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Full length article

A graded elastic modulus concept to eliminate stress or strain energy density singularity at sharp notches and cracks, with consequent elimination of size-scale effect on strength

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ABSTRACT

It has been recently suggested by the author that in the classical problem of a sharp wedge or crack loaded in plane (mode I and/or mode II), the stress singularity can be removed by grading the elastic properties of the underlying material from the notch tip by using a power law, $E \sim r^{\beta}$. While the treatment is extended to the case of mode III (antiplane shear) which permits closed form results, we also discuss two ways to deal with the likely effect of material's grading on strength. In one, already explored in the previous paper, the strength is a power law of the modulus, and we suggest an "optimal" design by keeping the dominant stress constantly equal to the strength. In a second method, we propose to cancel the singularity in the strain energy density, which requires a much *stronger* grading, and we also possibly take into account that the critical strain energy density is a power law of the modulus. Noticing that only in the presence of a singularity a length scale can be defined experimentally by testing a very large notch and a very small one, according to the Theory of Critical Distances (TCD), the effect of cancelling singularity also implies independence on size/scale and constant strength. It is concluded that the technique is much more powerful than drilling a hole or rounding the tip of the notch/crack. Moreover, if a "smart" material could be designed to damage itself as to reduce its modulus when near a high stress concentration according to our prescriptions, it would naturally self-heal, opening up interesting applications.

1. Introduction

Functionally Graded Materials (FGM), i.e. materials with spatially varying composition, have been observed in Nature (soils, bones, etc.) and have been intensively exploited in an attempt to ameliorate resistance to failure by various industries, including turbines and rocket engines, artificial bone implants, and various others (Li and Han, 2018). Zhang et al. (2018) show that additive manufacturing techniques permit to obtain functionally graded materials and structures with the greatest freedom. Applications extend also to tribology (Suresh et al., 1999).

Usually, in the literature crack problems in FGMs are considered relatively similar to the homogeneous counterpart, since the crack-tip has a regular square-root singularity and stress and displacement fields have the same form as those of the homogeneous materials (Pau, 2002; Eischen, 1987). If the grading of properties is weak, it is plausible to assume that also strength and other properties are unaffected, and mainly dealing with this problem through Linear Elastic Fracture Mechanics (LEFM) and thus determining the underlying Stress Intensity Factor (SIF), which is the only factor which depends on the grading. Similarly, for finite stress concentration problems, like that near a circular hole in an infinite plate subjected to uniform tension, many authors neglect the effect of grading on the strength properties of the material and address the stress concentration reduction (Huang et al., 2003; Kubair and Bhanu-Chandar, 2008; Kumar et al., 2023; Mohammadi et al., 2011; Abdalla et al., 2023).

Conversely, Huang et al. (2003) studied the plate with a hole problem considering also the "allowable" strength σ_{allow} class of power law of the modulus. This assumption may be justified from many materials (for example, cellular materials (Gibson and Ashby, 1997) or bones (Martin, 1991)), since the Young's modulus E varies as power law of the density ρ

$$E = P\rho^{p} \tag{1}$$

with P, p being material constants, and also the strength

$$\sigma_{allow} = Q\rho^{\delta} \tag{2}$$

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Nomenclature

σ_{allow}	"allowable" strength
E	Young's modulus
ν	Poisson's ratio
ρ	density
C, P, Q, δ, p, q	material constants
β	power law exponent in the graded modulus
r, θ	cylindrical coordinate system at the notch tip
x, y	cartesian coordinate system at the notch tip
γ	half-angle of the notch ($\gamma = \pi$ is the crack)
$\varphi(r, \theta)$	Airy stress function
α	power law exponent of the stresses in the mode I and mode II problems
u_x, u_y, u_z	displacements in the antiplane problem
$f(x, y)$	function of displacements u_z
e_{ij}	components of the strain tensor
σ_{ij}	components of the stress tensor
$\mu = \frac{E}{2(1+\upsilon)}$	shear modulus
$T(\theta)$	angular function of the stress function
$\eta^2 = (\lambda+1)\beta + (\lambda+1)^2 > 1$	coefficient in the solution of the characteristic ODE of the angular function $T(\theta)$
$\beta + \lambda_{III} =$	power law exponents of the stress in mode III
$\beta_{\min} =$	minimum coefficient of grading to remove stress singularity
$\beta_{\lim} =$	minimum coefficient of grading to define "optimal design" based on stress
$K^V_{Ic,0} =$	generalized critical stress intensity factor
$\Delta K^V_{th,0}(\beta) =$	generalized critical stress intensity factor threshold (fatigue)
$a_0(\beta) =$	critical length scale in TCD (static)
$a_{0,l}(\beta) =$	critical length scale in TCD (fatigue)
$\sigma_{allow,0} =$	flawless allowable strength (static)
$\sigma_{\lim,0}(\beta) =$	flawless allowable strength (fatigue limit)
$W =$	strain energy density
$W_{allow} =$	allowable strain energy density
$\beta'_{\min} =$	minimum coefficient of grading to remove strain energy density singularity
$\beta'_{\lim} =$	minimum coefficient of grading to define "optimal design" according to strain energy density

By cancelling density from Eqs. (1), (2), we obtain an equation for strength as power law of the modulus

$$\sigma_{allow} = C E^{\delta/p} \tag{3}$$

where C, P, Q, δ, p, q are all material constants. For cellular materials (Gibson and Ashby, 1997) or bones (Martin, 1991), some explicit equations are known for the constants. However, we can assume that this power law between strength and modulus could be more general, without having to postulate the intermediate step of the density.

