Complexity of Spatially Interconnected Systems

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TO THE SPIRIT OF MY FATHER

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Abstract

In contrast to lumped dynamical systems, which are defined with respect to the temporal variable only, spatially interconnected systems are defined with respect to temporal as well as spatial variables (spatio-temporal models). Spatially interconnected systems are obtained by applying a spatial discretization to Partial Differential Equations, such that the resulting system is represented as a spatial interconnection of subsystems. The discretization is usually induced by an array of actuator-sensor pairs. An important feature of spatially interconnected systems is that they are causal with respect to time, but non-causal with respect to space.

In practice, after the discretization the complexity of the resulting system often renders the associated analysis and synthesis problems intractable. Therefore, constructing reduced-complexity models without losing the characteristic features of the original model is of high practical importance. Standard model reduction methods do not preserve the structure of the system, while here the spatial interconnection structure of the system must be preserved in the reduced model.

Spatially interconnected systems can be distinguished into *time and space invariant*, and *time and space varying* systems.

This thesis studies and proposes methods for solving the reduction problem for both parameter invariant and parameter varying systems, where different kinds of complexities for such systems are considered. A trade-off between model accuracy and model complexity is considered when solving the reduction problem. The work is based on representing the system in Linear Fractional Transformation (LFT) form with respect to shift operators.

A model order reduction method based on balanced truncation for parameter-invariant interconnected systems is proposed via solving a pair of LMIs with non-convex rank constraint, which helps in constructing improved solutions (generalized Gramians). Methods to solve the latter non-convex condition are proposed as well. The balanced truncation is done by transforming the generalized Gramians via a balancing transformation constructed for non-causal systems.

In addition, the proposed model reduction method is extended to parameter-varying interconnected systems, which are varying with respect to time and space via scheduling parameters. A practical difficulty is that the reduction problem needs to be applied to all scheduling parameter variations. A method is proposed here to simplify the problem, such that the reduction problem is applied to the parameter-invariant part of the system after "pulling out" the time and space varying parameters. The method is based on the application of the full block S-procedure.

A joint model and scheduling order reduction method is proposed via transforming and truncating the LFT multipliers as well as the generalized Gramians. The reduction is performed using either constant generalized Gramians or parameter-dependent generalized Gramians with bounded rate of parameter variation. The reduction procedure with parameter-dependent generalized Gramians is done by reducing the scheduling order (the number of scheduling parameters) first, and then the state order (the number of states) of the system. In the case of constant generalized Gramians, the scheduling order and the state order are simultaneously reduced. A comparison between the use of constant generalized Gramians and parameter-dependent ones shows that the latter case improves the accuracy and reduces the conservatism.
For all above cases, error bounds (defined in terms of the induced $\mathcal{L}_2$-norm, between the original systems and the reduced ones) are proved. The proposed methods take into consideration the *non-causality* of the system's spatial dynamics.
In addition, the proposed methods preserve the spatial structure as well as the stability in the reduced model provided the original system is stable.

Theoretical results are illustrated on an experimentally identified model of an actuated beam.

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Chapter 1

Introduction

The use of large sensor-actor arrays for controlling physical variables distributed over spatially extended structures, made possible by recent technological advances, has created an interest in efficient and distributed control schemes for such arrays. When the dynamics of such systems are governed by partial differential equations (PDE's), a useful model can be obtained from a spatial discretisation induced by the distribution of sensors and actors, where when the systems are governed by PDEs, then due to spatial continuum, the system is modeled as infinite-dimensional [2] (in which case semigroup theory [3] is utilized). In [4] and [2] methods were proposed which avoid the need for spatial discretization and provide a reduction of an infinite dimensional problem to a problem in which only matrices of finite dimensions are involved. However, based on [5], in this thesis a spatial discretization is applied to PDE's such that the model is a spatial interconnection of a number of subsystems [5]. Each subsystem has its own sensing and actuation capability; such systems are known as *spatially interconnected systems (SIS)*. Spatially interconnected systems arise in several applications, such as Micro-Electro-Mechanical Systems (MEMS) arrays, vibrating cables [6], smart mechanical structures [7], etc. Such systems, unlike lumped systems, depend on time and space; the states of the system are defined with respect to temporal and spatial variables, so that we refer to such systems as multidimensional (MD) systems.

Since these systems are approximated by a discretization, one obtains high-order models. In [5], a framework was proposed to reduce the analysis and synthesis of the system to the size of a single subsystem. However, even when the model of a subsystem is simple, the interaction between neighboring subsystems leads to complex dynamics when viewed as a whole, which can make it difficult to design and analyse distributed control schemes. It may still turn out (and this problem is encountered in practical applications) that the subsystem models need to be reduced in order to render the synthesis problem tractable.

Usually, it is difficult to deal with high-order models; after designing a controller the order of the controller is equal to (or greater than) the order of the system. Furthermore, interconnected systems have a complexity due to their interconnection structure. Therefore, the generation of reduced-order models is of practical interest. This thesis addresses the complexity of SIS under two different aspects: the complexity of the **dynamic order** of the system (i.e. number of states), and the complexity of the **scheduling order** for spatial LPV systems i.e., the order of the scheduling parameters of the systems.

There is a trade off between model complexity and model accuracy/performance. This thesis provides new methods that reduce the complexity of SIS without losing the significant properties (such as the interconnection structure and the stability) of the original system.

This chapter presents an overview of the thesis, together with the objective, motivation and main contributions.

1.1 Classification of Spatially Interconnected Systems (SIS)

As mentioned, approximate models for systems governed by PDE's are often sought through spatial-discretization induced by the distribution of sensor-actor pairs, such that the resulting spatially distributed systems are represented as a spatial interconnection of subsystems exchanging information with neighbors via interconnection signals [5]. For a single-spatial dimension, a spatially interconnected system is shown in Fig. 1.1

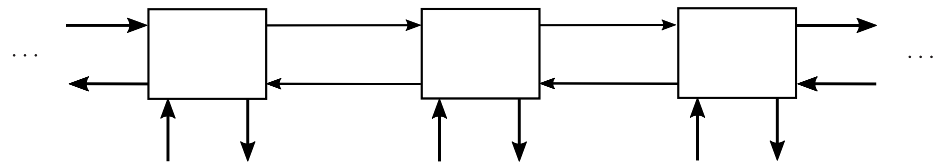

Figure 1.1: Part of a spatially interconnected model

As an example to such systems, consider an actuated flexible structure (an aluminum beam of length 4.8 m defined in a single-spatial dimension as shown in Fig 1.2) which has been constructed in [8]. The beam is equipped with 16 piezoelectric actuator-sensor pairs distributed along its length such that the distances between any two pairs are identical. The beam is accordingly discretized into 16 identical subsystems in order to define a spatially interconnected system similar to Fig. 1.1, such that each subsystem is equipped with its local actuator-sensor (input/output) pair. This beam example will be used to validate the performance of results in this thesis.

An important feature of such systems is that they are non-causal with respect to space, but causal with respect to time (where in contrast to the temporal variable which propagates only into the future, the spatial variable propagates both left and right). The system dynamics may (or may not) vary over temporal and/or spatial translations. Accordingly the system is classified as follows.

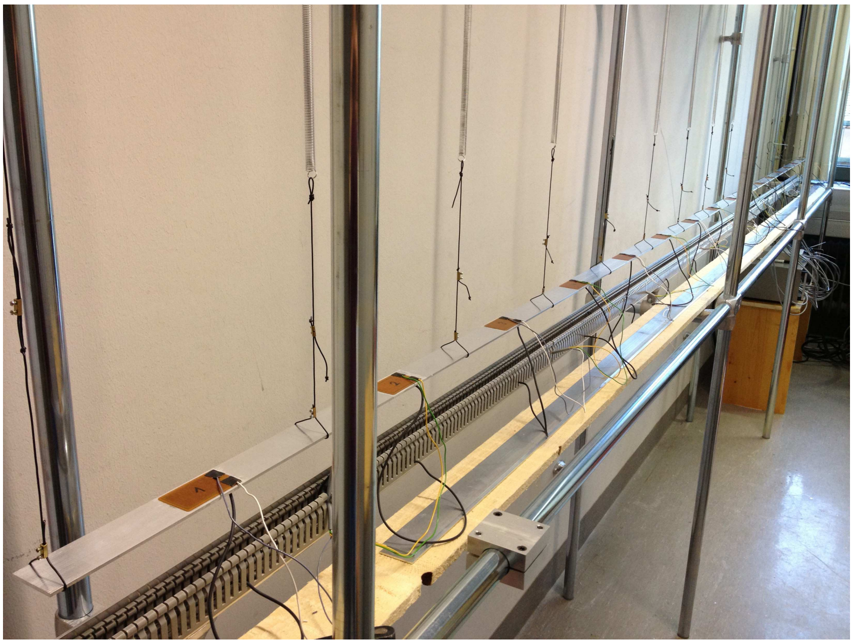

Figure 1.2: Aluminium beam actuated with collocated piezo actuators and sensors.

1.1.1 Time- and Space-Invariant Interconnected Systems (LTSI)

For systems with time- and space-invariant dynamic properties and evenly distributed sensors and actuators (the distance between any two pairs are identical), the model takes the form of a spatial interconnection of identical subsystems (Fig. 1.1), where each subsystem is modelled as an linear time invariant (LTI) system that has its own sensing and actuation capability [5].

1.1.2 Time- and Space-Varying Interconnected Systems (LTSV)

In practical application, the subsystems of SIS are often varying due to their physical properties or due to boundary effects, such that the subsystems are nonidentical, i.e. the dynamics of each subsystem are varying with respect to time/space variables. If the variation of the system properties with respect to time and space can be expressed in terms of suitable scheduling parameters (such that each subsystem is varying with respect to time and space via scheduling parameters), then we can represent the system as temporal- and spatial LPV systems, which is a direct extension of the concept of lumped (temporal) LPV systems, see e.g. [9]. Therefore, the systems considered in the previous section are extended now to parameter-varying interconnected systems [10], [11]. For example, if one considers unevenly distributed sensors and actuators (the distance between any two pairs are nonidentical) in Fig.1.2, then the beam has nonidentical subsystems ("spatial variation").

Keeping the scheduling parameters inside the subsystems leads to difficulties in analysis and synthesis problems. Note that it is possible to capture different properties of subsystems via an LPV representation and still have analysis problem with a complexity that is independent of the number of the subsystems (as considered in LTSI systems). For that reason, we pull out (from each subsystem) the scheduling parameters into local uncertainty blocks, such that the system is represented by LTSI subsystems (identical subsystems) each connected to its local time- and space-varying uncertainty block defined in Linear Fractional Transformation (LFT) form. We refer to such model representations as LPV/LFT systems [12].

1.2 Reduced System Architectures and Problem Formulation

Model reduction for SIS requires structure preserving methods, i.e. the reduced systems should have the same structure as the original ones, such that the spatial interconnection is preserved. Therefore, the question arises: How can we reduce systems that are defined as an interconnection of subsystems without losing the interconnection structure? This thesis will answer that question: the reduced versions of LTSI (LTSV) systems are LTSI (LTSV) systems as well. The reduced versions of SIS have the following structure.

- LTSI: reduced LTSI systems (defined in Section 1.1) have the same structure as the original systems, depicted in Fig. 1.1.
- LTSV: reduced LTSV systems are defined as spatial interconnections of reduced LTSI (identical) subsystems each connected to its local uncertainty blocks. For this class we consider two reduction dimensions as follows.
  - Dynamic Order Reduction (i.e. reducing the number of states): The reduced systems are defined as a spatial interconnection of reduced subsystems connected to the same uncertainty blocks as the original subsystems (i.e. the same number of scheduling parameters).

    Note that the complexity of the LTSV systems has two dimensions: the complexity caused by the state order (i.e. dynamic order) of the system and the scheduling parameter order. Therefore, a joint order reduction is considered:
  - Joint Dynamic and Scheduling Order Reduction: The reduced systems are defined as a spatial interconnection of reduced subsystems connected to reduced uncertainty blocks (i.e. reduced number of scheduling parameters and of states).

The reduction problem in this thesis is formulated as follows: given the original system, find a reduced complexity system that closely approximates the original one, such that the distance in the sense of the induced $\mathcal{L}_2$-norm between the original system and the reduced one is minimized.

1.3 Motivation, Relevant Work and Thesis Contribution

In system identification, if the aim is to achieve an accurate approximation, the order of the system should be high, which leads to computational complexity for controller synthesis. Furthermore, after implementing the controller design, the obtained controller has the same order as the original system (as experimentally shown in [8] for distributed systems). Reducing the state order (i.e. dynamic order) of the system leads to a simplification in computational complexity of controller implementation. When a system is parameter-varying, the dependence on scheduling parameters will increase the complexity of the system further; therefore, it would be desirable to reduce the scheduling order as well.

When a system is defined as an interconnection of subsystems, as in the case of Spatially Interconnected Systems, the interconnection signals are defined as spatial states inside each subsystem. Even when the model of a subsystem is simple, the interaction between neighboring subsystems leads to complex dynamics. The framework proposed in [5] makes it possible to reduce the analysis and synthesis problem for such interconnected systems to the size of a single subsystem. However, it may still turn out that the subsystem models need to be reduced.

According to the above discussion, it is desirable to find a low-complexity system that retains the main features of the original system. Model Order Reduction (**MOR**) techniques receive considerable interest, see e.g., [13], [14], [15] and [16], also [17] for a class of interconnected systems.

Spatially interconnected systems form a class of distributed systems. One could also consider such systems as large scale systems (lifted systems) [8]. Various techniques have been proposed to reduce large scale systems, e.g. [16] which proposes an approach of reducing systems without employing a transformation, also [13] which presents a combination of SVD- and Krylov-based reduction.
A variety of approaches to model order reduction for 1-D (lumped-parameter) parameter-invariant systems has been proposed in the literature, for example, optimal Hankel norm approximation model reduction with error bound between the original system and the reduced one [18], state residualization model reduction [19], Proper Orthogonal Decomposition (POD) which approximates the Gramians of the system [20], and the Krylov subspace model reduction [21]. Note that the Krylov subspace (or moment match) approach does not guarantee stability of the reduced models and no a priori error bound can be determined.

Arguably, the most popular method for model order reduction is balanced truncation, where a high-order system is approximated by a lower order system, that captures the significant properties of the original one, with a priori upper error bound, by removing states which have little influence both in terms of controllability and observability. Further, the reduction preserves the stablility of the system. Balanced truncation for LTI systems was first proposed in [22], [23], where the technique was based on solving a pair of Lyapunov equations, these solutions usually known as controllability and observability Gramians, which help in determining hard observable/controllable modes (which to be truncated). An improved "tighter" error bound can be derived by considering Lyapunov inequalities instead of equations [24], because in the former case infinitely many solutions can be obtained such that one can search for improved ones. For a survey of balanced model reduction for 1-D (lumped) systems see e.g. [25] and references therein. Balanced truncation may guarantee a lower error bound in some cases, e.g. [26] which considers a class of systems. See [27] for further different model order reduction approaches.

For 1-D (lumped) systems, the above techniques were extended to 1-D Linear Time-Varying (LTV) and Linear Parameter-Varying (LPV) systems, respectively, e.g. [28], [29], [30], [31], [32], [33], [34] and references therein, and for nonlinear systems [35] in which a comparison between balanced truncation and POD model reduction was presented. Also, model reduction with application to a nuclear power plant has been given in [36]. In [37] a model reduction for LPV systems was proposed using the extended balanced truncation approach. In addition, model reduction for structured and distributed models based on coprime factorization and frequency-weighted model reduction were presented in [38]. In [1] a technique for model order reduction of polytopic systems was given, and model reduction problem for LPV and uncertain systems based on coprime factorization was discussed in [39].

The model order reduction problem for 1-D (lumped) LPV/LFT systems has been studied in [40] using algebraic Riccati inequalities. Model order reduction for uncertain systems was discussed in [41]; in the latter two references conservative diagonal scaling matrices are considered.

Since the complexity of 1-D LPV systems depends on both the state order of the system and the number of scheduling parameters, joint state and scheduling order reduction methods for 1-D (lumped-parameter) LPV systems are proposed [42], [43] and [44].

The extension of 1-D model reduction results to multidimensional systems (with temporal and spatial dimensions) is based on an LFT representation of the model with differential and/or shift operators in the Δ-block. Representing the model in this form makes it possible to employ results on model reduction for uncertain systems. Results presented in [45], [46], [47], [48] provide conditions for exact reducibility of nonminimal uncertain/multidimensional systems, represented in LFT form with structured Gramians[1]. A necessary and sufficient condition for exact reducibility is the existence of a singular, structured positive semidefinite solution to either one of a pair of nonstrict Lyapunov inequalities, together with a coupling condition. In [49, 50] results on model reduction for uncertain systems are presented. A minimal realization based on the Hankel matrix has been discussed in [51] for a class of multidimensional systems.

Different model reduction approaches for the so called *Roesser Form* [52] (which is a *causal* 2-D system) have been discussed in the literature, see e.g. [53], [54] in which quasi-Gramians have been used; the method does not guarantee stability preservation, see also [55]. In [56, 57] the relationship between the controller synthesis problem and the model order reduction problem for spatially interconnected systems was pointed out. It is however not possible to directly solve a controller synthesis problem to obtain a reduced

[1]Exact reduction here refers to the reduction of a state space model to a minimal realization, i.e. the removal of uncontrollable or unobservable modes.

model. In addition, the problem of model order reduction in [57] is not solvable as a convex optimization problem, which considerably limits its practical value. Moreover, in [57, 56], and [58] only the causal (i.e. temporal) part of the system is considered when minimizing the rank of the Gramians; the practically important issue of reducing the model of the spatial dynamics is not addressed.

In [59] the model order reduction problem for spatially-varying but temporally-invariant systems was proposed by considering not a single subsystem but the overall "lifted system". The sequentially semi-separable matrix structure was used to represent the whole system as a 1-dimensional system. The approach suffers from the fact that the size of the problem is that of the overall system, which may be intractable when the number of subsystems is large.
It is worth pointing out that there is a fundamental difference between approaches that have been proposed recently, such as [60], [61] and [62] on one hand, and the approach in [5] on which the present work is based. Both approaches aim at reducing the complexity of analysing large-scale networks of dynamic systems. But while the former one considers the network as a finite entity and tries to reduce the size of analysis problems by either diagonalising the interconnection matrix or by exploiting its sparsity, the latter approach relies on a regular grid structure of the network and utilises spatial shift operators, which leads to a different framework for solving analysis (and synthesis) problems, and achieves a complexity that is independent of the size of the network.

The goal of this thesis is to generate reduced complexity spatially interconnected systems (*in more than one dimension*) for the two classes of LTSI and LTSV systems, which preserve the significant properties and behave similar to the original systems. Such significant properties are represented by

- Spatial structure
- Stability
- Non-causality.

The contributions of this thesis are briefly described as follows.

LTSI systems: In this thesis, the work in [57], [56] and [58] on parameter-invariant multidimensional systems is extended in the following sense.

- We show how balanced truncation with a guaranteed error bound can be applied to non-causal systems, thus allowing to reduce the spatial dynamics as well as the temporal dynamic, where *noncausal* Gramians are considered in this work.
- In order to reduce a model while ensuring a small error bound, we present a two-stage approach. The idea of the proposed approach is to balance the system as a first step. Following this, the result of the balanced truncation is used to initialize the next step, which utilizes the machinery of controller design

for SIS, and minimizes the rank of the Gramians and the error bound further. Then a trade-off is performed between error bound and a rank constraint, while maintaining small generalized singular values[2].

- Furthermore, we turn the results of [57] and [58] into practical tools, by extending them into the continuous domain. For systems with non-causal (spatial) dimensions, the reduction problem cannot be solved as a convex LMI problem when a discrete representation of the spatial dynamics is used. This difficulty is also encountered in the controller synthesis problem for such systems, see [5]. This is illustrated in this thesis with the application to a practical problem: model reduction for an actuated beam shown in Fig. 1.2.
- We improve the error bound for the reduced model by using efficient methods for solving the problem based on a log-det and cone complementarity approach, [63], [64]. In addition, using these approaches will guarantee singular solutions to either or both Lyapunov inequalities (meaning the system is exactly reducible) and will thus help to establish the minimality of the system.

LTSV systems: Again, taking the multidimensional aspect and the non-causality issue of the SIS into account, the contribution of the thesis in the case of LTSV systems is as follows.

- Model reduction of LTSV systems is based on a spatio-temporal LPV representation in LFT form. From reduction of lumped LPV systems it is known that this requires gridding of the parameter space, see e.g., [34] and [30]. Here we simplify the analysis problem via applying the full block S-procedure, which has the advantage of reducing the infinite dimensional problem into a finite one.
- Reducing both the state order as well as the scheduling order leads to further simplification. In addition, using parameter-dependent Gramians (rather than parameter-invariant ones) with the application of the FBSP gives an improved result and reduces the conservatism further.

1.4 Thesis Overview

The thesis is scheduled as follows.

Chapter 2: Spatially Interconnected Systems Formal Framework

This chapter introduces all preliminaries and definitions that are used through out the thesis. The spatially interconnected systems framework is defined together with a reduced version (which needs to have the same structure as the original system) for both parameter-invariant and varying systems. In addition, the chapter discusses

[2]The generalization of Hankel singular values [49].

the relationship between controller design and model reduction problem for SIS, where the machinery of controller design is utilized to generate the reduced models. Conditions for well-posedness, exponential stability and quadratic performance are discussed for the case of spatially interconnected systems rather than lumped systems.

Finally, the goal of the thesis is presented in this introductory chapter as well.

Chapter 3: Model Order Reduction for LTSI Systems

After presenting preliminaries, this chapter gives a technique for model order reduction of LTSI systems in two steps via applying the two-stage approach described above. Stability as well as the spatial interconnection structure are preserved in the reduced models while applying the reduction problem. In addition to the presentation of the exponential stability LMI condition, balanced truncation is extended such that it is applicable to LTSI systems which are non-causal with respect to space. Minimal realization construction based on the reachability and observability matrices is reviewed as well. The application to an actuated beam shows the efficiency of the results.

Chapter 4: Model Order Reduction for LTSV Systems

An extension of the result of the previous chapter and its application to the actuated beam, Fig. 1.2, to the case of parameter-varying (temporal- and spatial-LPV) interconnected systems is discussed in this chapter, where the considered systems are parameter-varying spatially interconnected systems represented as a spatial interconnection of LTSI (identical) subsystems each connected to a local uncertainty block in Linear Fractional Representation (LPV/LFT) form. A reduced state order system is generated and defined as a spatial interconnection of reduced subsystems; these reduced order subsystems are connected (in LFT form) to the same local uncertainties as for the original subsystems (i.e. the same number of scheduling parameters).

In contrast to results for 1D (lumped) systems in e.g., [34] and [30] the work here is based on the application of the full block S-procedure which reduces the complexity of the analysis problem.

Chapter 5: Joint Dynamic and Scheduling Order Reduction

In Chapter 4, reduced state order systems for temporal- and spatial-LPV systems are constructed such that the reduced subsystems are connected to the same uncertainty blocks as the original ones.

As already discussed, the complexity of LPV systems depends on both state and scheduling order. Therefore, this chapter extends the result of the previous chapter by proposing a method for reducing both state order (the number of states) and scheduling order (the number of scheduling parameters) of parameter-dependent spatially interconnected systems, such that the resulting reduced subsystems are connected to reduced uncertainty blocks. This allows a further simplification of analysis and synthesis problems.

The result of this chapter is based on a balanced truncation technique which is extended from LTSI systems (Chapter 3) to LTSV systems in Chapter 4. Here, it is extended further to reduce the number of scheduling parameters as well. The technique is proposed via transforming and truncating the LFT multipliers as well as the generalized Gramians.

A comparison between constant (parameter-independent) generalized Gramians and parameter-dependent generalized Gramians is given as well. An improved (and less conservative) result is obtained with the latter case. Worth mentioning that the reduction procedure with parameter-dependent generalized Gramians is done by reducing the scheduling order first, and then the state order of the system. While in the case of constant generalized Gramians, both the number of states as well as of scheduling parameters are simultaneously reduced. The result is again based on the application of the FBSP.

Chapter 6: Conclusion

Finally, Chapter 6 concludes the thesis and gives an outlook on potential future work.

Chapter 2

Spatially Interconnected Systems Formal Framework

This chapter reviews all basic definitions and system representations that are used in this thesis. More precisely, the structure of the LTSI, LTSV and LPV/LFT systems are stated with their reduced model architectures. Signal and induced norms, respectively, which are extended in [5] to multidimensional systems are defined in this chapter. In addition, some specific bounded operators, including shift operators, which are considered through out the work are defined as well. The controller structure for spatially interconnected systems is reviewed together with the relationship between the controller synthesis problem and the model order reduction problem.

2.1 Mathematical Preliminaries

The signal and system norms which are defined for one-dimensional (1D), lumped-parameter systems are extended in [5] to multidimensional (MD), distributed-parameter system and to *spatially interconnected systems*. The work in this thesis considers the case of single-spatial dimension interconnected systems, such that we focus on two-dimensional (spatio-temporal) signals, e.g., $x(t, s)$ with temporal variable $t \in \mathbb{Z}_+$ and spatial variable $s \in \mathbb{Z}$; both defined in discrete domain.

We define the extension of 1-D signal spaces as follows.

Definition 2.1 *($\mathcal{L}_2$ space, [5])*
The space $\mathcal{L}_2$ is the set of spatio-temporal functions $x : \mathbb{Z}_+ \times \mathbb{Z} \to \mathbb{R}^n$ that satisfy

$$\sum_{t=0}^{\infty} \sum_{s=-\infty}^{\infty} x(t, s)^T x(t, s) < \infty, \tag{2.1}$$

That is, for each fixed value t; $x(t, s)$ is square summable over s, (as provided in Definition 2.2 below), then relaxing the time such that the double sum is considered over t and s, defines the space $\mathcal{L}_2$.

The square root of the left-hand side of (2.1) is the corresponding $\mathcal{L}_2$-norm, and is denoted by $\|x(t,s)\|_2$.

Definition 2.2 *(l_2 space, [5])*
The space l_2 is the set of square summable functions that, ***for fixed value*** *t, satisfy*

$$\sum_{s=-\infty}^{\infty} x(t,s)^T x(t,s) < \infty, \tag{2.2}$$

over s.

The square root of the left-hand side of (2.2) is the corresponding l_2-norm, and is denoted by $\|x(t,s)\|_{l_2}$.

Definition 2.3 *(Induced Norm, [5])*
The induced 2-norm (denoted as $\|\bullet\|_{2\to 2}$) of an operator F on $\mathcal{L}_2$ is defined as

$$\|F\|_{2\to 2} = \sup_{0\neq x\in L_2} \frac{\|Fx\|_2}{\|x\|_2}. \tag{2.3}$$

If this norm is less than 1, then F is said to be contractive.

Next, we give the definition of the shift operators. The use of shift operators facilitates the representation of spatially interconnected systems presented in Section 2.2. The details will be discussed in the next section.

Definition 2.4 *(Shift Operator, [5])*
The forward and backward temporal shift operators are defined by

$$(T^+x)(t,s) = x(t+1,s), \quad (T^-x)(t,s) = x(t-1,s).$$

The forward and backward spatial shift operators are defined by

$$(S^+x)(t,s) = x(t,s+1), \quad \text{and} \quad (S^-x)(t,s) = x(t,s-1),$$

respectively.

Clearly, the inverse of the spatial forward shift operator is the spatial backward shift one, and vice versa.

Linear Fractional Transformation (LFT), [65], [66]

The upper linear fractional transformation (upper LFT) with respect to Δ, is defined as

$$F_u(M,\Delta) = M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} + M_{22} \tag{2.4}$$

we represent it as

$$F_u(M,\Delta) = \Delta \star M$$

for a matrix M which is partitioned as in Fig. 2.1, and Δ with compatible dimensions

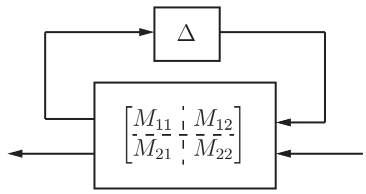

Figure 2.1: Upper linear fractional transformation form.

2.2 Spatially Interconnected Systems

As mentioned in Chapter 1, spatially interconnected systems are comprised of interconnected subsystems. The exchange of information between subsystems is represented via spatial shift operators. The advantage of modelling distributed systems in the spatially interconnected systems framework is that one can do analysis and control synthesis on a single subsystem instead of a whole large and complex system, see [5]. Neighbouring subsystems are interconnected via interconnection signals; these interconnection signals are interpreted as spatial state variables of each subsystem, using spatial shift operators.

In the following, we present in details the classification of the spatially interconnected systems that are mentioned briefly in Chapter 1.

2.2.1 Temporal- and Spatial-Invariant (LTSI) System

If the system is invariant with respect to temporal and spatial variables such that the system has identical subsystems, then it is referred to *time- and space-invariant (or, parameter-invariant) system.* As mentioned before, here we consider only a single spatial dimension; such that the system is defined as an interconnection of subsystems in a string of one spatial dimension as in Fig. 2.2, where G is defined in (2.5) later and the interconnection signals v_+, w_+, v_- and w_- are defined using the forward/backward spatial shift operators, see Definition 2.4.

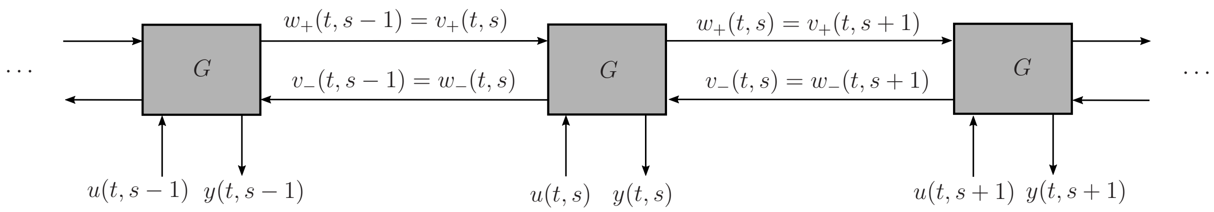

Figure 2.2: Part of a spatially interconnected model (LTSI).

For such distributed systems, we represent each subsystem as a state-space model with both temporal and spatial variables. Let $x(t, s) \in \mathbb{R}^{n_1}$ be the temporal states, and let $v_+(t, s), w_+(t, s) \in \mathbb{R}^{n_2}$; $v_-(t, s), w_-(t, s) \in \mathbb{R}^{n_3}$, be the spatial states in positive and negative directions, respectively; $u(t, s) \in \mathbb{R}^{n_u}$ and $y(t, s) \in \mathbb{R}^{n_y}$ represent external input and measured output, respectively. Then, we define a state-space model of a single subsystem G as (see Fig. 2.2)

$$\begin{bmatrix} x(t+1,s) \\ w_+(t,s) \\ w_-(t,s) \\ \hdashline y(t,s) \end{bmatrix} = \left[\begin{array}{ccc:c} A^{tt} & A^{ts}_+ & A^{ts}_- & B^t \\ A^{st}_+ & A^{ss}_{++} & A^{ss}_{+-} & B^s_+ \\ A^{st}_- & A^{ss}_{-+} & A^{ss}_{--} & B^s_- \\ \hdashline C^t & C^s_+ & C^s_- & D \end{array}\right] \begin{bmatrix} x(t,s) \\ v_+(t,s) \\ v_-(t,s) \\ \hdashline u(t,s) \end{bmatrix}. \tag{2.5}$$

Denote the system matrix in (2.5) by

$$M = \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] \in \mathbb{R}^{(n+n_y)\times(n+n_u)}$$

where $A \in R^{n\times n}$, $B \in R^{n\times n_u}$ and $C \in R^{n_y\times n}$ are partitioned according to temporal and forward/backward spatial parts, respectively; we refer to $n = (n_1 + n_2 + n_3)$ as the order of M.

Define the state vector

$$\xi(t,s) = \begin{bmatrix} x^T(t,s) & v^T_+(t,s) & v^T_-(t,s) \end{bmatrix}^T,$$

and the operator Δ_d as a block diagonal structure

$$\Delta_d = \begin{bmatrix} T^- I_{n_1} & 0 & 0 \\ 0 & S^- I_{n_2} & 0 \\ 0 & 0 & S^+ I_{n_3} \end{bmatrix} \tag{2.6}$$

where I_{n_i} is the $n_i \times n_i$ identity matrix, $i = 1, 2, 3$.

