520 | Januar 1992

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Shi-Jun Liao Hamburg, Technische Universität Hamburg-Harburg, 1992

ISBN: 3-89220-520-5

Schriftenreihe Schiffbau
Schwarzenbergstraße 95c
D-21073 Hamburg

http://www.tuhh.de/vss

INSTITUT FÜR SCHIFFBAU DER UNIVERSITÄT HAMBURG

Bericht Nr. 520

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Abstract

Neither numerical techniques nor analytical methods of nonlinear problems are satisfactory. The iterative techniques are sensitive not only to the initial solutions but also to the number of unknowns. On the other side, perturbation expansion method depends upon small or great parameters. These limit their applications.

This paper is a prelimilary attempt to improve both analytical and numerical techniques of general nonlinear problems, i.e., to overcome the limitations of perturbation methods and iterative techniques described above.

Based on a kind of linear property of continuous mapping (mathematically speaking, a kind of linearity-invariance under homotopy), a kind of analytical method for nonlinear problems, namely, Process Analysis Method, is described, which does not depend upon small or great parameters. And based on the same property of continuous mapping (homotopy), a kind of numerical technique, called Finite Process Method, is developed, which can avoid the use of iterative techniques to solve nonlinear problems.

The essence of solving nonlinear problems and the differences and relations of linear and nonlinear problems are also simply discussed.

Some simple nonlinear problems in mechanics, for instance, the falling of a ball in fluid, the motion of a simple pendulum, 2D nonlinear water waves and so on, are used to introduce and examine the both methods.

Acknowledgements

It is a great opportunity for me to be able to study and work in Institute of Shipbuilding, University of Hamburg. I believe that I will never forget the wonderful days in this beautiful country. Here, I would like to use this chance to express my thanks to all of people who give me helps.

I would like to express my sincere thanks to Professor H. Söding for his helpful suggestions in numerical computations and to Professor W.H. Isay for the discussions and suggestions of using Bessel functions to express the results derived in chapter 2. Thanks to professor Zhu Ji-mao for the discussions, when he was visiting in Hamburg. Thanks to professor Schultz for his interesting lecture on perturbation theory.

Thanks to the editors and reviewers of many journals, for instance, International Journal of Numerical Methods in Fluids, International Journal of Numerical Methods in Engineering, Journal of Applied Mechanics, Journal of Ship Research, Journal of Fluid Mechanics, Journal of Applied Mathematics and so on, for their helpful suggestions. Specially, I would like to express my sincere thanks to professor Cedric Taylor for his encouragements.

Mr. Fuhrmann, Mrs. Krüger, Mr. Schewe, Mr. Strey, Dr. Koh, Dr. Bertram, Mr. Cura Hochbau and Mr. Sander are acknowledged for their helps in the applications of the computer software system in Institute of Shipbuilding.

Professor Keil and Mrs. Zelt (DAAD) are acknowledged for their helps in the application of my stipend.

Thanks to German Academic Exchange Service (DAAD) for the financial support (Sandwich Program) and to the education ministry of People's Republic of China for the travel ticket. Thanks to Institute of Shipbuilding, University of Hamburg for the freely provided computer time.

Sincere thanks to my wife and to our parents for their supports and encouragements.

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Nomenclature

C	wave velocity
D	average water depth
$f_1(p), f_2(p)$	process functions
g	gravity acceleration
H_w	wave height
k	wave number
L_w	wave length
$\mathcal{L}$	auxiliary function
$\mathcal{N}$	auxiliary function
p	embedding parameter
Q	value of streamfunction on free surface
R_w	wave resistance
$\mathcal{R}$	auxiliary function
$\mathcal{S}$	auxiliary function
$\mathcal{T}$	auxiliary function
T	period of pendulum
t	time
u, v, w	velocity components
U, V	velocity
x, y, z	coordinates in Cartesian system
$\mathcal{Z}$	auxiliary function
β	initial angel of pendulum
ζ	wave elevation
θ	oscillation angel of pendulum
ν	kinematic viscocity
ρ	density
σ	singularity strength
τ	vortex strength
ϕ, Φ, φ	velocity potential function
ψ	streamfunction
ω	frequence of a simple pendulum
Γ	boundary of fluid
Ω	domain of fluid

Introduction

Since the appearence of computer, the numerical techniques have been developed very quickly. Now, Finite Difference Method (FDM), Finite Element Method (FEM) and Boundary Element Method (BEM) have been widely used as effective and useful tools to solve many complex problems in engineering.

After the discretization of a 2D or specially 3D problem, generally, a set of algebraic equations with a large number of unknowns will be obtained. If the considered problem is linear, the corresponding set of algebraic equations is also linear and can be solved without mathematical difficulties; if the researched problem is nonlinear, the obtained set of algebraic equations is generally also nonlinear. Up to now, there exists no effective numerical methods to solve this kind of nonlinear algebraic equations with a large number of unknowns. Generally, iterative techniques are used to solve these nonlinear problems; however, the convergence of iterations seems sometimes just a luck, specially when the nonlinearity of the considered problems is strong. This is mainly because nearly all iterative techniques are *sensitive* not only to initial solutions but also to the number of unknowns; but, it seems difficult to find a good enough initial solution, and the number of unknowns of a set of algebraic equations given by Boundary Element Method, or specially by Finite Difference Method or Finite Element Method is generally very large, specially for a 3D practical problem.

Similar to the numerical techniques of nonlinear problems, the analytical methods of them are also unsatisfactory. The widely used nonlinear analytical techniques are perturbation expansion methods, which are based on the *small* or *great* parameter supposition. But unfortunately, not every nonlinear problem has a small or great parameter. Even there exists such a parameter, the considered problem may be sometimes singular to general perturbation expansion method and then singular perturbation techniques (for example, method of multiplescale expansion, method of matched asymptotic expansion and so on) should be applied. In fact, the meaning of *small* and *great* is not very clear and it seems not easy to decide whether or not a parameter is small or great *enough* to perturbation methods. So, the small parameter supposition limits greatly the applications of perturbation techniques.

So, it seems that neither the numerical techniques nor the analytical methods of nonlinear problems are satisfactory. The iterative techniques are *sensitive* to the initial solutions and the number of unknowns; and perturbation methods depend on *small* or *great* parameter. Both of them seem kinds of arts, not science, because experience seems important in the applications of them.

In contrast to nonlinear problems, linear problems can be much more easily solved, either numerically or analytically. The success in solution of linear problems and the limitations in that of nonlinear problems are so obvious that it seems valuable to think deeply about the *relationships* and *difference* between the linearity and nonlinearity.

What are the true meanings of linearity and nonlinearity? Are there any relationships between them?

It seems that many things in our world have relations with and can be transformed to other things under some conditions. Therefore, there should exist some relations between the linearity and nonlinearity.

What operations *can* we do in arithmetic ? Just only *finite* number of foundational operations : addition, subtraction, multiplication, division, taking power, taking root, and transfering a term from one side of equation to another side. (We can do just only *finite* operations, because our humankind as a kind of common intelligent and courageous living beings in cosmos are limited in time, although we have been existing in this lovely blue planet for a long time and will exist still in a very, if we have luck, long time.) These *finite* operations are *enough* in solution of linear algrbraic equations. In fact, computer as an expansion of our ability can just only do these *finite* operations in arithmetic, too, although computer does them much faster and make much less errors than us. (Here, we do not consider the ability of logical analysis of computer.) The linear algebraic equations are simple (at least in theory) for us just only because the operations needed for solution of linear algebraic equations are those we *can* do. But unfortunately, there exist another kind of problems which seem be not able to be solved by *finite* foundational operations. These problems are very different in this point from the lovely linear equations and are called nonlinear problems. It is clear that our *natural* ability is limited. We can do only *finite* operations and therefore infinite is in essence superhuman. But, it has been proved that the solutions of an algebraic equations with power greater than five *can't* be expressed by *finite* operations of addition, substraction, multiplication, division, taking power and taking root. It means that there exist in mathematicas some things which are superhuman. Owing to these reason, we can find only approximate solutions of some nonlinear problems but can *never* give the *absolute exact* solution of it, because this is superhuman. Fortunately, we need not in practice the *absolute exact* solutions and the approximate results are generally enough. This is perhaps mainly because our practical world is in essence 'approximate', which is very different from the symbol world of mathematics created by us. If we inspect the methods developed for solutions of nonlinear problems, it is easy to see that all of them are based on such kinds of *transformations* that the *accurate enough* solutions of the considered problems can be solved by *finite* foundational arithmetical operations. Iteration is perhaps the most simple transformation, while perturbation expansion methods seem more complex. In fact, there exist many interest ideas in perturbation expansion methods for this kind of transformation. It seems that the key of developing numerical and analytical methods of nonlinear problems is to find kinds of transformations from nonlinear to linear problems which can be solved by *finite* foundational arithmetical operations. This seems be the essence of the solution of nonlinear problems.

As mentioned by Ortega and Rheinboldt [39], the continuous mapping technique, or sometimes called homotopy[1] method, has been generally used to widen the domain of convergence of a given method, or as a procedure to obtain sufficiently close starting points. The continuous mapping technique embeds a parameter that typically ranges from zero to unity; When the embedding parameter is zero, the equation is one of the linear system; When it is one, the equation is the same as the original. Traditionally, one *iterates* numerically along the solution path by incrementing the imbedding parameter from zero to one; this maps continuously the initial linear solutions into the solutions of the original equation. Note that iterative techniques are traditionally used at each step along the solution path, if the equation is nonlinear.

In a continuous mapping (homotopy), all variables are also dependent upon the embedding parameter. It is accidentally found that the relationshops between the derivatives of the variables with respect to embedding parameter are linear. This is an interesting and important property of continuous mapping (homotopy). Based on this property, a kind of transformation from nonlinearity to linearity can be obtained.

In present work, a pure mathematical proof of this linear property of continuous mapping has been given in Appendix A. Based on this property, a numerical technique, called Finite Process Method (FPM) and an analytical technique, namely Process Analysis Method (PAM), are derived. Both are for the nonlinear problems. For Finite Process Method (FPM), just only set of linear algebraic equations

[1]More precisely speaking, the mathematical base of this paper should be homotopy, which is an important part of algebraic topology (see reference [61] [62] [63]). But homotopy is unfamiliar to us; and in fact, only the concept of homotopy is needed for this paper. Owing to these two reasons, we use continuous mapping, which is more familiar to us, to express the ideas of homotopy – just the same as Ortega and Rheinboldt [39].

should be solved and then the iterative methods can be avoided so that it is *insensitive* to the initial solutions. Different from perturbation expansion method, Process Analysis Method (PAM) *does not* depend on small parameters and therefore can be applied to analyse more nonlinear problems in practice.

From the logical consideration, it seems better to have a detailed and clear understanding of the corresponding mathematical basis. Therefore, a simple mathematical proof is given in Appendix A, although mathematics is not the purpose of this paper. Its basic ideas are used to solve some simple but typical nonlinear problems in mechanics, either numericaly or analytically, in order to describe and examine the Finite Process Method and Process Analysis Method. This seems valuable and must be done, before they can be applied to solve complex practical problems in engineering.

Perturbation method is dependent on small parameters and iterative techniques are sensitive to the initial solutions. The main purpose of this paper is trying to find a new way for the analytical and numerical solution of general nonlinear problems so that these limitations could be overcomed.

Chapter 1

Limitations of perturbation expansion method and introduction to Process Analysis Method

Summary

Using the motion of a falling ball in fluid as a simple example, the limitations of perturbation expansion method are simply discussed and the basic ideas of a new kind of analytical method of nonlinear problems, namely Process Analysis Method, are described and examined.

1.1 limitations of perturbation method

The greatest limitation of perturbation method is that it depends on small or great parameters.

As a very simple example, let us consider the following problem: a ball falls in a kind of fluid. Let $V(t)$ denote the velocity of the ball and suppose that the fluid resistance force F acting on the ball is directly proportional to $V^2(t)$, i.e. , $F = -k_\nu V^2$ ($k_\nu > 0$). Thus, according to Newton's theory, one has:

$$m\frac{dV}{dt} = mg - k_\nu V^2 \tag{1.1}$$
$$V(t) = 0 \quad \text{for} \quad t = 0 \tag{1.2}$$

Then

$$\frac{dV}{dt} + \mu V^2 = g \tag{1.3}$$
$$V(t) = 0 \quad \text{for} \quad t = 0 \tag{1.4}$$

where, $\mu = k_\nu/m > 0$ and g is gravity acceleration, m is mass of the ball.

The above nonlinear first-order differential equation has the exact solution:

$$V(t) = \sqrt{\frac{g}{\mu}}\left(\frac{1 - e^{-2\sqrt{g\mu}\,t}}{1 + e^{-2\sqrt{g\mu}\,t}}\right) \tag{1.5}$$

When μ is small enough, perturbation method can be easily used to give an approximate solution. But, when μ is neither small nor great, for example,$\mu = 1.0$, then, it seems difficult to give a perturbation expansion solution of this simple nonlinear first-order differential equation.

Suppose that μ is small. Then, one can *suppose* that $V(t)$ can be written as follows:

$$V(t) = W_0(t) + \mu W_1(t) + \mu^2 W_2(t) + \mu^3 W_3(t) + \cdots \tag{1.6}$$

Substituting (1.6) in (1.3), one has:

$$\left\{\frac{dW_0}{dt} - g\right\} + \mu\left\{\frac{dW_1}{dt} + W_0^2\right\} + \mu^2\left\{\frac{dW_2}{dt} + 2W_0W_1\right\}$$
$$+ \quad \mu^3\left\{\frac{dW_3}{dt} + W_1^2 + 2W_0W_2\right\} + \mu^4\left\{\frac{dW_4}{dt} + 2W_1W_2 + 2W_0W_3\right\}$$

$$
\begin{aligned}
&+ \quad \mu^5 \left\{ \frac{d\,W_5}{d\,t} + W_2^2 + 2W_0W_4 + 2W_1W_3 \right\} + \mu^6 \left\{ \frac{d\,W_6}{d\,t} + 2W_0W_5 + 2W_1W_4 + 2W_2W_3 \right\} \\
&+ \quad \mu^7 \left\{ \frac{d\,W_7}{d\,t} + W_3^2 + 2W_0W_6 + 2W_1W_5 + 2W_2W_4 \right\} \\
&+ \quad \mu^8 \left\{ \frac{d\,W_8}{d\,t} + 2W_0W_7 + 2W_1W_6 + 2W_2W_5 + 2W_3W_4 \right\} \\
&+ \quad \mu^9 \left\{ \frac{d\,W_9}{d\,t} + W_4^2 + 2W_0W_8 + 2W_1W_7 + 2W_2W_6 + 2W_3W_5 \right\} \\
&+ \quad \mu^{10} \left\{ \frac{d\,W_{10}}{d\,t} + 2W_0W_9 + 2W_1W_8 + 2W_2W_7 + 2W_3W_6 + 2W_4W_5 \right\} \\
&+ \quad \cdots = 0
\end{aligned} \tag{1.7}
$$

Substituting (1.6) into (1.4) , one has:

$$
V(0) = W_0(0) + \mu\, W_1(0) + \mu^2\, W_2(0) + \mu^3\, W_3(0) + \cdots = 0 \tag{1.8}
$$

Suppose that the terms with $\mu^k (k = 0, 1, 2, \cdots)$ in equation (1.7) and (1.8) construct the kth-order approximate equation of perturbation method. Then, one has:

zero-order equation:

$$
\frac{d\,W_0(t)}{d\,t} = g \tag{1.9}
$$

$$
W_0(0) = 0 \tag{1.10}
$$

first-order equation:

$$
\frac{d\,W_1(t)}{d\,t} = -W_0^2(t) \tag{1.11}
$$

$$
W_1(0) = 0 \tag{1.12}
$$

second-order equation:

$$
\frac{d\,W_2(t)}{d\,t} = -2W_0(t)W_1(t) \tag{1.13}
$$

$$
W_2(0) = 0 \tag{1.14}
$$

third-order equation:

$$
\frac{d\,W_3(t)}{d\,t} = -W_1^2(t) - 2W_0(t)W_2(t) \tag{1.15}
$$

$$
W_3(0) = 0 \tag{1.16}
$$

forth-order equation:

$$
\frac{d\,W_4(t)}{d\,t} = -2W_0(t)W_3(t) - 2W_1(t)W_2(t) \tag{1.17}
$$

$$
W_4(0) = 0 \tag{1.18}
$$

fifth-order equation:

$$\begin{aligned} \frac{d\,W_5(t)}{d\,t} &= -W_2^2(t) - 2W_0(t)W_4(t) - 2W_1(t)W_3(t) & (1.19) \\ W_5(0) &= 0 & (1.20) \\ &\vdots \end{aligned}$$

From above linear equation, one can easily obtain $W_k(t)(k = 0, 1, 2, \cdots)$ as follows:

$$\begin{aligned} W_0(t) &= g\,t & (1.21) \\ W_1(t) &= -\frac{1}{3}g^2\,t^3 & (1.22) \\ W_2(t) &= \frac{2}{15}g^3\,t^5 & (1.23) \\ W_3(t) &= -\frac{17}{315}g^4\,t^7 & (1.24) \\ W_4(t) &= \frac{62}{2835}g^5\,t^9 & (1.25) \\ W_5(t) &= -\frac{1382}{155925}g^6\,t^{11} & (1.26) \\ W_6(t) &= 3.5921282 \times 10^{-3}g^7\,t^{13} & (1.27) \\ W_7(t) &= -1.4558344 \times 10^{-3}g^8\,t^{15} & (1.28) \\ W_8(t) &= 5.9002748 \times 10^{-4}g^9\,t^{17} & (1.29) \\ W_9(t) &= -2.3912913 \times 10^{-4}g^{10}\,t^{19} & (1.30) \\ W_{10}(t) &= 9.6915370 \times 10^{-5}g^{11}\,t^{21} & (1.31) \\ &\vdots \end{aligned}$$

Therefore, one has the perturbation expansion solution of equation (1.3)~(1.4) as follows : zero-order perturbation expansion approximation :

$$V_0(t) = W_0(t) = g\,t \qquad (1.32)$$

first-order perturbation expansion approximation :

$$\begin{aligned} V_1(t) &= W_0(t) + \mu\,W_1(t) \\ &= gt - \frac{1}{3}\mu g^2\,t^3 \\ &= \sqrt{\frac{g}{\mu}}\left[\sqrt{g\mu}t - \frac{1}{3}(\sqrt{g\mu}t)^3\right] & (1.33) \end{aligned}$$

second-order perturbation expansion approximation :

$$\begin{aligned} V_2(t) &= W_0(t) + \mu\,W_1(t) + \mu^2\,W_2(t) \\ &= gt - \frac{1}{3}\mu g^2\,t^3 + \frac{2}{15}\mu^2\,g^3\,t^5 \\ &= \sqrt{\frac{g}{\mu}}\left[\sqrt{g\mu}t - \frac{1}{3}(\sqrt{g\mu}t)^3 + \frac{2}{15}(\sqrt{g\mu}t)^5\right] & (1.34) \end{aligned}$$

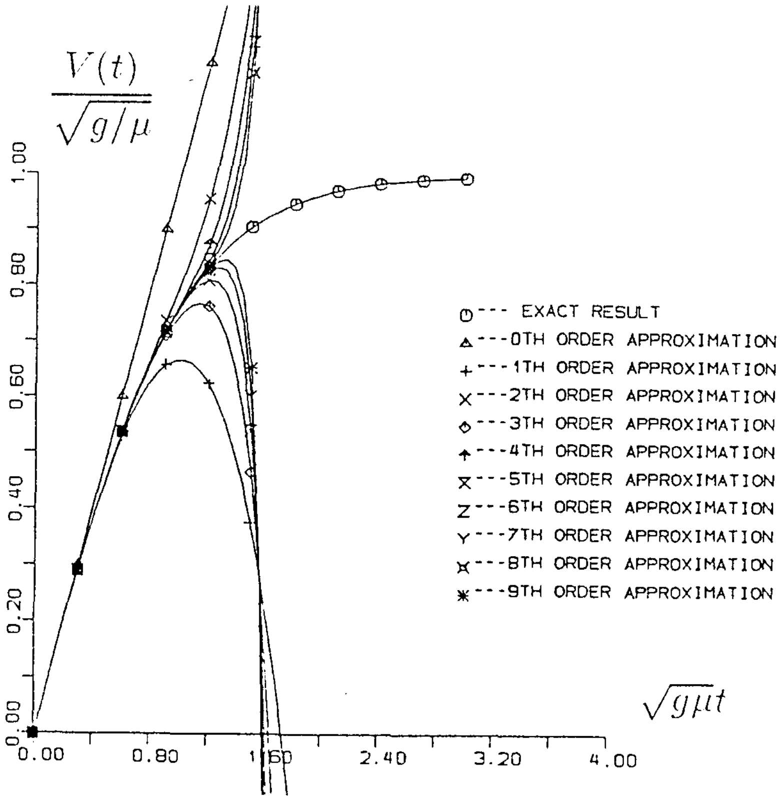

Figure 1.1: comparison of perturbation solutions to exact solution

third-order perturbation expansion approximation :

$$\begin{aligned} V_3(t) &= W_0(t) + \mu\, W_1(t) + \mu^2\, W_2(t) + \mu^3\, W_3(t) \\ &= gt - \frac{1}{3}\mu g^2\, t^3 + \frac{2}{15}\mu^2\, g^3\, t^5 - \frac{17}{315}\mu^3\, g^4\, t^7 \\ &= \sqrt{\frac{g}{\mu}}\left[\sqrt{g\mu}t - \frac{1}{3}(\sqrt{g\mu}t)^3 + \frac{2}{15}(\sqrt{g\mu}t)^5 - \frac{17}{315}(\sqrt{g\mu}t)^7\right] \end{aligned} \tag{1.35}$$

The comparison of perturbation solutions at up-to 9th-order of approximation to the exact solution is shown as figure 1.1. Here, we use $\sqrt{g\mu}t$ and $V(t)/\sqrt{g/\mu}$ as horizontal and perpendicular coordinates so that the curves don't change with μ. One can see that the perturbation expansion solution $V_k(t)$ $(k = 0, 1, 2, \cdots, 10)$ will converge to the exact solution $V(t)$ just only in region $0 \le \sqrt{g\mu}t \le 1.5$. If $\sqrt{g\mu}t > 1.5$, there exists very great difference between the perturbation expansion solutions $V_k(t)$ $(k = 0, 1, 2, \cdots, 10)$ and the exact solution $V(t)$. Specially, $\lim_{t\to\infty} V(t) = \sqrt{g/\mu}$; but, $\lim_{t\to\infty} V_k(t) = \infty$ $(k = 0, 1, 2, \cdots, 10)$. Therefore, the perturbation expansion solutions could be only used in region

$$0 \le t \le t_c \approx \frac{3}{2\sqrt{g\mu}}.$$

When $\mu = 0.00001$ (1/m), $t_c \approx 150$ (second), then the perturbation solutions are valid in about 2.5 minute. But, when $\mu = 1.0$ (1/m), then $t_c \approx 0.5$ (second); in this case, the perturbation solutions have nearly no meaning.

