Tribology Letters (2025) 73:55
https://doi.org/10.1007/s11249-025-01984-3

ORIGINAL PAPER

Some Remarks on a Recent Wear Theory

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Received: 18 December 2024 / Accepted: 17 March 2025

Abstract

Persson et al. in (Rubber wear: experiment and theory. arXiv:2411.07332, 2024) and Xu and Persson in (Sliding wear: role of plasticity. arXiv: abs/2412.13129, 2024) have recently proposed an interesting theory of wear which is based on particle formation due to fatigue crack growth at different scales of roughness. The theory perhaps is the first one to take into account of a full characterization of the roughness, and obtains semi-quantitative prediction of wear coefficients for rubber and PMMA, but in the original form, many details of actual roughness features and the material properties do not permit to elucidate general simple trends. We attempt to make general comments to show the main effects of the various macroscopic parameters in the theory, with qualitative comparisons having in mind the case of metals wear for which we found experimental trends, at least for the dependence on friction coefficient. It is found that wear rate in the elastic theory very strongly depends on friction coefficient and on rms roughness, showing even a regime of wearless behaviour below friction coefficient of about 0.2—which may indicate transition to other mechanisms, like adhesive wear. It is shown that an elasto-plastic theory probably mitigates these effects, as a fully plastic one depends only quadratically on friction coefficient, and has no dependence at all on any feature of roughness. However, the present oversimplistic perfectly plastic model truncating the elastic prediction, and the use of a crack propagation theory which is irrespective of large plastic flow can make the theory more hardly quantitative in general. In addition, hardness at asperity scale may increase due to size effect, so the elastic model may be the most appropriate choice in many cases. Along with many other complex effects known in wear (even limiting attention to fatigue wear), it remains, therefore, to be investigated how generally the Persson theory can result in quantitative predictions.

Keywords Wear · Roughness · Fatigue wear · Plasticity · Friction

1 Introduction

Reye in 1860 proposed [1] that the worn volume should be proportional to the work done by friction during sliding, hence, be proportional to the normal load and the sliding distance. This proposal did not go unnoticed in parts of Europe, but has been largely ignored in English and American literature, until Archard [2] reproposed it with some interesting experiments. The study of wear has large economic relevance in many areas of tribology, and the losses due to wear have serious impact in the GDP of developed nations. Moreover, in some biomechanical applications, wear losses also have medical serious implications, and wear needs to be limited[1].

There are various wear mechanisms (adhesive wear, abrasive wear, fatigue wear) although a clear distinction is not always possible, and moreover, the equation used for most of them remains similar, usually called Archard law, where for the worn volume V in a nominal area A_0 slid of an amount L, we have

$$\frac{V}{A_0 L} = K\frac{p}{3H} = \frac{k_a}{3}p \, , \tag{1}$$

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[1] There are, however, special cases where wear is beneficial, like in removing cracks that are forming in railways or wheels [3, 4], where in dry conditions, wear is very high and cracks do not really propagate very far, while in wet conditions, rolling contact fatigue crack prevail. Also, wear may be beneficial in mitigating thermoelastic instabilities in brakes and clutches [5].