The ratio δ/p is called *strength-modulus exponent ratio:* we can classify materials based on δ/p: for $\delta/p < 1$ strength increases slower than elastic modulus when the density is increased and vice versa. In the limit case $\delta/p = 0$, the structural optimization becomes *identical* to stress concentration minimization.

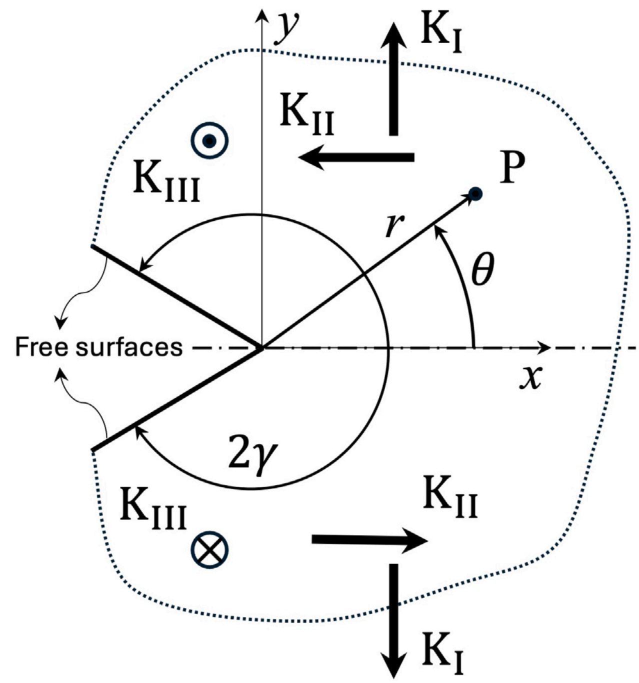

Fig. 1. An elastic plate having a notch in the form of a wedge defined by $0 \leq r \leq R, -\gamma \leq \theta \leq \gamma$ where $r = \sqrt{x^2+y^2}$ and $\gamma = \tan\frac{y}{x}$. The figure shows in plane loading in mode I or mode II (in plane loading) or mode III (antiplane).

Huang et al. (2003) show that the stress concentration of the hole is eliminated by locally stiffening areas that experience high stresses when $\delta/p > 1$ and the opposite for $\delta/p < 1$ as done by the Nature in optimized biomaterials such as bones. Indeed, in foramina holes of the skull, reducing the modulus near the hole boundary and only increasing it some distance away from it, Huang et al. (2003) describes an optimal solution in Nature.

In a previous paper, the author (Ciavarella, 2024) considered the classic problem of an elastic sharp wedge $-\gamma < \theta \leq \gamma$ (see Fig. 1) studied by Williams in the homogeneous case (see Barber's (2002) which for $\gamma = \pi$ becomes a crack, under the assumption of in plane loading, that is either mode I or mode II (or a combination of the two, see Fig. 1). In that paper, elastic modulus was considered varying as a power law of the distance from the notch tip,

$$E = E_0 r^{\beta} \tag{4}$$

By doing so, it was shown that the stress singularity was cancelled provided β was sufficiently large. Moreover, considering that this implies a zero modulus at the notch/crack tip, a strength which depends also as a power law of the modulus was postulated, and it was found that depending on the strength-modulus exponent ratio δ/p of the material, there are two regions of "optimal" design, the most likely one involving an elastic modulus increasing with distance (the case of $\beta > 0$).

As for a crack, the loading in a notch can activate in plane and out-of-plane responses. The out-of-plane one is called mode III (see Fig. 1), while in plane ones studied previously in Ciavarella (2024), are called mode I and mode II loading modes. In particular, the mode I singularity for the homogeneous material is dominant with respect to the mode II one, except in the limit case of the crack when they coincide. Mode III is an intermediate one.

In the present paper, while we extend the model for mode III, permitting closed form results and easier manipulations of results, we discuss more at length the issue of the likely "strength" of such notched/cracked structures, for the first time by considering both strain energy concepts and stress singularity.

2. Solution for the notch with power law modulus, and extension to mode III

The problem in mode I or II has been already solved in Ciavarella (2024), and only main results will be repeated here. It is solved by using an Airy stress function $\varphi(r,\theta)$ which defines stresses varying as $\sigma \sim r^{\alpha}$, and has separated variables form like in the homogeneous case (see (Barber, 2002), 11.1)

$$\varphi(r,\theta) = r^{\alpha+2} f(\theta) \tag{5}$$

Assuming a solution with a power law elastic modulus in the radial direction (4) leads to a fourth order characteristic ODE for the function $f(\theta)$

$$(2+\alpha)(\alpha-\beta)(\alpha(2+\alpha-\beta)+\beta(\nu-1)) f(\theta) + \\ +(4+2\alpha(2+\alpha-\beta)-\beta-\beta(1+\beta)\nu) f''(\theta) \\ +f^{IV}(\theta) = 0 \tag{6}$$

which depends on Poisson's ratio, contrary to the homogeneous case. Solving for a wedge with opening angle $\theta = \pm\gamma$ and imposing the free traction conditions $\sigma_{\theta\theta}(\pm\gamma) = 0$ and $\sigma_{r\theta}(\pm\gamma) = 0$ leads to a non-linear equation which has various solutions in terms of α. If $\alpha < 0$ the solution is *singular*, and for the homogeneous case, indeed, the dominant one is always the mode I singularity (which is $\alpha = -0.5$ for the crack case), while the mode II is less singular, except for the crack.