The above simple example shows that perturbation expansion method depends strongly on small parameter. This is a great disadvantage of it. Sometimes, even though there exists a small parameter, it is still complex and difficult to use perturbation method, and multiple scales method or matched asymptotic expansions method should be applied. For example, the following equation :

$$\mu \frac{d^2 y}{d x^2} + \frac{d y}{d x} + \mu y^2 = c(x), \qquad x \in [0,1] \tag{1.36}$$

$$y(0) = a \tag{1.37}$$

$$y(1) = b \tag{1.38}$$

has small parameter μ. If perturbation method is used, one can obtain the following zero-order approximate equation :

$$\frac{d y_0}{d x} = c(x) \tag{1.39}$$

$$y_0(0) = a \tag{1.40}$$

$$y_0(1) = b \tag{1.41}$$

The above first-order differential equation has two boundary conditions. Generally, a first-order differential equation needs only one boundary condition. Therefore, one boundary condition can not be satisfied and must be dropped. In this case, there exists often an inner perturbation expansion solution and an outer perturbation expansion solution which should be matched by the matching principle. This disadvantage of perturbation method comes from also the small parameter supposition.

From above discussion, one can see that the small parameter supposition brings some limitations of perturbation method. It seems that the small parameter supposition should be abandoned.

1.2 introduction of Process Analysis Method

In this section, the same example employed in above section is used to describe the basic idea of a new analytical method of nonlinear problems, namely Procass Analysis Method.

Let

$$f_1(p) = \begin{cases} 0 & \text{when } p = 0.0 \\ 1 & \text{when } p = 1.0 \end{cases} \tag{1.42}$$

where, $f_1(p) \in C^\infty$, called *first-sort of process function.*

Without loss of generality, a kind of continuous mapping (or more precisely speaking, a homotopy) can be constructed as follows:

$$\frac{d V(t;p)}{d t} + a\sqrt{g\mu}\, V(t;p) + f_1(p)\left\{\mu V^2(t;p) - a\sqrt{g\mu}\, V(t;p)\right\} = g \tag{1.43}$$

$$V(t;p) = 0 \quad \text{for} \quad t = 0 \tag{1.44}$$

Here, $V(t;p)$ denotes that V is not only a function of the *original independent variable* t, but also a function of the imbedding variable p; a is a positive parameter which should be determined.

When $p = 0$, owe have the *initial equation*:

$$\frac{d V(t;0)}{d t} + a\sqrt{g\mu}\, V(t;0) = g \tag{1.45}$$

$$V(t;0) = 0 \quad \text{for} \quad t = 0 \tag{1.46}$$

From equation (1.3), we have

$$\lim_{t\to\infty}\left(\frac{dV}{dt}+\mu V^2\right)=\mu V_\infty^2=g \tag{1.47}$$

then

$$V_\infty=\sqrt{\frac{g}{\mu}}$$

From equation (1.45), we have

$$\lim_{t\to\infty}\left\{\frac{dV(t;0)}{dt}+a\sqrt{g\mu}V(t;0)\right\}=a\sqrt{g\mu}V(\infty;0)=g \tag{1.48}$$

Let $V(\infty;0)=V_\infty=\sqrt{g/\mu}$ and substitute it in above equation, we have $a=1.0$

For simplicity, select $f_1(p)=p$. Then, the zero-order process equation is :

$$\frac{d\,V(t;p)}{d\,t}+\sqrt{g\mu}\,V(t;p)+p\left\{\mu\,V^2(t;p)-\sqrt{g\mu}\,V(t;p)\right\}=g \tag{1.49}$$

$$V(t;p)=0 \quad \text{for} \quad t=0 \tag{1.50}$$

and the corresponding initial equation is

$$\frac{d\,V(t;0)}{d\,t}+\sqrt{g\mu}\,V(t;0)=g \tag{1.51}$$

$$V(t;0)=0 \quad \text{for} \quad t=0 \tag{1.52}$$

which has the solution

$$V(t;0)=\sqrt{\frac{g}{\mu}}\left(1-e^{-\sqrt{g\mu}\,t}\right), \tag{1.53}$$

called *initial solution*, denoted as $V_0(t)$.

When $p=1.0$, we obtain the *final equation*:

$$\frac{d\,V(t;1)}{d\,t}+\mu\,V^2(t;1)=g \tag{1.54}$$

$$V(t;1)=0 \quad \text{for} \quad t=0 \tag{1.55}$$

The above equation is the same as the *original equation* (1.3) and (1.4). Its solution could be called *final solution*, denoted as $V_f(t)$.

From above analysis, we can see that equation (1.43) (1.44) give a kind of relation between the initial solution $V_0(t)$ and the final solution $V_f(t)$. The process of the change of p from zero to one is just the process of the continuous transformation from the initial solution $V_0(t)$ to the final solution $V_f(t)$. For simplicity, we call (1.43) (1.44) *zero-order process equation.*

Let

$$V^{[k]}(t;p)=\frac{\partial^k\,V(t;p)}{\partial\,p^{\,k}} \qquad (k=1,2,3,\cdots) \tag{1.56}$$

denote the *kth-order process derivative* of $V(t;p)$.

Suppose that

1. $V(t;p)$ has definition in $p \in [0,1]$;
2. $V^{[k]}(t;p)$ exist in $p \in [0,1], \forall k \in N$.

Then, according to Taylor's theory, one has:

$$V_f(t) = V_0(t) + \sum_{k=1}^{\infty} \left. \frac{V^{[k]}(t;p)}{k\,!} \right|_{p=0} \tag{1.57}$$

Deriving zero-order process equation (1.49)~(1.50) with respect to imbedding variable p, one can obtain the ***first-order process equation*** as follows:

$$\begin{aligned} & \frac{d\,V^{[1]}(t;p)}{d\,t} + \sqrt{g\mu}\,V^{[1]}(t;p) + \left\{\mu\,V^2(t;p) - \sqrt{g\mu}\,V(t;p)\right\} \\ + \quad & p\left\{2\mu V(t;p)V^{[1]}(t;p) - \sqrt{g\mu}\,V^{[1]}\right\} = 0 \end{aligned} \tag{1.58}$$

$$V^{[1]}(t;p) = 0 \quad \text{for } t = 0 \tag{1.59}$$

When $p = 0$, one has:

$$\frac{d\,V^{[1]}(t;0)}{d\,t} + \sqrt{g\mu}\,V^{[1]}(t;0) = \sqrt{g\mu}\,V_0(t) - \mu\,V_0^2(t) \tag{1.60}$$

$$V^{[1]}(t;0) = 0 \quad \text{for} \quad t = 0 \tag{1.61}$$

Substituting (1.53) in the above linear first-order differential equation, one has:

$$\frac{d\,V^{[1]}(t;0)}{d\,t} + \sqrt{g\mu}\,V^{[1]}(t;0) = g\left\{e^{-\sqrt{g\mu}\,t} - e^{-2\sqrt{g\mu}\,t}\right\} \tag{1.62}$$

$$V^{[1]}(t;0) = 0 \quad \text{for} \quad t = 0 \tag{1.63}$$

which has the following solution:

$$V_0^{[1]}(t) = V^{[1]}(t;0) = \sqrt{\frac{g}{\mu}}\left\{(\sqrt{g\mu}\,t - 1)\,e^{-\sqrt{g\mu}\,t} + e^{-2\sqrt{g\mu}\,t}\right\} \tag{1.64}$$

Deriving the first-order process equation (1.58)~(1.59) with respect to p, one can obtain the *second-order process equation*:

$$\begin{aligned} & \frac{d\,V^{[2]}(t;p)}{d\,t} + \sqrt{g\mu}\,V^{[2]}(t;p) + \left\{2\mu V(t;p)V^{[1]}(t;p) - \sqrt{g\mu}\,V^{[1]}(t;p)\right\} \\ + \quad & p\left\{2\mu\left[V^{[1]}(t;p)\right]^2 + 2\mu V(t;p)V^{[2]}(t;p) - \sqrt{g\mu}\,V^{[2]}(t;p)\right\} = 0 \end{aligned} \tag{1.65}$$

$$V^{[2]}(t;p) = 0 \quad \text{for} \quad t = 0 \tag{1.66}$$

When $p = 0$, one has:

$$\frac{d\,V_0^{[2]}(t)}{d\,t} + \sqrt{g\mu}\,V_0^{[2]}(t) = 2\left\{\sqrt{g\mu}\,V_0^{[1]}(t) - 2\mu V_0(t)V_0^{[1]}(t)\right\} \tag{1.67}$$

$$V_0^{[2]}(t) = 0 \quad \text{for} \quad t = 0 \tag{1.68}$$

where, $V_0^{[2]}(t) = V^{[2]}(t;0)$.

Substituting $V_0(t), V_0^{[1]}(t)$ in the above linear equation, one has:

$$\frac{d\,V_0^{[2]}(t)}{d\,t} + \sqrt{g\mu}\,V_0^{[2]}(t) = 2g\left\{(1-\sqrt{g\mu}\,t)\,e^{-\sqrt{g\mu}\,t} + (2\sqrt{g\mu}\,t-3)\,e^{-2\sqrt{g\mu}\,t} + 2e^{-3\sqrt{g\mu}\,t}\right\} \tag{1.69}$$

$$V_0^{[2]}(0) = 0 \tag{1.70}$$

The above linear equation has following solution:

$$V_0^{[2]}(t) = -\sqrt{\frac{g}{\mu}}\left\{\sqrt{g\mu}\,t\,(\sqrt{g\mu}\,t-2)\,e^{-\sqrt{g\mu}\,t} + 2\,(2\sqrt{g\mu}\,t-1)\,e^{-2\sqrt{g\mu}\,t} + 2e^{-3\sqrt{g\mu}\,t}\right\} \tag{1.71}$$

Deriving the second-order process equation (1.65)~(1.66) with respect to p, one can obtain the *third-order process equation* as follows:

$$\begin{aligned} & \frac{d\,V^{[3]}(t;p)}{d\,t} + \sqrt{g\mu}\,V^{[3]}(t;p) \\ + \; & 3\left\{2\mu\left[V^{[1]}(t;p)\right]^2 + 2\mu V(t;p)V^{[2]}(t;p) - \sqrt{g\mu}\,V^{[2]}(t;p)\right\} \\ + \; & p\left\{2\mu V(t;p)V^{[3]}(t;p) + 6\mu V^{[1]}(t;p)V^{[2]}(t;p) - \sqrt{g\mu}\,V^{[3]}(t;p)\right\} = 0 \end{aligned} \tag{1.72}$$

$$V^{[3]}(t;p) = 0 \quad \text{for} \quad t = 0 \tag{1.73}$$

When $p = 0$, one has:

$$\frac{d\,V_0^{[3]}(t)}{d\,t} + \sqrt{g\mu}\,V_0^{[3]}(t) = 3\left\{\sqrt{g\mu}\,V_0^{[2]}(t) - 2\mu\left[V_0^{[1]}(t)\right]^2 - 2\mu V_0(t)V_0^{[2]}(t)\right\} \tag{1.74}$$

$$V_0^{[3]}(t) = 0 \qquad \text{for} \qquad t = 0 \tag{1.75}$$

where, $V_0^{[3]}(t) = V^{[3]}(t;0)$.

Substituting $V_0(t), V_0^{[1]}(t)$ and $V_0^{[2]}(t)$ in the above equation, one has:

$$\begin{aligned} & \frac{d\,V_0^{[3]}(t)}{d\,t} + \sqrt{g\mu}\,V_0^{[3]}(t) \\ = \; & 3g\left\{\sqrt{g\mu}\,t\,(\sqrt{g\mu}\,t-2)\,e^{-\sqrt{g\mu}\,t} - 4\left[(\sqrt{g\mu}\,t)^2 - 3\sqrt{g\mu}\,t + 1\right]e^{-2\sqrt{g\mu}\,t}\right. \\ + \; & \left.(-12\sqrt{g\mu}\,t+10)\,e^{-3\sqrt{g\mu}\,t} - 6e^{-4\sqrt{g\mu}\,t}\right\} \end{aligned} \tag{1.76}$$

$$V_0^{[3]}(0) = 0 \tag{1.77}$$

The above linear equation has the following solution:

$$\begin{aligned} V_0^{[3]}(t) \; = \; & \sqrt{\frac{g}{\mu}}\left\{(\sqrt{g\mu}\,t)^2\,(\sqrt{g\mu}\,t-3)\,e^{-\sqrt{g\mu}\,t} + 12\sqrt{g\mu}\,t\,(\sqrt{g\mu}\,t-1)\,e^{-2\sqrt{g\mu}\,t}\right. \\ + \; & \left.6\,(3\sqrt{g\mu}\,t-1)\,e^{-3\sqrt{g\mu}\,t} + 6e^{-4\sqrt{g\mu}\,t}\right\} \end{aligned} \tag{1.78}$$

Therefore, one has:

zero-order process analysis approximation:

$$V_0(t) = \sqrt{\frac{g}{\mu}}\left(1 - e^{-\sqrt{g\mu}\,t}\right) \tag{1.79}$$

first-order process analysis approximation:

$$\begin{aligned} V_1(t) &= V_0(t) + \frac{V_0^{[1]}(t)}{1\,!} \\ &= \sqrt{\frac{g}{\mu}}\left\{1 + (\sqrt{g\mu}\,t - 2)\,e^{-\sqrt{g\mu}\,t} + e^{-2\sqrt{g\mu}\,t}\right\} \end{aligned} \tag{1.80}$$

(1.81)

second-order process analysis approximation;

$$\begin{aligned} V_2(t) &= V_0(t) + \frac{V_0^{[1]}(t)}{1\,!} + \frac{V_0^{[2]}(t)}{2\,!} \\ &= \sqrt{\frac{g}{\mu}}\left\{1 - \left[2 - 2\sqrt{g\mu}\,t + \frac{1}{2}(\sqrt{g\mu}\,t)^2\right]e^{-\sqrt{g\mu}\,t}\right. \\ &+ \left. 2\,(1 - \sqrt{g\mu}\,t)\,e^{-2\sqrt{g\mu}\,t} - e^{-3\sqrt{g\mu}\,t}\right\} \end{aligned} \tag{1.82}$$

third-order process analysis approximation:

$$\begin{aligned} V_3(t) &= V_0(t) + \frac{V_0^{[1]}(t)}{1\,!} + \frac{V_0^{[2]}(t)}{2\,!} + \frac{V_0^{[3]}(t)}{3\,!} \\ &= \sqrt{\frac{g}{\mu}}\left\{1 - \left[2 - 2\sqrt{g\mu}\,t + (\sqrt{g\mu}\,t)^2 - \frac{1}{6}(\sqrt{g\mu}\,t)^3\right]e^{-\sqrt{g\mu}\,t}\right. \\ &+ \left. 2\left[1 - 2\sqrt{g\mu}\,t + (\sqrt{g\mu}\,t)^2\right]e^{-2\sqrt{g\mu}\,t} - (2 - 3\sqrt{g\mu}\,t)\,e^{-3\sqrt{g\mu}\,t} + e^{-4\sqrt{g\mu}\,t}\right\} \end{aligned} \tag{1.83}$$

Note that the first-order process equation (1.58) $\sim$ (1.59), the second-order process equation (1.65) $\sim$ (1.66) and the third-order process equation (1.72) $\sim$ (1.73) are *linear* in $V^{[1]}(t;p)$, $V^{[2]}(t;p)$, $V^{[3]}(t;p)$, respectively. This is a general property of continuous mapping (or more precisely speaking, a kind of linearity-invariance under homotopy) and a mathematical proof of it has been given in appendix A. The concept of invariance was first used by Langrage and then developed mainly by Cayley in mathematics and was introduced by A. Einstein to engineers in his theory of relativity. It is interesting that this linearity-invariance under homotopy can give a kind of transformation from nonlinearity to linearity, which is the main subject we will discuss and research in this paper.

The results of up to third-order process analysis approximation of equation (1.3)$\sim$(1.4) are shown as figure 1.2. One can see that the third-order process analysis approximation agrees very good with the exact solution in region $0 \le t < +\infty$. With comparison to figure 1.1, these approximate results are much better than those obtained from perturbation method.

It should be emphasized that Process Analysis Method does not use the small parameter supposition of the perturbation method. Therefore, Process Analysis Method does not depend on small parameter. From figure 1.2, one can see that the *process analysis solution* at third order of approximation give a good agreement with exact solution even if μ is very great. This can be easily understood

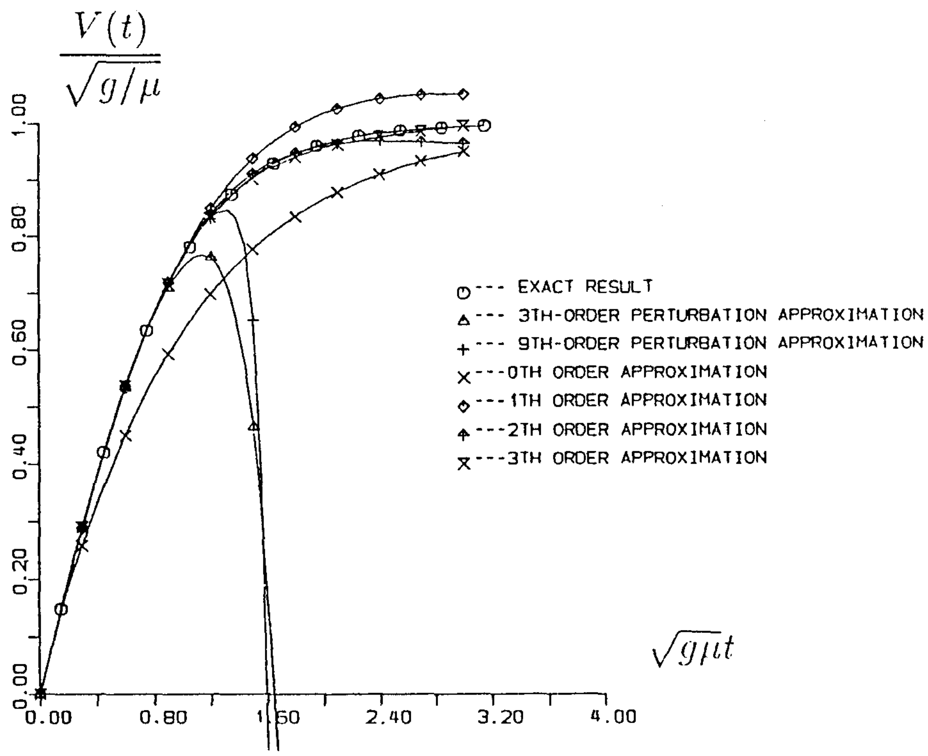

Figure 1.2: comparison of process analysis solutions to exact result

from the view point of Process Analysis Method, because Process Analysis Method needs not the small parameter supposition.

For the original equation (1.36)~(1.38) , one can construct a continuous mapping as following:

$$\mu \frac{d^2 y(x;p)}{d x^2} + \frac{d y(x;p)}{d x} + \sqrt{c\mu}\, y(x;p) + f_1(p) \left\{\mu\, y^2(x;p) - \sqrt{c\mu}\, y(x;p)\right\} = c(x), \quad x \in [0,1] \tag{1.84}$$

$$y(x;p) = a \quad \text{for} \quad x = 0 \tag{1.85}$$

$$y(x;p) = b \quad \text{for} \quad x = 1 \tag{1.86}$$

When $p = 0$, one has the following initial equation:

$$\mu \frac{d^2 y_0(x)}{d x^2} + \frac{d y_0(x)}{d x} + \sqrt{c\mu}\, y_0(x) = c(x) \tag{1.87}$$

$$y_0(x) = a \quad \text{for} \quad x = 0 \tag{1.88}$$

$$y_0(x) = b \quad \text{for} \quad x = 1 \tag{1.89}$$

This is a linear second-order differential equation with two boundary conditions. Therefore, both boundary conditions can be satisfied and neither needs be dropped. This is also an advantage of Process Analysis Method.

1.3 relation between Process Analysis Method and Perturbation method

The *general zero-order process equation* of the original equation (1.3) $\sim$ (1.4) are as follows:

$$\frac{d\,V(t;p)}{d\,t} + a\sqrt{g\mu}\,V(t;p) + f_1(p)\left\{\mu V^2(t;p) - a\sqrt{g\mu}\,V\right\} = g \tag{1.90}$$

$$V(t;p) = 0 \quad \text{for} \quad t = 0 \tag{1.91}$$

where, a is a selected parameter. For simplicity, select $f_1(p) = p$.

When $p = 0$, one can obtain the *general initial equation* of it:

$$\frac{d\,V_0(t)}{d\,t} + a\sqrt{g\mu}\,V_0(t) = g \tag{1.92}$$

$$V_0(0) = 0 \tag{1.93}$$

where, $V_0(t) = V(t;0)$.

If $a = 0$, the initial equation is:

$$\frac{d\,V_0}{d\,t} = g \tag{1.94}$$

$$V_0(0) = 0 \tag{1.95}$$

It is very interesting that the above initial equation (1.94) $\sim$ (1.95) is the same as the zero-order approximate equation (1.9) and (1.10) of perturbation method.