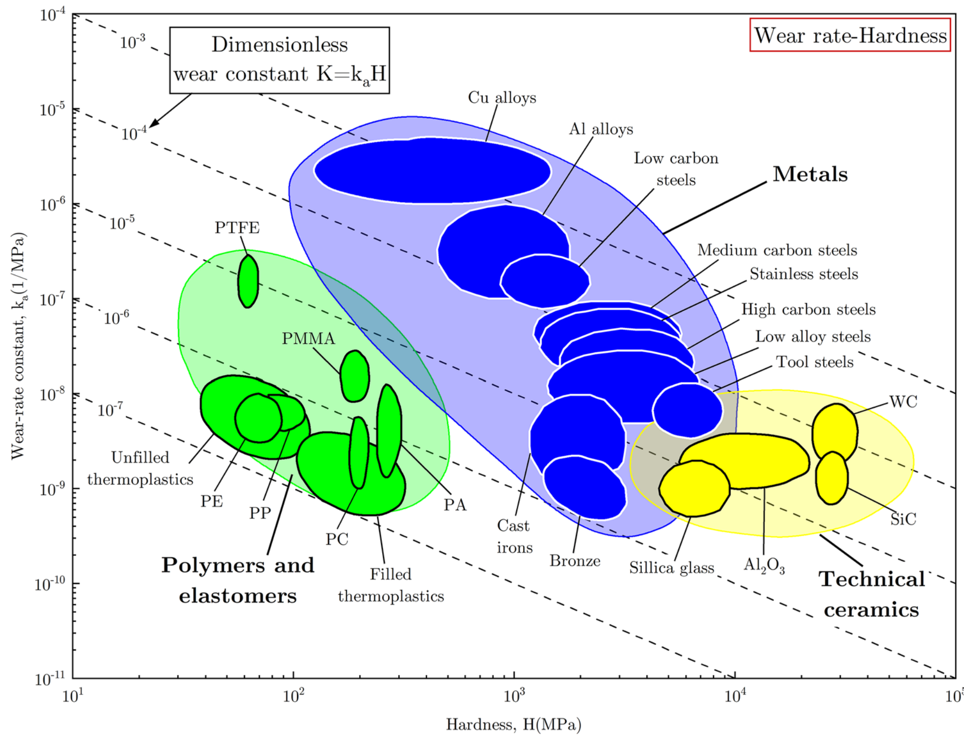

Fig. 1 Wear-rate constant $k_a = K/H$ according to Ashby [6] for many materials. Notice that the dimensionless constant K is reported with the decaying dashed lines

where p is the nominal pressure, H is Brinell hardness [MPa], and K is dimensionless wear coefficient, whereas $k_a[1/MPa]$ as reported in Fig. 1 adapted from Ashby [6] spans almost 5 orders of magnitude for metals, technical ceramics and polymers and elastomers. For steels only, almost 2 orders of magnitude can be observed changing from tool steels with the lower wear rates, to the mild low carbon steels with the highest ones. This large variation is not well understood and most often the wear coefficient is so sensitive to tribological conditions and other variables not included in this map that there is no alternative to the measurement.

Fatigue wear occurs where there is no "sharp" roughness and multiple contacts are needed to remove particles, as in a fatigue process. Abrasive wear is usually referred to the case with sharp asperities which induce micro-cutting or scratching and finally adhesive wear occurs generally in metallic systems where "cold welded" junctions form and then break due to a fracture process, leading to either particle formation or transfer of material from one surface to the other. Adhesive wear is anyway one of the most prevalent types of wear.

In another not very popular but very pioneeristic papers, Rabinowicz [7] defined a critical length scale, distinguishing the size for formation of wear particles in a contact of homogeneous sliding bodies. This idea was much later also further developed and extended in mesoparticle simulations by Aghababaei et al. [8], who explain also history and surface roughness evolution are important—incidentally, these effect cannot be predicted except by a full numerical simulation, which is not what we shall discuss here.

In two recent papers, indeed, Persson et al. [9, 10] have proposed an interesting theory which elaborate on the Rabinowicz idea further, within the context of a fatigue process where the strain energy only drives a certain crack advancement per contact. The theory, originally developed for rubber wear where elastic strains dominate, is an attempt to include the effect of roughness to a full extent, and was more recently developed for PMMA sliding on various surfaces, where the effect of plasticity is taken into account at least in a very crude manner. The results seem to indicate a semi-quantitative prediction of wear volume rate and distribution of wear particles, where by semi-quantitative we mean that despite efforts to use detailed material properties and roughness characteristics, the wear coefficient may differ from the experimental one by 2 orders of magnitude.

We shall briefly comment on this recent theory, trying to develop some simplification from the original full form which are not very accurate but permit to understand the role of the basic roughness and material properties, so as to discuss the possible implications, and comparison with existing experimental evidence. In doing this, we are limited by the fact that most wear studies do not report sufficient data to use Persson's theory and the required input material and roughness properties, so we mainly discuss the possibility of qualitative agreement with existing evidence.

2 Persson's Wear Theory

We assume that the process occurs at various magnifications $\zeta = q/q_0$ (where q is wavevector in roughness, and q_0 is the lowest wavevector present in the roughness spectrum) which defines the radius of asperity $r_0 = \pi/q$. If the process were static, the critical pressure σ_c above which there is wear according to the Rabinowicz criterion is found imposing that the elastic strain energy release rate $G = \gamma$ where γ is a material constant tearing energy, and hence, $\mu^2\sigma_c^2 r_0 = 2\gamma E^*$ or

$$\sigma_c = \frac{1}{\mu}\sqrt{\frac{2E^*\gamma}{r_0}}, \tag{2}$$

where $E^* = E/\left(1-\nu^2\right)$ is plane strain elastic modulus, ν is Poisson's ratio, and μ is friction coefficient. This equation recognizes that it is shear stresses $\mu\sigma_c$ which drives crack formation, as with zero friction, the stress is mainly compressive, and simply comes from the balance of the strain energy in the particle that can be released upon formation of fracture, and the surface energy.

