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MARITIME COLLISION AVOIDANCE AS A DIFFERENTIAL GAME

by

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4 MARITIME COLLISION AVOIDANCE AS A DIFFERENTIAL GAME¹
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8 T. MILOH and S.D. SHARMA
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1
2 Collision avoidance during a two-ship encounter in the open sea is treated as
3 a problem of optimal control using the theory of differential games. Each ship
4 is assumed to have two controls corresponding to rudder angle and engine
5 setting. Different versions of the game are obtained under assumptions such as
6 (a) one ship evades, while the other stands on, (b) both ships evade in col-
7 laboration, and (c) one ship evades while being pursued by the other. Optimal
8 evasive maneuvers and limiting conditions for ship encounters beyond which
9 collision becomes unavoidable are determined analytically. Numerical examples
10 are given for encounters between two ships, one of which has a higher speed
1 and the other a smaller turning radius. Results are presented in a graphical
2 form suitable for display on a radar screen.
3

4 INTRODUCTION

5
6 This paper is an offshoot of a broader project seeking to determine how exactly
7 the frequency of collisions, and hence the economy and safety of ship operation,
8 depends on the maneuvering capabilities of ships among other things, see
9 Krappinger (1972). In order to solve this problem it has been found necessary
10 to develop a mathematical model of collision avoidance and occurrence. A natu-
1 ral by-product of this effort was the determination of optimal evasive ma-
2 neuvers and of limiting conditions for ship encounters beyond which collision
3 cannot be avoided, a concept we adopted from Kenan (1972) and Webster (1974).
4 However, instead of their numerical trial-and-error approach we have used an
5 analytical technique based on the theory of *Differential Games* as formulated
6 by Isaacs (1965). Although this theory was originally developed for solving
7 military problems of guided pursuit, the same techniques can be profitably
8 applied also to problems of collision avoidance, as pointed out explicitly by
9 Isaacs himself in the preface to his book. In fact we have done no more than
10 to apply several modified versions of Isaacs' *Game of Two Cars*. As the par-
1 ticipants of a Ship Control Systems Symposium² may be expected to be inter-
2 ested in both collision avoidance and differential games, which have much in
3 common with optimal system control, this would seem to be a suitable occasion
4 to present a summary of our work. Those interested in a fuller account are
5 referred to an institute report by Miloh (1974) and to a previous paper ad-
6 dressed to an audience of navigators, cf. Miloh and Sharma (1975). Meanwhile,
7 we have discovered that work along similar lines has been done elsewhere by
8 Vincent et al. (1972) and Merz (1973). However, since Vincent was concerned
9 mainly with aircraft collision avoidance and Merz treated only the special
10 case of two identical ships, our work seems to complement rather than dupli-
1 cate theirs.
2
3

4 ¹This work was done within the framework of the *Sonderforschungsbereich 98*
5 "*Schiffbau und Schiffstechnik*" at the *Institut für Schiffbau, Hamburg*, with
6 the financial support of the *Deutsche Forschungsgemeinschaft*.

²This paper is a slightly revised version of a paper originally presented at the
Fourth Ship Control Systems Symposium, The Hague, Netherlands, 27-31 Oct. 1975.

COLLISION AVOIDANCE MODEL

Rules of the Road

In order to be able to formulate a mathematical model of maritime collision avoidance let us consider how an encounter with another ship in the open sea is handled by a ship master. The complex chain of decision making subject to the International Rules of the Nautical Road has been ably summarized in logic flow diagrams by Luse (1972) and Kwik (1973), from whom our Fig. 1 is adapted. Whenever another ship enters the range of observation of own ship we speak of an encounter, which may be said to last until the other ship again passes out of our range of observation. In general, both ships will hold course and speed during the encounter so that the range will steadily decrease upto the closest point of approach (CPA) and then increase. The range at CPA is called the miss distance. If the miss distance predicted from continuous or successive observations of range and relative bearing at the beginning of the encounter is less than a certain amount regarded as safe, the encounter is said to constitute a risk of collision. Depending upon the type of ships involved and upon whether it is a meeting, crossing or overtaking situation, either both ships are burdened, i.e. required to evade, or one is burdened while the other is privileged, i.e. required to hold course and speed (also called *standing on*) upto the so-called *last minute* (also called ships in extremis), when it is also allowed and required to evade. The last minute is presumed to have arrived when it becomes clear that the burdened ship cannot avoid collision by its action alone. Obviously, a collision is always preceded by a last minute situation and, in general, can occur only when both ships have failed to take appropriate action.

Barriers and Critical Maneuvers

Consider, as depicted in Fig. 2, the initial or steady-state phase of a two-ship encounter during which both ships stand on. This is the phase of situation assessment as expertly described by Luse (1972). Let us draw an imaginary circle of radius r_m about the center of own ship O, such that a miss distance less than r_m would constitute an undesirable event in some suitable sense. A burdened ship in the open sea might choose an r_m of two or three miles to avoid any hazardous approach, whereas a privileged ship in congested waters, not being free to take early action, may have to choose an r_m on the order of a ship length so that a miss distance less than r_m would be tantamount to a physical collision. Since the particular interpretation is of no consequence to the following analysis, which can be carried out for any numerical value of r_m , we shall for the sake of brevity call the undesirable event a *collision* and the circle of radius r_m a *terminal* circle.

Consider now the set of all possible encounters with fixed initial velocities but arbitrary relative positions of the two ships. In relative coordinates attached to ship O the ship A will seem to move with relative velocity V_r .

Let us draw tangents to the terminal circle in a direction parallel but opposite to velocity V_r . The inside of the band formed by these two tangents (from B and C) we call the *risk* zone, and the outside we call the *safe* zone. For it is clear from Fig. 2 that a risk of collision in the above sense exists if and only if A is initially observed inside this band. Of course, this risk can, in general, be obviated by a timely evasive maneuver of either ship. It can be shown that there exists a curve BD such that if A is still beyond this curve, a miss distance larger than r_m can be attained by a hard right turn of O, indicated by O_R in Fig. 2. Similarly, there is a limiting curve CE for a hard left turn of O. Hence, if right and left turns were the only permissible maneuvers, we

1 could call the area enclosed by the arcs BC, CF and FB the *collision* zone. If O
2 is responsible for collision avoidance, it must evade before A reaches the
3 boundary BFC. For if O waits until A (initially inside the risk zone) has al-
4 ready penetrated the collision zone, the risk transforms into certainty and col-
5 lision becomes imminent, even though some further time may elapse before A fi-
6 nally hits the terminal circle irrespective of how O now tries to evade. We may
7 therefore call a boundary such as BFC⁴ the *barrier* and the evasive maneuvers
8 that would prevent collision marginally from initial positions on the barrier
9 the *critical* maneuvers.

10 Evidently, a family of barriers and critical maneuvers determined for systemati-
1 cally varied values of the minimum desired miss distance would constitute a com-
2 plete set of optimal evasive maneuvers for maximizing miss distance starting
3 from any initial position. It is also clear that the shape and size of the col-
4 lision and risk zones will depend on the maneuvering capabilities and objectives
5 of the ships involved. Typically, one ship would evade, while the other stands
6 on. However, a best case analysis assuming optimal evasion in collaboration and
7 a worst case analysis assuming optimal evasion in face of optimal pursuit (pre-
8 sumably owing to ignorance or error) might also be meaningful in the context of
9 collision avoidance. We shall see that by formulating the problem as a differ-
10 ential game various combinations can be treated by a single uniform approach.

1 DIFFERENTIAL GAME FORMULATION

2 Basic Concepts

3 The essential requisite of a differential game is a dynamic system controlled by
4 two or more players. State variables characterize the current state of the sys-
5 tem. Its dynamic behavior is expressed by kinematic equations relating the time
6 rate of change of state variables to the state variables themselves and to con-
7 trol variables, which are at the volition of the players. Quantities which do
8 not change with time during a particular play of the game, but which we might
9 wish to vary from play to play are called parameters of the game. The end of
10 any play is marked by predefined terminal conditions of the system. The game is
1 characterized by a payoff comprising a terminal component as a function of the
2 terminal conditions and/or an integral component as a function of the path along
3 which the terminal conditions are reached. The objective of an individual play-
4 er is to maximize (or minimize) the payoff. A function relating a control vari-
5 able to the state variables is called a strategy. All players are assumed to
6 have complete information on the current state of the system and the range of
7 controls available to all other players. If all players play their mutually
8 optimal strategies, there exists for every starting state of the system a con-
9 ceptually predetermined unique payoff called the value. The theoretical solu-
10 tion of the game comprises the optimal strategies, the optimal paths leading to
1 the terminal conditions and the value as a function of the starting states of
2 the system. A continuous payoff function yields a game of degree with a con-
3 tinuous value function in state space. A payoff function defined to have only
4 discrete values yields a game of kind with the state space subdivided into
5 zones of different value separated by barriers.