In the same way described in above section and using the genaral zero-order process equation (1.90)$\sim$(1.91), one can obtain the general process analysis solutions of equation (1.3) and (1.4) as follows:

(A) if $a > 0$, then:

(a) zero-order general process analysis approximation:

$$\mathcal{V}_0(t) = \frac{1}{a}\sqrt{\frac{g}{\mu}}\left(1 - e^{-a\sqrt{g\mu}\,t}\right) \tag{1.96}$$

(b) first-order general process analysis approximation:

$$\mathcal{V}_1(t) = \frac{1}{a}\sqrt{\frac{g}{\mu}}\left\{\left(2 - \frac{1}{a^2}\right) + \left[-\left(1 - \frac{2}{a^2}\right)(a\sqrt{g\mu}\,t) - 2\right]e^{-a\sqrt{g\mu}\,t} + \frac{1}{a^2}e^{-2a\sqrt{g\mu}\,t}\right\} \tag{1.97}$$

(c) second-order genaral process analysis approximation:

$$\begin{aligned}\mathcal{V}_2(t) &= \frac{1}{a}\sqrt{\frac{g}{\mu}}\left\{\left(3 - \frac{4}{a^2} + \frac{2}{a^4}\right)\right.\\ &- \left[\frac{1}{2}\left(1 - \frac{2}{a^2}\right)^2(a\sqrt{g\mu}\,t)^2 + 2\left(1 - \frac{3}{a^2} + \frac{1}{a^4}\right)(a\sqrt{g\mu}\,t) + \left(3 - \frac{1}{a^4}\right)\right]e^{-a\sqrt{g\mu}\,t}\\ &+ \left.\frac{2}{a^2}\left[\left(1 - \frac{2}{a^2}\right)(a\sqrt{g\mu}\,t) + \left(2 - \frac{1}{a^2}\right)\right]e^{-2a\sqrt{g\mu}\,t} - \frac{1}{a^4}e^{-3a\sqrt{g\mu}\,t}\right\}\end{aligned} \tag{1.98}$$

(d) third-order general process analysis approximation:

$$\begin{aligned}\mathcal{V}_3(t) &= \frac{1}{a}\sqrt{\frac{g}{\mu}}\left\{\left(4-\frac{10}{a^2}+\frac{12}{a^4}-\frac{5}{a^6}\right)\right.\\
&+ \left[-\frac{1}{6}\left(1-\frac{2}{a^2}\right)^3(a\sqrt{g\mu}\,t)^3-\left(1-\frac{6}{a^2}+\frac{10}{a^4}-\frac{4}{a^6}\right)(a\sqrt{g\mu}\,t)^2\right.\\
&+ \left.-\left(3-\frac{12}{a^2}+\frac{9}{a^4}-\frac{2}{a^6}\right)(a\sqrt{g\mu}\,t)-2\left(2-\frac{3}{a^4}+\frac{2}{a^6}\right)\right]e^{-a\sqrt{g\mu}\,t}\\
&+ \frac{2}{a^2}\left[\left(1-\frac{2}{a^2}\right)^2(a\sqrt{g\mu}\,t)^2+2\left(2-\frac{6}{a^2}+\frac{3}{a^4}\right)(a\sqrt{g\mu}\,t)+\left(5-\frac{6}{a^2}+\frac{2}{a^4}\right)\right]e^{-2a\sqrt{g\mu}\,t}\\
&+ \left.\frac{1}{a^4}\left[-3\left(1-\frac{2}{a^2}\right)(a\sqrt{g\mu}\,t)+\left(\frac{4}{a^2}-6\right)\right]e^{-3a\sqrt{g\mu}\,t}+\frac{1}{a^6}e^{-4a\sqrt{g\mu}\,t}\right\}\end{aligned} \tag{1.99}$$

(B) if $a = 0$, then:

(a) zero-order process analysis solution:

$$\mathcal{V}_0(t) = gt \tag{1.100}$$

(b) first-order process analysis solution:

$$\mathcal{V}_1(t) = gt - \frac{1}{3}\mu\, g^2\, t^3 \tag{1.101}$$

(c) second-order process analysis solution:

$$\mathcal{V}_2(t) = gt - \frac{1}{3}\mu\, g^2\, t^3 + \frac{2}{15}\mu^2 g^3\, t^5 \tag{1.102}$$

(c) third-order process analysis solution:

$$\begin{aligned}\mathcal{V}_3(t) &= gt - \frac{1}{3}\mu\, g^2\, t^3 + \frac{2}{15}\mu^2 g^3\, t^5 - \frac{17}{315}\mu^3 g^4\, t^7 \\ &\vdots\end{aligned} \tag{1.103}$$

It is very interesting that when $a = 0$, the process analysis solutions are the same as the perturbation expansion solutions. Please compare the expressions (1.100)~(1.103) to the expressions (1.32)~(1.35).

Let $\mathcal{V}_k(t)$ (k=0,1,2,3) denotes the general solution at kth-order of approximation given by Process Analysis Method, and let

$$\mathcal{E}_k(a) = \lim_{T\to\infty}\frac{\int_0^T\left\{\frac{d\mathcal{V}_k}{dt}+\mu\mathcal{V}_k^2(t)-g\right\}^2 dt}{g^2 T} \tag{1.104}$$

denotes the nondimensional error of $\mathcal{V}_k(t)$ in region $t \in [0,\infty)$. It is easy to know that, smaller $\mathcal{E}_k(a)$ is, a better approximation $\mathcal{V}_k(t)$ is to the exact solution. From (1.96),(1.97),(1.98) and (1.99), one has

$$\mathcal{E}_0(a) = \left(1-\frac{1}{a^2}\right)^2 \tag{1.105}$$

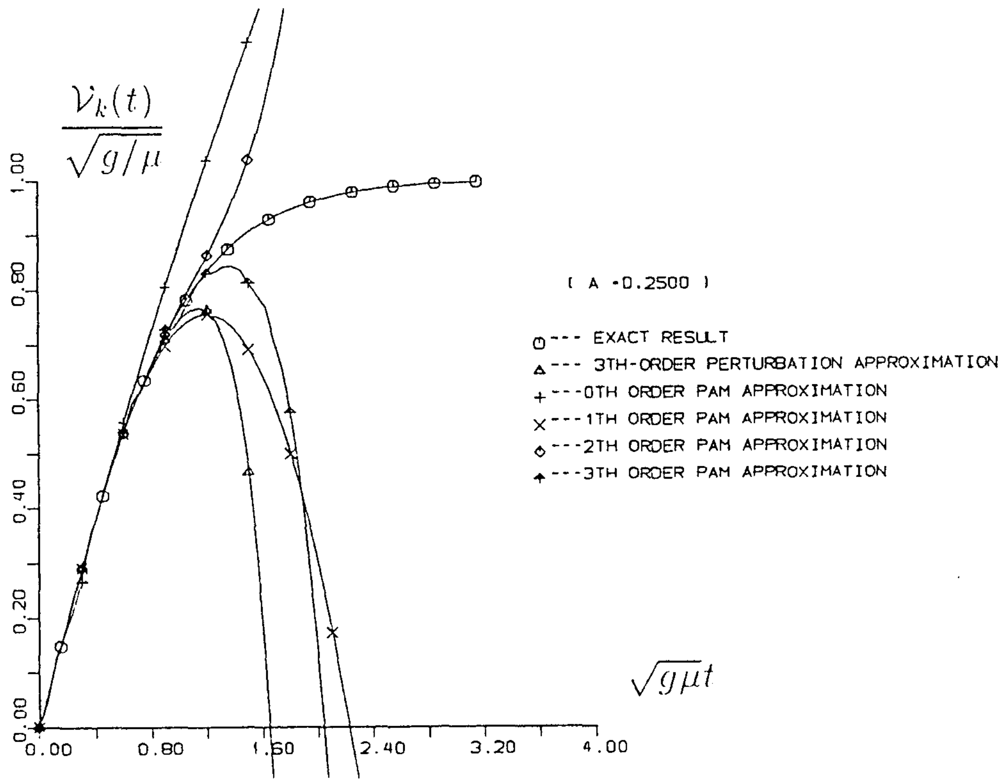

Figure 1.3: up to third-order general process analysis solution in case $a = 0.25$

$$\mathcal{E}_1(a) = \left[1 - \frac{1}{a^2}\left(2 - \frac{1}{a^2}\right)^2\right]^2 \tag{1.106}$$

$$\mathcal{E}_2(a) = \left[1 - \frac{1}{a^2}\left(3 - \frac{4}{a^2} + \frac{2}{a^4}\right)^2\right]^2 \tag{1.107}$$

$$\mathcal{E}_3(a) = \left[1 - \frac{1}{a^2}\left(4 - \frac{10}{a^2} + \frac{12}{a^4} - \frac{5}{a^6}\right)^2\right]^2 \tag{1.108}$$

From

$$\frac{d\mathcal{E}_0(a)}{da} = \frac{4}{a^3}\left(1 - \frac{1}{a^2}\right) = 0 \tag{1.109}$$

one has $a = 1.0$. Therefore, $a = 1$ gives the *best* initial solution. In the same way, it is easy to know that $a = 1$ gives the *best* solutions at first, second and third-order of approximation. At each order of approximation, $a = 0$, corresponding to the perturbation solution, gives the worst solution, because in case $a = 0$, $\mathcal{E}_k(0) = +\infty$ (k=0,1,2,3).

Figure 1.3 , figure 1.4 , figure 1.5 and figure 1.6 show general process analysis solutions at up-to third order of approximation in case $a = 0.25, a = 0.5, a = 0.75, a = 4$, respectively. Figure 1.7 shows the influence of the parameter a in general process analysis approximate solution at third order.

According to the analysis given above, a crude conclusion can be given as follows:

1. in case $0 \le a < 1$, greater a is, closer the general process analysis solution is to the exact result; where, $a = 0$ gives the worst approximation which is corresponding to perturbation expansion solutions.

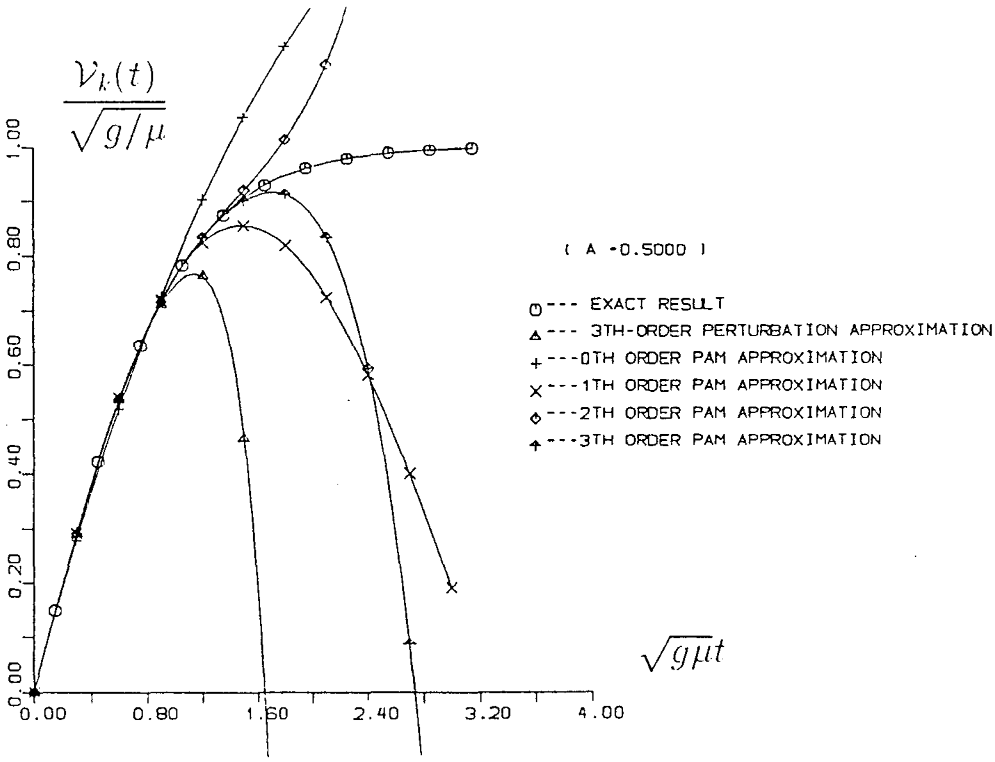

Figure 1.4: up to third-order general process analysis solution in case $a = 0.5$

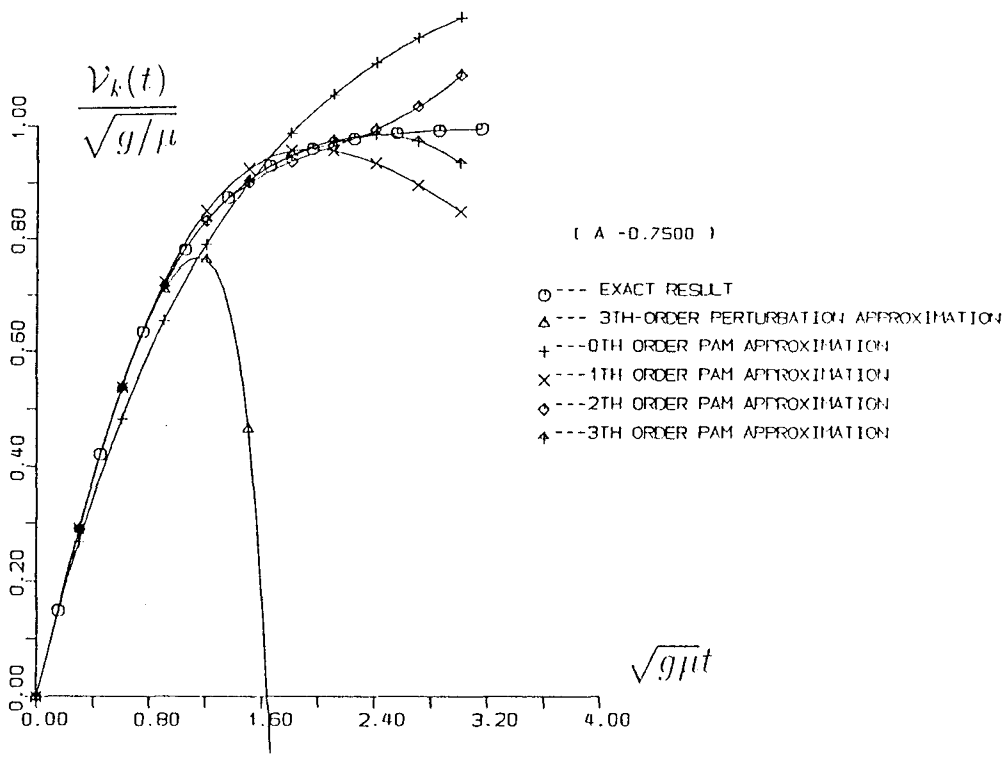

Figure 1.5: up to third-order general process analysis solution in case $a = 0.75$

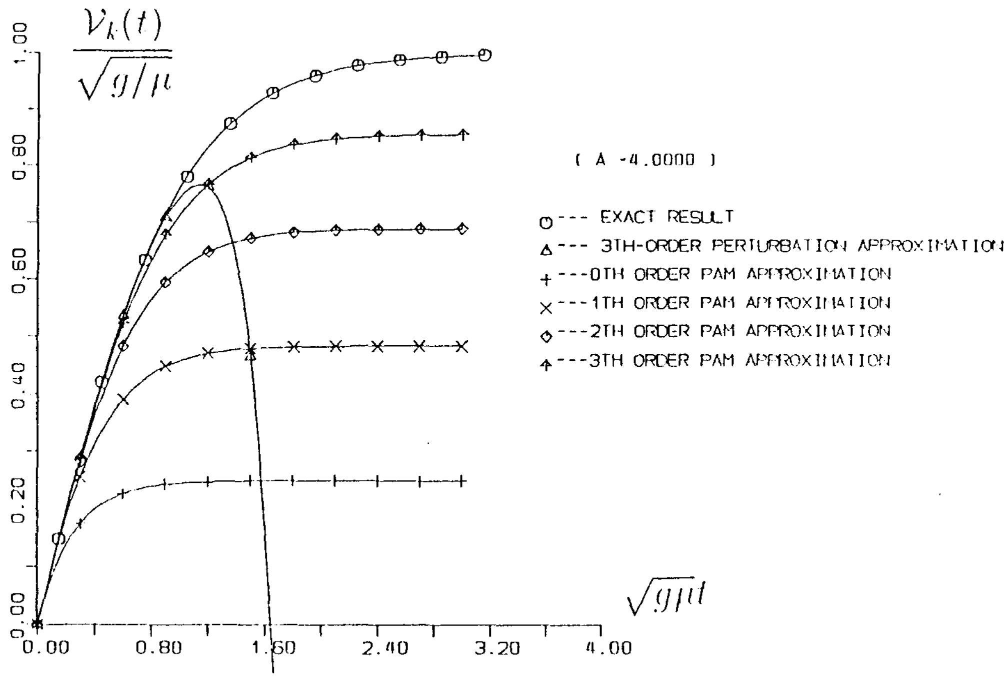

Figure 1.6: up to third-order general process analysis solution in case $a = 4.0$

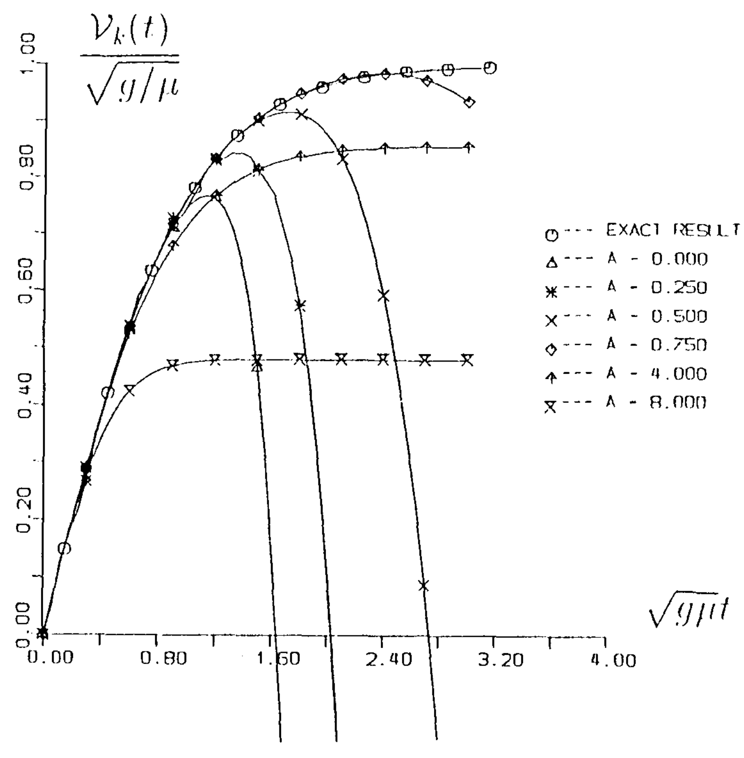

Figure 1.7: third-order general process analysis solution in case of different a

2. in case $1 \leq a \leq \sqrt{2}$, the third-order general process analysis solution is in a good agreement with the exact result.

3. in case $a > \sqrt{2}$, greater a will give worse process analysis solutions. But it is important that, for a definite a, the solution at $(k+1)$th-order of approximation is closer to the exact result than the solutions at kth-order of approximation. i.e., the solutions at (very) high-order of approximation seem converge to the exact solution. (please see figure 1.6).

4. $a = 1$ gives the best approximations.

From above discussion, one can see that Process Analysis Method can produce a *space of initial equation*, denoted as **E** and the corresponding *space of analysis solution* denoted as **S**. And there exist the best approximations in space S and the best initial equations in space **E**. In contrast with Process Analysis Method, perturbation method can give only one or limited number of initial equations and corresponding approximate solutions, which are corresponding to some points of space **E** and space S given by Process Analysis Method in this example. Therefore, Process Analysis Method seems more flexible than perturbation method.

1.4 conclusion

Perturbation expansion method is dependent upon small parameters. Generally, perturbation method can give only one or limited number of initial equations and the corresponding solutions, and results given by perturbation expansion method converge to the true solutions just only in a limited region.

In contrast with perturbation expansion method, Process Analysis Method is undependent upon small or great parameters. Process Analysis Method can give infinite number of initial equations and infinite number of corresponding solutions among which best approximations exist. For the simple example used here, process analysis solutions in case $a \geq 1$ will converge to the exact solution, and the perturbation expansion solution is corresponding to the worst of all given by Process Analysis Method in case $0 \leq a < 1$.

The considered example in this chapter is indeed very simple and exact analytical solution of it can be easily obtained. But, it is interesting that the analytical method described above needs *not* the supposition of small parameter and much better results can be obtained by it. So, it seems that there should exist some reasonable things in this method. In following chapters, we will do our best to use the basic idea described in this chapter to solve a little complex problems in mechanics, in order to develop these reasonable things.

Chapter 2

A second-order approximate analytical solution of a simple pendulum by Process Analysis Method

Summary

In this chapter, Process Analysis Method is used to give a second-order approximate solution of a simple pendulum. They are compared to perturbation solutions and it appears that even the first-order solutions are more accurate than the perturbation solutions at second-order of approximation.

2.1 introduction

It is well-known that the motion of a simple pendulum is periodical, which can be described in mathematics as follows:

$$\begin{cases} \frac{d^2\theta}{dt^2} + \omega_0^2 \, sin\theta = 0 \\ \theta(t) = \beta \quad \text{for} \quad t = 0 \\ \theta'(t) = 0 \quad \text{for} \quad t = 0 \end{cases} \tag{2.1}$$

where, θ is the angel of swing, t is the time, β is the initial angel of swing, $\omega_0 = \sqrt{\frac{g}{l}}$ is the initial frequence; here, g is the gravity acceleration, l is the length of the simple pendulum.

It is easy to know that $|\theta(t)| \le \beta$. If the initial angle β is small enough, then θ is a small quantity and $sin(\theta) \approx \theta$ is a good approximation; thus the above equation has solution :

$$\theta(t) = \beta \, cos(\omega_0 \, t) \tag{2.2}$$

with the period :

$$T_0 = \frac{2\pi}{\omega_0} \tag{2.3}$$

But, if β is not small, the above solutions are not accurate. For example, when $\beta = 5\pi/9$, the numerical result of the period is $1.232\, T_0$. Therefore, higher order approximate solutions should be given in this case. However, this is not easy, because equation (2.1) is nonlinear and has *no* small parameters in this case.

In this chapter, we will try to use Process Analysis Method, which has been described in chapter one, to solve this nonlinear problem.

2.2 basic ideas and deriving:

Let

$$\omega = \frac{2\pi}{T},$$

and

$$z = \omega t.$$

where,T is the period and ω is the frequence of a simple pendulum, respectively.

Then, equation (2.1) is transformed into:

$$\begin{cases} \frac{d^2\theta}{dz^2} + \lambda^2 sin\theta = 0 \\ \theta(z) = \beta \quad \text{for } z = 0 \\ \theta'(z) = 0 \quad \text{for } z = 0 \end{cases} \tag{2.4}$$

Here,

$$\lambda = \frac{\omega_0}{\omega} = \frac{T}{T_0} \tag{2.5}$$

denotes the non-dimensional frequence or period of a simple pendulum.

A kind of continuous mapping (homotopy) can be constructed as follows:

$$\begin{cases} \frac{\partial^2\theta(z;p)}{\partial z^2} + \lambda^2(p)\theta(z;p) + p\lambda^2(p)\left\{sin[\theta(z;p)] - \theta(z;p)\right\} = 0 \\ \theta(z;p) = \beta \quad \text{for } z = 0 \\ \frac{\partial\theta(z;p)}{\partial z} = 0 \quad \text{for } z = 0 \end{cases} \tag{2.6}$$

where,$p \in [0,1]$, called imbedding variable.