Then, Fig. 2.3 shows system (2.5) represented in LFT form, $G = F_u(M, \Delta_d)$, where $\check{\xi}(t,s) = {\Delta_d}^{-1}\, \xi(t,s)$. So, we have

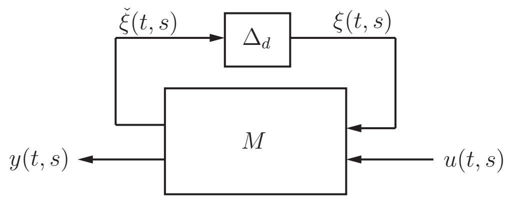

Figure 2.3: Upper LFT form of (2.5).

$$\begin{aligned} \check{\xi}(t,s) &= A\, \xi(t,s) + B\, u(t,s), \\ y(t,s) &= C\, \xi(t,s) + D\, u(t,s). \end{aligned} \tag{2.7}$$

and

$$G = \Delta_d \star M = C\Delta_d(I - A\Delta_d)^{-1}B + D. \tag{2.8}$$

2.2.2 Temporal- and Spatial-Varying (LTSV) System

In the previous section, we consider the conservative case of temporal- and spatial-invariant systems. In more realistic situations, due to variation in physical properties, the dynamic of the system can be varying with respect to temporal and spatial variables (the latter implies that the system has non-identical subsystems). If the variation of the system properties with respect to temporal and spatial variables can be expressed in terms of suitable scheduling parameters, then we can represent the system as *temporal- and spatial-LPV system*, which is an extension of the concept of 1-D LPV systems [9]. Here, we extend the system representation of Section 2.2.1 to a spatially interconnected system which is varying with respect to temporal variable (time $t \in \mathbb{Z}_+$) and spatial variable (space $s \in \mathbb{Z}$) via scheduling parameters $\delta(t), \rho(s)$ defined as functions of time and space, respectively[1]. The vector dimension of the temporal scheduling parameter $\delta(t)$ is n_δ, whereas the vector dimension of the spatial scheduling parameter $\rho(s)$ is n_ρ. The variation ranges and rates of these scheduling parameters are bounded to the compact sets $\mathcal{F}_\delta$ and $\mathcal{F}_\rho$, respectively.

Then, each subsystem G in (2.5) is extended now to

$$\begin{bmatrix} x(t+1,s) \\ w_+(t,s) \\ w_-(t,s) \\ \hdashline y(t,s) \end{bmatrix} = M(\delta_t, \rho_s) \begin{bmatrix} x(t,s) \\ v_+(t,s) \\ v_-(t,s) \\ \hdashline u(t,s) \end{bmatrix} \tag{2.9}$$

where

$$M(\delta_t,\rho_s) = \left[\begin{array}{c:c} A(\delta_t,\rho_s) & B(\delta_t,\rho_s) \\ \hdashline C(\delta_t,\rho_s) & D(\delta_t,\rho_s) \end{array}\right] = \left[\begin{array}{ccc:c} A^{tt}(\delta_t,\rho_s) & A^{ts}_+(\delta_t,\rho_s) & A^{ts}_-(\delta_t,\rho_s) & B^t(\delta_t,\rho_s) \\ A^{st}_+(\delta_t,\rho_s) & A^{ss}_{++}(\delta_t,\rho_s) & A^{ss}_{+-}(\delta_t,\rho_s) & B^s_+(\delta_t,\rho_s) \\ A^{st}_-(\delta_t,\rho_s) & A^{ss}_{-+}(\delta_t,\rho_s) & A^{ss}_{--}(\delta_t,\rho_s) & B^s_-(\delta_t,\rho_s) \\ \hdashline C^t(\delta_t,\rho_s) & C^s_+(\delta_t,\rho_s) & C^s_-(\delta_t,\rho_s) & D(\delta_t,\rho_s) \end{array}\right]$$

where, $A(\delta_t,\rho_s) : \mathbb{R}^{n_\delta} \times \mathbb{R}^{n_\rho} \to \mathbb{R}^{n\times n}$; $B(\delta_t,\rho_s) : \mathbb{R}^{n_\delta} \times \mathbb{R}^{n_\rho} \to \mathbb{R}^{n\times n_u}$; $C(\delta_t,\rho_s) : \mathbb{R}^{n_\delta} \times \mathbb{R}^{n_\rho} \to \mathbb{R}^{n_y\times n}$ and $D(\delta_t,\rho_s) : \mathbb{R}^{n_\delta} \times \mathbb{R}^{n_\rho} \to \mathbb{R}^{n_y\times n_u}$.

In this thesis, the spatial scheduling parameter $\rho(s)$ is assumed known a priori and fixed, therefore the induced norm, as defined in (2.3), is extended to LTSV systems by defining it as

$$\left(\underset{\delta(t)\in F_\delta}{sup} \; \underset{0\neq u\in L_2}{sup} \frac{\|y(t,s)\|_2}{\|u(t,s)\|_2} \right). \tag{2.10}$$

Subsystem (2.9) has a state-space model varying with respect to time and space via scheduling parameters. Keeping these scheduling parameters inside system matrices could render some difficulties through analysis. In this section, we redefine the system in a more convenient way by pulling out (from each subsystem) the scheduling parameters into two

[1]For simplicity of presentation, sometimes we use δ_t and ρ_s instead of $\delta(t)$ and $\rho(s)$.

(local) uncertainty blocks, with varying parameters $\Delta^t(t) \in \Theta^t$ and varying parameters[2] $\Delta^s(s) \in \Theta^s$, after scaling the variation ranges to [-1,1], such that

$$\begin{aligned} \Theta^t &= \{\Delta^t : \text{diag}\left(\delta_1(t) I_{n_1^t}, \cdots, \delta_{n_\delta}(t) I_{n^t_{n_\delta}}\right), |\delta_k(t)| \leq 1, k = 1, \cdots, n_\delta\} \\ \Theta^s &= \{\Delta^s : \text{diag}\left(\rho_1(s) I_{n_1^s}, \cdots, \rho_{n_\rho}(s) I_{n^s_{n_\rho}}\right), |\rho_k(s)| \leq 1, k = 1, \cdots, n_\rho\}. \end{aligned} \tag{2.11}$$

where n_k^t and n_k^s denote the multiplicity of scheduling parameters $\delta_k(t)$ and $\rho_k(s)$, respectively, and $n^t = \sum_{k=1}^{n_\delta} n_k^t$ and $n^s = \sum_{k=1}^{n_\rho} n_k^s$.
This leaves a "nominal" LTSI system G^0, shown in Fig. 2.4 (for a single subsystem) where p^t and $q^t \in \mathbb{R}^{n^t}$, p^s and $q^s \in \mathbb{R}^{n^s}$ are the uncertainty input/output channels to the nominal system $G^0 \in \mathbb{R}^{(n+n^t+n^s+n_y)\times(n+n^t+n^s+n_u)}$.

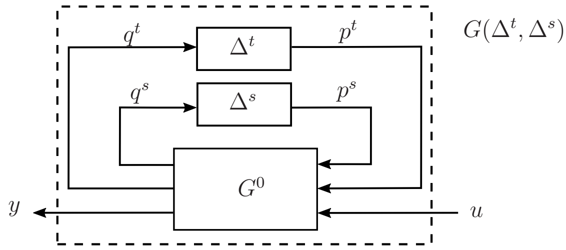

Figure 2.4: A single subsystem $G(\Delta^t, \Delta^s)$ defined in LPV/LFT form.

Fig. 2.4 shows that each subsystem is represented as a nominal system G^0 connected with its local temporal- and spatial-varying uncertainty blocks in LFT form, [12], [67], [8]. Note that G^0 is identical for all subsystems. Accordingly, the system in Fig. 2.2 of Section 2.2.1 is extended now to Fig 2.5.

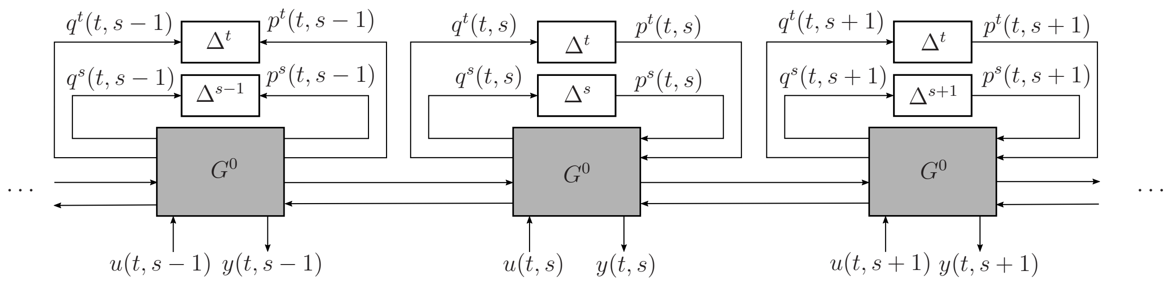

Figure 2.5: Part of LTSV model represented in LPV/LFT form.

Temporal- and spatial-LPV interconnected system in LFT representation The LFT rep-

[2]Time- and space-dependence of scheduling parameters are dropped sometimes for the sake of brevity, e.g. $\Delta^t = \Delta^t(t)$ and $\Delta_s = \Delta^s(s)$.

resentation of the subsystem $G(\Delta^t, \Delta^s)$ in Fig. 2.4 takes the form [12, 68],

$$G(\Delta^t,\Delta^s):\left\{\begin{array}{l}
\left[\begin{array}{c}\check{\xi}(t,s)\\ \hline q^t(t,s)\\ q^s(t,s)\\ \hline y(t,s)\end{array}\right]=\left[\begin{array}{ccc|cc|c}
A^{tt} & A^{ts}_{+} & A^{ts}_{-} & B^{0,tt} & B^{0,ts} & B^{1,t}\\
A^{st}_{+} & A^{ss}_{++} & A^{ss}_{+-} & B^{0,st}_{+} & B^{0,ss}_{+} & B^{1,s}_{+}\\
A^{st}_{-} & A^{ss}_{-+} & A^{ss}_{--} & B^{0,st}_{-} & B^{0,ss}_{-} & B^{1,s}_{-}\\ \hline
C^{0,tt} & C^{0,ts}_{+} & C^{0,ts}_{-} & D^{00,t} & 0 & D^{01,t}\\
C^{0,st} & C^{0,ss}_{+} & C^{0,ss}_{-} & 0 & D^{00,s} & D^{01,s}\\ \hline
C^{1,t} & C^{1,s}_{+} & C^{1,s}_{-} & D^{10,t} & D^{10,s} & D^{11}
\end{array}\right]\left[\begin{array}{c}\xi(t,s)\\ \hline p^t(t,s)\\ p^s(t,s)\\ \hline u(t,s)\end{array}\right]\\
\qquad:=\left[\begin{array}{ccc} A & B^0 & B^1\\ C^0 & D^{00} & D^{01}\\ C^1 & D^{10} & D^{11}\end{array}\right]\left[\begin{array}{c}\xi(t,s)\\ \hline p^t(t,s)\\ p^s(t,s)\\ \hline u(t,s)\end{array}\right];\\
\left[\begin{array}{c} p^t(t,s)\\ p^s(t,s)\end{array}\right]=\left[\begin{array}{cc}\Delta^t & \\ & \Delta^s\end{array}\right]\left[\begin{array}{c} q^t(t,s)\\ q^s(t,s)\end{array}\right],\ \Delta_t\in\Theta^t,\Delta_s\in\Theta_s,
\end{array}\right. \tag{2.12}$$

where $\xi(t,s)\in\mathbb{R}^n$, $\quad p^t(t,s), q^t(t,s)\in\mathbb{R}^{n^t}\quad$ and $\quad p^s(t,s), q^s(t,s)\in\mathbb{R}^{n^s}$.

An explicit representation of a temporal- and spatial-LPV system (as upper LFT, (2.4)) can be written as

$$G(\Delta^t,\Delta^s)=F_u\left(G^0,\left[\begin{array}{cc}\Delta^t & \\ & \Delta^s\end{array}\right]\right)=\left[\begin{array}{c|c} A(\Delta^t,\Delta^s) & B(\Delta^t,\Delta^s)\\ \hline C(\Delta^t,\Delta^s) & D(\Delta^t,\Delta^s)\end{array}\right],\Delta^t\in\Theta^t,\Delta^s\in\Theta^s \tag{2.13}$$

where

$$\left[\begin{array}{cc} A(\Delta^t,\Delta^s) & B(\Delta^t,\Delta^s)\\ C(\Delta^t,\Delta^s) & D(\Delta^t,\Delta^s)\end{array}\right]=\left[\begin{array}{cc} A+B^0\,\Phi\,C^0 & B^1+B^0\,\Phi\,D^{01}\\ C^1+D^{10}\,\Phi\,C^0 & D^{11}+D^{10}\,\Phi\,D^{01}\end{array}\right], \tag{2.14}$$

$$\text{and}\quad \Phi=\left[\begin{array}{cc}\Delta^t & \\ & \Delta^s\end{array}\right]\left(I-D^{00}\left[\begin{array}{cc}\Delta^t & \\ & \Delta^s\end{array}\right]\right)^{-1}.$$

Remark 2.1 *In general, model reduction techniques for LPV systems without pulling out the scheduling parameters into an uncertainty block leads to difficulties when solving the reduction problem, since it requires infinitely many conditions to be solved unless it is based on defining a grid on the admissible parameter range. Whereas, for the case of LPV/LFT systems, there are ways to avoid this, e.g. the full block S-procedure [69] with D-G scales [70]. This is one of the motivations for considering the LPV/LFT system representations in this thesis.*

2.3 Reduced Model Representations

In order to construct a reduced model for a spatially interconnected system without losing the structure of the original system, the spatial interconnection structure must be preserved while applying the reduction.

The framework proposed in [5] allows to do analysis and control synthesis on a single subsystem instead of considering the whole system. Therefore, the reduction problem will be considered for a single subsystem.

In the following, we will present reduced model forms for the two classes of system representation given in Section 2.2; LTSI and LTSV system, respectively.

Note that the reduced version of each system representation, retains the system class, e.g., the reduced LTSI system is an LTSI system as well.

2.3.1 Reduced Order LTSI System

The reduced version of an LTSI system is an LTSI system that behaves similar to the original model and has the same properties but with reduced complexity. The reduced model is represented in Fig. 2.6 (the reduced version of Fig. 2.2).

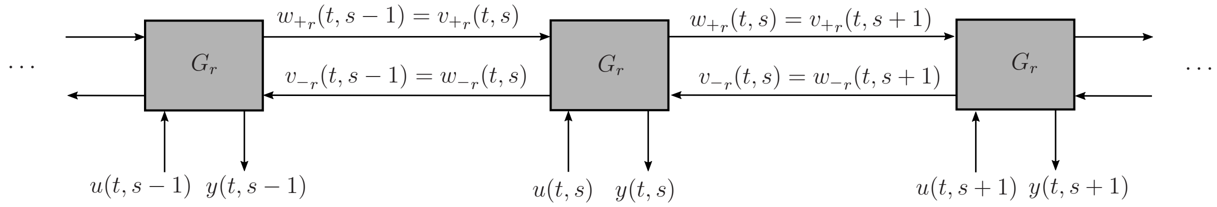

Figure 2.6: Reduced order LTSI system.

A state-space representation for each reduced subsystem G^r is defined as

$$\begin{bmatrix} \check{\xi}_r(t,s) \\ \hdashline y(t,s) \end{bmatrix} = \left[\begin{array}{ccc:c} A_r^{tt} & A_{+r}^{ts} & A_{-r}^{ts} & B_r^t \\ A_{+r}^{st} & A_{++r}^{ss} & A_{+-r}^{ss} & B_{+r}^s \\ A_{-r}^{st} & A_{-+r}^{ss} & A_{--r}^{ss} & B_{-r}^s \\ \hdashline C_t^r & C_{+r}^s & C_{-r}^s & D \end{array}\right] \begin{bmatrix} \xi_r(t,s) \\ \hdashline u(t,s) \end{bmatrix}, \tag{2.15}$$

where $\check{\xi}_r(t,s) = \Delta_r^{-1}\xi_r(t,s)$ such that for a reduced order $n_r = n_{r_1} + n_{r_2} + n_{r_3} < n$, $\xi_r(t,s) \in \mathbb{R}^{n_r}$ and Δ_r is the reduced version of Δ_d defined in (2.6), where Δ_r is defined as

$$\Delta_r = \begin{bmatrix} T^- I_{n_{r_1}} & 0 & 0 \\ 0 & S^- I_{n_{r_2}} & 0 \\ 0 & 0 & S^+ I_{n_{r_3}} \end{bmatrix}. \tag{2.16}$$

Denote the system matrix in (2.15) by

$$M_r = \left[\begin{array}{c:c} A_r & B_r \\ \hdashline C_r & D \end{array}\right] \in \mathbb{R}^{(n_r+n_y)\times(n_r+n_u)}.$$

Then the reduced subsystem (which is the reduced version of (2.8)) is given as

$$G_r = \Delta_r \star M_r. \tag{2.17}$$

2.3.2 Reduced Complexity LTSV System defined as LPV/LFT Form

As is clear from the reduced model representation (2.15) the reduced model has a reduced number of states n_r. In this section, complexity reduction is considered in two dimensions.

For the case of an LTSV system defined in Section 2.2.2, each subsystem G^0 is interconnected with its local uncertainty blocks $\Delta^t \in \Theta^t$ and $\Delta^s \in \Theta^s$ governing the subsystem $G(\Delta^t, \Delta^s)$ as in Fig. 2.5. The reduced version of such an LTSV system is an LTSV system itself, but with reduced complexity. The complexity of these systems can be seen in two dimensions: first the complexity of the state order of the system n, second the complexity of the temporal and spatial scheduling orders (which are the dimensions of the uncertainty channels p^t, q^t, p^s and q^s), i.e., n^t and n^s, see (2.12). This thesis will discuss both cases in details.

Reduced State Order LTSV Systems

The state order is reduced by reducing the number of states (i.e., n) of the system, such that $n_r = n_{r_1} + n_{r_2} + n_{r_3} < n$. The reduced order system is shown in Fig. 2.7, where each reduced nominal subsystem G_r^0 is connected with the same local uncertainty blocks ($\Delta^t \in \Theta^t$ and $\Delta^s \in \Theta^s$, respectively) as for the original system.

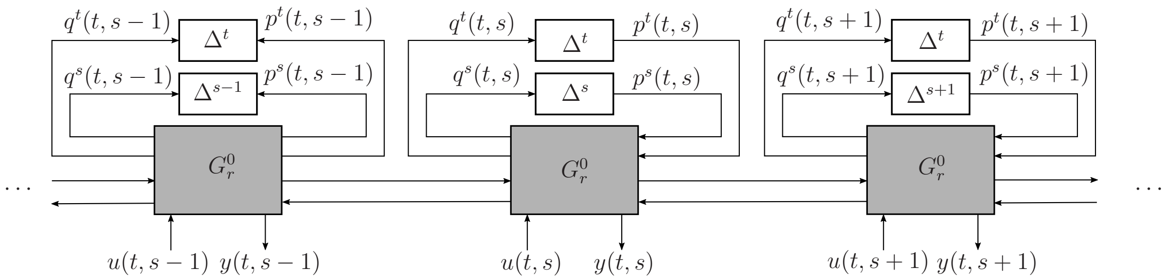

Figure 2.7: Reduced state order.

The LFT representation of each single subsystem $G_r(\Delta^t, \Delta^s)$ is defined as

$$
G_r(\Delta^t, \Delta^s) : \left\{
\begin{array}{l}
\begin{bmatrix} \check{\xi}_r(t,s) \\ \hline q^t(t,s) \\ q^s(t,s) \\ \hline y(t,s) \end{bmatrix} = \left[\begin{array}{c|c|c} A_r & B_r^0 & B_r^1 \\ \hline C_r^0 & D_{00} & D_{01} \\ \hline C_r^0 & D_{10} & D_{11} \end{array}\right] \begin{bmatrix} \xi_r(t,s) \\ \hline p^t(t,s) \\ p^s(t,s) \\ \hline d(t,s) \end{bmatrix}; \\[2em]
\begin{bmatrix} p^t(t,s) \\ p^s(t,s) \end{bmatrix} = \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \begin{bmatrix} q^t(t,s) \\ q^s(t,s) \end{bmatrix}, \ \Delta^t \in \Theta^t, \Delta^s \in \Theta^s,
\end{array}
\right.
\tag{2.18}
$$

where $\xi^r(t,s) \in \mathbb{R}^{n_r}$, $\quad p^t(t,s), q^t(t,s) \in \mathbb{R}^{n^t}$ and $\quad p^s(t,s), q^s(t,s) \in \mathbb{R}^{n^s}$.

Such that

$$
G_r(\Delta^t,\Delta^s) = F_u\left(G_r^0, \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\right) = \left[\begin{array}{c:c} A_r(\Delta^t,\Delta^s) & B_r(\Delta^t,\Delta^s) \\ \hdashline C_r(\Delta^t,\Delta^s) & D(\Delta^t,\Delta^s) \end{array}\right],\ \Delta_t \in \Theta_t, \Delta_s \in \Theta_s, \tag{2.19}
$$

$$
\text{where} \quad \left[\begin{array}{c:c} A_r(\Delta^t,\Delta^s) & B_r(\Delta^t,\Delta^s) \\ \hdashline C_r(\Delta^t,\Delta^s) & D(\Delta^t,\Delta^s) \end{array}\right] = \begin{bmatrix} A_r + B_r^0\,\Phi\,C_r^0 & B_r^1 + B_r^0\,\Phi\,D_{01} \\ C_r^1 + D_{10}\,\Phi\,C_r^0 & D_{11} + D_{10}\,\Phi\,D_{01} \end{bmatrix},
$$

with the same Φ as defined in (2.14).

Joint Reduced State and Scheduling Order

Clearly from (2.18) and Fig. 2.7, the system complexity is dependent on the number of scheduling parameters (the dimensions n^t and n^s of the uncertainty channels) as well as on the number of states of the system. That is, complexity is also reduced by reducing the dimensions of the uncertainty blocks $\Delta^t \in \Theta^t$ and $\Delta^s \in \Theta^s$.

Here we consider the joint reduction of state and scheduling order as shown in Fig. 2.8,

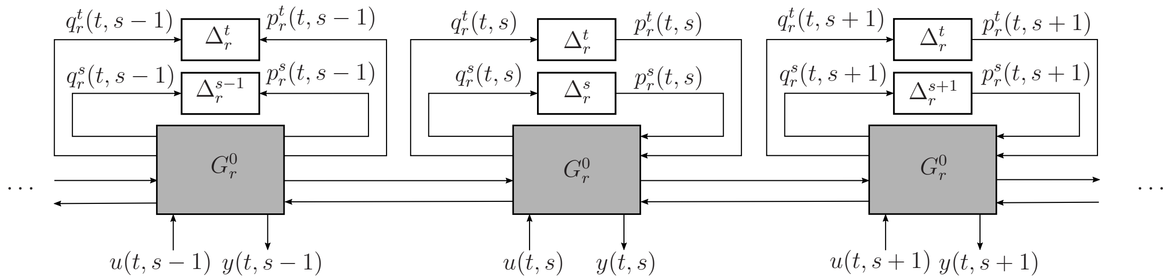

Figure 2.8: joint state and scheduling order reduction.

such that for a reduced state order $n_r = n_{r_1} + n_{r_2} + n_{r_r} < n$ (as in (2.18)) and reduced scheduling orders $n_r^t < n^t$, $n_r^s < n^s$, respectively, we define the joint state and scheduling order reduced system as

$$
G_r(\Delta_r^t,\Delta_r^s): \begin{cases} \left[\begin{array}{c} \check{\xi}_r(t,s) \\ \hdashline q_r^t(t,s) \\ q_r^s(t,s) \\ \hdashline y(t,s) \end{array}\right] := \left[\begin{array}{c:c:c} A_r & B_r^0 & B_r^1 \\ \hdashline C_r^0 & D_r^{00} & D_r^{01} \\ \hdashline C_r^1 & D_r^{10} & D_{11} \end{array}\right] \left[\begin{array}{c} \xi_r(t,s) \\ \hdashline p_r^t(t,s) \\ p_r^s(t,s) \\ \hdashline d(t,s) \end{array}\right]; \\ \begin{bmatrix} p_r^t(t,s) \\ p_r^s(t,s) \end{bmatrix} = \begin{bmatrix} \Delta_r^t & \\ & \Delta_r^s \end{bmatrix} \begin{bmatrix} q_r^t(t,s) \\ q_r^s(t,s) \end{bmatrix},\ \Delta_r^t \in \Theta_r^t \subset \Theta^t, \Delta_r^s \in \Theta_r^s \subset \Theta^s, \end{cases} \tag{2.20}
$$

where $\xi^r(t,s) \in \mathbb{R}^{n_r}$, $\quad p_r^t(t,s), q_r^t(t,s) \in \mathbb{R}^{n_r^t}$ and $p_r^s(t,s), q_r^s(t,s) \in \mathbb{R}^{n_r^s}$,

such that

$$G_r(\Delta_r^t, \Delta_r^s) =$$
$$F_u\left(G_r^0, \begin{bmatrix} \Delta_r^t & \\ & \Delta_r^s \end{bmatrix}\right) = \left[\begin{array}{c|c} A_r(\Delta_r^t, \Delta_r^s) & B_r(\Delta_r^t, \Delta_r^s) \\ \hline C_r(\Delta_r^t, \Delta_r^s) & D_r(\Delta_r^t, \Delta_r^s) \end{array}\right], \Delta_r^t \in \Theta_r^t \subset \Theta^t,\ \Delta_r^s \in \Theta_r^s \subset \Theta^s, \tag{2.21}$$

where

$$\left[\begin{array}{c|c} A_r(\Delta_r^t, \Delta_r^s) & B_r(\Delta_r^t, \Delta_r^s) \\ \hline C_r(\Delta_r^t, \Delta_r^s) & D_r(\Delta_r^t, \Delta_r^s) \end{array}\right] = \begin{bmatrix} A_r + B_r^0\,\Phi_r\,C_r^0 & B_r^1 + B_r^0\,\Phi_r\,D_r^{01} \\ C_r^1 + D_r^{10}\,\Phi_r\,C_r^0 & D^{11} + D_r^{10}\,\Phi_r\,D_r^{01} \end{bmatrix},$$
$$\Phi_r = \begin{bmatrix} \Delta_r^t & \\ & \Delta_r^s \end{bmatrix}\left(I - D_r^{00}\begin{bmatrix} \Delta_r^t & \\ & \Delta_r^s \end{bmatrix}\right)^{-1}.$$

Note that in (2.19) the reduced model is represented in LFT form with the same temporal and spatial varying parameters as for the original system, i.e. $\Delta_t \in \Theta_t$ and $\Delta_s \in \Theta_s$ still as they are, while in (2.21) the reduced model is represented in LFT form with respect to $\Delta_r^t \in \Theta_r^t \subset \Theta_t$ and $\Delta_r^s \in \Theta_r^s \subset \Theta_s$.

The accuracy of generating the reduced models (2.15), (2.18) and (2.20) can be measured via the induced 2-norm of the *error system (the difference between the original system and the reduced one)*. We search for a reduced system such that *the induced 2-norm of the error system is minimized.*

Before we discuss the measure of the accuracy in the model reduction problem, we present, in the next section the controller synthesis problem for spatially interconnected systems and its relation to the reduced model generation. Here, we are discussing the case of LTSI systems only. The case of LTSV systems will be discussed in Chapter 4 when model reduction for LTSV systems is considered.

2.4 Relationship Between Error System and Controlled System Configurations

In this section, first we recall the distributed controller construction and the closed loop system configuration for LTSI systems. Then, we clarify the role of the controller construction problem in this context, by recalling the well known relationship between controller synthesis and model order reduction (MOR), see e.g. [57], [71], [1], [72].

Of interest here is the design of a controller for a distributed system that inherits a distributed structure (i.e. distributed controller) [5], [12], [10]. In contrast, a centralized [73], [74] has disadvantages, because it has to be designed for a complex multi-input/multi-output (MIMO) systems with many input/output channels and large system order, see [8] for a full discussion.

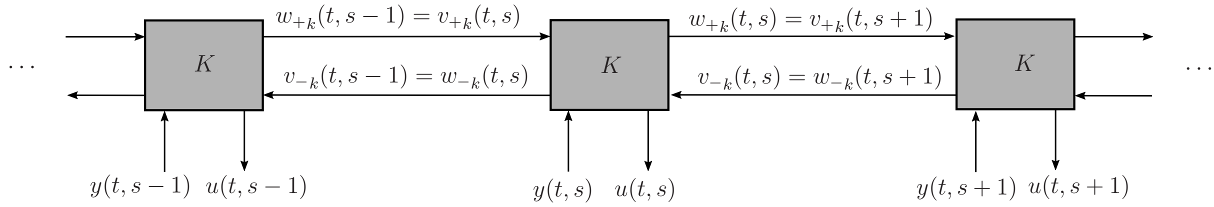

Figure 2.9: Part of an LTSI distributed controller.

Here we consider a distributed controller, where the controller is defined as a spatial interconnection of subsystems, as in Fig. 2.9 , [5]. The LTSI distributed controller itself is an LTSI system, each subsystem K in Fig. 2.9 is defined as

$$\check{\xi}_k(t,s)\left\{\begin{bmatrix} x_k(t+1,s) \\ w_{+k}(t,s) \\ w_{-k}(t,s) \\ \hline u(t,s) \end{bmatrix}\right. = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \left.\begin{bmatrix} x_k(t,s) \\ v_{+k}(t,s) \\ v_{-k}(t,s) \\ \hline y(t,s) \end{bmatrix}\right\}\xi_k(t,s) \tag{2.22}$$

where

$$M_k = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] = \left[\begin{array}{ccc|c} A_k^{tt} & A_{+k}^{ts} & A_{-k}^{ts} & B_k^t \\ A_{+k}^{st} & A_{++k}^{ss} & A_{+-k}^{ss} & B_{+k}^s \\ A_{-k}^{st} & A_{-+k}^{ss} & A_{--k}^{ss} & B_{-k}^s \\ \hline C_k^t & C_{+k}^s & C_{-k}^s & D_k \end{array}\right].$$

For the controller K, we represent its order by $c = c_1 + c_2 + c_3$, such that $\xi_k(t,s) \in \mathbb{R}^c$. Regarding the distributed controller design and the corresponding closed loop system configuration, we will build on the results of [8] and [5].

For

$$\Delta_k = \begin{bmatrix} T^- I_{c_1} & & \\ & S^- I_{c_2} & \\ & & S^+ I_{c_3} \end{bmatrix},$$

we define

$$K = \Delta_k \star M_k. \tag{2.23}$$

The resulting closed-loop system representation is depicted in Fig. 2.10, where G is the plant and K is the controller; d and z represent the performance channels [8], [5].

Fig.2.11 shows the closed-loop configuration for each subsystem,

A state-space realization of the closed-loop (controlled) subsystem shown in Fig. 2.11 is defined as

$$\check{\xi}_{cl}(t,s)\left\{\begin{bmatrix} x_{cl}(t+1,s) \\ w_{+cl}(t,s) \\ w_{-cl}(t,s) \\ \hline d(t,s) \end{bmatrix}\right. = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] \left.\begin{bmatrix} x_{cl}(t,s) \\ v_{+cl}(t,s) \\ v_{-cl}(t,s) \\ \hline z(t,s) \end{bmatrix}\right\}\xi_{cl}(t,s). \tag{2.24}$$

It can be represented as

$$G_{cl} = \Delta_{cl} \star M_{cl}, \tag{2.25}$$

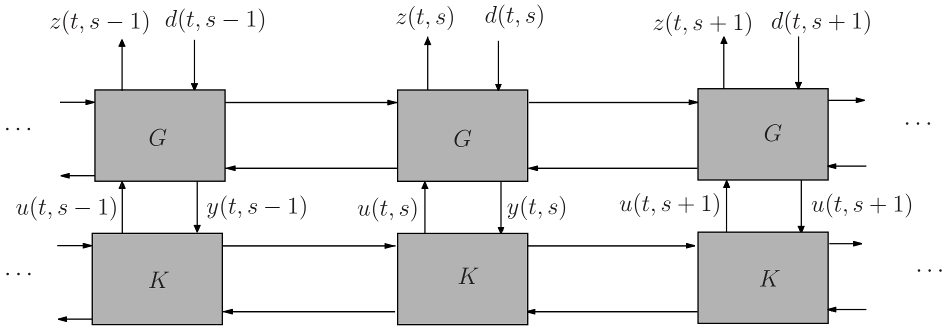

Figure 2.10: Closed-loop configuration.

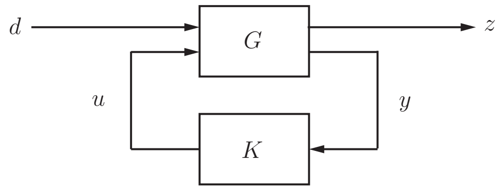

Figure 2.11: Closed loop configuration for a single subsystem.

where a permutation has been applied to (2.24), (2.25) in order to group the temporal and spatial variables of system and controller together, such that

$$M_{cl} = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] = \left[\begin{array}{ccc|c} A^{tt}_{cl} & A^{ts}_{+cl} & A^{ts}_{-cl} & B^{t}_{cl} \\ A^{st}_{+cl} & A^{ss}_{++cl} & A^{ss}_{+-cl} & B^{s}_{+cl} \\ A^{st}_{-cl} & A^{ss}_{-+cl} & A^{ss}_{--cl} & B^{s}_{-cl} \\ \hline C^{t}_{cl} & C^{s}_{+cl} & C^{s}_{-cl} & D^{cl} \end{array}\right]$$

and

$$\Delta_{cl} = \begin{bmatrix} T^{-} I_{n_1+c_1} & & \\ & S^{-} I_{n_2+c_2} & \\ & & S^{+} I_{n_3+c_3} \end{bmatrix}.$$

The objective when designing the controller (2.22) is to obtain a closed-loop system (2.24) that is

- exponentially stable
- well-posed,
- and achieves a the performance level $\gamma > 0$ such that

$$\sum_{t=0}^{\infty} \sum_{s=-\infty}^{\infty} \begin{bmatrix} d(t,s) \\ z(t,s) \end{bmatrix}^{T} \begin{bmatrix} -\gamma I & \\ & \gamma^{-1} I \end{bmatrix} \begin{bmatrix} d(t,s) \\ z(t,s) \end{bmatrix} \leq 0. \tag{2.26}$$

These three requirements are the main objectives in system analysis. Their definitions are given next. Furthermore, LMI conditions for the above three requirements will be given in the next chapters when they will be needed.