6 Kinematic Equations

7 Our dynamic system consists of two ships O and A maneuvering on a sea surface
8 assumed to be homogeneous, isotropic, unbounded and undisturbed. We choose to
9 describe it by the following set of kinematic equations (KE) involving five
10 state variables, four control variables and eight parameters:

1 ⁴If it is required that O be able to avoid collision while retaining the choice
2 of a right *or* left turn, e.g. in congested or restricted waters, then the
3 curve BEFDC will have to be regarded as the barrier.

$$dx/dt = V_a \cos\theta + y\phi_1 V_o/R_o - V_o \quad (1)$$

$$dy/dt = V_a \sin\theta - x\phi_1 V_o/R_o \quad (2)$$

$$d\theta/dt = \psi_1 V_a/R_a - \phi_1 V_o/R_o \quad (3)$$

$$dV_o/dt = \phi_2 T_o/m_o - V_o |V_o| k_o/m_o \quad (4)$$

$$dV_a/dt = \psi_2 T_a/m_a - V_a |V_a| k_a/m_a \quad (5)$$

The state variables are the rectangular coordinates (x, y) and the course angle θ of A relative to O, and the absolute forward speeds V_o, V_a of O and A. Whenever convenient we shall also use the polar coordinates (range r and bearing α) instead of (x, y) . The two are related by

$$x = r \cos\alpha, \quad y = r \sin\alpha \quad (6)$$

The four fixed parameters for each ship are the minimum turning radius (R_o, R_a) , the maximum propulsive thrust (T_o, T_a) , the effective mass (m_o, m_a) and the coefficient of resistance (k_o, k_a) . The two controls allotted to each ship $(\phi_1, \phi_2$ for O; ψ_1, ψ_2 for A) can be interpreted as normalized rates of turn and thrusts respectively, being idealizations of the two basic controls (rudder angle and engine setting) available to any ship. The control variables may instantaneously assume any value within the standard interval $[-1, 1]$. Eq. (1-3) are purely kinematic. Eq. (4-5) imply the assumption that resistance to forward motion is proportional to speed squared; all other hydrodynamic and inertial effects are ignored.

Payoff and Termination

We shall consider two alternative but essentially equivalent ways of defining payoff and termination in the game of collision avoidance. (I) We can define the closest approach at first pass, characterized by

$$dr/dt = 0, \quad d^2r/dt^2 > 0, \quad (7)$$

as the terminal condition and use the terminal range r_f as payoff yielding a game of degree. (II) We can define collision, characterized by

$$r = r_m, \quad (8)$$

as the terminal condition and use a discrete payoff (say +1 for collision avoidance and -1 for collision occurrence) yielding a game of kind. In the latter case not every point of the terminal surface can be reached from the outside. The useable part is defined by the condition of positive penetration velocity, i.e. negative range rate:

$$dr/dt \leq 0 \quad (9)$$

The boundary of the useable part (BUP) is delineated by the equality sign in Eq. (9) which again yields the condition of closest approach, Eq. (7). Since in a game of kind only the marginal plays ending at the BUP are of concern, we conclude that in either case the only terminal conditions of interest are those given by Eq. (7). The terminal parameter r_f in our game of degree is identical with the game parameter r_m in our game of kind. Substitution of the KE yields

1 a definite relation between the terminal bearing α_f and the terminal course
 2 angle θ_f on the BUP:
 3

$$4 \sin\alpha_f = \varepsilon(V_{of} - V_{af}\cos\theta_f)/V_{rf}, \quad \cos\alpha_f = \varepsilon(V_{af}\sin\theta_f)/V_{rf}, \quad \varepsilon = \pm 1 \quad (10)$$

7 For any given θ_f and terminal speed ratio V_{af}/V_{of} we obtain two solutions
 8 α_f^u, α_f^l corresponding to the upper and lower sign in Eq. (10) such that if we
 9 go around the terminal circle once in the clockwise sense, the useable part
 10 begins at α_f^l (lower BUP) and ends at α_f^u (upper BUP). Note that in the excep-
 1 tional case $V_{af} = V_{of}$ and $\theta_f = 0$ any value of α_f satisfies Eq. (7) so that the
 2 entire terminal circle becomes the useable part!
 3
 4
 5
 6

7 Player Objectives

8 To complete the formulation of the differential game we must now specify the
 9 objectives of our two players O and A. We have already seen that in a typical
 10 encounter one of the two ships will be obligated to evade and the other to
 1 stand on. However, the *best* case of evasion in collaboration and the *worst* case
 2 of evasion in face of pursuit are also worth studying. We can cover all possible
 3 combinations by considering the following nine versions of the game:
 4

$$5 O_E A_E, O_N A_E, O_P A_E; O_E A_N, O_N A_N, O_P A_N; O_E A_P, O_N A_P, O_P A_P.$$

8 Here E stands for evasion, N for not maneuvering (i.e. standing on) and P for
 9 pursuit. For instance, $O_E A_E$ implies evasion by O and A in collaboration. More
 10 exactly, the subscript E implies that the ship chooses its controls so as to
 1 maximize payoff (i.e. range at CPA), N implies keeping controls fixed so as to
 2 hold absolute course and speed, and P implies choosing controls so as to mini-
 3 mize payoff (i.e. range at CPA). Note that only the versions $O_P A_E$ and $O_E A_P$
 4 are genuine games with a conflict of objectives. The others are degenerate or
 5 one-sided games which may also be treated as ordinary problems of optimal con-
 6 trol. Game $O_N A_N$ is trivial and $O_P A_P$ is a case of rendezvous rather than
 7 collision avoidance.
 8
 9

10 One might legitimately argue that in normal operation collision avoidance is a
 1 side condition rather than the primary objective of a ship so that return to
 2 original course rather than CPA should be regarded as the terminal condition
 3 and that evasion should aim at minimizing time loss rather than maximizing miss
 4 distance. That would indeed generate a useful game for the timely evasive ma-
 5 neuver of a burdened ship. However, this paper is concerned mainly with last
 6 minute maneuvers, where collision avoidance is the primary objective and hence
 7 miss distance is an appropriate payoff.
 8
 9

9 ANALYTICAL SOLUTION

10 Main Equation

1 Following Isaacs (1965, p. 67) optimal play is governed by the equation

$$2 M \left\{ W_x dx/dt + W_y dy/dt + W_\theta d\theta/dt + W_{V_o} dV_o/dt + W_{V_a} dV_a/dt \right\} = 0 \quad (11)$$

3 where the operator M reflects the players' objectives. Explicitly, M is

$$\max_{\phi_1, \phi_2} \max_{\psi_1, \psi_2} \text{ for game } O_{E^A E}; \quad \max_{\phi_1, \phi_2} \text{ for game } O_{E^A N}; \quad \max_{\phi_1, \phi_2} \min_{\psi_1, \psi_2} \text{ for game } O_{E^A P} \text{ etc.}$$

In the game of degree W stands for the unknown value function with the subscripts denoting partial derivatives with respect to the state variables. In the game of kind W_x, W_y etc. can be interpreted as the components of the outward normal vector on the barrier surface in the state space. Since $dx/dt, dy/dt$ etc. are the components of the vector along which play proceeds in state space, Eq. (11) implies that in a game of terminal payoff optimal paths lie along surfaces of constant value in a game of degree and along barrier surfaces in a game of kind. It is called the Main Equation (ME) of differential games by Isaacs. Its analogue in control theory is called the Bellman Equation, see Bryson and Ho (1969, p. 135). Its solution comprises the solution of the game.

Adjoint Equations

The ME is a first-order partial differential equation of the Hamilton-Jacobi type with $\{ \}$ as the Hamiltonian and W_x, W_y etc. as costate variables. It can be solved by integrating its characteristic equations which consist of the KE and the adjoint equations obtained by differentiating the ME with respect to the state variables and rearranging the terms, see Isaacs (1965, p. 80):

$$dW_x/d\tau = -W_y \bar{\phi}_1 V_o / R_o \quad (12)$$

$$dW_y/d\tau = W_x \bar{\phi}_1 V_o / R_o \quad (13)$$

$$dW_\theta/d\tau = (W_y \cos\theta - W_x \sin\theta) V_a \quad (14)$$

$$dW_{V_o}/d\tau = (W_x y - W_y x - W_\theta) \bar{\phi}_1 / R_o - W_x - 2W_{V_o} |V_o| k_o / m_o \quad (15)$$

$$dW_{V_a}/d\tau = W_x \cos\theta + W_y \sin\theta + W_\theta \bar{\psi}_1 / R_a - 2W_{V_a} |V_a| k_a / m_a \quad (16)$$

Here the derivatives have been taken with respect to retrograde time τ to emphasize an idea fundamental to game theory, namely the *retrogression* principle, which requires that the characteristic equations be solved retrogressively, i.e. starting from the terminal conditions and working backward into state space. The bars over ϕ and ψ indicate that optimal controls satisfying the ME are to be used.