In a rough contact, Persson's theory has shown that $P(\sigma,\zeta)$ is the probability distribution of pressures in the contact area:

$$P(\sigma,\zeta) = \frac{1}{\sqrt{4\pi G(\zeta)}}\left(e^{-\frac{(\sigma-p)^2}{4G(\zeta)}} - e^{-\frac{(\sigma+p)^2}{4G(\zeta)}}\right), \tag{3}$$

where p is the applied nominal pressure and $G = \frac{1}{2}V_{fc}/p^2$ where V_{fc} is the variance of the full contact pressures (see also [11]).

Recognizing however that wear occurs over many passages because of fatigue, we introduce a crack advancement per cycle $\Delta x(\sigma,\zeta)$ which will come from empirical laws, and the wear volume per unit area and sliding length (in short, wear rate) is then found at given magnification as $V(\zeta)$ according to Persson's theory integrating over the distribution of contact stresses (see [10] for details where prefactor 1/3 is changed from prefactor 1/2 used in [9], due to a slightly different description of the particle removal process):

$$\frac{V(\zeta)}{A_0 L} = \frac{1}{3}\int_{\sigma_{th}(\zeta)}^{\infty} d\sigma \frac{P(\sigma,\zeta)}{1 + r_0(\zeta)/\Delta x(\sigma,\zeta)}, \tag{4}$$

where $r_0(\zeta)/\Delta x(\sigma,\zeta)$ gives the number of contacts (cycles) needed to remove a particle, NS the lower critical stress in order to have some wear cracks (the threshold value $\sigma_{th}(\zeta)$) is a modification of (2):

$$\sigma_{th}(\zeta) = \frac{1}{\mu}\sqrt{\frac{2E\gamma_{th}}{r_0}} = \frac{1}{\mu}\sqrt{\frac{2E\gamma_{th}}{\pi}q_0\zeta}, \tag{5}$$

where γ_{th} is a threshold energy for fatigue crack propagation. Notice one assumption in the theory is that geometrical factors taking into account of the relative size of the loaded part of the contact vs the crack size are not taken into account, as they may be difficult to estimate. However, this may be quite a strong assumption when we are dealing with such a multiscale process.

As the magnification increases, G, V_{fc} also increase and the probability distribution function spreads more uniform across all possible (positive) pressures. We then use a Paris "law" for the crack advancement per cycle Δx : the denomination of "law" is really inappropriate since it is an empirical fit using stress intensity factor or energy release rate as driving the crack growth and comes not only in power law forms. It is clearly also different in nature in rubber (where there is viscoelastic dissipation mainly) than in metals (where dislocation emission and motion is a main mechanism). What suggested for PMMA by [12, 13], including the effect of the fatigue γ_{th} in the curve, has some generality also for many metals

$$\Delta x \simeq \frac{16}{E}\left(\gamma - \gamma_{th}\right), \tag{6}$$

although [9] use a slightly different expression for rubber, and [10] also a slightly different version for PMMA. In our paper, we shall use (6) for all simulations having in mind no specific material, because of its simplicity and the hope of a certain qualitative generality for many materials.

Notice that if we write $\sigma = \beta\sigma_{th}(\zeta)$, where β is an amplification factor from the threshold stress, using (6) as the Stress Intensity Factor (omitting geometrical effects) $K_I = \mu\sigma\sqrt{r_0}$ and $\gamma = K_I^2/2E^*$

$$\Delta x \simeq \frac{16}{E^*}\left(\beta^2 - 1\right)\gamma_{th}, \tag{7}$$

which for steel having $\gamma_{th} = 500J/m^2$ and $E^* = 200GPa$ leads to $\Delta x \simeq 4\times 10^{-8}\left(\beta^2-1\right)m/cycle$. In general, fatigue crack propagation in a power law regime occurs for $\Delta x \simeq 10^{-8} - 10^{-6}m/cycles$ for most materials [14], so (6) (7) do give correct orders of magnitude, although they have some obvious limitations since the large literature on fatigue crack growth in metals has endless studies to find more accurate laws. In any case, we cannot cover above the intermediate regime of propagation. But this is not very important in Persson's theory, since in the denominator of (4), Persson has introduced the "1" exactly for the purpose to make very fast propagation result in the crack formation—notice that in the contact theory, stresses up to infinite exist and these are disregarded as giving infinite rate of propagation in general: [9, 10] have an attempt to include fast propagation near the static limit, which we disregard in our discussion.