Exponential stability definition of lumped systems is extended to the spatially interconnected systems as follows.

Definition 2.5 *(Exponential Stability, [12])*
The system (2.24) is exponentially stable if given any initial condition $x_{cl}(0,s)$, the states $x_{cl}(t,s)$ converge to zero exponentially as $t \to \infty$ for all integer s.

While, a system is well-posed if it is physically realizable, [5]; we present the following lemma.

Lemma 2.1 *(Well-posedness, [5])*
The system (2.24) is well-posed if and only if $(\Delta^{ss}_{cl} - A^{ss}_{cl})$ is invertible on l_2, where

$$\Delta^{ss}_{cl} = \begin{bmatrix} S^{+} I_{n_2+c_2} & \\ & S^{-} I_{n_3+c_3} \end{bmatrix} \quad \text{and} \quad A^{ss}_{cl} = \begin{bmatrix} A^{ss}_{++cl} & A^{ss}_{+-cl} \\ A^{ss}_{-+cl} & A^{ss}_{--cl} \end{bmatrix}.$$

The third requirement is the most relevant one in the context of model reduction; it is defined with respect to the induced 2-norm (2.3), as follows.

Definition 2.6 *(Quadratic Performance, [5], [8])*
The system (2.24) is said to have quadratic performance $\gamma > 0$, if γ is an upper bound on the induced 2-norm of the closed-loop configuration in Fig. 2.11, such that for $d \in \mathcal{L}_2$ and $z \in \mathcal{L}_2$, we have

$$\|z\|_{\mathcal{L}_2} < \gamma \, \|d\|_{\mathcal{L}_2}, \tag{2.27}$$

which is equivalent to (2.26).

The closed-loop system is said to be contractive, if γ in (2.27) is less than 1, see Definition 2.3.

A matrix inequality condition for the closed loop system to be exponentially stable, well-posed and to have quadratic performance or γ has been derived in [5] via the application of the bounded real lemma; it will be presented later.

So far, we reviewed the distributed controller construction and the closed-loop system configuration for LTSI systems, with an admissible controller that achieves the three above objectives (well-posedness, exponential stability and performance index).

Next, we discuss the link between the controller synthesis problem and the model order reduction (MOR) problem.

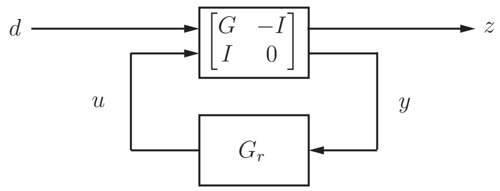

Figure 2.12: Error System Configuration for a single subsystem.

The relationship between the controller synthesis problem and the MOR problem for LTSI systems was pointed out in [57], [56], and [71].

If we augment the subsystem G (2.5) in Fig. 2.11 with specific zero and identity blocks between channels $u \in \mathcal{L}_2$ and $y \in \mathcal{L}_2$, see Fig. 2.12, such that we define the channels $d \in \mathcal{L}_2$ and $z \in \mathcal{L}_2$ to represent the physical inputs and outputs of the system, while the channels $u \in \mathcal{L}_2$ and $y \in \mathcal{L}_2$ have zero and identity blocks, as

$$\begin{bmatrix} \breve{\xi}(t,s) \\ \hline z(t,s) \\ \hline y(t,s) \end{bmatrix} = \left[\begin{array}{c|c|c} A & B & 0 \\ \hline C & D & -I \\ \hline 0 & I & 0 \end{array}\right] \begin{bmatrix} \xi(t,s) \\ \hline d(t,s) \\ \hline u(t,s) \end{bmatrix}. \tag{2.28}$$

Then utilizing the machinery of controller design [5] will render a reduced order model (Fig. 2.12) rather than a controller design (Fig. 2.11). More precisely, if we close the loop in Fig. 2.12 (in a similar way as in Fig. 2.11), then we will have $(G - G_r)$ which is the error system G_e defined as the difference between the original system and the reduced one. The resulting error system G_e for each subsystem in Fig. 2.12 is defined as

$$\begin{bmatrix} \breve{\xi}(t,s) \\ \breve{\xi}_r(t,s) \\ \hline z(t,s) \end{bmatrix} = \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right] \begin{bmatrix} \xi(t,s) \\ \xi_r(t,s) \\ \hline d(t,s) \end{bmatrix} \tag{2.29}$$

$$G_e = \Delta_e \star E \tag{2.30}$$

where

$$E = \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A_r & B_r \\ \hline C & -C_r & D - D_r \end{array}\right] \quad \text{and} \quad \Delta_e = \begin{bmatrix} \Delta_d & \\ & \Delta_r \end{bmatrix},$$

which is the difference between two LFT representations: $G = \Delta_d \star M$ (defined in (2.8)) and $G_r = \Delta_r \star M_r$ (defined in (2.17)), [65].

Similar to the closed-loop system, a permutation matrix (we refer to it by P) should be applied to the error system matrices E and Δ_e as

$$\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} E \begin{bmatrix} P^T & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad P\Delta_e P^T; \tag{2.31}$$

such that the temporal and forward- backward-spatial variables of the error system are preserved, while

$$\|\Delta_e \star E\|_{2\to 2} = \|P\ (\Delta_e \star E)\ P^T\|_{2\to 2}.$$

The performance index for the model reduction problem is the induced 2-norm of the error system. Accordingly, γ defined in (2.26) represents the overall approximation error of the system (see (2.3)) which should be kept small.

Therefore, the goal is:

Given $G = \Delta_d \star M$ with order n, find a reduced model $G_r = \Delta_r \star M_r$ with order $n_r < n$, such that

$$\|(\Delta_d \star M) - (\Delta_r \star M_r)\|_{2\to 2} = \|\Delta_e \star E\|_{2\to 2} \quad \text{is} \quad \text{minimized.} \tag{2.32}$$

In other words, the error system $G_e = \Delta^e * E$ must satisfy the performance index given in (2.26) for a small $\gamma > 0$.

However, it is not possible to directly solve the controller synthesis problem to obtain a reduced model; we need to enforce some constraints on the conditions for the reduction problem. That will be discussed in Chapter 3.

The above discussion will be taken up again when MOR for LTSI systems based on the Bounded Real Lemma [5] is considered in Chapter 3; MOR for LTSV systems is discussed in Chapter 4.

The expression (2.32) represents the main goal of this thesis, i.e., we seek a reduced model such that the induced norm in (2.32) is minimized.

2.5 Summary

Spatially interconnected systems have been defined in this chapter, where two classes of system representations (LTSI, LTSV) have been discussed, together with their corresponding reduced representations. Also, basic definitions and norms which are used through out the thesis have been given in this chapter. In addition to controller construction for spatially interconnected systems such that the closed-loop system satisfies the three objectives (Well-posed, exponential stability and performance index), the controller synthesis problem with its relation to MOR problem has been introduced as well. The main goal of the thesis has been presented with respect to the induced 2-norm of the error system.

Chapter 3

Model Order Reduction for LTSI Systems

3.1 Introduction

This chapter presents a technique for model reduction of LTSI systems. Due to the advantages of a balanced realization, the proposed technique of this chapter is based on balancing the system via a balanced transformation which is constructed using the solutions of a pair of Lyapunov inequalities with specific constraints; these solutions are structured and known as the *generalized Gramians* [49] which are a generalization of the usual Gramians where the corresponding conditions to be solved are Lyapunov equations rather than Lyapunov inequalities. The resulting balanced model is used to initialize a second step in the model reduction problem, which utilizes the machinery of controller design for LTSI systems, as discussed in Chapter 2. The non-causality of the system with respect to space leads to indefinite generalized Gramians; we will take that into account when solving the reduction problem. The proposed technique is applicable to exponentially stable LTSI systems and preserves stability, such that the reduced LTSI system is exponentially stable as well.

This chapter is organized as follows. Section 3.2 starts with some preliminaries and recalls the definition of exponential stability for LTSI systems and an LMI condition, also it defines the representation of the structured generalized Gramians, such that the structure of the system is preserved when solving the reduction problem. Section 3.3 presents a balanced truncation model reduction for exponentially stable LTSI systems with guaranteed error bound. An improved error bound is obtained in Section 3.4 via utilizing the machinery of controller design for the considered system. In addition, two methods (log-determinant [63] and a cone complementarity approach [64]) are presented in Sections 3.3 and 3.4, respectively; these two methods are used to linearize the problem of computing low rank generalized Gramians. The reduction technique is validated in Section 3.5 on an experimentally identified actuated beam model [8] which is discussed and depicted in Chapter 1, Fig 1.2. The results of this chapter are based on [66].

3.2 Preliminaries

As discussed in Chapter 2, the problem addressed here is: given $G = \Delta_d \star M$ (2.8) with order $n = n_1 + n_2 + n_3$, find a representation $G_r = \Delta_r \star M_r$ with

$$M_r = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \tag{3.1}$$

of order $n_r = n_{r1} + n_{r2} + n_{r3} < n$ and

$$\Delta_r = \begin{bmatrix} T^- I_{n_{r1}} & 0 & 0 \\ 0 & S^- I_{n_{r2}} & 0 \\ 0 & 0 & S^+ I_{n_{r3}} \end{bmatrix}$$

such that $\| G - G_r \|_{2\to 2}$ is minimised, where $G - G_r$ is the error system defined in (2.30).

In order to preserve the temporal and forward/backward structure of the system, we need to define a set of matrices that commute with Δ_d. For system (2.7) in LFT form, we define the set of structured matrices as follows.

Definition 3.1 *The set $\mathcal{X}$ of structured matrices with respect to temporal and forward/backward spatial components is defined as*

$$\mathcal{X} = \{X = X^* : X\,\Delta_d = \Delta_d\,X, X = \text{diag}\,(X_1, X_2, X_3),\ X_1 > 0\} \tag{3.2}$$

where $X_1 \in \mathbb{C}^{n_1 \times n_1}$, $X_2 \in \mathbb{C}^{n_2 \times n_2}$ and $X_3 \in \mathbb{C}^{n_3 \times n_3}$, such that the matrix X_1 corresponds to the temporal part, and X_2, and X_3 correspond to the forward and backward spatial parts, respectively.

Remark 3.1 *In order to be able to solve the reduction problem as LMI problem, the model needs to be transformed from the discrete domain (temporal- and spatial-discrete) into the continuous domain (temporal- and spatial-continuous) because non-causality prevents the use of the Schur complement, a problem also known for controller synthesis, see [5]. The identified model is defined in discrete-time and -space, so we have to apply a bilinear transformation [5] (see Appendix E) before we can define the results of this thesis. After a bilinear transformation, the delta block in (2.6) will be defined as*

$$\Delta = \begin{bmatrix} \int dt\, I_{n_1} & 0 & 0 \\ 0 & \int ds\, I_{n_2} & 0 \\ 0 & 0 & \frac{d}{ds} I_{n_3} \end{bmatrix}. \tag{3.3}$$

Before defining the generalized Gramians of LTSI systems, we recall the sufficient exponential [5] stability condition, which is given here as an LMI condition. This result is a simplified version of the exponential stability condition given in [12] for temporal- and spatial-varying interconnected systems.

Theorem 3.1 *(Exponential Stability, [12])*
The LTSI system $(\Delta * M)$ *is exponentially stable if there exists a matrix* $X \in \mathcal{X}$ *such that* $\begin{bmatrix} I & A^* \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} < 0.$

Note that not $X > 0$, but only $X_1 > 0$, enforced by (3.2), is required for stability.

Definition 3.2 *(Generalized Gramians, [49])*
For system (2.7), we define generalised controllability and generalised observability Gramians $X, Y \in \mathcal{X}$*, respectively, as matrices satisfying*

$$AX + XA^* + BB^* < 0; \qquad A^*Y + YA + C^*C < 0. \tag{3.4}$$

Finally, we conclude this section by defining a minimal realisation [48] of a state-space model of a spatially interconnected system. For this purpose, we give the definition of the reachability matrix R and its dual version (the observability matrix O) [48], which are based on the following block product definition.

Definition 3.3 *(Block Product, [48])*
Suppose $P \in \mathbb{C}^{n \times n}$ *and* $Q \in \mathbb{C}^{n \times J}$ *are block partitioned as follows:*

$$P = [P_1 \;\; P_2 \;\; P_3], \qquad Q = [Q_1^* \;\; Q_2^* \;\; Q_3^*]^*$$

where[1] $dim(P_i) = n \times n_i$, $\;dim(Q_i) = n_i \times J\;$ *and* $\;\sum_i n_i = n$, $\;i = 1, 2, 3$. *Then define the block products*

$$\beta^0[P, Q] = Q \tag{3.5}$$
$$\beta^1[P, Q] = [P_1Q_1 \;\; P_2Q_2 \;\; P_3Q_3] \tag{3.6}$$
$$\beta^2[P, Q] = \beta^1[P, \beta^1[P, Q]] \tag{3.7}$$
$$\beta^k[P, Q] = \beta^1[P, \beta^{k-1}[P, Q]]. \tag{3.8}$$

Then the reachability matrix R is defined as

$$R = \begin{bmatrix} \beta^0[A, B] & \beta^1[A, B] & \cdots & \beta^{\hat{n}-1}[A, B] \end{bmatrix} \tag{3.9}$$

where $\hat{n} = max\ n_i, i = 1, 2, 3$.

The dual of R is the observability matrix O, it is defined as

$$O = \begin{bmatrix} \beta_*^0[A^*, C^*] \\ \beta_*^1[A^*, C^*] \\ \vdots \\ \beta_*^{\hat{n}-1}[A^*, C^*] \end{bmatrix}$$

[1]As mentioned before, here we are dealing with systems of one spatial dimension. Therefore, we partition P and Q into three blocks according to the temporal and forward/backward spatial state variables of the system.

where $\beta_*^k[P,Q] = \left(\beta^k[P,Q]\right)^*$.

In order to establish a relationship between minimality and the rank of the observability and reachability matrices, respectively, a block partitioning for the latter matrices compatible with Δ needs to be taken in account. Checking the rank of each block of the reachability and observability matrices, respectively, allows to establish the minimality of the system with respect to time and space, see [75], [48], [51] and [76].

Partition R (O) into block rows (columns), such that $R = \begin{bmatrix} R^1 \\ R^2 \\ R^3 \end{bmatrix}$ and $O = \begin{bmatrix} O^1 & O^2 & O^3 \end{bmatrix}$; each block R^i has n_i rows and each block O^i has n_i columns, $i = 1, 2, 3$.

Now, we are ready to present the following.

Lemma 3.1 *(Minimal Realization, [48])*
A state-space model $(\Delta * M)$ is said to be minimal if each block of its reachability and observability matrices ($R^i, O^i, i = 1, 2, 3$) has full rank.

A relationship between the singularity of the generalized Gramians and deficiency rank of R and/or O is given next; the following Theorem is a continuous-domain version of results given in [48], [47].

Theorem 3.2 *Given an exponentially stable spatially distributed system $(\Delta \star M)$, there exists a lower order representation $(\Delta_m \star M_m)$, $m < n$, such that $\|(\Delta \star M) - (\Delta_m \star M_m)\|_{2\to 2} = 0$, (where $n - m$ is the number of zero-valued eigenvalues of XY), if there exists a singular $X \in \mathcal{X}$ satisfying*

(i) $AX + XA^* + BB^* \leq 0$

or a singular $Y \in \mathcal{X}$ satisfying

(ii) $A^*Y + YA + C^*C \leq 0$.

Furthermore, if there exists a singular $X \in \mathcal{X}$ satisfying (i), then rank$(R^i) < n_i$, for some or all $i = 1, 2, 3$, where R^i is a block of the reachability matrix. Also, if there exists a singular $Y \in \mathcal{X}$ satisfying (ii), then rank$(O^i) < n_i$, for some or all $i = 1, 2, 3$, where O^i is a block of the observability matrix.

3.3 Balanced Realisation and Balanced Truncation

As already discussed in Chapter 1, in order to define a balanced realization, we need a transformation that transforms the system matrices and the generalized Gramians into a balanced realization. Such a transformation that preserves the structure of the system is defined next.

Definition 3.4 *(Structured Transformation)*
The set $\mathcal{T}$ of structured transformation matrices is defined as

$$\mathcal{T} = \{T \in \mathbb{C}^{n\times n} : \Delta\, T = T\Delta \text{ and } \det T \neq 0\} \tag{3.10}$$

where Δ is defined in (3.3).

The commuting condition [49] for Δ and T in (3.10) allows to preserve the structure of the system: if we include T and $T^{-1} \in \mathcal{T}$ in the structure in diagram, Figure 3.1, then $T^{-1}\Delta T = T^{-1}T\Delta = \Delta$.

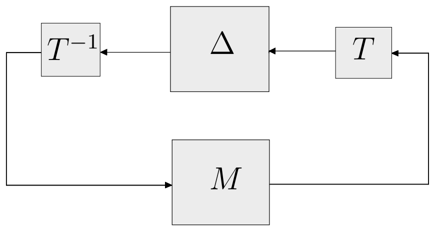

Figure 3.1: Preserving the structure of the system

Now we can present a procedure for transforming a *non-causal* system into a balanced realisation. Suppose there exists a block diagonal matrix $T \in \mathcal{T}$ such that $\tilde{Y} = T^*YT$, $\tilde{X} = T^{-1}XT^{-*}$ satisfy

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^* + \tilde{B}\tilde{B}^* < 0 \qquad\qquad \tilde{A}^*\tilde{Y} + \tilde{Y}\tilde{A} + \tilde{C}^*\tilde{C} < 0 \tag{3.11}$$

with $\tilde{X} = \tilde{Y} = \Sigma$ diagonal, where

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & D \end{bmatrix} = \begin{bmatrix} T^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T & \\ & I \end{bmatrix},$$

the transformed model $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ is balanced.
Note that the generalised singular values are preserved under the transformation $T \in \mathcal{T}$.

We can construct a balanced state-space realisation for an exponentially stable system (Δ, M) using the following procedure:

Algorithm 3.3

step 1. Solve (3.4) for feasible $X, Y \in \mathcal{X}$

step 2. Compute the Cholesky decomposition for each block of $X \in \mathcal{X}$, and $Y \in \mathcal{X}$, such that

$$\begin{aligned} X_1 &= S_1^* S_1, & X_2 &= S_2^* S_2, & X_3 &= -S_3^* S_3, \\ Y_1 &= R_1^* R_1, & Y_2 &= R_2^* R_2, & Y_3 &= -R_3^* R_3. \end{aligned}$$

step 3. Compute the SVD of $R_1 S_1^*$, SVD of $R_2 S_2^*$ and the SVD of $R_3(-S_3^*)$, such that

$$R_1 S_1^* = U_1 \Sigma_1 V_1^*, \quad R_2 S_2^* = U_2 \Sigma_2 V_2^* \quad \text{and} \quad R_3(-S_3^*) = U_3 \Sigma_3 V_3^*$$

step 4. Define

$$\begin{aligned} T_1 &= S_1^* V_1 \Sigma_1^{-(1/2)}, & T_2 &= S_2^* V_2 \Sigma_2^{-(1/2)}, & T_3 &= (-S_3^*) V_3 \Sigma_3^{-(1/2)} \\ T_1^{-1} &= \Sigma_1^{-(1/2)} U_1^* R_1, & T_2^{-1} &= \Sigma_2^{-(1/2)} U_2^* R_2, & T_3^{-1} &= \Sigma_3^{-(1/2)} U_3^* R_3. \end{aligned}$$

Define $T = \mathrm{diag}(\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3)$, and $T^{-1} = \mathrm{diag}(\mathrm{T}_1^{-1}, \mathrm{T}_2^{-1}, \mathrm{T}_3^{-1})$. Note that $T \in \mathcal{T}$.

step 5. Utilising the block diagonal transformation matrix $T \in \mathcal{T}$, calculate a balanced realisation as: $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$, $\tilde{C} = CT$, $\tilde{D} = D$, and $\Sigma = \tilde{X} = \tilde{Y} = \mathrm{diag}(\Sigma_1, \Sigma_2, -\Sigma_3)$, where since X_3, Y_3 are negative definite matrices, the simultaneously diagonalisable matrices $\tilde{X}, \tilde{Y}$ still have negative definite submatrices.

Remark 3.2 *We need $(-X_3)$ and $(-Y_3)$ in the Cholesky decomposition in step 2, because X_3, Y_3 are negative definite (they are associated with the backward spatial part), see Appendix A.*

Since exponential stability follows from either of the Lyapunov inequalities (from Theorem 3.1), the balanced system is exponentially stable.

Note that if the system has a singular solution either X or Y, then Algorithm 3.3 will not work, since we will have zero values in some or all of $\Sigma_i, i = 1, 2, 3$, which prevents computing the inverse of Σ_i, (in case of very small singular values, a similar problem may happen). In this case, we can apply the same idea as in [77] for 1D systems with unstructured Gramians by defining the reduced system directly, truncating the zero singular values in Σ_i and removing the corresponding rows and columns in U_i, and V_i for each i, respectively, then defining T, and T^{-1} as in step 4 of Algorithm 3.3.

3.3.1 Balanced Truncation

After applying the structured balanced transformation, we can partition each $\Sigma_i, i = 1, 2, 3$ into two diagonal blocks corresponding to significant (s) and non-significant (ns) generalised singular values, respectively, as $\Sigma_i = \begin{bmatrix} \Sigma_i^s & \\ & \Sigma_i^{ns} \end{bmatrix}$ and truncate the states corresponding to Σ_i^{ns} to obtain a reduced-order balanced system. To see that, partition the system matrices according to $\Sigma_i = \begin{bmatrix} \Sigma_i^s & \\ & \Sigma_i^{ns} \end{bmatrix}, i = 1, 2, 3$ as follows.

$$\tilde{A} = \left[\begin{array}{cc:cc:cc} (A^{tt})_{11} & (A^{tt})_{12} & (A^{ts}_{+})_{11} & (A^{ts}_{+})_{12} & (A^{ts}_{-})_{11} & (A^{ts}_{-})_{12} \\ (A^{tt})_{21} & (A^{tt})_{22} & (A^{ts}_{+})_{21} & (A^{ts}_{+})_{22} & (A^{ts}_{-})_{21} & (A^{ts}_{-})_{22} \\ \hdashline (A^{st}_{+})_{11} & (A^{st}_{+})_{12} & (A^{ss}_{++})_{11} & (A^{ss}_{++})_{12} & (A^{ss}_{+-})_{11} & (A^{ss}_{+-})_{12} \\ (A^{st}_{+})_{21} & (A^{st}_{+})_{22} & (A^{ss}_{++})_{21} & (A^{ss}_{++})_{22} & (A^{ss}_{+-})_{21} & (A^{ss}_{+-})_{22} \\ \hdashline (A^{st}_{-})_{11} & (A^{st}_{-})_{12} & (A^{ss}_{-+})_{11} & (A^{ss}_{-+})_{12} & (A^{ss}_{--})_{11} & (A^{ss}_{--})_{12} \\ (A^{st}_{-})_{21} & (A^{st}_{-})_{22} & (A^{ss}_{-+})_{21} & (A^{ss}_{-+})_{22} & (A^{ss}_{--})_{21} & (A^{ss}_{--})_{22} \end{array}\right], \quad \tilde{B} = \left[\begin{array}{c} (B^t)_1 \\ (B^t)_2 \\ \hdashline (B^s_{+})_1 \\ (B^s_{+})_2 \\ \hdashline (B^s_{-})_1 \\ (B^s_{-})_2 \end{array}\right] \tag{3.12}$$

$$\text{and} \quad \tilde{C} = \left[\begin{array}{cc:cc:cc} (C^t)_1 & (C^t)_2 & (C^s_{+})_1 & (C^s_{+})_2 & (C^s_{-})_1 & (C^s_{-})_2 \end{array}\right].$$

Truncate the blocks which are corresponding to $\Sigma_i^{ns}, i = 1, 2, 3$, such that

$$A_r = \begin{bmatrix} (A^{tt})_{11} & (A^{ts}_{+})_{11} & (A^{ts}_{-})_{11} \\ (A^{st}_{+})_{11} & (A^{ss}_{++})_{11} & (A^{ss}_{+-})_{11} \\ (A^{st}_{-})_{11} & (A^{ss}_{-+})_{11} & (A^{ss}_{--})_{11} \end{bmatrix}, B_r = \begin{bmatrix} (B^t)_1 \\ (B^s_{+})_1 \\ (B^s_{-})_1 \end{bmatrix} \quad \text{and} \quad C_r = \begin{bmatrix} (C^t)_1 & (C^s_{+})_1 & (C^s_{-})_1 \end{bmatrix}. \tag{3.13}$$

Then $G_r = \Delta_r \star \begin{bmatrix} A_r & B_r \\ C_r & D \end{bmatrix}$, where Δ_r is the reduced version of Δ given in (3.3) and $M_r = \begin{bmatrix} A_r & B_r \\ C_r & D \end{bmatrix}$.

Using the error bound of Theorem 3.3 below as an initial step, and since the solutions X and Y of (3.4) are not unique (because the defining conditions are inequalities, but not equations), it is reasonable to search for X and Y with as many as possible small singular values, in order to reduce the order as far as possible. Therefore, a rank constraint on $X, Y \in \mathcal{X}$ has to be imposed while solving (3.4). Note that this is a non-convex condition. In previous work (e.g., [49, 58]), this problem was addressed using a trace heuristic algorithm. Here, we propose to use the log-det heuristic, which is a smooth surrogate for matrix rank minimisation, see [63]. The log-det heuristic uses the result of the trace heuristic as starting point and minimises the rank further, thus giving more accurate results, and as a consequence an improved error bound will be obtained. Here, we briefly describe the log-det heuristic: define a small positive regularisation constant δ (to ensure invertibility), take the objective function as log-det($X+\delta I$) + log-det($Y+\delta I$), subject to (3.4), where I is the identity matrix.

In [63], an efficient method is proposed to solve the rank condition of a matrix as LMI problem by utilising the first-order Taylor series expansion of log-det($X+\delta I$) and of log-det($Y+\delta I$). Here, we propose the steps in Algorithm 3.3.1 below.

Algorithm 3.3.1 together with Algorithm 3.3 yields generalized Gramians with small generalized singular values.

Theorem 3.2 of Section 3.3 provides a result for exact reducibility (i.e. zero error bound); the next theorem presents an error bound for model reduction.

Algorithm 3.3.1
step 1. Initialise (3.4) with
$X_k = I_{n_1}$, $X_{kk} = I_{n_2}$, $X_{kkk} = I_{n_3}$, and $Y_k = I_{n_1}$, $Y_{kk} = I_{n_2}$, $Y_{kkk} = I_{n_3}$, set $\alpha = \infty$, fix $\epsilon > 0$

step 2. Minimise (over $X, Y \in \mathcal{X}$)

$$\begin{aligned}\beta = \{&s_1 \cdot \text{ trace } (X_k + \delta I)^{-1} X_1 + s_2 \cdot \text{ trace } (X_{kk} + \delta I)^{-1} X_2 + s_3 \cdot \text{ trace } (X_{kkk} + \delta I)^{-1}(-X_3) \\ &+ s_4 \cdot \text{ trace } (Y_k + \delta I)^{-1} Y_1 + s_5 \cdot \text{ trace } (Y_{kk} + \delta I)^{-1} Y_2 + s_6 \cdot \text{ trace } (Y_{kkk} + \delta I)^{-1}(-Y_3)\};\end{aligned}$$

subject to (3.4), where $\sum_{l=1}^{6} s_l = 1$.

step 3. if $|(\beta - \alpha)/\beta| > \epsilon$;
set $X_k = X_1$, $X_{kk} = X_2$, $X_{kkk} = X_3$, $Y_k = Y_1$, $Y_{kk} = Y_2$, $Y_{kkk} = Y_3$, and $\delta = \delta/\eta$, and set $\alpha = \beta$, go to step 2, else stop.

Theorem 3.3 *Suppose $(\Delta_r \star M_r)$ is the reduced model obtained by truncation from a balanced realisation of the exponentially stable spatially interconnected model $(\Delta \star M)$. Then the reduced model is exponentially stable, balanced and*

$$\| (\Delta \star M) - (\Delta_r \star M_r) \|_{2\to 2} \leq 2 \left(\sum_{i=1}^{3} \sum_{j=n_{ri}+1}^{m_i} \sigma_{i,j} \right), \tag{3.14}$$

where m is as defined in Theorem 3.2; and $\sigma_{i,j}$ are the absolute values of the diagonal entries of $\Sigma^{ns} = \text{diag}(\Sigma_1^{ns}, \Sigma_2^{ns}, -\Sigma_3^{ns})$.

Proof The exponential stability condition in Theorem 3.1 is preserved in either of the Lyapunov inequalities. Clearly, the transformed (balanced) system is exponentially stable according to (3.11). Therefore, truncating the non-significant states and substitute (3.13) and $\Sigma^s = \text{diag}(\Sigma_1^s, \Sigma_2^s, -\Sigma_3^s)$, then the resulting reduced model is still exponentially stable and balanced.
For the proof of the error bound (3.14) see Appendix B. ■

Remark 3.3 *In [58], a matlab toolbox for model reduction of discrete domain multidimensional systems using the balanced truncation method has been presented. As already mentioned, [58, 56] and [57] consider only the reduction of the temporal dynamics, which are associated with positive definite blocks in the generalized Gramians. Here, we consider continuous domain systems, and include both temporal and spatial dynamics; in this case, we encounter positive and negative diagonal matrix blocks when we minimise the rank of the generalized Gramians. The balanced truncation and its error bound presented in this section are used as an initial step for applying the results proposed in the next section.*

3.4 Model Order Reduction with Guaranteed Error Bound

In the previous section, balanced truncation for LTSI systems was considered. In this section, we continue with the model reduction problem via applying the bounded real lemma. As discussed in Section 3.2, in order to preserve the temporal- and forward/backward spatial-structure of the system, a structured set $\mathcal{X}$ has to be defined. In the same way, define

$$\mathcal{Y} = \left\{ Y_e = Y_e^* : Y_e = \begin{bmatrix} Y & Y^{12} \\ Y^{12*} & Y^{22} \end{bmatrix} \right\} \tag{3.15}$$

where $Y \in \mathcal{X}$, $Y^{k2} = \text{diag}\ (Y_1^k, Y_2^k, Y_3^k)$, $Y_1^k > 0, k = 1, 2$; $Y_i^1 \in \mathbb{C}^{n_i \times n_{ri}}$, $Y_i^2 \in \mathbb{C}^{n_{ri} \times n_{ri}}, i = 1, 2, 3$.

Based on the bounded real lemma [5], in order to find an exponentially stable reduced model while at the same time ensuring a small error bound γ on the difference between the original system and the reduced one, we formulated the reduction problem as follows.

Reduction Problem: Given γ, find a reduced system M_r and $Y_e \in \mathcal{Y}$, such that

$$\begin{bmatrix} A_e^*\, Y_e + Y_e A_e & Y_e B_e & C_e^* \\ B_e^*\, Y_e & -\gamma I & D_e^* \\ C_e & D_e & -\gamma I \end{bmatrix} < 0, \tag{3.16}$$

where (A_e, B_e, C_e, D_e) denotes a state space realization of the error model (2.29).
Note that (3.16) is a necessary and sufficient condition for the error system to be exponentially stable, well-posed and for the error norm to be less than γ, i.e. $\| (\Delta_e \star E) \|_{2\to 2} \leq \gamma$, if condition (3.16) is solvable for $Y_e \in \mathcal{Y}$ and M_r.
Note that in the discrete-domain version of inequality (3.16), a term quadratic in A_e arises; in order to render the problem convex, a bilinear transformation is required that transforms the problem into continuous-domain. Actually, (3.16) is still nonlinear in the variables; however, applying the elimination lemma (see Appendix E.2) leads to the convex conditions (3.18)–(3.20) in Theorem 3.4 below.

In order to apply the elimination lemma, the above condition (3.16) is rewritten as

$$R + U M_r V^* + V M_r^* U^* < 0 \tag{3.17}$$

where

$$R = \begin{bmatrix} \begin{bmatrix} A^* & 0 \\ 0 & 0 \end{bmatrix} Y_e + Y_e \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} & Y_e \begin{bmatrix} B \\ 0 \end{bmatrix} & \begin{bmatrix} C^* \\ 0 \end{bmatrix} \\ \begin{bmatrix} B^* & 0 \end{bmatrix} Y_e & -\gamma I & D^* \\ \begin{bmatrix} C & 0 \end{bmatrix} & D & -\gamma I \end{bmatrix}, \quad U = \begin{bmatrix} Y_e \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 0 \\ 0 & -I \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}.$$

Defining

$$V_\perp^* = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \quad \text{and} \quad U_\perp^* = \begin{bmatrix} [I \quad 0]\, Y_e^{-1} & 0 & 0 \\ 0 & I & 0 \end{bmatrix},$$

and combining the result of [56] and [1], we have the following result.

Theorem 3.4 *Given the representation $(\Delta \star M)$ of a LTSI system, and $\gamma > 0$, there exists a reduced order representation $(\Delta_r \star M_r)$ such that $\|(\Delta \star M) - (\Delta_r \star M_r)\|_{2\to 2} \leq \gamma$, if there exist $X, Y \in \mathcal{X}$, and a scalar constant $\nu > 0$ satisfying* [2]

$$\begin{bmatrix} A^*Y + YA & C^* \\ C & -\gamma I \end{bmatrix} < -\nu I \tag{3.18}$$

$$\begin{bmatrix} A^*X + XA & XB \\ B^*X & -\gamma I \end{bmatrix} < -\nu I \tag{3.19}$$

$$Y_1 - X_1 \geq 0, \tag{3.20}$$

$$\text{rank } (Y - X) \leq r. \tag{3.21}$$

Here, γ represents the overall approximation error, expressed in the induced spatio-temporal 2-norm of the error system. Note that in contrast to (3.18)–(3.20), condition (3.21) is non-convex; this will be discussed below.