Optimal Controls

Substitution of the KE (1-5) into the ME (11) immediately yields the following expressions for the optimal controls as functions of the state and costate variables:

$$\bar{\phi}_1 = \gamma \text{sgn} \left\{ (W_x y - W_y x - W_\theta) V_o / R_o \right\} \quad (17)$$

$$\bar{\phi}_2 = \gamma \text{sgn} \left\{ W_{V_o} T_o / m_o \right\} \quad (18)$$

$$\bar{\psi}_1 = \delta \text{sgn} \left\{ W_\theta V_a / R_a \right\} \quad (19)$$

$$\bar{\psi}_2 = \delta \text{sgn} \left\{ W_{V_a} T_a / m_a \right\} \quad (20)$$

1 with

2
3 $\gamma = +1$ if 0 evades and -1 if 0 pursues, (21)

4
5 $\delta = +1$ if A evades and -1 if A pursues, (22)

6
7 where we have made use of the fact that the control variables were normalized
8 to lie within the standard interval $[-1, 1]$. If a ship stands on, instead of
9 optimal controls we just have fixed controls:

10
1 $\bar{\phi}_1 = 0$ and $\bar{\phi}_2 = k_o |V_o| V_o / T_o$, if 0 stands on, (23)

2
3 $\bar{\psi}_1 = 0$ and $\bar{\psi}_2 = k_a |V_a| V_a / T_a$, if A stands on. (24)

4
5 Eq. (17-22) imply that the optimal controls are at least piecewise *constant*
6 and, in general, the extreme permissible values, with the possible exception of
7 singular cases where the expressions in braces vanish. This is a direct con-
8 sequence of our choice of the dynamic model such that the control variables
9 occur linearly in the KE (1-5) and is known as the *bang-bang* principle in con-
10 trol theory. It greatly simplifies the integration of the characteristic equa-
1 tions.

2 Optimal Terminal Maneuvers

3 Since the game has to be solved retrogressively we first seek to evaluate the
4 optimal controls at the terminal conditions. We have a four-parametric terminal
5 manifold⁵:

6
7 $0 \leq r_f < \infty$, $0 \leq \theta_f \leq 2\pi$, $0 \leq V_{of} \leq \sqrt{T_o/k_o}$, $0 \leq V_{af} \leq \sqrt{T_a/k_a}$ (25)

8
9 as α_f follows from Eq. (10). The terminal values of the state and costate vari-
0 ables have the parametric representations:

1
2 $x = r_f \cos \alpha_f$, $y = r_f \sin \alpha_f$, $\theta = \theta_f$, $V_o = V_{of}$, $V_a = V_{af}$ (26)

3
4 $W_x = \cos \alpha_f$, $W_y = \sin \alpha_f$, $W_\theta = 0$, $W_{V_o} = 0$, $W_{V_a} = 0$ (27)

5
6 Substituting Eq. (26-27) into Eq. (17-20) shows to our disgrace that the argu-
7 ments of the signum functions are all zero! The situation can be saved, however,
8 by replacing the expressions in braces by their retrograde time derivatives with
9 the final result:

10
1 $\bar{\phi}_1 = \gamma \operatorname{sgn}(-\sin \alpha_f) = \gamma \epsilon \operatorname{sgn}(\cos \theta_f - V_{of}/V_{af})$ (28)

2
3 $\bar{\phi}_2 = \gamma \operatorname{sgn}(-\cos \alpha_f) = -\gamma \epsilon \operatorname{sgn}(\sin \theta_f)$ (29)

4
5 $\bar{\psi}_1 = \delta \operatorname{sgn}[\sin(\alpha_f - \theta_f)] = \delta \epsilon \operatorname{sgn}(\cos \theta_f - V_{af}/V_{of})$ (30)

6
7 $\bar{\psi}_2 = \delta \operatorname{sgn}[\cos(\alpha_f - \theta_f)] = \delta \epsilon \operatorname{sgn}(\sin \theta_f)$ (31)

8
9 ⁵For the sake of simplicity we assume positive terminal speeds. A speed reversal
- during the play would then imply a negative initial speed which can also be
- interpreted as a positive speed along the opposite course.

Recalling that $(\pi+\alpha-\theta)$ is the relative bearing of 0 as seen from A, the optimal terminal maneuvers turn out to have a very simple and intuitively appealing interpretation: At CPA the evader is turning and accelerating *away* from his target, while the pursuer is turning and accelerating *toward* his target! In other words, if the target is to port, the evader is turning to starboard and vice versa; if the target is forward of abeam, the evader is applying backward thrust and vice versa. Unfortunately, no such simple rules exist in terms of conditions prevailing *before* the maneuver is executed, and hence the need to solve the game retrogressively. In passing we note that from the above we may expect singular behavior when the target is dead ahead, abeam or abaft at CPA. This is indeed what happens. So-called singular surfaces originate from such points as we shall presently see.

Optimal Paths

We could now determine optimal paths by integrating the characteristic equations (1-5 and 12-16), working retrogressively from the terminal conditions (26-27). The union of all optimal paths for a given value of terminal parameter r_f would produce a curved surface in state space that can be interpreted as a surface of constant value r_f in our game of degree or as a barrier of miss distance r_f in our game of kind. Different parts of the barrier emanating from the lower and upper BUP (and even from different segments of the same BUP) will in general intersect and enclose a finite collision zone in state space. The intersections are so-called *dispersal* curves separating regions of different optimal strategies (e.g. left turns on one side, right turns on the other). If Eq. (17-20) produce discontinuous controls along some optimal paths, we may have *switching* curves on barriers implying multi-step maneuvers.

For the sake of closed form integration we shall now sacrifice some generality of our kinematic model by assuming *constant* forward speeds, which is probably not too unrealistic in view of the short duration of a collision avoidance maneuver and the rather low accelerations of which ships are capable. (A deceleration is generally considered undesirable in practice due to the attendant loss of rudder effectiveness.) In any case, assuming constant forward speeds (V_o, V_a) and thanks to constant optimal controls $(\bar{\phi}_1, \bar{\psi}_1)$ the characteristic equations can be analytically integrated to yield the following retrograde path equations:

$$\theta = \theta_f + (\bar{\phi}_1 V_o / R_o - \bar{\psi}_1 V_a / R_a) \tau \quad (32)$$

$$x = r_f \cos(\alpha_f + \tau \bar{\phi}_1 V_o / R_o) + R_o \sin(\tau \bar{\phi}_1 V_o / R_o) / \bar{\phi}_1 + R_a \left\{ \sin \theta - \sin(\theta_f + \tau \bar{\phi}_1 V_o / R_o) \right\} / \bar{\psi}_1 \quad (33)$$

$$y = r_f \sin(\alpha_f + \tau \bar{\phi}_1 V_o / R_o) + R_o \left\{ 1 - \cos(\tau \bar{\phi}_1 V_o / R_o) \right\} / \bar{\phi}_1 - R_a \left\{ \cos \theta - \cos(\theta_f + \tau \bar{\phi}_1 V_o / R_o) \right\} / \bar{\psi}_1 \quad (34)$$

$$W_x = \cos(\alpha_f + \tau \bar{\phi}_1 V_o / R_o) \quad (35)$$

$$W_y = \sin(\alpha_f + \tau \bar{\phi}_1 V_o / R_o) \quad (36)$$

$$W_\theta = R_a \left\{ \cos(\alpha_f - \theta_f) - \cos(\alpha_f - \theta_f + \tau \bar{\psi}_1 V_a / R_a) \right\} / \bar{\psi}_1 \quad (37)$$