So far, we have obtained that at any particular radius $\sigma(r) = \sigma_0 r^{\alpha(\beta)}$: by invoking the allowable local strength formulation as a function of the elastic modulus (Eq. (3)) and the spatial distribution of E as per Eq. (4), it is possible to express the allowable local strength as a function of the distance from the notch tip as

$$\sigma_{allow}(r) = C E^{\delta/p} = C E_0^{\delta/p} r^{\beta\delta/p} \tag{7}$$

and hence

$$\sigma(r)/\sigma_{allow}(r) = \frac{\sigma_0 r^{\alpha(\beta)}}{C E_0^{\delta/p} r^{\beta\delta/p}} = C_1 r^{\alpha(\beta)-\beta\delta/p} \tag{8}$$

where C_1 is a constant. We concluded already in Ciavarella (2024) that we can keep $\sigma/\sigma_{allow} < 1$ when approaching the notch apex if

$$\alpha(\beta) > \beta\delta/p \tag{9}$$

which led to the results of Fig. 2, where the region of "optimal" power law β of the elastic modulus variation depending on the strength-modulus exponent ratio δ/p are drawn as shadowed regions for an example case of a sharp notch with $\gamma = 3/4\pi$. Notice that there are two regions of optimal design.

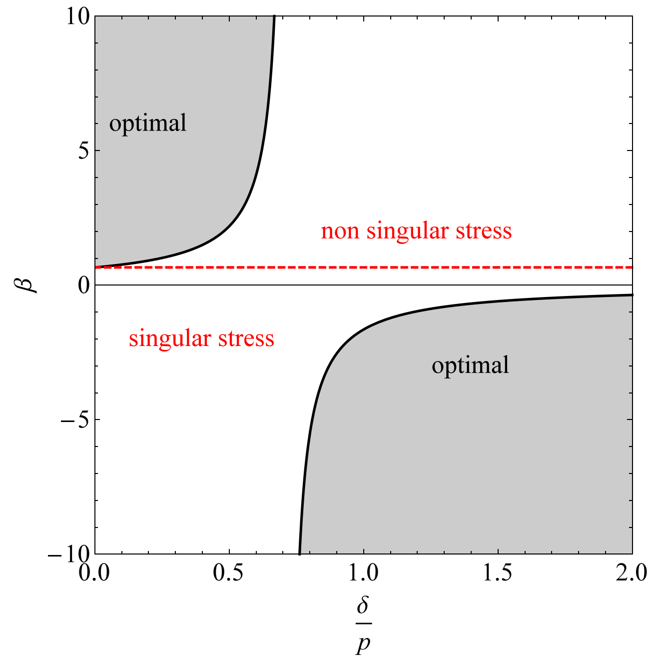

Fig. 2. The two regions of "optimal" power law β of the elastic modulus variation depending on the strength-modulus exponent ratio δ/p for a sharp notch with $\gamma = 3/4\pi$. The sharp notch can be "eliminated" not necessarily the cancelling of the singularity but a value of β satisfying the inequality (9).

3. Antiplane notch problem

In the antiplane problem, displacements in the cartesian coordinate system are

$$u_x = u_y = 0 \quad ; \quad u_z = f(x,y) \tag{10}$$

and hence strains from the strain–displacement relations are

$$e_{xx} = e_{yy} = e_{zz} = e_{xy} = 0 \tag{11}$$

$$e_{yz} = \frac{1}{2}\frac{\partial f}{\partial y} \quad ; \quad e_{zx} = \frac{1}{2}\frac{\partial f}{\partial x} \tag{12}$$

and using Hooke's constitutive law the shear stresses are

$$\sigma_{yz} = \mu(x,y)\frac{\partial f}{\partial y} \quad ; \quad \sigma_{xz} = \mu(x,y)\frac{\partial f}{\partial x} \tag{13}$$

where $\mu = \frac{E}{2(1+\upsilon)}$ is shear modulus.