At low magnifications, asperities/cracks are large and give the largest contributions to wear rate, since they also have larger advancement per cycle for a given stress. The full

Persson's theory then obtains a double integral making some choice of integration interval over ζ in steps of factors of 2 in order not to compute particles twice (alternative choices do not lead to significant differences), using $q = q_0 e^{\xi}$ and obtaining

$$\frac{V}{A_0 L} \simeq \frac{1}{3\ln 2}\int_0^{\xi_1} d\xi \int_{\sigma_c(\zeta)}^{\infty} d\sigma \frac{P(\sigma,\zeta)}{1 + r_0(\zeta)/\Delta x(\sigma,\zeta)}. \tag{8}$$

We will present results for the full theory in what follows, but in order to simplify Persson's full theory (8) for help discussion, we now make an approximation taking $\Delta x(\sigma,\zeta)$ at the intermediate value for values of tearing energy between the threshold and the critical value Δx^*, which gives for the majority of materials [14], $\Delta x^* = 10^{-7} m/cycle$, which incidentally corresponds to (7) when $\beta = 1.87$ for the chosen material constants, leading to

$$\int_{\sigma_{th}(\zeta)}^{\infty} d\sigma \frac{P(\sigma,\zeta)}{1 + \pi/(q\Delta x^*)} = \frac{1}{2(1 + \pi/(q\Delta x^*))}\frac{A_{\sigma_{th}}}{A_0}, \tag{9}$$

where $A_{\sigma_{th}}$ is the real contact area with stress higher than the threshold stress. In our calculations, we shall assume indeed $\Delta x^* = 10^{-7} m/cycle$.

Using a simplified expression for $P(\sigma,\zeta)$ (3) which can be obtained for light pressures $V_{fc}/p^2 > 1$ not too close to full contact (this is correct except perhaps at very low magnifications),

$$\begin{aligned} \frac{A_{\sigma_{th}}(\zeta)}{A_0} &\simeq 2\sqrt{\frac{1}{\pi}}\frac{p}{\sqrt{2V_{fc}(\zeta)}}\exp\left(-\frac{\sigma_{th}^2(\zeta)}{2V_{fc}(\zeta)}\right) \\ &= \frac{A(\zeta)}{A_0}\exp\left(-\frac{\sigma_{th}^2(\zeta)}{2V_{fc}(\zeta)}\right), \end{aligned} \tag{10}$$

where $\frac{A(\zeta)}{A_0}$ is the real area ratio at magnification ζ. Summing all contributions from different scales in the form suggested by Persson, we get

$$\frac{V}{A_0 L} \simeq \frac{1}{3\ln 2}\int_0^{\xi_1} d\xi \frac{A_{\sigma_{th}(\xi)}}{1 + \frac{\pi}{q_0 \Delta x^* e^{\xi}}}. \tag{11}$$

For a typical power law spectrum of roughness $C(q) = C_0 q^{-2(1+H)}$, where H is Hurst exponent (generally close to $H = 0.8$, see Persson (2014)) we have $V_{fc} = \frac{1}{2}E^2 h_{rms}^{\prime 2} \simeq \frac{1}{2}E^2 \frac{\pi C_0}{2-2H} q^{2-2H}$ while the rms roughness is $h_{rms} = \sqrt{\pi C_0/H} q_0^{-H}$ and h'_{rms} is rms slope. Hence,

$$\frac{\sigma_{th}^2(\zeta)}{2V_{fc}(\zeta)} = \frac{1}{\pi\mu^2}\frac{2\gamma_{th}(2-2H)/H}{E h_{rms}^2 q_0 \exp(2-2H)}, \tag{12}$$

which does not depend on magnification and, hence, can be taken out of the integral (11).

Hence, (11) can be written as follows:

$$\frac{V}{A_0 L} \simeq \frac{2\sqrt{\frac{1}{\pi}}}{3\ln 2}\frac{p}{E}\frac{\sqrt{\frac{2-2H}{H}}}{h_{rms} q_0}\exp\left(-\frac{1}{\pi\mu^2}\frac{2\gamma_{th}(2-2H)/H}{E h_{rms}^2 q_0 \exp(2-2H)}\right) I(q_0, H), \tag{13}$$

where

$$I = \int_0^{\xi_1} \frac{d\xi}{\exp(\xi(1-H))\left(1 + \frac{\pi}{q_0\Delta x^* e^{\xi}}\right)} \simeq \frac{q_0 \Delta x^*}{2.3}\exp(0.8\xi_1), \tag{14}$$

where we have assumed $H = 0.8$ in the approximation. It turns out that the convergence in the full theory with ξ_1 is much faster than what expected from (14) because when asperity size r_0 becomes small, only very large stress can propel the crack at rate of $\Delta x^* = 10^{-7} m/cycle$, which we have assumed. Therefore, it is seen in the full theory that small-scale contribution becomes negligible rapidly. However, our simplified model (13) shows the proportionality of the wear rate on the pressure as in Reye–Archard law, and the main dependence on material properties (after we have simplified the effect of Paris' law constants) is on the elastic modulus E^* and the threshold γ_{th}. The critical tribological features that emerge in the theory are then friction coefficient and amplitude of roughness. Hence, in our simplified Persson's model, Archard's law (1) can be written with a coefficient writing for example $\exp(0.8\xi_1) = \exp(0.8 \times 2) \simeq 5$