Here α_f is to be substituted from Eq. (10) and $\bar{\phi}_1, \bar{\psi}_1$ are to be taken for the appropriate game from Eq. (28, 30) in conjunction with Eq. (21, 22); if one of the ships stands on, the limit $\bar{\phi}_1 \rightarrow 0$ or $\bar{\psi}_1 \rightarrow 0$ can be taken analytically. The barrier is now a curved surface in the reduced three-dimensional (x, y, θ)

state space. For its graphical representation on plane paper (or on a radar screen for that matter) it seems convenient to compute cross-sections at constant θ . Moreover, it is numerically expedient to use nondimensional coordinates and retrograde time:

$$\bar{x} = x/R_0, \quad \bar{y} = y/R_0, \quad \bar{\tau} = \tau V_0/R_0 \quad (38)$$

as well as nondimensional game and terminal parameters:

$$\zeta = R_a/R_0, \quad \eta = V_a/V_0, \quad \bar{r}_f = r_f/R_0 \quad (39)$$

After the necessary substitutions the barrier cross-sections generated by Eq. (32-34) with $\bar{\tau}$ as the parameter are found to be

$$\bar{x} = \varepsilon \bar{r}_f \left\{ \eta \sin(\theta + \mu \bar{\psi}_1 \bar{\tau}) - \sin(\bar{\phi}_1 \bar{\tau}) \right\} / \omega + \sin(\bar{\phi}_1 \bar{\tau}) / \bar{\phi}_1 + \zeta \left\{ \sin \theta - \sin(\theta + \mu \bar{\psi}_1 \bar{\tau}) \right\} / \bar{\psi}_1 \quad (40)$$

$$\bar{y} = \varepsilon \bar{r}_f \left\{ \cos(\bar{\phi}_1 \bar{\tau}) - \eta \cos(\theta + \mu \bar{\psi}_1 \bar{\tau}) \right\} / \omega + \left\{ 1 - \cos(\bar{\phi}_1 \bar{\tau}) \right\} / \bar{\phi}_1 - \zeta \left\{ \cos \theta - \cos(\theta + \mu \bar{\psi}_1 \bar{\tau}) \right\} / \bar{\psi}_1 \quad (41)$$

where we have used two redundant symbols

$$\mu = \eta / \zeta, \quad \omega = \left\{ 1 + \eta^2 - 2\eta \cos[\theta + (\mu \bar{\psi}_1 - \bar{\phi}_1) \bar{\tau}] \right\}^{1/2} \quad (42)$$

in order to abbreviate the expressions. Note that valid barrier cross-sections are generated only in the time interval $0 < \bar{\tau} < \bar{\tau}_m$, where $\bar{\tau}_m$ is the smallest value of $\bar{\tau}$ at which the argument of the signum function in Eq. (17) or (19) vanishes after having been nonzero.

Secondary Paths

In general we have exactly two optimal paths leading to any value of θ_f in the interval $[0, 2\pi]$, i.e. one to lower BUP ($\varepsilon = -1$) and the other to upper BUP ($\varepsilon = +1$). However, there exist singular values of θ_f from which emanate (retrogressively) whole families of paths which we may call secondary paths for the sake of distinction from the primary paths given above.

We first consider the singular case mentioned following Eq. (10):

$$\eta = 1, \quad \theta_f = 0, \quad 0 \leq \alpha_f \leq 2\pi \quad (43)$$

Path equations (32-34) still apply, but for obvious reasons barrier cross-sections ($\theta = \text{const}$) should be calculated using α_f rather than θ_f or τ as the parameter.

Next we recall having anticipated singular turning controls when the relative bearing of target at CPA becomes zero or π . Evidently, this can happen only to the slower ship. Accordingly, we distinguish the cases $\eta \cong 1$. If $\eta > 1$, Eq. (28) predicts singular behavior of ship 0 at

$$\sin \alpha_f = 0, \quad \theta_f = \pm \theta_s \quad \text{with} \quad \theta_s = \arccos(1/\eta) \quad (44)$$

Closer scrutiny reveals that there are no optimal paths (but dispersal surfaces) leading to the singular points $\theta_f = \gamma \varepsilon \theta_s$, whereas there are two families of

optimal paths (so-called *tributaries*) leading to the singular points $\theta_f = -\gamma\epsilon\theta_s$ via so-called *universal* surfaces (US) by means of two-step maneuvers ($\bar{\phi}_1 = \pm 1$ along tributaries and zero along US). The barrier cross-sections ($\theta = \text{const}$) generated by $\bar{\phi}_1$ -tributaries are found to be

$$\bar{x} = (\bar{\tau}_u - \gamma\bar{r}_f)\cos(\bar{\phi}_1\bar{\tau}_t) + \sin(\bar{\phi}_1\bar{\tau}_t)/\bar{\phi}_1 + \zeta\left\{\sin\theta - \sin(\bar{\phi}_1\bar{\tau}_t - \gamma\epsilon\theta_s)\right\}/\bar{\psi}_1 \quad (45)$$

$$\bar{y} = (\bar{\tau}_u - \gamma\bar{r}_f)\sin(\bar{\phi}_1\bar{\tau}_t) + \left\{1 - \cos(\bar{\phi}_1\bar{\tau}_t)\right\}/\bar{\phi}_1 - \zeta\left\{\cos\theta - \cos(\bar{\phi}_1\bar{\tau}_t - \gamma\epsilon\theta_s)\right\}/\bar{\psi}_1 \quad (46)$$

with the retrograde time $\bar{\tau}$ divided between tributary and US as

$$\bar{\tau}_t = (\theta + \gamma\epsilon\theta_s + \bar{\psi}_1\bar{\tau}\eta/\zeta)/\bar{\phi}_1, \quad \bar{\tau}_u = \bar{\tau} - \bar{\tau}_t \quad (47)$$

both of which must be positive. Of course, $\bar{\psi}_1$ depends on the game and follows from Eq. (30).

Similarly if $\eta < 1$, Eq. (30) predicts singular behavior of A at

$$\sin(\alpha_f - \theta_f) = 0, \quad \theta_f = \pm\theta_s \quad \text{with} \quad \theta_s = \arccos\eta \quad (48)$$

Now there are dispersal surfaces leading to $\theta_f = -\delta\epsilon\theta_s$ and tributaries via US leading to $\theta_f = \delta\epsilon\theta_s$. The barrier cross-sections generated by $\bar{\psi}_1$ -tributaries are

$$\bar{x} = (\delta\bar{r}_f - \eta\bar{\tau}_u)\cos(\delta\epsilon\theta_s + \bar{\phi}_1\bar{\tau}) + \sin(\bar{\phi}_1\bar{\tau})/\bar{\phi}_1 + \zeta\left\{\sin\theta - \sin(\bar{\phi}_1\bar{\tau} + \delta\epsilon\theta_s)\right\}/\bar{\psi}_1 \quad (49)$$

$$\bar{y} = (\delta\bar{r}_f - \eta\bar{\tau}_u)\sin(\delta\epsilon\theta_s + \bar{\phi}_1\bar{\tau}) + \left\{1 - \cos(\bar{\phi}_1\bar{\tau})\right\}/\bar{\phi}_1 - \zeta\left\{\cos\theta - \cos(\bar{\phi}_1\bar{\tau} + \delta\epsilon\theta_s)\right\}/\bar{\psi}_1 \quad (50)$$

$$\text{with} \quad \bar{\tau}_t = \zeta(\bar{\phi}_1\bar{\tau} - \theta + \delta\epsilon\theta_s)/(\eta\bar{\psi}_1), \quad \bar{\tau}_u = \bar{\tau} - \bar{\tau}_t \quad (51)$$

Here $\bar{\psi}_1 = \pm 1$ and $\bar{\phi}_1$ depends on the game according to Eq. (28). This essentially completes the analytical solution of our game.

NUMERICAL EXAMPLES

Choice of Parameters

Lack of space prohibits the presentation of comprehensive computations with a systematic variation of all relevant parameters. The following numerical examples have been chosen to illustrate just a few salient features of the solution. They all deal with encounters between two ships one of which is twice as fast as the other but also has twice as large a minimum turning radius. The results are presented in a graphical form similar to Fig. 2 showing calculated barrier cross-sections for selected values of relative course angle θ , but with the miss distance r_f varied in steps from 1/3 to twice the minimum turning

radius of the faster ship. Nondimensional scales have been used so that the graphs apply to any combination of absolute numbers within the above constraints. For convenience of interpretation the faster ship may be regarded as a container carrier ($L = 300$ m, $V = 30$ kn, $R = 900$ m) and the slower as a tanker ($L = 300$ m, $V = 15$ kn, $R = 450$ m) with the miss distance r_f varied from 300 m to 1800 m,

i.e. from one ship length (necessary to avoid physical collision) up to about a mile (considered appropriate for a safe pass). The games treated are $O_E A_N$ (own

1 ship evades, target stands on), $O_E A_E$ (own ship and target evade in collabora-
2 tion), $O_E A_P$ (own ship evades in face of pursuit by target) and $O_N A_E$ (own ship
3 stands on, target evades) as these are considered typical problems of collision
4 avoidance.
5