The only equilibrium equation to write gives (Barber 2010, Chap. 2)

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \tag{14}$$

Hence the fact that we have a modulus which varies in plane gives a more general equation than the Laplacian one of the homogeneous case, namely substituting (13) into (14)

$$\frac{\partial \mu(x,y)}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial \mu(x,y)}{\partial y}\frac{\partial f}{\partial y} + \mu(x,y)\nabla^2 f(x,y) = 0 \tag{15}$$

Hence, changing to cylindrical coordinates using $\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$, we obtain

$$\left(\cos\theta\frac{\partial\mu}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\mu}{\partial\theta}\right)\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) + \\ \left(\sin\theta\frac{\partial\mu}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\mu}{\partial\theta}\right)\left(\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}\right) + \\ +\mu\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right) \\ = 0 \tag{16}$$

Now, before moving further, let us make the assumption of a power law shear elastic modulus in the radial direction in the coordinate system centered on the notch apex (in analogy with (4)) (see Fig. 4)

$$\mu = \mu_0 r^{\beta} \tag{17}$$

and Eq. (16) simplifies into

$$\frac{\partial f}{\partial r}\mu_0\beta r^{\beta-1} + \mu\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right) = 0 \tag{18}$$

Further, given the considered geometry we must assume a separate variable solution, i.e.

$$f(r,\theta) = r^{\lambda+1} T(\theta) \tag{19}$$

which corresponds to stresses in cylindrical coordinates

$$\sigma_{zr} = \mu(r)\frac{\partial f}{\partial r} = \mu(r)(\lambda+1) r^{\lambda} T(\theta) \tag{20}$$

$$\sigma_{z\theta} = \frac{\mu(r)}{r}\frac{\partial f}{\partial\theta} = \mu(r) r^{\lambda}\frac{\partial T}{\partial\theta} \tag{21}$$

Substituting the ansatz (19) into the equilibrium Eq. (18) the following simple second order ODE is obtained

$$T(\theta)\eta^2 + T''(\theta) = 0 \tag{22}$$

where $\eta^2 = (\lambda+1)(\beta+\lambda+1) = (\lambda+1)\beta + (\lambda+1)^2$ (we are considering the case $\beta > 0$ as most interesting, first). Hence, the solution is $T(\theta) = C\cos\eta\theta + K_{III}\sin\eta\theta$ (the meaning of the constant K_{III} is rather obvious given the analogy with the stress intensity factors in Linear Elastic Fracture Mechanics). Therefore, the stresses can be obtained from Eq. ((20),(21))

$$\sigma_{zr} = \mu_0(\lambda+1)r^{\beta+\lambda}\left(C\cos\eta\theta + K_{III}\sin\eta\theta\right) \tag{23}$$

$$\sigma_{z\theta} = \mu_0 r^{\beta+\lambda}\left(-C\sin\eta\theta + K_{III}\cos\eta\theta\right) \tag{24}$$

We now set the traction $\sigma_{z\theta}(\pm\gamma)$ to zero, giving

$$-C\sin\gamma\eta + K_{III}\cos\gamma\eta = 0 \tag{25}$$

$$C\sin\gamma\eta + K_{III}\cos\gamma\eta = 0 \tag{26}$$

The characteristic equations are therefore

$$C\sin\gamma\eta = 0 \tag{27}$$

$$K_{III}\cos\gamma\eta = 0 \tag{28}$$

The symmetric solution $\sin\gamma\eta = 0$ (27) gives no solution of η less than unity since $\gamma < \pi$, and therefore $\eta^2 = (\lambda+1)\beta + (\lambda+1)^2 > 1$ leads to power law exponents of the stress $\beta+\lambda$ which are already non singular in the homogeneous case, and a fortiori in the graded case. Instead, in the homogeneous case the antisymmetric solution does have singularity. In our case, the relevant root of (28) is given by $\eta = \frac{\pi}{2\gamma}$ or

$$(\lambda+1)\beta + (\lambda+1)^2 = \left(\frac{\pi}{2\gamma}\right)^2 \tag{29}$$

This leads to the dominant *antisymmetric* root which corresponds to the power law exponents of the stress

$$\beta + \lambda_{III} = \frac{1}{2}\beta - 1 + \sqrt{\frac{\beta^2}{4} + \frac{\pi^2}{4\gamma^2}} \tag{30}$$

In order to make the stress non singular we have to find the locus $\beta + \lambda_{III} = 0$ and this gives

$$\beta > \beta_{\min} = 1 - \frac{\pi^2}{4\gamma^2} \tag{31}$$

The power law exponents of the stress $\beta + \lambda_{III}$ is plotted as a function of β and for various δ in Fig. 3 , while the region where singularity is cancelled for mode III notch is shown in Fig. 4 according to Eq. (31).

Further, repeating the reasoning of the mode I,II problem in the previous paragraph,

$$\sigma(r)/\sigma_{allow}(r) = C_1 r^{\frac{1}{2}\beta - 1 + \sqrt{\frac{\beta^2}{4} + \frac{\pi^2}{4\gamma^2}} - \beta\delta/p} \tag{32}$$

Therefore, in order to avoid failure, the following inequality must be true

$$\frac{1}{2}\beta - 1 + \sqrt{\frac{\beta^2}{4} + \frac{\pi^2}{4\gamma^2}} \geq \beta\delta/p \tag{33}$$

The equality sign gives a $\beta_{\lim}$

$$\beta_{\lim} = \frac{\gamma - 2\frac{\delta}{p}\gamma^2 - \sqrt{\gamma^4 - \frac{d}{p}\pi^2\gamma}}{2\gamma^2\frac{\delta}{p}\left(\frac{\delta}{p} - 1\right)} \tag{34}$$

The region where stress singularity is cancelled is plotted in Fig. 4, while the regions of optimal design where $\sigma/\sigma_{allow} < 1$ are plotted in Fig. 5.