$$\begin{aligned} k_a = &\frac{2\sqrt{\frac{1}{\pi}}}{\ln 2}\sqrt{\frac{2-2H}{H}}\frac{1}{E h_{rms}}\frac{5\Delta x^*}{2.3} \\ &\exp\left(-\frac{1}{\pi\mu^2}\frac{2\gamma_{th}(2-2H)/H}{E h_{rms}^2 q_0 \exp(2-2H)}\right). \end{aligned} \tag{15}$$

Assuming for steels, for example, $E = 200GPa$. As reported by [15], most polished steel surfaces, when measured on a length of $1mm$, show h_{rms} of the order of a micron so we assume $h_{rms} = 1\mu m$, $q_0 = 10^3 m^{-1}$, $H = 0.8$, and $\gamma_{th} = \frac{K_{th}^2}{E} = 500\frac{J}{m^2}$ to obtain

$$k_a = 1.25 \times 10^{-6}\exp\left(-\frac{0.525}{\mu^2}\right), \tag{16}$$

which is plotted in Fig. 2 as a solid line, together with the full theory which is indicated as blue dots, and a line corresponding to a power law taken for reference since Rabinowicz [16] reports a dependence of wear rate on power five for a large range of experiments on metals (lubricated and unlubricated): to extract this particular line, we had to assume that all data corresponded to a hardness of $1GPa$. As it can be seen, some conclusions are as follows:

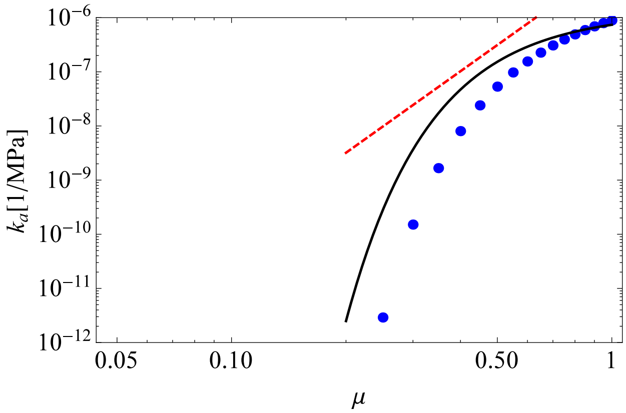

Fig. 2 Wear-rate constant $k_a$as function of friction coefficient μ for our example case steels with $E = 200GPa$, $h_{rms} = 1\mu m$, $q_0 = 10^3 m^{-1}$, $\gamma_{th} = 500\frac{J}{m^2}$ and the "universal" crack propagation law as described in the text. Solid line is our approximate qualitative analytical prediction, and blue dots is the full Persson's theory. Dashed red line is a Raboniwicz [16] data experimental power five dependence of wear rate on friction coefficient assuming metals had 1GPa hardness

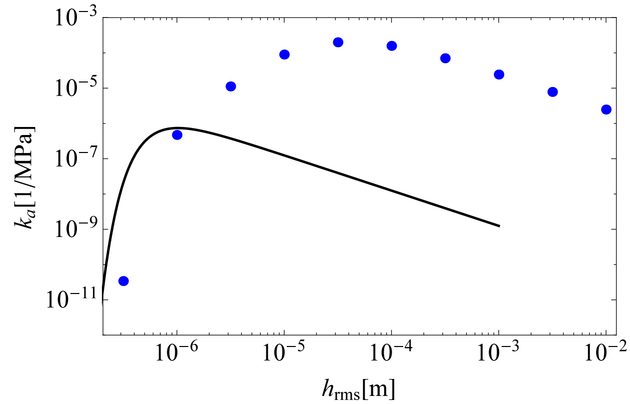

Fig. 3 Wear-rate constant $k_a$as function of rms amplitude of roughness h_{rms} with $\mu = 1$ and other constants as Fig. 2. Solid line is our approximate qualitative analytical prediction, and blue dots are the full Persson theory

