over $y \in [n] \setminus \{x\}$ and using $\sum_{y \in [n]\setminus\{x\}} W_y \leq L_n$. As all edges in the paths are distinct and therefore conditionally independent, we obtain

$$\begin{aligned}
\mathbf{E}_{\mathcal{W}}[|\mathcal{P}_k^{(n)}(x, y, A)|] &\leq \mathbf{E}_{\mathcal{W}}\Bigg[\sum_{(v_1,\dots,v_k)\in[n]^k_{\neq}} \mathbf{1}\{v_1 = x, v_k = y\} \prod_{i=2}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\}\Bigg] \\
&\leq \sum_{(v_1,\dots,v_k)\in[n]^k_{\neq}} \mathbf{1}\{v_1 = x, v_k = y\} \prod_{i=2}^{k} \mathbf{E}_{\mathcal{W}}[E_n\{v_i, v_{i-1}\}] \\
&= \sum_{(v_1,\dots,v_k)\in[n]^k_{\neq}} \mathbf{1}\{v_1 = x, v_k = y\} \prod_{i=2}^{k} \frac{W_{v_i} W_{v_{i-1}}}{L_n} \\
&\leq \frac{W_x W_y}{L_n}\left(\mathbf{1}\{k = 2\} + \sum_{v_2,\dots,v_{k-1}=1}^{n} \prod_{i=2}^{k-1} \frac{W_{v_i}^2}{L_n}\right) = \frac{W_x W_y}{L_n}\left(\frac{L_n^{(2)}}{L_n}\right)^{k-2},
\end{aligned}$$

which is the first inequality. □

The next lemma essentially says that vertices with large weights are typically not connected or, equivalently, every component has at most one vertex with large weight. This may seem counterintuitive as vertices with larger weights are more likely to be connected. However, we increase the threshold for having a large weight, i.e. larger than $aq(n)$ for $a > 0$, such that there will be, on average, just a constant number of such vertices.

Lemma 6 *For $a > 0$ and $n \in \mathbb{N}$ define the event*

$$\mathcal{A}_n = \{\exists x, y \in [n] \colon x \neq y, x \in \mathcal{C}_n(y), W_x \geq W_y \geq aq(n)\}^c.$$

Under assumption (W) *we have $\mathbf{P}(\mathcal{A}_n) \to 1$ as $n \to \infty$.*

Proof For all $\ell, n \in \mathbb{N}$ we introduce the event

$$\mathcal{E}_{n,\ell} = \left\{\sum_{y=1}^{n} \mathbf{1}\{W_y > aq(n)\} \leq \ell\right\}.$$

Note that $\mathcal{E}_{n,\ell}$ is $\mathcal{W}$-measurable. It holds that

$$\mathbf{P}(\mathcal{A}_n^c) \leq \mathbf{P}(\mathcal{E}_{n,\ell}^c) + \mathbf{P}(\mathcal{A}_n^c \cap \mathcal{E}_{n,\ell}). \tag{13}$$

Here the Markov inequality yields

$$\mathbf{P}(\mathcal{E}_{n,\ell}^c) = \mathbf{P}\left(\sum_{y=1}^{n} \mathbf{1}\{W_y > aq(n)\} > \ell\right) \leq \frac{\mathbf{E}\left[\sum_{y=1}^{n} \mathbf{1}\{W_y > aq(n)\}\right]}{\ell}$$

$$= \frac{n}{\ell}\mathbf{P}\left(W > aq(n)\right).$$

By Theorem 3.6 and Remark 3.3(a) in Resnick (2007) one has

$$\lim_{n\to\infty} \frac{n}{\ell}\mathbf{P}\left(W > aq(n)\right) = \lim_{n\to\infty} \frac{n}{\ell} \cdot \frac{a^{-\beta}}{n} = \frac{a^{-\beta}}{\ell}.$$

Now we address the second summand in (13) and show that it converges to zero for any fixed $\ell \in \mathbb{N}$. For x, y as demanded in $\mathcal{A}_n$ there must exist a path consisting of at least two vertices which connects x and y. Summing over the possible path lengths, we obtain via Lemma 5

$$\begin{aligned}
&\mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{E}_{n,\ell}}\mathbf{1}_{\mathcal{A}_n^c}] \leq \mathbf{E}_{\mathcal{W}}\left[\mathbf{1}_{\mathcal{E}_{n,\ell}} \sum_{x,y=1}^{n} \mathbf{1}\{x \neq y\} \sum_{k=2}^{n} |\mathcal{P}_k^{(n)}(x, y, \emptyset)| \mathbf{1}\{W_x \geq W_y \geq aq(n)\}\right] \\
&\leq \mathbf{1}_{\mathcal{E}_{n,\ell}} \sum_{x,y=1}^{n} \mathbf{1}\{W_x \geq W_y \geq aq(n)\} \sum_{k=2}^{n} \frac{W_x W_y}{L_n}\left(\frac{L_n^{(2)}}{L_n}\right)^{k-2} \\
&\leq \mathbf{1}\left\{\sum_{y=1}^{n} \mathbf{1}\{W_y > aq(n)\} \leq \ell\right\} \sum_{x=1}^{n} \frac{W_x^2 \mathbf{1}\{W_x > aq(n)\}}{L_n} \\
&\quad \times \sum_{y=1}^{n} \mathbf{1}\{W_y > aq(n)\} \sum_{k=2}^{n} \left(\frac{L_n^{(2)}}{L_n}\right)^{k-2} \\
&\leq \sum_{x=1}^{n} \frac{W_x^2 \mathbf{1}\{W_x > aq(n)\}}{L_n} \ell\, S_{n,\mathcal{W}}.
\end{aligned}$$

The first sum above converges almost surely to zero due to the finite second moment of W and the strong law of large numbers since $q(n) \to \infty$ as $n \to \infty$. By (11), $S_{n,\mathcal{W}}$ converges almost surely to a constant so that the whole term converges almost surely to zero. For the conditional expectation we have the upper bound 1 so that the dominated convergence theorem implies $\mathbf{P}(\mathcal{A}_n^c \cap \mathcal{E}_{n,\ell}) \to 0$ as $n \to \infty$. From (13) we conclude $\limsup_{n\to\infty} \mathbf{P}(\mathcal{A}_n^c) \leq a^{-\beta} \cdot \ell^{-1}$ for all $\ell \in \mathbb{N}$. Letting $\ell \to \infty$, we get $\lim_{n\to\infty} \mathbf{P}(\mathcal{A}_n^c) = 0$. ☐

We introduce a slightly different variant of assumption (A2) here.

(A2') Let $n \in \mathbb{N}$, $v \in [n]$ and $x \in \mathcal{C}_n(v)$. Then $v \notin \mathcal{X}_n(v)$ and checking whether $x \in \mathcal{X}_n(v)$ only depends on all paths that start in v and contain x.

In contrast to (A2) we do not allow v itself to lie in $\mathcal{X}_n(v)$. This ensures that all $x \in \mathcal{X}_n(v)$ have positive distance to v, which simplifies the phrasing and application of the following lemmas. One could also work with (A2), but the notation would become more involved. Eventually, we will argue in the proof of Theorem 1 that we can assume (A2') instead of (A2) without loss of generality.

For the following lemma we introduce $L_n^{(3)} = \sum_{v=1}^{n} W_v^3$. Note that assumption (W) ensures that L_n and $L_n^{(2)}$ grow linearly in n, whereas the third moment of the weights could be infinite, resulting in a superlinear growth of $L_n^{(3)}$.

Lemma 7 *Assume* (W), (A2') *and define*

$$\tilde{S}_{n,\mathcal{W}} = \sum_{k=0}^{n} (k+3)^3 \left(\frac{L_n^{(2)}}{L_n} \right)^k$$

as well as

$$X_n = 4\tilde{S}_{n,\mathcal{W}}^2 \left(1 + \frac{W_{(n)}^2}{L_n} \tilde{S}_{n,\mathcal{W}} + \tilde{S}_{n,\mathcal{W}}^2 \right) \text{ and } Y_n = 4\frac{n}{L_n} \tilde{S}_{n,\mathcal{W}}^5$$

for $n \in \mathbb{N}$. *Then* X_n *and* Y_n *converge in probability to positive constants as* $n \to \infty$ *and for all* $n \in \mathbb{N}$ *and* $v \in [n]$,

$$\mathbf{Var}_{\mathcal{W}}\left(S_n(v)\right) \le W_v X_n + W_v \frac{L_n^{(3)}}{n} Y_n.$$

Proof Almost sure convergence of $\tilde{S}_{n,\mathcal{W}}$ as $n \to \infty$ was established in (12). The convergence of X_n and of Y_n in probability as $n \to \infty$ in turn follows from the respective convergence of $\tilde{S}_{n,\mathcal{W}}$, of $W_{(n)}^2/L_n$ in (8) and of n/L_n.

Now we address the actual variance bound. Conditionally on the weights $\mathcal{W}$, the quantity $S_n(v)$ depends on the independent random variables $(E_n\{i,j\})_{1\le i\le j\le n}$, which form a Poisson process on the discrete space $\{\{i,j\} \colon 1 \le i \le j \le n\}$ with intensity measure $\lambda(\{i,j\}) = W_i W_j / L_n$ for $1 \le i \le j \le n$. This means that $S_n(v)$ is a Poisson functional which is in particular square-integrable as it is bounded by n. We use the Poincaré inequality for Poisson functionals, see e.g. Theorem 18.7 in Last and Penrose (2018), to derive

$$\mathbf{Var}_{\mathcal{W}}\left(S_n(v)\right) \le \sum_{1\le i\le j\le n} \mathbf{E}_{\mathcal{W}}\left[\left(D_{\{i,j\}} S_n(v)\right)^2 \right] \frac{W_i W_j}{L_n}, \tag{14}$$

where $D_{\{i,j\}}$ denotes the difference operator

$$D_{\{i,j\}} S_n(v) = S_{n,i,j}(v) - S_n(v)$$

and $S_{n,i,j}(v)$ is the number of elements in $\mathcal{X}_n(v)$ after increasing the number of edges between the vertices i and j by 1. By assumption (A2'), $v \notin \mathcal{X}_n(v)$ and for a vertex $x \in \mathcal{C}_n(v)$ with $x \ne v$ the property $x \in \mathcal{X}_n(v)$ only depends on all paths starting in v and containing the vertex x. For $i = j$ we conclude that $D_{\{i,i\}} S_n(v) = 0$ as a path is self-avoiding by definition and therefore not allowed to contain loops. Therefore, we do not alter any paths by adding a loop and hence do not change $S_n(v)$. For $i \ne j$ we obtain

$$\begin{aligned} |D_{\{i,j\}} S_n(v)| \le |\{x \in [n] \setminus \{v\} \colon & \text{There is a path starting in } v \text{ that contains } x \\ & \text{and the edge } \{i,j\}\}|, \end{aligned}$$

where the existence of the path is to be checked after increasing the number of edges between i and j by 1. We obtain four different scenarios before the addition of one edge between i and j (independent of the number of edges between i and j before the addition). Figure 2 contains pictures of the different cases. We write $v \dots x$ for a path that starts in v and ends in x and introduce the following four sets which decompose the set on the right-hand side of the bound above into the respective cases:

$$\mathcal{M}_1(i,j) = \{x \in [n] \setminus \{v\} \colon \text{There is a path } v \dots i \text{ that includes } x \text{ but not } j.\}$$
$$\mathcal{M}_2(i,j) = \{x \in [n] \setminus \{v\} \colon \text{There is a path } v \dots j \text{ that includes } x \text{ but not } i.\}$$
$$\mathcal{M}_3(i,j) = \{x \in [n] \setminus \{v\} \colon \text{There are two disjoint paths } v \dots i \text{ and } j \dots x.\}$$
$$\mathcal{M}_4(i,j) = \{x \in [n] \setminus \{v\} \colon \text{There are two disjoint paths } v \dots j \text{ and } i \dots x.\}$$

By disjoint paths we mean that they do not share a single vertex. Also, we allow a path to consist of only a single vertex, e.g. in $\mathcal{M}_3(i,j)$ the special case $v = i$ immediately yields the existence of a path from v to i. We obtain

$$|D_{\{i,j\}} S_n(v)| \le \sum_{k=1}^{4} |\mathcal{M}_k(i,j)|$$

and with Jensen's inequality

$$(D_{\{i,j\}} S_n(v))^2 \le 4 \sum_{k=1}^{4} |\mathcal{M}_k(i,j)|^2 = 4 \sum_{k=1}^{4} |\mathcal{M}_k(i,j)^2|.$$

Observe that $\mathcal{M}_1(i,j) = \mathcal{M}_2(j,i)$ as well as $\mathcal{M}_3(i,j) = \mathcal{M}_4(j,i)$. Recall that the case $i = j$ has no contribution so that (14) simplifies to

$$\mathbf{Var}_{\mathcal{W}}(S_n(v)) \le 4 \sum_{(i,j)\in[n]^2_{\neq}} \mathbf{E}_{\mathcal{W}}[|\mathcal{M}_1(i,j)^2|] \frac{W_i W_j}{L_n} + 4 \sum_{(i,j)\in[n]^2_{\neq}} \mathbf{E}_{\mathcal{W}}[|\mathcal{M}_3(i,j)^2|] \frac{W_i W_j}{L_n}, \tag{15}$$

where $[n]^2_{\neq}$ denotes the set of ordered pairs with distinct entries from $[n]$. We start with bounding the first sum and consider $\mathbf{E}_{\mathcal{W}}[|\mathcal{M}_1(i,j)^2|]$. Let $(x,y) \in \mathcal{M}_1(i,j)^2$. This means that there is a first path P from v through x to i and a second path from v through y to i. Note that P must contain at least two vertices because x is not allowed to equal v. We distinguish two cases, see also Fig. 3 for a visualisation:

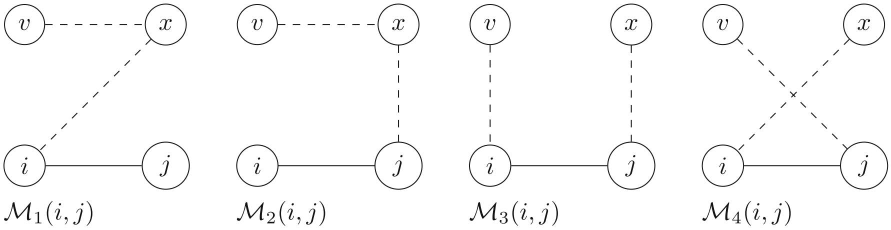

Fig. 2 Visualisation of $x \in \mathcal{M}_\ell(i,j)$ for $\ell \in \{1,2,3,4\}$

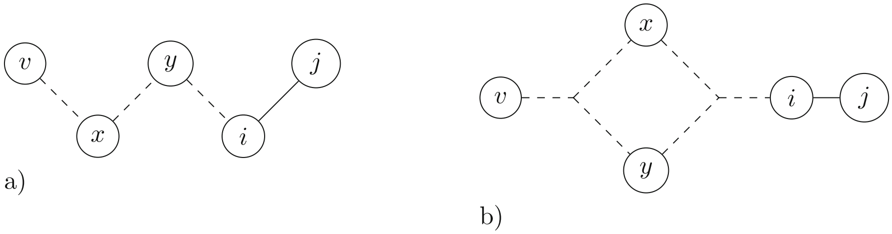

Fig. 3 Scenarios for $(x, y) \in \mathcal{M}_1(i, j)^2$

a) y lies on P,
b) y does not lie on P.

We write $M_{1,\mathrm{a}}(i, j)$ for the number of all $(x, y) \in \mathcal{M}_1(i, j)^2$ of type a), similarly $M_{1,\mathrm{b}}(i, j)$ for b), so that $|\mathcal{M}_1(i, j)^2| \leq M_{1,\mathrm{a}}(i, j) + M_{1,\mathrm{b}}(i, j)$. We have

$$M_{1,\mathrm{a}}(i, j) \leq \sum_{k=2}^{n} (k-1)^2 \cdot |\mathcal{P}_k^{(n)}(v, i, \{i, j\})|$$

because x and y need to lie on any path from v to i which does not include the vertices i, j as inner vertices. From Lemma 5 we conclude that

$$\mathbf{E}_{\mathcal{W}}[M_{1,\mathrm{a}}(i, j)] \leq \sum_{k=2}^{n} (k-1)^2 \frac{W_v W_i}{L_n} \left(\frac{L_n^{(2)}}{L_n} \right)^{k-2} \leq W_v \frac{W_i}{L_n} \tilde{S}_{n,\mathcal{W}}.$$

In case b) we use a similar argument to obtain

$$\begin{aligned} M_{1,\mathrm{b}}(i, j) \leq \sum_{k=2}^{n} & \sum_{(p_1,\ldots,p_k) \in \mathcal{P}_k^{(n)}(v,i,\{i,j\})} (k-1) \\ & \times \sum_{1 \leq c < d \leq k} \sum_{\ell=3}^{n} (\ell - 2) \cdot |\mathcal{P}_\ell^{(n)}(p_c, p_d, \{i, j, p_1, \ldots, p_k\})| \end{aligned}$$

because x needs to lie one some path P from v to i whereas y needs to lie on another path Q which splits off P and merges with P again. Since y does not lie on P due to case b), Q contains at least three vertices and y can be neither its starting point nor its endpoint, denoted by p_c and p_d above. Note that the path Q can be chosen in such a way that it only intersects P in p_c and p_d. We therefore obtain conditional independence of all paths above and use Lemma 5 to obtain

$$\begin{aligned} \mathbf{E}_{\mathcal{W}}[M_{1,\mathrm{b}}(i, j)] &\leq \sum_{k=2}^{n} (k-1) \frac{W_v W_i}{L_n} \left(\frac{L_n^{(2)}}{L_n} \right)^{k-2} \sum_{1 \leq c < d \leq k} \sum_{\ell=3}^{n} (\ell - 2) \frac{W_{(n)}^2}{L_n} \left(\frac{L_n^{(2)}}{L_n} \right)^{\ell-2} \\ &\leq W_v \frac{W_i}{L_n} \frac{W_{(n)}^2}{L_n} \sum_{k=2}^{n} k^3 \left(\frac{L_n^{(2)}}{L_n} \right)^{k-2} \sum_{\ell=1}^{n} \ell \left(\frac{L_n^{(2)}}{L_n} \right)^{\ell} \leq W_v \frac{W_i}{L_n} \frac{W_{(n)}^2}{L_n} \tilde{S}_{n,\mathcal{W}}^2. \end{aligned}$$

We conclude

$$\mathbf{E}_{\mathcal{W}}[|\mathcal{M}_1(i,j)^2|] \le W_v \frac{W_i}{L_n} \tilde{S}_{n,\mathcal{W}} \left(1 + \frac{W_{(n)}^2}{L_n} \tilde{S}_{n,\mathcal{W}}\right).$$

Therefore, the first sum in (15) is bounded by

$$\sum_{(i,j)\in[n]_{\neq}^2}^{n} W_v \frac{W_i}{L_n} \tilde{S}_{n,\mathcal{W}} \left(1 + \frac{W_{(n)}^2}{L_n} \tilde{S}_{n,\mathcal{W}}\right) \frac{W_i W_j}{L_n} \le W_v \tilde{S}_{n,\mathcal{W}}^2 \left(1 + \frac{W_{(n)}^2}{L_n} \tilde{S}_{n,\mathcal{W}}\right), \tag{16}$$

where we used that $L_n^{(2)}/L_n \le \tilde{S}_{n,\mathcal{W}}$.

We proceed with a similar strategy for $\mathcal{M}_3(i,j)^2$. There are some more cases due to the fact that v itself may be equal to i whereas this was impossible in the case of $\mathcal{M}_1(i,j)^2$. Let $(x,y) \in \mathcal{M}_3(i,j)^2$. This means that we can find a path P from v to i via $p_2, \ldots, p_{u-1}$ where $u = 1$ corresponds to the case $v = i$. We fix such a path P. Additionally, there is a path from j to x and another path from j to y. We distinguish the following cases, see also Fig. 4. By definition of $\mathcal{M}_3(i,j)$ all paths Q and R in the cases below may be chosen disjoint from the path P from v to i:

a) $x = y = j$,
b) $|\{x, y, j\}| \ge 2$ and there is a path Q that starts in j and contains x and y,
c) $|\{x, y, j\}| = 3$ and there is a path Q from j to x and a path R from j to y such that j is their only common vertex,
d) $|\{x, y, j\}| = 3$, there is a path Q from j to x and there is a second path R from one of Q's inner vertices to y that is disjoint from Q apart from its first vertex.

For fixed i, j, P we write $M_{3,\mathrm{z}}(i,j,P)$ for the number of all $(x,y) \in \mathcal{M}_3(i,j)^2$ considered in case z) for z $\in$ {a, b, c, d}. We have

$$\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{a}}(i,j,P) = \mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\}$$

and by counting all possibilities to place x and y on Q we obtain

$$\begin{aligned} &\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{b}}(i,j,P) \\ &\le \mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} \sum_{k=2}^{n} |\mathcal{P}_k^{(n)}(j, \{v, i, j, p_2, \ldots, p_{u-1}\})| k^2, \end{aligned}$$

so that Lemma 5 yields

$$\begin{aligned} &\mathbf{E}_{\mathcal{W}}[\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{b}}(i,j,P)] \\ &\le \mathbf{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j \sum_{k=2}^{n} k^2 \left(\frac{L_n^{(2)}}{L_n}\right)^{k-2} \\ &\le \mathbf{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j \tilde{S}_{n,\mathcal{W}}. \end{aligned}$$

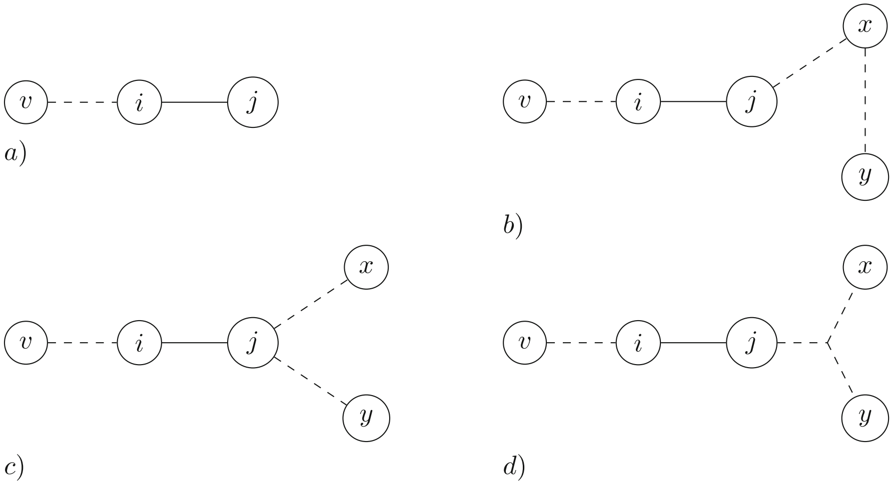

Fig. 4 Different scenarios for $(x, y) \in \mathcal{M}_3(i, j)^2$

For c) we obtain similarly

$$\begin{aligned}
&\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{c}}(i, j, P)\\
&\le \mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} \sum_{k=2}^{n} k \sum_{(q_1,\ldots,q_k)\in\mathcal{P}_k^{(n)}(j,\{v,p_2,\ldots,p_{u-1},i\})}\\
&\quad \times \sum_{\ell=2}^{n} \ell \cdot |\mathcal{P}_\ell^{(n)}(j, \{v, p_2, \ldots, p_{u-1}, i, q_1, \ldots, q_k\})|.
\end{aligned}$$

We derive due to conditional independence

$$\begin{aligned}
&\mathbf{E}_{\mathcal{W}}[\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{c}}(i, j, P)]\\
&\le \mathbf{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j^2 \left(\sum_{k=2}^{n} k \left(\frac{L_n^{(2)}}{L_n}\right)^{k-2}\right)^2\\
&\le \mathbf{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j^2 \tilde{S}_{n,\mathcal{W}}^2.
\end{aligned}$$

In case d) we have

$$\begin{aligned}
&\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{d}}(i, j, P)\\
&\le \mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} \sum_{k=3}^{n} k \sum_{(q_1,\ldots,q_k)\in\mathcal{P}_k^{(n)}(j,\{v,p_2,\ldots,p_{u-1},i\})}\\
&\quad \times \sum_{\ell=2}^{n} \sum_{1<t<k} \ell \cdot |\mathcal{P}_\ell^{(n)}(q_t, \{q_1, \ldots, q_k, v, p_2, \ldots, p_{u-1}, i\})|.
\end{aligned}$$

When calculating the conditional expectation, we again use conditional independence. This time, the weight W_{q_t} of the vertex in which the second path originates will appear with a third power. It already has a second power since it is an inner vertex of the first path and as starting point of the second path, we obtain another factor. In total, we get

$$\begin{aligned}
&\mathbf{E}_{\mathcal{W}}[\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{d}}(i,j,P)] \\
&\quad \le \mathbf{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j \frac{L_n^{(3)}}{L_n} \sum_{k=3}^{n} k^2 \left(\frac{L_n^{(2)}}{L_n}\right)^{k-3} \sum_{\ell=2}^{n} \ell \left(\frac{L_n^{(2)}}{L_n}\right)^{\ell-2} \\
&\quad \le \mathbf{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j \frac{L_n^{(3)}}{L_n} \tilde{S}_{n,\mathcal{W}}^2.
\end{aligned}$$