For simplicity, call $\theta(z;p)$ and $\lambda(p)$ *process*, or more precisely, *zero-order process*. Then, equation (2.6) is *zero-order process equation.*

When $p = 0$, from zero-order process equation (2.6), one has the *initial equation* :

$$\begin{cases} \frac{\partial^2\theta(z;0)}{\partial z^2} + \lambda^2(0)\theta(z;0) = 0 \\ \theta(z;0) = \beta \quad \text{for } z = 0 \\ \frac{\partial\theta(z;0)}{\partial z} = 0 \quad \text{for } z = 0 \end{cases} \tag{2.7}$$

Denote $\lambda_0 = \lambda(0)$ and $\theta_0(z) = \theta(z;0)$. For simplicity, select $\lambda_0 = 1.0$, called *initial solution* of $\lambda(p)$. It is easy to know that the above *linear* equation (2.7) has the solution:

$$\theta_0(z) = \beta cos(z) \tag{2.8}$$

when $p = 1.0$, from zero-order process equation (2.6), one has the *final equation*:

$$\begin{cases} \frac{\partial^2\theta(z;1.0)}{\partial z^2} + \lambda^2(1.0) sin\theta(z;1.0) = 0 \\ \theta(z;1.0) = \beta \quad \text{for} \quad z = 0 \\ \frac{\partial\theta(z;1.0)}{\partial z} = 0 \quad \text{for} \quad z = 0 \end{cases} \tag{2.9}$$

The above equation (2.9) are the same as (2.4). Denote $\theta_f(z) = \theta(z;1.0)$ and $\lambda_f = \lambda(1.0)$, called *final solution.* It is easy to understand that $\theta_f(z)$ and λ_f are just what we want to know.

From above analysis, we can see that the zero-order process equation (2.6) gives a kind of relation between the initial solutions $\theta_0 = \beta\, cos(z), \lambda_0 = 1.0$ and the final solutions θ_f, λ_f. But, this kind of relation is nonlinear, because the zero-order process equation (2.6) is generally a nonlinear one. In the following part of this section, a linear relation between θ_0, λ_0 and θ_f, λ_f will be introduced and used to give a kind of solutions at second-order of approximation.

Define

$$\begin{aligned} \theta^{[k]}(z;p) &= \frac{\partial^k \theta(z;p)}{\partial p^{\,k}} \\ \lambda^{[k]}(p) &= \frac{\partial^k \lambda(p)}{\partial p^{\,k}} \end{aligned}$$

as the kth-order *process derivative* of $\theta(z;p)$ and $\lambda(p)$, respectively.

Suppose :

1. $\theta(z;p), \lambda(p)$ have definition in $p \in [0,1]$, $0 \le z < \infty$
2. there exist $\theta^{[k]}(z;p)$ and $\lambda^{[k]}(p)$ in $p \in [0,1]$, $0 \le z < \infty$ for $k \ge 1$

then, according to Taylor's theory, $\theta_0(z)$, λ_0 and $\theta_f(z)$, λ_f have the following relations:

$$\theta_f(z) = \theta_0(z) + \sum_{k=1}^{\infty} \left. \frac{\theta^{[k]}(z;p)}{k!} \right|_{p=0} \tag{2.10}$$

$$\lambda_f = \lambda_0 + \sum_{k=1}^{\infty} \left. \frac{\lambda^{[k]}(p)}{k!} \right|_{p=0} \tag{2.11}$$

Where, $k\,! = 1 \times 2 \times \cdots \times (k-1) \times k$ is the factorial of k. $\theta^{[k]}(z;p), \lambda^{[k]}(p)$ at $p = 0$ can be obtained in the following way.

Deriving the zero-order process equation (2.6) with respect to p, one can obtain the *first-order process equation* as follows:

$$\left\{ \begin{array}{l} \frac{\partial^2 \theta^{[1]}(z;p)}{\partial z^2} + \lambda^2(p)\theta^{[1]}(z;p) + 2\lambda(p)\lambda^{[1]}(p)\theta(z;p) + \lambda^2(p)\left\{sin[\theta(z;p)] - \theta(z;p)\right\} \\ +2p\lambda(p)\lambda^{[1]}(p)\left\{sin\left[\theta(z;p)\right] - \theta(z;p)\right\} + p\lambda^2(p)\left\{cos\theta(z;p) - 1\right\}\theta^{[1]}(z;p) = 0 \\ \theta^{[1]}(z;p) = 0 \qquad \text{for } z = 0 \\ \frac{\partial \theta^{[1]}(z;p)}{\partial z} = 0 \qquad \text{for } z = 0 \end{array} \right. \tag{2.12}$$

When $p = 0.0$, from first-order process equation (2.12), one has:

$$\left\{ \begin{array}{l} \frac{\partial^2 \theta^{[1]}(z;0)}{\partial z^2} + \lambda_0^2 \theta^{[1]}(z;0) = \lambda_0^2\left\{\theta_0(z) - sin\theta_0\right\} - 2\lambda_0\lambda^{[1]}(0)\theta_0 \\ \theta^{[1]}(z;0) = 0 \qquad \text{for } z = 0 \\ \frac{\partial \theta^{[1]}(z;0)}{\partial z} = 0 \qquad \text{for } z = 0 \end{array} \right. \tag{2.13}$$

It can be derived that

$$sin(\theta_0) = 2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(\beta)cos(2m+1)z \tag{2.14}$$

and

$$cos(\theta_0) = J_0(\beta) + 2\sum_{m=1}^{\infty}(-1)^m J_{2m}(\beta)cos(2mz) \tag{2.15}$$

where, $J_n(\beta)$ is first-sort of Bessel function denoted as

$$J_n(\beta) = \sum_{k=0}^{\infty}\frac{(-1)^k(\beta/2)^{2k+n}}{k!(k+n)!}$$

The detailed deriving of expression (2.14) and (2.15) are given in Appendix B.

Substituting (2.14) into (2.13), one has the first-order process equation at $p = 0$ as follows:

$$\begin{cases} \frac{\partial^2\theta^{[1]}(z;0)}{\partial z^2} + \theta^{[1]}(z;0) = \left\{\beta - 2J_1(\beta) - 2\beta\lambda_0^{[1]}\right\} cos(z) \\ \qquad\qquad -2\sum_{m=1}^{\infty}(-1)^m J_{2m+1}(\beta)cos(2m+1)z \\ \theta^{[1]}(z;0) \;\; = 0 \quad , \quad \text{for } z = 0 \\ \frac{\partial\theta^{[1]}(z;0)}{\partial z} = 0 \quad , \quad \text{for } z = 0 \end{cases} \tag{2.16}$$

In order to let the above equation (2.16) have finite solution (i.e., the secular terms should be eliminated), it must be satisfied that:

$$\begin{aligned} \lambda^{[1]}(0) &= \frac{1}{2} - \frac{J_1(\beta)}{\beta} \\ &= \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}(\frac{\beta}{2})^{2k}\frac{1}{k!(k+1)!} \\ &= \frac{\beta^2}{16} - \frac{1}{24}\left(\frac{\beta}{2}\right)^4 + \cdots \end{aligned} \tag{2.17}$$

Thus, the *linear* differential equation (2.16) has solution:

$$\theta^{[1]}(z;0) = \sum_{m=0}^{\infty} a_m cos(2m+1)z \tag{2.18}$$

Where

$$a_0 = -\sum_{m=1}^{\infty} a_m \tag{2.19}$$

$$a_m = \frac{(-1)^m J_{2m+1}(\beta)}{2m(m+1)} \qquad (m \geq 1) \tag{2.20}$$

Deriving the first-order process equation (2.12) with respect to p and then let $p = 0$, one can obtain the *second-order process equation* at $p = 0$ as follows:

$$\begin{cases} \frac{\partial^2 \theta^{[2]}(z;0)}{\partial z^2} + \theta^{[2]}(z;0) = 4\lambda^{[1]}(0)\{\theta_0 - sin\theta_0\} + 2\{1 - cos\theta_0\}\theta^{[1]}(z;0) \\ \qquad -4\lambda^{[1]}(0)\theta^{[1]}(z;0) - 2\left\{\left(\lambda^{[1]}\right)^2 + \lambda^{[2]}(0)\right\}\theta_0 \\ \theta^{[2]}(z;0) = 0 \qquad \text{for } z = 0 \\ \frac{\partial \theta^{[2]}(z;0)}{\partial z} = 0 \qquad \text{for } z = 0 \end{cases} \tag{2.21}$$

The above equation is linear with respect to $\theta^{[2]}(z;0)$. Eliminating the secular terms, we have

$$\begin{aligned} \lambda_0^{[2]} &= 3\left[\frac{1}{2} - \frac{J_1(\beta)}{\beta}\right]^2 - \frac{J_2(\beta)J_3(\beta)}{4\beta} \\ &\quad -\frac{1}{\beta}\left[\frac{J_1(\beta)}{\beta} + \frac{J_2(\beta) - J_0(\beta)}{2}\right]\sum_{m=1}^{\infty}\frac{(-1)^m J_{2m+1}(\beta)}{m(m+1)} \\ &\quad -\sum_{m=2}^{\infty}\frac{J_{2m}(\beta)}{2m\beta}\left[\frac{J_{2m+1}(\beta)}{m+1} - \frac{J_{2m-1}(\beta)}{m-1}\right] \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \theta^{[2]}(z;0) &= \sum_{m=0}^{\infty} b_m cos(2m+1)z + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} cos[2(n+m)+1]z \\ &\quad + \sum_{m=1}^{\infty}\left(\sum_{\substack{n=1 \\ n\neq m \\ n\neq m-1}}^{\infty}\right) d_{mn} cos[2(n-m)+1]z \end{aligned} \tag{2.23}$$

where

$$b_0 = -\sum_{m=1}^{\infty} b_m - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} - \sum_{m=1}^{\infty}\left(\sum_{\substack{n=1 \\ n\neq m \\ n\neq m-1}}^{\infty}\right) d_{mn} \tag{2.24}$$

$$\begin{aligned} b_m &= \frac{(-1)^m J_{2m+1}(\beta)}{2m(m+1)}\left\{2 - \frac{4J_1(\beta)}{\beta} - \frac{1}{m(m+1)}\left[\frac{J_1(\beta)}{\beta} - \frac{J_0(\beta)}{2}\right]\right\} \\ &\quad + \left(\frac{a_0}{2}\right)\frac{(-1)^m[J_{2m}(\beta) - J_{2m+2}(\beta)]}{m(m+1)} \qquad (m \geq 1) \end{aligned} \tag{2.25}$$

$$c_{mn} = \frac{(-1)^{m+n} J_{2m}(\beta) J_{2n+1}(\beta)}{4n(n+1)(n+m)(n+m+1)} \qquad (m \geq 1, n \geq 1) \tag{2.26}$$

$$d_{mn} = \frac{(-1)^{m+n} J_{2m}(\beta) J_{2n+1}(\beta)}{4n(n+1)(n-m)(n-m+1)} \qquad (m \geq 1; n \geq 1, n \neq m, n \neq m-1) \tag{2.27}$$

2.2.1 approximation of frequence ω

According to

$$\omega(p) = \frac{\omega_0}{\lambda(p)}$$

and Taylor's theory, one has the first-order approximation of frequence ω as

$$\omega_1 = \omega_0 \left\{ \frac{1}{\lambda(p)} - \frac{\lambda^{[1]}(p)}{\lambda^2(p)} \right\}\Bigg|_{p=0} = \omega_0(1 - \lambda_0^{[1]}) \tag{2.28}$$

and second-order approximation of frequence as

$$\begin{aligned} \omega_2 &= \omega_0 \left\{ \frac{1}{\lambda(p)} - \frac{\lambda^{[1]}(p)}{\lambda^2(p)} + \frac{\left[\lambda^{[1]}(p)\right]^2}{\lambda^3(p)} - \frac{\lambda^{[2]}(p)}{2\,\lambda^2(p)} \right\}\Bigg|_{p=0} \\ &= \omega_0 \left\{ 1 - \lambda_0^{[1]} - \frac{1}{2}\lambda_0^{[2]} + \left(\lambda_0^{[1]}\right)^2 \right\} \end{aligned} \tag{2.29}$$

2.2.2 approximation of period T:

According to $T/T_0 = \omega_0/\omega$, one has the first-order approximation of period T as

$$\frac{T_1}{T_0} = \frac{1}{1 - \lambda_0^{[1]}} = 1 + \frac{\beta^2}{16} + \frac{1}{48}\left(\frac{\beta}{2}\right)^4 + \cdots \tag{2.30}$$

and the second-order approximation of period as

$$\frac{T_2}{T_0} = \frac{1}{1 - \lambda_0^{[1]} - \frac{1}{2}\lambda_0^{[2]} + \left(\lambda_0^{[1]}\right)^2} \tag{2.31}$$

2.2.3 approximation of $\theta(t)$:

$\theta(t)$ at first-order of approximation is

$$\begin{aligned} & \theta_1(t) \\ = \; & \theta_0(z) + \theta^{[1]}(z;0) \\ = \; & (\beta + a_0)cosz + \sum_{m=1}^{\infty} a_m cos(2m+1)z \\ = \; & (\beta + a_0)cosz - \frac{J_3(\beta)}{4}cos(3z) + \frac{J_5(\beta)}{12}cos(5z) - \frac{J_7(\beta)}{24}cos(7z) + \cdots \\ = \; & \left\{ \beta + \frac{\beta^3}{192} - \frac{1}{90}\left(\frac{\beta}{2}\right)^5 + \cdots \right\} cos(\omega_1 t) \\ & - \left\{ \frac{\beta^3}{192} - \frac{1}{96}\left(\frac{\beta}{2}\right)^5 + \cdots \right\} cos(3\omega_1 t) \\ & + \left\{ \frac{1}{1440}\left(\frac{\beta}{2}\right)^5 + \cdots \right\} cos(5\omega_1 t) + \cdots \end{aligned} \tag{2.32}$$

and $\theta(t)$ at second-order of approximation is

$$\begin{aligned} & \theta_2(t) \\ = & \theta_0(z) + \theta^{[1]}(z;0) + \theta^{[2]}(z;0)/2 \\ = & (\beta + a_0 + \frac{b_0}{2})\cos z + \sum_{m=1}^{\infty}(a_m + \frac{b_m}{2})\cos[(2m+1)z] + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{c_{mn}}{2}cos[2(n+m)+1]z \\ & + \sum_{m=1}^{\infty}\left(\sum_{\substack{n=1 \\ n\neq m \\ n\neq m-1}}^{\infty}\right)\frac{d_{mn}}{2}cos[2(n-m)+1]z \end{aligned} \tag{2.33}$$

As the last part of this section, let us discuss simply the expression (2.10) and (2.11). The expression (2.10) (2.11) describe a kind of relation between the initial solution $\theta_0 = \beta\ cos(z), \lambda_0 = 1.0$ and the final solution θ_f, λ_f by means of kth-order process derivatives $\theta^{[k]}(z;p)$ and $\lambda^{[k]}(p)$ at $p = 0$. The first-order process equation (2.15) and the second-order precess equation (2.20) are *linear* with respect to $\theta^{[1]}(z;0)$ and $\theta^{[2]}(z;0)$, respectively. One can prove easily that the kth-order process equation ($k = 1, 2, 3, \cdots$) is *always* linear with respect to $\theta^{[k]}(z;0)$ (see appendix A). Therefore, $\theta^{[1]}(z;0), \theta^{[2]}(z;0), \cdots, \theta^{[k]}(z;0), \cdots$, can be obtained without great difficulties. It means that the expression (2.10) (2.11) give a kind of linear relation between the initial solution and final solution, although the zero-order process equation (2.6) is generally nonlinear. Using process derivatives, a nonlinear problem with respect to $\theta(z)$ can be transformed into an infinite number of linear problems with respect to $\theta^{[k]}(z;p)\,(k = 1, 2, 3, \cdots, \infty)$ at $p = 0$. But, only finite number of linear problems with respect to $\theta^{[k]}(z;p)(k = 1, 2, \cdots, n_p)$ at $p = 0$ can be solved. It means that a nonlinear problem can be approximated by finite number of linear problems. One would say that a nonlinear problem could be discretized into n_p linear problems with respect to kth-order process derivatives ($k = 1, 2, \cdots, n_p$). Greater n_p, exacter this approximation. This does be the basic ideas of Process Analysis Method.

2.3 comparisons to the numerical and perturbation results

In order to examine the solutions given by Process Analysis Method, it is valuable to compare them to the numerical results and perturbation solutions at the same order of approximation.

The original equation (2.1) can be solved numerically by means of Runge-Kutta's method. In numerical computation, we select $\Delta\, t = 0.0001$ second and use double precision variables in computer program.

On the other side, subsititute

$$sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

into (2.4) and *suppose* that β be *small enough* and θ and λ could be described respectively as

$$\theta = \beta\left\{\Theta^{(0)} + \beta\Theta^{(1)} + \beta^2\Theta^{(2)} + \cdots\right\}, \tag{2.34}$$

$$\lambda = 1 + \beta\Lambda^{(1)} + \beta^2\Lambda^{(2)} + \cdots, \tag{2.35}$$

then, we can obtain

(a) perturbation solutions at first-order of approximation:

$$\theta(t) = \beta\cos(\omega_0 t) \tag{2.36}$$

$$\frac{T_1}{T_0} = 1 \tag{2.37}$$

(b) perturbation solutions at second-order of approximation:

$$\theta(t) = \left(\beta + \frac{\beta^3}{192}\right)\cos\left(\frac{\omega_0 t}{1+\frac{\beta^2}{16}}\right) - \frac{\beta^3}{192}\cos\left(\frac{3\omega_0 t}{1+\frac{\beta^2}{16}}\right) \tag{2.38}$$

$$\frac{T_2}{T_0} = 1 + \frac{\beta^2}{16} \tag{2.39}$$

2.3.1 comparison of non-dimensional period T/T_0 :

The numerical and analytical results show that the non-dimensional period T/T_0 of a simple pendulum is only dependent on the initial angel β. The detailed comparison of numerical and analytical results of T/T_0 obtained respectively by Process Analysis Method and perturbation method is given in Table 2.1. It seems that PAM solutions even at first-order of appximation agree generally better with the numerical results than perturbation solutions at second-order of approximation. Even in case of great initial angel $\beta = 130$ degree, the second-order PAM solution can also give good enough approximate value of period, but the perturbation solution at same-order of approximation has about 15% relative error.

2.3.2 spectrum analysis:

It is well-known that the solution $\theta(t)$ of the original equation (2.1) is periodical function. So, $\theta(t)$ can be expressed in form of Fourier serie.

According to (2.32) and (2.33), the PAM solutions at both first and second order of approximation can be written in the form of

$$\theta(t) = \sum_{k=1}^{\infty} A_{2k-1}\cos\left[(2k-1)\omega t\right] \quad (k = 1, 2, \cdots) \tag{2.40}$$

Numerically, we have

$$\overline{A}_n = \frac{4}{T}\int_0^{T/2} \theta(t)\cos(n\omega t)\,dt \quad (n = 1, 2, \cdots) \tag{2.41}$$

where, the numerical solution of $\theta(t)$ is used and thus a numerical integral over $[0, T/2]$ is needed.

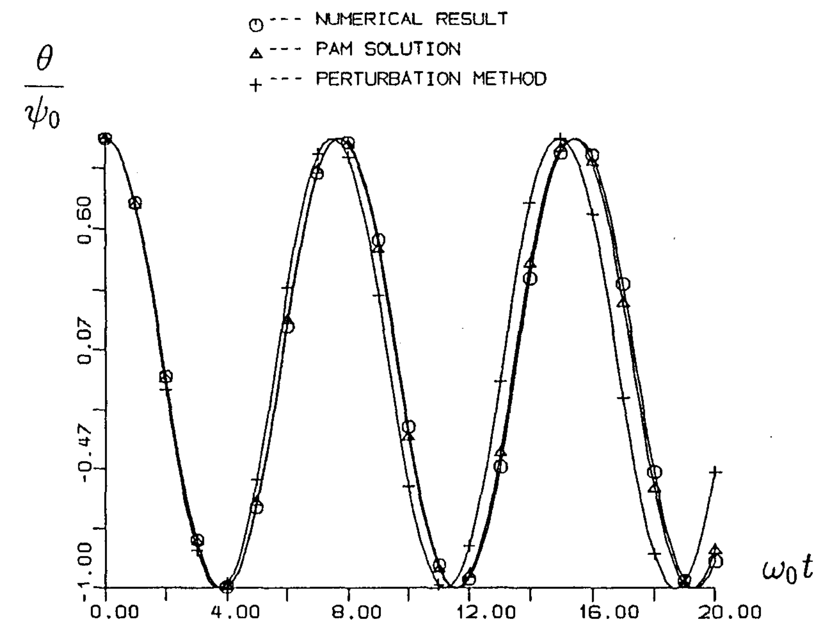

Figure 2.1: Comparision of the analytical solutions at second-order of approximation to the numerical results

The numerical values $\overline{A}_k(k = 1,2,\cdots)$ and their corresponding analytical values at first and second order of approximation given by Process Analysis Method are shown as Table 2.2, Table 2.3 and Table 2.4, respectively. These results show that the second-order PAM solutions agree better with the numerical solutions than the first-order PAM results. Note that the analytical values of $A_{2k}(k = 1,2,\cdots)$ are zero and the corresponding numerical values of $\overline{A}_2, \overline{A}_4$ and $\overline{A}_6$ given in Table 2.2 are so small that they can be as the numerical errors in integral. It appears that Process Analysis Method captures the leading nonlinear harmonics found in the numerical solutions.

The numerical results and analytical solutions at second-order of approximation given respectively by perturbation method and Process Analysis Method in case of $\beta = 5\pi/9$ are shown as Figure 2.1.

The analytical and numerical valus of $A_k(k > 1)$ are so smaller than A_1 that

$$\theta(t) = \beta\cos(\omega_2\, t)$$

can give a good enough approximation of origial equation (2.1). This simplified expression give essentially the same results as the second-order PAM solutions (2.33).

With comparison of expressions (2.30), (2.32) to expressions (2.39) and (2.38), it appears that the first-order PAM solutions include the terms of the perturbation solutions at second-order of approximation.