- There is a threshold friction coefficient of about 0.2 below which there seems to be no wear predicted. This conclusion is perhaps too strong to correspond to actual experimental evidence, as surely friction coefficients below 0.2 occur and wear cannot be zero nevertheless: however, this may indicate a transition to a different wear mechanism.
- Jjust changing friction coefficient from 0.2 to 1 seems to change the wear coefficient by about *six* orders of magnitude, which is perhaps even too much. Results are consistent with Fig. 19 of [10] which change of five orders of magnitude across the same range of friction coefficient, confirming the validity of our approximations and independence on details of roughness and crack propagation law. However, Rabinowicz [16] reports rather 4 orders of magnitude variation across the same range of friction coefficient.
- Our approximate result is only qualitatively correct.
- A plateau is reached at about $k_a = 3 \times 10^{-7}$, which is about the highest value for low carbon steels in Ashby map (Fig. 1). However, this result is obtained here with high grade surface finish which requires very close control to produce, whereas a standard value is of the order of $3\mu m$. We shall explore the effect of rms roughness next, since it will dramatically increase the wear rate.
- If the full theory is explored above $\mu = 1$, it seems that the wear rate starts to decrease beyond certain values of friction coefficients, and then perhaps reach a new wearless state: it is unclear if this result is due to integration errors, but it is certainly counterintuitive and should be explored further.

Assuming the same quantities except now $\mu = 1$ and h_{rms} is varying, we obtain from the qualitative model:

$$k_a = \frac{1.2}{10^{12} h_{rms}} \exp\left(-0.21 \frac{5}{2 \times 10^{12}} \frac{1}{h_{rms}^2}\right), \quad (17)$$

which is plotted in Fig. 3 together with the full theory in blue dots:

- That wear rate coefficient first grows very abruptly until it peaks at about $1\mu m$ (but full theory peaks much later at about $20\mu m$) and then it seems to decrease.
- These results are partly consistent with those reported in [10], namely plotted in Fig. 18 which show a very fast increase of wear rate (a very large 7 orders of magnitude with roughness changing from 1 to 4 μm.
- Most experimental studies find a trend of increased wear with roughness [17–20], or sometimes independence on roughness [21]. The trend can be abrupt, but *hardly as abrupt as predicted*: Federici et al. [20] indeed find almost no change in wear rate up to $1\mu m$ and then an increase of 3 orders of magnitude of wear rate coefficient changing roughness form $1\mu m$ to $5\mu m$. Jiang and Arnell [18] find a modest increase with roughness.
- The decrease for large roughness is not really expected, and needs further investigation.
- Our approximate result is again only qualitatively correct, but there are more significant errors in the dependence on roughness.

Figure 4 shows finally an extended set of results (only with the full theory now), as function of the friction coefficient

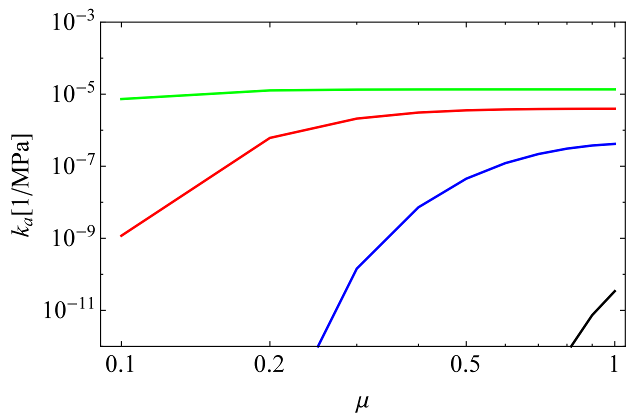

Fig. 4 Wear-rate constant k_a as function of friction coefficient for rms amplitude of roughness $h_{rms} = 3.2 \times 10^{-7}$, 10^{-6}, 3.2×10^{-6}, and $10^{-5}[m]$ (black, blue, red, green curve) and other constants as Fig. 2. Solid lines are obtained with the full Persson theory

for rms amplitude of roughness $h_{rms} = 3.2 \times 10^{-7}$, 10^{-6}, 3.2×10^{-6}, $10^{-5}[m]$ (black, blue, red, green curve). As it can be seen, for high roughness results become a lot less sensitive to friction and eventually for $10\mu m$ we have no sensitivity at all, with a constant $k_a = 10^{-5}[1/MPa]$ which is almost 2 orders of magnitude higher than values reported by Ashby for steels (see Fig. 1). Although we cannot rule out that Ashby's map does not include extreme cases, it requires more investigation that the friction dependence becomes very weak for high roughness, contrary to what found for example by Rabinowicz [16].

The role of plastic deformations in Persson's theory is certainly to cut the stresses to the yield strength, and the effect on wear rate is less obvious. We shall discuss this effect in the next paragraph.