Summing over all different choices for P to connect v and i via $p_2, \ldots, p_{u-1}$ yields

$$\begin{aligned}
&\mathbf{E}_{\mathcal{W}}[|\mathcal{M}_3(i,j)^2|] \\
&\le \left(\mathbf{1}\{v=i\} + \mathbf{P}_{\mathcal{W}}(v \leftrightarrow i) + \sum_{u=3}^{n} \sum_{(p_2,\ldots,p_{u-1}) \in ([n]\setminus\{v,i\})_{\neq}^{u-2}} \mathbf{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i)\right) \\
&\quad \times \left(1 + W_j \tilde{S}_{n,\mathcal{W}} + W_j^2 \tilde{S}_{n,\mathcal{W}}^2 + W_j \frac{L_n^{(3)}}{L_n} \tilde{S}_{n,\mathcal{W}}^2\right) \\
&\le \left(1\{v=i\} + \sum_{k=2}^{n} \mathbf{E}_{\mathcal{W}}[|\mathcal{P}_k^{(n)}(v,i,\emptyset)|]\right)\left(1 + W_j \tilde{S}_{n,\mathcal{W}} + W_j^2 \tilde{S}_{n,\mathcal{W}}^2 + W_j \frac{L_n^{(3)}}{L_n} \tilde{S}_{n,\mathcal{W}}^2\right) \\
&\le \left(1\{v=i\} + \frac{W_v W_i}{L_n} \tilde{S}_{n,\mathcal{W}}\right)\left(1 + W_j \tilde{S}_{n,\mathcal{W}} + W_j^2 \tilde{S}_{n,\mathcal{W}}^2 + W_j \frac{L_n^{(3)}}{L_n} \tilde{S}_{n,\mathcal{W}}^2\right),
\end{aligned}$$

where the last inequality uses Lemma 5. This bounds the second sum in (15) by

$$\begin{aligned}
&\sum_{(i,j)\in[n]_{\neq}^2} \left(1\{v=i\} + \frac{W_v W_i}{L_n} \tilde{S}_{n,\mathcal{W}}\right)\left(1 + W_j \tilde{S}_{n,\mathcal{W}} + W_j^2 \tilde{S}_{n,\mathcal{W}}^2 + W_j \frac{L_n^{(3)}}{L_n} \tilde{S}_{n,\mathcal{W}}^2\right) \frac{W_i W_j}{L_n} \\
&\le W_v \left(1 + \frac{L_n^{(2)}}{L_n} \tilde{S}_{n,\mathcal{W}}\right)\left(1 + \frac{L_n^{(2)}}{L_n} \tilde{S}_{n,\mathcal{W}} + \frac{L_n^{(3)}}{L_n} \tilde{S}_{n,\mathcal{W}}^2 + \frac{L_n^{(2)} L_n^{(3)}}{L_n^2} \tilde{S}_{n,\mathcal{W}}\right) \\
&\le W_v \tilde{S}_{n,\mathcal{W}}^2 \left(\tilde{S}_{n,\mathcal{W}}^2 + \frac{L_n^{(3)}}{L_n} \tilde{S}_{n,\mathcal{W}}^3\right),
\end{aligned} \tag{17}$$

where we used $1 + L_n^{(2)} \tilde{S}_{n,\mathcal{W}} / L_n \le \tilde{S}_{n,\mathcal{W}}^2$. Combining (15) with (16) and (17), we obtain

$$\begin{aligned}
\mathbf{Var}_{\mathcal{W}}(S_n(v)) &\le 4 W_v \left(\tilde{S}_{n,\mathcal{W}}^2 \left(1 + \frac{W_{(n)}^2}{L_n} \tilde{S}_{n,\mathcal{W}}\right) + \tilde{S}_{n,\mathcal{W}}^2 \left(\tilde{S}_{n,\mathcal{W}}^2 + \frac{L_n^{(3)}}{L_n} \tilde{S}_{n,\mathcal{W}}^3\right)\right) \\
&= 4 W_v \tilde{S}_{n,\mathcal{W}}^2 \left(1 + \frac{W_{(n)}^2}{L_n} \tilde{S}_{n,\mathcal{W}} + \tilde{S}_{n,\mathcal{W}}^2\right) + 4 W_v \frac{L_n^{(3)}}{L_n} \tilde{S}_{n,\mathcal{W}}^5 = W_v X_n + W_v \frac{L_n^{(3)}}{n} Y_n,
\end{aligned}$$

which finishes the proof. □

The following lemma essentially shows that the conditional m-th moment of the number of vertices in a component is bounded by a polynomial of degree m in the largest weight of the component, up to a term which converges almost surely.

Lemma 8 *Assume* (W), (A2'), *let* $m \in \mathbb{N}$ *and define for* $n \in \mathbb{N}$,

$$R_{n,m} = m^m \sum_{t=1}^{m} t! \left(\sum_{k=0}^{n} \left(\frac{L_n^{(2)}}{L_n} \right)^k (k+3)^t \right)^t .$$

Then $R_{n,m}$ *converges almost surely to a constant as* $n \to \infty$ *and for all* $v \in [n]$ *it holds that*

$$\mathbf{E}_{\mathcal{W}} \left[\mathbf{1}\{v \in V_n^{\max}\} S_n(v)^m \right] \leq R_{n,m} \sum_{t=1}^{m} W_v^t .$$

Proof For $m \in \mathbb{N}$, the almost sure convergence of $R_{n,m}$ as $n \to \infty$ follows from (12). To simplify notation we assume without loss of generality that $v = n$. For the moment bound, recall $v \notin \mathcal{X}_n(v)$ by (A2') so that

$$\begin{aligned}
&\mathbf{E}_{\mathcal{W}} \left[\mathbf{1}\{v \in V_n^{\max}\} S_n(v)^m \right] \leq \mathbf{E}_{\mathcal{W}} \left[\mathbf{1}\{v \in V_n^{\max}\} \Bigg(\sum_{x \in [n-1]} \mathbf{1}\{x \in \mathcal{C}_n(v)\} \Bigg)^m \right] \\
&= \sum_{(x_1,\ldots,x_m) \in [n-1]^m} \mathbf{E}_{\mathcal{W}} \left[\mathbf{1}\{v \in V_n^{\max}\} \prod_{i=1}^{m} \mathbf{1}\{x_i \in \mathcal{C}_n(v)\} \right] \\
&\leq \sum_{t=1}^{m} m^m \sum_{(x_1,\ldots,x_t) \in [n-1]^t_{\neq}} \mathbf{E}_{\mathcal{W}} \left[\mathbf{1}\{v \in V_n^{\max}\} \prod_{i=1}^{t} \mathbf{1}\{x_i \in \mathcal{C}_n(v)\} \right],
\end{aligned}$$

where the last step accounts for equal entries of a vector $(x_1, \ldots, x_m) \in [n-1]^m$ by the additional factor m^m and instead only sums over the $t \in [m]$ distinct entries of x. Hence it suffices to show for all $t \in [m]$ that

$$\sum_{(x_1,\ldots,x_t) \in [n-1]^t_{\neq}} \mathbf{E}_{\mathcal{W}} \left[\mathbf{1}\{v \in V_n^{\max}\} \prod_{i=1}^{t} \mathbf{1}\{x_i \in \mathcal{C}_n(v)\} \right] \leq W_v^t t! \Bigg(\sum_{k=0}^{n} \left(\frac{L_n^{(2)}}{L_n} \right)^k (k+3)^t \Bigg)^t .$$

We order the vertices $x_1, \ldots, x_t$ in such a way that the graph distance of x_i and v is non-decreasing in i which gives us the $t!$ on the right-hand side of the inequality above.

We apply an iterative argument to bound the product on the left-hand side above. If $x_1, \ldots, x_t \in \mathcal{C}_n(v)$, there exists a shortest path P_1 starting in v and ending in x_1, consisting of $k_1 \geq 2$ vertices because $x_1 \neq v$. There may be several shortest paths and if so, we always choose the smallest one with respect to the lexicographic order of its vertices. Now consider a shortest path from v to x_2 and remove the part leading from v to its last intersection with the already existing path P_1. The remaining part, connecting P_1 and x_2, is called P_2 and contains $k_2 \geq 2$ vertices. Applying this construction iteratively, we connect for $i = 2, \ldots, t$ a vertex x_i via a path P_i of $k_i \geq 2$ vertices to some vertex a_i of the previously added paths $P_1, \ldots, P_{i-1}$. Note that we obtain a tree structure without cycles by choosing the shortest path by lexicographic order (if necessary).

For $i \in [t]$ and any of these paths P_i, write a_i for the starting vertex of P_i and recall that the weight W_v is maximal in $\mathcal{C}_n(v)$. Note that $a_1 = n$. For $i \geq 2$, there are at most $k_1 + \ldots + k_{i-1} \leq \prod_{j=1}^{i-1}(k_j + 1)$ possibilities to choose the point a_i on any of the paths $P_1, \ldots, P_{i-1}$ where P_i may be attached to. We write $\cup_{j=1}^{i-1} P_j$ for the set of all vertices of $P_1, \ldots, P_{i-1}$. Once one has chosen any such a starting vertex a_i, the conditional expectation of the number of such paths P_i is bounded by

$$\mathbf{E}_{\mathcal{W}}\Bigg[\sum_{k_i=2}^{n} |\mathcal{P}_{k_i}^{(n)}(a_i, x_i, \cup_{j=1}^{i-1} P_j)| \cdot \mathbf{1}\{W_{a_i} \leq W_v\}\Bigg] \leq W_v \frac{W_{x_i}}{L_n} \sum_{k_i=2}^{n} \left(\frac{L_n^{(2)}}{L_n}\right)^{k_i-2},$$

where we used Lemma 5 and the fact that the weight of W_v is maximal in its component, so in particular not smaller than the weight W_{a_i}. By construction the paths $P_1, \ldots, P_t$ do not share an edge and are therefore conditionally independent.

Combining this bound with the number of possible choices for a_i we obtain

$$\begin{aligned}
&\sum_{(x_1,\ldots,x_t)\in[n-1]^t_{\neq}} \mathbf{E}_{\mathcal{W}}\Bigg[\mathbf{1}\{v \in V_n^{\max}\} \prod_{i=1}^{t} \mathbf{1}\{x_i \in \mathcal{C}_n(v)\}\Bigg] \\
&\leq W_v^t t! \sum_{(x_1,\ldots,x_t)\in[n-1]^t_{\neq}} \prod_{i=1}^{t} \frac{W_{x_i}}{L_n} \sum_{k_i=2}^{n} \left(\frac{L_n^{(2)}}{L_n}\right)^{k_i-2} (k_i + 1)^{t-i} \\
&\leq W_v^t t! \left(\sum_{k=0}^{\infty} \left(\frac{L_n^{(2)}}{L_n}\right)^{k} (k+3)^t\right)^t,
\end{aligned}$$

which concludes the proof. □

Now we have collected all auxiliary lemmas and proceed with the proof of the main theorem.

Proof of Theorem 1 For $n \in \mathbb{N}$ we compare the point processes

$$\Theta_n = \sum_{v=1}^{n} \delta_{W_v q(n)^{-1}} \quad \text{and} \quad \Xi_n = \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}\} \delta_{S_n(v) q(n)^{-1} \xi^{-1}}.$$

For $n \to \infty$, the convergence of Θ_n, the point process of the rescaled weights, to the Poisson process η_β in $M_p((0, \infty])$ has been established in Lemma 3.6 in Bhattacharjee and Schulte (2022). We show that Ξ_n behaves asymptotically like Θ_n. To be more precise, due to the proof of Theorem 2.1 in Bhattacharjee and Schulte (2022) it suffices to show for all $a > 0$ that

$$\Xi_n((a,\infty]) - \Theta_n((a,\infty]) \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty \tag{18}$$

in order to conclude our main theorem. Since $q(n) \to \infty$ as $n \to \infty$, it does not matter whether we consider $S_n(v)$ or $S_n(v) - 1$ in Ξ_n. Therefore, we may assume without loss of generality that (A2') is satisfied, i.e. that $v \notin \mathcal{X}_n(v)$ for all $v \in [n]$. It holds that

$$\begin{aligned}
&|\Xi_n((a,\infty]) - \Theta_n((a,\infty])| \\
&\le \sum_{v=1}^{n} \mathbf{1}\{W_v > aq(n)\}\mathbf{1}\{v \notin V_n^{\max}\} \\
&\quad + \sum_{v=1}^{n} \mathbf{1}\{W_v > aq(n)\}\mathbf{1}\{v \in V_n^{\max}, S_n(v) \le aq(n)\xi\} \\
&\quad + \sum_{v=1}^{n} \mathbf{1}\{W_v \le aq(n)\}\mathbf{1}\{v \in V_n^{\max}, S_n(v) > aq(n)\xi\} =: I_1 + I_2 + I_3.
\end{aligned}$$

We show that I_1, I_2 and I_3 converge to zero in probability. For I_1 it follows from Lemma 6 that

$$\mathbf{P}(I_1 \ne 0) = \mathbf{P}\left(\exists x, y \in [n] : x \ne y, W_x \ge W_y \ge aq(n), x \in \mathcal{C}_n(y)\right) = \mathbf{P}(\mathcal{A}_n^c) \to 0,$$

as $n \to \infty$. We continue with decomposing I_2 and I_3. To this end, let $\varepsilon \in (0, a)$. It holds that

$$\begin{aligned}
I_2 &= \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}, S_n(v) \le aq(n)\xi < W_v\xi\} \\
&\le \sum_{v=1}^{n} \mathbf{1}\{aq(n) < W_v \le (a+\varepsilon)q(n)\} + \sum_{v=1}^{n} \mathbf{1}\{W_v > (a+\varepsilon)q(n), S_n(v) \le aq(n)\xi\} \\
&=: I_{2,1} + I_{2,2}.
\end{aligned}$$

For I_3 we fix some small positive γ satisfying

$$0 < \gamma < \beta^{-1} \quad \text{and define} \quad \tilde{q}(n) = n^{-\gamma} q(n) \quad \text{for } n \in \mathbb{N}.$$

Note that $\tilde{q}(n) \le q(n)$, $\tilde{q}(n)q(n)^{-1} \to 0$ and, by (7), $\tilde{q}(n) \to \infty$ as $n \to \infty$. For $n \in \mathbb{N}$ we have

$$I_3 = \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}, W_v\xi \le aq(n)\xi < S_n(v)\}$$

$$
\begin{aligned}
&\leq \sum_{v=1}^{n} \mathbf{1}\{(a-\varepsilon)q(n) < W_v \leq aq(n)\} \\
&\quad + \sum_{v=1}^{n} \mathbf{1}\{a\tilde{q}(n) < W_v \leq (a-\varepsilon)q(n), S_n(v) > aq(n)\xi\} \\
&\quad + \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}, W_v \leq a\tilde{q}(n), S_n(v) > aq(n)\xi\} =: I_{3,1} + I_{3,2} + I_{3,3}.
\end{aligned}
$$

By Theorem 3.6 and Remark 3.3(a) in Resnick (2007) we get

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}\Bigg[\sum_{v=1}^{n} \mathbf{1}\{(a-\varepsilon)q(n) < W_v \leq (a+\varepsilon)q(n)\}\Bigg] \\
&= \lim_{n\to\infty} n\mathbf{P}((a-\varepsilon)q(n) < W \leq (a+\varepsilon)q(n)) = (a-\varepsilon)^{-\beta} - (a+\varepsilon)^{-\beta},
\end{aligned}
$$

so that $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbf{E}\left[I_{2,1}\right] = 0$ and $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbf{E}\left[I_{3,1}\right] = 0$.