6 General Features

7
8 Fig. 3 shows barrier cross-sections at the extreme values $\theta = 0^\circ$ and 180° for
9 the game $O_E A_N$ (which is probably the most common task in collision avoidance)
10 with the faster ship evading. As one might expect intuitively, the collision
1 zone extends forward of abeam, is smallest for overtaking ($\theta = 0^\circ$) and largest
2 for head-on meeting ($\theta = 180^\circ$). The barrier cross-sections are nearly semi-
3 circular at large ranges but become elongated at short ranges, i.e. maneuvering
4 capability is of increasing importance for evasion at short ranges. Here the
5 x -axis is a dispersal curve separating areas of different optimal strategy. If
6 target A is to port, evader O must turn right and vice versa.
7

8 Before passing on to the next figures a few general comments are in order.
9 First, the critical (or optimal, depending on interpretation) maneuvers are
10 marked in the figures by the self-explanatory symbols L, N and R. For instance,
1 the mark $O_R A_N$ denotes that while A stands on, O executes a hard right turn
2 beginning at the instant A hits the barrier and terminating at CPA⁶. The marks
3 LN and RN denote two-step maneuvers, i.e. the ship so marked executes a hard
4 left or right turn respectively beginning at the barrier and terminating at the
5 US, at which time it switches to straight course and holds it at least upto CPA.
6 Second, note that although the barrier consists entirely of optimal paths, its
7 cross-sections with $\theta = \text{const}$ are, in general, not optimal paths but only
8 starting points of optimal paths which leave the plane $\theta = \text{const}$ on their way
9 to a point θ_f on the BUP. Moreover, barriers are not static, but change with
10 time when either ship maneuvers during an encounter. Finally, note that barrier
1 cross-sections need be calculated only for $0 \leq \theta \leq \pi$. By virtue of symmetry,
2 cross-sections for negative θ are obtained by taking mirror images about the
3 x -axis and interchanging L and R.
4
5

6 Advantage of Speed

7
8 Fig. 4 has been chosen mainly to illustrate the complicated strategies required
9 of the slower ship (whether evading or pursuing) as compared to the relatively
10 simple strategies of the faster ship in a two-ship encounter. Consider first
1 Fig. 4 (top) showing barrier cross-sections at $\theta = 90^\circ$ with the faster ship
2 evading and the slower standing on. In conformity with intuition we still have
3 only two areas of different strategies, although the collision zone is now
4 tilted and the dispersal curve (dashed line D) is neither a straight line nor
5 pointing ahead. Now look at Fig. 4 (bottom) showing essentially the same en-
6 counter but now with the slower ship evading and the faster standing on. The
7 scales have been so chosen that the two figures are directly comparable. Not
8 only is the collision zone larger when the slower ship is burdened (despite its
9 smaller turning radius), but also it must discriminate five different regions
10 (including two requiring two-step maneuvers RN) separated by dashed lines D_1 - D_4 .
1
2

3
4 ⁶If from CPA both ships hold their current courses, the range cannot decrease
5 again. In fact it will always increase except in the special case $V_a = V_o$,
6 $\theta_f = 0$. Original courses may be resumed after awaiting a safe separation. The
required amount of turn (i.e. change of course angle upto CPA or US) is also in-
dicated in degrees for each ship. Lines of equal turn (short dashed lines in the
figures) happen to be straight lines orthogonal to the barrier cross-sections
(solid lines).

Here D_2 and D_4 are true dispersal curves formed by intersecting optimal paths. D_1 , D_3 only mark the transition between barrier segments formed by primary and secondary paths. D_2 separates regions of similar strategy leading to upper and lower BUP. Of special interest are also singular points E_1 - E_3 with three different but equivalent optimal strategies. The inner barriers have fewer different segments and can be obtained by a local orthogonal shift of the outer barriers by virtue of Eq. (35-36).

Collaboration versus Conflict

Fig. 5 is designed to display the difference between collaborative and conflicting maneuvers. Barrier cross-sections at $\theta = 90^\circ$ are shown with the faster ship O always evading, and the slower ship A either evading (top) or pursuing (bottom). As expected, compared to the case of A standing on (Fig. 4 top) the collision zone for collaboration is smaller, while that for pursuit is larger. That the differences are small means that the slower ship is rather ineffective while the faster ship commands the game. Note also that there are two genuine dispersal curves D_1 , D_2 in the game $O_E A_E$. But in game $O_E A_P$ only D_3 is a true dispersal curve, D_1 marks the transition from primary paths to secondary paths and D_2 from left-turn tributaries to right-turn tributaries. Hence D_2 is the intersection of a US with the plane $\theta = \text{const}$ and represents points at which the optimal strategy of A is simply to stand on up to CPA. Finally, we note that in games of pursuit the collision zone is not necessarily closed unless the evader is faster, the exact criterion for closure being known only for the special case $r_f = 0$, see Cockayne (1967).

Slower Ship's Dilemma

The previous figures all show the collision zone from the evader's point of view. However, for an objective determination of the "last minute" according to the Rules of the Road the stand-on vessel must consult the barriers for the game $O_N A_E$. This is shown in Fig. 6 with O as the faster ship. Comparison with Fig. 4 (top) shows that the faster stand-on ship can afford to let the slower burdened ship approach up to the "last minute" and still avoid collision by own action alone. If we reverse the situation we see that the slower stand-on ship cannot afford the same since the collision zone of game $O_E A_N$ is then larger than for $O_N A_E$. Hence if the faster burdened ship does not do her the favor of collaborating, the slower stand-on ship may be forced either to break the "last minute" rule or to risk collision. This might be called the slower ship's dilemma. The Rules of the Road were apparently designed for similar ships for which the problem does not arise.

CONCLUSIONS

It has been shown how the powerful analytical theory of differential games can be applied to determine mathematically optimal evasive maneuvers for two-ship encounters in the open sea, using a simple but realistic kinematic model of maneuvering. Various generalizations of this model and of side conditions (e.g. more than two ships, restricted waters etc.) are conceivable and much work remains to be done before the problem of maritime collision avoidance can be considered completely solved.

Perhaps the most obvious criticism of our model is that it ignores two typical dynamical effects of a real ship maneuver: the time lag Δt from a rudder command up to the onset of an actual turn, and the loss of speed ΔV due to increased resistance in a turn. Pending further calculations using a more elaborate dynamic model we recommend the following approximate way of accounting for

1 these effects: Just pretend that the target on the radar screen is at a distance
2 $V \Delta t$ in advance of its true position and use barriers based on a reduced speed
3 $(V-\Delta V)$ instead of the approach speed V .

4 Comparing our results with the traditional work in the field of anti-collision
5 maneuvers such as that of Calvert (1960) and Jones (1971) we find that the prob-
6 lem is much deeper than previously thought. Optimal evasive maneuvers depend not
7 only on relative bearing of target, but also on its range, course angle, speed
8 ratio, maneuvering capability and objectives. This naturally raises the question
9 of how all this information is to be acquired in practice onboard. Lacking mutu-
10 al communication, even the accurate determination of the other ship's speed and
1 course by radar is a formidable problem in face of noise. Fortunately, this
2 problem is being tackled by various people, see e.g. Strobel and Richter
3 (1974/75).
4

5 The theory of differential games is full of surprises. We close with one final
6 observation in apparent contradiction with intuition. It is generally believed
7 that the so-called perfect collision course (absolutely steady initial bearing)
8 is the most crucial condition for collision avoidance, cf. Kenan (1972). Simple
9 inspection of calculated barriers shows that this is not necessarily so, spe-
10 cially for the slower ship, see Fig. 4-6. For almost any θ (except zero and π)
1 the longest critical range occurs along a bearing other than that corresponding
2 to a perfect collision.
3

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9
-

LIST OF SYMBOLS

Abbreviations

A	Another ship
BUP	Boundary of useable part
CPA	Closespt point of approach
E	Evasion (used as subscript to identify game)
KE	Kinematic equations
L	Hard left turn upto CPA (used as subscript to identify maneuver)
LN	Two-step maneuver: first L upto US, then N upto CPA (")
ME	Main Equation
N	No maneuver, i.e. straight course at constant speed
O	Own ship (also origin of relative coordinates)
P	Pursuit (used as subscript to identigy game)
R	Hard right turn upto CPA (used as subscript to identify maneuver)
RN	Two-step maneuver: first R upto US, then N upto CPA (")
UP	Useable part of terminal surface
US	Universal surface