3 The Role of Plasticity

The role of plasticity has remained controversial in the contact of rough surfaces. While it was the basic model to understand Amonton-Coulomb's law in the classical view of Bowden and Tabor, it was less enthusiastically accepted by Archard [22] who first demonstrated the possibility of the linear dependence of frictional load on the real contact area in a fully elastic model. The argument was that even ductile materials, after run-in, show relatively little damage, due to hardening of asperities and shakedown to elastic state. Classical experiments suggested the "persistence of asperities" (see [23] and references therein, as due to an increase in indentation hardness at smaller length scales, which has been confirmed in micro- and nanoindentation tests and explained by with models of dislocations or strain gradient plasticity. Therefore, as explained in [23], what appears as plastic at intermediate scale of magnification may return to an elastic behaviour at a small scale, consistent to the concept of "asperity persistence" and their experimental observation. Therefore, the inclusion of a realistic plasticity model is challenging. And as shown by mesoparticle simulations by Aghababaei et al. [8], history and surface roughness evolution may lead to wear mechanism change over time, which can only be predicted except by a full numerical simulations, which are still computationally extremely demanding at macroscopic scale.

The theory of Persson takes into account of plasticity [10] but in a rather crude way, to make the problem tractable. First of all, a simple perfectly plastic model is assumed. Secondly, there is no account of the change of the roughness due to plastic deformation. Third, the same Paris' curve is used as for elastic loading, although this is clearly an oversimplification in the presence of bulk plasticity: indeed, Chandran [24] shows that crack growth rates are much higher than what expected from elastic strain energy models, and one should rather consider plastic strain energy, which however requires a much more sophisticated model of plasticity and of the contact problem.

To observe the qualitative behaviour of the correction for plasticity, let us assume the contact is fully plastic at stress σ_Y. This limit in [10] is clearly possible when material has extremely low hardness, and otherwise would be correct at high enough magnifications, but the approximation here will be to assume that this is also valid at low magnifications. The critical stress σ_c in Persson's theory above which there is wear is uniquely defined by σ_Y. Since we are dealing with a fatigue model, γ for crack propagation is a value between the threshold inferior limit when cracks start to propagate γ_{th} and the true static limit γ_c. Hence a generalization of the Rabinowicz criterion follows that particles can form only in between the two sizes

$$r_{0\,\min} = \frac{2E\gamma_{th}}{(\mu\sigma_Y)^2}; \qquad r_{0\,\max} = \frac{2E\gamma_c}{(\mu\sigma_Y)^2}. \tag{18}$$

Summing directly on all magnifications/scales, one obtains that the distribution of pressures is a delta function which integrates gives the plastic area $A_{pl}/A_0 = F/(A_0\sigma_Y)$ and, hence,

$$\frac{V}{A_0 L} = \frac{1}{3\ln 2}\int_{q_0}^{q_1}\int_{\sigma_c(\zeta)}^{\infty} d\sigma \frac{P(\sigma,\zeta)}{q + \pi/\Delta x(\sigma,\zeta)} = \frac{F/(A_0\sigma_Y)}{3\ln 2}\int_{q_0}^{q_1}\frac{\alpha\frac{\pi}{q}}{\alpha\pi + \pi}dq, \tag{19}$$

where $\alpha = \frac{16}{E}\frac{(\mu\sigma_Y)^2}{2E}$, where have assumed a simplified form of the crack propagation law $\Delta x \simeq \frac{16}{E}\gamma$, i.e. removing the effect of the threshold.

Then we obtain

$$\frac{V}{A_0 L} \simeq \frac{F/(A_0\sigma_Y)}{3\ln 2} 8\left(\mu\frac{\sigma_Y}{E}\right)^2 \log\zeta, \tag{20}$$

since $\frac{\sigma_Y}{E} << 1$, which shows a weak dependence on the magnification and an Archard-Reye coefficient which has clear dependences on the two material constants in the model (E, σ_Y), and *no dependence at all* on roughness. Considering that steels range in $\sigma_Y = 250 \div 1500MPa$ while $E = 200GPa$, $\frac{\sigma_Y}{E}$ is in the range $[1.2\times10^{-3}, 7.5\times10^{-3}]$. We can assume as an order of magnitude $\log\zeta = 10$. Hence, we get

$$\begin{aligned} k_a &= 7.2\times10^{-7}\mu^2 \text{ low carbon steels} \\ &= 4.3\times10^{-6}\mu^2 \text{ high strength steels.} \end{aligned} \tag{21}$$

Figure 5 reports wear-rate constant $k_a$as function of friction coefficient μ for elastic model in case of Fig. 2 (blue dots as before for full elastic theory) or plastic bounds for low carbon steel (red-dashed curve) and high strength steel (blue-dashed curve). Clearly the dependence of the curves on friction is much less dramatic of that of the elastic model (only quadratic dependence), and values are not too far from the values reported by Ashby (see Fig. 1). However, the dependence on the yield strength doesn't seem to be correct. Low carbon steels, having low yield strength, should have higher wear rates, rather than lower ones. Looking at the results of Xu and Persson [10] (see for example Fig. 17, it seems likely that our fully plastic model only covers the case of very low yield strength, where indeed also in Fig. 17 of [10] there is an increasing trend, but beyond a certain value of the yield strength, the trend with hardness/yield strength becomes decreasing, which is what we expect from Ashby map. Moreover, depending on the actual case, the elasto-plastic correction may be very small with respect to the original fully elastic model. For example, in the PMMA theory presented in [10], the strong dependencies of wear rate on friction and roughness remain almost unaffected by the plasticity correction.