We easily see that

$$
I_4 := \sum_{v=1}^{n} \mathbf{1}\left\{a\tilde{q}(n) < W_v, |S_n(v) - W_v\xi| > \varepsilon q(n)\xi\right\}
$$

is an upper bound for $I_{2,2}$ and $I_{3,2}$. It remains to show that $I_{3,3} \xrightarrow{\mathbf{P}} 0$ and $I_4 \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$. Note that both these random variables take values in $\mathbb{N}_0$. For any sequence of $\mathbb{N}_0$-valued random variables $(\zeta_n)_{n\in\mathbb{N}}$ and any sequence of events $(Q_n)_{n\in\mathbb{N}}$ with $\mathbf{P}(Q_n) \to 1$ as $n \to \infty$ it holds that

$$
\begin{aligned}
\mathbf{P}(\zeta_n \neq 0) &= \mathbf{P}(\mathbf{1}_{Q_n^c}\zeta_n \neq 0) + \mathbf{P}(\mathbf{1}_{Q_n}\zeta_n \neq 0) \leq \mathbf{P}(Q_n^c) + \mathbf{E}\left[\mathbf{1}\{\mathbf{1}_{Q_n}\zeta_n \neq 0\}\right] \\
&= \mathbf{P}(Q_n^c) + \mathbf{E}\left[\min\left(1, \mathbf{1}_{Q_n}\zeta_n\right)\right] = \mathbf{P}(Q_n^c) + \mathbf{E}\left[\mathbf{E}_{\mathcal{W}}\left[\min\left(1, \mathbf{1}_{Q_n}\zeta_n\right)\right]\right] \\
&\leq \mathbf{P}(Q_n^c) + \mathbf{E}\left[\min\left(1, \mathbf{E}_{\mathcal{W}}\left[\mathbf{1}_{Q_n}\zeta_n\right]\right)\right], \qquad (19)
\end{aligned}
$$

where we used Jensen's inequality in the last step. Thus, for $\zeta_n \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$ it suffices to show $\mathbf{E}_{\mathcal{W}}[\mathbf{1}_{Q_n}\zeta_n] \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$.

We start with the summand $I_{3,3}$. As discussed above it suffices to show that $\mathbf{E}_{\mathcal{W}}[I_{3,3}] \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$. For $m \in \mathbb{N}$ we compute

$$
\begin{aligned}
\mathbf{E}_{\mathcal{W}}[I_{3,3}] &= \mathbf{E}_{\mathcal{W}}\Bigg[\sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}, W_v \leq a\tilde{q}(n), S_n(v) > aq(n)\xi\}\Bigg] \\
&= \sum_{v=1}^{n} \mathbf{1}\{W_v \leq a\tilde{q}(n)\}\mathbf{P}_{\mathcal{W}}\Big(\mathbf{1}\{v \in V_n^{\max}\} \cdot S_n(v) > aq(n)\xi\Big) \\
&\leq \sum_{v=1}^{n} \mathbf{1}\{W_v \leq a\tilde{q}(n)\} \frac{\mathbf{E}_{\mathcal{W}}\left[\mathbf{1}\{v \in V_n^{\max}\} \cdot S_n(v)^m\right]}{(aq(n)\xi)^m},
\end{aligned}
$$

where the last inequality follows from the Markov inequality. Let n be large enough such that $a\tilde{q}(n) > 1$. We can bound the conditional expectation by applying Lemma 8 which leads to

$$\begin{aligned}\mathbf{E}_{\mathcal{W}}[I_{3,3}] &\leq \left(\frac{1}{aq(n)\xi}\right)^m \sum_{v=1}^{n} \mathbf{1}\{W_v \leq a\tilde{q}(n)\} R_{n,m} \sum_{t=1}^{m} W_v^t \\ &\leq R_{n,m} \left(\frac{1}{aq(n)\xi}\right)^m \sum_{v=1}^{n} m(a\tilde{q}(n))^m = m\xi^{-m} R_{n,m} \cdot n^{1-m\gamma},\end{aligned}$$

where the last step uses $\tilde{q}(n) = n^{-\gamma} q(n)$. Since $R_{n,m}$ converges almost surely to a constant by Lemma 8, choosing $m > \gamma^{-1}$ ensures that $\mathbf{E}_{\mathcal{W}}[I_{3,3}]$ converges almost surely to zero.

In order to deal with I_4 we define the event

$$\mathcal{G}_{n,\varepsilon} = \left\{ \sup_{v=1,\ldots,n} |\mathbf{E}_{\mathcal{W}}[S_n(v)] - W_v\xi| \leq \frac{\varepsilon q(n)\xi}{2} \right\},$$

which satisfies $\mathbf{P}(\mathcal{G}_{n,\varepsilon}) \to 1$ as $n \to \infty$ due to (A1). By the discussion after (19) we are left to show that $\mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{G}_{n,\varepsilon}} I_4] \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$. We use the $\mathcal{W}$-measurability of $\mathcal{G}_{n,\varepsilon}$ to compute

$$\begin{aligned}&\mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{G}_{n,\varepsilon}} I_4] = \mathbf{1}_{\mathcal{G}_{n,\varepsilon}} \sum_{v=1}^{n} \mathbf{P}_{\mathcal{W}}\left(W_v > a\tilde{q}(n), |S_n(v) - W_v\xi| > \varepsilon q(n)\xi\right) \\ &\leq \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \mathbf{P}_{\mathcal{W}}\left(|\mathbf{E}_{\mathcal{W}}[S_n(v)] - W_v\xi| \leq \frac{\varepsilon q(n)\xi}{2}, |S_n(v) - W_v\xi| > \varepsilon q(n)\xi\right) \\ &\leq \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \mathbf{P}_{\mathcal{W}}\left(|S_n(v) - \mathbf{E}_{\mathcal{W}}[S_n(v)]| > \frac{\varepsilon q(n)\xi}{2}\right).\end{aligned}$$

We use the Chebyshev inequality and Lemma 7 to bound this further by

$$\begin{aligned}\mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{G}_{n,\varepsilon}} I_4] &\leq \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \frac{\mathbf{Var}_{\mathcal{W}}\left(S_n(v)\right)}{\left(\frac{\varepsilon q(n)\xi}{2}\right)^2} \\ &\leq \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \frac{W_v X_n + W_v n^{-1} L_n^{(3)} Y_n}{\left(\frac{\varepsilon q(n)\xi}{2}\right)^2} \\ &= \frac{4}{(\varepsilon\xi)^2} X_n \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \frac{W_v}{q(n)^2} + \frac{4}{(\varepsilon\xi)^2} Y_n \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \frac{W_v L_n^{(3)}}{nq(n)^2} \\ &=: \frac{4}{(\varepsilon\xi)^2}(X_n I_X + Y_n I_Y).\end{aligned}$$

From Lemma 7 we know that X_n and Y_n converge in probability to positive constants as $n \to \infty$. By Slutsky's theorem it suffices to show that $I_X + I_Y \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$.

If $\beta \in (2, 3]$, we choose some $\tau \in (0, \beta)$. We define

$$p = \begin{cases} 1, & \text{for } \beta > 3, \\ \frac{\beta-\tau}{3}, & \text{for } \beta \in (2, 3]. \end{cases}$$

Since then $\mathbf{E}[W^{3p}] < \infty$, it follows from the Marcinkiewicz-Zygmund strong law of large numbers, see e.g. Theorem 5.23 in Kallenberg (2021), that

$$\frac{L_n^{(3)}}{n^{1/p}} \xrightarrow{a.s.} \begin{cases} \mathbf{E}[W^3], & \text{for } \beta > 3, \\ 0, & \text{for } \beta \in (2, 3], \end{cases} \quad \text{as} \quad n \to \infty.$$

Together with Slutsky's theorem, we see that

$$I_X + I_Y \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty$$

if

$$\frac{1 + n^{1/p-1}}{q(n)^2} \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} W_v \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty.$$

In the following we prove this by showing that its expectation

$$h(n) = \frac{n + n^{1/p}}{q(n)^2} \mathbf{E}[\mathbf{1}\{W > a\tilde{q}(n)\} W]$$

vanishes for $n \to \infty$. For a function $g\colon (0, \infty) \to [0, \infty)$ and $\rho \in \mathbb{R}$ we write $g \in \mathrm{RV}_\rho$ if g is regularly varying with index ρ at infinity, i.e. if there exists a slowly varying function $L\colon (0, \infty) \to [0, \infty)$ such that $g(t) = L(t)t^\rho$ for all $t \in (0, \infty)$.

From Lemma 3.3 in Bhattacharjee and Schulte (2022) we know that $q \in \mathrm{RV}_{1/\beta}$, which implies $\tilde{q} \in \mathrm{RV}_{1/\beta-\gamma}$. For $u > 0$ we have

$$\mathbf{E}[\mathbf{1}\{W > u\}W] = \int_0^\infty \mathbf{P}(\mathbf{1}\{W > u\}W > t)dt = u\mathbf{P}(W > u) + \int_u^\infty \mathbf{P}(W > t)dt.$$

Since, by assumption (W), $u \mapsto \mathbf{P}(W > u)$ belongs to $\mathrm{RV}_{-\beta}$, $u \mapsto u\mathbf{P}(W > u)$ is from $\mathrm{RV}_{1-\beta}$ and, by Karamata's theorem, see Theorem 2.1 in Resnick (2007), we derive that $u \mapsto \int_u^\infty \mathbf{P}(W > t)dt$ is also from $\mathrm{RV}_{1-\beta}$. Together with $\tilde{q} \in \mathrm{RV}_{1/\beta-\gamma}$ and Proposition 2.6 in Resnick (2007), we obtain that

$$n \mapsto \mathbf{E}[\mathbf{1}\{W > a\tilde{q}(n)\} W]$$

belongs to $\mathrm{RV}_{(1-\beta)(1/\beta-\gamma)}$. As $1/p \geq 1$ and $q \in \mathrm{RV}_{1/\beta}$, we have $h \in \mathrm{RV}_{(1-\beta)(1/\beta-\gamma)+1/p-2/\beta}$.

If $\beta > 3$,

$$(1-\beta)(1/\beta - \gamma) + 1/p - 2/\beta = -\frac{1+\beta}{\beta} + 1 + \gamma(\beta - 1),$$

while for $\beta \in (2, 3]$,

$$(1-\beta)(1/\beta - \gamma) + 1/p - 2/\beta = -\frac{1+\beta}{\beta} + \frac{3}{\beta - \tau} + \gamma(\beta - 1).$$

Now we can choose γ and τ sufficiently small so that the expressions become negative for both cases. Then, we have $h(n) \to 0$ as $n \to \infty$, which implies

$$\mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{G}_{n,\varepsilon}} I_4] \xrightarrow{\mathbf{P}} 0 \quad \text{and} \quad I_4 \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty.$$

This concludes the proof. □

2.2 Proof of Theorem 3

It is clear that all four classes of vertices in Theorem 3 satisfy condition (A2). The idea to prove (A1) is to explore the component $\mathcal{C}_n(v)$ starting from the vertex v. We start with a lemma for the connection probabilities.

Lemma 9 *For $x, y \in [n]$,*

$$\frac{W_x W_y}{L_n}\left(1 - \min\left(1, \frac{W_{(n)}^2}{L_n}\right)\right) \le \mathbf{P}_{\mathcal{W}}(x \leftrightarrow y) \le \frac{W_x W_y}{L_n}.$$

Proof By a Taylor expansion we have

$$\mathbf{P}_{\mathcal{W}}(x \leftrightarrow y) = 1 - \exp\left(-\frac{W_x W_y}{L_n}\right) = \frac{W_x W_y}{L_n} - \frac{e^{-z}}{2}\left(\frac{W_x W_y}{L_n}\right)^2$$

for some $z > 0$. With

$$0 \le \frac{e^{-z}}{2}\left(\frac{W_x W_y}{L_n}\right)^2 \le \frac{W_x W_y}{L_n}\frac{W_{(n)}^2}{L_n}$$

and $\mathbf{P}_{\mathcal{W}}(x \leftrightarrow y) \ge 0$, the claim follows. □

For a vertex $v_0 \in [n]$ we recall from (9) the random variable

$$T_n(v_0) = \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\}\mathbf{1}\{v_k \in \mathcal{X}_n(v_0)\},$$

which checks for all possible paths $v_0 \dots v_k$ whether they exist or not and if the endpoint belongs to $\mathcal{X}_n(v_0)$. If the component of v_0 is a tree, i.e. it has no cycles, then there is a unique path between any two of its vertices so that $T_n(v_0)$ equals $S_n(v_0)$ – if $v_0 \notin \mathcal{X}_n(v_0)$. However, if there is a cycle, it is possible that $T_n(v_0)$ counts some vertices more than once. We define the event

$$\mathcal{B}_n(v_0) = \left\{ \exists k \geq 3, (s_1, \dots, s_k) \in \mathcal{C}_n(v_0)^k_{\neq} : s_1 \leftrightarrow \dots \leftrightarrow s_k \leftrightarrow s_1 \right\}^c,$$

which means that there are no cycles in v_0's component, and the random variable

$$\overline{T}_n(v_0) = \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\},$$

which is an upper bound for $T_n(v_0)$. The following lemma essentially shows that cycles are unlikely.