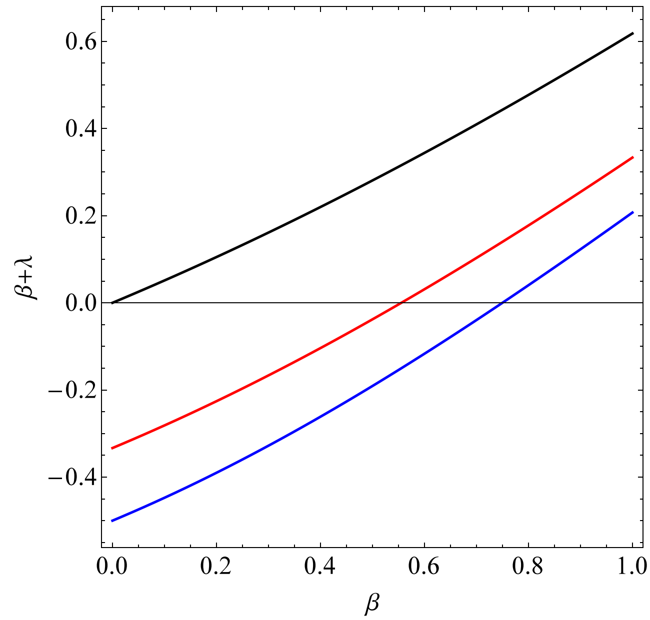

Fig. 3. The power law exponents of the stress $\beta + \lambda_{III}$ in stresses $\sigma \sim r^{\beta+\lambda_{III}}$ with a modulus varying as $\mu \sim r^{\beta}$ for a notch geometry ($\gamma = \pi/2, 3/4\pi, \pi$ (crack) respectively as black, red and blue solid line). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

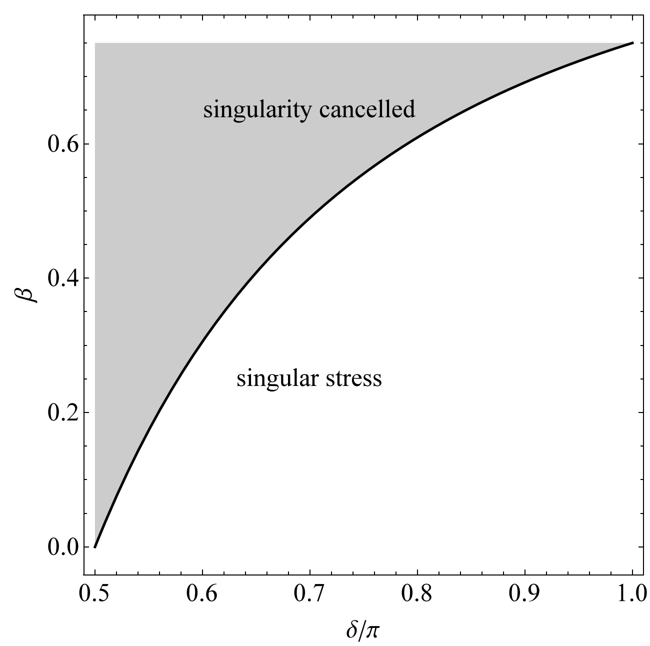

Fig. 4. The region where stress singularity is cancelled for mode III notch (see Eq. (33)).

4. Fracture mechanics and strain energy density criteria

Crack problems are usually dealt with energetic approaches, since the times of Griffith (Griffith, 1921). The case of a notch has also been dealt with energy criteria, although in this case the infinitesimal propagation of a crack is not self-similar and hence it requires a different treatment when initiation of a crack is considered. We have seen that, as in the previous paper on the subject about mode I and II (Ciavarella, 2024), we can remove the singularity in stress, but since the modulus goes to zero at the notch root, the strains will grow

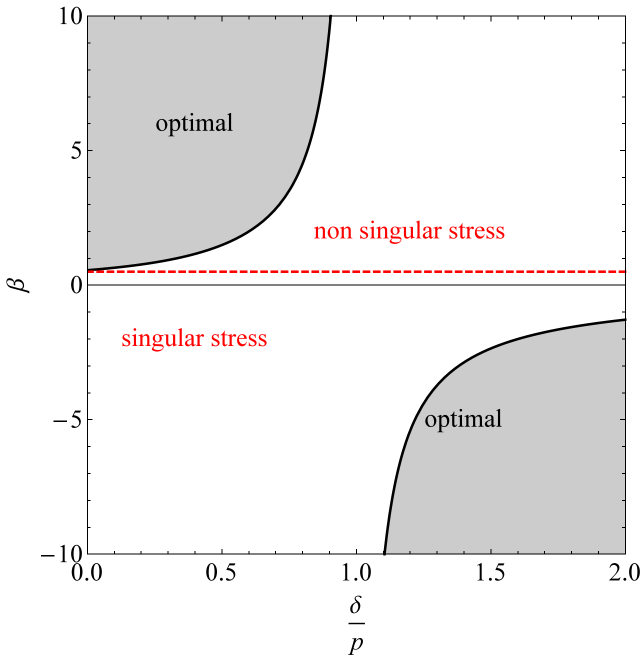

Fig. 5. The regions of optimal design based on stress-based criterion for mode III notch.