Table 2.1: comparison of theoretical and numerical results of T/T_0

β	numerical method	PAM		perturbation method (second-order)
		first-order	second-order	
20°	1.008	1.008	1.008	1.008
30°	1.017	1.017	1.017	1.017
40°	1.031	1.031	1.031	1.030
50°	1.050	1.048	1.050	1.048
60°	1.073	1.070	1.073	1.069
70°	1.102	1.096	1.101	1.093
80°	1.138	1.127	1.136	1.122
90°	1.180	1.162	1.177	1.154
100°	1.232	1.201	1.225	1.190
110°	1.295	1.246	1.282	1.230
120°	1.373	1.296	1.347	1.274
130°	1.471	1.351	1.424	1.322

Table 2.2: Spectrum analysis : numerical result

β	30°	60°	90°	120°
A_1/β	1.001	1.006	1.015	1.030
A_2/β	3.29×10^{-6}	-2.21×10^{-7}	4.77×10^{-8}	3.38×10^{-6}
A_3/β	-1.45×10^{-3}	-6.14×10^{-3}	-1.53×10^{-2}	-3.20×10^{-2}
A_4/β	5.90×10^{-7}	-8.39×10^{-8}	-4.24×10^{-8}	1.43×10^{-7}
A_5/β	3.43×10^{-6}	6.62×10^{-5}	3.96×10^{-4}	1.65×10^{-3}
A_6/β	2.25×10^{-7}	-6.85×10^{-8}	-4.86×10^{-8}	1.41×10^{-7}
A_7/β	-1.61×10^{-7}	-7.84×10^{-7}	-1.22×10^{-5}	-1.01×10^{-4}

Table 2.3: Spectrum Analysis : first-order PAM solutions

β	30°	60°	90°	120°
A_1/β	1.001	1.005	1.011	1.017
A_3/β	-1.40×10^{-3}	-5.33×10^{-3}	-1.10×10^{-2}	-1.72×10^{-2}
A_5/β	1.61×10^{-6}	2.49×10^{-5}	1.19×10^{-4}	3.47×10^{-4}
A_7/β	-1.32×10^{-9}	-8.23×10^{-8}	-8.95×10^{-7}	-4.70×10^{-6}

Table 2.4: Spectrum Analysis : second-order PAM solutions

β	30°	60°	90°	120°
A_1/β	1.001	1.006	1.014	1.025
A_3/β	-1.45×10^{-3}	-6.03×10^{-3}	-1.41×10^{-2}	-2.60×10^{-2}
A_5/β	1.96×10^{-6}	4.57×10^{-5}	3.29×10^{-4}	1.34×10^{-3}
A_7/β	-1.88×10^{-9}	-2.22×10^{-7}	-4.27×10^{-6}	-3.57×10^{-5}

2.4 conclusion and discussion

In this chapter, Process Analysis Method is used to give a second-order approximate solutions of a simple pendulum. These solutions are compared to the numerical and perturbation solutions. The comparison shows that even the first-order PAM solutions agree better with the numerical solutions than the second-order perturbation solutions. So, we have reason to believe that Process Analysis Method can give indeed more accurate analytical results than perturbation method. Note that small or great parameter supposition is *not* needed for Process Analysis Method. This is an advantage of PAM.

Experiences seem important for perturbation method. Specially in case of singular perturbation problems, one must use different techniques, for example, method of multiple-scale expansions, method of matched asymptotic expansions and so on, to solve different problems. But, in contrast to perturbation techniques, Process Analysis Method has the simplicity in logic. This is another advantage of Process Analysis Method.

Process Analysis Method is based on the two concept:*process* and *process derivatives.* Process is a kind of continuous mapping (or more precisely speaking, homotopy) which connects the initial solutions with the soluions of the original nonlinear problem. But, this continuous mapping described by zero-order process equation is generally nonlinear for a nonlinear problem. It is interesting and important that the kth-order process equations are *always* linear with respect to the kth-order process derivatives[1]. Then, according to Taylor's theory, the final solution and initial solution can be connected by the kth-order process derivatives $(k = 1, 2, \cdots,)$. That is why the process derivatives should be used.

Process Analysis Method is also based on suppositions, i.e., there should exist zero-order process and kth-order process derivatives, and the corresponding Taylor's serie should converge to the solution of the original equations. Therefore, Process Analysis Method is also dependent on suppositions, but these suppositions are *not* small parameter suppositions. Owing to this reason, one can use it to solve more nonlinear problems, specially those without small or great parameters.

However, it should be pointed out that Taylor's serie has generally a *finite* radius of convergence (Only few functions have converged Taylor's serie with infinite radius of convergence; for example $sin(x), \cos(x)$ and so on). Process Analysis Method requires the kth-order process derivatives $(k = 1, 2, \cdots)$ to exist, and to be well behaved and be bounded (i.e., they must not have turing points or diverge to infinity in $p \in [0, 1]$), and converge in $p \in [0, 1]$ (i.e., the radius of convergence of the corresponding Taylor's serie should be greater than or equal to one). This is the limitations of Process Analysis Method.

[1]Mathematically speaking, this is a kind of invariance under homotopy, as proved in appendix A. But, mathematics is not the purpose of this paper, so we will not here discuss it.

Chapter 3

Application of Process Analysis Method in solution of 2D nonlinear progressive gravity waves

Summary

Process Analysis Method is used to solve the 2D nonlinear progressive gravity waves. The solutions at forth-order of approximation are obtained. Comparisons to stokesian waves are given.

3.1 introduction

The 2D progressive gravity waves is a typical nonlinear problem. The difficulty for the solution of this problem would perhaps be that the two boundary conditions of free surface are nonlinear and must be satisfied on the unknown free surface. In 1847, Stokes [50] used the method of perturbation expansion in a small parameter to study this problem and gave a kind of solution at third-order of approximation. In 1883, he reconsidered this problem and gave a kind of solution at forth-order of approximation. In 1974, Schwartz [48] used computer to develop the stokesian method and gave a kind of solution at 117th-order of approximation for deep water waves. Many other authors researched theoretically this problem by means of perturbation expansion method. It is very interesting and also puzzled that different analytical solutions are obtained, if different *small parameters* are selected. "There exist in the American Literature at least five different irrotational stokesian wave theories at a fifth order of appoximation; all of them are based on the same assumption, but the mathematical formulations are different." (Bernard Le Méhauté [30])

In this chapter, approximate analytical solutions of 2D progressive gravity waves is derived by Process Analysis Method, which are different from the traditional stokesian waves.

3.2 mathematical describtion:

A 2D progressive gravity wave is moving with phase velocity C. Suppose that the fluid is inviscid, incompressible and without surface tensen. A 2D coordinate system, moving with the same velocity C in the same direction as wave motion, is used with y-axis pointing upward and x-axis located in the still water level with x positive in the direction of the wave motion. Suppose the water depth is infinite.

In this coordinate system , the 2D progressive gravity waves can be described mathematically as follows:

$$\nabla^2\phi(x,y) = 0.0 \quad , \qquad \text{in } \Omega \tag{3.1}$$

with boundary conditions:

$$C^2\phi_{xx} + g\phi_y + \frac{1}{2}\nabla\phi\,\nabla(\nabla\phi\,\nabla\phi) - 2C\nabla\phi\,\nabla\phi_x = 0 \qquad \text{on } y = \zeta(x) \tag{3.2}$$

$$\lim_{y\to-\infty}\frac{\partial\phi}{\partial y} = 0.0 \tag{3.3}$$

and

$$\zeta(x) = \frac{1}{g}\left(C\phi_x - \frac{1}{2}\nabla\phi\,\nabla\phi\right) \qquad \text{on } y = \zeta(x) \tag{3.4}$$

where, $\phi(x,y)$ is the velocity-potential, $\zeta(x)$ is the wave-elevation and g is the gravity acceleration.

(3.4) is the dynamic condition of free surface. (3.2) is obtained from the dynamic condition and kinematic condition of free surface, its detailed deriving is given in Appendix C. Note that C in expression (3.2) and (3.4) is phase velocity of the 2D progressive wave and in this paper we call it simply wave-velocity.

For simplicity , define:

$$\mathcal{L}[\phi(x,y)] = C^2\phi_{xx} + g\phi_y \quad , \quad \text{on } y = \zeta(x) \tag{3.5}$$

$$\mathcal{N}[\phi(x,y)] = \tfrac{1}{2}\nabla\phi\,\nabla(\nabla\phi\,\nabla\phi) - 2C\nabla\phi\,\nabla\phi_x \quad , \quad \text{on } y = \zeta(x) \tag{3.6}$$

$$\mathcal{Z}[\phi(x,y)] = \tfrac{1}{g}\left(C\phi_x - \tfrac{1}{2}\nabla\phi\,\nabla\phi\right) \quad , \quad \text{on } y = \zeta(x) \tag{3.7}$$

where, $\mathcal{L}[\phi(x,y)]$ is the linear term, $\mathcal{N}[\phi(x,y)]$ is the nonlinear term of nonlinear free surface condition (3.2), respectively.

Thus, the problem considered in this chapter can be described as follows:

$$\nabla^2\phi(x,y) = 0.0, \qquad \text{in } \Omega \tag{3.8}$$

with boundary conditions:

$$\mathcal{L}[\phi(x,y)] + \mathcal{N}[\phi(x,y)] = 0 \quad , \qquad \text{on } y = \zeta(x) \tag{3.9}$$

$$\lim_{y\to-\infty} \frac{\partial\phi(x,y)}{\partial y} = 0 \tag{3.10}$$

where

$$\zeta(x) = \mathcal{Z}[\phi(x,y)] \quad , \qquad \text{on } y = \zeta(x) \tag{3.11}$$

3.3 basic ideas and deriving

A continuous mapping (or more precisely speaking, homotopy) can be constructed as follows:

$$\nabla^2\phi(x,y;p) = 0, \qquad \text{in } \Omega(p) \tag{3.12}$$

with boundary conditions:

$$\mathcal{L}[\phi(x,y;p)] + p\,\mathcal{N}[\phi(x,y;p)] = 0.0 \quad , \quad \text{on } y = \zeta(x;p) \tag{3.13}$$

$$\lim_{y\to-\infty} \frac{\partial\phi(x,y;p)}{\partial y} = 0.0 \quad , \tag{3.14}$$

whereas:

$$\zeta(x;p) = p\,\mathcal{Z}[\phi(x,y;p)] + (1-p)\,\zeta_0(x) \quad , \quad \text{on } y = \zeta(x;p) \tag{3.15}$$

where

$$\mathcal{L}[\phi(x,y;p)] = C^2(p)\phi_{xx}(x,y;p) + g\phi_y(x,y;p) \quad , \text{ on } y = \zeta(x;p) \tag{3.16}$$

$$\begin{aligned}\mathcal{N}[\phi(x,y;p)] = {} & \tfrac{1}{2}\nabla\phi(x,y;p)\,\nabla\left[\nabla\phi(x,y;p)\,\nabla\phi(x,y;p)\right] \\ & -2C(p)\nabla\phi(x,y;p)\,\nabla\phi_x(x,y;p) \quad , \text{ on } y = \zeta(x;p)\end{aligned} \tag{3.17}$$

$$\mathcal{Z}[\phi(x,y;p)] = \tfrac{1}{g}\left[C(p)\phi_x(x,y;p) - \tfrac{1}{2}\nabla\phi(x,y;p)\,\nabla\phi(x,y;p)\right] \quad , \text{ on } y = \zeta(x;p) \tag{3.18}$$

Here, $\phi(x, y; p), \zeta(x; p)$, $C(p)$ and $\Omega(p)$ mean that they are not only the functions of original independent quantities x, y, but also the functions of imbedding variable p. $\zeta_0(x)$ is a freely selected initial wave-elevation.

For simplicity, select $\zeta_0(x) = 0$ in this chapter. Then, one has the *zero-order process equation* as follows:

$$\nabla^2\phi(x, y; p) = 0 \quad , \quad \text{in } \Omega(p) \tag{3.19}$$

with boundary conditions:

$$\mathcal{L}\left[\phi(x, y; p)\right] + p\mathcal{N}\left[\phi(x, y; p)\right] = 0 \quad , \quad \text{on } y = \zeta(x; p) \tag{3.20}$$

$$\lim_{y\to-\infty} \frac{\partial\phi(x, y; p)}{\partial y} = 0 \quad , \tag{3.21}$$

where

$$\zeta(x; p) = p\mathcal{Z}\left[\phi(x, y; p)\right] \quad , \quad \text{on } y = \zeta(x; p) \tag{3.22}$$

When $p = 0$, from zero-order process equation (3.19)~(3.22), one has the *initial equation*:

$$\nabla^2\phi(x, y; 0) = 0 \quad , \quad \text{in } \Omega(0) \tag{3.23}$$

with boundary conditions:

$$\mathcal{L}\left[\phi(x, y; 0)\right] = C^2(0)\phi_{xx}(x, y; 0) + g\phi_y(x, y; 0) = 0 \quad , \quad \text{on } y = 0 \tag{3.24}$$

$$\lim_{y\to-\infty} \frac{\partial\phi(x, y; 0)}{\partial y} = 0 \quad , \tag{3.25}$$

where

$$\zeta(x; 0) = 0 \tag{3.26}$$

The boundary condition (3.24) is linear and satisfied on the known initial wave-elevation $y = 0$. Therefore, the initial equation (3.23)~(3.26) can be easily solved. The solution of the initial equation (3.23)~(3.26) could be called initial velocity-potential, denoted as $\phi_0(x, y)$; and $C(0)$ is denoted as C_0. It is easy to know that

$$\phi_0(x, y) = AC_0e^{ky}sin(kx) \tag{3.27}$$

$$C_0 = \sqrt{\frac{g}{k}} \tag{3.28}$$

where, A is the initial wave-amplitude, $k = \frac{2\pi}{\lambda}$ is the wave-number, λ is the wave-length.

When p=1.0, from zero-order process equation (3.19)~(3.22), one can obtain the *final equation* as follows:

$$\nabla^2\phi(x, y; 1) = 0 \quad , \quad \text{in } \Omega(1) \tag{3.29}$$

with boundary conditions:

$$\mathcal{L}\left[\phi(x, y; 1)\right] + \mathcal{N}\left[\phi(x, y; 1)\right] = 0 \quad , \quad \text{on } y = \zeta(x; 1) \tag{3.30}$$

$$\lim_{y\to-\infty} \frac{\partial\phi(x, y; 1)}{\partial y} = 0 \quad , \tag{3.31}$$

where

$$\zeta(x;1) = \mathcal{Z}\left[\phi(x,y;1)\right] \quad , \quad \text{on } y = \zeta(x;1) \tag{3.32}$$

The final equations (3.29)~(3.32) are the same as the original equations (3.8)~(3.11) described in **3.2** and are just what one wants to solve. The solution of the final equation $\phi(x,y;1)$ could be called final velocity-potential, denoted as $\phi_f(x,y)$;$\zeta(x;1)$ called the final wave-elevation, denoted as $\zeta_f(x)$;and $C(1)$ is denoted as C_f.

One can see that the zero-order process equation (3.19)~(3.22) connects the linear initial equation (3.23)~(3.26) with the nonlinear final equation (3.29)~(3.32). The process of change from $p = 0.0$ into $p = 1.0$ is just the process of transformation from the initial equation into the final equation. Analysing this process of transformation in region $p \in [0,1]$, a kind of relation between the initial solutions $\phi_0(x,y), \zeta_0(x), C_0$ and the final solutions $\phi_f(x,y), \zeta_f(x), C_f$ can be given in the following way.

Define:

$$\phi^{[m]}(x,y;p) = \frac{\partial^m \phi(x,y;p)}{\partial p^m} \tag{3.33}$$

$$\zeta^{[m]}(x;p) = \frac{\partial^m \zeta(x;p)}{\partial p^m} \tag{3.34}$$

$$C^{[m]}(p) = \frac{\partial^m C(p)}{\partial p^m} \tag{3.35}$$

as the *mth-order process derivatives* of velocity-potential $\phi(x,y;p)$, wave-elevation $\zeta(x;p)$ and wave velocity $C(p)$, respectively.

For simplicity, define

$$\phi_0^{[m]}(x,y) = \phi^{[m]}(x,y;0) \tag{3.36}$$

$$\zeta_0^{[m]}(x) = \zeta^{[m]}(x;0) \tag{3.37}$$

$$C_0^{[m]} = C^{[m]}(0) \tag{3.38}$$

Then, according to Taylor's theorem, the initial solutions and final solutions have the following relations:

$$\begin{aligned}
\phi_f(x,y) &= \phi(x,y;1) \\
&= \phi(x,y;0) + \sum_{m=1}^{\infty} \left.\frac{\phi^{[m]}(x,y;p)}{m!}\right|_{p=0} \\
&= \phi_0(x,y) + \sum_{m=1}^{\infty} \frac{\phi_0^{[m]}(x,y)}{m!}
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
\zeta_f(x) &= \zeta(x;1) \\
&= \zeta(x;0) + \sum_{m=1}^{\infty} \left.\frac{\zeta^{[m]}(x;p)}{m!}\right|_{p=0} \\
&= \zeta_0(x) + \sum_{m=1}^{\infty} \frac{\zeta_0^{[m]}(x)}{m!}
\end{aligned} \tag{3.40}$$

$$\begin{aligned} C_f &= C(1) \\ &= C(0) + \sum_{m=1}^{\infty} \left. \frac{C^{[m]}(p)}{m!} \right|_{p=0} \\ &= C_0 + \sum_{m=1}^{\infty} \frac{C_0^{[m]}}{m!} \end{aligned} \tag{3.41}$$

$\phi_0^{[m]}(x,y), \zeta_0^{[m]}(x)$ and $C_0^{[m]}$ can be obtained from the mth-order process equation which can be obtained in the following way.

Deriving (3.19) and (3.21) m times with respect to p, one has:

$$\frac{\partial^m}{\partial p^m}\left[\nabla^2 \phi(x,y;p)\right] = \nabla^2 \phi^{[m]}(x,y;p) = 0 \quad , \quad \text{in } \Omega(p) \tag{3.42}$$

$$\frac{\partial^m}{\partial p^m}\left[\lim_{y\to-\infty} \frac{\partial \phi(x,y;p)}{\partial y}\right] = \lim_{y\to-\infty} \frac{\partial \phi^{[m]}(x,y;p)}{\partial y} = 0 \tag{3.43}$$

The nonlinear free surface condition (3.20), (3.22) must be satisfied on the unknown wave-elevation $y = \zeta(x;p)$, which is also the function of p. Therefore, deriving (3.20) and (3.22) m times with respect to p respectively, one has:

$$\begin{aligned} &\frac{d^m}{dp^m}\left\{\mathcal{L}\left[\phi(x,y;p)\right] + p\mathcal{N}\left[\phi(x,y;p)\right]\right\} \\ &= \frac{d^m \mathcal{L}\left[\phi(x,y;p)\right]}{dp^m} + \frac{d^m}{dp^m}\left\{p\mathcal{N}\left[\phi(x,y;p)\right]\right\} \qquad \text{on } y = \zeta(x;p) \end{aligned} \tag{3.44}$$

$$\frac{d^m \zeta(x;p)}{dp^m} = \zeta^{[m]}(x;p) = \frac{d^m}{dp^m}\left\{p\mathcal{Z}\left[\phi(x,y;p)\right]\right\} \qquad \text{on } y = \zeta(x;p) \tag{3.45}$$

When $p = 0$,one obtains from (3.44) and (3.45) that

$$\begin{aligned} &\left.\frac{d^m}{dp^m}\left\{\mathcal{L}\left[\phi(x,y;p)\right] + p\mathcal{N}\left[\phi(x,y;p)\right]\right\}\right|_{p=0} \\ &= \left.\frac{d^m \mathcal{L}\left[\phi(x,y;p)\right]}{dp^m}\right|_{p=0} + m\left.\frac{d^{m-1}\mathcal{N}\left[\phi(x,y;p)\right]}{dp^{m-1}}\right|_{p=0} \qquad \text{on } y = 0 \end{aligned} \tag{3.46}$$

and

$$\zeta_0^{[m]}(x) = m\left.\frac{d^{m-1}\mathcal{Z}\left[\phi(x,y;p)\right]}{dp^{m-1}}\right|_{p=0} \qquad \text{on } y = 0 \tag{3.47}$$

Where

$$\begin{aligned} \frac{d}{dp} &= \frac{\partial}{\partial p} + \frac{\partial}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial}{\partial y}\frac{\partial y}{\partial p} \\ &= \frac{\partial}{\partial p} + \frac{\partial \zeta}{\partial p}\frac{\partial}{\partial y} \\ &= \frac{\partial}{\partial p} + \zeta^{[1]}\frac{\partial}{\partial y} \quad , \\ \frac{d^2}{dp^2} &= \left\{\frac{\partial}{\partial p} + \zeta^{[1]}\frac{\partial}{\partial y}\right\}\left\{\frac{\partial}{\partial p} + \zeta^{[1]}\frac{\partial}{\partial y}\right\} \\ &= \frac{\partial^2}{\partial p^2} + \zeta^{[2]}\frac{\partial}{\partial y} + 2\zeta^{[1]}\frac{\partial^2}{\partial y \partial p} + \left(\zeta^{[1]}\right)^2\frac{\partial^2}{\partial y^2} \quad , \end{aligned} \tag{3.48}$$

Then, one has the *mth-order process equation* ($m = 1, 2, \cdots$) at $p = 0$, $\zeta_0(x) = 0$ as follows:

$$\nabla^2 \phi_0^{[m]}(x,y) = 0 \tag{3.49}$$

with boundary conditions:

$$\left.\frac{d^m \mathcal{L}}{dp^m}\right|_{p=0} + m \left.\frac{d^{m-1}\mathcal{N}}{dp^{m-1}}\right|_{p=0} = 0 \quad , \quad \text{on } y = 0 \tag{3.50}$$

$$\lim_{y \to -\infty} \frac{\partial \phi_0^{[m]}(x,y)}{\partial y} = 0 \quad , \tag{3.51}$$

where

$$\zeta_0^{[m]}(x) = m \left.\frac{d^{m-1}\mathcal{Z}}{dp^{m-1}}\right|_{p=0} \quad , \quad \text{on } y = 0 \tag{3.52}$$

The above equations (3.50)~(3.53) are linear with respect to $\phi_0^{[m]}(x,y)$ and $\zeta_0^{[m]}(x)$, because $\mathcal{L}[\phi(x,y)]$ is linear with respect to $\phi(x,y)$; and $\left.\frac{d^{m-1}\mathcal{N}}{dp^{m-1}}\right|_{p=0}$, $\left.\frac{d^{m-1}\mathcal{Z}}{dp^{m-1}}\right|_{p=0}$ are only the function of $C_0^{[k]}$, $\phi_0^{[k]}(x,y)$ for $k = 0, 1, 2, \cdots, m-1$. Therefore, $\phi_0^{[m]}(x,y)$, ,$\zeta_0^{[m]}(x)$, $C_0^{[m]}$ can be obtained from the above mth-order process equation, if $\phi_0^{[k]}(x,y)$, $\zeta_0^{[k]}(x)$ and $C_0^{[k]}$ ($k = 0, 1, 2, \cdots\cdots, m-1$) are known.