Lemma 10 *Assume* (W). *For $n \in \mathbb{N}$ and $v_0 \in [n]$ we have*

$$\mathbf{E}_{\mathcal{W}}\left[\mathbf{1}_{\mathcal{B}_n(v_0)^c}\overline{T}_n(v_0)\right] \leq W_{v_0}\frac{W_{(n)}^2}{L_n}U_n, \text{ where } U_n = 2\left(\sum_{k=0}^{\infty}(k+2)^2\left(\frac{L_n^{(2)}}{L_n}\right)^k\right)^2.$$

Proof Assume without loss of generality that $v_0 = n$. Then

$$\mathbf{1}_{\mathcal{B}_n(v_0)^c}\overline{T}_n(v_0) = \mathbf{1}_{\mathcal{B}_n(v_0)^c}\sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n-1]^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\}.$$

For $\mathbf{1}_{\mathcal{B}_n(v_0)^c}\overline{T}_n(v_0)$ to be non-zero, there must be a cycle somewhere in the component of v_0 by the definition of $\mathcal{B}_n(v_0)$. Consider the path $v_0 \dots v_k$ currently counted in $\overline{T}_n(v_0)$. For the existence of a cycle in the component of v_0 we obtain two possible cases:

1. Two vertices of the path, say v_p and v_q for $p < q$, can be connected by a different path $s_1 \dots s_\ell$ with $s_1 = v_p$ and $s_\ell = v_q$, resulting in a cycle. We may assume that this new path has no further intersections with the originally considered path by shortening it if needed.
2. There is a path $s_1 \dots s_\ell$ with $\ell \geq 3$ starting in $s_1 = v_p$ for some $0 \leq p \leq k$ such that $s_\ell \leftrightarrow s_q$ for some $q < \ell - 1$. We may assume that $s_1 \dots s_\ell$ has no intersections other than s_1 with $v_0 \dots v_k$ by shortening $s_1 \dots s_\ell$ or ending up in the first case.

The two cases above yield the following upper bound for $\mathbf{1}_{\mathcal{B}_n(v_0)^c}\overline{T}_n(v_0)$, where we use the notation from Lemma 5,

$$\sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n-1]^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\} \sum_{0\leq p<q\leq k} \sum_{\ell=2}^{n} |\mathcal{P}_\ell^{(n)}(v_p, v_q, \{v_0, \dots, v_k\})|$$

$$+\sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n-1]^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\} \sum_{p=0}^{k}\sum_{\ell=3}^{n} \sum_{x\in[n]\setminus\{v_0,\dots,v_k\}}$$

$$\times \sum_{(s_1,\dots,s_\ell)\in\mathcal{P}_\ell^{(n)}(v_p,x,\{v_0,\dots,v_k\})} \sum_{q=1}^{\ell-2} \mathbf{1}\{x \leftrightarrow s_q\} =: I_1 + I_2.$$

Without any shared edges, we can use conditional independence and apply Lemma 9 and Lemma 5 to obtain

$$\begin{aligned}\mathbf{E}_{\mathcal{W}}[I_1] &\leq \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n-1]_{\neq}^{k}} \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \sum_{p,q=0}^{k} \sum_{\ell=2}^{n} \left(\frac{L_n^{(2)}}{L_n}\right)^{\ell-2} \frac{W_{(n)}^2}{L_n} \\ &\leq W_{v_0}\frac{W_{(n)}^2}{L_n} \sum_{k=1}^{n} \left(\frac{L_n^{(2)}}{L_n}\right)^{k-1} (k+1)^2 \sum_{\ell=2}^{n} \left(\frac{L_n^{(2)}}{L_n}\right)^{\ell-2} \leq W_{v_0}\frac{W_{(n)}^2}{L_n}\frac{U_n}{2}.\end{aligned}$$

For the second summand we use

$$\mathbf{E}_{\mathcal{W}}\left[\sum_{q=1}^{\ell-2} \mathbf{1}\{x \leftrightarrow s_q\}\right] \leq \ell \frac{W_x W_{(n)}}{L_n}$$

so that we can bound $\mathbf{E}_{\mathcal{W}}[I_2]$ by

$$\begin{aligned}&\sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n-1]_{\neq}^{k}} \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \sum_{p=0}^{k}\sum_{\ell=3}^{n} \sum_{x\in[n]\setminus\{v_0,\dots,v_k\}} \frac{W_{v_p}W_x}{L_n} \left(\frac{L_n^{(2)}}{L_n}\right)^{\ell-2} \ell \frac{W_x W_{(n)}}{L_n} \\ &\leq W_{v_0}\frac{W_{(n)}^2}{L_n} \sum_{k=1}^{n} \left(\frac{L_n^{(2)}}{L_n}\right)^{k-1} (k+1) \sum_{\ell=3}^{n} \left(\frac{L_n^{(2)}}{L_n}\right)^{\ell-2} \ell \frac{\sum_{x=1}^{n} W_x^2}{L_n} \leq W_{v_0}\frac{W_{(n)}^2}{L_n}\frac{U_n}{2},\end{aligned}$$

where the last inequality uses $\sum_{x=1}^{n} W_x^2 = L_n^{(2)}$ and an index shift. Summing up both bounds yields the claim. □

The following technical lemma provides sufficient conditions for verifying assumption (A1). The intuition is that one wants to determine $S_n(v_0)$ by counting paths of length k, checking locally whether the endpoint belongs to $\mathcal{X}_n(v_0)$ and summing over all possible path lengths.

Lemma 11 *Let* $\xi > 0$. *Assume* (W) *and that there exists for all* $n \in \mathbb{N}, k \in [n]$ *and* $(v_0,\dots,v_k) \in [n]_{\neq}^{k+1}$ *an event* $\mathcal{I}_n(v_0,\dots,v_k)$ *such that*

$$\mathcal{I}_n(v_0,\dots,v_k) \textit{ is conditionally on } \mathcal{W} \textit{ independent of } \{v_0 \leftrightarrow v_1\},\dots,\{v_{k-1} \leftrightarrow v_k\}, \tag{20}$$

$$\mathbf{1}_{\mathcal{B}_n(v_0)}\mathbf{1}\{v_0 \leftrightarrow \dots \leftrightarrow v_k, v_k \in \mathcal{X}_n(v_0)\} = \mathbf{1}_{\mathcal{B}_n(v_0)}\mathbf{1}\{v_0 \leftrightarrow \dots \leftrightarrow v_k\}\mathbf{1}_{\mathcal{I}_n(v_0,\dots,v_k)} \tag{21}$$

and

$$\frac{1}{q(n)} \sup_{v_0\in[n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})_{\neq}^{k}} \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - W_{v_0}\xi \right| \xrightarrow{\mathbf{P}} 0, \tag{22}$$

as $n \to \infty$. *Then, assumption* (A1) *is satisfied for that choice of* ξ.

Proof We need to show

$$\frac{1}{q(n)} \sup_{v=1,\dots,n} |\mathbf{E}_{\mathcal{W}}[S_n(v)] - W_v \xi| \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty.$$

For $v \in [n]$, adding v to $\mathcal{X}_n(v)$ or removing v from $\mathcal{X}_n(v)$ changes the value of $S_n(v)$ by one. Due to the triangle inequality and $q(n) \to \infty$ as $n \to \infty$, this does not change whether the statement above is true or false. Therefore, we may assume without loss of generality that $v \notin \mathcal{X}_n(v)$ for all $v \in [n]$. For all $v_0 \in [n]$ we write

$$\widetilde{T}_n(v_0) = \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\} \mathbf{1}_{\mathcal{I}_n(v_0,\dots,v_k)}$$

and obtain with (21) and $v \notin \mathcal{X}_n(v)$ for all $v \in [n]$ that

$$\mathbf{1}_{\mathcal{B}_n(v)} S_n(v) = \mathbf{1}_{\mathcal{B}_n(v)} T_n(v) = \mathbf{1}_{\mathcal{B}_n(v)} \widetilde{T}_n(v).$$

We conclude

$$\begin{aligned} \mathbf{E}_{\mathcal{W}}[S_n(v)] &= \mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)} S_n(v)] + \mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} S_n(v)] \\ &= \mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)} \widetilde{T}_n(v)] + \mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} S_n(v)] \\ &= \mathbf{E}_{\mathcal{W}}[\widetilde{T}_n(v)] - \mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} \widetilde{T}_n(v)] + \mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} S_n(v)] \end{aligned}$$

so that $S_n(v), \widetilde{T}_n(v) \le \overline{T}_n(v)$ and Lemma 10 yield

$$|\mathbf{E}_{\mathcal{W}}[S_n(v)] - \mathbf{E}_{\mathcal{W}}[\widetilde{T}_n(v)]| \le 2\mathbf{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} \overline{T}_n(v)] \le 2W_v \frac{W_{(n)}^2}{L_n} U_n \le 2\frac{W_{(n)}^3}{L_n} U_n.$$

This in turn provides us with

$$\frac{1}{q(n)} \sup_{v \in [n]} |\mathbf{E}_{\mathcal{W}}[S_n(v)] - W_v \xi| \le \frac{1}{q(n)} \sup_{v \in [n]} |\mathbf{E}_{\mathcal{W}}[\widetilde{T}_n(v)] - W_v \xi| + \frac{2W_{(n)}^3}{q(n) L_n} U_n.$$

Here the last summand converges in probability to zero as $n \to \infty$, see (12) for U_n and (6) as well as (8) for the other factors. Next, we bound the supremum on the right-hand side. By (20) we have

$$\begin{aligned} \mathbf{E}_{\mathcal{W}}[\widetilde{T}_n(v_0)] &= \mathbf{E}_{\mathcal{W}}\left[\sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\} \mathbf{1}_{\mathcal{I}_n(v_0,\dots,v_k)}\right] \\ &= \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{P}_{\mathcal{W}}(v_i \leftrightarrow v_{i-1}) \mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) \end{aligned}$$

so that

$$\begin{aligned}
&\frac{1}{q(n)}\sup_{v_0\in[n]}|\mathbf{E}_{\mathcal{W}}[\widetilde{T}_n(v_0)]-W_{v_0}\xi| \\
&\le \frac{1}{q(n)}\sup_{v_0\in[n]}\left|\sum_{k=1}^{n}\sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}}\prod_{i=1}^{k}\frac{W_{v_i}W_{v_{i-1}}}{L_n}\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k))-W_{v_0}\xi\right| \\
&\quad+\frac{1}{q(n)}\sup_{v_0\in[n]}\sum_{k=1}^{n}\sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}}\left|\prod_{i=1}^{k}\mathbf{P}_{\mathcal{W}}(v_i\leftrightarrow v_{i-1})-\prod_{i=1}^{k}\frac{W_{v_i}W_{v_{i-1}}}{L_n}\right|.
\end{aligned}$$

By assumption (22), the first term on the right-hand side converges in probability to zero as $n\to\infty$. For the second summand we obtain from Lemma 9 for all $v_0,\dots,v_k\in[n]$,

$$\begin{aligned}
&\left|\prod_{i=1}^{k}\mathbf{P}_{\mathcal{W}}(v_i\leftrightarrow v_{i-1})-\prod_{i=1}^{k}\frac{W_{v_i}W_{v_{i-1}}}{L_n}\right|\le\prod_{i=1}^{k}\frac{W_{v_i}W_{v_{i-1}}}{L_n}\left(1-\left(1-\min\left(1,\frac{W_{(n)}^2}{L_n}\right)\right)^k\right) \\
&=W_{v_0}\prod_{i=1}^{k-1}\frac{W_{v_i}^2}{L_n}\frac{W_{v_k}}{L_n}\left(1-\left(1-\min\left(1,\frac{W_{(n)}^2}{L_n}\right)\right)^k\right)\le W_{v_0}\prod_{i=1}^{k-1}\frac{W_{v_i}^2}{L_n}\frac{W_{v_k}}{L_n}k\frac{W_{(n)}^2}{L_n},
\end{aligned}$$

where the last inequality uses that $1-y^k\le k(1-y)$ for all $y\in[0,1]$ by the mean value theorem. We obtain

$$\begin{aligned}
&\frac{1}{q(n)}\sup_{v_0\in[n]}\sum_{k=1}^{n}\sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}}\left|\prod_{i=1}^{k}\mathbf{P}_{\mathcal{W}}(v_i\leftrightarrow v_{i-1})-\prod_{i=1}^{k}\frac{W_{v_i}W_{v_{i-1}}}{L_n}\right| \\
&\le\frac{1}{q(n)}\sup_{v_0\in[n]}\sum_{k=1}^{n}\sum_{(v_1,\dots,v_k)\in[n]^k}W_{v_0}\prod_{i=1}^{k-1}\frac{W_{v_i}^2}{L_n}\frac{W_{v_k}}{L_n}k\frac{W_{(n)}^2}{L_n}=\frac{W_{(n)}^3}{q(n)L_n}\sum_{k=1}^{n}k\left(\frac{L_n^{(2)}}{L_n}\right)^{k-1},
\end{aligned}$$

which converges in probability to zero as $n\to\infty$ due to (6), (8) and (12). □