Variables

\bar{k}	Coefficient of ship resistance
L	Length of ship
m	Mass of ship (including hydrodynamic mass)
R	Minimum turning radius of ship
r	Range between A and O; polar coordinate of A
\bar{r}	Nondimensional range, see Eq. (39)
r_m	Minimum required range
T	Maximum thrust of ship propeller
t	Physical time
x	Distance of A from O measured along velocity of O, see Fig. 1
\bar{x}	Nondimensional coordinate, see Eq. (38)
y	Distance of A from O normal to velocity of O, see Fig. 1
\bar{y}	Nondimensional coordinate, see Eq. (38)
V	Absolute speed of ship
V_r	Speed of A relative to O
W	Value function
α	Relative bearing of A taken clockwise from velocity of O
γ	Game parameter (+1 if O evades and -1 if O pursues)
δ	Game parameter (+1 if A evades and -1 if A pursues)
ϵ	Path parameter (+1 if leading to upper BUP and -1 to lower BUP)
ζ	Ratio of minimum turning radii, see Eq. (39)
η	Ratio of ship speeds, see Eq. (39)
θ	Relative course angle of A taken clockwise from velocity of O
θ_s	Singular value of terminal course angle, see Eq. (44, 48)
τ	Retrograde time, i.e. time to reach CPA
$\bar{\tau}$	Nondimensional retrograde time, see Eq.(38)
τ_t	Time travelled along tributary to reach US
τ_u	Time travelled along US to reach CPA
ϕ, ψ	Control variable of ship O, A
ϕ_1, ψ_1	Normalized rate of turn of O, A (positive to right)
ϕ_2, ψ_2	Normalized thrust of O, A (positive for acceleration)
$\bar{\phi}, \bar{\psi}$	Optimal value of control ϕ, ψ

Subscripts

a	Of ship A	<u>Apply to</u> k, L, m, R, T, V
f	Final (or terminal) value, i.e. value at CPA	r, α, θ, V
o	Of ship O	k, L, m, R, T, V

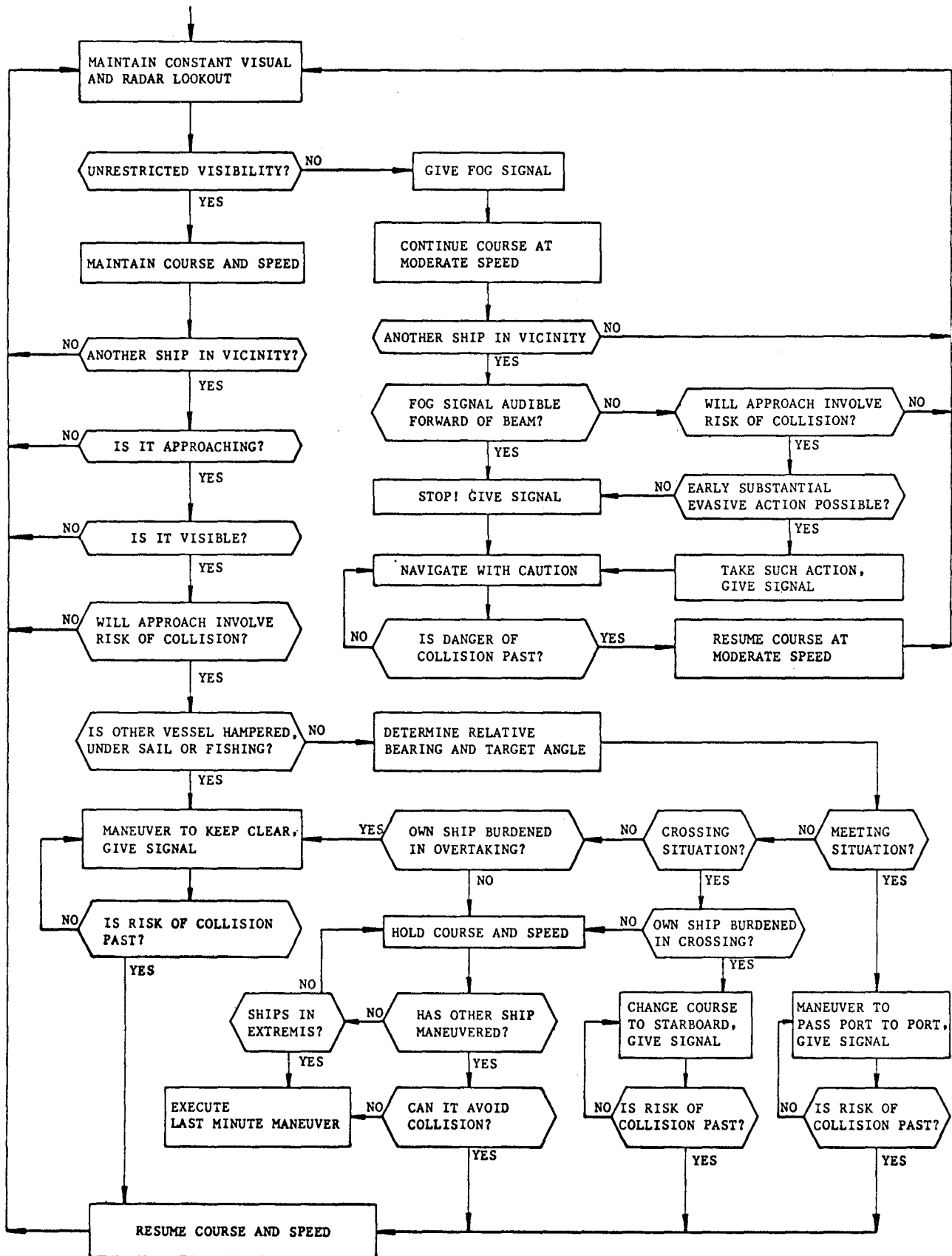


Fig. 1 - Logic Flow Diagram for Two-Ship Encounter in Open Sea following International Rules of the Nautical Road

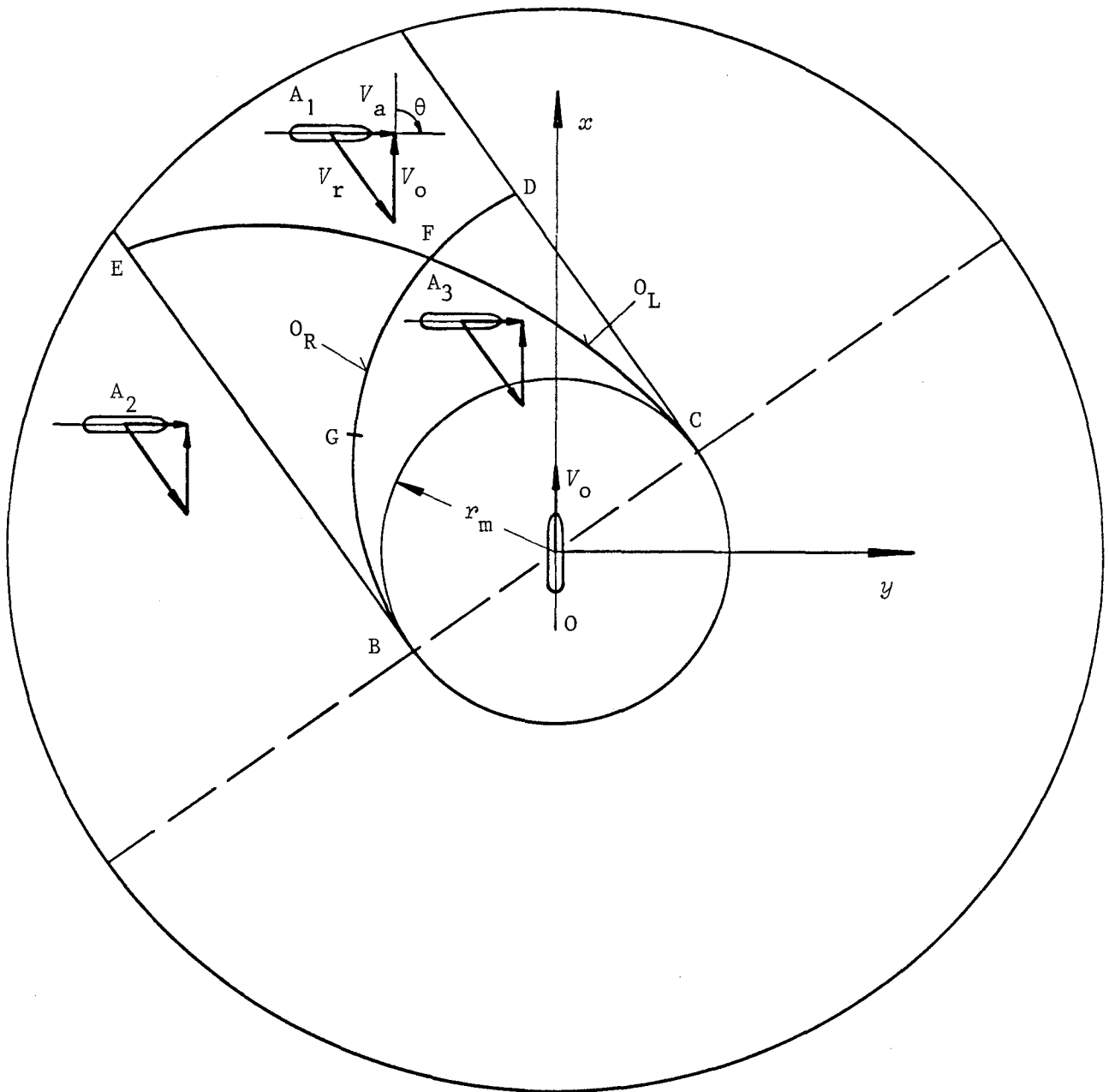


Fig. 2 - Sample Illustration of Barriers and Critical Maneuvers for Collision Avoidance in a Two-Ship Encounter