It should be emphasized that the wave-velocity C must be considered also as the function of imbedding variable p. $C_0^{[m]}$ can be obtained from linear partial-differential equation constructed by (3.50), (3.51), (3.52) by using the condition that the solution $\phi_0^{[m]}(x,y)$ should be finite.

Although, from pure mathematical viewpoint, the analytical expressions of $\phi_0^{[m]}(x,y)$, $\zeta_0^{[m]}(x)$, $C_0^{[m]}$ for any m can be obtained from mth-order process equation (3.50)~(3.53), it will be very complex if m is great. In this chapter, we will give the results at forth-order of approximation of 2D deep progressive gravity waves. The first-order and higher-order process derivatives of $\phi(x,y;p)$, $\zeta(x;p)$ and $C(p)$ at $p = 0$ are given as follows:

$$\begin{aligned}
\phi_0^{[1]}(x,y) &= 0 && (3.53)\\
\phi_0^{[2]}(x,y) &= C_0 k^3 A^4 e^{2ky} sin(2kx) && (3.54)\\
\phi_0^{[3]}(x,y) &= \frac{1}{2} C_0 k^4 A^5 \left\{ e^{3ky} sin(3kx) - 9kA\, e^{2ky} sin(2kx) \right\} && (3.55)\\
\phi_0^{[4]}(x,y) &= \frac{1}{2} C_0 k^5 A^6 \left\{ 4e^{4ky} sin(4kx) + 10kA\, e^{3ky} sin(3kx) \right.\\
&\quad \left. + \left(92 + 45k^2A^2\right) e^{2ky} sin(2kx) \right\} && (3.56)\\
\zeta_0^{[1]}(x) &= A \left\{ cos(kx) - \frac{1}{2} kA \right\} && (3.57)\\
\zeta_0^{[2]}(x) &= kA^2 \left\{ cos(2kx) - 2kA\, cos(kx) + 1 + k^2A^2 \right\} && (3.58)\\
\zeta_0^{[3]}(x) &= \frac{3}{4} k^2 A^3 \left\{ 3cos(3kx) - 4kAcos(2kx) + 3\left(3 + 2k^2A^2\right) cos(kx) \right.\\
&\quad \left. -6kA\left(2 + k^2A^2\right) \right\} && (3.59)\\
\zeta_0^{[4]}(x) &= k^3 A^4 \left\{ 8cos(4kx) - 7kAcos(3kx) + \left(32 - 24k^2A^2\right) cos(2kx) \right.
\end{aligned}$$

$$-kA\left(81-10k^2A^2\right)cos(kx)+\left(24+54k^2A^2+32k^4A^4\right)\} \quad (3.60)$$

$$C_0^{[1]} = \frac{1}{2}C_0k^2A^2 \quad (3.61)$$

$$C_0^{[2]} = -\frac{5}{4}C_0k^4A^4 \quad (3.62)$$

$$C_0^{[3]} = \frac{3}{2}C_0k^4A^4\left(5+\frac{17}{4}k^2A^2\right) \quad (3.63)$$

$$C_0^{[4]} = -\frac{1}{2}C_0k^6A^6\left(177+\frac{815}{8}k^2A^2\right) \quad (3.64)$$

The detailed deriving of first and second-order process derivatives $\phi_0^{[1]}(x,y)$,$\phi_0^{[2]}(x,y)$,$\zeta_0^{[1]}(x)$, $\zeta_0^{[2]}(x)$,$C_0^{[1]}$ and $C_0^{[2]}$ have been given in Appendix D. In order to decrease the length of this paper, the detailed deriving of third and forth-order process derivatives are neglected.

Thus, according to (3.39),(3.40) and (3.41), one has:

3.3.1 first-order process analysis approximation

$$\begin{aligned} \phi_1(x,y) &= \phi_0(x,y)+\frac{\phi_0^{[1]}(x,y)}{1!} \\ &= AC_0e^{ky}sin(kx) \quad (3.65) \end{aligned}$$

$$\begin{aligned} \zeta_1(x) &= \frac{\zeta_0^{[1]}(x)}{1!} \\ &= A\left\{cos(kx)-\frac{1}{2}kA\right\} \quad (3.66) \end{aligned}$$

$$\begin{aligned} C_1 &= C_0+\frac{C_0^{[1]}}{1!} \\ &= C_0\left(1+\frac{1}{2}k^2A^2\right) \quad (3.67) \end{aligned}$$

$$\begin{aligned} (H_w)_1 &= \zeta(0)-\zeta\left(\frac{\pi}{k}\right) \\ &= 2A \quad (3.68) \end{aligned}$$

where, H_w is wave-height.

3.3.2 second-order process analysis approximation

$$\begin{aligned} \phi_2(x,y) &= \phi_0(x,y)+\frac{\phi_0^{[1]}(x,y)}{1!}+\frac{\phi_0^{[2]}(x,y)}{2!} \\ &= AC_0\left\{e^{ky}sin(kx)+\frac{1}{2}k^3A^3e^{2ky}sin(2kx)\right\} \quad (3.69) \end{aligned}$$

$$\zeta_2(x) = \frac{\zeta_0^{[1]}(x)}{1!}+\frac{\zeta_0^{[2]}(x)}{2!}$$

$$= A\left\{(1-k^2A^2)cos(kx)+\frac{1}{2}kAcos(2kx)+\frac{1}{2}k^3A^3\right\} \tag{3.70}$$

$$C_2 = C_0+\frac{C_0^{[1]}}{1!}+\frac{C_0^{[2]}}{2!}$$

$$= C_0\left(1+\frac{1}{2}k^2A^2-\frac{5}{8}k^4A^4\right) \tag{3.71}$$

$$(H_w)_2 = 2A\left(1-k^2A^2\right) \tag{3.72}$$

3.3.3 third-order process analysis approximation

$$\phi_3(x,y) = \phi_0(x,y)+\frac{\phi_0^{[1]}(x,y)}{1!}+\frac{\phi_0^{[2]}(x,y)}{2!}+\frac{\phi_0^{[3]}(x,y)}{3!}$$

$$= AC_0\left\{e^{ky}sin(kx)+k^3A^3\left(\frac{1}{2}-\frac{3}{4}k^2A^2\right)e^{2ky}sin(2kx)+\frac{1}{12}k^4A^4e^{3ky}sin(3kx)\right\} \tag{3.73}$$

$$\zeta_3(x) = \frac{\zeta_0^{[1]}(x)}{1!}+\frac{\zeta_0^{[2]}(x)}{2!}+\frac{\zeta_0^{[3]}(x)}{3!}$$

$$= A\left\{\left(1+\frac{1}{8}k^2A^2+\frac{3}{4}k^4A^4\right)cos(kx)+\frac{1}{2}kA\left(1-k^2A^2\right)cos(2kx)+\frac{3}{8}k^2A^2cos(3kx)-k^3A^3\left(1+\frac{3}{4}k^2A^2\right)\right\} \tag{3.74}$$

$$C_3 = C_0+\frac{C_0^{[1]}}{1!}+\frac{C_0^{[2]}}{2!}+\frac{C_0^{[3]}}{3!}$$

$$= C_0\left(1+\frac{1}{2}k^2A^2+\frac{5}{8}k^4A^4+\frac{17}{16}k^6A^6\right) \tag{3.75}$$

$$(H_w)_3 = 2A\left(1+\frac{1}{2}k^2A^2+\frac{3}{4}k^4A^4\right) \tag{3.76}$$

3.3.4 forth-order process analysis approximation

$$\phi_4(x,y) = \phi_0(x,y)+\frac{\phi_0^{[1]}(x,y)}{1!}+\frac{\phi_0^{[2]}(x,y)}{2!}+\frac{\phi_0^{[3]}(x,y)}{3!}+\frac{\phi_0^{[4]}(x,y)}{4!}$$

$$= AC_0\left\{e^{ky}sin(kx)+k^3A^3\left(\frac{1}{2}+\frac{7}{6}k^2A^2+\frac{15}{16}k^4A^4\right)e^{2ky}sin(2kx)+k^4A^4\left(\frac{1}{12}+\frac{5}{24}k^2A^2\right)e^{3ky}sin(3kx)+\frac{1}{12}k^5A^5e^{4ky}sin(4kx)\right\} \tag{3.77}$$

$$\zeta_4(x) = \frac{\zeta_0^{[1]}(x)}{1!}+\frac{\zeta_0^{[2]}(x)}{2!}+\frac{\zeta_0^{[3]}(x)}{3!}+\frac{\zeta_0^{[4]}(x)}{4!}$$

$$= A\left\{\left(1+\frac{1}{8}k^2A^2-\frac{21}{8}k^4A^4+\frac{5}{12}k^6A^6\right)cos(kx)+\frac{1}{2}kA\left(1+\frac{5}{3}k^2A^2-\frac{1}{2}k^4A^4\right)cos(2kx)+\frac{3}{8}k^2A^2\left(1-\frac{7}{9}k^2A^2\right)cos(3kx)+\frac{1}{3}k^3A^3cos(4kx)+\frac{3}{2}k^5A^5\left(1+\frac{8}{9}k^2A^2\right)\right\} \tag{3.78}$$

$$\begin{aligned} C_4 &= C_0 + \frac{C_0^{[1]}}{1!} + \frac{C_0^{[2]}}{2!} + \frac{C_0^{[3]}}{3!} + \frac{C_0^{[4]}}{4!} \\ &= C_0\left(1 + \frac{1}{2}k^2A^2 + \frac{5}{8}k^4A^4 - \frac{21}{8}k^6A^6 - \frac{815}{384}k^8A^8\right) \end{aligned} \tag{3.79}$$

$$(H_w)_4 = 2A\left(1 + \frac{1}{2}k^2A^2 - \frac{35}{12}k^4A^4 + \frac{5}{12}k^6A^6\right) \tag{3.80}$$

3.4 comparison with perturbation expension solution

The stokesian waves at up-to third-order of approximation are as follows:

first-order approximation :

$$\phi_1(x,y) = C_0Ae^{ky}sin(kx) \tag{3.81}$$

$$\zeta_1(x) = Acos(kx) \tag{3.82}$$

$$C_1 = \sqrt{\frac{g}{k}} = C_0 \tag{3.83}$$

$$(H_w)_1 = 2A \tag{3.84}$$

second-order approximation :

$$\phi_2(x,y) = C_0Ae^{ky}sin(kx) \tag{3.85}$$

$$\zeta_2(x) = A\left\{cos(kx) + \frac{1}{2}kAcos(2kx)\right\} \tag{3.86}$$

$$C_2 = C_0 \tag{3.87}$$

$$(H_w)_2 = 2A \tag{3.88}$$

third-order approximation :

$$\phi_3(x,y) = C_0Ae^{ky}sin(kx) \tag{3.89}$$

$$\zeta_3(x) = A\left\{\left(1 + \frac{1}{8}k^2A^2\right)cos(kx) + \frac{1}{2}kAcos(2kx) + \frac{3}{8}k^2A^2cos(3kx)\right\} \tag{3.90}$$

$$C_3 = C_0\left(1 + \frac{1}{2}k^2A^2\right) \tag{3.91}$$

$$(H_w)_3 = 2A\left(1 + \frac{1}{2}k^2A^2\right) \tag{3.92}$$

Comparing the stokesian waves to solutions derived by Process Analysis Method, one can see that, at the same order of approximation, the process analysis solutions have higher-order items of kA. For example, the wave-velocity C at third-order of approximation given by Stokes has only k^2A^2 item, but the second-order approximate wave-velocity obtained by Process Analysis Method has not only the same item k^2A^2, but also the item k^4A^4. The wave-elevation $\zeta(x)$ and the velocity-potential $\phi(x, y)$ are similar to wave-velocity in this point.

Select wave-length $\lambda = 10.0$ meter, initial wave-amplitude $A = 0.5$ meter. The corresponding wave-elevations at up-to forth-order of approximation given in this chapter are as shown in figure 3.1. If kA is small, the solutions given by the two different methods will be in a good agreement, because the lower items of kA are the same. So, it is interesting to compare the wave-elevations given by the two methods at great enough kA. Figure 3.2 show the comparison of stokesian wave-elevation at third-order of approximation to the wave-elevation given by Process Analysis Method at third and forth-order of approximation for $\lambda = 10.0$ meter, $A = 0.7$ meter, corresponding to $kA = 0.4398$. The wave-elevation obtained by Process Analysis Metyhod has a constant item of kA which is neglected in figure 3.2.

Clearly, the ratio of wave height-to-length $\left(\frac{H_w}{L_w}\right)$ is a function of kA; in another word, kA is also a function of $\left(\frac{H_w}{L_w}\right)$. We know that $\left(\frac{H_w}{L_w}\right)$ has a limit value. Schwartz's [48] perturbation solution of deep water at 117th order of approximation gave $\left(\frac{H_w}{L_w}\right)_{max} = 0.1412$ and many other researchers obtained similar result. According to $\left(\frac{H_w}{L_w}\right)_{max}$, there exists a critical kA, denoted as $(kA)_c$. For solutions at different order of approximation, $(kA)_c$ is generally different but should tend to a definite value when the order of approximation is great enough. $(kA)_c$ can be determined by using wave-breaking condition: the fluid at crest is stagnant. For the solutions obtained by Process Analysis Method, we have

$$(kA)_c^{(1)} = 0.8258 \qquad \left(\frac{H_w}{L_w}\right)_{max}^{(1)} = 0.2628$$

$$(kA)_c^{(2)} = 0.5127 \qquad \left(\frac{H_w}{L_w}\right)_{max}^{(2)} = 0.1203$$

$$(kA)_c^{(3)} = 0.5226 \qquad \left(\frac{H_w}{L_w}\right)_{max}^{(3)} = 0.1984$$

$$(kA)_c^{(4)} = 0.4487 \qquad \left(\frac{H_w}{L_w}\right)_{max}^{(4)} = 0.1408$$

At forth-order of approximation, the maximum ratio of wave height-to-length given by Process Analysis Method is 0.1408, which is very close to that given by Schwartz.

Clearly, it should be satisfied

$$\int_0^{L_w} \zeta(x)dx = 0 \tag{3.93}$$

Let

$$h_m = \frac{\int_0^{L_w} \zeta_m(x)dx}{L_w} \tag{3.94}$$

then, for perturbation solutions, we have $h_m = 0$ $(m = 1, 2, 3, \cdots)$. For solutions given by Process Analysis Method, we have

$$h_1 = -\frac{1}{2}kA\left(\frac{A}{L}\right)$$

$$\begin{aligned}
h_2 &= \frac{1}{2}(kA)^3\left(\frac{A}{L}\right)\\
h_3 &= -(kA)^3\left[1+\frac{3}{4}(kA)^2\right]\left(\frac{A}{L}\right)\\
h_4 &= \frac{3}{2}(kA)^5\left[1+\frac{8}{9}(kA)^2\right]\left(\frac{A}{L}\right)
\end{aligned}$$

Because $kA < (kA)_c < 1$, we have reason to believe that

$$\lim_{m\to\infty} h_m = 0$$

It means that if the order of approximation is high enough, the condition

$$\begin{aligned}
&\int_0^{L_w} \zeta(x)dx\\
&= \sum_{m=1}^{\infty} \frac{\int_0^{L_w} \zeta^{[m]}(x;0)dx}{m!}
\end{aligned} \tag{3.95}$$

can be satisfied fine enough, although each item

$$\int_0^{L_w} \zeta^{[m]}(x;0)dx \neq 0 \qquad (m \geq 1)$$

3.5 discussion and conclusion

Based on continuous mapping, an analytical method for nonlinear problems is described and used to give a kind of solution at forth-order of approximation of 2D nonlinear progressive gravity waves in deep water. Comparisons to the stokesian waves show that the results given by Process Analysis Method are reasonable. At the same order of approximation, the solutions given by Process Analysis Method have not only the same items of kA as, but also the higher-order items of kA than the traditional stokesian waves. So, it would be possible that Process Analysis Method could give an approximate serie which would converge faster than stokesian waves. In order to research the convergence of these process analytical solutions of 2D nonlinear progressive gravity waves, the solutions at much higher order of approximation should be given.

It is interesting that the equations of mth-order process derivatives ($m > 0$) are linear. This is an important property of continuous mapping (homotopy). Using Taylor's expansion, the final solutions can be connected with the initial solutions by mth-order $(m \geq 1)$ process derivatives which can be obtained from corresponding linear equations. In this way, a nonlinear problem can be transformed into infinite number of linear problems about process derivatives at different order of approximation. This is the basic idea of Process Analysis Method.

Although there exist many solutions of 2D nonlinear progressive gravity waves, all of them are in the meaning of perturbation. Note that the solutions given in this chapter is different from them in this point so that they would have, perhaps, some new properties. Note that Taylor's serie should be also used to perturbation method in solution of 2D nonlinear progressive water waves. So, stokes waves are based not only on the small parameter supposition but also on the convergence of Taylor's serie. It is doubtful whether or not there exists indeed a small *enough* quantity for waves near limiting

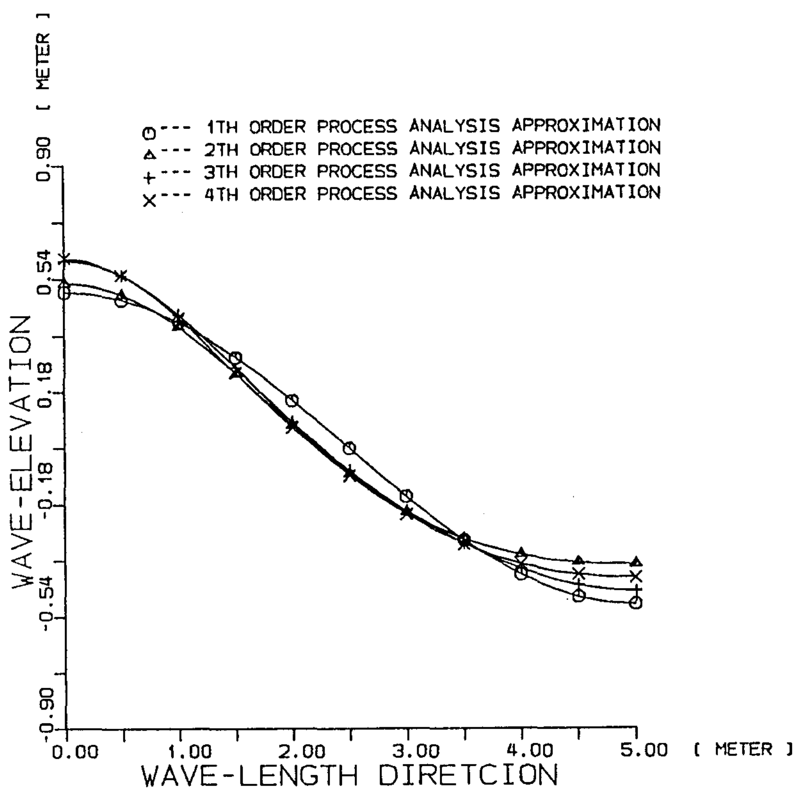

Figure 3.1: wave-elevations at up-to forth-order of approximation given by Process Analysis Method ($A = 0.5$ meter, $\lambda = 10$ meter)

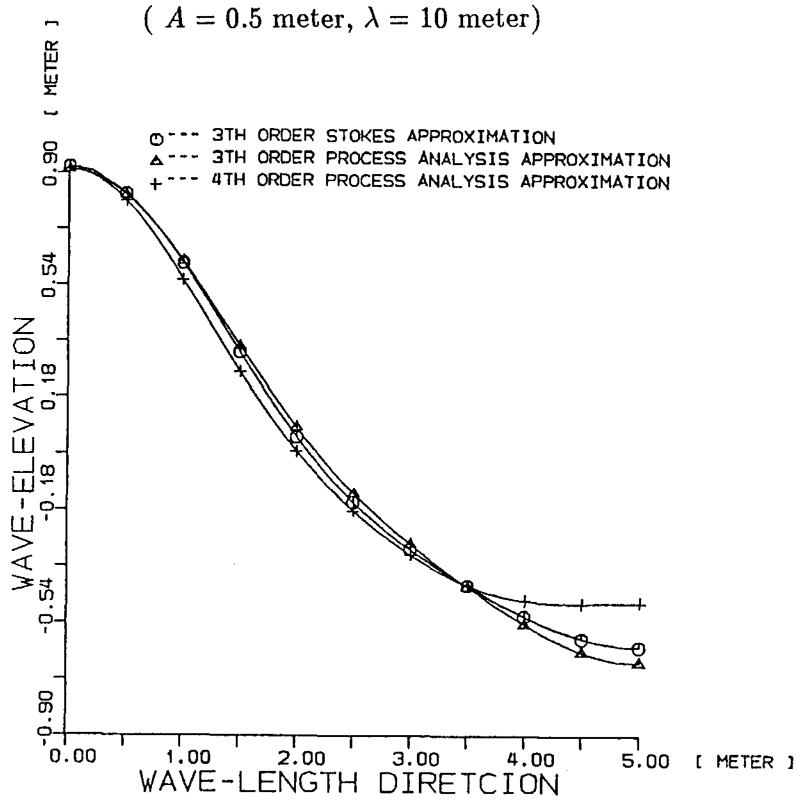

Figure 3.2: comparison of wave-elevation ($A = 0.7$ meter, $\lambda = 10$ meter)

state (we suppose that wave will break in limiting state). If it is not sure that there exists indeed such a small enough parameter in this case, then the solutions obtained by perturbation methods seem doubtful, specially in case of very strong nonlinearity[1]. Different from perturbation method, Process Analysis Method is *independent* upon small parameters, although, same as perturbation method in this special case, it is also based on the convergence of Taylor's serie. From view point of this, Process Analysis Method seems more reasonable, at least in logic, than perturbation method.

Although we have tried to compare the results given in this chapter to the traditional stokesian solutions, it seems difficult at forth-order of approximation to obtain any clear conclusions about which method or which solutions are better. The process analysis solutions at much higher order of approximation should be given in order to compare to the perturbation solutions at 117th-order of approximation given by Schwartz [48]. In spite of this, the results obtained in this chapter seems reasonable and as an attempt to overcome the limitations of perturbation method, it seems valuable to give deeper theoretical research and more applications of Process Analysis Method.