Lemma 12 *Assume* (W). *Suppose that for all* $n\in\mathbb{N},k\in[n]$ *and* $(v_0,\dots,v_k)\in[n]^{k+1}_{\neq}$ *there exists an event* $\mathcal{I}_n(v_0,\dots,v_k)$ *such that* (20) *as well as* (21) *hold. Moreover, assume that there exist a bounded, measurable function* $g\colon[0,\infty)\times\mathbb{N}\to[0,\infty)$, *a polynomial* p *of finite degree and random variables* $(R_n)_{n\in\mathbb{N}}$ *such that for all* $n\in\mathbb{N},k\in[n]$ *and* $(v_0,\dots,v_k)\in[n]^{k+1}_{\neq}$,

$$|\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k))-g(W_{v_k},k)|\le p(k)R_n \tag{23}$$

and

$$R_n\xrightarrow{\mathbf{P}}0\quad as\quad n\to\infty. \tag{24}$$

Then, (A1) *is satisfied with*

$$\xi = \sum_{k=1}^{\infty} \left(\frac{\mathbf{E}[W^2]}{\mathbf{E}[W]} \right)^{k-1} \frac{\mathbf{E}[Wg(W,k)]}{\mathbf{E}[W]}.$$

Proof By Lemma 11 it suffices to show (22). We have

$$\begin{aligned}
&\frac{1}{q(n)} \sup_{v_0\in[n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - W_{v_0}\xi \right| \\
&\le \frac{1}{q(n)} \sup_{v_0\in[n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \right. \\
&\quad \left. \times \Big(\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - g(W_{v_k},k) \Big) \right| \\
&\quad + \frac{1}{q(n)} \sup_{v_0\in[n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n]^k\setminus([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} g(W_{v_k},k) \right| \\
&\quad + \frac{1}{q(n)} \sup_{v_0\in[n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n]^k} \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} g(W_{v_k},k) - W_{v_0}\xi \right| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Due to (23) we have

$$I_1 \le \frac{W_{(n)}}{q(n)} \sum_{k=1}^{n} \left(\frac{L_n^{(2)}}{L_n} \right)^{k-1} p(k) R_n,$$

which converges to zero in probability as $n \to \infty$, by (6), (12) and (24). For I_2, we consider the elements $(v_1,\dots,v_k) \in [n]^k \setminus ([n]\setminus\{v_0\})^k_{\neq}$ for a fixed $v_0 \in [n]$, which means that $(v_1,\dots,v_k)$ has two equal entries or contains v_0. There are $\binom{k}{2}$ choices for $i, j \in [k]$ with $v_i = v_j$ for $i < j$. In this case, we may bound one factor by $W_{(n)}^2/L_n$ and omit the summation over v_i. On the other hand, it may be that there exists $i \in [k]$ such that $v_i = v_0$. If $i < k$, we have $(k-1)$ choices and can once more omit the summation over v_i and bound the respective factor by $W_{(n)}^2/L_n$. For $i = k$, we obtain a different scenario due to the factor $g(W_{v_k},k)$. We derive

$$\begin{aligned}
I_2 \le{}& \frac{W_{(n)}^3}{L_n q(n)} \sum_{k=2}^{n} \left(\binom{k}{2} + k \right) \left(\frac{L_n^{(2)}}{L_n} \right)^{k-2} \frac{\sum_{i=1}^{n} W_i g(W_i,k)}{L_n} \\
&+ \sum_{k=1}^{n} \frac{\sup_{v_0\in[n]} W_{v_0}^2 g(W_{v_0},k)}{L_n q(n)} \left(\frac{L_n^{(2)}}{L_n} \right)^{k-1}.
\end{aligned}$$

In the first summand the first factor vanishes as $n \to \infty$ due to (6) and (8), while the two sums are bounded because of (12) as well as the boundedness of g and the law of large numbers. For the second summand, the first fraction converges in probability to zero as $n \to \infty$ by the boundedness of g, (7) and (8), whereas the remaining sum converges almost surely as $n \to \infty$ by (11).

For I_3 we obtain

$$I_3 = \frac{W_{(n)}}{q(n)} \left| \sum_{k=1}^{n} \left(\frac{L_n^{(2)}}{L_n} \right)^{k-1} \frac{\sum_{i=1}^{n} W_i g(W_i, k)}{L_n} - \xi \right|.$$

From the strong law of large numbers and the boundedness of g we deduce

$$\sum_{k=1}^{n} \left(\frac{L_n^{(2)}}{L_n} \right)^{k-1} \frac{\sum_{i=1}^{n} W_i g(W_i, k)}{L_n} \xrightarrow{a.s.} \sum_{k=1}^{\infty} \left(\frac{\mathbf{E}[W^2]}{\mathbf{E}[W]} \right)^{k-1} \frac{\mathbf{E}[W g(W, k)]}{\mathbf{E}[W]} = \xi,$$

as $n \to \infty$, which finishes the proof. □

We show that all our classes of vertices in Theorem 3 meet the assumptions of Lemma 12.

Proof that all vertices satisfy (A1) For $n \in \mathbb{N}$, $k \in [n]$ and $(v_0, \ldots, v_k) \in [n]_{\neq}^{k+1}$ we choose $\mathcal{I}_n(v_0, \ldots, v_k)$ as an event with probability one, $g = 1$ as well as $p = R_n = 0$. In this case it is easy to see that all assumptions of Lemma 12 are met. □

Proof that vertices in a fixed distance m to v_0 satisfy (A1) For $n \in \mathbb{N}$, $k \in [n]$ as well as $(v_0, \ldots, v_k) \in [n]_{\neq}^{k+1}$ we choose $\mathcal{I}_n(v_0, \ldots, v_k) = \{k = m\}$, $g = \mathbf{1}\{k = m\}$ and $p = R_n = 0$. We see that all assumptions of Lemma 12 are met. □

Proof that vertices with a fixed degree m satisfy (A1) We start with the case $m = 1$, i.e. we consider leaves. We define for $n \in \mathbb{N}$, $k \in [n]$ and $(v_0, \ldots, v_k) \in [n]_{\neq}^{k+1}$,

$$\mathcal{I}_n(v_0, \ldots, v_k) = \{v_k \text{ has no neighbour in } [n] \setminus \{v_{k-1}, v_k\}\}.$$

Then $\mathcal{I}_n(v_0, \ldots, v_k)$ satisfies (20) and (21) since the path $v_0 \ldots v_k$ ensures that v_k already has one neighbour. Additionally, let $g(x, k) = e^{-x}$ and $p = 1$. For (23) we calculate

$$\begin{aligned} \left| \mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \ldots, v_k)) - e^{-W_{v_k}} \right| &= \left| \mathbf{E}_{\mathcal{W}} \left[\prod_{x \in [n] \setminus \{v_{k-1}, v_k\}} \mathbf{1}\{v_k \not\leftrightarrow x\} \right] - e^{-W_{v_k}} \right| \\ &= \left| \exp\left(- W_{v_k} \frac{\sum_{x \in [n] \setminus \{v_{k-1}, v_k\}} W_x}{L_n} \right) - e^{-W_{v_k}} \right| \\ &\le e^{-W_{v_k}} \left(e^{2W_{(n)}/L_n} - 1 \right) \le e^{2W_{(n)}/L_n} - 1 =: R_n \end{aligned}$$

so that (24) holds due to (8).

For vertices of degree $m \geq 2$ we proceed similarly. We define

$$\mathcal{I}_n(v_0, \ldots, v_k) = \{v_k \text{ has exactly } m-1 \text{ neighbours in } [n] \setminus \{v_{k-1}, v_k\}\}$$

and see that (20) and (21) hold. We rewrite $\mathbf{1}_{\mathcal{I}_n(v_0,\ldots,v_k)}$ as

$$\frac{1}{(m-1)!} \sum_{(a_1,\ldots,a_{m-1}) \in ([n]\setminus\{v_{k-1},v_k\})^{m-1}_{\neq}} \prod_{i=1}^{m-1} \mathbf{1}\{v_k \leftrightarrow a_i\} \prod_{x \in [n]\setminus\{a_1,\ldots,a_{m-1},v_{k-1},v_k\}} \mathbf{1}\{v_k \not\leftrightarrow x\}$$

and define $g(x, k) = x^{m-1}e^{-x}/(m-1)!$, which is a bounded function as required. For (23) we provide upper and lower bounds for $\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \ldots, v_k))$. We compute with the equality above and Lemma 9

$$\begin{aligned}
&\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \ldots, v_k)) \\
&\leq \frac{1}{(m-1)!} \sum_{a_1,\ldots,a_{m-1}=1}^{n} \prod_{i=1}^{m-1} \frac{W_{a_i} W_{v_k}}{L_n} \exp\left(- W_{v_k} \sum_{x \in [n]\setminus\{a_1,\ldots,a_{m-1},v_{k-1},v_k\}} \frac{W_x}{L_n} \right) \\
&\leq \frac{W_{v_k}^{m-1} e^{-W_{v_k}}}{(m-1)!} \sum_{a_1,\ldots,a_{m-1}=1}^{n} \prod_{i=1}^{m-1} \frac{W_{a_i}}{L_n} \exp\left(\frac{(m+1)W_{(n)}^2}{L_n} \right) \\
&= g(W_{v_k}, k) \exp\left(\frac{(m+1)W_{(n)}^2}{L_n} \right) =: g(W_{v_k}, k)\overline{I}_n,
\end{aligned}$$

where the upper bound $\overline{I}_n$ satisfies $\overline{I}_n \xrightarrow{\mathbf{P}} 1$ as $n \to \infty$ by (8). Moreover, by Lemma 9

$$\begin{aligned}
\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \ldots, v_k)) &\geq \frac{W_{v_k}^{m-1} e^{-W_{v_k}}}{(m-1)!} \sum_{(a_1,\ldots,a_{m-1}) \in ([n]\setminus\{v_{k-1},v_k\})^{m-1}_{\neq}} \prod_{i=1}^{m-1} \frac{W_{a_i}}{L_n} \left(1 - \frac{W_{(n)}^2}{L_n}\right)^{m-1} \\
&= g(W_{v_k}, k)\left(1 - \frac{W_{(n)}^2}{L_n}\right)^{m-1} \left(1 - \sum_{(a_1,\ldots,a_{m-1}) \in [n]^{m-1}\setminus([n]\setminus\{v_{k-1},v_k\})^{m-1}_{\neq}} \prod_{i=1}^{m-1} \frac{W_{a_i}}{L_n}\right).
\end{aligned}$$

The last sum above involves all tuples a of length $m-1$ which either contain the same entry twice or contain at least one entry equal to v_{k-1} or v_k. This leads to

$$\sum_{(a_1,\ldots,a_{m-1}) \in [n]^{m-1}\setminus([n]\setminus\{v_{k-1},v_k\})^{m-1}_{\neq}} \prod_{i=1}^{m-1} \frac{W_{a_i}}{L_n} \leq \frac{W_{(n)}}{L_n}\left(\binom{m}{2} + 2m\right) \leq 3m^2 \frac{W_{(n)}}{L_n}$$

so that

$$\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \ldots, v_k)) \geq g(W_{v_k}, k)\left(1 - \frac{W_{(n)}^2}{L_n}\right)^{m-1} \left(1 - 3m^2 \frac{W_{(n)}}{L_n}\right) =: g(W_{v_k}, k)\underline{I}_n,$$

where the lower bound $\underline{I}_n$ satisfies $\underline{I}_n \xrightarrow{\mathbf{P}} 1$ as $n \to \infty$ by (8). For $k \in [n]$, $p = \sup_{x\in[0,\infty)} g(x,k)$ we obtain

$$\begin{aligned}\left|\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - g(W_{v_k},k)\right| &\le g(W_{v_k},k)(|1-\overline{I}_n| + |1-\underline{I}_n|)\\ &\le p(|1-\overline{I}_n| + |1-\underline{I}_n|) =: pR_n,\end{aligned}$$

which shows (23). From $\underline{I}_n, \overline{I}_n \xrightarrow{\mathbf{P}} 1$ as $n \to \infty$ it follows $R_n \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$ as demanded in (24). ☐

Proof that terminal trees satisfy (A1) We consider a tree T with m vertices $1,\dots,m$ and root 1. We write $V(T) = [m]$ for its vertices and $E(T)$ for its edges. For $n \in \mathbb{N}$, $k \in [n]$ and $(v_0,\dots,v_k) \in [n]^{k+1}_{\neq}$ let

$$\begin{aligned}\mathcal{I}_n(v_0,\dots,v_k) = \{&\exists(a_1,\dots,a_m) \in ([n]\setminus\{v_0,\dots,v_{k-1}\})^m_{\neq} \text{ with } a_1 = v_k \text{ such that the}\\ &\text{graph induced by } a_1,\dots,a_m \text{ with } a_1 \text{as root is isomorphic to } T\\ &\text{and, after deleting all edges between } v_{k-1} \text{ and } v_k, \text{ the vertices}\\ &a_1,\dots,a_m \text{ form a component in } G_n\},\end{aligned}$$