Remark 3.4 *Having obtained solutions $X, Y \in \mathcal{X}$ (of Theorem 3.4) with satisfaction of the rank constraint (*rank $(Y_i - X_i) \leq r_i$*, for all or some i), one can compute $Y_e \in \mathcal{Y}$ as follows.*

1) Recall that the solutions $Y \in \mathcal{X}$ and $X^{-1} \in \mathcal{X}$ are the upper left blocks of $Y_e \in \mathcal{Y}$ and $Y_e^{-1} \in \mathcal{Y}$, respectively, see (3.15). Then, according to the matrix inversion Lemma [78] we have $X = Y - Y^{12}Y^{22^{-1}}Y^{12*}$, which gives $Y - X = Y^{12}Y^{22^{-1}}Y^{12*}$.

2) Define [79] $Y^{22} = \begin{bmatrix} I_{n_{r1}+n_{r2}} & \\ & -I_{n_{r3}} \end{bmatrix}$, i.e., decompose $Y_i - X_i = Y_i^1 Y_i^2 Y_i^{1*}$, $i = 1, 2, 3$, (see (3.15)) such that $Y_i^2 = I_{r_i}$, for *$i = 1, 2$, and $Y_i^2 = -I_{r_i}$,* for *$i = 3$. Note that Y_i^1 are tall matrices as desired. To see this, consider the eigen decomposition for $(Y_i - X_i)$ such that Y_i^2 contains the eigenvalues of $(Y_i - X_i)$, and Y_i^1 contains the eigenvectors of $(Y_i - X_i)$ for each i; then pulling out the eigenvalues (without their signs) from Y_i^2 into Y_i^1 gives the desired decomposition.*

Finally, having the complete $Y_e \in \mathcal{Y}$, one can find M_r that satisfies condition (3.16), which is an LMI in M_r (a more efficient way to calculate M_r is to construct it explicitly, see [80]).

According to Remark 3.4, the new order r of the reduced model $(\Delta_r * M_r)$ is determined by the rank of the solutions of (3.18) – (3.21). In order to reduce the rank, which is a non-convex problem, we use the cone complementarity method [64] in order to linearize this problem. The procedure is summarized in Algorithm 3.4 below.

[2] To insure the strict feasibility of conditions (3.18) and (3.19), we use a regularization $\nu > 0$.

Algorithm 3.4
step 1. Solve (3.18) – (3.20) for feasible initial $X_0, Y_0 \in \mathcal{X}$. Set $g = 0$.
step 2. Since $X, Y \in \mathcal{X}$ are block diagonal, for $i = 1, 2, 3$, set $V_{gi} = Y_{gi}$, $W_{gi} = X_{gi}$. Find $X_{(g+1)i}$ and $Y_{(g+1)i}$ that solve the LMI problem

$$\min_{X_i, Y_i} \sum_i trace(V_{gi}X_i + W_{gi}Y_i)$$

subject to conditions of Theorem 3.4.
step 3. If a stopping criterion is satisfied, stop. Otherwise, set $g = g + 1$ and go back to step 2.

Now, we summarize our proposed model order reduction (MOR) scheme which allows a trade-off between minimising the rank of the Gramians, and improving the error bound γ. The approach is validated in the next section as an effective approach to obtain a reduced model with improved error bound.

Model order reduction scheme

step 1. Apply the balanced realisation procedure (Algorithm 3.3) to the spatially interconnected system using the log-det method (Algorithm 3.3.1), and set the initial [3] upper bound to $\gamma < 2\sum_{i=1}^{3}\sum_{j=n_{ri}+1}^{m_i}\sigma_{ij}$, where m is defined as in Theorem 3.2. Note that for this step, we determine the smallest generalised singular values without truncating them. The initial new order $r = \sum_i r_i$ is selected for the reduced model.

step 2. Solve three LMIs with one non-convex rank constraint (Theorem 3.4), using the cone complementarity method (Algorithm 3.4), and obtain the generalised controllability and observability Gramians including small singular values.

step 3. Check the rank of $Y_i - X_i$ for $i = 1, 2, 3$; if rank$(Y_i - X_i)$ is greater than r_i for some i, then increase the value of γ, if not then decrease the value of γ. Go to ***step 2*** until satisfactory results are obtained.

3.5 Application to An Actuated Beam

The results of this chapter are validated on an experimentally validated model of an actuated beam that was introduced in Chapter 1; see also [8].

First we illustrate the representation of a distributed system with sensor-actuator array by spatially and temporally discretizing a distributed model of an actuated beam.

[3] Note that we set the initial γ to a value which is strictly less than the balanced truncation error bound, i.e., in step 2 we start with this initial value and we try to minimize it further.

Thus, consider the Euler-Bernoulli beam equation

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 y(k,x)}{\partial x^2}\right)+\frac{\partial^2 y(k,x)}{\partial t^2}=u(k,x) \tag{3.22}$$

where $u(k,x)$ represents the force applied to the beam (for continuous time k and space x) and $y(k,x)$ describes the deflection of the beam as a response to the force. Here, we ignore physical dimensions and assume for simplicity that all physical constants have value 1.

Then, applying temporal- and spatial-discretization (using finite difference approximation) to (3.22), leads to the spatially interconnected system (2.5) as follows.
The discretization of (3.22) is

$$\begin{aligned}&\frac{y(t,s-2)-4y(t,s-1)+6y(t,s)-4y(t,s+1)+y(t,s+2)}{h^4}\\&\qquad+\frac{y(t-1,s)-2y(t,s)+y(t+1,s)}{l^2}=u(t,s)\end{aligned} \tag{3.23}$$

where l is the temporal sampling period, h is the spatial sampling period and we use the central finite difference, i.e.

$$\frac{\partial^2 y(k,x)}{\partial x^2}\approx\frac{y(t,s-1)-2y(t,s)+y(t,s+1)}{h^2}.$$

Define coefficients (a and b) suitably, such that (3.23) yields

$$\begin{aligned}y(t,s)&=a_{1,2}\,y(t-1,s-2)+a_{1,1}\,y(t-1,s-1)+a_{1,-1}\,y(t-1,s+1)\\&+a_{1,-2}\,y(t-1,s+2)+a_{2,0}\,y(t-2,s)+a_{1,0}\,y(t-1,s)+b_{1,0}\,u(t-1,s).\end{aligned} \tag{3.24}$$

The general form of (3.24) is

$$y(t,s)=\sum_{i_t,i_s\in Y_{mask}}a_{i_t,i_s}\;y(t-i_t,s-i_s)+\sum_{i_t,i_s\in U_{mask}}b_{i_t,i_s}\;u(t-i_t,s-i_s) \tag{3.25}$$

The input and output masks U_{mask}, Y_{mask} (given in Figure 3.2) indicate the temporally- and spatially-shifted inputs and outputs required for calculating the output $y(t,s)$ in (3.24). Note how the fact the system is causal in time and non-causal in space results in the masks to be confined to the lower half plane.

Define the temporal state vector as $\begin{bmatrix}y(t-2,s) & y(t-1,s)\end{bmatrix}^T$, the spatial state vector as $\begin{bmatrix}y(t-1,s-1) & y(t-1,s-2) & y(t-1,s+1) & y(t-1,s+2)\end{bmatrix}^T$ and recall the definition of temporal and spatial shift operators (Definition 2.4). Then, a state-space model of the Euler-Bernoulli beam is

$$\begin{array}{l}T^+x\left\{\vphantom{\begin{matrix}a\\a\end{matrix}}\right.\\ w_+\left\{\vphantom{\begin{matrix}a\\a\end{matrix}}\right.\\ w_-\left\{\vphantom{\begin{matrix}a\\a\end{matrix}}\right.\\ \\ \end{array}\left[\begin{array}{c}y(t-1,s)\\y(t,s)\\\hline y(t-1,s)\\y(t-1,s-1)\\y(t-1,s)\\y(t-1,s+1)\\\hline y(t,s)\end{array}\right]=\left[\begin{array}{cc|cccc|c}0&1&0&0&0&0&0\\a_{2,0}&a_{1,0}&a_{1,1}&a_{1,2}&a_{1,-1}&a_{1,-2}&1\\\hline 0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&1&0&0\\\hline 0&b_{1,0}&0&0&0&0&0\end{array}\right]\left[\begin{array}{c}y(t-2,s)\\y(t-1,s)\\\hline y(t-1,s-1)\\y(t-1,s-2)\\y(t-1,s+1)\\y(t-1,s+2)\\\hline u(t,s)\end{array}\right]\begin{array}{l}\left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\}x\\\left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\}v_+\\\left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\}v_-\\ \\ \end{array} \tag{3.26}$$

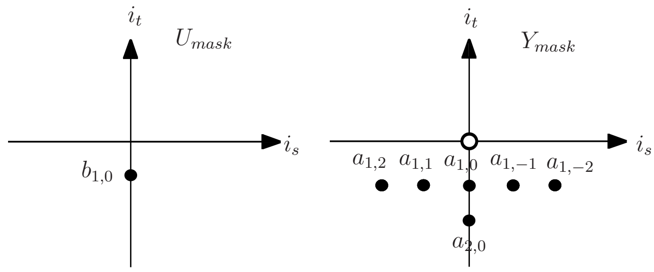

Figure 3.2: Input and output masks for Euler-Bernoulli beam.

3.5.1 Experimental Setup

The framework presented in Section 2.2.1 has been used to model an experimentally identified actuated beam [8]. The setup is shown in Figure 1.2 of Chapter 1. An aluminium beam of length = 4.8 m, width = 0.04 m, thickness = 0.003 m is considered. It is equipped with 16 evenly distributed collocated piezoelectric actuator and sensor pairs, i.e., the distances between any two neighboring sensors (actuators) are identical. Soft springs have been used to suspend the beam, such that it has free-free boundary conditions.

The beam is sufficiently thin to be modeled in a single-spatial dimension. Also, it is long enough to be approximated by an infinite model according to [8] and [81], see Remark 3.5 below.
As discussed above, the locations of the piezoelectric actuator/sensor pairs (i.e., the identical distance between between neighboring pairs) and the fact that the beam is long enough, suggest identical subsystems which are used to capture the spatially-invariant structure of the system (see [8],[81]), where the beam is discretised into 16 identical parts according to the 16 piezoelectric actuators and sensors as shown in Figure 3.3. Here, $y(t, s)$ represents the measured curvature, and $u(t, s)$ a moment generated by the actuators.

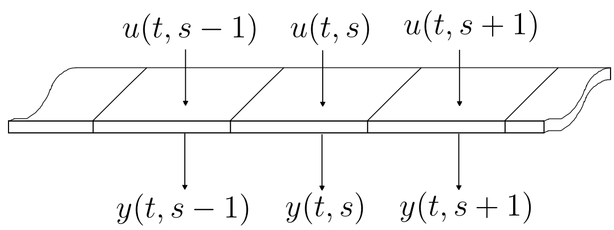

Figure 3.3: Part of a long beam.

A single subsystem is represented by the partial difference equation

$$y(t, s) = \sum_{i_t, i_s \in Y_{mask}} a_{i_t, i_s}\; y(t - i_t, s - i_s) + \sum_{i_t, i_s \in U_{mask}} b_{i_t, i_s}\; u(t - i_t, s - i_s) \tag{3.27}$$

where U_{mask} and Y_{mask} are the masks for the input and output coefficients a_{i_t,i_s} and b_{i_t,i_s} as shown in Figure 3.4. Note that their masks are different from the ones shown in Fig.

3.2. This reflects the fact that the experimental beam is not fully represented by the model (3.24) but includes additional dynamics, e.g. due to piezo actuators and sensors.

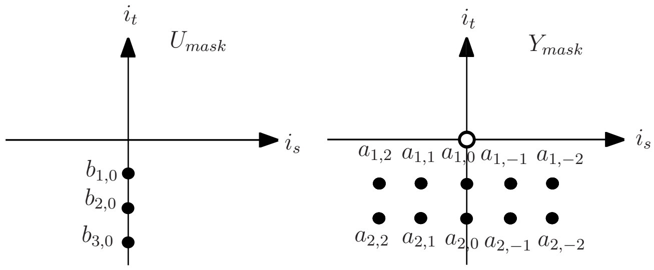

Figure 3.4: Input and output masks.

A state-space model of a single subsystem is given in Appendix D.1, where the temporal and spatial state vectors are chosen suitably.

The system has order $n = n_1 + n_2 + n_3 = 11$; $n_1 = 3$ due to a three steps temporal shift (according to Figure 3.4), $n_2 = n_3 = 4$ due to four steps spatial shifts. The number of inputs and outputs of the system are $n_u = n_y = 1$.

3.5.2 Application to The Experimental Beam

The results of this chapter are illustrated by applying them to the experimentally identified model of the actuated beam.

The results of Section 3.3 and Section 3.4 have been applied to the beam model (D.1) of order $n = 11$. We obtain a reduced model with order $n_r = 6$, where $n_{r1} = 2$, $n_{r2} = 2$ and $n_{r3} = 2$, and $\gamma = 0.0018$.

Applying the results of Section 3.3 yields a reduced system with order $m = 7$, $m_1 = 3$, $m_2 = 2$ and $m_3 = 2$; this completes step 1 of the proposed model-order reduction scheme and provides the start values for step 2 with initial error bound $\gamma < 0.1798$.
Table 3.5.2 shows a comparison between the generalised singular values when we minimise the rank of only the temporal part of the Gramians as done in [57, 56] (using the trace heuristic), and when we minimise the rank of each block of the Gramians (using Algorithm 3.3.1).

In step 2 of the proposed method, the order of the system is further reduced by applying the results of Section 3.4 we arrive at the reduced system of order $r = 6$, and error bound $\gamma = 0.0018$. The motion of the beam in response to a disturbance unit step at subsystem 8 is simulated using the multidimensional (MD)-toolbox [80]. Figure 3.5 shows the response over time of the fourth and thirteenth subsystems (i.e. sensor 4 and sensor 13), respectively, for the original system ($\Delta * M$) of order $n = 11$ (blue) and the reduced

Table 3.5.2:
Generalized Singular Values

	only temporal (Trace heuristic)	spatial-temporal (Log-det)
Σ_1	diag(0.3913, 0.1890, 0.0403)	diag(0.4552, 0.2846, 0.0899)
Σ_2	diag(0.3448, 0.2074, 0.0126, 0.0070)	diag(0.1250, 0.1103, 0, 0)
Σ_3	diag(0.3183, 0.2548, 0.2157, 0.1935)	diag(0.0958, 0.0156, 0, 0)

system $(\Delta_r * M_r)$ of order $r = 6$ (dark green).

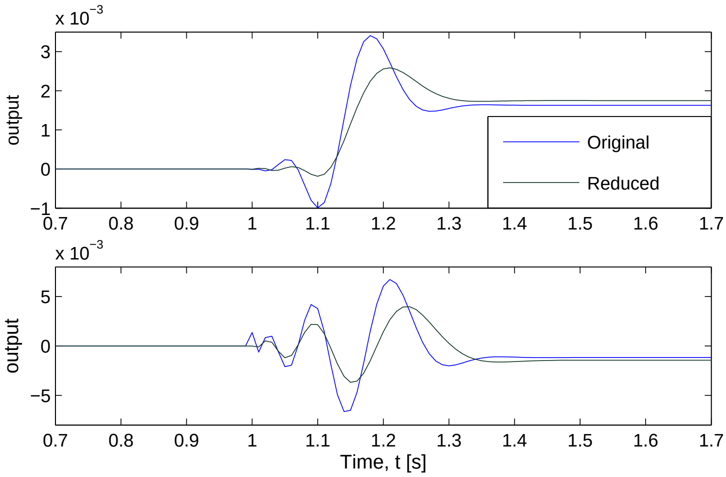

Figure 3.5: Simulated response over time of the 4^{th} (top) and 13^{th} (bottom) subsystem, to a disturbance step at the 8^{th} subsystem.

In addition, the overall response of all subsystems is shown as 3D-plot in Fig. 3.6.

Finally, the experimental input with 16 noises signals has been applied here as well in order to excite 16 actuators simultaneously. The measured and simulated responses of the original model and the response of the reduced model to the experimental input are shown in Figure 3.7 over time for subsystems 3rd and 11th, respectively.

Remark 3.5 *For spatially interconnected systems with finite spatial extension, boundary effects play an important role; one way of dealing with them within the framework of spatially interconnected systems was proposed in [82] and is based on the spatial reversibility property.*
There is an alternative way of dealing with boundary effects that has been taken in the present work: experimental results reveal that when the spatial extension of the structure under consideration is sufficiently long, then a spatially-invariant model can be experimentally identified that captures the system's dynamic properties with reasonable accuracy, whereas structures with short extension require (due to dominant boundary effects)

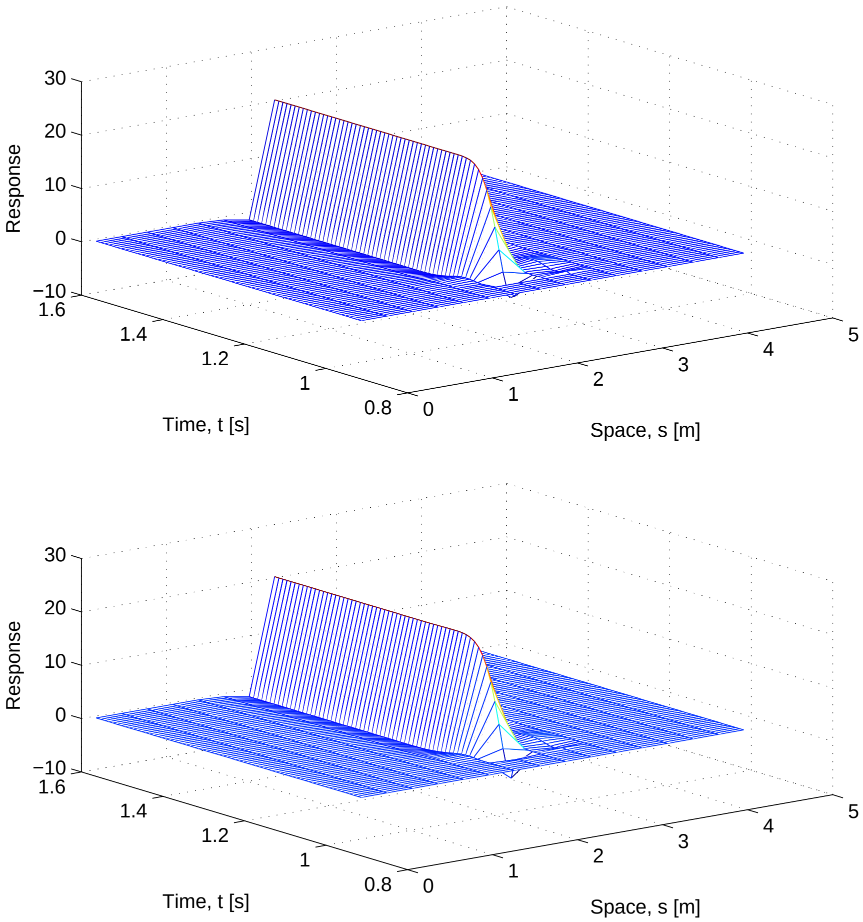

Figure 3.6: Response of all subsystems (Original model, $n = 11$ top), and (reduced model, $r = 6$ bottom).

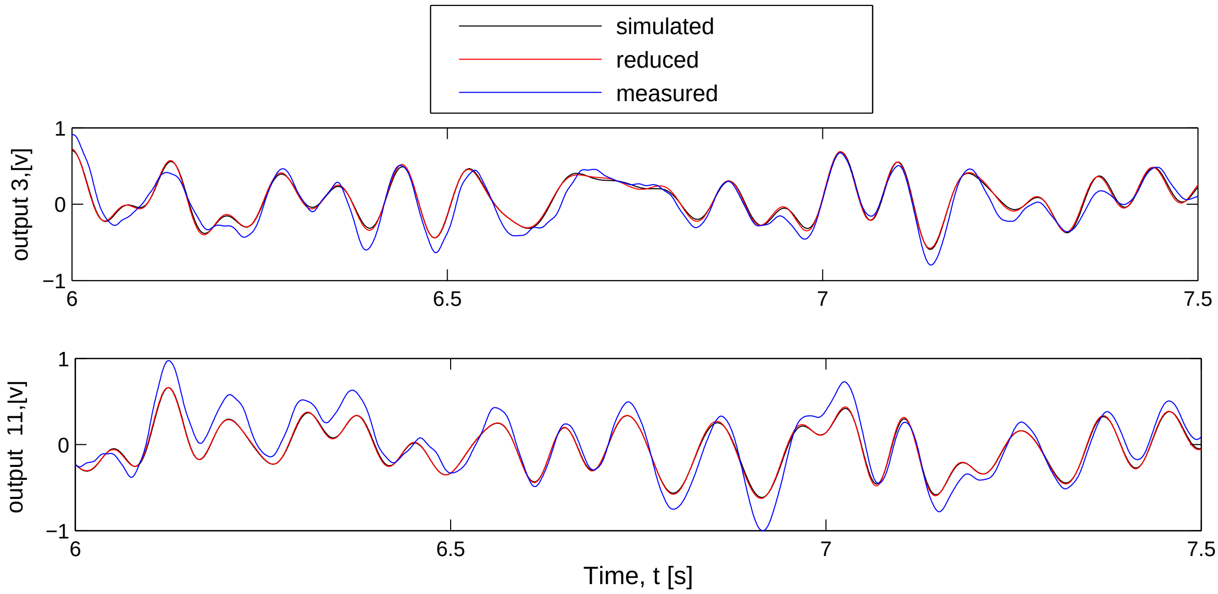

Figure 3.7: Measured and simulated (full and reduced order) responses over time of the 3rd (top) and 11th (bottom) subsystem, respectively, to 16 noises signals applied in parallel to 16 actuators.

a spatially-varying model to represent their dynamics. A case study illustrating this was reported in [83]: For the 4.8 m beam which is also considered in this thesis, a spatially-invariant model has been identified experimentally that captures the dynamic behavior with reasonable accuracy. In contrast, a short (0.5m) beam could only be represented accurately by a spatially parameter-varying model, see also [11]. This is one of the motivations for considering spatial-LPV model and its order reduction, which is considered next.

3.6 Summary

Model order reduction for LTSI systems has been presented in this chapter, based on generalized Gramians. LMI conditions for exponential stability of LTSI systems have been stated. Moreover, minimal state space realizations and their relation to the reachability and the observability matrices have been established.

Based on generalized Gramians, a reduced model has been constructed via truncating small generalized singular values. A MOR approach for exponentially stable LTSI systems is proposed in Section 3.4.

The exponential stability and the spatial structure of the system are preserved in the reduced model. An error bound in terms of truncated generalized singular values between the original model and the reduced one is derived for LTSI systems with non-causal Gramians. The practicality of the theoretical results is demonstrated with their application to an actuated beam.

The results of this chapter which are derived for time- and space-invariant systems, will next be extended to the more realistic situation of temporal- and spatial-varying parameters. That is discussed in the following chapters.

Chapter 4

Model Order Reduction for LTSV Systems

4.1 Introduction

In contrast to the rather restrictive assumption of parameter-invariant spatially interconnected systems (or LTSI) considered in the last chapter, a model order reduction technique is provided here for the more general and realistic case of parameter-varying spatially interconnected systems (time- and space-varying interconnected systems).

Model order reduction based on balanced truncation for LPV systems was first proposed in [34] for 1D lumped systems varying with respect to time via temporal scheduling parameters. The technique was based on defining a grid on the set of the admissible scheduling parameters. Extended result to the case of LTSV systems was presented in [66] with traditional error bound which has been improved in [84]. The result was based on gridding. However, defining a grid on the admissible parameter range considerably increases the complexity of solving the reduction problem. Here we propose an effective way that avoids the gridding requirement.

This chapter extends the MOR technique given in the previous chapter to the case of LTSV systems with a novel representation of a pair of Lyapunov inequalities. The system is represented as (temporal and spatial) LPV model in LFT form. The representation in LFT form basically means a decomposition of the model into a parameter varying and an LTSI part; the latter is referred to as the nominal system. As mentioned in the previous chapters, the temporal and spatial scheduling parameters are pulled out into two uncertainty blocks which allows to apply the full block S-procedure to the reduction problem such that the reduction problem is applied to the nominal system.

As shown in Chapter 3, in order to improve the result it would be reasonable to balance the system as a first step. The balancing transformation is applied to the nominal system. The reduced order system will be connected to the same local uncertainties.

After presenting some preliminaries, MOR for LPV/LFT systems based on balanced truncation is presented in Section 4.3 with a classical balanced truncation error bound.

An improved error bound result based on the equivalence of model reduction and controller design is shown in Section 4.4, where a value less than the resulting balanced truncation error bound is used as an initial value for this step. The efficiency of the proposed method is examined with the same long actuated beam that is considered in the previous chapter, but after deactivating some sensor/actuator pairs to realize a spatially-varying interconnected system rather than spatially-invariant one, [8]. In addition, a comparison between the proposed technique and the gridding-based technique is discussed as well. The chapter includes results presented in [85] and [68].

4.2 Preliminaries

Recall the system (2.13) of state order n, where for $G^0 = \begin{bmatrix} A & B^0 & B^1 \\ C^0 & D^{00} & D^{01} \\ C^1 & D^{10} & D^{11} \end{bmatrix}$, we have

$$G(\Delta^t, \Delta^s) = F_u\left(G^0, \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\right) = \left[\begin{array}{c|c} A(\Delta^t, \Delta^s) & B(\Delta^t, \Delta^s) \\ \hline C(\Delta^t, \Delta^s) & D(\Delta^t, \Delta^s) \end{array}\right], \Delta^t \in \Theta^t, \Delta^s \in \Theta^s, \tag{4.1}$$

where

$$\begin{bmatrix} A(\Delta^t, \Delta^s) & B(\Delta^t, \Delta^s) \\ C(\Delta^t, \Delta^s) & D(\Delta^t, \Delta^s) \end{bmatrix} = \begin{bmatrix} A + B^0\,\Phi\,C^0 & B_1 + B^0\,\Phi\,D^{01} \\ C^1 + D^{10}\,\Phi\,C^0 & D^{11} + D^{10}\,\Phi\,D^{01} \end{bmatrix},$$

$$\text{and} \quad \Phi = \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\left(I - D^{00}\begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\right)^{-1}.$$

In this chapter we search for a reduced version (with reduced state order $n_r < n$) of the form (see Fig. 4.1 below)

$$G_r(\Delta^t, \Delta^s) = F_u\left(G_r^0, \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\right) = \left[\begin{array}{c|c} A_r(\Delta^t, \Delta^s) & B_r(\Delta^t, \Delta^s) \\ \hline C_r(\Delta^t, \Delta^s) & D_r(\Delta^t, \Delta^s) \end{array}\right], \ \Delta^t \in \Theta^t, \Delta^s \in \Theta^s, \tag{4.2}$$

where for $n_r = n_{r1} + n_{r2} + n_{r3} < n$, and a given constant $\gamma > 0$, we have

$$\|G(\Delta^t, \Delta^s) - G_r(\Delta^t, \Delta^s)\|_{2\to 2} \leq \gamma.$$

Following the assumptions stated in [8], we assume that $D(\Delta^t, \Delta^s) = 0$ and $D^{00} = 0$.

Our proposed technique is based on the application of the full block S-procedure Lemma, which is stated next.

Lemma 4.1 *(Full block S-procedure, [86])*
Given a quadratic matrix inequality

$$G^*(\Delta^t, \Delta^s)\; N\; G(\Delta^t, \Delta^s) < 0 \tag{4.3}$$

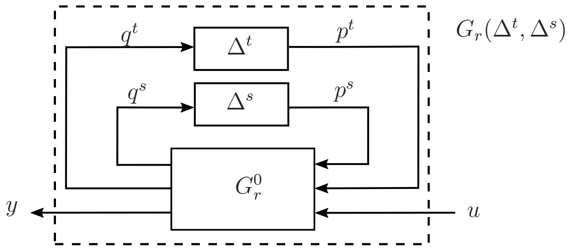

Figure 4.1: LFT representation of temporal- and spatial-LPV reduced model.

with two scheduling parameter blocks Δ^t *and* Δ^s*, where* G *and* N *are real-valued matrices,* $\Delta^t \in \Theta^t$*,* $\Delta^s \in \Theta^s$*, and* $G(\Delta^t, \Delta^s)$ *can be written in a linear fractional transformation (LFT) form*

$$G(\Delta^t, \Delta^s) = \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \star \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

the inequality (4.3) holds $\forall \Delta^t \in \Theta^t, \Delta^s \in \Theta^s$ *if and only if the following two conditions are satisfied:*
1. There exists a real symmetric matrix Π *such that*

$$[*]^* \left[\begin{array}{c|c} \Pi & \\ \hline & N \end{array}\right] \left[\begin{array}{cc} G_{11} & G_{12} \\ I & 0 \\ \hline G_{21} & G_{22} \end{array}\right] < 0$$

2. For any $\Delta^t \in \Theta^t$*,* $\Delta^s \in \Theta^s$

$$[*]^* \, \Pi \left[\begin{array}{cc} I & \\ & I \\ \hline \Delta^t & \\ & \Delta^s \end{array}\right] \geq 0.$$

Let us define the set $\mathcal{P}$ of the symmetric matrices Π (referred to as multipliers) as

$$\begin{aligned} \mathcal{P} = \{ & \Pi \in \mathbb{R}^{2(n^t+n^s)\times 2(n^t+n^s)} : \Pi^T = \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix}, \Pi_{mk} = \begin{bmatrix} \Pi_{mk}^t & \\ & \Pi_{mk}^s \end{bmatrix}, \\ & \Pi_{mk}^t \in \mathbb{R}^{n^t\times n^t}, \Pi_{mk}^s \in \mathbb{R}^{n^s\times n^s}, \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \Pi_{mk} = \Pi_{mk} \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}, \ m, k = 1, 2 \}. \end{aligned} \tag{4.4}$$

4.3 Balanced Truncation

In order to construct a balanced LTSV realization for the original system (4.1), we define the following.

Definition 4.1 .
For system (4.1), the generalized controllability and observability Gramians [49], respectively are defined as $Y, X \in \mathcal{X}$*, satisfying* $\forall \Delta^t \in \Theta^t, \Delta^s \in \Theta^s$

$$\begin{aligned} A(\Delta^t,\Delta^s)Y + YA^*(\Delta^t,\Delta^s) + B(\Delta^t,\Delta^s)B^*(\Delta^t,\Delta^s) < 0 \\ A^*(\Delta^t,\Delta^s)X + XA(\Delta^t,\Delta^s) + C^*(\Delta^t,\Delta^s)C(\Delta^t,\Delta^s) < 0. \end{aligned} \tag{4.5}$$

In Chapter 3 we presented a technique for balancing LTSI systems. Here, we extend it to the case of LTSV system (4.1).
First, we have to construct a balanced transformation $T \in \mathcal{T}$ (as defined in (3.10)) via extended version of Algorithm 3.3, such that we have (see (4.1))

$$\begin{aligned} \begin{bmatrix} \tilde{A}(\Delta^t,\Delta^s) & \tilde{B}(\Delta^t,\Delta^s) \\ \tilde{C}(\Delta^t,\Delta^s) & D(\Delta^t,\Delta^s) \end{bmatrix} &= \begin{bmatrix} T^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} A(\Delta^t,\Delta^s) & B(\Delta^t,\Delta^s) \\ C(\Delta^t,\Delta^s) & D(\Delta^t,\Delta^s) \end{bmatrix} \begin{bmatrix} T & \\ & I \end{bmatrix} \\ &= \begin{bmatrix} T^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} A + B^0\,\Phi\,C^0 & B^1 + B^0\,\Phi\,D^{01} \\ C^1 + D^{10}\,\Phi\,C^0 & D^{11} + D^{10}\,\Phi\,D^{01} \end{bmatrix} \begin{bmatrix} T & \\ & I \end{bmatrix} \\ &\qquad \forall \Delta^t \in \Theta^t, \text{ and } \Delta^s \in \Theta^s, \end{aligned} \tag{4.6}$$

and $\tilde{X} = T^{-1}YT^{-*} = T^*YT = \tilde{Y}$.
According to the above representation (4.6), clearly the nominal system matrices are multiplied by the balancing transformation $T \in \mathcal{T}$, such that

$$\tilde{G}(\Delta^t,\Delta^s) = \begin{bmatrix} \tilde{A}(\Delta^t,\Delta^s) & \tilde{B}(\Delta^t,\Delta^s) \\ \tilde{C}(\Delta^t,\Delta^s) & D(\Delta^t,\Delta^s) \end{bmatrix} = \begin{bmatrix} \tilde{A} + \tilde{B}^0\,\Phi\,\tilde{C}^0 & \tilde{B}^1 + \tilde{B}^0\,\Phi\,D^{01} \\ \tilde{C}^1 + D^{10}\,\Phi\,\tilde{C}^0 & D^{11} + D^{10}\,\Phi\,D^{01} \end{bmatrix}, \tag{4.7}$$

where $\Phi = \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \left(I - D^{00} \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \right)^{-1}$.