[1] Indeed, Schwartz's perturbation solutions at 117th-order of approximation converge. In chapter one, we give also a kind of perturbation solutions which *converges to the true solution* just only in a limited region $0 \le \sqrt{g\mu}t \le 1.5$. Do the perturbation solutions of 2D progressive waves converge indeed to the ture solution of it in case of strong nonlinearity? What is the region of kA in which the perturbation solutions hold ?

Chapter 4

Numerical Solution of Nonlinear Gravity Waves Part 1 : 2D Progressive Waves in Shallow Water

Summary

2D steep gravity waves in shallow water is used to describe and examine a kind of numerical method for solution of nonlinear problems, called Finite Process Method. Based on velocity potential function and Finite Process Method, a numerical scheme for 2D nonlinear gravity waves in shallow water is described, which can be expanded to solve 3D problems. The convergence is examined and the comparison to the results of other authers is given.

4.1 introduction

We have proved a kind of linearity-invariance under homotopy in appendix A and shown in chapter one, chapter two and chapter three some applications of it in obtaining analytical solutions of nonlinear problems. In this chapter, we will begin to show some applications of this linearity-invariance under homotopy in numerical computions of nonlinear problems. For simplicity, we will use in this chapter the 2D progressive nonlinear water waves in shallow water as a simple example to describe and examine some basic ideas.

For study of gravity waves, it is generally supposed that the fluid is inviscid. If the flow is irrotational, there exists a velocity potential function $\phi(x,y,z;t)$, from which the velocity components (u,v,w) can be obtained. If the fluid is also imcompressible, the velocity potential function $\phi(x,y,z;t)$ satisfies the Laplace's equation

$$\nabla^2\phi = 0$$

throughout the fluid.

Although the above governing equation of Φ is a linear Laplace's equation, its two boundary conditions on the free surface, called dynamic condition and kinematic condition of free surface, are unfornunately nonlinear, which should be satisfied on the unknown free surface. Therefore, gravity wave problems are generally nonlinear. The nonlinearity leads the difficulties of the numerical computation of these sort of problems. The special difficulty for solution of gravity waves, which is perhaps the main difficulty, is that the free surface elevation is unknown but the two nonlinear boundary conditions must be satisfied on it.

Generally, a nonlinear discretized algebraic equation with a large number of unknown quantities will be obtained after numerical discretization of a nonlinear problem. The general methods used widely to solve these kind of nonlinear algebraic equations are the iterative methods. But, the convergence of iterative methods depends greatly on not only the selection of initial values but also the number of unknown quantities. Another way is the semidirect methods which can overcome partly the limitation of iterative methods. But, semiderict methods use still the iterative techniques to solve nonlinear algebraic equations in each step of its computation. That means the semidirect method can not avoid the application of the iterative methods.

In this chapter, it is tried to use 2D gravity wave in shallow water as an example to describe a kind of numerical method of general nonlinear problems, called Finite Process Method. Finite Process Method can avoid the use of iterative methods. From view point of this, We can call it a kind of direct method of solution of nonlinear problems.

In 1979, Vanden-Broeck & Schwartz [55] gave a report about the numerical computation of 2D nonlinear gravity waves in shallow water. They used an analytical function of a complex variable z to represent the solution and obtained a nonlinear integrodifferential equation for the free surface which was solved by Newton's iterative techniques. In 1981, Rienecker & Fenton [44] researched also this problem from a different way. They applied a finite Fourier series to give a set of nonlinear equations which was also solved by Newton's iterative method. Many other authors, e.g., Chaplin [6], Chappelear [7], Cokelet [8], Dean [15], Eagleson [18], Le Méhauté [29], Longuet-Higgins [32], Stokes [50] [51] [52] and so on, have researched the 2D gravity waves experimently,theoretically or numerically. The numerical results given by different authors with different methods agree good with the theoretical results and even with the experiment results.

The works described above are excellent. If only from the view point of numerical computation of 2D gravity waves in shallow water, the present work described in this chapter would have no meaning. But, perhaps, it should be emphasized that the authors mentioned above used the Newton's iterative method to solve the corresponding nonlinear algebraic equations. On the other hand, their methods can not be applied to solve 3D gravity wave problems, e.g., to compute the wave resistance of a ship moving with a constant velocity in shallow water.

The purpose of present work is not mainly to discuss more deeply the numerical method of 2D gravity waves in shallow water, but to describe a new kind of numerical method which can avoid the use of iterative methods.

4.2 mathematical description

The problem considered here is that of 2D periodic waves moving without change of form over a layer of fluid on a horizontal bed. With horizontal co-ordinate x and vertical co-ordinate y, the origin is on the bed and moves with the same velocity C as the waves. Therefore, all motion is steady in the frame of reference. Let D denote the average depth of the water.

Suppose the flow is irrotational. Then, there exists a velocity potential function $\phi(x, y)$ such that the velocity components (u, v) can be given by

$$u = \frac{\partial \phi}{\partial x} \qquad , \qquad v = \frac{\partial \phi}{\partial y} \qquad .$$

Suppose the fluid is imcompressible. Then, $\phi(x, y)$ satisfies Laplace's equation in the field:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{in} \qquad \Omega$$

Suppose the fluid is inviscid and without surface tensen. Then, on the free surface, $\phi(x, y)$ satisfies :

$$C^2 \phi_{xx} + g\phi_y + \frac{1}{2}\nabla\phi\nabla(\nabla\phi\,\nabla\phi) - 2C\nabla\phi\,\nabla\phi_x = 0 \qquad , \qquad \text{on} \qquad y = \zeta(x) \qquad (4.1)$$

where

$$\zeta(x) = \frac{1}{g}\left(C\,\phi_x - \frac{1}{2}\nabla\,\phi\nabla\,\phi\right) \qquad , \qquad \text{on } y = \zeta(x) \tag{4.2}$$

Here, g is gravity acceleration; $\zeta(x)$ denotes the wave elevation, C is wave-velocity.

The expression (4.2) is the dynamic condition of free surface. The expression (4.1) is obtained from the dynamic and kinematic conditions of free surface, where, the relation

$$\frac{\partial}{\partial t} = -C\,\frac{\partial}{\partial x}$$

is used.

On the bed, $\phi(x, y)$ satisfies :

$$\frac{\partial\,\phi}{\partial\,y} = 0 \qquad , \qquad \text{on } y = 0 \tag{4.3}$$

For simplicity in deriving, define

$$\begin{aligned}
\mathcal{R}\left[\phi(x,y)\right] &= C^2\phi_{xx} + g\phi_y + \frac{1}{2}\nabla\phi\nabla(\nabla\phi\,\nabla\phi) - 2C\nabla\phi\,\nabla\phi_x \\
&= C^2\phi_{xx} + g\phi_y + (\phi_x)^2\,\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + (\phi_y)^2\,\phi_{yy} \\
&\quad -2C\left(\phi_x\phi_{xx} + \phi_y\phi_{yx}\right) \qquad , \qquad \text{on} \qquad y = \zeta(x) \qquad (4.4) \\
\mathcal{Z}\left[\phi(x,y)\right] &= \frac{1}{g}\left(C\,\phi_x - \frac{1}{2}\nabla\,\phi\nabla\,\phi\right) \\
&= \frac{1}{g}\left[C\phi_x - \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right)\right] \qquad , \qquad \text{on} \qquad y = \zeta(x) \qquad (4.5)
\end{aligned}$$

Then, the problem considered can be described as :

$$\nabla^2\phi(x, y) = 0 \qquad \text{in} \qquad \Omega \tag{4.6}$$

with boundary conditions:

$$\mathcal{R}\left[\phi(x,y)\right] = 0 \qquad , \qquad \text{on } y = \zeta(x) \tag{4.7}$$

$$\frac{\partial\,\phi}{\partial\,y} = 0 \qquad , \qquad \text{on } y = 0 \tag{4.8}$$

where

$$\zeta(x) = \mathcal{Z}\left[\phi(x,y)\right] \qquad , \qquad \text{on } y = \zeta(x) \tag{4.9}$$

4.3 basic idea of Finite Process Method

A continuous mapping (homotopy) can be constructed as follows:

$$\nabla^2\phi(x, y; p) = 0 \quad , \quad \text{in } \Omega(p) \tag{4.10}$$

with boundary conditions:

$$f_1(p)\mathcal{R}\left[\phi(x,y;p)\right] + f_2(p)\left\{\mathcal{R}\left[\phi(x,y;p)\right] - \mathcal{R}\left[\phi_0(x,y)\right]\right\} = 0 \quad , \quad \text{on } y = \zeta(x;p) \tag{4.11}$$

$$\frac{\partial\phi(x,y;p)}{\partial y} = 0 \quad , \quad \text{on } y = 0 \tag{4.12}$$

Where:

$$\zeta(x;p) = f_1(p)\mathcal{Z}[\phi(x,y;p)] + f_2(p)\zeta_0(x) \quad , \quad \text{on } y = \zeta(x;p) \tag{4.13}$$

Here, $f_1(p), f_2(p)$ are respectively the first-sort and second-sort of process function defined in section 1.1, p is the process independent variable or called embedding variable, $\zeta_0(x)$ is a freely selected initial wave-elevation, ϕ_0 is a freely selected initial velocity-potential,which should satisfy :

$$\nabla^2\phi_0(x,y) = 0 \quad , \quad \text{in } \Omega_0 \tag{4.14}$$

with boundary conditions:

$$\frac{\partial\phi_0}{\partial y} = 0 \quad , \quad \text{on } y = 0 \tag{4.15}$$

where,the initial definition domain Ω_0 of fluid field is defined by $y = \zeta_0(x)$ and $y = 0$.

$\phi(x,y;p)$ and $\zeta(x;p)$ are zero-order process and equation (4.10)~(4.13) are zero-order process equation.

When $p = 0$,from the zero-order process equation (4.10)~(4.13), one can obtain the initial equation:

$$\nabla^2\phi(x,y;0) = 0 \quad , \quad \text{in } \Omega_0 \tag{4.16}$$

with boundary conditions:

$$\mathcal{R}(\phi) = \mathcal{R}(\phi_0) \quad , \quad \text{on } y = \zeta_0(x) \tag{4.17}$$

$$\frac{\partial\phi}{\partial y} = 0 \quad , \quad \text{on } y = 0 \tag{4.18}$$

where

$$\zeta(x;0) = \zeta_0(x) \tag{4.19}$$

One can see that the initial velocity-potential $\phi_0(x,y)$ which satisfies the linear equation (4.14) and (4.15) is just the solution of the nonlinear initial equation (4.16)~(4.19),i.e.,

$$\phi(x,y;0) = \phi_0(x,y).$$

When $p = 1.0$,from the zero-order process equation (4.10)~(4.13), one can obtain the final equation :

$$\nabla^2\phi(x,y;1) = 0 \quad , \quad \text{in } \Omega_f \tag{4.20}$$

with boundary conditions:

$$\mathcal{R}[\phi(x,y;1)] = 0 \quad , \quad \text{on } y = \zeta(x;1) \tag{4.21}$$

$$\frac{\partial\phi(x,y;1)}{\partial y} = 0 \quad , \quad \text{on } y = 0 \tag{4.22}$$

where

$$\zeta(x;1) = \mathcal{Z}[\phi(x,y;1)] \quad , \quad \text{on } y = \zeta(x;1) \tag{4.23}$$

The final equation described by (4.20)~(4.23) is the same as the original equation (4.6)~(4.9) described in section 4.2, which are just what one wants to solve. Here,$\phi(x,y;1)$ are final velocity-potential,denoted as $\phi_f(x,y)$;$\zeta(x;1)$ are the final wave-elevation, denoted as $\zeta_f(x)$.

From above analysis, one can see that the zero-order process equation (4.10)~(4.13) connects the freely selected initial velocity potential $\phi_0(x,y)$ and initial wave-elevation $\zeta_0(x)$ with solutions of the original equation (4.6)~(4.9) described in section 4.2. The change process of variable p from zero to one is just the process of the transformation from the selected initial solution to the final solution. The zero-order process equation (4.10) ~(4.13) is a special continuous mapping which gives a kind of relation between the selected initial solution $\phi_0(x,y),\zeta_0(x)$ and the final solution $\phi_f(x,y),\zeta_f(x)$. But, this kind of relation is also nonlinear to a nonlinear problems. In the following part of this section, we would try to give a linear relation between initial solution $\phi_0(x,y),\zeta_0(x)$ and final solution $\phi_f(x,y),\zeta_f(x)$ by means of using the concept *process derivatives.*

Define

$$\phi^{[1]}(x,y;p) = \frac{\partial\phi(x,y;p)}{\partial p} \tag{4.24}$$

$$\zeta^{[1]}(x;p) = \frac{\partial\zeta(x;p)}{\partial p} \tag{4.25}$$

be the first-order process derivatives of $\phi(x,y;p)$ and $\zeta(x;p)$.Then,the freely selected initial solutions $\phi_0(x,y),\zeta_0(x)$ and the final solutions $\phi_f(x,y),\zeta_f(x)$ have the following integral relation:

$$\begin{aligned}\phi_f(x,y) &= \phi(x,y;1)\\ &= \phi(x,y;0) + \int_0^1 \phi^{[1]}(x,y;p)dp\\ &= \phi_0(x,y) + \int_0^1 \phi^{[1]}(x,y;p)dp \end{aligned}\tag{4.26}$$

$$\begin{aligned}\zeta_f(x) &= \zeta(x;1)\\ &= \zeta(x;0) + \int_0^1 \zeta^{[1]}(x;p)dp\\ &= \zeta_0(x) + \int_0^1 \zeta^{[1]}(x;p)dp \end{aligned}\tag{4.27}$$

The equations of $\phi^{[1]}(x,y;p)$ and $\zeta^{[1]}(x;p)$ can be obtained from the zero-order process equation (4.10)~(4.13) in the following way.

From (4.10) and (4.12),one has:

$$\frac{\partial}{\partial p}\left(\nabla^2\phi\right) = \nabla^2\phi^{[1]}(x,y;p) = 0 \quad , \quad \text{in } \Omega(p) \tag{4.28}$$

$$\frac{\partial}{\partial p}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial\phi^{[1]}(x,y;p)}{\partial y} = 0 \quad , \quad \text{on } y = 0 \tag{4.29}$$

The dynamic and kinematic conditions of free surface must be satisfied on the wave-elevation $y = \zeta(x;p)$,which is also a function of process independent variable p. Therefore, deriving (4.11) and

(4.13) with respect to process independent variable p, one can obtain :

$$\begin{aligned} & f_1'(p)\mathcal{R}[\phi(x,y;p)] + f_2'(p)\left\{\mathcal{R}[\phi(x,y;p)] - \mathcal{R}[\phi_0(x,y)]\right\} \\ + & \quad \{f_1(p)+f_2(p)\}\,\frac{d\,\mathcal{R}[\phi(x,y;p)]}{d\,p} = 0 \ , \ \text{on } y=\zeta(x;p) \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} \zeta^{[1]}(x;p) &= f_1'(p)\,\mathcal{Z}[\phi(x,y;p)] + f_1(p)\frac{d\mathcal{Z}[\phi(x,y;p)]}{dp} + f_2'(p)\,\zeta_0(x) \\ & \quad \text{on } y=\zeta(x;p) \end{aligned} \tag{4.31}$$

On wave-elevation $y=\zeta(x;p)$, one has:

$$\frac{d\mathcal{R}}{dp} = \frac{\partial\mathcal{R}}{\partial p} + \zeta^{[1]}\frac{\partial\mathcal{R}}{\partial y} \quad , \quad \text{on } y=\zeta(x;p) \tag{4.32}$$

$$\frac{d\mathcal{Z}}{dp} = \frac{\partial\mathcal{Z}}{\partial p} + \zeta^{[1]}\frac{\partial\mathcal{Z}}{\partial y} \quad , \quad \text{on } y=\zeta(x;p) \tag{4.33}$$

Substituting (4.33) into (4.31),one has:

$$\zeta^{[1]}(x;p) = \frac{f_1'(p)\mathcal{Z}[\phi(x,y;p)] + f_2'(p)\zeta_0(x) + f_1(p)\frac{\partial\mathcal{Z}[\phi(x,y;p)]}{\partial p}}{1 - f_1(p)\frac{\partial\mathcal{Z}[\phi(x,y;p)]}{\partial y}} \ , \ \text{on } y=\zeta(x;p) \tag{4.34}$$

For simplicity,define

$$\mathcal{S}[\phi(x,y;p);p] = \frac{\frac{\partial\mathcal{R}[\phi(x,y;p)]}{\partial y}}{1 - f_1(p)\frac{\partial\mathcal{Z}[\phi(x,y;p)]}{\partial y}} \ , \ \text{on } y=\zeta(x;p) \tag{4.35}$$

Substituting (4.32) into (4.30) and using (4.34), (4.35), one has:

$$\frac{\partial\mathcal{R}[\phi(x,y;p)]}{\partial p} + f_1(p)\mathcal{S}[\phi(x,y;p);p]\frac{\partial\mathcal{Z}[\phi(x,y;p)]}{\partial p} = \mathcal{T}[\phi(x,y;p);p] \quad , \quad \text{on } y=\zeta(x;p) \tag{4.36}$$

where,

$$\begin{aligned} & \mathcal{T}[\phi(x,y;p);p] \\ = & \ -\frac{\{f_1'(p)+f_2'(p)\}\,\mathcal{R}[\phi(x,y;p)] - f_2'(p)\mathcal{R}[\phi_0(x,y)]}{f_1(p)+f_2(p)} \\ & -\mathcal{S}[\phi(x,y;p);p]\left\{f_1'(p)\mathcal{Z}[\phi(x,y;p)] + f_2'(p)\zeta_0(x)\right\} \ , \ \text{on } y=\zeta(x;p) \end{aligned} \tag{4.37}$$

According to (4.4), one has :

$$\begin{aligned} \frac{\partial\mathcal{R}}{\partial p} &= \frac{\partial\mathcal{R}}{\partial\phi_x}\frac{\partial\phi_x}{\partial p} + \frac{\partial\mathcal{R}}{\partial\phi_y}\frac{\partial\phi_y}{\partial p} + \frac{\partial\mathcal{R}}{\partial\phi_{xx}}\frac{\partial\phi_{xx}}{\partial p} + \frac{\partial\mathcal{R}}{\partial\phi_{xy}}\frac{\partial\phi_{xy}}{\partial p} + \frac{\partial\mathcal{R}}{\partial\phi_{yy}}\frac{\partial\phi_{yy}}{\partial p} \\ &= \frac{\partial\mathcal{R}}{\partial\phi_x}\phi_x^{[1]} + \frac{\partial\mathcal{R}}{\partial\phi_y}\phi_y^{[1]} + \frac{\partial\mathcal{R}}{\partial\phi_{xx}}\phi_{xx}^{[1]} + \frac{\partial\mathcal{R}}{\partial\phi_{xy}}\phi_{xy}^{[1]} + \frac{\partial\mathcal{R}}{\partial\phi_{yy}}\phi_{yy}^{[1]} \ , \ \text{on } y=\zeta(x;p) \end{aligned} \tag{4.38}$$

In the same way, according to (4.5), one can obtain :

$$\begin{aligned}\frac{\partial \mathcal{Z}}{\partial p} &= \frac{\partial \mathcal{Z}}{\partial \phi_x}\frac{\partial \phi_x}{\partial p} + \frac{\partial \mathcal{Z}}{\partial \phi_x}\frac{\partial \phi_x}{\partial p} \\ &= \frac{\partial \mathcal{Z}}{\partial \phi_x}\phi_x^{[1]} + \frac{\partial \mathcal{Z}}{\partial \phi_y}\phi_y^{[1]} \quad , \quad \text{on } y = \zeta(x;p) \end{aligned} \tag{4.39}$$

Thus, substituting (4.38),(4.39) into (4.36), one has :

$$\begin{aligned} &\frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial p} + f_1(p)\mathcal{S}[\phi(x,y;p);p]\frac{\partial \mathcal{Z}[\phi(x,y;p)]}{\partial p} \\ =& \left\{\frac{\partial \mathcal{R}}{\partial \phi_x} + f_1(p)\mathcal{S}[\phi(x,y;p);p]\frac{\partial \mathcal{Z}}{\partial \phi_x}\right\}\phi_x^{[1]} + \left\{\frac{\partial \mathcal{R}}{\partial \phi_y} + f_1(p)\mathcal{S}[\phi(x,y;p);p]\frac{\partial \mathcal{Z}}{\partial \phi_y}\right\}\phi_y^{[1]} \\ & \left(\frac{\partial \mathcal{R}}{\partial \phi_{xx}} - \frac{\partial \mathcal{R}}{\partial \phi_{yy}}\right)\phi_{xx}^{[1]} + \frac{\partial \mathcal{R}}{\partial \phi_{xy}}\phi_{xy}^{[1]} \\ =& \mathcal{T}[\phi(x,y;p);p] \quad , \quad \text{on } y = \zeta(x;p) \end{aligned} \tag{4.40}$$

Where,the relation

$$\phi_{xx}^{[1]} + \phi_{yy}^{[1]} = 0 \quad , \quad \text{in } \Omega(p) \tag{4.41}$$

has been used.

Therefore, according to (4.28), (4.29), (4.34) and (4.40), the $\phi^{[1]}(x,y;p)$ and $\zeta^{[1]}(x;p)$ satisfy the following partial-differential equation:

$$\nabla^2\phi^{[1]}(x,y;p) = 0 \quad , \quad \text{in } \Omega(p) \tag{4.42}$$

with boundary conditions:

$$\begin{aligned} & \left\{\frac{\partial \mathcal{R}}{\partial \phi_x} + f_1(p)\mathcal{S}[\phi(x,y;p);p]\frac{\partial \mathcal{Z}}{\partial \phi_x}\right\}\phi_x^{[1]} + \left\{\frac{\partial \mathcal{R}}{\partial \phi_y} + f_1(p)\mathcal{S}[\phi(x,y;p);p]\frac{\partial \mathcal{Z}}{\partial \phi_y}\right\}\phi_y^{[1]} \\ & + \left\{\frac{\partial \mathcal{R}}{\partial \phi_{xx}} - \frac{\partial \mathcal{R}}{\partial \phi_{yy}}\right\}\phi_{xx}^{[1]} + \frac{\partial \mathcal{R}}{\partial \phi_{xy}}\phi_{xy}^{[1]} \\ & = \mathcal{T}[\phi(x,y;p);p] \quad , \quad \text{on } y = \zeta(x;p) \end{aligned} \tag{4.43}$$

$$\frac{\partial \phi^{[1]}(x,y;p)}{\partial y} = 0 \ , \ \text{on } y = 0 \tag{4.44}$$

where

$$\zeta^{[1]}(x;p) = \frac{f_1'(p)\mathcal{Z}[\phi(x,y;p)] + f_2'(p)\zeta_0(x) + f_1(p)\frac{\partial \mathcal{Z}[\phi(x,y;p)]}{\partial p}}{1 - f_1(p)\frac{\partial \mathcal{Z}[\phi(x,y;p)]}{\partial y}} \quad , \quad \text{on } y = \zeta(x;p) \tag{4.45}$$

The above equation (4.42)~(4.45) could be called first-order process equation.