$p(k) = k+1$ and with $c(T)$ as in Theorem 3

$$g(x,k) = \frac{x^{\deg_T(1)}e^{-x}}{c(T)} \prod_{i=2}^{m} \frac{\mathbf{E}[W^{\deg_T(i)}e^{-W}]}{\mathbf{E}[W]}.$$

We note that (20) and (21) are satisfied by the choice of $\mathcal{I}_n(v_0,\dots,v_k)$. We write $\varphi\colon V(T) \to [n]$ for the map $i \mapsto a_i$ which assigns a vertex of T to its corresponding vertex in G_n. Consequently, $\varphi(E(T)) = \{\{\varphi(i),\varphi(j)\}\colon i,j \in V(T)\}$ denotes the edges of T embedded into G_n via φ. For $\varphi(V(T))$ to be isomorphic to T, we require all edges in $\varphi(E(T))$ to exist in G_n and no further connections between any vertices in $\varphi(V(T))$ are allowed. Furthermore, all edges between $\varphi(V(T))$ and $[n]\setminus\varphi(V(T))$ are forbidden since the tree is supposed to be terminal, up to the exception of $a_1 = v_k$ being connected to v_{k-1}. We gather all these forbidden edges in the set

$$F_a = \{\{a_x, y\}\colon x \in V(T),\, y \in [n]\setminus\{a_x\}\} \setminus (\varphi(E(T)) \cup \{v_k, v_{k-1}\}).$$

Here we do not have $\{x,x\} \in F_a$ for any $x \in V(T)$ since we chose the convention to ignore loops. Considering all possible choices for $a_1,\dots,a_m$ and permutations thereof yields

$$\mathbf{1}_{\mathcal{I}_n(v_0,\dots,v_k)} = c(T)^{-1} \sum_{\substack{(a_1,\dots,a_m)\in([n]\setminus\{v_0,\dots,v_{k-1}\})^m_{\neq}\\ a_1=v_k}} \prod_{\{i,j\}\in E(T)} \mathbf{1}\{a_i \leftrightarrow a_j\} \prod_{\{i,j\}\in F_a} \mathbf{1}\{i \not\leftrightarrow j\}. \tag{25}$$

All occurring factors are independent when conditioning on $\mathcal{W}$. For (23) we provide upper and lower bounds for $\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k))$. We start with the last product concerning the forbidden edges. Let $a_1 = v_k$ and $(a_2, \ldots, a_m) \in ([n] \setminus \{v_0, \ldots, v_k\})^{m-1}_{\neq}$. As $E_n\{x, y\}$ follows a Poisson distribution, we get

$$\mathbf{E}_{\mathcal{W}}\Bigg[\prod_{\{i,j\}\in F_a} \mathbf{1}\{i \nleftrightarrow j\}\Bigg] = \prod_{\{i,j\}\in F_a} \exp\bigg(-\frac{W_i W_j}{L_n}\bigg).$$

Since all factors are bounded by one, we may add similar factors for a lower bound. Since $F_a \subseteq \{\{a_x, y\} \colon x \in V(T),\, y \in [n]\}$ this yields

$$\mathbf{E}_{\mathcal{W}}\Bigg[\prod_{\{i,j\}\in F_a} \mathbf{1}\{i \nleftrightarrow j\}\Bigg] \geq \prod_{x=1}^{m}\prod_{y=1}^{n} \exp\bigg(-\frac{W_{a_x} W_y}{L_n}\bigg) = \prod_{x=1}^{m} \exp(-W_{a_x}). \tag{26}$$

For an upper bound we observe that $F_a \supseteq \{\{a_x, y\} \colon x \in V(T),\, y \in [n] \setminus (\varphi(V(T)) \cup \{v_{k-1}, a_x\})\}$. In comparison to $\{\{a_x, y\} \colon x \in V(T),\, y \in [n]\}$, there are at most $m(m+2)$ fewer elements. Therefore,

$$\begin{aligned}\mathbf{E}_{\mathcal{W}}\Bigg[\prod_{\{i,j\}\in F_a} \mathbf{1}\{i \nleftrightarrow j\}\Bigg] &\leq \prod_{x=1}^{m}\prod_{y=1}^{n} \exp\bigg(-\frac{W_{a_x} W_y}{L_n}\bigg) \exp\bigg(m(m+2)\frac{W_{(n)}^2}{L_n}\bigg)\\ &= \prod_{x=1}^{m} \exp(-W_{a_x}) \exp\bigg(m(m+2)\frac{W_{(n)}^2}{L_n}\bigg) =: \prod_{x=1}^{m} \exp(-W_{a_x})\overline{I}_n,\end{aligned} \tag{27}$$

where $\overline{I}_n$ converges in probability to 1 as $n \to \infty$ due to (8). Next, we consider the first product in (25). From Lemma 9 and the fact that a tree with m vertices has $m-1$ edges we get the upper bound

$$\mathbf{E}_{\mathcal{W}}\Bigg[\prod_{\{i,j\}\in E(T)} \mathbf{1}\{a_i \leftrightarrow a_j\}\Bigg] \leq \prod_{\{i,j\}\in E(T)} \frac{W_{a_i} W_{a_j}}{L_n} = W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \tag{28}$$

and similarly the lower bound

$$\begin{aligned}\mathbf{E}_{\mathcal{W}}\Bigg[\prod_{\{i,j\}\in E(T)} \mathbf{1}\{a_i \leftrightarrow a_j\}\Bigg] &\geq W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \times \Big(1 - \min(1, W_{(n)}^2/L_n)\Big)^{m-1}\\ &=: W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \underline{I}_n,\end{aligned} \tag{29}$$

where $\underline{I}_n$ converges in probability to 1 as $n \to \infty$ by (8). Using the upper bounds (27) and (28) for the conditional expectation of (25) yields

$$\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \ldots, v_k)) \leq c(T)^{-1} \sum_{\substack{(a_1,\ldots,a_m)\in([n]\setminus\{v_0,\ldots,v_{k-1}\})^m_{\neq}\\ a_1 = v_k}} W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \prod_{x=1}^{m} \exp(-W_{a_x})\overline{I}_n$$

$$\leq \frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} \prod_{i=2}^{m} \frac{\sum_{a_i=1}^{n} W_{a_i}^{\deg_T(i)} e^{-W_{a_i}}}{L_n} \overline{I}_n =: \frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} X_n \overline{I}_n,$$

where the weak law of large numbers gives us

$$X_n \xrightarrow{\mathbf{P}} \prod_{i=2}^{m} \frac{\mathbf{E}[W^{\deg_T(i)} e^{-W}]}{\mathbf{E}[W]} =: X \quad \text{as} \quad n \to \infty.$$

The lower bounds (26) and (29) provide

$$\begin{aligned}
\mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \dots, v_k)) &\geq c(T)^{-1} \sum_{\substack{(a_1,\dots,a_m) \in ([n] \setminus \{v_0,\dots,v_{k-1}\})_{\neq}^{m} \\ a_1 = v_k}} W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \prod_{x=1}^{m} \exp(-W_{a_x}) \underline{I}_n \\
&= \frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} \underline{I}_n \Bigg(X_n - \sum_{(a_2,\dots,a_m) \in [n]^{m-1} \setminus ([n] \setminus \{v_0,\dots,v_k\})_{\neq}^{m-1}} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)} e^{-W_{a_i}}}{L_n} \Bigg).
\end{aligned}$$

In the last term we sum over all tuples $(a_2, \dots, a_m)$ which contain two equal entries or one entry from the set $\{v_0, \dots, v_k\}$. The number of such tuples is bounded by

$$\binom{m-1}{2} n^{m-2} + (k+1)(m-1) n^{m-2},$$

where the first summand accounts for two equal entries whereas the second one considers the case where one entry lies in $\{v_0, \dots, v_k\}$. Noting that $C := \max_{j=1,\dots,m} \sup_{x \in [0,\infty)} x^j e^{-x} < \infty$ as well as $\underline{I}_n \leq 1$, $\deg_T(i) \leq m$ we obtain

$$\begin{aligned}
&\frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} \underline{I}_n \sum_{(a_2,\dots,a_m) \in [n]^{m-1} \setminus ([n] \setminus \{v_0,\dots,v_k\})_{\neq}^{m-1}} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)} e^{-W_{a_i}}}{L_n} \\
&\leq \frac{C^m}{c(T)} \frac{n^{m-2}}{L_n^{m-1}} \left(\binom{m-1}{2} + (k+1)(m-1) \right) \\
&\leq (k+1) \frac{C^m}{c(T)} \frac{n^{m-2}}{L_n^{m-1}} \left(\binom{m-1}{2} + (m-1) \right) =: (k+1) J_n = p(k) J_n,
\end{aligned}$$

where J_n converges in probability to 0 as $n \to \infty$. The lower and upper bound provide

$$\begin{aligned}
&\left| \mathbf{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \dots, v_k)) - g(W_{v_k}, k) \right| \\
&\leq p(k) J_n + \frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} \Big(|X - X_n \overline{I}_n| + |X - X_n \underline{I}_n| \Big)
\end{aligned}$$

$$\leq p(k)\left(J_n + \frac{C}{c(T)}\left(|X - X_n\overline{I}_n| + |X - X_n\underline{I}_n|\right)\right) =: p(k)R_n,$$

which is condition (23). Since $J_n, X_n, \overline{I}_n$ and $\underline{I}_n$ do not depend on k and converge in probability to 0, X, 1 and 1 as $n \to \infty$, respectively, we conclude that $R_n \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$, showing (24) and finishing the proof. ☐

2.3 Proof of Theorem 4

In this subsection we use results from Janson (2010). The first notion is that two sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of random graphs are asymptotically equivalent if one can couple them such that $\mathbf{P}(X_n \neq Y_n) \to 0$ as $n \to \infty$, see Definition 1.1 and Theorem 4.2 in Janson (2010).

Lemma 13 *Under assumption (W) the following hold:*

1. $\mathrm{ENR}(n)$, $\mathrm{CL}(n)$ *and* $\mathrm{GRG}(n)$ *are asymptotically equivalent.*
2. $\mathrm{ENR}'(n)$, $\mathrm{CL}'(n)$ *and* $\mathrm{GRG}'(n)$ *are asymptotically equivalent.*

Proof 1. The statement follows directly from Example 3.6 in Janson (2010) since (W) implies that $\mathbf{P}(W > t) = o(t^{-2})$.

2. Example 3.1 in Janson (2010) shows that $\mathrm{ENR}'(n)$, $\mathrm{CL}'(n)$ and $\mathrm{GRG}'(n)$ are asymptotically equivalent if

$$\sum_{1\leq i<j\leq n}\left(\frac{W_iW_j}{n\mathbf{E}[W]}\right)^3 \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty,$$

see Equation (3.6) in Janson (2010). Under assumption (W) we find that

$$\sum_{1\leq i<j\leq n}\left(\frac{W_iW_j}{n\mathbf{E}[W]}\right)^3 \leq \frac{W_{(n)}^2}{n\mathbf{E}[W]^3}\left(\frac{L_n^{(2)}}{n}\right)^2 \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty$$

since the first factor converges in probability to zero as $n \to \infty$ by (8) whereas the second factor remains bounded by the strong law of large numbers and the finite second moment of W. ☐

We also require a concept from Definition 1.1 in Janson (2010), which is formulated in a more abstract setting. For $n \in \mathbb{N}$ we consider some measurable space $(\mathcal{X}_n, \mathcal{A}_n)$ with two probability measures $\mathbf{P}_n$ and $\mathbf{Q}_n$ defined on it. One calls $(\mathbf{P}_n)_{n\in\mathbb{N}}$ contiguous with respect to $(\mathbf{Q}_n)_{n\in\mathbb{N}}$ if

$$\mathbf{Q}_n(A_n) \to 0 \quad \text{as} \quad n \to \infty \quad \Longrightarrow \quad \mathbf{P}_n(A_n) \to 0 \quad \text{as} \quad n \to \infty$$

for all sequences of sets $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathcal{A}_n$. Since we need to keep track of the underlying weights on top of the generated graphs, we think of the random graphs as probability measures on $\mathcal{X}_n = (0, \infty)^n \times \mathcal{G}_n$, where $\mathcal{G}_n$ is the set of all graphs with vertex set $[n]$. Then, $\mathcal{X}_n$ is equipped with a suitable σ-field $\mathcal{A}_n$ and contains elements of the form $(\mathbf{w}_n, G_n)$ with a graph G_n having vertex set $[n]$ and weights given by $\mathbf{w}_n = (w_1, \dots, w_n)$. We couple our models in such a way that the vertices have the same weights $\mathbf{W}_n = (W_1, \dots, W_n)$ for all models, but the generation of the graph G_n, given $\mathbf{W}_n$, uses the connection probability from the respective model.

Lemma 14 *Under assumption (W),* $(\mathbf{W}_n, \mathrm{ENR}'(n))$ *is contiguous with respect to* $(\mathbf{W}_n, \mathrm{ENR}(n))$.