Then, the balanced LTSV system (4.1) is defined as (see Fig. 4.2)

$$\tilde{G}(\Delta^t,\Delta^s) = F_u\left(\tilde{G}^0, \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\right), \ \Delta^t \in \Theta^t, \Delta^s \in \Theta^s, \tag{4.8}$$

where, the nominal system matrices are transformed via the balanced transformation

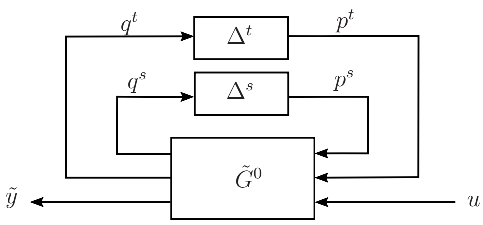

Figure 4.2: LFT representation of temporal- and spatial-LPV balanced system.

$T \in \mathcal{T}$; note that they are connected to the same uncertainty blocks $\Delta^t \in \Theta^t$ and $\Delta^s \in \Theta^s$.

According to Algorithm 3.3, the first step to construct the balanced transformation $T \in \mathcal{T}$ for an LTSV system is to solve (4.5) for $X, Y \in \mathcal{X}$. A practical difficulty is that (4.5) needs to be checked for all $\Delta^t \in \Theta^t, \Delta^s \in \Theta^s$, i.e. at infinitely many points.

If we rewrite (4.5) as $\forall \Delta^t \in \Theta^t, \Delta^s \in \Theta^s$

$$
\begin{aligned}
&\left[\begin{array}{c} I \\ A^*(\Delta^t,\Delta^s) \\ \hline B^*\Delta^t,\Delta^s) \end{array}\right]^* \left[\begin{array}{cc|c} & Y & \\ Y & & \\ \hline & & I \end{array}\right] \left[\begin{array}{c} I \\ A^*(\Delta^t,\Delta^s) \\ \hline B^*(\Delta^t,\Delta^s) \end{array}\right] < 0 \\
&\left[\begin{array}{c} I \\ A(\Delta^t,\Delta^s) \\ \hline C(\Delta^t,\Delta^s) \end{array}\right]^* \left[\begin{array}{cc|c} & X & \\ X & & \\ \hline & & I \end{array}\right] \left[\begin{array}{c} I \\ A(\Delta^t,\Delta^s) \\ \hline C(\Delta^t,\Delta^s) \end{array}\right] < 0,
\end{aligned}
\tag{4.9}
$$

we can apply the full block S-procedure. Using Lemma 4.1 for each of the inequalities in (4.9) individually, we have the following result.

Lemma 4.2 .
The matrix inequalities (4.9) hold for $X, Y \in \mathcal{X}$ iff there exist symmetric matrices $\Pi_x, \Pi_y \in \mathcal{P}$ such that the following conditions hold $\forall\ \Delta^t \in \Theta^t, \Delta^s \in \Theta^s$

$$
[*]^* \left[\begin{array}{cc|c|c} & Y & & \\ Y & & & \\ \hline & & \Pi_y & \\ \hline & & & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A^* & C^{0*} \\ \hline 0 & I \\ B^{0*} & D^{00*} \\ \hline B^{1*} & D^{01*} \end{array}\right] < 0 \tag{4.10}
$$

$$
[*]^*\, \Pi_y \left[\begin{array}{c} \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \\ \hline I \end{array}\right] \geq 0 \tag{4.11}
$$

$$
[*]^* \left[\begin{array}{cc|c|c} & X & & \\ X & & & \\ \hline & & \Pi_x & \\ \hline & & & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B^{0} \\ \hline 0 & I \\ C^{0} & D^{00} \\ \hline C^{1} & D^{10} \end{array}\right] < 0 \tag{4.12}
$$

$$
[*]^*\, \Pi_x \left[\begin{array}{c} \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \\ \hline I \end{array}\right] \geq 0. \tag{4.13}
$$

However, the resulting conditions (4.10), (4.11), (4.12) and (4.13) still have to hold at infinitely many points, (4.11) and (4.13). Using D scales (4.15) below or D-G scales (4.14) [70] has the advantage of converting the problem into a finite dimensional one, possibly at the expense of conservatism.

For that reason, we define the set $\mathcal{P}_{DG}$ of specific symmetric matrices Π that have a D-G scale structure as

$$\mathcal{P}_{DG} = \left\{ \Pi \in \mathcal{P} : \Pi_{11} < 0, \Pi_{22} = -\Pi_{11}, \text{ and } \Pi_{12} = -\Pi_{12}^T \right\}. \tag{4.14}$$

While the set $\mathcal{P}_D$ of symmetric matrices Π that have D scale as

$$\mathcal{P}_D = \{ \Pi \in \mathcal{P} : \Pi_{11} < 0, \Pi_{22} = -\Pi_{11}, \Pi_{12} = 0 \}. \tag{4.15}$$

In order to reduce the number of LMI conditions to be solved (in Lemma 4.2) to a finite number, here we impose a D-G scaling structure on Π_x and Π_y.
Note that imposing a D-G scale structure on the matrices $\Pi \in \mathcal{P}$ results in a conservative solution. However, if only one scheduling parameter is considered, then there is no conservatism [70], which is the case in the beam example, which considers only one spatial scheduling parameter, as will be discussed in Section 4.5.

Therefore, imposing $\Pi_x, \Pi_y \in \mathcal{P}_{DG}$ on conditions (4.10), (4.11), (4.12) and (4.13), we see that the conditions (4.11) and (4.13) are always fulfilled and therefore can be removed, so that we have to satisfy the LMIs (4.10) and (4.12) only; this fact will be used in the rest of the chapter.

Note that stability of (4.1) is preserved in conditions (4.10)–(4.13), [12], [69], [8], as given in the next Theorem which is a direct application of Lemma 4.1 with the introduction of the set $\mathcal{P}_{DG}$.

Theorem 4.1 *(Exponential Stability for LPV/LFT Systems)*
The LTSV system (2.12) is exponentially stable if one of the following two conditions is satisfied

(1) There exists a matrix $X \in \mathcal{X}$, such that $\forall \Delta^t \in \Theta^t, \Delta^s \in \Theta^s$

$$\begin{bmatrix} I \\ A(\Delta^t, \Delta^s) \end{bmatrix}^* \begin{bmatrix} & X \\ X & \end{bmatrix} \begin{bmatrix} I \\ A(\Delta^t, \Delta^s) \end{bmatrix} < 0. \tag{4.16}$$

(2) There exists a matrix $X \in \mathcal{X}$ and a symmetric matrix $\Pi \in \mathcal{P}_{DG}$ such that

$$\left[\begin{array}{cc} I & 0 \\ A & B^0 \\ \hline 0 & I \\ C^0 & D^{00} \end{array}\right]^* \left[\begin{array}{cc|c} & X & \\ X & & \\ \hline & & \Pi \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B^0 \\ \hline 0 & I \\ C^0 & D^{00} \end{array}\right] < 0. \tag{4.17}$$

Based on the above discussion, we propose the following definition

Definition 4.2 *(Balanced LPV/LFT Realization)*
Consider a pair of matrices $X, Y \in \mathcal{X}$ and $\Pi_x, \Pi_y \in \mathcal{P}_{DG}$ satisfying (4.10) and (4.12),

and a transformation matrix $T \in \mathcal{T}$. We call T a balancing transformation if $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2, -\Sigma_3) = T^{-1}Y(T^{-1})^T = T^T X T$ is a diagonal matrix [1]*, i.e. the transformed system is balanced: $\tilde{A} = T^{-1}AT$, $\tilde{B}^0 = T^{-1}B^0$, $\tilde{B}^1 = T^{-1}B^1$, $\tilde{C}^0 = C^0T$, $\tilde{C}^1 = C^1T$, see (4.8) and Fig. 4.2.*

The matrix Σ contains the generalized singular values in descending order along its diagonal. For the integer $n_r = \sum_{i=1}^{3} n_{ri}$ where $n_{ri} < n_i$ for all or some i, we partition $\Sigma = \begin{bmatrix} \Sigma^s & \\ & \Sigma^{ns} \end{bmatrix}$, $\Sigma^s = \operatorname{diag}(\Sigma_1^s, \Sigma_2^s, -\Sigma_3^s)$ and $\Sigma^{ns} = \operatorname{diag}(\Sigma_1^{ns}, \Sigma_2^{ns}, -\Sigma_3^{ns})$ according to the significant (s) and non-significant (ns) generalized singular values such that Σ^s has dimension $n_r \times n_r$ and Σ^{ns} has dimension $(n - n_r \times n - n_r)$. Truncate the states corresponding to Σ^{ns} by partitioning $\tilde{A}, \tilde{B}^0, \tilde{B}^1, \tilde{C}^0, \tilde{C}^1$ conformably with $\begin{bmatrix} \Sigma_i^s & \\ & \Sigma_i^{ns} \end{bmatrix}$, $i = 1, 2, 3$, as (see (2.12))

$$
\tilde{A} = \left[\begin{array}{cc:cc:cc}
(A^{tt})_{11} & (A^{tt})_{12} & (A^{ts}_{+})_{11} & (A^{ts}_{+})_{12} & (A^{ts}_{-})_{11} & (A^{ts}_{-})_{12} \\
(A^{tt})_{21} & (A^{tt})_{22} & (A^{ts}_{+})_{21} & (A^{ts}_{+})_{22} & (A^{ts}_{-})_{21} & (A^{ts}_{-})_{22} \\ \hdashline
(A^{st}_{+})_{11} & (A^{st}_{+})_{12} & (A^{ss}_{++})_{11} & (A^{ss}_{++})_{12} & (A^{ss}_{+-})_{11} & (A^{ss}_{+-})_{12} \\
(A^{st}_{+})_{21} & (A^{st}_{+})_{22} & (A^{ss}_{++})_{21} & (A^{ss}_{++})_{22} & (A^{ss}_{+-})_{21} & (A^{ss}_{+-})_{22} \\ \hdashline
(A^{st}_{-})_{11} & (A^{st}_{-})_{12} & (A^{ss}_{-+})_{11} & (A^{ss}_{-+})_{12} & (A^{ss}_{--})_{11} & (A^{ss}_{--})_{12} \\
(A^{st}_{-})_{21} & (A^{st}_{-})_{22} & (A^{ss}_{-+})_{21} & (A^{ss}_{-+})_{22} & (A^{ss}_{--})_{21} & (A^{ss}_{--})_{22}
\end{array}\right], \quad
\tilde{B}^0 = \left[\begin{array}{cc}
(B^{0,tt})_1 & (B^{0,ts})_1 \\
(B^{0,tt})_2 & (B^{0,ts})_2 \\ \hdashline
(B^{0,st}_{+})_1 & (B^{0,ss}_{+})_1 \\
(B^{0,st}_{+})_2 & (B^{0,ss}_{+})_2 \\ \hdashline
(B^{0,st}_{-})_1 & (B^{0,ss}_{-})_1 \\
(B^{0,st}_{-})_2 & (B^{0,ss}_{-})_2
\end{array}\right],
$$

$$
\tilde{B}^1 = \left[\begin{array}{c}
(B^{1,t})_1 \\
(B^{1,t})_2 \\ \hdashline
(B^{1,s}_{+})_1 \\
(B^{1,s}_{+})_2 \\ \hdashline
(B^{1,s}_{-})_1 \\
(B^{1,s}_{-})_2
\end{array}\right], \quad
\tilde{C}^0 = \left[\begin{array}{cc:cc:cc}
(C^{0,tt})_1 & (C^{0,tt})_2 & (C^{0,ts}_{+})_1 & (C^{0,ts}_{+})_2 & (C^{0,ts}_{-})_1 & (C^{0,ts}_{-})_2 \\
(C^{0,st})_1 & (C^{0,st})_2 & (C^{0,ss}_{+})_1 & (C^{0,ss}_{+})_2 & (C^{0,ss}_{-})_1 & (C^{0,ss}_{-})_2
\end{array}\right],
$$

$$
\tilde{C}^1 = \left[\begin{array}{cc:cc:cc}
(C^{1,t})_1 & (C^{1,t})_2 & (C^{1,s}_{+})_1 & (C^{1,s}_{+})_2 & (C^{1,s}_{-})_1 & (C^{1,s}_{-})_2
\end{array}\right], \tag{4.18}
$$

where the dimension of each sub-matrix is given by the dimension of partitioned Σ. Then we end up with the reduced nominal system

$$
G_r^0 = \left[\begin{array}{ccc:cc:c}
(A^{tt})_{11} & (A^{ts}_{+})_{11} & (A^{ts}_{-})_{11} & (B^{0,tt})_1 & (B^{0,ts})_1 & (B^{1,t})_1 \\
(A^{st}_{+})_{11} & (A^{ss}_{++})_{11} & (A^{ss}_{+-})_{11} & (B^{0,st}_{+})_1 & (B^{0,ss}_{+})_1 & (B^{1,s}_{+})_1 \\
(A^{st}_{-})_{11} & (A^{ss}_{-+})_{11} & (A^{ss}_{--})_{11} & (B^{0,st}_{-})_1 & (B^{0,ss}_{-})_1 & (B^{1,s}_{-})_1 \\ \hdashline
(C^{0,tt})_1 & (C^{0,ts}_{+})_1 & (C^{0,ts}_{-})_1 & \multicolumn{2}{c:}{\multirow{2}{*}{D^{00}}} & \multirow{2}{*}{D^{01}} \\
(C^{0,st})_1 & (C^{0,ss}_{+})_1 & (C^{0,ss}_{-})_1 & & & \\ \hdashline
(C^{1,t})_1 & (C^{1,s}_{+})_1 & (C^{1,s}_{-})_1 & \multicolumn{2}{c:}{D^{10}} & D^{11}
\end{array}\right] = \begin{bmatrix} A_r & B_r^0 & B_r^1 \\ C_r^0 & D^{00} & D^{01} \\ C_r^1 & D^{10} & D^{11} \end{bmatrix} \tag{4.19}
$$

[1] In some cases, inverses of $X, Y \in \mathcal{X}$ are required rather than the original matrices, [40].

such that

$$G_r(\Delta^t, \Delta^s) = F_u\left(G_r^0, \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\right), \; \Delta^t \in \Theta^t, \Delta^s \in \Theta^s \tag{4.20}$$

Remark 4.1 *Note that as in (3.2), according to (4.5), $X, Y \in \mathcal{X}$ have both positive and negative eigenvalues, so does the matrix Σ, where $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2, -\Sigma_3)$.*

Theorem 4.2 *Suppose that $G_r(\Delta^t, \Delta^s)$ in the form of (4.20) is obtained from the exponentially stable $G(\Delta^t, \Delta^s)$ in the form of (4.1) according to Definition 4.2. Then $G_r(\Delta^t, \Delta^s)$ is balanced, exponentially stable and*

$$\|G(\Delta^t, \Delta^s) - G_r(\Delta^t, \Delta^s)\|_{2\to 2} \leq 2\left(\sum_{i=1}^{3} \sum_{g=r_i+1}^{n_i} \sigma_{i,g}\right) \tag{4.21}$$

where $\sigma_{i,g}$ are the absolute values of the diagonal entries of $\operatorname{diag}(\Sigma_1^{ns}, \Sigma_2^{ns}, -\Sigma_3^{ns})$.

Proof According to (4.7) and (4.8), the transformed nominal system matrices (4.18) define the balanced system $\tilde{G}(\Delta^t, \Delta^s)$ as in Fig. 4.2. Since the original system $G(\Delta^t, \Delta^s)$ satisfies (4.10) and (4.12) with $X, Y \in \mathcal{X}$ and $\Pi_x, \Pi_y \in \mathcal{P}_{DG}$. Clearly, the balanced system $\tilde{G}(\Delta^t, \Delta^s)$ still satisfies (4.10),(4.12) with Σ and $\Pi_x, \Pi_y \in \mathcal{P}_{DG}$.

Therefore, if we truncate the non-significant states and substitute (4.19) and Σ^s, then the resulting reduced model is still balanced with Σ^s and exponentially stable, see Theorem 4.1.
The error bound (4.21) can be proved following a similar way as in Theorem 3.3 or Theorem 5.1. ■

Clearly from Theorem 4.2 the error bound in (4.21) is based on the truncated generalized singular values of the generalized Gramians, so it would be reasonable to minimize their rank as stated in Chapter 3. Therefore, rank constraints on X and $Y \in \mathcal{X}$ (i.e. minimize rank(X) and rank (Y)) are imposed on conditions (4.10) and (4.12), as in Algorithm 3.3.1.
Minimizing the rank of $X, Y \in \mathcal{X}$ while solving the reduction problem, improves the balanced truncation error bound. However, the error bound in (4.21) can be improved further as shown in the next subsection.

4.4 Improved Error Bound Model Order Reduction Problem

As already discussed and shown in Chapters 2 and 3, respectively (in Section 2.4 and Section 3.4), a bound on the induced 2-norm of the stable error system can be represented by the Integral Quadratic Constraint (IQC)

$$\int_{t=0}^{\infty}\int_{s=-\infty}^{\infty} \begin{pmatrix} d(t,s) \\ z(t,s) \end{pmatrix}^T \begin{pmatrix} -\gamma I & \\ & \gamma^{-1} I \end{pmatrix} \begin{pmatrix} d(t,s) \\ z(t,s) \end{pmatrix} \leq 0, \tag{4.22}$$

which is equivalent to the existence of a matrix $X_e \in \mathcal{Y}$ (as defined in (3.15)) and a matrix Π_e such that [12], [8] $\forall \Delta_e^{t,s} \in \Theta_e^{t,s}$

$$[*]^* \left[\begin{array}{cc|c|cc} & X_e & & & \\ X_e & & & & \\ \hline & & \Pi_e & & \\ \hline & & & -\gamma I & \\ & & & & \gamma^{-1} I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ A_e & B_e^0 & B_e^1 \\ \hline 0 & I & 0 \\ C_e^0 & D_e^{00} & D_e^{01} \\ \hline 0 & 0 & I \\ C_e^1 & D_e^{10} & D_e^{11} \end{array}\right] < 0 \tag{4.23}$$

$$[*]^T \, \Pi_e \left[\begin{array}{c} \Delta_e^{t,s} \\ \hline I \end{array}\right] \geq 0, \tag{4.24}$$

where $\Theta_e^{t,s} = \{\Delta_e^{t,s} : \Delta_e^{t,s} = \mathrm{diag}(\Delta^t, \Delta^s, \Delta^t, \Delta^s), \Delta^t \in \Theta^t, \Delta^s \in \Theta^s\}$ where we assume that the reduced system is connected to the same uncertainty blocks that for the original system.

Accordingly, we define a symmetric matrix $\Pi_e = \begin{bmatrix} \Pi_{11}^e & \Pi_{12}^e \\ \Pi_{12}^{eT} & \Pi_{22}^e \end{bmatrix}$, where

$$\Pi_{mk}^e = \begin{bmatrix} \Pi_{mk}^{11} & \Pi_{mk}^{12} \\ \Pi_{mk}^{12T} & \Pi_{mk}^{11} \end{bmatrix}, m, k = 1, 2, \quad \text{such that} \quad \begin{bmatrix} \Pi_{11}^{11} & \Pi_{12}^{11} \\ \Pi_{12}^{11T} & \Pi_{22}^{11} \end{bmatrix} = \Pi \in \mathcal{P}. \tag{4.25}$$

The nominal error system matrices are defined as

$$A_e = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \quad B_e^0 = \begin{bmatrix} B^0 & 0 \\ 0 & B_r^0 \end{bmatrix}, \quad C_e^0 = \begin{bmatrix} C^0 & 0 \\ 0 & C_r^0 \end{bmatrix}, \quad D_e^{00} = \begin{bmatrix} D^{00} & 0 \\ 0 & D_r^{00} \end{bmatrix},$$
$$B_e^1 = \begin{bmatrix} B^1 \\ B_r^1 \end{bmatrix}, \quad D_e^{01} = \begin{bmatrix} D^{01} \\ D_r^{01} \end{bmatrix}, \quad C_e^1 = \begin{bmatrix} C^1 & -C_r^1 \end{bmatrix}, \quad D_e^{10} = \begin{bmatrix} D^{10} & -D_r^{10} \end{bmatrix} \quad \text{and}$$
$$D_e^{11} = D^{11} - D_r^{11}.$$

That is done following a similar result for LTSI systems (see (2.28)); here we define $\hat{G}(\Delta^t, \Delta^s) = F_u\left(\begin{bmatrix} G^0 & -I \\ I & 0 \end{bmatrix}, \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\right)$, see Fig 4.3, such that

$$\hat{G}_{\Delta^t,\Delta^s} : \begin{cases} \left[\begin{array}{c} \tilde{\xi}(t,s) \\ \hline q^t(t,s) \\ q^s(t,s) \\ \hline z(t,s) \\ \hline y(t,s) \end{array}\right] = \left[\begin{array}{c|c|c|c} A & B^0 & B^1 & 0 \\ \hline C^0 & D^{00} & D^{01} & 0 \\ \hline C^1 & D^{10} & D^{11} & -I \\ \hline 0 & 0 & I & 0 \end{array}\right] \left[\begin{array}{c} \xi(t,s) \\ \hline p^t(t,s) \\ p^s(t,s) \\ \hline d(t,s) \\ \hline u(t,s) \end{array}\right] \\ \begin{bmatrix} p^t(t,s) \\ p^s(t,s) \end{bmatrix} = \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \begin{bmatrix} q^t(t,s) \\ q^s(t,s) \end{bmatrix}, \ \Delta^t \in \Theta^t, \Delta^s \in \Theta^s. \end{cases} \tag{4.26}$$

Then, based on (4.23) and (4.24), in order to find a reduced LPV/LFT system while ensuring a small error bound γ, we have the following result.

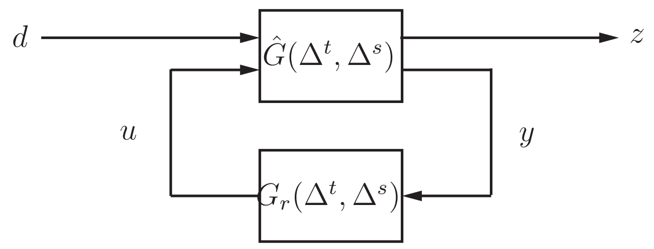

Figure 4.3: Error system configuration

Reduction Problem: Given γ, find a reduced model (4.2) and $X_e \in \mathcal{Y}$, Π_e such that (4.23) and (4.24) hold for all $\Delta_e^{t,s} \in \Theta_e^{t,s}$.

In order to be able to solve (4.23), (4.24) for the reduced system, we have again to apply the elimination lemma as in Chapter 3. Therefore, we redefine the error system matrices as [12]

$$\begin{bmatrix} A_e & B_e^0 & B_e^1 \\ C_e^0 & D_e^{00} & D_e^{01} \\ C_e^1 & D_e^{10} & D_e^{11} \end{bmatrix} = \left[\begin{array}{cc|cc|c} A & 0 & B^0 & 0 & B^1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline C^0 & 0 & D^{00} & 0 & D^{01} \\ 0 & 0 & 0 & 0 & 0 \\ \hline C^1 & 0 & D^{10} & 0 & D^{11} \end{array}\right] + U \begin{bmatrix} A_r & B_r^1 & B_r^0 \\ C_r^1 & D_r^{11} & D_r^{10} \\ C_r^0 & D_r^{01} & D_r^{00} \end{bmatrix} V^*,$$

$$\text{where } U = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & I \\ \hline 0 & -I & 0 \end{array}\right], \quad \text{and} \quad V = \left[\begin{array}{ccc} 0 & 0 & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & I \\ \hline 0 & I & 0 \end{array}\right].$$

We then construct $U_\perp^* = \left[\begin{array}{cc|cc|c} N_{y1} & 0 & N_{y2} & 0 & N_{y3} \end{array}\right]$, $V_\perp^* = \left[\begin{array}{cc|cc|c} N_{x1} & 0 & N_{x2} & 0 & N_{x3} \end{array}\right]$, where

$$N_y = \begin{bmatrix} N_{y1} & N_{y2} & N_{y3} \end{bmatrix}^* = \text{Ker} \begin{bmatrix} 0 & 0 & -I \end{bmatrix} \quad \text{and} \quad N_x = \begin{bmatrix} N_{x1} & N_{x2} & N_{x3} \end{bmatrix}^* = \text{Ker} \begin{bmatrix} 0 & 0 & I \end{bmatrix}.$$

Clearly, $N_y = N_x = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}^T$.

Then we have the result of Theorem 4.3 below.

Before we present the next result, in order to decrease the value of γ together with the rank constraint (as mentioned before), we summarize the proposed model order reduction scheme (the extended version of the one proposed in Chapter 3 for LTSI systems), which allows a trade-off between minimizing the rank of the generalized Gramians and improving the error bound γ.

Algorithm 4.4: *Model Order Reduction Scheme for LPV/LFT*

1) Apply the balanced realization procedure, Definition 4.2, and set the initial upper bound to value less than the balanced truncation error bound ($\gamma < 2\sum_{i=1}^{3}\sum_{g=n_{ri}+1}^{n_i}\sigma_{ig}$). We determine the smallest generalized singular values, the initial new order $n_r = \sum_{i=1}^{3} n_{ri}$ is selected for the reduced system. The resulting balanced system and γ are used to initialize the next step.

2) Solve LMIs (4.27), (4.29), (4.31) and condition (4.32) of Theorem 4.3 below for $X, Y \in \mathcal{X}$ and $\Pi_x, \Pi_y \in \mathcal{P}_{DG}$, and obtain X and Y including the small generalized singular values.

3) Check the rank of $Y_i X_i - I$ for $i = 1, 2, 3$; if $\mathrm{rank}(Y_i X_i - I)$ is greater than n_{ri}, for some i, then increase the value of γ, if not then decrease the value of γ (if necessary). Go to step 2) until satisfactory results are obtained.

The following Theorem is inspired by [8].

Theorem 4.3 *Given $G(\Delta^t, \Delta^s)$ and $\gamma > 0$, there exists $G_r(\Delta^t, \Delta^s)$ (as defined in (4.2)) such that $\|G(\Delta^t, \Delta^s) - G_r(\Delta^t, \Delta^s)\|_{2\to 2} \leq \gamma$, if there exist $X, Y \in \mathcal{X}, \Pi_x, \Pi_y \in \mathcal{P}$, such that $\forall \Delta^t \in \Theta^t, \Delta^s \in \Theta^s$, we have*

$$N_y^* \, [*]^* \left[\begin{array}{cc|c|cc} & Y & & & \\ Y & & & & \\ \hline & & \Pi_y & & \\ \hline & & & -\gamma^{-1} I & \\ & & & & \gamma I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ A^* & C^{0*} & C^{1*} \\ \hline 0 & I & 0 \\ B^{0*} & D^{00*} & D^{10*} \\ \hline 0 & 0 & I \\ B^{1*} & D^{01*} & D^{11*} \end{array}\right] N_y < 0 \tag{4.27}$$

$$[*]^* \, \Pi_y \left[\begin{array}{c} \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \\ \hline I \end{array}\right] \geq 0 \tag{4.28}$$

$$N_x^* \, [*]^* \left[\begin{array}{cc|c|cc} & X & & & \\ X & & & & \\ \hline & & \Pi_x & & \\ \hline & & & -\gamma I & \\ & & & & \gamma^{-1} I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ A & B^{0} & B^{1} \\ \hline 0 & I & 0 \\ C^{0} & D^{00} & D^{01} \\ \hline 0 & 0 & I \\ C^{1} & D^{10} & D^{11} \end{array}\right] N_x < 0 \tag{4.29}$$

$$[*]^* \, \Pi_x \left[\begin{array}{c} \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \\ \hline I \end{array}\right] \geq 0 \tag{4.30}$$

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} \geq 0 \tag{4.31}$$

$$\mathrm{rank}(YX - I) \leq n_r \tag{4.32}$$

where $N_y = N_x = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}^T$.

The D-G structure (4.14) as in Section 4.3 can be imposed on the multipliers Π_x and Π_y here in order to reduce the problem to finite dimension, where as mentioned at the beginning of this section, we consider that the reduced model is connected to the same uncertainty blocks that for the original model.
In addition, a cone complementarity method (as in Algorithm 3.4) is used here to solve the non-convex condition (4.32).

Having obtained solutions $X, Y \in \mathcal{X}$ and $\Pi_x, \Pi_y \in \mathcal{P}_{DG}$ (by satisfying Theorem 4.3), a reduced order LPV/LFT spatially interconnected system can be constructed in the following steps.

(i) Use X and Y to compute X_e (which satisfies (4.23)) in the same way as in the LTSI systems case proposed in Chapter 3 , Remark 3.4.

(ii) Use $\Pi_x, \Pi_y \in \mathcal{P}_{DG}$ to compute Π_e (in (4.23)) as in [69], [8], where for skew symmetric Π_{12}^e

$$\Pi_e = \begin{bmatrix} -\Pi_{11}^e & \Pi_{12}^e \\ \Pi_{12}^{eT} & \Pi_{11}^e \end{bmatrix}, \Pi_{mk}^e = \begin{bmatrix} -\Pi_{mk}^{11} & \Pi_{mk}^{12} \\ \Pi_{mk}^{12T} & \Pi_{mk}^{11} \end{bmatrix}, \quad \Pi_{mk}^{12} \text{ is skew symmetric }, m, k = 1, 2. \tag{4.33}$$

(iii) Solve the LMI (4.23) for the reduced nominal system matrices.

(iv) Define the reduced system as in (4.2), with the same uncertainty blocks as the original system, i.e. $\Delta^t \in \Theta^t$ and $\Delta^s \in \Theta^s$.

4.5 Application to an Actuated Beam

To demonstrate the results of this chapter, the proposed model order reduction technique has been applied again to the same experimentally identified model [8] of the actuated beam shown in Fig.1.2 in Chapter 1 and used in Chapter 3. Now, six (arbitrarily located) actuator-sensor pairs have been deactivated, whereas the remaining ten *unevenly* spaced actuator-sensor pairs (piezo patches) are still active and distributed along the length of the aluminium beam, such that the beam has unequal length subsystems. The beam is accordingly divided into 10 nonidentical subsystems as in Fig. 4.4; each subsystem has its own actuator-sensor capabilities.

Since the beam has been discretized into ten unequal length subsystems, the dynamic behavior of the system is now represented by a spatial LPV model with one spatial scheduling parameter, $\rho(s)$. [8].

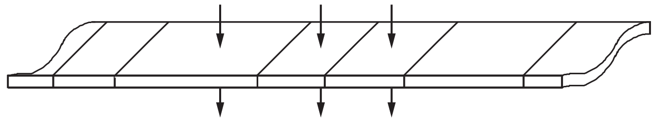

Figure 4.4: Aluminium beam with unevenly spaced actuator-sensor pairs

Because the actuator-sensor locations s are unevenly spaced, we have $s \in \mathbb{R}$ and introduce a mapping $q : \mathbb{R} \to \mathbb{Z}$, so that we replace $\rho(s)$ by $\rho(q)$ in order to be able to apply spatial shift operations.

A single subsystem is represented by a two-dimensional difference equation

$$y(t,q) = -\sum_{i_t,i_s \in Y_{mask}} \alpha_{(i_t,i_s)}(\rho(q))y(t-i_t, q-i_s) + \sum_{i_t,i_s \in U_{mask}} \beta_{(i_t,i_s)}(\rho(q))u(t-i_t, q-i_s) \quad (4.34)$$

where U_{mask} and Y_{mask} are the input and output masks, respectively, determined at the subsystem level as shown in Fig. 4.5, indicating which temporally and spatially shifted inputs and outputs contribute to the current output, $\alpha_{(i_t,i_s)}(\rho)$ and $\beta_{(i_t,i_s)}(\rho)$ are the coefficients varying with respect to space.
If one compares (4.34) with (3.27) in Chapter 3, we can see that here the coefficients $\alpha_{(i_t,i_s)}(\rho)$ and $\beta_{(i_t,i_s)}(\rho)$ are allowed to vary with respect to space, while in (3.27) the coefficients a_{i_t,i_s} and b_{i_t,i_s} are constants.

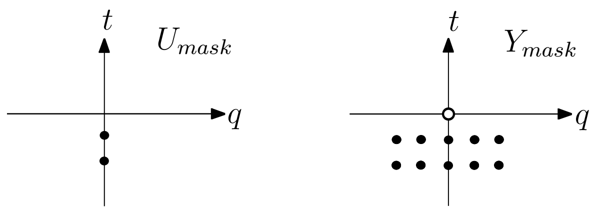

Figure 4.5: Input and output mask for spatial LPV model

The state space model of a single subsystem is defined in Appendix D.2 as in (2.9) according to (4.34), with suitably chosen temporal and spatial state vectors.

Each subsystem has state order $n = n_1 + n_2 + n_3 = 10$, where $n_1 = 2$ and $n_2 = n_3 = 4$, see Fig. 4.5.