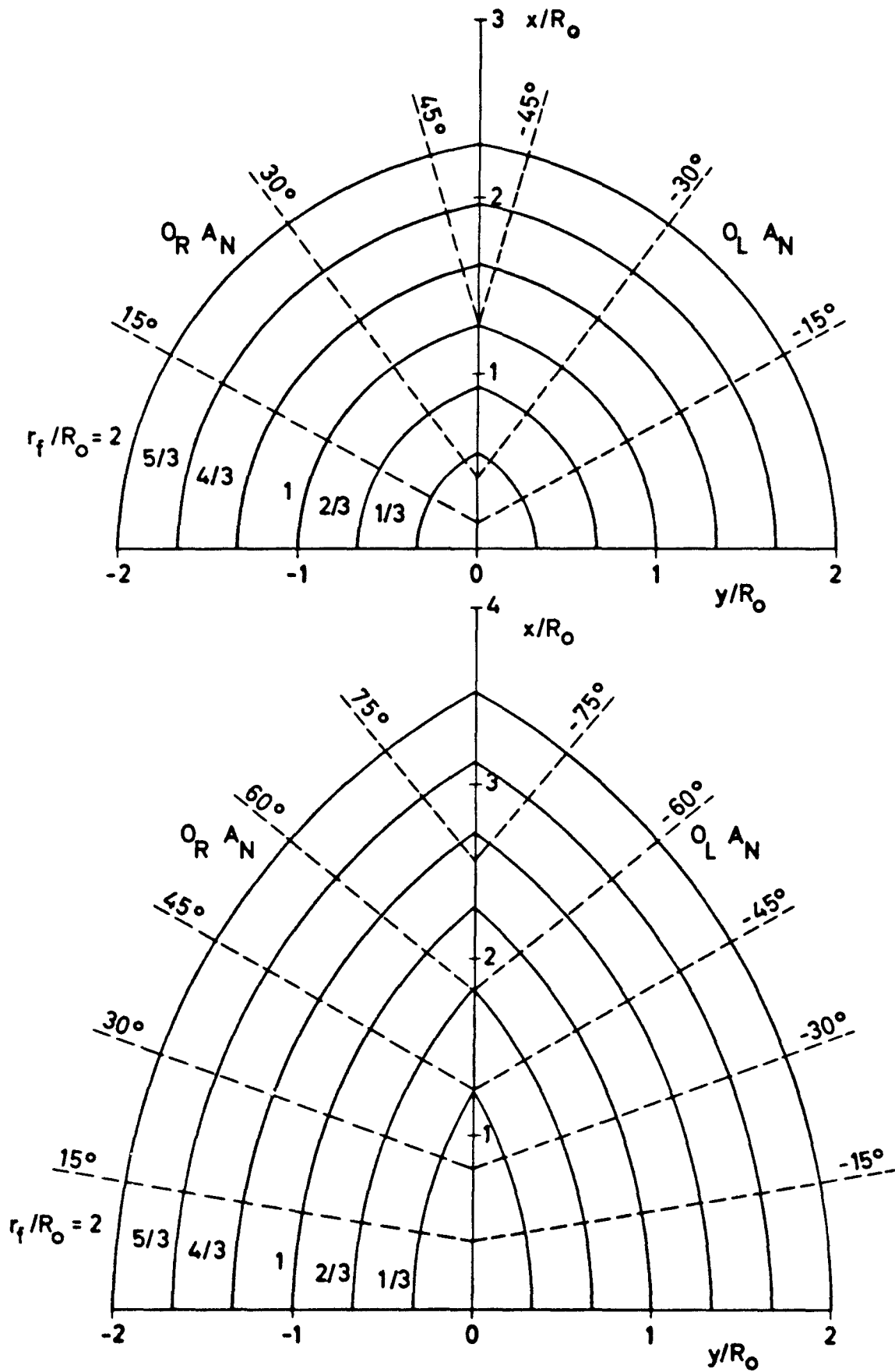


Fig. 3 - Barrier Cross-Sections at $\theta = 0^\circ$ (top) and $\theta = 180^\circ$ (bottom) for Game $O_E A_N$ with $V_a = V_o/2$ and $R_a = R_o/2$

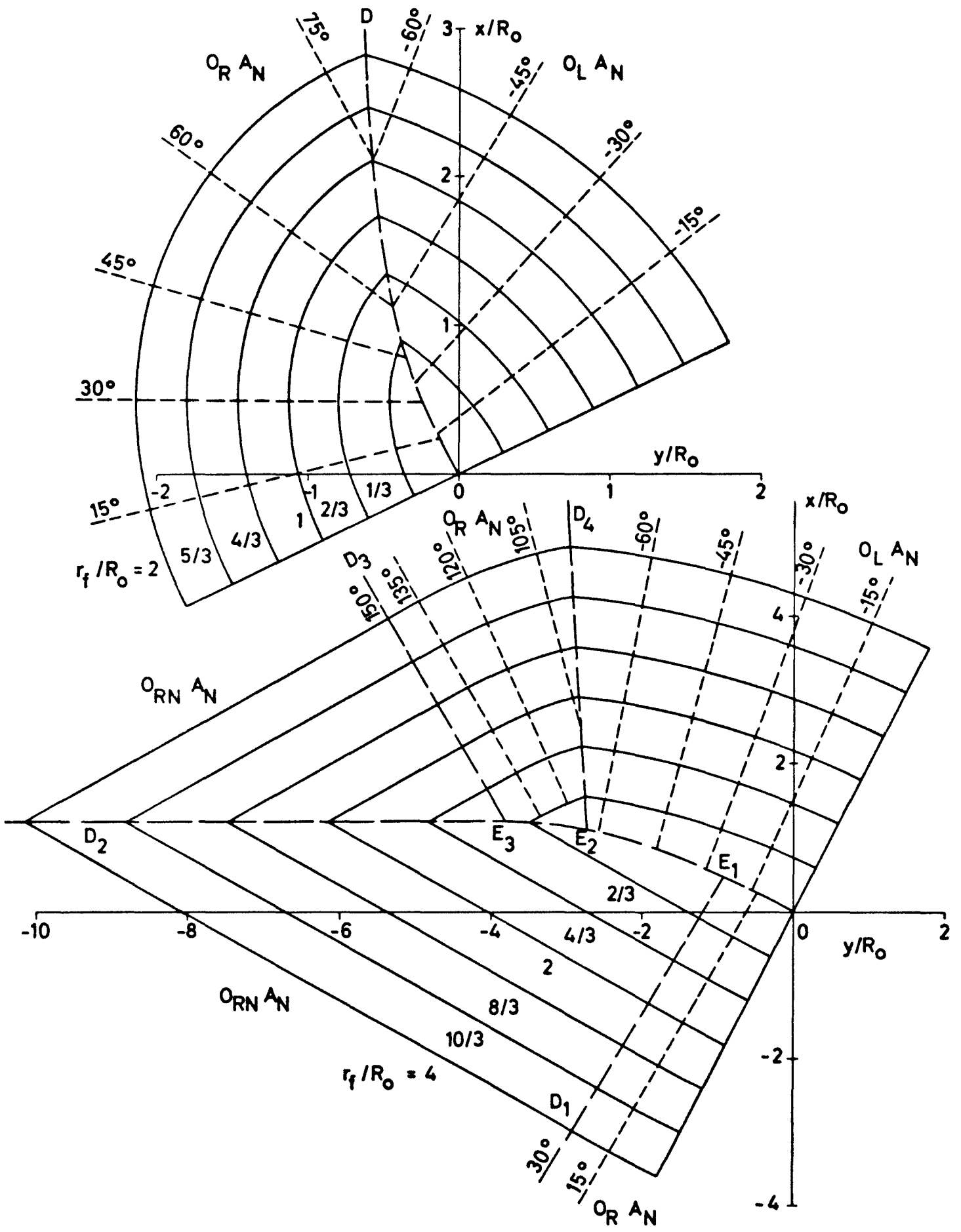


Fig. 4 - Barrier Cross-Sections at $\theta = 90^\circ$ for Game $O_E A_N$ with $V_a = V_o/2, R_a = R_o/2$ (top) and $V_a = 2V_o, R_a = 2R_o$ (bottom)