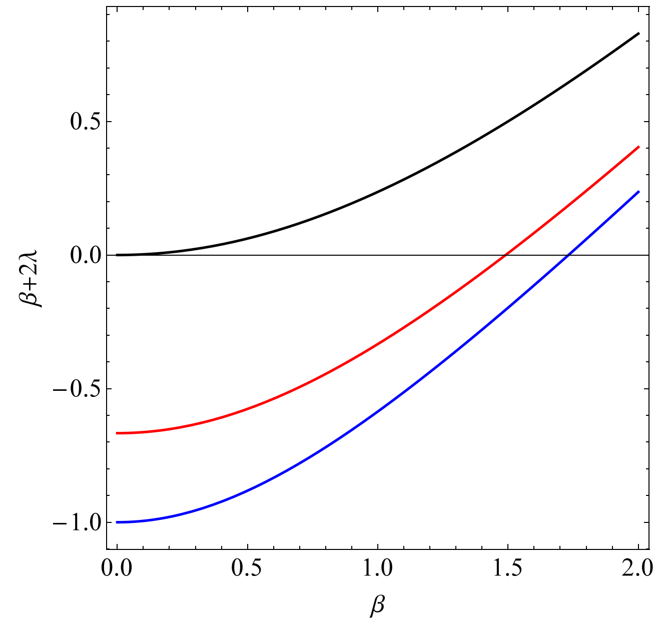

Fig. 6. The power law exponents of the strain energy density $W \sim r^{\beta+2\lambda_{III}}$ with a modulus varying as $\mu \sim r^{\beta}$ for a notch geometry ($\gamma = \pi/2, 3/4\pi, \pi$ (crack) respectively as black, red and blue solid line). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

unbounded, making impossible to define a local strength. In particular, supposing to use the minimum amount of grading to remove singularity in mode III, according to Eq. (31), strains will go as $\varepsilon \sim r^{-\beta}$. For example considering the crack case for simplicity, then $\varepsilon \sim r^{-3/4}$ and strain energy density would go as $\sigma\varepsilon \sim r^{-3/4}$ and therefore be higher than the actual homogeneous crack, for which $\sigma\varepsilon \sim r^{-1/4}$. Therefore, the methodology as defined in the previous paper (Ciavarella, 2024) and so far in the present article, makes sense from a classical strength perspective, but is less certain from a fracture mechanics perspective.

A more fracture mechanics oriented methodology therefore would need to take into account also of toughness of the material. The onset of fracture initiation in the presence of notches, especially in fatigue loading scenarios, is widely accepted to be dependent on a length scale also known as structural support length, as first proposed by Neuber (1968), and later developed and generalized in a number of approaches, such as fictitious notch rounding (Radaj et al., 2013), Theory of Critical Distances (TCD) (Taylor, 2008), Averaged Strain Energy Density (ASED) (Berto and Lazzarin, 2009), implicit gradient (Tovo and Livieri, 2011; Livieri et al., 2016). In this context of TCD (Taylor, 2008) we include both strength and fracture mechanics standard ideas, by looking at strength obtained when the specimen has a very small notch or crack, and the strength obtained when the specimen has a very large notch or crack, as explained below.

This principle must necessarily work from an experimental point of view even when the material is graded, although the toughness and strength will depend on the level of grading. For a given grading β, and for a very small crack, the material is considered as flaw-insensitive and therefore its strength is dominated by $\sigma_{allow,0}$. On the other hand, with large notch/crack a generalized critical stress intensity factor $K^V_{Ic,0}$ (with dimension stress×length$^{-\alpha}$ where $\alpha = -0.5$ for a homogeneous material crack, as we have seen). This immediately defines a length scale

$$a_0(\beta) \sim \left(\frac{K^V_{Ic,0}(\beta)}{\sigma_{allow,0}(\beta)}\right)^{1/\alpha} \tag{35}$$

and hence this defines a control volume size, because it defines the size of the notch/crack for which there is transition between strength control and toughness control. For example, in the simplest version of the TCD, then simply the stress at a distance $a_0/2$ from the notch tip is the appropriate stress which is to be compared to the nominal strength of the unnotched specimen.[1]

To find $a_0(\beta)$, there is no real alternative than the experimental one. But notice immediately that if we consider the main case of interest for the present article, namely that which cancels the singularity, then there are no length scales $a_0(\beta_{\min})$ and $a_{0,l}(\beta_{\min})$ because there is no size effect by definition. Most likely, what happens is that $a_0(\beta), a_{0,l}(\beta)$ tend to infinity as $\beta \to \beta_{\min}$, which is an alternative way to look at the effect of cancelling the singularity. More speculations about a_0 can be found in Appendix.

It seems we can conclude that there would be no size-scale effect in the strength of such specimen with cancelled singularity.