After pulling out the scheduling parameter ρ into a delta block, a state space model of a single subsystem is defined in LFT form with respect to $\Delta^s \in \Theta^s$ as in (2.12) with constant Δ^t. Its LFT representation is shown in Fig. 4.6 as $F_u(G^0, \Delta^s)$, where

$$G^0 = \begin{bmatrix} A & B^0 & B^1 \\ C^0 & D^{00} & D^{01} \\ C^1 & D^{10} & D^{11} \end{bmatrix} \quad \text{and} \quad \Delta^s = \rho I_2.$$

Note that $n^s = 2$, since ρ has multiplicity 2 (see Appendix D.2, where ρ appears in $A(\rho)$

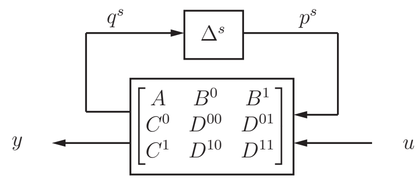

Figure 4.6: Spatial LPV/LFT for Actuated Beam

and $C(\rho)$).
As mentioned before, the system has order $n = 10$. The application of the approach proposed in Section 4.4 (see Algorithm 4.4) gives a reduced system of order $n_r = 6$; $(n_{r1} = n_{r2} = n_{r3} = 2)$, and provides an initial error bound (set to $\gamma < 0.017$) which is improved to $\gamma = 0.0026$ in Section 4.4.

First, a balanced truncation in Section 4.3 has been applied by solving conditions (4.10) and (4.12) (with rank constraint) for $\Pi_x, \Pi_y \in \mathcal{P}_{DG}$ and $X, Y \in \mathcal{X}$ with small generalized singular values. Following the procedure of Definition 4.2, we get a reduced system of order $(n_{r1} = 2, n_{r2} = n_{r3} = 3)$; the selected initial error bound is $\gamma = 0.017$. This completes step 1 of the proposed MOR scheme (Algorithm 4.4) and provides the start values for step 2.

In step 2, starting with an error bound $\gamma < 0.017$ we try to reduce this error bound and the order of the system further by solving conditions (4.27), (4.29), (4.31) and (4.32) for $\Pi_x, \Pi_y \in \mathcal{P}_{DG}$ and $X, Y \in \mathcal{X}$ with small generalized singular values. Then, following the four steps given at the very end of Section 4.4 gives the required reduced system with order $(n_{r1} = n_{r2} = n_{r3} = 2)$ and $\gamma = 0.0026$.

Fig. 4.7 shows the response over time of the first, fifth, seventh and tenth subsystems (i.e. sensors 1,5,7 and 10), respectively, to a step disturbance applied on actuator 5, comparing the original identified model (original) and the reduced model based on the full block S-procedure (FBSP, CGs). Furthermore, Fig.4.7 includes a comparison with the MOR approach based on gridding; this result has been presented in [66] and [84]. It is worth mentioning that the result based gridding is obtained using parameter-dependent generalized Gramians, i.e. $X(\rho)$ and $Y(\rho)$; whereas here we are using constant generalized Gramians (a comparison between the using of parameter-dependent generalized Gramians and constant generalized Gramians is considered in Chapter 5). Even though, the results are of comparable accuracy. This suggests that it is more efficient to solve the reduction problem for systems defined in LPV/LFT representation with the application of the full block S-procedure and utilizing D-G scaling, rather than using a grid representation, because in the former case, only a single pair of LMIs have to be solved instead of solving one at each grid point.

In addition, based on the FBSP, the response to a step disturbance over space at time $t = 1.05$, and $t = 4.05$ seconds, respectively, is shown in Fig.4.8, which compares the original identified model (original) and the reduced model.

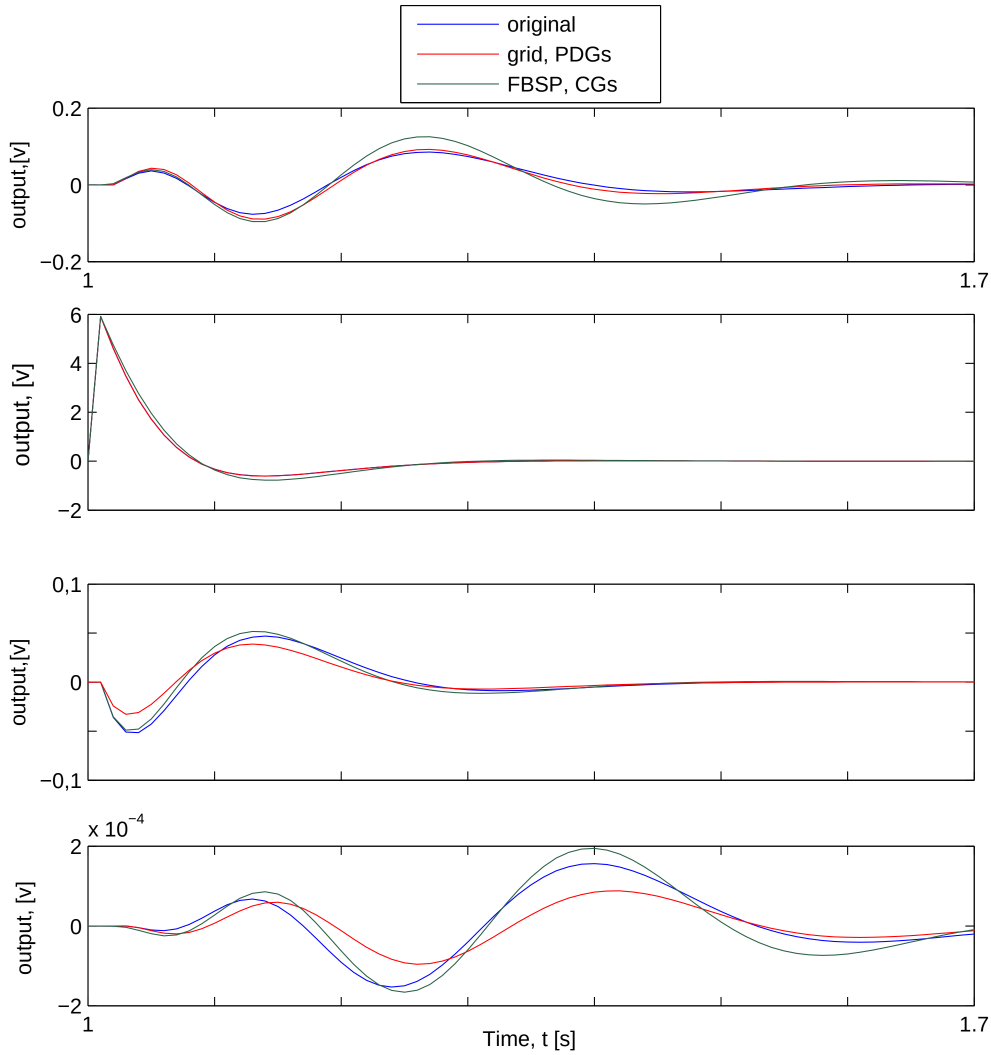

Figure 4.7: Response over time of (from top to bottom) the 1^{st}, 5^{th}, 7^{th} and 10^{th} LPV subsystems to a disturbance step

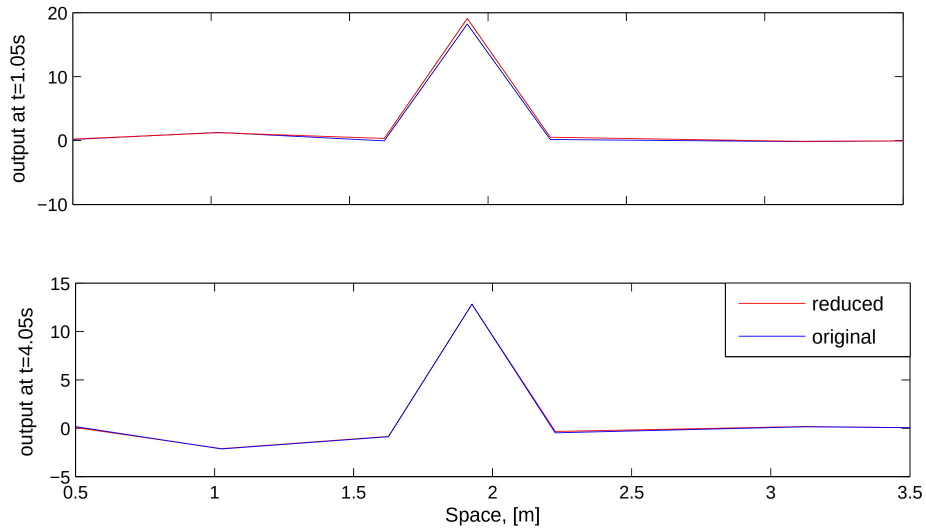

Figure 4.8: Response over space of the beam at time (from top to bottom) $t = 1.05\ s$, and $t = 4.05\ s$ to a disturbance step

Finally, experimental measurements have been used as well, with the injection of 10 noise signals in parallel into 10 actuators as inputs to the original and reduced system, respectively. The measured outputs (measured), simulated outputs using the original model (full), and the outputs of the reduced model (reduced) for all subsystems are shown in Fig. 4.9.

4.6 Conclusion

In this chapter, an extension of the MOR technique proposed in Chapter 3 has been presented for temporal- and spatial-LPV interconnected systems. The nominal system matrices are reduced and connected again with the same uncertainty blocks as for the original system, while the stability of the system is preserved in the reduced model. In order to reduce an exponentially stable LPV/LFT spatially interconnected systems, the full block S-procedure has been utilized, which leads to efficient solutions, reduces conservatism and simplifies the MOR problem in the sense of avoiding gridding. Based on the result of the previous chapter, a novel way of dealing with the non-causality of the spatial dynamics (when solving the MOR for temporal- and spatial-LPV interconnected systems) is proposed as well. An application of the proposed procedure to an experimentally identified actuated beam, demonstrated its practicality.

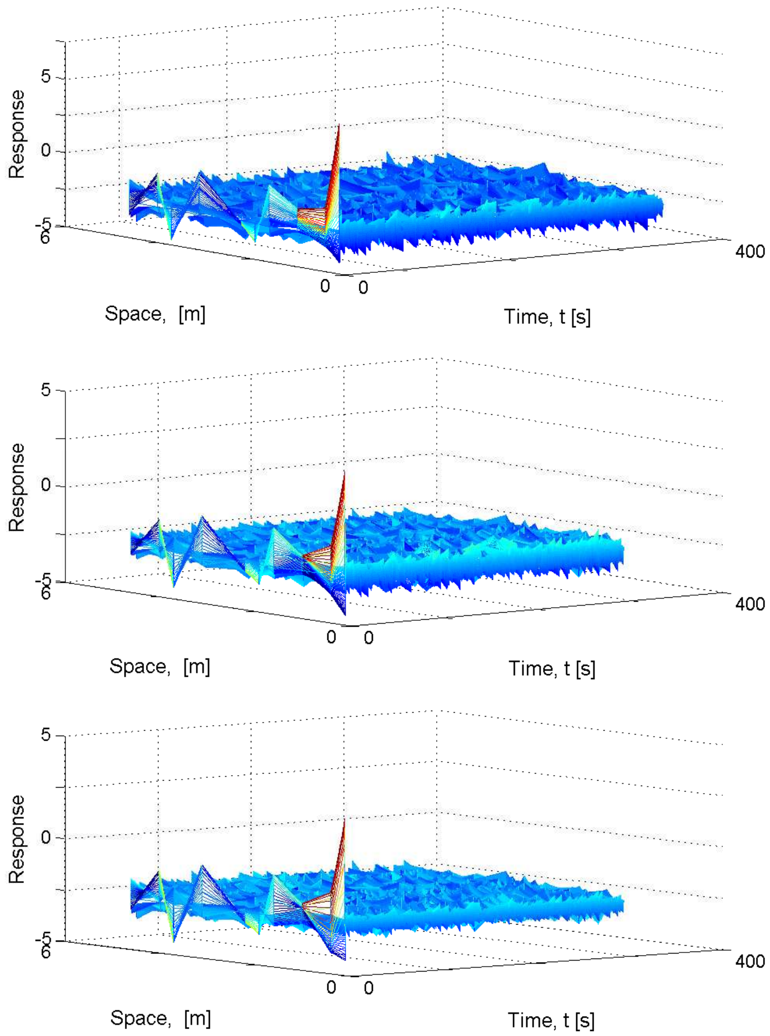

Figure 4.9: Measured response of the beam (measured, top), simulated response of the beam (full, middle), response of the reduced model (reduced, bottom) to 10 noises signals applied in parallel to 10 actuators.

Chapter 5

Joint Dynamic and Scheduling Order Reduction

5.1 Introduction

Usually, model order reduction refers to the number of states, but for LPV systems this is not always the case, because the complexity of LPV systems is not determined by its state order only, but also by the order of the scheduling parameters that represent the system.

In this chapter, a technique for joint dynamic and scheduling order reduction (i.e. reduction of the number of states as well as of scheduling parameters) of exponentially stable LTSV systems is presented, the technique is based on balanced truncation. Again, the full block S-procedure is applied here. The reduced models preserve exponential stability and the spatial structure of the system.

The key idea of joint dynamic and scheduling order reduction is to simultaneously diagonalize the Gramians X and Y as well as the multipliers Π_x and Π_y, then apply the usual balanced truncation technique.

In contrast to the previous chapter, where the nominal system has been reduced and connected with the same uncertainty blocks as for the original system (i.e. the same number of scheduling parameters is kept), here both the nominal system as well as the uncertainty blocks are reduced (i.e. the number of scheduling parameters and of states is reduced), such that an error bound for the uncertainty block reduction has to be considered as well as the error bound for the state order reduction. This chapter considers both constant generalized Gramians (CGs) and Parameter-Dependent generalized Gramians (PDGs). Where for the latter case, a parameter-dependent balanced transformation is constructed. Note when PDGs are used, then the reduction procedure should be done in two steps, first reduce the scheduling order, and second reduce the state order of the system. While when CGs are used, then the scheduling order and the state order are simultaneously reduced. A comparison between the use of CGs and PDGs is presented as well.

The same actuated beam (Fig. 4.4) is used to demonstrate the proposed method. The results of this chapter are based on [68].

The chapter is structured as follows. In Section 5.2, the joint order reduction via balanced truncation for LTSV systems is presented. A less conservative result based on the use of PDGs is given in Section 5.3, where a generalization of the result of the previous section is derived. The efficiency of the proposed approach is demonstrated on the actuated beam in Section 5.4 with a comparison between using CGs and PDGs.

5.2 Joint Order Reduction Based on Balanced Truncation

Problem Statement

The model reduction problem considered here can be formulated as follows: given $G(\Delta^t, \Delta^s)$ as in (2.12), (2.13) with state order n and temporal and spatial scheduling orders n^t and n^s, respectively, defined as

$$G(\Delta^t, \Delta^s) = F_u\left(G^0, \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\right) = \left[\begin{array}{c:c} A(\Delta^t, \Delta^s) & B(\Delta^t, \Delta^s) \\ \hdashline C(\Delta^t, \Delta^s) & D(\Delta^t, \Delta^s) \end{array}\right], \Delta^t \in \Theta^t, \Delta^s \in \Theta^s, \tag{5.1}$$

where

$$\begin{aligned} \begin{bmatrix} A(\Delta^t, \Delta^s) & B(\Delta^t, \Delta^s) \\ C(\Delta^t, \Delta^s) & D(\Delta^t, \Delta^s) \end{bmatrix} &= \begin{bmatrix} A + B_0\, \Phi\, C_0 & B_1 + B_0\, \Phi\, D_{01} \\ C_1 + D_{10}\, \Phi\, C_0 & D_{11} + D_{10}\, \Phi\, D_{01} \end{bmatrix}, \\ \Phi &= \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} \left(I - D_{00} \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}\right)^{-1}. \end{aligned} \tag{5.2}$$

find a reduced order model (2.20), (2.21) with reduced state order $n_r = n_{r1}+n_{r2}+n_{r3} < n$, reduced temporal and spatial scheduling orders $n^t_r < n^t$ and $n^s_r < n^s$, respectively, defined as

$$\begin{aligned} &G_r(\Delta^t_r, \Delta^s_r) = \\ &F_u\left(G^0_r, \begin{bmatrix} \Delta^t_r & \\ & \Delta^s_r \end{bmatrix}\right) = \left[\begin{array}{c:c} A_r(\Delta^t_r, \Delta^s_r) & B_r(\Delta^t_r, \Delta^s_r) \\ \hdashline C_r(\Delta^t_r, \Delta^s_r) & D_r(\Delta^t_r, \Delta^s_r) \end{array}\right], \Delta^t_r \in \Theta^t_r \subset \Theta^t,\ \Delta^s_r \in \Theta^s_r \subset \Theta^s, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} \left[\begin{array}{c:c} A_r(\Delta^t_r, \Delta^s_r) & B_r(\Delta^t_r, \Delta^s_r) \\ \hdashline C_r(\Delta^t_r, \Delta^s_r) & D_r(\Delta^t_r, \Delta^s_r) \end{array}\right] &= \begin{bmatrix} A_r + B^0_r\, \Phi_r\, C^0_r & B^1_r + B^0_r\, \Phi_r\, D^{01}_r \\ C^1_r + D^{10}_r\, \Phi_r\, C^0_r & D^{11} + D^{10}_r\, \Phi_r\, D^{01}_r \end{bmatrix}, \\ \Phi_r &= \begin{bmatrix} \Delta^t_r & \\ & \Delta^s_r \end{bmatrix} \left(I - D^{00}_r \begin{bmatrix} \Delta^t_r & \\ & \Delta^s_r \end{bmatrix}\right)^{-1}. \end{aligned} \tag{5.4}$$

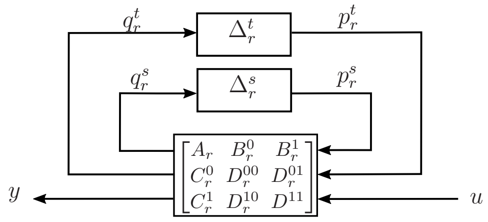

Figure 5.1: A joint reduced state and scheduling order subsystem $G_r(\Delta_r^t, \Delta_r^s)$.

as shown in Fig. 5.1, such that

$$\|G(\Delta^t, \Delta^s) - G_r(\Delta_r^t, \Delta_r^s)\|_{2\to 2} \quad \text{is} \quad \text{minimized.}$$

In this section, for simplicity of presentation, we concentrate on the single parameter case, i.e., $n_\delta = n_\rho = 1$, see (2.11). Also, as in Chapter 4, for simplicity we consider $D^{00} = 0$.

In order to solve the above problem, as in Chapter 4, with the introduction of the balanced transformation $T \in \mathcal{T}$ (as in (3.10)), system (5.1) is transformed to its balanced realization. Here, in addition to the transformation $T \in \mathcal{T}$, we need another transformation $W \in \mathcal{W}$ where

$$\mathcal{W} = \left\{ W = \begin{bmatrix} W_{\Delta^t} & \\ & W_{\Delta^s} \end{bmatrix} : \Delta^t\, W_{\Delta^t} = W_{\Delta^t}\, \Delta^t,\ \Delta^s\, W_{\Delta^s} = W_{\Delta^s}\, \Delta^s,\ \det(W) \neq 0 \right\}. \tag{5.5}$$

Using both transformation matrices, i.e. $T \in \mathcal{T}$ and $W \in \mathcal{W}$, the change of coordinates in (5.1) gives (see Fig. 5.2)[1]

$$\begin{aligned} \tilde{G}(\tilde{\Delta}^t, \tilde{\Delta}^s) &= \left[\begin{array}{c|c} \tilde{A}(\tilde{\Delta}^t, \tilde{\Delta}^s) & \tilde{B}(\tilde{\Delta}^t, \tilde{\Delta}^s) \\ \hline \tilde{C}(\tilde{\Delta}^t, \tilde{\Delta}^s) & \tilde{D}(\tilde{\Delta}^t, \tilde{\Delta}^s) \end{array}\right] \\ &= \begin{bmatrix} T^{-1} & \\ & I \end{bmatrix} \left[\begin{array}{c|c} A(\tilde{\Delta}^t, \tilde{\Delta}^s) & B(\tilde{\Delta}^t, \tilde{\Delta}^s) \\ \hline C(\tilde{\Delta}^t, \tilde{\Delta}^s) & D(\tilde{\Delta}^t, \tilde{\Delta}^s) \end{array}\right] \begin{bmatrix} T & \\ & I \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A} + \tilde{B}^0 \tilde{\Phi} \tilde{C}^0 & \tilde{B}^1 + \tilde{B}^0 \tilde{\Phi} \tilde{D}^{01} \\ \tilde{C}^1 + \tilde{D}^{10} \tilde{\Phi} \tilde{C}^0 & D^{11} + \tilde{D}^{10} \tilde{\Phi} \tilde{D}^{01} \end{bmatrix} \end{aligned} \tag{5.6}$$

[1] For simplicity of presentation in Fig. 5.2, we combine the two uncertainty blocks Δ^t and Δ^s together in one block. This will be used in figures throughout this chapter. Also, we combine the uncertainty temporal channel p^t with the uncertainty spatial channel p^s in one vector $p = \begin{bmatrix} p^t \\ p^s \end{bmatrix}$. This vector p will be used also in the proof of **Theorem 5.1**. The same is true for the vector q as well.

$$\text{with} \quad \tilde{\Phi} = \begin{bmatrix} \tilde{\Delta}^t & \\ & \tilde{\Delta}^s \end{bmatrix} \left(I - \tilde{D}_{00} \begin{bmatrix} \tilde{\Delta}^t & \\ & \tilde{\Delta}^s \end{bmatrix} \right)^{-1},$$

$$\begin{aligned} \text{where} \quad \tilde{A} &= T^{-1}AT, \\ \tilde{B}^0 &= T^{-1}B^0W, \qquad \tilde{B}^1 = T^{-1}B^1, \\ \tilde{C}^0 &= W^{-1}C^0T, \qquad \tilde{C}^1 = C^1T, \\ \tilde{D}^{00} &= W^{-1}D^{00}W, \quad \tilde{D}^{01} = W^{-1}D^{01}, \quad \tilde{D}^{10} = D^{10}W, \end{aligned}$$

and $\begin{bmatrix} \tilde{\Delta}^t & \\ & \tilde{\Delta}^s \end{bmatrix} = W^{-1} \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} W = \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}$, $\forall\, \Delta^t \in \Theta^t$, $\Delta^s \in \Theta^s$.

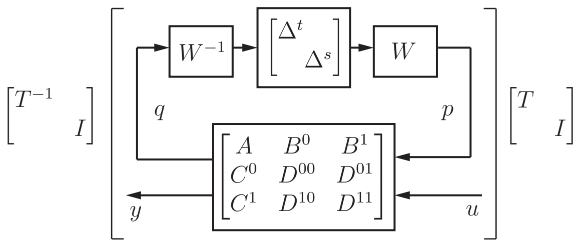

Figure 5.2: Balanced Realization

System (5.1) has been transformed into (5.6), but is not reduced yet. In order to construct system (5.3) of reduced complexity, we have to apply a specific partition (according to the significant and non-significant singular values) and truncation (the blocks relating to the non-significant parts) to (5.6). To do that, (i.e. to construct $T \in \mathcal{T}$ (as defined in (3.10)) and $W \in \mathcal{W}$), again, as in Chapter 4 we use the full block S-procedure such that the inequalities in (4.9) hold for $X, Y \in \mathcal{X}$ for all $\Delta^t \in \Theta^t$ and $\Delta^s \in \Theta^s$ if and only if there exist symmetric matrices $\Pi_x, \Pi_y \in \mathcal{P}_D$ satisfying (5.7) and (5.8) below. Then, we have the following result.

Joint Order Reduction Scheme

Given $G(\Delta^t, \Delta^s)$ defined in (5.1) with state order n, temporal and spatial scheduling orders n^t and n^s, respectively. A reduced complexity system (5.3) can be constructed with reduced state order $n_r < n$, reduced temporal and spatial scheduling orders $n_r^t < n^t$ and $n_r^s < n^s$, respectively, by following these steps:

1. Find $X, Y \in \mathcal{X}$ and $\Pi_x, \Pi_y \in \mathcal{P}_D$ (defined in (4.15)) that satisfy

$$[*]^* \left[\begin{array}{cc|c|c} & Y & & \\ Y & & & \\ \hline & & \Pi_y & \\ \hline & & & I \end{array} \right] \left[\begin{array}{cc} I & 0 \\ A^* & C^{0*} \\ \hline 0 & I \\ B^{0*} & D^{00*} \\ \hline B^{1*} & D^{01*} \end{array} \right] < 0 \tag{5.7}$$

$$[*]^* \left[\begin{array}{cc:c:c} & X & & \\ X & & & \\ \hdashline & & \Pi_x & \\ \hdashline & & & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B^0 \\ \hdashline 0 & I \\ C^0 & D^{00} \\ \hdashline C^1 & D^{10} \end{array}\right] < 0. \tag{5.8}$$

According to the definition of the set P_D in (4.15), the D-scale structure is imposed here to $\Pi_x = \begin{bmatrix} -\Pi_{x_{11}} & \\ & \Pi_{x_{11}} \end{bmatrix}$ and $\Pi_y = \begin{bmatrix} -\Pi_{y_{11}} & \\ & \Pi_{y_{11}} \end{bmatrix}$; this fact will be used in the rest of this section.

2. Construct a transformation $T \in \mathcal{T}$ (as defined in (3.10) via the extended version of Algorithm 3.3), that

$$T^{-1}YT^{-*} = T^TXT = \operatorname{diag}(\Sigma_1, \Sigma_2, -\Sigma_3) = \Sigma$$

is diagonal and contains the generalized singular values along its diagonal in descending order.

3. Find a transformation[2] $W \in \mathcal{W}$ (via applying the Cholesky/SVD decomposition to $\Pi_x, \Pi_y \in \mathcal{P_D}$ in a similar way as in constructing $T \in \mathcal{T}$, via applying the Cholesky/SVD decomposition to $X, Y \in \mathcal{X}$, see Algorithm 3.3) that

$$\begin{bmatrix} W^{-1} & \\ & W^{-1} \end{bmatrix} \underbrace{\begin{bmatrix} -\Pi_{y_{11}} & \\ & \Pi_{y_{11}} \end{bmatrix}}_{\Pi_y} \begin{bmatrix} W^{-T} & \\ & W^{-T} \end{bmatrix} = \begin{bmatrix} -\Psi & \\ & \Psi \end{bmatrix}$$

and

$$\begin{bmatrix} W^{T} & \\ & W^{T} \end{bmatrix} \underbrace{\begin{bmatrix} -\Pi_{x_{11}} & \\ & \Pi_{x_{11}} \end{bmatrix}}_{\Pi_x} \begin{bmatrix} W & \\ & W \end{bmatrix} = \begin{bmatrix} -\Psi & \\ & \Psi \end{bmatrix},$$

and Ψ is structured according to the temporal and spatial uncertainty blocks as

$$\Psi = \begin{bmatrix} \Psi_{\Delta^t} & \\ & \Psi_{\Delta^s} \end{bmatrix},$$

where Ψ_{Δ^t} and Ψ_{Δ^s} are diagonal matrices.

4. Calculate $\tilde{A}(\tilde{\Delta}^t, \tilde{\Delta}^s), \tilde{B}(\tilde{\Delta}^t, \tilde{\Delta}^s), \tilde{C}(\tilde{\Delta}^t, \tilde{\Delta}^s)$ and $\tilde{D}(\tilde{\Delta}^t, \tilde{\Delta}^s)$ as in (5.6).

5. Partition each $\Sigma_i,\ i = 1, 2, 3$, into two blocks according to the significant (s) and non-significant (ns) singular values as $\Sigma_i = \begin{bmatrix} \Sigma_i^s & \\ & \Sigma_i^{ns} \end{bmatrix}$, $i = 1, 2, 3$, such that Σ_i^s has dimension $n_{ri} \times n_{ri}$ and Σ_i^{ns} has dimension $(n_i - n_{ri}) \times (n_i - n_{ri}), i = 1, 2, 3$.

[2] Note that a sufficient condition for the existence of such a transformation W is that the multiplication $(\Pi_{y_{11}}\Pi_{x_{11}})$ has distinct eigenvalues [87], where if $W^{-1}\Pi_{y_{11}}\Pi_{x_{11}}W = \Psi^2$, then $W^{-1}\Pi_{y_{11}} \underbrace{W^{-T}W^{T}}_{I} \Pi_{x_{11}}W = \Psi^2$ which means $W^{-1}\Pi_{y_{11}}W^{-T} = W^T\Pi_yW = \Psi$.

Also, partition $\Psi = \mathrm{diag}(\Psi^s, \Psi^{ns})$, where $\Psi^s = \mathrm{diag}(\Psi^s_{\Delta^t}, \Psi^s_{\Delta^s})$ and $\Psi^{ns} = \mathrm{diag}(\Psi^{ns}_{\Delta^t}, \Psi^{ns}_{\Delta^s})$ according to the significant and non-significant values, where $\Psi^s \in \mathbb{R}^{(n_r^t+n_r^s)\times(n_r^t+n_r^s)}$ and $\Psi^{ns} \in \mathbb{R}^{((n^t-n_r^t)+(n^s-n_r^s))\times((n^t-n_r^t)+(n^s-n_r^t))}$.

6. Partition the transformed system (5.6) conformably with Σ_i, $i = 1, 2, 3$ and Ψ. Then truncate the non-significant parts from each block of Σ_i, Ψ and accordingly from $\tilde{A}(\tilde{\Delta}^t, \tilde{\Delta}^s)$, $\tilde{B}(\tilde{\Delta}^t, \tilde{\Delta}^s)$, $\tilde{C}(\tilde{\Delta}^t, \tilde{\Delta}^s)$ and $\tilde{D}(\tilde{\Delta}^t, \tilde{\Delta}^s)$ to get $A_r(\Delta_r^t, \Delta_r^s)$, $B_r(\Delta_r^t, \Delta_r^s)$, $C_r(\Delta_r^t, \Delta_r^s)$ and $D_r(\Delta_r^t, \Delta_r^s)$ as in (5.3).

Now, we present the following Theorem.

Theorem 5.1 *Consider a reduced model $G_r(\Delta_r^t, \Delta_r^s)$, defined in (5.3), that has been constructed from the exponentially stable $G(\Delta^t, \Delta^s)$ (according to the above scheme). Then, the reduced model is exponentially stable and we have an upper error bound γ that satisfies*

$$\|G(\Delta^t, \Delta^s) - G_r(\Delta_r^t, \Delta_r^s)\|_{2\to 2} \leq 2\left(\mathrm{trace}(\bar{\Sigma}^{ns}) + \mathrm{trace}(\Psi^{ns})\right) = \gamma \tag{5.9}$$

where $\bar{\Sigma}^{ns}$ contains (along its diagonal) the absolute values of the diagonal entries of $\Sigma^{ns} = \mathrm{diag}(\Sigma_1^{ns}, \Sigma_2^{ns}, -\Sigma_3^{ns})$.

Proof The proof of the exponential stability of the reduced model follows the same line in Theorem 4.2.
With a simple permutation (and dropping the ˜), rewrite the transformed versions of (5.8) as

$$[*]^* \begin{bmatrix} & & \Sigma & \\ & \Psi & & \\ \Sigma & & & \\ & & & -\Psi \end{bmatrix} \begin{bmatrix} A & B^0 \\ C^0 & D^{00} \\ I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} C^{1*} \\ D^{10*} \end{bmatrix} \begin{bmatrix} C^{1*} \\ D^{10*} \end{bmatrix}^* < 0 \tag{5.10}$$

and (5.7) as (see Appendix C)

$$[*]^* \begin{bmatrix} & & \Sigma^{-1} & \\ & \Psi^{-1} & & \\ \Sigma^{-1} & & & \\ & & & -\Psi^{-1} \end{bmatrix} \begin{bmatrix} A & B^0 & B^1 \\ C^0 & D^{00} & D^{01} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} < \begin{bmatrix} 0 & & \\ & 0 & \\ & & I \end{bmatrix} \tag{5.11}$$

Then, a similar way as in Theorem 3.3 can be followed, to show how an error bound on the induced 2-norm of the error system can be computed.
Following the same line as in the proof of Theorem 3.3, with state vector ξ partitioned according to the significant (s) and non-significant (ns) singular values as $\xi = \begin{bmatrix} \xi^s \\ \xi^{ns} \end{bmatrix}$ with reduced state vector ξ^r, the same is true for $p = \begin{bmatrix} p^s \\ p^{ns} \end{bmatrix}$ with reduced p^r.

We divide the proof into two parts.

- Consider first that only the state order is reduced, i.e. $n_r < n$ and $(n_r^t = n^t, n_r^s = n_s)$:

Multiply (5.10) by $[(\xi^s - \xi^r)^* \quad \xi^{ns*} \quad p^*]$ from the left and its transpose from the right, and add this to the inequality obtained by multiplying inequality (5.11) by $[(\xi^s + \xi^r)^* \quad \xi^{ns*} \quad p^* \quad 2u^*]$ from the left and its transpose from the right.

- Consider that only the scheduling orders are reduced, i.e. $n_r = n$ and $(n_r^t < n^t, n_r^s < n^s)$:

  Multiply (5.10) by $[\xi^* \quad (p^s - p^r)^* \quad p^{ns*}]$ from the left and its complex conjugate from the right, and adding this to the inequality obtained by multiplying inequality (5.11) by $[\xi^* \quad (p^s + p^r)^* \quad p^{ns*} \quad 2u^*]$ from the left and its complex conjugate from the right.

The proof is then complete by following the same reasoning as in Theorem 3.3. ■

Remarks:

1 As already mentioned, the error bound in (5.9) *(a related result is given in [88] and [89] for uncertain 1-D lumped systems with a specific scaling)*, is based on the truncated singular values of the matrices $\Pi_x, \Pi_y \in \mathcal{P}_D$ as well as on those of the Gramians $X, Y \in \mathcal{X}$. Therefore, it would be reasonable to minimize their rank when solving the reduction problem. Again, this nonconvex condition can been resolved by applying a cone complemetarity method [64] to the blocks $\begin{bmatrix} X & \\ & \Pi_{x_{11}} \end{bmatrix}$ and $\begin{bmatrix} Y & \\ & \Pi_{y_{11}} \end{bmatrix}$, instead of X and Y in Algorithm 3.4.