Our framework containing graphs *and* the weights is not explicitly covered in Janson (2010), but the techniques employed therein generalise to this setting as shown in the following. In order to prepare the proof of Lemma 14, we formulate a lemma. Similarly to the notation in Janson (2010) we write for $p, q \in [0, 1]$,

$$\rho(p, q) = (\sqrt{p} - \sqrt{q})^2 + \big(\sqrt{1-p} - \sqrt{1-q}\big).$$

By Equation (2.5) in Janson (2010) there exists some constant $C_0 > 0$ such that for all $p < 0.9$ and $q \in [0, 1]$,

$$\rho(p, q) \le C_0 \frac{(p-q)^2}{p}. \tag{30}$$

Given distinct $i, j \in [n]$, we denote the conditional connection probabilities in $\mathrm{ENR}(n)$ and $\mathrm{ENR}'(n)$, respectively, by

$$p_{ij} = 1 - \exp\left(-\frac{W_i W_j}{L_n}\right) \quad \text{and} \quad p'_{ij} = 1 - \exp\left(-\frac{W_i W_j}{n\mathbf{E}[W]}\right).$$

Lemma 15 *Let assumption (W) hold. Then, for all $\gamma > 0$ there exist $C_1, C_2 > 0$ and $N \in \mathbb{N}$ such that for all $n \ge N$,*

$$\mathbf{P}\bigg(\sum_{1 \le i < j \le n} \rho(p_{ij}, p'_{ij}) > C_1 \text{ or } \max_{i,j \in [n]} \frac{p'_{ij}}{p_{ij}} > C_2 \text{ or } \max_{i,j \in [n]} \frac{1 - p'_{ij}}{1 - p_{ij}} > C_2\bigg) \le \gamma.$$

Proof For $C_1, C_2 > 0$ we have

$$\begin{aligned}
&\mathbf{P}\bigg(\sum_{1 \le i < j \le n} \rho(p_{ij}, p'_{ij}) > C_1 \text{ or } \max_{i,j \in [n]} \frac{p'_{ij}}{p_{ij}} > C_2 \text{ or } \max_{i,j \in [n]} \frac{1 - p'_{ij}}{1 - p_{ij}} > C_2\bigg) \\
&\le \mathbf{P}\bigg(\sum_{1 \le i < j \le n} \rho(p_{ij}, p'_{ij}) > C_1, \frac{W_{(n)}^2}{L_n} < 0.9\bigg) + \mathbf{P}\bigg(\frac{W_{(n)}^2}{L_n} \ge 0.9\bigg) \\
&\quad + \mathbf{P}\bigg(\max_{i,j \in [n]} \frac{p'_{ij}}{p_{ij}} > C_2\bigg) + \mathbf{P}\bigg(\max_{i,j \in [n]} \frac{1 - p'_{ij}}{1 - p_{ij}} > C_2\bigg) =: R_1 + R_2 + R_3 + R_4.
\end{aligned}$$

By (8) we see that $R_2 \to 0$ as $n \to \infty$. In particular, there exists $N \in \mathbb{N}$ such that $R_2 \le \gamma/4$ for all $n \ge N$. In the following, all statements that hold for large enough n

could increase N, which we will not mention explicitly. For R_3 we employ Lemma 9 to obtain a lower bound on p_{ij} and an analogue bound for p'_{ij},

$$\max_{i,j\in[n]} \frac{p'_{ij}}{p_{ij}} \le \max_{i,j\in[n]} \frac{\frac{W_iW_j}{n\mathbf{E}[W]}}{\frac{W_iW_j}{L_n}\big(1-\min(1, W_{(n)}^2/L_n)\big)} = \frac{L_n}{n\mathbf{E}[W]}\frac{1}{1-\min(1, W_{(n)}^2/L_n)} \xrightarrow{\mathbf{P}} 1$$

as $n\to\infty$ due to the law of large numbers and (8). Similarly, we have

$$\max_{i,j\in[n]} \frac{1-p'_{ij}}{1-p_{ij}} \le \max_{i,j\in[n]} \frac{1}{\exp(-W_iW_j/L_n)} = \frac{1}{\exp(-W_{(n)}^2/L_n)} \xrightarrow{\mathbf{P}} 1$$

as $n\to\infty$. Both convergences together yield the existence of some $C_2>0$ such that for n large enough both $R_3\le\gamma/4$ and $R_4\le\gamma/4$. It remains to address R_1. From (30) we derive

$$\begin{aligned} R_1 &= \mathbf{P}\Bigg(\sum_{1\le i<j\le n} \rho(p_{ij}, p'_{ij}) > C_1, \frac{W_{(n)}^2}{L_n} < 0.9\Bigg) \\ &\le \mathbf{P}\Bigg(\sum_{1\le i<j\le n} \frac{(p_{ij}-p'_{ij})^2}{p_{ij}} > \frac{C_1}{C_0}\Bigg). \end{aligned} \tag{31}$$

The mean value theorem yields

$$\exp(x)-\exp(y) = \exp(z)(x-y)$$

for some z lying between x and y for $x, y\in\mathbb{R}$. With $\exp(0)=1$ we obtain

$$\sum_{1\le i<j\le n} \frac{(p_{ij}-p'_{ij})^2}{p_{ij}} \le \sum_{i,j=1}^{n} \frac{\left(\exp\left(-\frac{W_iW_j}{L_n}\right)-\exp\left(-\frac{W_iW_j}{n\mathbf{E}[W]}\right)\right)^2}{1-\exp\left(-\frac{W_iW_j}{L_n}\right)}$$

$$= \sum_{i,j=1}^{n} \frac{\left(\left(\frac{W_iW_j}{L_n}-\frac{W_iW_j}{n\mathbf{E}[W]}\right)\exp(z_{ij})\right)^2}{\left(\frac{W_iW_j}{L_n}\right)\exp(z'_{ij})}$$

for some z_{ij} between $-W_iW_j/L_n$ and $-W_iW_j/n\mathbf{E}[W]$ as well as some z'_{ij} between $-W_iW_j/L_n$ and 0. We have $\exp(z_{ij})<1$ since $z_{ij}<0$. Together with the relation $z'_{ij}\ge -W_{(n)}^2/L_n$ we bound the expression above further by

$$\exp\left(\frac{W_{(n)}^2}{L_n}\right)\sum_{i,j=1}^{n} \frac{\left(\frac{W_iW_j}{L_n}-\frac{W_iW_j}{n\mathbf{E}[W]}\right)^2}{\frac{W_iW_j}{L_n}} = \exp\left(\frac{W_{(n)}^2}{L_n}\right)\sum_{i,j=1}^{n} W_iW_jL_n\left(\frac{n\mathbf{E}[W]-L_n}{n\mathbf{E}[W]L_n}\right)^2$$

$$= \exp\left(\frac{W_{(n)}^2}{L_n}\right)\frac{L_n}{n\mathbf{E}[W]^2}\left(\frac{L_n - n\mathbf{E}[W]}{\sqrt{n}}\right)^2.$$

This expression converges in distribution as $n \to \infty$ by Slutsky's theorem: The first and second factor converge in probability due to (8) and the law of large numbers whereas the last factor converges in distribution due to the central limit theorem and the continuous mapping theorem. Together with (31) we can choose $C_1 > 0$ large enough such that $R_1 \leq \gamma/4$ for all n large enough. This concludes the proof. □

We are now ready to prove Lemma 14.

Proof of Lemma 14 Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of measurable $A_n \subseteq \mathcal{X}_n$ such that

$$\mathbf{P}\big((\mathbf{W}_n, \mathrm{ENR}(n)) \in A_n\big) \longrightarrow 0 \quad \text{as} \quad n \to \infty, \tag{32}$$

where $\mathbf{W}_n = (W_1, \ldots, W_n)$ denotes the underlying random weights. We have to show that

$$\mathbf{P}\big((\mathbf{W}_n, \mathrm{ENR}'(n)) \in A_n\big) \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{33}$$

For $u > 0$ the Markov inequality yields

$$\mathbf{P}\big(\mathbf{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{ENR}(n)) \in A_n) \geq u\big) \leq \frac{1}{u}\mathbf{P}\big((\mathbf{W}_n, \mathrm{ENR}(n)) \in A_n\big) \longrightarrow 0$$

as $n \to \infty$ by (32), where we use the notation $\mathcal{W}$ for the conditional expectation with respect to the weights as before. Therefore,

$$\mathbf{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{ENR}(n)) \in A_n) \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty. \tag{34}$$

We will show that the same statement holds for $\mathrm{ENR}'(n)$, i.e.

$$\mathbf{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{ENR}'(n)) \in A_n) \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty. \tag{35}$$

As the conditional probabilities are at most one, this yields (33) and thus the claim. In order to show (35), let $\gamma > 0$. By Lemma 15 there exist $C_1, C_2 > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mathbf{P}(B_n) \leq \gamma$ with

$$B_n = \left\{ \sum_{1\leq i<j\leq n} \rho(p_{ij}, p'_{ij}) > C_1 \text{ or } \max_{i,j\in[n]} \frac{p'_{ij}}{p_{ij}} > C_2 \text{ or } \max_{i,j\in[n]} \frac{1-p'_{ij}}{1-p_{ij}} > C_2 \right\}.$$

Now let $\varepsilon > 0$. By applying Lemma 5.2 in Janson (2010), there exists some $\delta > 0$ such that we have on B_n^c,

$$\mathbf{P}_{\mathcal{W}}(\mathrm{ENR}(n) \in \tilde{A}) < \delta \quad \Longrightarrow \quad \mathbf{P}_{\mathcal{W}}(\mathrm{ENR}'(n) \in \tilde{A}) < \varepsilon$$

for all $\tilde{A} \subseteq \mathcal{G}_n$, which in turn yields

$$\mathbf{P}_{\mathcal{W}}\big((\mathbf{W}_n, \mathrm{ENR}(n)) \in A_n\big) < \delta \implies \mathbf{P}_{\mathcal{W}}\big((\mathbf{W}_n, \mathrm{ENR}'(n)) \in A_n\big) < \varepsilon. \qquad (36)$$

For $n \geq N$ we conclude with the contraposition of (36) that

$$\begin{aligned}\mathbf{P}\big(\mathbf{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{ENR}'(n)) \in A_n) \geq \varepsilon\big) &\leq \mathbf{P}(B_n) + \mathbf{P}\big(B_n^c, \mathbf{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{ENR}'(n)) \in A_n) \geq \varepsilon\big)\\ &\leq \gamma + \mathbf{P}\big(B_n^c, \mathbf{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{ENR}(n)) \in A_n) \geq \delta\big)\\ &\leq \gamma + \mathbf{P}\big(\mathbf{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{ENR}(n)) \in A_n) \geq \delta\big).\end{aligned}$$

By (34), the second summand above converges to zero as $n \to \infty$. As $\gamma > 0$ can be chosen arbitrarily small, we obtain (35), which yields the claim. □

Proof of Theorem 4 We start with showing the first claim, i.e. that the point process convergence in (1) extends to all models $G(n)$ for $G \in \mathcal{G}$ if the studied sets of vertices $(\mathcal{X}_n(v))_{v \in [n], n \in \mathbb{N}}$ coincide for the Norros-Reittu model $\mathrm{NR}(n)$ and the erased Norros-Reittu model $\mathrm{ENR}(n)$. This assumption immediately yields the claim for $\mathrm{ENR}(n)$. The statement transfers to $\mathrm{CL}(n)$ and $\mathrm{GRG}(n)$ due to the asymptotic equivalence in the first part of Lemma 13.

For the second claim, it suffices to show transferability of the convergence in (1), as the convergence in Corollary 2 is an immediate consequence thereof. We observe that assumption (A2) does not allow the counted vertices $(\mathcal{X}_n(v))_{v \in [n], n \in \mathbb{N}}$ to depend on loops or multiple edges. Therefore, the first part of Theorem 4 shows that convergence in (1) extends from $\mathrm{NR}(n)$ to $\mathrm{ENR}(n)$, $\mathrm{CL}(n)$ and $\mathrm{GRG}(n)$. We are left to transfer the result to $\mathrm{ENR}'(n)$, $\mathrm{CL}'(n)$ and $\mathrm{GRG}'(n)$. Due to the second part of Lemma 13 it suffices to transfer the result to $\mathrm{ENR}'(n)$. We proved Theorem 1 by comparing the point processes

$$\Theta_n = \sum_{v=1}^{n} \delta_{W_v q(n)^{-1}} \quad \text{and} \quad \Xi_n = \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}\} \delta_{S_n(v) q(n)^{-1} \xi^{-1}}$$

for $n \in \mathbb{N}$ and showing for all $a > 0$ that

$$\Xi_n((a, \infty]) - \Theta_n((a, \infty]) \xrightarrow{\mathbf{P}} 0 \quad \text{as} \quad n \to \infty. \qquad (37)$$

In particular, we have shown the statement above for $\mathrm{NR}(n)$. By assumption (A2), Ξ_n and Θ_n coincide for $\mathrm{NR}(n)$ and $\mathrm{ENR}(n)$ as $S_n(v)$ may not depend on multiple edges or loops. Therefore, the convergence in (37) also holds for $\mathrm{ENR}(n)$. Since $(\mathbf{W}_n, \mathrm{ENR}'(n))$ is contiguous with respect to $(\mathbf{W}_n, \mathrm{ENR}(n))$ by Lemma 14, it remains true for $\mathrm{ENR}'(n)$. Here, it is important that we extended the concept of contiguity to incorporate the weights, which determine the point process Θ_n. The second assertion follows.

For the third claim it suffices to note that we have shown Theorem 3 by using Theorem 1. Thus, it follows from the second claim. This concludes the proof. □

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Declarations

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