4.1. Strain energy density criterion

We have mentioned an alternative criterion could be strain energy density. The strain energy in a region near the root apex given by the product of stresses $\sigma_{ij} \sim r^{\beta+\lambda_{III}}$ and strains $\varepsilon_{ij} \sim r^{\lambda_{III}}$, and hence strain energy density goes as

$$W = \sigma_{ij}\varepsilon_{ij} \sim r^{\beta+2\lambda_{III}} \tag{37}$$

or

$$W = \sigma_{ij}\varepsilon_{ij} \sim r^{-2+2\sqrt{\frac{\beta^2}{4}+\frac{\pi^2}{4\gamma^2}}} \tag{38}$$

The required grading coefficient β to cancel the singularity in strain energy density is *much higher than that needed to cancel the singularity of stress*, and is plotted in Fig. 6.

[1] In fatigue, the equivalent definition is

$$a_{0,l}(\beta) \sim \left(\frac{\Delta K^V_{th,0}(\beta)}{\sigma_{\lim,0}(\beta)}\right)^{1/\alpha} \tag{36}$$

where $\Delta K^V_{th,0}(\beta)$ is V notch generalized fatigue threshold and $\sigma_{\lim,0}(\beta)$ is fatigue limit.

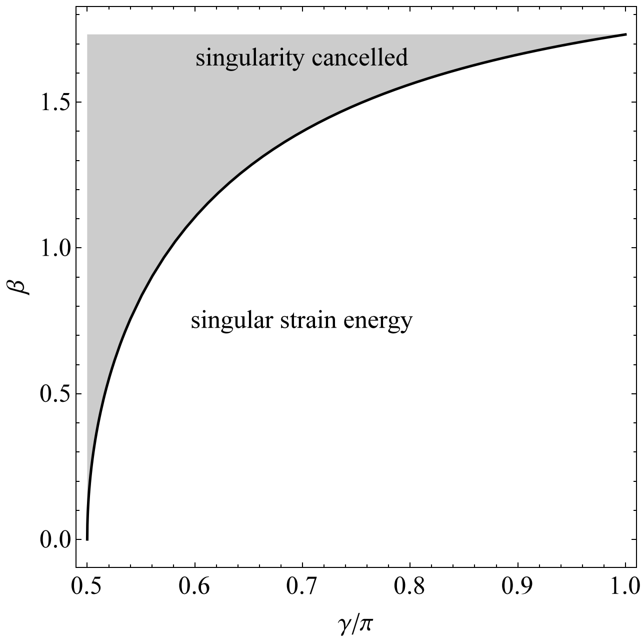

Fig. 7. The region where strain energy density singularity is cancelled for mode III notch.

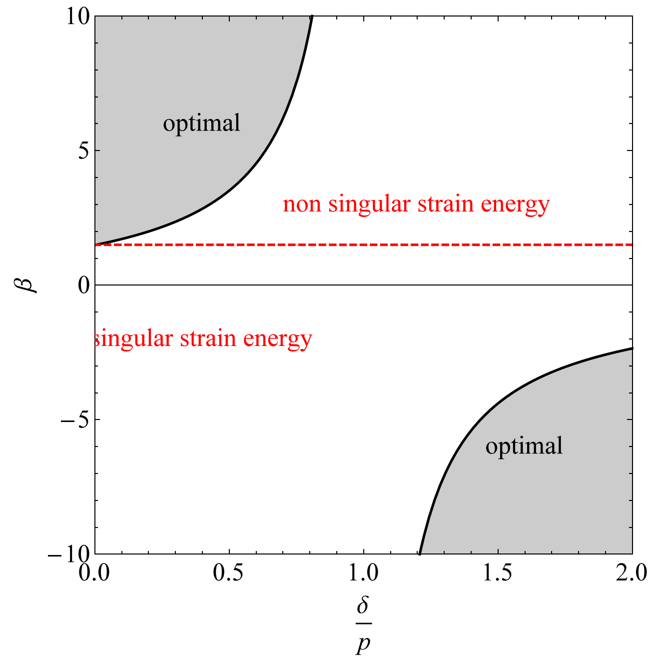

Fig. 8. The regions of optimal design according to strain energy density criterion $W(r)/W_{allow}(r) < 1$ for a notch with $\gamma = 3/4\pi$ (see Eq. (42)).

Further, repeating the reasoning that the allowable strain energy density is likely to be dependent on the modulus (here, δ/p is a different number than that used for strength-to-modulus ratio)

$$W_{allow}(r) \sim E(r)^{\delta/p} \sim r^{\beta\delta/p} \tag{39}$$

and hence the ratio

$$W(r)/W_{allow}(r) \sim \frac{r^{\beta+2\lambda_{III}}}{r^{\beta\delta/p}} = r^{\beta(1-\delta/p)+2\lambda_{III}} \tag{40}$$

we can conclude that we can keep $W(r)/W_{allow}(r) < 1$ when approaching the notch apex if

$$\beta(1-\delta/p) + 2\lambda_{III} \geq 0 \tag{41}$$

The equality sign gives a $\beta'_{\lim}$. Using the results obtained for λ_{III} (30), we get that the previous Eq. (41) becomes

$$-2 + 2\sqrt{\frac{\beta^2}{4} + \frac{\pi^2}{4\gamma^2}} \geq \beta\delta/p \tag{42}$$

and leads to the two regions where strain energy density singularity is cancelled for mode III V notch with $\gamma = 3/4\pi$ in Fig. 7, and finally Fig. 8 gives the region of optimal design according to strain energy density criterion $W(r)/W_{allow}(r) < 1$.