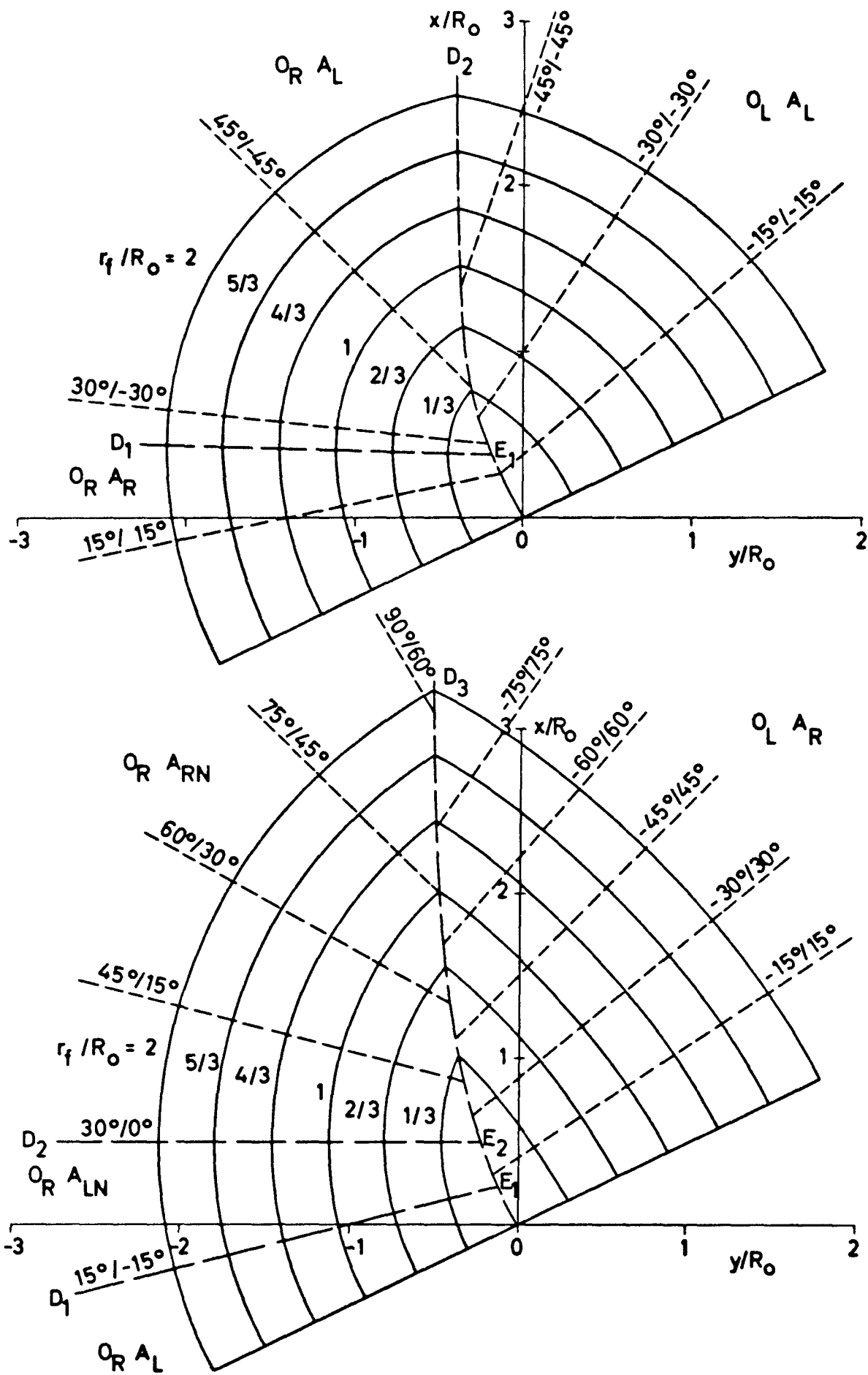


Fig. 5 - Barrier Cross-Sections at $\theta = 90^\circ$ for Game $O_E A_E$ (top) and $O_E A_P$ (bottom), each time with $V_a = V_0/2$, $R_a = R_0/2$

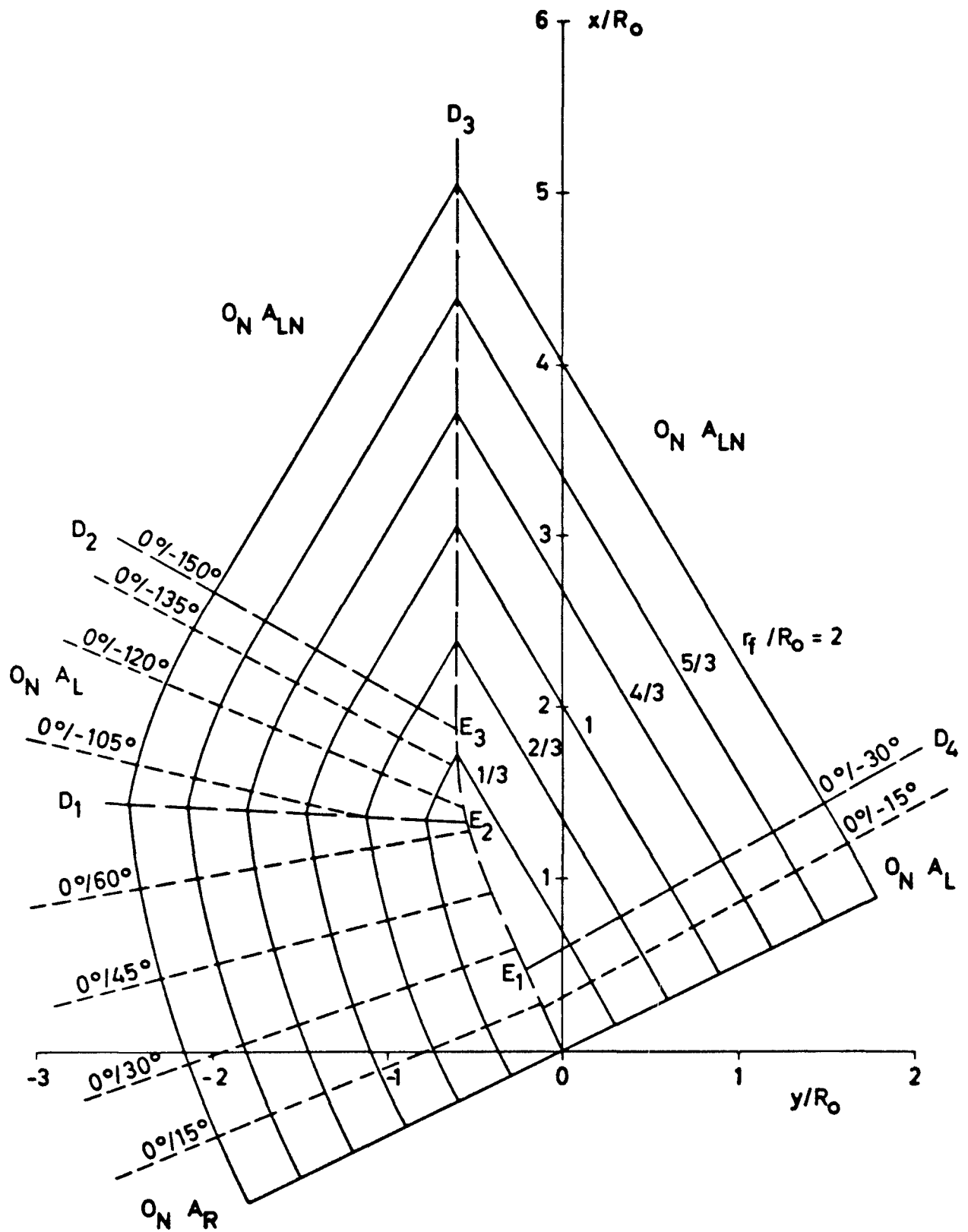


Fig. 6 - Barrier Cross-Sections at $\theta = 90^\circ$ for Game $O_N A_E$
 with $V_a = V_o/2$ and $R_a = R_o/2$