2 Exponential stability is preserved in the reduced model when solving the reduction problem as in Chapter 4.

3 Reducing the scheduling order takes two aspects into consideration: first reducing the number of the scheduling parameters, and second reducing their multiplicity. According to the second term in the error bound (defined in (5.9)), there is a direct relationship between the error bound and the number of scheduling parameters. Scheduling parameters that do not have a significant impact on the model will not have a significant effect on the error bound either.

4 Using constant generalized Gramians (CGs) (parameter-independent generalized Gramians) will be conservative (see Chapter 4) due to allowing parameter variations of infinite rate. Therefore, in the next section, we extend the result of the current section to the case of using Parameter-Dependent generalized Gramians (PDGs) rather than CGs, where an improved result is obtained as shown in Section 5.4.

5.3 Joint Order Reduction Using PDGs

In this section, an improved and less conservative result (based on the use of PDGs) compared with the previous section is presented. Due to the use of PDGs rather than CGs, the balanced transformation has to be parameter-varying; consequently the differentiability of such transformation has to be checked [30]. While for discrete-domain systems this is not required, which makes the problem easier. Therefore, we return here to the original discrete-time and -space system. We will not distinguish in notation between the representation of the discrete-domain system here and the continuous-domain system in the previous chapters.

For solving the problem of joint order reduction using PDGs, we first define a structured set

$$\mathcal{S} = \{X(\Delta^t, \Delta^s) = \text{diag}\,(X_1(\Delta^t), X_2(\Delta^s), X_3(\Delta^s)),\ X = X^*, X_1 > 0\} \tag{5.12}$$

where $X_1 \in \mathbb{C}^{n_1 \times n_1}$ corresponds to the temporal part, and $X_2 \in \mathbb{C}^{n_2 \times n_2}$ and $X_3 \in \mathbb{C}^{n_3 \times n_3}$ to the forward and backward spatial parts, respectively.

Also, define $\partial X(\Delta^t, \Delta^s)$ as the variation rate of $X(\Delta^t, \Delta^s) \in \mathcal{S}$. Let the set of temporal and spatial variation rates of uncertainties be denoted by Θ^t_∂ and Θ^s_∂, such that $(\Delta^t, \partial\Delta^t) \in \Theta^t \times \Theta^t_\partial$ and $(\Delta^s, \partial\Delta^s) \in \Theta^s \times \Theta^s_\partial$.
Next, we define the PDGs for LTSV system defined in discrete time and space as follows.

Definition 5.1 *The parameter-dependent generalized controllability and observability Gramians are defined as $Y(\Delta^t, \Delta^s), X(\Delta^t, \Delta^s) \in \mathcal{S}$, which satisfy*

$$\begin{aligned} A(\Delta^t, \Delta^s)\, Y(\Delta^t, \Delta^s)\, A^*(\Delta^t, \Delta^s) - \partial Y(\Delta^t, \Delta^s) + B(\Delta^t, \Delta^s) B^*(\Delta^t, \Delta^s) < 0, \\ A^*(\Delta^t, \Delta^s)\, \partial X(\Delta^t, \Delta^s)\, A(\Delta^t, \Delta^s) - X(\Delta^t, \Delta^s) + C^*(\Delta^t, \Delta^s) C(\Delta^t, \Delta^s) < 0, \end{aligned} \tag{5.13}$$

$\forall(\Delta^t, \partial\Delta^t) \in \Theta^t \times \Theta^t_\partial$ *and* $(\Delta^s, \partial\Delta^s) \in \Theta^s \times \Theta^s_\partial$*, where*

$$\begin{aligned} \partial X(\Delta^t, \Delta^s) &= \text{diag}\,(X_1(\Delta^t(t+1)), X_2(\Delta^s(s+1)), X_3(\Delta^s(s-1))), \\ \partial Y(\Delta^t, \Delta^s) &= \text{diag}\,(Y_1(\Delta^t(t+1)), Y_2(\Delta^s(s+1)), Y_3(\Delta^s(s-1))). \end{aligned}$$

The inequalities in (5.13) can be rewritten as

$$\begin{aligned} [*]^* \left[\begin{array}{cc|c} -\partial Y(\Delta^t, \Delta^s) & & \\ & Y(\Delta^t, \Delta^s) & \\ \hline & & I \end{array}\right] \left[\begin{array}{c} I \\ A^*(\Delta^t, \Delta^s) \\ \hline B^*(\Delta^t, \Delta^s) \end{array}\right] < 0 \\ [*]^* \left[\begin{array}{cc|c} -X(\Delta^t, \Delta^s) & & \\ & \partial X(\Delta^t, \Delta^s) & \\ \hline & & I \end{array}\right] \left[\begin{array}{c} I \\ A(\Delta^t, \Delta^s) \\ \hline C(\Delta^t, \Delta^s) \end{array}\right] < 0, \end{aligned} \tag{5.14}$$

$\forall(\Delta^t, \partial\Delta^t) \in \Theta^t \times \Theta^t_\partial$ and $(\Delta^s, \partial\Delta^s) \in \Theta^s \times \Theta^s_\partial$.

Define the set Γ consisting of structured block diagonal matrices that have bounded inverses and are partitioned (conformably with temporal and spatial forward/backward variables) as

$$\Gamma = \{T(\Delta^t, \Delta^s) : T(\Delta^t, \Delta^s) = \text{diag}\,(T_1(\Delta^t), T_2(\Delta^s), T_3(\Delta^s)).$$

We also introduce $\partial T(\Delta^t, \Delta^s) = \text{diag}\,(T_1(\Delta^t(t+1)), T_2(\Delta^s(s+1)), T_3(\Delta^s(s-1)))\}.$ (5.15)

In Section 5.2, it was shown that the change of coordinates in (5.1) could be done via the transformations $T \in \mathcal{T}$ and $W \in \mathcal{W}$, see (5.6). Here since we are using PDGs rather than CGs, we have to use the parameter-dependent transformation $T(\Delta^t, \Delta^s) \in \Gamma$ rather than the constant one (i.e., $T \in \mathcal{T}$ defined in (3.10)), such that we have (see Fig. 5.3, compare with Fig. 5.2)

$$\begin{aligned}
\tilde{G}(\tilde{\Delta}^t, \tilde{\Delta}^s) &= \left[\begin{array}{c|c} \tilde{A}(\tilde{\Delta}^t, \tilde{\Delta}^s) & \tilde{B}(\tilde{\Delta}^t, \tilde{\Delta}^s) \\ \hline \tilde{C}(\tilde{\Delta}^t, \tilde{\Delta}^s) & \tilde{D}(\tilde{\Delta}^t, \tilde{\Delta}^s) \end{array}\right] \\
&= \begin{bmatrix} \partial T^{-1}(\tilde{\Delta}^t, \tilde{\Delta}^s) & \\ & I \end{bmatrix} \left[\begin{array}{c|c} A(\tilde{\Delta}^t, \tilde{\Delta}^s) & B(\tilde{\Delta}^t, \tilde{\Delta}^s) \\ \hline C(\tilde{\Delta}^t, \tilde{\Delta}^s) & D(\tilde{\Delta}^t, \tilde{\Delta}^s) \end{array}\right] \begin{bmatrix} T(\tilde{\Delta}^t, \tilde{\Delta}^s) & \\ & I \end{bmatrix} \\
&= \begin{bmatrix} \tilde{A} + \tilde{B}^0 \tilde{\Phi} \tilde{C}^0 & \tilde{B}^1 + \tilde{B}^0 \tilde{\Phi} \tilde{D}^{01} \\ \tilde{C}^1 + \tilde{D}^{10} \tilde{\Phi} \tilde{C}^0 & D^{11} + \tilde{D}^{10} \tilde{\Phi} \tilde{D}^{01} \end{bmatrix}
\end{aligned} \tag{5.16}$$

$$\text{with} \quad \tilde{\Phi} = \begin{bmatrix} \tilde{\Delta}^t & \\ & \tilde{\Delta}^s \end{bmatrix} \left(I - \tilde{D}_{00} \begin{bmatrix} \tilde{\Delta}^t & \\ & \tilde{\Delta}^s \end{bmatrix} \right)^{-1},$$

$$\begin{aligned}
\text{where} \quad \tilde{A} &= \partial T^{-1}(\tilde{\Delta}^t, \tilde{\Delta}^s) A T(\tilde{\Delta}^t, \tilde{\Delta}^s), \\
\tilde{B}^0 &= \partial T^{-1}(\tilde{\Delta}^t, \tilde{\Delta}^s) B^0 W, \quad \tilde{B}^1 = \partial T^{-1}(\tilde{\Delta}^t, \tilde{\Delta}^s) B^1, \\
\tilde{C}^0 &= W^{-1} C^0 T(\tilde{\Delta}^t, \tilde{\Delta}^s), \quad \tilde{C}^1 = C^1 T(\tilde{\Delta}^t, \tilde{\Delta}^s), \\
\tilde{D}^{00} &= W^{-1} D^{00} W, \quad \tilde{D}^{01} = W^{-1} D^{01}, \quad \tilde{D}^{10} = D^{10} W,
\end{aligned}$$

and $\begin{bmatrix} \tilde{\Delta}^t & \\ & \tilde{\Delta}^s \end{bmatrix} = W^{-1} \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix} W = \begin{bmatrix} \Delta^t & \\ & \Delta^s \end{bmatrix}$, $\forall\ \Delta^t \in \Theta^t,\ \Delta^s \in \Theta^s$, where $W \in \mathcal{W}$.

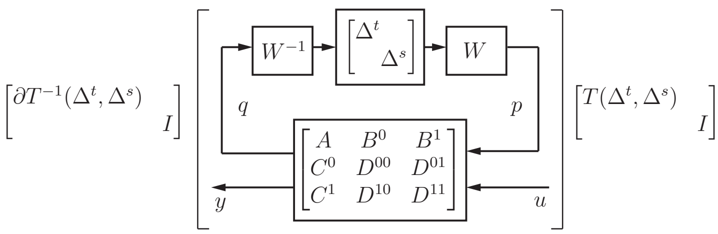

Figure 5.3: Balanced Realization Using PDGs

Before we present the generalization of the joint order reduction scheme (given in Section 5.3), we give the following result which is a direct application of Lemma 4.1 (in Chapter 4) with the set $\mathcal{P}_D$ defined in (4.15).

Lemma 5.1 *The matrix inequalities (5.14) hold for $X(\Delta^t, \Delta^s), Y(\Delta^t, \Delta^s) \in \mathcal{S}$ if and only if there exist symmetric matrices $\Pi_x, \Pi_y \in \mathcal{P}_D$ such that the following conditions hold $\forall(\Delta^t, \partial\Delta^t) \in \Theta^t \times \Theta^t_\partial$ and $(\Delta^s, \partial\Delta^s) \in \Theta^s \times \Theta^s_\partial$*

$$[*]^* \left[\begin{array}{cc|c|c} -\partial Y(\Delta^t, \Delta^s) & & & \\ & Y(\Delta^t, \Delta^s) & & \\ \hline & & \Pi_y & \\ \hline & & & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A^* & C^{0*} \\ \hline 0 & I \\ B^{0*} & D^{00*} \\ \hline B^{1*} & D^{01*} \end{array}\right] < 0 \tag{5.17}$$

$$[*]^* \left[\begin{array}{cc|c|c} -X(\Delta^t, \Delta^s) & & & \\ & \partial X(\Delta^t, \Delta^s) & & \\ \hline & & \Pi_x & \\ \hline & & & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B^0 \\ \hline 0 & I \\ C^0 & D^{00} \\ \hline C^1 & D^{10} \end{array}\right] < 0. \tag{5.18}$$

Next, we generalize the joint order reduction scheme of Section 5.3 to the the case of using PDGs rather than CGs. Note that in the case of using PDGs the joint order reduction scheme has to be applied in two steps: first, transform (and truncate) the model $G(\Delta^t, \Delta^s)$ via a transformation $W \in \mathcal{W}$ (as in (5.16)) in order to reduce the scheduling order (i.e., the uncertainty blocks) , such that a reduced scheduling order model (defined with $\Delta^t_r \in \Theta^t_r \subset \Theta^t$ and $\Delta^s_r \in \Theta^s_r \subset \Theta^s$) is obtained; this completes steps 1–4 of the joint order reduction scheme which is given below. Second, proceed with the reduction procedure by applying the transformation $T(\Delta^t_r, \Delta^s_r)$ in order to reduce the state order of the system as well, such that finally we obtain a jointly reduced dynamic and scheduling order model. Note that when using CGs, the reduction procedure is applied directly to simultaneously reduce the scheduling order and the state order of the system. That is because the transformation $T \in \mathcal{T}$ is constant (does not depend on (Δ^t, Δ^s) or (Δ^t_r, Δ^s_r)).

Joint Order Reduction Scheme Using PDGs
Begin with system $G(\Delta^t, \Delta^s)$ defined in (5.1) with state order n, temporal and spatial scheduling orders n^t and n^s, respectively; follow the steps:

1. Find $X(\Delta^t, \Delta^s), Y(\Delta^t, \Delta^s) \in \mathcal{S}$ and $\Pi_x, \Pi_y \in \mathcal{P}_D$ that satisfy (5.17) and (5.18) for all $(\Delta^t, \partial\Delta^t) \in \Theta^t \times \Theta^t_\partial$ and $(\Delta^s, \partial\Delta^s) \in \Theta^s \times \Theta^s_\partial$ and

$$\text{minimize} \quad \text{rank}(\Pi_x) \quad \text{and} \quad \text{rank}(\Pi_y). \tag{5.19}$$

2. Construct a transformation $W \in \mathcal{W}$ that

$$\begin{bmatrix} W^{-1} & \\ & W^{-1} \end{bmatrix} \Pi_y \begin{bmatrix} W^{-T} & \\ & W^{-T} \end{bmatrix} = \begin{bmatrix} \Psi & \\ & -\Psi \end{bmatrix}$$

and

$$\begin{bmatrix} W^T & \\ & W^T \end{bmatrix} \Pi_x \begin{bmatrix} W & \\ & W \end{bmatrix} = \begin{bmatrix} \Psi & \\ & -\Psi \end{bmatrix}, \text{where } \Psi = \begin{bmatrix} \Psi_{\Delta^t} & \\ & \Psi_{\Delta^s} \end{bmatrix} \quad \text{is diagonal.}$$

3. Define $\tilde{A}(\tilde{\Delta}^t, \tilde{\Delta}^s), \tilde{B}(\tilde{\Delta}^t, \tilde{\Delta}^s), \tilde{C}(\tilde{\Delta}^t, \tilde{\Delta}^s)$ and $\tilde{D}(\tilde{\Delta}^t, \tilde{\Delta}^s)$ as in (5.16) (see Fig. 5.3) for $T(\tilde{\Delta}^t, \tilde{\Delta}^s) = \partial T^{-1}(\tilde{\Delta}^t, \tilde{\Delta}^s) = I$.

4. Partition $\Psi = \begin{bmatrix} \Psi^s & \\ & \Psi^{ns} \end{bmatrix}$, $\Psi^s = \text{diag}(\Psi^s_{\Delta^t}, \Psi^s_{\Delta^s})$ and $\Psi^{ns} = \text{diag}(\Psi^{ns}_{\Delta^t}, \Psi^{ns}_{\Delta^s})$ according to the significant and non-significant singular values, where $\Psi^s \in \mathbb{R}^{(n^t_r+n^s_r)\times(n^t_r+n^s_r)}$ and $\Psi^{ns} \in \mathbb{R}^{((n^t-n^t_r)+(n^s-n^s_r))\times((n^t-n^t_r)+(n^s-n^t_r))}$.
   Consequently, partition the transformed system matrices conformably with Ψ. Then truncate the non-significant parts from Ψ and accordingly from $\tilde{A}(\tilde{\Delta}^t, \tilde{\Delta}^s)$, $\tilde{B}(\tilde{\Delta}^t, \tilde{\Delta}^s)$, $\tilde{C}(\tilde{\Delta}^t, \tilde{\Delta}^s)$ and $\tilde{D}(\tilde{\Delta}^t, \tilde{\Delta}^s)$ to get $A(\Delta^t_r, \Delta^s_r), B(\Delta^t_r, \Delta^s_r), C(\Delta^t_r, \Delta^s_r)$ and $D(\Delta^t_r, \Delta^s_r)$.

   This completes the scheduling order reduction steps and provides the starting for the next four steps of reducing the state order of the system.

5. Solve the resulting reduced version of (5.17) and (5.18) for $X(\Delta^t_r, \Delta^s_r), Y(\Delta^t_r, \Delta^s_r) \in \mathcal{S}$, and $\Pi_{xr}, \Pi_{y_r} \in \mathcal{P}_{Dr}$ (where $\mathcal{P}_{Dr}$ is the reduced version of $\mathcal{P}_D$ defined in (4.15)) for all $(\Delta^t_r, \partial\Delta^t_r) \in \Theta^t_r \times \Theta^t_{\partial r}$ and $(\Delta^s_r, \partial\Delta^s_r) \in \Theta^s_r \times \Theta^s_{\partial r}$ that

$$\text{minimize} \quad \text{rank}(X(\Delta^t_r, \Delta^s_r)) \quad \text{and} \quad \text{rank}(Y(\Delta^t_r, \Delta^s_r)). \tag{5.20}$$

6. Construct the transformation operator $T(\Delta^t_r, \Delta^s_r) \in \Gamma$, that for all $\Delta^t_r \in \Theta^t_r$, $\Delta^s_r \in \Theta^s_r$ simultaneously diagonalizes $X(\Delta^t_r, \Delta^s_r), Y(\Delta^t_r, \Delta^s_r) \in \mathcal{S}$, i.e.

$$\begin{aligned} T^{-1}(\Delta^t_r, \Delta^s_r)Y(\Delta^t_r, \Delta^s_r)T^{-*}(\Delta^t_r, \Delta^s_r) &= T^*(\Delta^t_r, \Delta^s_r)X(\Delta^t_r, \Delta^s_r)T(\Delta^t_r, \Delta^s_r) \\ &= \text{diag}(\Sigma_1(\Delta^t_r), \Sigma_2(\Delta^s_r), -\Sigma_3(\Delta^s_r)) = \Sigma(\Delta^t_r, \Delta^s_r) \end{aligned}$$

   is diagonal for all $\Delta^t_r \in \Theta^t_r$, $\Delta^s_r \in \Theta^s_r$, and Σ contains the generalized singular values along its diagonal in descending order.

7. Define $\tilde{A}(\Delta^t_r, \Delta^s_r), \tilde{B}(\Delta^t_r, \Delta^s_r), \tilde{C}(\Delta^t_r, \Delta^s_r)$ and $\tilde{D}(\Delta^t_r, \Delta^s_r)$ as in (5.16) (see Fig. 5.3) for $W = W^{-1} = I$.

8. Partition each $\Sigma_i(\Delta^t_r, \Delta^s_r)$, $i = 1, 2, 3$, into two blocks according to the significant (s) and non-significant (ns) singular values as $\Sigma_i(\Delta^t_r, \Delta^s_r) = \begin{bmatrix} \Sigma^s_i(\Delta^t_r, \Delta^s_r) & \\ & \Sigma^{ns}_i(\Delta^t_r, \Delta^s_r) \end{bmatrix}$, $i = 1, 2, 3$, such that Σ^s_i has dimension $n_{ri} \times n_{ri}$ and Σ^{ns}_i has dimension $(n_i - n_{ri}) \times (n_i - n_{ri}), i = 1, 2, 3$.
   Consequently, partition $\tilde{A}(\Delta^t_r, \Delta^s_r)$, $\tilde{B}(\Delta^t_r, \Delta^s_r)$, $\tilde{C}(\Delta^t_r, \Delta^s_r)$ and $\tilde{D}(\Delta^t_r, \Delta^s_r)$ conformably with $\Sigma_i(\Delta^t_r, \Delta^s_r)$, $i = 1, 2, 3$. Then truncate the non-significant parts from each block of $\Sigma_i(\Delta^t_r, \Delta^s_r)$ and accordingly from the transformed matrices to get $A_r(\Delta^t_r, \Delta^s_r)$, $B_r(\Delta^t_r, \Delta^s_r)$, $C_r(\Delta^t_r, \Delta^s_r)$ and $D_r(\Delta^t_r, \Delta^s_r)$, see (5.4).

Note that conditions (5.17), (5.18) need to be checked at infinitely many points inside $\Theta^t \times \Theta^t_\partial$ and $\Theta^s \times \Theta^s_\partial$, due to the introduction of PDGs rather than CGs. The following two approaches can be used to solve the above problem, such that the conditions are reduced to finite dimensions

- Define a grid on the admissible parameter ranges $\Theta^t \times \Theta^t_\partial$ and $\Theta^s \times \Theta^s_\partial$, such that we solve the inequalities at each grid point.
- Before applying the FBSP, parameterize $X(\Delta^t, \Delta^s), Y(\Delta^t, \Delta^s) \in \mathcal{S}$ (in (5.14)) in quadratic LFT form of Δ^t, Δ^s [90], [8] i.e.,

$$X(\Delta^t, \Delta^s) = L_x^*(\Delta^t, \Delta^s)\ X\ L_x(\Delta^t, \Delta^s),$$
$$Y(\Delta^t, \Delta^s) = L_y^*(\Delta^t, \Delta^s)\ Y\ L_y(\Delta^t, \Delta^s).$$

  where $L_x(\Delta^t, \Delta^s), L_y(\Delta^t, \Delta^s)$ are pre-specified LFT functions of Δ^t, Δ^s and $X, Y \in \mathcal{X}$ are constant (parameter-independent) matrices defined as decision variables. This approach suffers from the difficulty of imposing the rank minimization constraint on $X(\Delta^t, \Delta^s), Y(\Delta^t, \Delta^s) \in \mathcal{S}$ for this case. Experience with practical examples suggests however that it performs reasonably well.

Remarks:

1. The non-convex rank constraints (5.19) and (5.20) are solved using Algorithm 3.3.1, or Algorithm 3.4.
2. Due to the use of PDGs, the reduced model is defined with respect to $(\Delta^t_r, \partial\Delta^t_r) \in \Theta^t_r \times \Theta^t_{\partial r}$ and $(\Delta^s_r, \partial\Delta^s_r) \in \Theta^s_r \times \Theta^s_{\partial r}$. We suppress here the dependence on $\partial\Delta^t_r$ and $\partial\Delta^s_r$ for simplicity of presentation. In addition, the reduced system is not defined in LFT form (as in Fig. 5.1) any more, in contrast to the case where CGs are used.

5.4 Application to an Actuated Beam

The actuated beam in Fig. 4.4 of Chapter 4 is used here to demonstrate the efficiency of the proposed results. As mentioned, the system has state order $n = 2 + 4 + 4 = 10$ and spatial scheduling order $n^s = 2$, see Section 4.5.

The application of the proposed approach gives a reduced model of state order $n_r = 2+2+2 = 6$ and reduced spatial scheduling order $n^s_r = 1$, and an error bound $\gamma = 0.0972$.

Fig. 5.4 shows the simulated responses over time of the third, fifth and ninth subsystem to a step disturbance (applied to subsystem 5), comparing the original identified model and the reduced models obtained by using the results presented in Section 5.2 (CG) and in Section 5.3 (PDG). According to Fig.5.4, clearly using CG leads to conservatism, while using PDG reduces this conservatism.

In addition, a comparison is made using experimental measurements with the injection of 10 noise signals in parallel to 10 actuators as inputs. Fig.5.5 shows the measured output (Measured), simulated output using the original model (Full), and the output of the reduced model (Reduced) for the first and eighth subsystem, respectively.

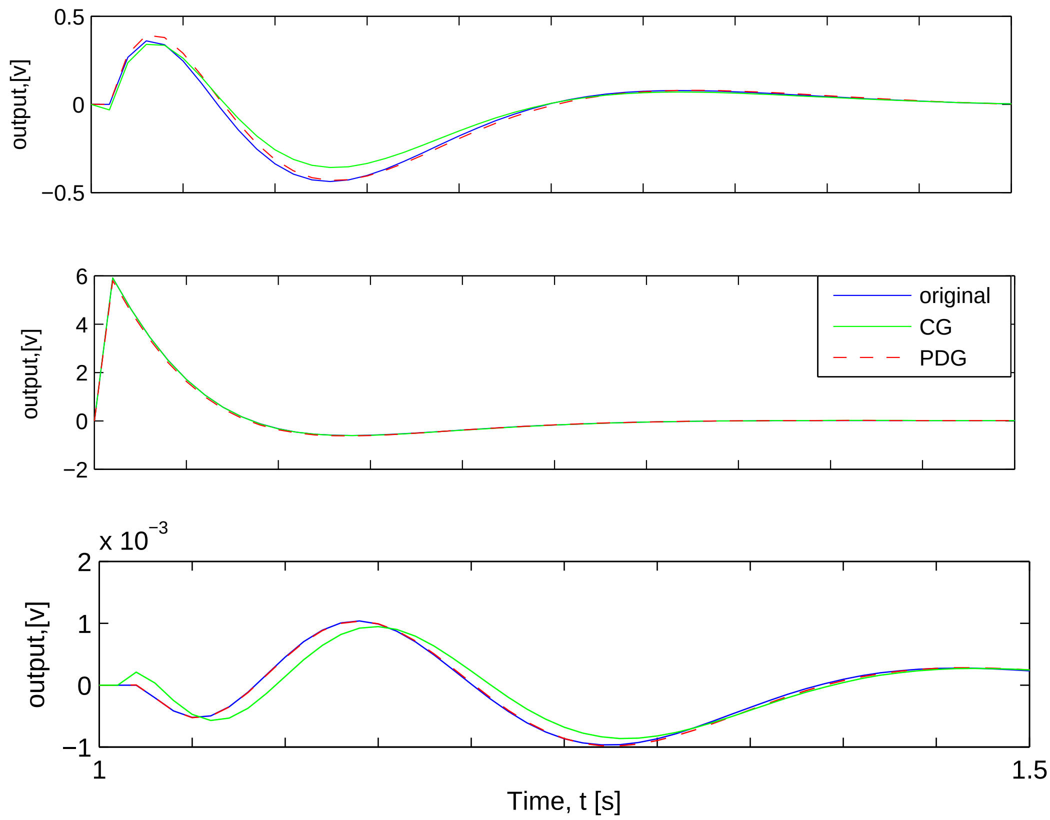

Figure 5.4: Simulated responses over time of the (from top to bottom) $3^{rd}, 5^{th}$ and 9^{th} LPV subsystem to a disturbance step

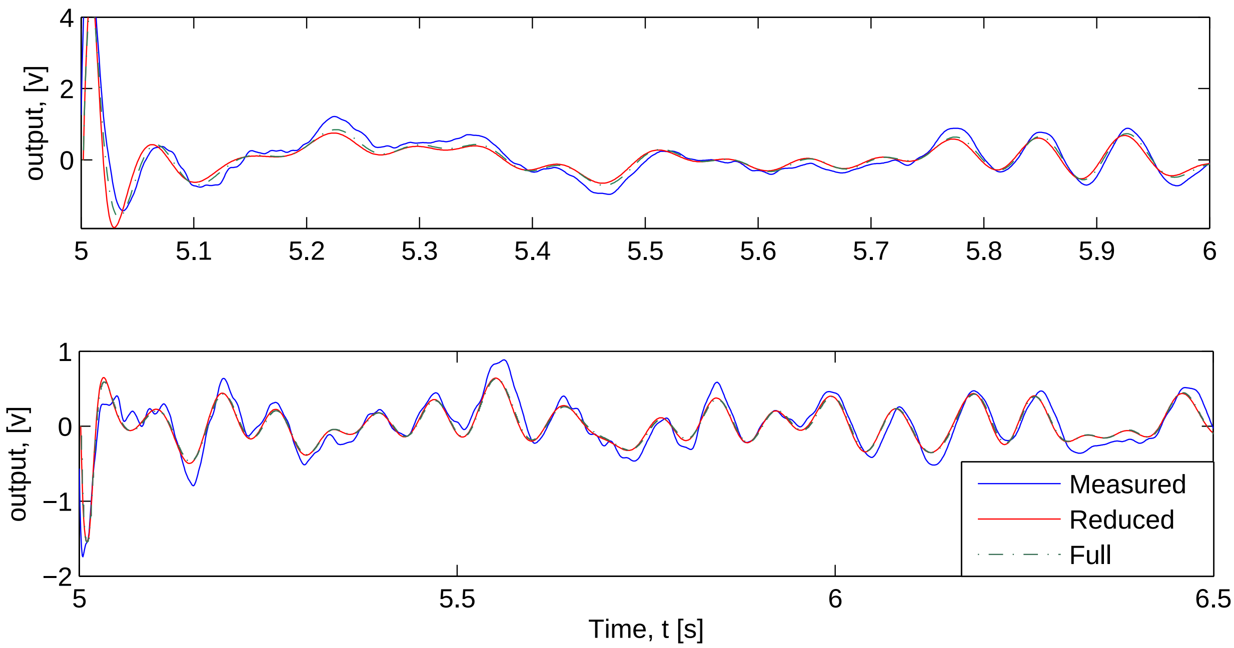

Figure 5.5: Responses over time of the 1^{st} (top) and 8^{th} (bottom) subsystems to 10 noises signals applied in parallel to 10 actuators, experimentally.

5.5 Conclusion

In this chapter, a practical method for reducing both the state and the scheduling order of exponentially stable parameter-dependent spatially interconnected systems has been presented, and a guaranteed error bound has been established. Utilizing the full block S-procedure leads to efficient solutions and reduced conservatism. Constant as well as parameter-dependent generalized Gramians have been considered. Application of the proposed procedures to an experimentally identified actuated beam shows its practicality, and demonstrates the reduction of conservatism when PDGs are employed.

Chapter 6

Conclusion

6.1 Summary

This thesis considers complexity reduction for spatially interconnected systems. Such systems are governed by a spatial-discretization of a Partial Differential Equation, such that the system is represented as a spatial interconnection of subsystems; each subsystem is defined as a two-dimensional model with respect to time and space.
Such large-scale systems can be complex due to several reasons, such as the interconnection structure, a high order representation, or the dependence on several time and space varying parameters (in the case of parameter-varying systems). This thesis addresses the problem of reducing the complexity of such systems and proposes new methods for solving the reduction problem for such multidimensional systems. The spatial interconnection of the system is preserved when solving the reduction problem.

The accuracy of a reduced model is measured via the induced norm of the error system (the difference between the original system and the reduced one); bounds on this error have been established as well.

The work is based on representing the system in Linear Fractional Transformation form with respect to shift operators, which allows to employ results on model reduction for uncertain systems and helps to extend results on lumped systems to spatially interconnected systems.

Considered are both *parameter-invariant (LTSI)* and *parameter-varying (LTSV)* systems.

Model Order Reduction (MOR) for LTSI systems considered in chapter 3 is based on balanced truncation (through balancing the Gramians) via solving a pair of Lyapunov inequalities with one rank constraint, which is non-convex; efficient methods for linearizing the non-convex rank constraint are used. The reduced model preserves the spatial-structure as well as the stability of the system, provided that the original model is stable. The proposed method can cope with the fact that the considered systems are non-causal with respect to space.

Model reduction for LTSV systems is considered in Chapters 4 and 5 using Constant

Gramians (CG) as well as Parameter-Dependent Gramians (PDG) to reduce conservatism. Due to parameter variation with respect to time and space, the reduction problem has an infinite number of conditions to be solved. In order to reduce conditions to a finite number, two approaches are considered: first, defining a grid on the admissible parameter range (in the application part, where it has been used for a comparison) and second, using the full block S-procedure (FBSP). A comparison between the latter two approaches is discussed as well. This thesis shows that the FBSP is applicable to solve the model reduction problem efficiently and avoids gridding.

The complexity of parameter-varying systems is not only determined by the state order of the system, but also by the scheduling order. Therefore, the problem of joint state and scheduling order reduction for LTSV systems is proposed in Chapter 5 (through balancing the Gramians as well as the multipliers). In this chapter, a comparison between the use of CGs and PDGs is also discussed.

For all the above cases, error bounds have been established. In addition, the performance of the proposed methods has been demonstrated with the application to an experimentally identified piezoelectric actuated beam.

6.2 Outlook

An outlook on possible future research is summarized by the following aspects.

- Using PDGs in model reduction of LTSV systems (such as in Section 5.3) raises some issues;
  - The error bound between the original system and the reduced one is dependent on $\Delta^t \in \Theta^t$ and $\Delta^s \in \Theta^s$. A way to define that error bound is to consider it for admissible value of frozen scheduling parameters (which is conservative). In order to reduce the conservatism, the (time and space) varying scheduling signals should be taken into consideration while defining the error bound. In this case, we have to consider twice the sum of all truncated generalized singular values along time and space as shown in e.g., [28], [91], [92] and [93] for lumped LTV systems which is generalized to LTSV systems in [66] and then improved in [84] in order to avoid the sum of *all* truncated generalized singular values along time and space, but instead only *some* of them have been considered. That what already has been done, further improvement is an interesting research.
  - After applying balanced truncation, the reduced system matrices are dependent on the rate of change of the scheduling parameters (due to the introduction of the parameter-dependent transformation) which would complicate the resulting reduced system unless a balanced transformation which does not depend on the rate of the scheduling parameters is obtained. Work in this direction would be of interest. Even though several methods proposed in the literature address this issue for 1-D (lumped) systems see e.g. [94], [95], [96], [97], unfortunately these

methods lead to unbalanced models, in addition no a priori error bound can be guaranteed. In these approaches, after defining several local (at each grid point) balanced transformations, a global transformation has to be constructed (which is parameter-independent) and used to transform the original system matrices and constructing a reduced system, which is then no longer guaranteed to be balanced even at each local point.