5. Discussion

We have seen that stress or strain energy density singularities can be cancelled, the latter requiring a stronger level of grading than the former. Once the singularity is removed, we have effectively removed also size effects and therefore it is irrelevant to define the "critical distance" often used by the Theory of Critical Distances (Taylor, 2008) to encompass the concepts of strength and toughness. We have effectively removed the effect of the crack/notch, and therefore there is only a single strength criterion to be used.

In the previous paper (Ciavarella, 2024), we introduced a concept for which the strength is a power law of the modulus, and we suggested an "optimal" design by keeping the dominant stress constantly equal to the strength. Here, having also introduced the concept of cancelling strain energy density, we have extended the same ideas to the case of strain energy density.

The theory to estimate the effect of the graded notch is at present an hypothesis, and needs experimental verification, but it has been suggested before for finite stress concentration problems, although again without experimental verification (Huang et al., 2003). There is no point in thinking about averaging the stress, as done often in static or fatigue from stress concentrators (Taylor, 2008): here, the stress is not widely varying, and averaging would not lead to much change. However, we can imagine that the effect of cancelling the singularity can be partly diminished by the fact that resistance of the low modulus material is lower than the resistance of the high modulus material. This effect could even overcompensate the cancelling of singularity. For example, assuming that we create the "optimal" solution where stress coincides with resistance point by point, we are left anyway with a resistance that is lower than in a unnotched specimen in a homogeneous material with the net section of the notched specimen. Indeed, the load carried by the ligament of such a specimen will be the integral of the strength which varies point by point, but which is certainly lower than that of the (harder) homogeneous material. In turn a specimen with a very blunt notch can have the strength of this homogeneous specimen, and hence the advantage of the graded V-notch specimen would be dubious. Perhaps under fatigue loading the beneficial effect could be larger, because the stress concentration effect is generally more damaging in fatigue than in static conditions. A vaguely similar effect has been examined for the decrease of stress intensity near a crack due to the presence of microcracks in ceramic materials. It has been shown that this effect is generally overcompensated by the decreased resistance of the microcracked material (Ortiz and Giannakopoulos, 1989).

We have conducted FEM investigations to confirm the analytical results obtained, but we do not report them here in the interest of brevity, as it is rather obvious that the singularity corresponds to what obtained with the exact solution.

Finally, notice that an equivalence is possible between grading the elastic modulus, and grading the thickness of the plane stress (or plane strain) problem. In other words, to make the notch stronger we could think of increasing the thickness near the tip. Strictly speaking the zero elastic modulus corresponds to an infinite thickness at notch tip.

6. Conclusions

We have extended a recent model suggested by the author for a sharp wedge or crack loaded in mode I or mode II, where the singularity in the stresses is removed with a modulus varying as a power law of the distance from the notch apex, $E \sim r^{\beta}$. Here, we have dealt with the even simpler case of mode III, but we have also postulated another criterion for cancelling the notch/crack by making the strain energy density uniform which requires a stronger power law coefficient for modulus grading. Extending the Theory of Critical Distances (TCD) to functionally graded materials, toughness and strength of a graded notch can be defined by looking a very large and very small notch, respectively. However, by removing the singularity (either on stress or strain energy density) we find independence on the critical distance, and hence no size-scale effect. Hence, the application of the present work could be significant, certainly for the notch geometry, but even for the cracked one, if a material could be intelligently "designed" as to reduce its modulus when near a high stress concentration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Some more observation on the critical distance

If one wanted to estimate the value of a_0 based on homogeneous materials properties, as we have done for strength in (3), one could assume for example both toughness and strength would depend on modulus in general in a point wise sense, and that the same law holds for toughness and strength as in Eq. (3)

$$K_{IC}(r) = C_K E^{\delta/p} \tag{43}$$

in which case hence we could define a length scale

$$a_0(r) \sim \left(\frac{K_{Ic}(r)}{\sigma_{allow}(r)}\right)^2 \sim \left(\frac{C_K E^{\delta/p}}{C E^{\delta/p}}\right)^2 \sim \left(\frac{C_K}{C}\right)^2 = const \tag{44}$$

which has the good property that does not depend on radius, but is also independent on β, in contradiction with our expectation. Assuming instead a different power for strength and toughness as function of modulus would lead to a_0 not a constant in contradiction with obvious experimental evidence that there must be a toughness and a strength for a given material.

A more general assumption could be to take an average modulus, namely make the assumption on toughness more general

$$K_{IC}(r) = C_K E^{(\delta/p)'} \tag{45}$$

which would define a length scale

$$a_0 \sim \left(\frac{C_K}{C} E^{(\delta/p)' - (\delta/p)}\right)^2 \tag{46}$$

and now in order for this quantity not to depend on distance r we could take an average value of the modulus over the control distance a_0 itself. Even simpler still, assume the modulus at the control distance itself $E(a_0)$, resulting in

$$a_0 \sim \left(\frac{C_K}{C} E_0 a_0^{\beta[(\delta/p)' - (\delta/p)]}\right)^2 \tag{47}$$

This would result in an implicit equation for a_0, and would not depend on β.

Data availability

No data was used for the research described in the article.

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