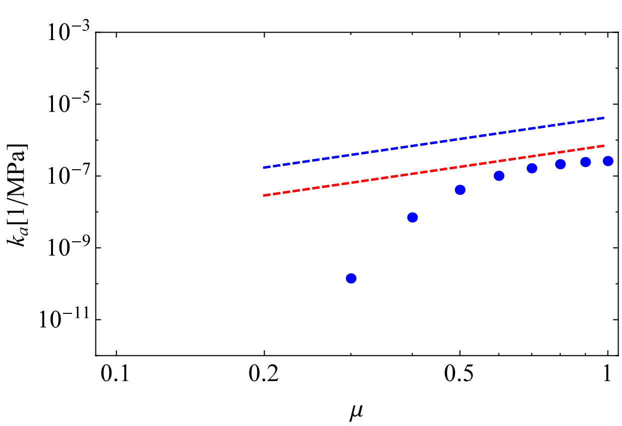

Fig. 5 Wear-rate constant $k_a$as function of friction coefficient μ for elastic model in case of Fig. 2 (blue dots for full elastic Persson's theory) or plastic bounds for low carbon steel (red dashed curve) and high strength steel (blue-dashed curve)

Hence, there is potential for an elasto-plastic model to give realistic results, although we don't expect quantitative predictions, which would require a more detailed plasticity model, and of the driving force for crack propagation to take into account of large-scale plasticity [24].

4 Discussion

In a recent paper [25], we have discussed that one possible criticism of Persson's wear theory is that initial crack sizes are assumed to be existing wavelenghts in the roughness spectrum. This may look rather arbitrary, as there is no strong motivation to assume that initial defects correspond to "asperity scale." However, we also explained that this may not have a large influence for the type of fatigue crack growth curves found for rubber. Of course, it is not easy to make alternative hypothesis if we want to keep a theory simple, and not introduce, for example, true initial defects in the material. Clearly, amplitude of roughness also seems to be an indicator of initial defects, and as we have seen, it does come out from the results as strongly affecting wear rate. In the recent paper [25], we introduce an alternative treatment based on crack initiation rather than fatigue crack propagation, in which the problem of initial crack size seems avoided. Results look qualitatively similar to the original Persson's theory discussed in the present theory, and only a detailed comparison with an extensive experimental campaign would help shed light into the best approach. An additional problem in many materials is that the fatigue threshold is not well defined for short cracks. This is clear for metals, for which the recent paper suggests the use of known equations, but less clear for rubbers, where there is lack of investigation.

5 Conclusion

We have commented a recent proposal by Persson and coworkers for a semi-quantitative model of wear taking into account detailed roughness description and contact stresses, together with crack propagation laws in fatigue. We have observed that the fully elastic theory shows strong dependence on friction coefficient and even more on roughness amplitude, and both effects are much milder in a fully plastic theory (in particular, no effect at all of roughness). Therefore, an elasto-plastic theory shows potential for metals and most wear conditions, but it then becomes questionable if a simple perfectly plastic model and the use of a crack propagation theory which doesn't include the effect of large plastic flow can be made quantitative. In some cases shown by [10] for PMMA, the strong dependencies of wear rate on friction coefficient or rms amplitude of roughness remain practically unaffected by the plasticity correction. Moreover, plasticity at asperity scale is known to show size effects, so the elastic model may be a valid choice in many cases. It remains to be fully explored therefore by what extent the Persson theory can result quantitative more in general, considering even in the original Persson's theory the agreement between detailed and careful experiments and theory was only within two orders of magnitude of wear coefficient prediction.

Acknowledgements MC acknowledges support from the Italian Ministry of Education, University and Research (MIUR) under the program "Departments of Excellence" (L.232/2016). Also, discussion with Prof. Bo Persson.

Author Contributions The author being a sole author has made contribution to all the parts of the process of the paper.

Funding Open access funding provided by Politecnico di Bari within the CRUI-CARE Agreement. The author declares that no funds, grants, or other support were received during the preparation of this manuscript.

Data Availability The data that support the findings of this study are available from the corresponding author, Michele Ciavarella, upon reasonable request.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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