- Using dynamic multipliers [98], [99] rather than static ones (such as Π_x and Π_y which have been used in the thesis) helps in reducing the conservatism of the model reduction problem. While the FBSP reduces the complexity of the analysis problem, dynamic multipliers further reduce the conservatism. Therefore, improved results may be obtained based on dynamic multipliers rather than constant ones.

A starting point for doing this could be the following observation.
First let us simplify the problem by considering the case of 1D *lumped* LPV systems. Also, assume that the LPV system is defined as with only one scheduling parameter, such that the considered system can be represented as in Fig. 6.1. Extension to the more general case of more than one scheduling parameter is straightforward.

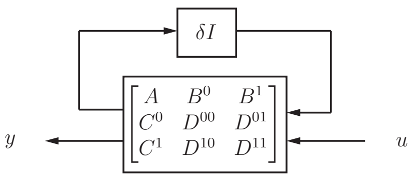

Figure 6.1: LPV system with one scheduling parameter

For both cases (using *static* and *dynamic* multipliers), conditions (which to be solved in order to construct a balancing transformation T) for generating a reduced model based on balanced truncation are given next.

- **Static Multipliers**:

  The conditions (6.1) and (6.2) have to be solved for $Y > 0$, $X > 0$, and symmetric multipliers Π_y and Π_x

$$[*]^* \left[\begin{array}{cc|c|c} & Y & & \\ Y & & & \\ \hline & & \Pi_y & \\ \hline & & & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A^* & C^{0*} \\ \hline 0 & I \\ B^{0*} & D^{00*} \\ \hline B^{1*} & D^{01*} \end{array}\right] < 0, \quad [*]^* \, \Pi_y \left[\begin{array}{c} \delta I \\ \hline I \end{array}\right] \geq 0 \qquad (6.1)$$

$$[*]^* \left[\begin{array}{cc:c:c} & X & & \\ X & & & \\ \hdashline & & \Pi_x & \\ \hdashline & & & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B^0 \\ \hdashline 0 & I \\ C^0 & D^{00} \\ \hdashline C^1 & D^{10} \end{array}\right] < 0, \quad [*]^* \, \Pi_x \left[\begin{array}{c} \delta I \\ \hdashline I \end{array}\right] \geq 0. \tag{6.2}$$

– **Dynamic Multipliers**:

In this case one can define (and factorize) the frequency dependent multipliers $\Pi_y = \Psi^* N \Psi$ and $\Pi_x = \Phi^* M \Phi$, where N and M are static matrices, while $\Psi = \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \end{array}\right]$ and $\Phi = \left[\begin{array}{c|c} A_\phi & B_\phi \\ \hline C_\phi & D_\phi \end{array}\right]$.

We have (directly using the Kalman-Yakubovich Lemma, [100]) to solve conditions (6.3) and (6.4) for symmetric $\mathbf{Y}, N$, S and $\mathbf{X}, M$, R with $Y_{22} > 0$ and $X_{22} > 0$

$$[*]^* \left[\begin{array}{cc:c:c} & \mathbf{Y} & & \\ \mathbf{Y} & & & \\ \hdashline & & N & \\ \hdashline & & & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \mathcal{A}^* & \mathcal{C}^* \\ \hdashline 0 & I \\ \mathcal{B}^* & \mathcal{D}^* \\ \hdashline \mathcal{B}^{1*} & D^{01*} \end{array}\right] < 0, \quad [*]^* \left[\begin{array}{cc:c} & S & \\ S & & \\ \hdashline & & N \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A_\psi^* & C_\psi^* \\ \hdashline B_\psi^* & D_\psi^* \\ 0 & I \end{array}\right] > 0 \tag{6.3}$$

$$[*]^* \left[\begin{array}{cc:c:c} & \mathbf{X} & & \\ \mathbf{X} & & & \\ \hdashline & & M & \\ \hdashline & & & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hdashline 0 & I \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \\ \hdashline \mathcal{C}^1 & D^{10} \end{array}\right] < 0, \quad [*]^* \left[\begin{array}{cc:c} & R & \\ R & & \\ \hdashline & & M \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A_\phi & C_\phi \\ \hdashline B_\phi & D_\phi \\ 0 & I \end{array}\right] > 0 \tag{6.4}$$

where $\left[\begin{array}{c:c} \mathcal{A} & \mathcal{B} \\ \hdashline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc:c} A_\psi & B_\psi C^0 & B_\psi D^{00} \\ 0 & A & B^0 \\ \hdashline C_\psi & D_\psi C^0 & D_\psi D^{00} \end{array}\right]$, $\left[\begin{array}{c:c} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hdashline \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{array}\right] = \left[\begin{array}{cc:c} A_\phi & B_\phi C^0 & B_\phi D^{00} \\ 0 & A & B^0 \\ \hdashline C_\phi & D_\phi C^0 & D_\phi D^{00} \end{array}\right]$

and $\mathcal{B}^1 = \begin{bmatrix} 0 \\ B^1 \end{bmatrix}$, $\mathcal{C}^1 = \begin{bmatrix} 0 & C^1 \end{bmatrix}$.

Also $\mathbf{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ are partitioned according to $\mathcal{A}$ and $\hat{\mathcal{A}}$.

The positive definiteness constraint on Y_{22} or/and X_{22} is to ensure the stability of the system [101], as is clear from the first block of first inequality in (6.3) or/and (6.4).

Then construct a balancing transformation T such that the transformed (balanced) system is defined as

$$\left[\begin{array}{c:c}\tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \hdashline \tilde{\mathcal{C}} & \mathcal{D}\end{array}\right] = \begin{bmatrix} I & & \\ & T^{-1} & \\ & & I \end{bmatrix} \left[\begin{array}{cc:c} A^{\psi} & B^{\psi}C^{0} & B^{\psi}D^{00} \\ 0 & A & B^{0} \\ \hdashline C^{\psi} & D^{\psi}C^{0} & D^{\psi}D^{00} \end{array}\right] \begin{bmatrix} I & & \\ & T & \\ & & I \end{bmatrix}$$

$$= \left[\begin{array}{cc:c} A^{\psi} & B^{\psi}\tilde{C}^{0} & B^{\psi}D^{00} \\ 0 & \tilde{A} & \tilde{B}^{0} \\ \hdashline C^{\psi} & D^{\psi}\tilde{C}^{0} & D^{\psi}D^{00} \end{array}\right]$$

and

$$T^{-1}Y_{22}T^{-T} = T^{T}X_{22}T = \Sigma.$$

Clearly, by comparing the above conditions (6.1), (6.2) with (6.3), (6.4), the latter ones are less conservative than the first ones, due to the presence of extra free variables.

Appendix A

Lyapunov Stability for Non-Causal Systems

Consider the 1D discrete-time system

$$x(t+1) = Ax(t), \;\; x(t) \in \mathbb{R}^n \tag{A.1}$$

We consider the Lyapunov stability of such a system, by defining a candidate Lyapunov function $V(x(t)) = x^T(t)Px(t), \; P > 0$, and the Lyapunov difference as

$$x^T(t+1)\; P\; x(t+1) - x^T(t)\; P\; x(t) < 0, \; \forall x \neq 0. \tag{A.2}$$

System (A.1) is stable if $x(t+1) < x(t), \forall t \geq 0$ is satisfied. Then, $P > 0$ satisfying (A.2) will ensure the stability of the system.
If we suppose that system (A.1) is non-causal and consider $-\infty < t < \infty$, then $P < 0$ will satisfy (A.2), such that $x(t) < x(t+1), \text{ for } -\infty < t \leq 0$ which is exactly what is required for the backward shift such that:

$$x(t) = Ax(t+1), \tag{A.3}$$

see Figure A.1.

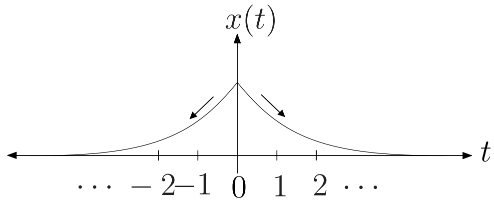

Figure A.1: Decreasing states in both directions

Appendix B

Proof of Theorem 3.3

This proof is an extension of the 1D systems case given in [102], [28] to *non-causal* MD systems.
Define the continuous-time and -space domain system of (2.7) as:

$$\begin{aligned} z(t,s) &= A\,\xi(t,s) + B\,u(t,s) \\ y(t,s) &= C\,\xi(t,s) + D\,u(t,s) \end{aligned}$$

where[1] $z(t,s) = \Delta^{-1}\xi(t,s) = \begin{bmatrix} \dot{x}(t,s) \\ w_+(t,s) \\ w_-(t,s) \end{bmatrix}$; $\xi(0,s) = 0$ and $\xi(t,0) = 0$. (For simplicity of presentation, sometimes we drop the dependence on t and s.
Partition $\xi = \begin{bmatrix} \xi^s \\ \xi^{ns} \end{bmatrix}$ according to significant (s) and non-significant (ns) states such that $\xi^s = \begin{bmatrix} x^s \\ v_+^s \\ v_-^s \end{bmatrix}$, $\xi^{ns} = \begin{bmatrix} x^{ns} \\ v_+^{ns} \\ v_-^{ns} \end{bmatrix}$, and accordingly $z = \begin{bmatrix} z^s \\ z^{ns} \end{bmatrix}$.
Introduce a corresponding partition of the matrices A, B and C as

$$A = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_r \\ B_2 \end{bmatrix} \text{ and } C = \begin{bmatrix} C_r & C_2 \end{bmatrix} \tag{B.1}$$

such that the dimensions of $A_r, A_{12}, A_{21}, A_{22}$ are $(n_r \times n_r), (n_r \times n - n_r), (n - n_r \times n_r), (n - n_r \times n - n_r)$, respectively; the dimensions of B_r, B_2 are $(n_r \times n_u), (n - n_r \times n_u)$, respectively; and the dimensions of C_r, C_2 are $(n_y \times n_r), (n_y \times n - n_r)$, respectively.
Matrices A_r, B_r and C_r are defined in (3.13) and the other matrices are defined accordingly.

[1] Note that we present the differential and the integral of $v_+(t,s), v_-(t,s)$ respectively, as $w_+(t,s), w_-(t,s)$; see (3.3).

According to the above partitions, we define the reduced system as

$$\begin{aligned} z^r &= A_r\, \xi^r + B_r\, u \\ y_r &= C_r\, \xi^r + D\, u, \\ \xi^r(0,s) = 0;&\ \xi^r(t,0) = 0. \end{aligned}$$

Define the auxiliary signal

$$\hat{z} = A_{21}\xi^r + B_2 u.$$

Suppose that $\Sigma = \begin{bmatrix} \Sigma^s & \\ & \begin{bmatrix} \sigma_{n_1-1} & \\ & \sigma_{n_1} \end{bmatrix} \end{bmatrix}$, such that $\Sigma^{ns} = \begin{bmatrix} \sigma_{n_1-1} & \\ & \sigma_{n_1} \end{bmatrix}$ and $n_r = n-2$. Start by removing the state with the generalized singular value σ_{n_1}, then proceed iteratively and remove σ_{n_1-1}. For simplicity of presentation, we use σ.
Rewrite the non-strict inequalities (dropping the $\tilde{}$) in (3.11) as

$$\begin{bmatrix} A \\ I \end{bmatrix}^* \begin{bmatrix} & \Sigma \\ \Sigma & \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} + C^*C \leq 0 \tag{B.2}$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} & \Sigma^{-1} \\ \Sigma^{-1} & \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \leq 0. \tag{B.3}$$

Multiplying inequality (B.2) with the row vector $[(\xi^s - \xi^r)^* \quad \xi^{ns*}]$ from the left and its complex conjugate from the right, and adding this to the inequality obtained by multiplying inequality (B.3) with the row vector $\sigma[(\xi^s + \xi^r)^* \quad \xi^{ns*} \quad 2u^*]$ from the left and its complex conjugate from the right, we obtain

$$\begin{aligned} &\begin{bmatrix} z^s - z^r \\ z^{ns} - \hat{z} \\ \xi^s - \xi^r \\ \xi^{ns} \end{bmatrix}^* \begin{bmatrix} & & \Sigma^s & \\ & & & \sigma \\ \Sigma^s & & & \\ & \sigma & & \end{bmatrix} \begin{bmatrix} z^s - z^r \\ z^{ns} - \hat{z} \\ \xi^s - \xi^r \\ \xi^{ns} \end{bmatrix} + (y - y_r)^*(y - y_r) \\ &+\sigma^2 \left(\begin{bmatrix} z^s + z^r \\ z^{ns} + \hat{z} \\ \xi^s + \xi^r \\ \xi^{ns} \end{bmatrix}^* \begin{bmatrix} & & (\Sigma^s)^{-1} & \\ & & & \sigma^{-1} \\ (\Sigma^s)^{-1} & & & \\ & \sigma^{-1} & & \end{bmatrix} \begin{bmatrix} z^s + z^r \\ z^{ns} + \hat{z} \\ \xi^s + \xi^r \\ \xi^{ns} \end{bmatrix} \right) - 4\sigma^2 \ \ u^*u \leq 0. \end{aligned} \tag{B.4}$$

Then double-integrating the resulting inequality (B.4) over the time interval $[0, T]$ and space interval $[\underline{s}, \overline{s}]$ we obtain

$$\int_0^T \int_{\underline{s}}^{\overline{s}} [*]^* \begin{bmatrix} & & \Sigma^s & \\ & & & \sigma \\ \Sigma^s & & & \\ & \sigma & & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{x}^s \\ w_+^s \\ w_-^s \end{bmatrix} - \begin{bmatrix} \dot{x}^r \\ w_+^r \\ w_-^r \end{bmatrix} \\ \begin{bmatrix} \dot{x}^{ns} \\ w_+^{ns} \\ w_-^{ns} \end{bmatrix} - \hat{z} \\ \begin{bmatrix} x^s \\ v_+^s \\ v_-^s \end{bmatrix} - \begin{bmatrix} x^r \\ v_+^r \\ v_-^r \end{bmatrix} \\ \begin{bmatrix} x^{ns} \\ v_+^{ns} \\ v_-^{ns} \end{bmatrix} \end{bmatrix} ds\, dt + \int_0^T \int_{\underline{s}}^{\overline{s}} (y - y_r)^*(y - y_r)\ ds\ dt$$

$$+\ \sigma^2 \left(\int_0^T \int_{\underline{s}}^{\overline{s}} [\ *\ ]^* \begin{bmatrix} & & (\Sigma^s)^{-1} & \\ & & & \sigma^{-1} \\ (\Sigma^s)^{-1} & & & \\ & \sigma^{-1} & & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{x}^s \\ w_+^s \\ w_-^s \end{bmatrix} + \begin{bmatrix} \dot{x}^r \\ w_+^r \\ w_-^r \end{bmatrix} \\ \begin{bmatrix} \dot{x}^{ns} \\ w_+^{ns} \\ w_-^{ns} \end{bmatrix} + \hat{z} \\ \begin{bmatrix} x^s \\ v_+^s \\ v_-^s \end{bmatrix} + \begin{bmatrix} x^r \\ v_+^r \\ v_-^r \end{bmatrix} \\ \begin{bmatrix} x^{ns} \\ v_+^{ns} \\ v_-^{ns} \end{bmatrix} \end{bmatrix} ds\, dt \right) - 4\ \sigma^2 \int_0^T \int_{\underline{s}}^{\overline{s}} u^* u\, ds\, dt \leq 0.$$

Recall that $\Sigma^s = \text{diag}(\Sigma_1^s, \Sigma_2^s, -\Sigma_3^s)$. We can rewrite the above inequality as

$$\int_0^T \left(- \int_0^{\underline{s}} [\ *\ ]^* \begin{bmatrix} & -\Sigma_3^s \\ -\Sigma_3^s & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} w_-^s - w_-^r \\ w_-^{ns} \end{bmatrix} \\ \begin{bmatrix} v_-^s - v_-^r \\ v_-^{ns} \end{bmatrix} \end{bmatrix} ds + \int_0^{\overline{s}} [*]^* \begin{bmatrix} & & & \Sigma_1^s & & \\ & & & & \Sigma_{2^s} & \\ & & & & & \sigma \\ \Sigma_1^s & & & & & \\ & \Sigma_2^s & & & & \\ & & \sigma & & & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{x}^s \\ w_+^s \end{bmatrix} - \begin{bmatrix} \dot{x}^r \\ w_+^r \end{bmatrix} \\ \begin{bmatrix} \dot{x}^{ns} \\ w_+^{ns} \end{bmatrix} \\ \begin{bmatrix} x^s \\ v_+^s \end{bmatrix} - \begin{bmatrix} x^r \\ v_+^r \end{bmatrix} \\ \begin{bmatrix} x^{ns} \\ v_+^{ns} \end{bmatrix} \end{bmatrix} ds \right) dt$$

$$
\begin{aligned}
&-\sigma \int_0^T \int_{\underline{s}}^{\overline{s}} 2\hat{z}^* \xi^{ns} ds\, dt + \int_0^T \int_{\underline{s}}^{\overline{s}} (y-y_r)^*(y-y_r)\ \ ds\ \ dt \\
&+ \sigma^2 \int_0^T \left(- \int_0^{\underline{s}} [\ * \ ]^* \begin{bmatrix} & (-\Sigma_3^s)^{-1} \\ (-\Sigma_3^s)^{-1} & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} w_-^s + w_-^r \\ w_-^{ns} \end{bmatrix} \\ \begin{bmatrix} v_-^s + v_-^r \\ v_-^{ns} \end{bmatrix} \end{bmatrix} ds \right. \\
&\left. + \int_0^{\overline{s}} [*]^* \begin{bmatrix} & & & (\Sigma_1^s)^{-1} & & \\ & & & & (\Sigma_2^s)^{-1} & \\ & & & & & \sigma^{-1} \\ (\Sigma_1^s)^{-1} & & & & & \\ & (\Sigma_2^s)^{-1} & & & & \\ & & \sigma^{-1} & & & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{x}^s \\ w_+^s \end{bmatrix} + \begin{bmatrix} \dot{x}^r \\ w_+^r \end{bmatrix} \\ \begin{bmatrix} \dot{x}^{ns} \\ w_+^{ns} \end{bmatrix} \\ \begin{bmatrix} x^s \\ v_+^s \end{bmatrix} + \begin{bmatrix} x^r \\ v_+^r \end{bmatrix} \\ \begin{bmatrix} x^{ns} \\ v_+^{ns} \end{bmatrix} \end{bmatrix} ds \right) dt \\
&+ \sigma \int_0^T \int_{\underline{s}}^{\overline{s}} 2\hat{z}^* \xi^{ns} ds\, dt - 4\,\sigma^2 \int_0^T \int_{\underline{s}}^{\overline{s}} u^* u\ \ ds\ \ dt \leq 0.
\end{aligned}
$$

Note that the third and seventh term in the latter inequality cancel each other; moreover, the remaining terms are positive and the result follows by letting $T \to \infty$, $\overline{s} \to \infty$ and $\underline{s} \to -\infty$,

$$\| \ y - y_r \ \|_{L_2}^2 \leq 4\ \sigma^2 \ \| \ u \ \|_{L_2}^2,$$

which completes the proof.

Appendix C

The Equivalence Between (5.11) and (5.7)

Inequality (5.11) is equivalent to the transformed version of (5.7), where expanding (5.11) and applying the Schure complement yield

$$\begin{bmatrix} A^*\Sigma^{-1} + \Sigma^{-1}A & \Sigma^{-1}B^0 & \Sigma^{-1}B^1 \\ B^{0*}\Sigma^{-1} & -\Psi^{-1} & 0 \\ B^{1*}\Sigma^{-1} & 0 & -I \end{bmatrix} + \begin{bmatrix} C^{0*} \\ 0 \\ D^{01*} \end{bmatrix} \Psi^{-1} \begin{bmatrix} C^0 & 0 & D^{01} \end{bmatrix} < 0. \tag{C.1}$$

Pre- and post-multiplying (C.1) by $\mathrm{diag}(\Sigma, I, I)$ and applying the Schur complement twice gives the equivalence; we obtain

$$\begin{bmatrix} \Sigma A^* + A\Sigma + B^0\Psi B^{0*} & B^1 \\ B^{1*} & -I \end{bmatrix} + \begin{bmatrix} \Sigma C^{0*} \\ D^{01*} \end{bmatrix} \Psi^{-1} \begin{bmatrix} C^0\Sigma & D^{01} \end{bmatrix} < 0.$$

Appendix D

State-space Models of Actuated Beam

D.1 Spatially Invariant System

A state-space model for a single subsystem of the experimentally identified beam as in (2.5) is

$$
\begin{array}{c}
T^{+}x\left\{\begin{array}{c}\\ \\ \\ \end{array}\right.\\
w_{+}\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.\\
w_{-}\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.\\
\\
\end{array}
\left[\begin{array}{c}
y(t-2,s)\\
y(t-1,s)\\
y(t,s)\\ \hdashline
y(t-2,s-1)\\
y(t-2,s)\\
y(t-1,s-1)\\
y(t-1,s)\\
y(t-2,s+1)\\
y(t-2,s)\\
y(t-1,s+1)\\
y(t-1,s)\\ \hdashline
y(t,s)
\end{array}\right]
=
\left[\begin{array}{ccc:cccccccc:c}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_{2,0} & a_{1,0} & a_{2,2} & a_{2,1} & a_{1,2} & a_{1,1} & a_{2,-2} & a_{2,-1} & a_{1,-2} & a_{1,-1} & 1\\ \hdashline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \hdashline
b_{3,0} & b_{2,0} & b_{1,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
y(t-3,s)\\
y(t-2,s)\\
y(t-1,s)\\ \hdashline
y(t-2,s-2)\\
y(t-2,s-1)\\
y(t-1,s-2)\\
y(t-1,s-1)\\
y(t-2,s+2)\\
y(t-2,s+1)\\
y(t-1,s+2)\\
y(t-1,s+1)\\ \hdashline
u(t,s)
\end{array}\right]
\begin{array}{c}
\left.\begin{array}{c}\\ \\ \\ \end{array}\right\}x\\
\left.\begin{array}{c}\\ \\ \\ \\ \end{array}\right\}v_{+}\\
\left.\begin{array}{c}\\ \\ \\ \\ \end{array}\right\}v_{-}\\
\\
\end{array}
\tag{D.1}
$$

In this case, $n = 11; n_u = n_y = 1$; n_u, n_y are the number of inputs and outputs of the system, respectively. Here, $n = n_1 + n_2 + n_3$; $n_1 = 3$ due to a three-step temporal shift (Figure 3.4), $n_2 = n_3 = 4$ due to four-step spatial forward and backward shifts, respectively. The values of the a and b coefficients are as follows.

$a_{1,2}$	$a_{1,1}$	$a_{1,0}$	$a_{1,-1}$	$a_{1,-2}$	$a_{2,2}$	$a_{2,1}$	$a_{2,0}$	$a_{2,-1}$	$a_{2,-2}$	$b_{1,0}$	$b_{2,0}$	$b_{3,0}$
-0.1456	-0.0451	-0.0098	0.2016	0.2288	0.0822	0.0453	-0.0223	-0.1061	-0.0215	-0.0071	-0.0720	-0.039

D.2 Spatially Varying System

A state-space model of a spatial-LPV subsystem of the experimentally identified beam is

$$
\begin{array}{l}
T^{+}x\left\{\begin{array}{l} \\ \\ \end{array}\right. \\
w_{+}\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \\
w_{-}\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \\
\\
\end{array}
\begin{bmatrix} y(t-1,q) \\ y(t,q) \\ \hline y(t-2,q-1) \\ y(t-2,q) \\ y(t-1,q-1) \\ y(t-1,q) \\ y(t-2,q+1) \\ y(t-2,q) \\ y(t-1,q+1) \\ y(t-1,q) \\ \hline y(t,q) \end{bmatrix}
= M(\rho))
\begin{bmatrix} y(t-2,q) \\ y(t-1,q) \\ \hline y(t-2,q-2) \\ y(t-2,q-1) \\ y(t-1,q-2) \\ y(t-1,q-1) \\ y(t-2,q+2) \\ y(t-2,q+1) \\ y(t-1,q+2) \\ y(t-1,q+1) \\ \hline u(t,q) \end{bmatrix}
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \end{array}\right\} x \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} v_{+} \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} v_{-} \\
\\
\end{array}
\tag{D.2}
$$

where $M(\rho) =$

$$
\left[\begin{array}{cc|cccccccc|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_8(\rho) & \alpha_3(\rho) & \alpha_6(\rho) & \alpha_7(\rho) & \alpha_1(\rho) & \alpha_2(\rho) & \alpha_{10}(\rho) & \alpha_9(\rho) & \alpha_5(\rho) & \alpha_4(\rho) & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\beta_2(\rho) & \beta_1(\rho) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

In this case, $n = 10$; $n_u = n_y = 1$; $n = n_1 + n_2 + n_3$, $n_1 = 2$, $n_2 = n_3 = 4$.
The α and β coefficients are affine functions of the spatial scheduling parameter $\rho(q)$:

$$
\begin{aligned}
\alpha_d(\rho(q)) &= \alpha_{d1} + \rho(q)\,\alpha_{d2}, \;\; d = 1, \cdots, 10, \\
\beta_z(\rho(q)) &= \beta_{z1} + \rho(q)\,\beta_{z2}, \;\; z = 1, 2,
\end{aligned}
$$

where $\rho(q)$ varies along $q = 1, \cdots, 10$.

Appendix E

Auxiliary Results

E.1 Bilinear Transformation

Here, in order to clarify the presentation, we refer to the discrete-domain system matrices as: A_d, B_d, C_d and D_d, while the continuous system matrices are A, B, C and D.

With $H = \begin{bmatrix} I_{n_1} & & \\ & I_{n_2} & \\ & & -I_{n_3} \end{bmatrix}$, the transformation from temporal- and spatial-discrete domain into temporal- and spatial-continuous domain is given by [12]

$$\begin{aligned} A &= H\left(A_d - I\right)\left(A_d + I\right)^{-1} \\ B &= \sqrt{2}\, H\left(A_d + I\right)^{-1} B_d \\ C &= \sqrt{2}\, C_d\left(A_d + I\right)^{-1} \\ D &= D_d - C_d\left(A_d + I\right)^{-1} B_d. \end{aligned} \tag{E.1}$$

E.2 Elimination Lemma [1]

Lemma E.1 *Given a matrix $R = R^* \in \mathbb{C}^{m\times m}$ and given full column rank matrices $U \in \mathbb{C}^{m\times l}$ and $V \in \mathbb{C}^{m\times k}$. Let $U_\perp, V_\perp$ denote the matrices such that $[U \quad U_\perp]$, $[V \quad V_\perp]$ are square and invertible with $U_\perp^* U = 0$, $V_\perp^* V = 0$. Then, there exists a matrix $K \in \mathbb{C}^{l\times k}$, such that*

$$R + UKV^* + VK^*U^* < 0 \tag{E.2}$$

if and only if

$$U_\perp^* R U_\perp < 0 \quad \text{and} \quad V_\perp^* R V_\perp < 0. \tag{E.3}$$

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List of Symbols and Abbreviations

$\mathbb{Z}$	Set of integers
$\mathbb{Z}_+$	Set of nonnegetive integers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
n	Size of the state vector
n_1	Size of the temporal state vector
n_2, n_3	Size of the forward/backward spatial state vectors
i	Temporal, forward/backward spatial index
n_r	Size of the reduced state vector
n^t, n^s	Size of the temporal/spatial uncertainty
n_r^t, n_r^s	Size of the reduced temporal/spatial uncertainty
n_u	Size of the physical input
n_y	Size of the physical output
$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]$	Shorthand for parameter-invariant state space realization
$\left[\begin{array}{c:c} A(\Delta) & B(\Delta) \\ \hdashline C(\Delta) & D(\Delta) \end{array}\right]$	Shorthand for state space realization varying on Δ
diag	Diagonal matrix
δ	Temporal scheduling parameter
ρ	Spatial scheduling parameter
$\partial(.)$	Variation rate
n_δ, n_ρ	Number of temporal/spatial scheduling parameters
X	Structured generalized Gramian
X_1, X_2, X_3	Temporal, forward/backward spatial blocks of generalized Gramian
$\mathcal{X}$	Set of structured generalized Gramian

$\mathcal{Y}$	Set of structured generalized Gramian of the error system
$X(\Delta)$	Structured generalized Gramian varying on Δ
$\mathcal{S}$	Set of structured generalized Gramian varying on Δ
T	Structured state space transformation matrix
$T(\Delta)$	Structured state space transformation matrix varying on Δ
$\mathcal{T}$	Set of structured state space transformation matrix
Γ	Set of structured state space transformation matrix varying on Δ
$W = \begin{bmatrix} W_{\Delta^t} & \\ & W_{\Delta^s} \end{bmatrix}$	Structured uncertainty transformation matrix
$\mathcal{W}$	Set of structured uncertainty transformation matrix
$\star$	Star product
$*$	Symmetric terms in LMI
$(.)^T, (.)^*$	Transpose/complex conjugate transpose
$\ker(.)$	Null space
O, R	Observability and Reachability matrices
a, b	Coefficients in partial difference equation (and state space) of parameter-invariant models
α, β	Coefficients of parameter-varying models
n_k^t, n_k^s	Multiplicity of temporal/spatial scheduling parameters $\delta_k(t)$ and $\rho_k(s)$
T^-	Backward temporal shift operator
S^+, S^-	Forward/backward spatial shift operator
$\xi(t, s)$	State vector of the state space realization
$x(t, s)$	Temporal state vector
$v_+(t, s), v_-(t, s)$	Spatial states in the positive/negative direction
$w_+(t, s), w_-(t, s)$	Shifted spatial states in the positive/negative direction
$p^t(t, s), q^t(t, s)$	Input and output of the temporal uncertainty channel
$p^s(t, s), q^s(t, s)$	Input and output of the spatial uncertainty channel
$p = \begin{bmatrix} p^t \\ p^s \end{bmatrix}$	Input of the temporal and spatial uncertainty channels
$q = \begin{bmatrix} q^t \\ q^s \end{bmatrix}$	output of the temporal and spatial uncertainty channels
U_{mask}, Y_{mask}	Input/output mask
γ	Error bound

Δ^t, Δ^s	Structured temporal/spatial uncertainty
Θ^t, Θ^s	Compact set of structured temporal/spatial uncertainty
$\Theta^t_\partial, \Theta^s_\partial$	Set of the temporal/spatial variation rate
Π	Multiplier
$\mathcal{P}$	Multiplier set
$\mathcal{P}_{DG}$	Subset of $\mathcal{P}$
$\mathcal{P}_D$	Subset of $\mathcal{P}_{DG}$
M	State space realization of parameter-invariant system
Δ_d	Diagonal block of temporal and spatial shift operators
G	Star product (in upper linear fractional transformation) of M and Δ_d
Δ	Continuous (time and space) domain version of Δ_d block
G^0	Nominal system of parameter-varying model

Superscripts

s, ns	Significant/non-significant part
$\tilde{}$	Transformed version

Subscripts

r	Reduced version
k	Controller
cl	Closed loop system
e	Error system
$\lVert\cdot\rVert_{2\to 2}$	Induced 2-norm
$\left[\begin{array}{c:c} (\cdot)_{11} & (\cdot)_{12} \\ \hdashline (\cdot)_{21} & (\cdot)_{22} \end{array}\right]$	Partitioning according to significant and non-significant blocks
$\perp$	Perpendicular complement

Abbreviations

PDEs	Partial Differential Equations
CGs	Constant generalized Gramians
PDGs	Parameter-Dependent generalized Gramians
SVD	Singular Value Decomposition
LTSI	Linear Time- and Space- Invariant
LTSV	Linear Time- and Space- Varying
LPV	Linear Parameter-Varying
LFT	Linear Fractional Transformation
LPV/LFT	Linear Parameter-Varying/Linear Fractional Transformation
MOR	Model Order Reduction
1D	One-Dimensional system
MD	Multi-Dimensional system

List of Publications

Published

F. Al-Taie and H. Werner, ”Structure-preserving model reduction for spatially interconnected systems with experimental validation on an actuated beam,” *International Journal of Control*, vol. 89, no. 6, pp. 1248–1268, 2016.

F. Al-Taie and H. Werner, ”Balanced truncation error bound for temporally- and spatially-varying interconnected systems,” *in 22nd International Symposium. on Mathematical Theory of Networks and Systems*, 2016.

F. Al-Taie, Q. Liu and H. Werner, ”Joint model and scheduling order reduction via balanced truncation for parameter-varying spatially interconnected systems,” *in Proc. Amer. Control Conf.*, 2016.

Submitted

F. Al-Taie and H. Werner, ”Model order reduction for temporal- and spatial-LPV interconnected systems based on full block S-procedure,” Submitted to *International Journal of Control*, 2016.

F. Al-Taie and H. Werner, ”Extended balanced truncation for parameter-varying multidimensional systems,” Submitted to *Proc. Amer. Control Conf.*, 2017.