It is very interesting that the equation (4.42)~(4.44) is a linear partial-differential equation with respect to $\phi^{[1]}(x,y;p)$. If $\phi(x,y;p), \zeta(x;p)$ are known,$\phi^{[1]}(x,y;p)$ and $\zeta^{[1]}(x;p)$ can be obtained by solving this linear partial differential equation. Then, according to (4.26) and (4.27), one has :

$$\begin{aligned} \phi(x,y;p+\Delta p) &= \phi(x,y;p) + \phi^{[1]}(x,y;p)\Delta p \\ \zeta(x;p+\Delta p) &= \zeta(x;p) + \zeta^{[1]}(x;p)\Delta p \end{aligned}$$

In this way, one can obtain the final solution $\phi_f(x,y)$ and $\zeta_f(x)$. Generally, the first order process equation is linear,although the zero-order process is nonlinear. This is the most important property of continuous mapping (homotopy). A pure mathematical proof of this property has been given in Appendix A.

If we solve directly the nonlinear zero-order process equation (4.10)∼(4.13), it seems difficult to avoid using iterative methods. But, by solving the n_p *linear* first-order process equation (4.42)∼(4.45) of $\phi^{[1]}(x,y;p)$ and $\zeta^{[1]}(x;p)$ ($n_p \geq 1, \Delta p = 1/n_p$), we can obtain the solution of original equation (4.6)∼(4.9) without using iterative methods. In another word, a nonlinear problem can be transformed into a complex system of linear problem by means of using the concept *process-derivatives*.

From (4.4) and (4.5), One can obtain :

$$\frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_x} = 2\left\{(\phi_x - C)\,\phi_{xx} + \phi_y\phi_{xy}\right\} \tag{4.46}$$

$$\frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_y} = g + 2\left\{(\phi_x - C)\,\phi_{xy} + \phi_y\phi_{yy}\right\} \tag{4.47}$$

$$\frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_{xx}} = (\phi_x - C)^2 \tag{4.48}$$

$$\frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_{xy}} = 2\,(\phi_x - C)\,\phi_y \tag{4.49}$$

$$\frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_{yy}} = \phi_y^2 \tag{4.50}$$

$$\frac{\partial \mathcal{Z}[\phi(x,y;p)]}{\partial \phi_x} = -\frac{1}{g}\,(\phi_x - C) \tag{4.51}$$

$$\frac{\partial \mathcal{Z}[\phi(x,y;p)]}{\partial \phi_y} = -\frac{1}{g}\phi_y \tag{4.52}$$

$$\tag{4.53}$$

$$\begin{aligned}\frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial y} &= \frac{\partial \mathcal{R}}{\partial \phi_x}\phi_{xy} + \frac{\partial \mathcal{R}}{\partial \phi_y}\phi_{yy} + \frac{\partial \mathcal{R}}{\partial \phi_{xx}}\phi_{xxy} \\ &\quad + \frac{\partial \mathcal{R}}{\partial \phi_{xy}}\phi_{xyy} + \frac{\partial \mathcal{R}}{\partial \phi_{yy}}\phi_{yyy} \\ &= \frac{\partial \mathcal{R}}{\partial \phi_x}\phi_{xy} - \frac{\partial \mathcal{R}}{\partial \phi_y}\phi_{xx} + \left(\frac{\partial \mathcal{R}}{\partial \phi_{xx}} - \frac{\partial \mathcal{R}}{\partial \phi_{yy}}\right)\phi_{xxy} + \frac{\partial \mathcal{R}}{\partial \phi_{xy}}\phi_{xyy} \qquad (4.54)\\ \frac{\partial \mathcal{Z}[\phi(x,y;y)]}{\partial y} &= \frac{\partial \mathcal{Z}}{\partial \phi_x}\phi_{xy} + \frac{\partial \mathcal{Z}}{\partial \phi_y}\phi_{yy} \\ &= \frac{\partial \mathcal{Z}}{\partial \phi_x}\phi_{xy} - \frac{\partial \mathcal{Z}}{\partial \phi_y}\phi_{xx} \qquad (4.55)\end{aligned}$$

The process of numerical computation can be described simply as follows:

I select the initial wave-elevation $\zeta_0(x)$ and initial velocity-potential $\phi_0(x,y)$

II let $p_i = i\Delta p$ for $i = 0,1,2,\cdots,n_p-1$,where,$\Delta p = \frac{1}{n_p}$;

III solve the linear partial-differential equation described by (4.42) ∼(4.44) to obtain the value of $\phi^{[1]}(x,y;p_i)$;

IV compute the value of $\zeta^{[1]}(x;p_i)$ by means of the expression (4.34);

V compute the value of wave-elevation and velocity-potential in next step p_{i+1}:

$$\begin{aligned} \phi(x,y;p_{i+1}) &= \phi(x,y;p_i) + \phi^{[1]}(x,y;p_i)\Delta p \qquad (4.56) \\ \zeta(x;p_{i+1}) &= \zeta(x;p_i) + \zeta^{[1]}(x;p_i)\Delta p \qquad (4.57) \end{aligned}$$

VI if $p_{i+1} < 1.0$,then,go back to step II.

Here, the definition region $p \in [0,1]$ of process independent variable p is discretized. That is why this method would be called Finite Process Method.

One can see that in the above numerical computation, only n_p linear partial-differential equations (4.42)~(4.44) need be solved $(n_p \geq 1, \Delta p = 1/n_p)$. It means that the original nonlinear problem can be transformed into a set of n_p linear problems $(n_p \geq 1)$.Theoretically, by means of large or huge computer, almost all linear problems could be solved without great difficulties. Thus, from the view point of Finite Process Method, theoretically, it seems that a nonlinear problem could be also solved by large or huge computers, because a nonlinear problem can be approximated by a set of n_p linear problems $(n_p \geq 1)$. In another word, a nonlinear problem is just a very complex linear one, if from the view point of Finite Process Method.

In the following sections, the detailed numerical formulas and numerical results of 2D progressive gravity waves in shallow water will be given.It should be emphasized that the numerical method described here can be also used to solve problems of 3D gravity waves,e.g.,the wave-resistance of a ship moving in a deep or shallow water,if only a simple expansion is made and Hess-Smith's method is used.

4.4 numerical solution of gravity waves in shallow water

4.4.1 numerical formulae

According to the deriving of the above section,the zero-order process equation of velocity potential function $\phi(x,y)$ of 2D progressive gravity waves in shallow water can be described as follows:

$$\nabla^2\phi(x,y;p) = 0 \quad , \quad \text{in } \Omega(p) \qquad (4.58)$$

with boundary conditions:

$$\begin{aligned} f_1(p)\mathcal{R}[\phi(x,y;p)] + f_2(p)\{\mathcal{R}[\phi(x,y;p)] - \mathcal{R}[\phi_0(x,y)]\} &= 0 \quad , \quad \text{on } y = \zeta(x;p) \qquad (4.59) \\ \frac{\partial\phi(x,y;p)}{\partial y} &= 0 \quad , \quad \text{on } y = 0 \qquad (4.60) \end{aligned}$$

where

$$\zeta(x;p) = f_1(p)\mathcal{Z}(x,y;p) + f_2(p)\zeta_0(x) \qquad \text{on } y = \zeta(x;p) \qquad (4.61)$$

And the first-order process equation of gravity waves in shallow water is as follows:

$$\nabla^2\phi^{[1]}(x,y;p) = 0 \quad , \quad \text{in } \Omega(p) \qquad (4.62)$$

with boundary conditions:

$$\begin{aligned} & b_1(x,y;p)\phi_x^{[1]}(x,y;p) + b_2(x,y;p)\phi_y^{[1]}(x,y;p) \\ + & b_3(x,y;p)\phi_{xx}^{[1]}(x,y;p) + b_4(x,y;p)\phi_{xy}^{[1]}(x,y;p) \\ = & \mathcal{T}\left[\phi(x,y;p);p\right] \quad , \quad \text{on } y = \zeta(x;p) \end{aligned} \tag{4.63}$$

$$\frac{\partial \phi^{[1]}(x,y;p)}{\partial y} = 0 \quad , \quad \text{on } y = 0 \tag{4.64}$$

where

$$b_1(x,y;p) = \frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_x} + f_1(p)\mathcal{S}\left[\phi(x,y;p);p\right]\frac{\partial \mathcal{Z}[\phi(x,y;p)]}{\partial \phi_x} \quad , \quad \text{on } y = \zeta(x;p) \tag{4.65}$$

$$b_2(x,y;p) = \frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_y} + f_1(p)\mathcal{S}\left[\phi(x,y;p);p\right]\frac{\partial \mathcal{Z}[\phi(x,y;p)]}{\partial \phi_y} \quad , \quad \text{on } y = \zeta(x;p) \tag{4.66}$$

$$b_3(x,y;p) = \frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_{xx}} - \frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_{yy}} \quad , \quad \text{on } y = \zeta(x;p) \tag{4.67}$$

$$b_4(x,y;p) = \frac{\partial \mathcal{R}[\phi(x,y;p)]}{\partial \phi_{xy}} \quad , \quad \text{on } y = \zeta(x;p) \tag{4.68}$$

and $\zeta^{[1]}(x;p)$ satisfies:

$$\zeta^{[1]}(x;p) = \frac{f_1'(p)\mathcal{Z}[\phi(x,y;p)] + f_2'(p)\zeta_0(x) + f_1(p)\frac{\partial \mathcal{Z}[\phi(x,y;p)]}{\partial p}}{1 - f_1(p)\frac{\partial \mathcal{Z}[\phi(x,y;p)]}{\partial y}} \quad , \quad \text{on } y = \zeta(x;p) \tag{4.69}$$

Let $k = \frac{2\pi}{L_w}$ denote the wave-number.Then,it is easy to understand that:

$$\phi(x,y;p) = \sum_{j=1}^{\infty} a_j(p)\varphi_j(x,y) \tag{4.70}$$

with

$$\varphi_j(x,y) = \frac{cosh(jky)sin(jkx)}{cosh(jkD)} \tag{4.71}$$

satisfies the Laplace's equation (4.58) and the bed condition (4.60).Here,$a_j(p)(j = 1,2,\cdots)$ are just only dependent on the process independent variable p.

Then,from (4.70),one has:

$$\begin{aligned} \frac{\partial \phi(x,y;p)}{\partial p} &= \phi^{[1]}(x,y;p) \\ &= \sum_{j=1}^{\infty} \frac{\partial a_j(p)}{\partial p}\varphi_j(x,y) \\ &= \sum_{j=1}^{\infty} a_j^{[1]}(p)\varphi_j(x,y) \end{aligned} \tag{4.72}$$

It can be proved that $\phi^{[1]}(x,y;p)$ described above satisfies also the Laplace's equation (4.62) and the corresponding bed condition (4.64).

In practice,only finite terms of $\varphi_j(x,y)$ can be selected.Similar to method of Rienecker & Fenton [44], this is the only approximation made in this numerical scheme. Suppose M terms of $\varphi_j(x,y)$ $(j = 1, 2, \cdots, M)$ are selected. Then, one has :

$$\phi(x,y;p) = \sum_{j=1}^{M} a_j(p)\varphi_j(x,y) \tag{4.73}$$

$$\phi^{[1]}(x,y;p) = \sum_{j=1}^{M} a_j^{[1]}(p)\varphi_j(x,y) \tag{4.74}$$

For simplicity,define

$$\begin{aligned} \mathbf{a}(p) &= \{a_j(p)\} \\ \mathbf{a}^{[1]}(p) &= \left\{a_j^{[1]}(p)\right\} \\ & \qquad (j = 1, 2, \cdots, M) \end{aligned}$$

Substituting (4.73) and (4.74) into equation (4.63), one can obtain the equation of the first-order process derivatives $\mathbf{a}^{[1]}(p)$:

$$\begin{aligned} \sum_{j=1}^{M} \quad & \{b_1(x,y,\mathbf{a};p)(\varphi_j)_x + b_2(x,y,\mathbf{a};p)(\varphi_j)_y \\ & + b_3(x,y,\mathbf{a};p)(\varphi_j)_{xx} + b_4(x,y,\mathbf{a};p)(\varphi_j)_{xy}\}\, a_j^{[1]} \\ & = \mathcal{T}(x,y,\mathbf{a};p) \quad , \quad \text{on } y = \zeta(x;p) \end{aligned} \tag{4.75}$$

where

$$(\varphi_j)_x = \frac{(jk)cosh(jky)cos(jkx)}{cosh(jkD)} \tag{4.76}$$

$$(\varphi_j)_y = \frac{(jk)sinh(jky)sin(jkx)}{cosh(jkD)} \tag{4.77}$$

$$(\varphi_j)_{xx} = \frac{-(jk)^2cosh(jky)sin(jkx)}{cosh(jkD)} \tag{4.78}$$

$$(\varphi_j)_{xy} = \frac{(jk)^2sinh(jky)cos(jkx)}{cosh(jkD)} \tag{4.79}$$

Select $\mathbf{a}_0 = \{\overline{a}_j\}$,$(j = 1, 2, \cdots, M)$ as the initial value of $\mathbf{a}(p)$, where,

$$\overline{a}_1 = \frac{H_{w0}}{2}\sqrt{\frac{g}{k}}cosh(kD) \tag{4.80}$$

$$\overline{a}_j = 0 \qquad \text{for } j = 2, 3, \cdots, M \tag{4.81}$$

then,one has the initial velocity-potential $\phi_0(x,y)$:

$$\begin{aligned} \phi_0(x,y) &= \phi(x,y;0) \\ &= \overline{a}_1\varphi_1(x,y) \end{aligned} \tag{4.82}$$

The initial wave-elevation $\zeta_0(x)$ can be given in the following form:

$$\zeta_0(x) = \frac{H_{w0}}{2} cos(kx) \tag{4.83}$$

In order to obtain the value of $\mathbf{a}^{[1]}(p) = \left\{a_j^{[1]}(p)\right\}$ $(j = 1, 2, \cdots, M)$ from the linear equation (4.75),M points (x_i, ζ_i) $(i = 1, 2, \cdots, M)$ on the free-surface $y = \zeta(x; p)$ should be given. Owing to the symmetry, one can distribute these M points on the region $[\frac{L_w}{2}, L_w]$ in the following way :

$$\begin{aligned} x_j &= \frac{L_w}{2}\left\{1 + \left(\frac{j}{M-1}\right)^\alpha\right\} \\ y_j &= \frac{H_{w0}}{2} cos(kx_j) \\ & \qquad (j = 0, 1, 2, \cdots, M-1) \end{aligned}$$

where, L_w is the wave length.Here, parameter α is used to determine the distribution form of the M points. When $\alpha = 1$, the M points are equally spaced in the horizontal direction. When $\alpha < 1$, points are spaced coarsely near the crest ($x = L_w$), similar to the Vanden-Broeck & Schwartz [55] method.

Then,according to (4.75), a linear equation with M unknown quantities $a_j^{[1]}(p_m)(j = 1, 2, \cdots, M)$ for $p_m = m\Delta p$ can be obtained:

$$\mathbf{E}(p_m)\mathbf{a}^{[1]}(p_m) = \mathbf{t}(p_m) \tag{4.84}$$

where

$$\mathbf{t}(p_m) = \{\mathcal{T}_i(x_i, y_i, \mathbf{a}(p_m); p_m)\} \quad (i = 1, 2, \cdots, M) \tag{4.85}$$

$$\mathbf{E}(p_m) = [e_{ij}(x, y; p_m)] \tag{4.86}$$

with

$$\begin{aligned} e_{ij}(x_i, y_i; p_m) &= b_1[x_i, y_i, \mathbf{a}(p_m); p_m][\varphi_j(x_i, y_i)]_x + b_2[x_i, y_i, \mathbf{a}(p_m); p_m][\varphi_j(x_i, y_i)]_y \\ & \quad b_3[x_i, y_i, \mathbf{a}(p_m); p_m][\varphi_j(x_i, y_i)]_{xx} + b_4[x_i, y_i, \mathbf{a}(p_m); p_m][\varphi_j(x_i, y_i)]_{xy} \\ & \quad (i, j = 1, 2, \cdots, M) \end{aligned} \tag{4.87}$$

If $\mathbf{a}^{[1]}(p_m)$ is known, one has:

$$\mathbf{a}(p_m + \Delta p) = \mathbf{a}(p_m) + \mathbf{a}^{[1]}(p_m)\Delta p \quad \text{for } p_m + \Delta p \le 1.0 \tag{4.88}$$

and

$$\phi^{[1]}\left[x, y, \mathbf{a}^{[1]}(p_m); p_m\right] = \sum_{j=1}^{M} a_j^{[1]}(p_m)\varphi_j(x, y) \quad \text{for } p_m + \Delta p \le 1.0 \tag{4.89}$$

Then, $\zeta^{[1]}(x_i; p_m)$ can be obtained :

$$\begin{aligned} \zeta^{[1]}(x_i; p_m) &= \frac{f_1'(p_m)\mathcal{Z}[x_i, y_i, \mathbf{a}(p_m)] + f_2'(p_m)\zeta_0(x_i) + f_1(p_m)\frac{\partial \mathcal{Z}[x_i, y_i, \mathbf{a}(p_m)]}{\partial p}}{1 - f_1(p_m)\frac{\partial \mathcal{Z}[x_i, y_i, \mathbf{a}(p_m)]}{\partial y}} \\ & \qquad \text{on } y_i = \zeta(x_i; p_m) \end{aligned} \tag{4.90}$$

where,

$$\begin{aligned}
\frac{\partial \mathcal{Z}\left[x_i, y_i, \mathbf{a}(p_m)\right]}{\partial p} &= \frac{\partial \mathcal{Z}\left[x_i, y_i, \mathbf{a}(p_m)\right]}{\partial \phi_x} \phi_x^{[1]}\left[x_i, y_i, \mathbf{a}^{[1]}(p_m)\right] \\
&+ \frac{\partial \mathcal{Z}\left[x_i, y_i, \mathbf{a}(p_m)\right]}{\partial \phi_y} \phi_y^{[1]}\left[x_i, y_i, \mathbf{a}^{[1]}(p_m)\right] \qquad (4.91)\\
\frac{\partial \mathcal{Z}\left[x_i, y_i, \mathbf{a}(p_m)\right]}{\partial y} &= \frac{\partial \mathcal{Z}\left[x_i, y_i, \mathbf{a}(p_m)\right]}{\partial \phi_x} \phi_{xy}\left[x_i, y_i, \mathbf{a}(p_m)\right] \\
&- \frac{\partial \mathcal{Z}\left[x_i, y_i, \mathbf{a}(p_m)\right]}{\partial \phi_y} \phi_{xx}\left[x_i, y_i, \mathbf{a}(p_m)\right] \qquad (4.92)
\end{aligned}$$

Thus, the new free-surface elevation can be obtained :

$$\zeta(x_i; p_m + \Delta p) = \zeta(x_i; p_m) + \zeta^{[1]}(x_i; p_m)\Delta p \quad \text{for } p_m + \Delta p \leq 1.0 \qquad (4.93)$$

Therefore,owing to the above numerical scheme, one can obtain the value of $\mathbf{a}(\Delta p)$ and $\zeta(x; \Delta p)$, if the initial wave-elevation $\zeta_0(x)$ and the initial velocity-potential $\phi_0(x, y)$ are given by selecting freely the initial wave-height H_{w0} . And in the same way, from $\mathbf{a}(\Delta p)$ and $\zeta(x; \Delta p)$, one can obtain $\mathbf{a}(2\Delta p)$ and $\zeta(x; 2\Delta p)$; $\cdots$. If $\mathbf{a}(n_p\Delta p)$ and $\zeta(x; n_p\Delta p)$ are known for $\Delta p = \frac{1}{n_p}$, then , the solution of the original equation (4.6)~(4.9) has been obtained.

The error of this numerical method is caused by two reasons. One is that the finite terms of $\varphi_j(x, y)$ $(j = 1, 2, \cdots, M)$ has been used. On the other side,the expressions (4.88) and (4.93) are obtained by dispersing the integral relation (4.26) and (4.27). It means that the process region $p \in [0, 1]$ is discretized.

It is very interesting that the numerical scheme described above can avoid successfully the use of the iterative methods which has been traditionally applied to solve nonlinear problems. Therefore, it is possible for the above method to overcome the disadvantages of the iterative methods.

For the above numerical method, the wave-length L_w ,water-depth D and wave velocity C should be as known quantities and a selected initial wave-hight H_{w0} should be given. One can select wave-depth D and the gravity acceleration g to let all quantities non-dimensioned. Thus, only the ratio of wave-length to water-depth L_w/D and non-dimensioned wave-velocity kC^2/g should be known quantities and the initial ratio of wave height-to-length H_{w0}/L_w should be selected.

4.4.2 numerical result

The numerical scheme described above is based on velocity potential function $\phi(x, y)$, but the methods of Vanden-Broeck & Schwartz [55] (1979) and Rienecker & Fenton [44] (1981) were based on stream-function $\psi(x, y)$. Using stream-function $\psi(x, y)$, the kinematic condition of free surface can be represented easily as

$$\psi(x, y) = -Q \quad , \quad \text{on} \quad y = \zeta(x).$$

Therefore, fast all results of other authors were given under a constant value of $exp(-kQ/C)$, e.g. , $exp(-kQ/C) = 0.5$ or $exp(-kQ/C) = 0.9$. The $exp(-kQ/C)$ is a function of L_w/D and kC^2/g. For a constant $exp(-kQ/C)$, the value of L_w/D is a function of kC^2/g. But, the numerical scheme described above does not use directly $-Q$, the value of stream-function on the wave free surface. On the other hand, the non-dimensioned wave-velocity kC^2/g should be as a known quantities and the wave height-to-length ratio H_w/L_w is the numerical result by means of Finite Process Method. This is very different from the methods of other authors. Therefore, it is rather more difficult to compare directly the results obtained by Finite Process Method to those of other authors. In spite of these, approximate comparisons should be given in order to examine this numerical method, although the direct comparison could not be obtained.

The solutions of original equation (4.6)$\sim$(4.9) of gravity wave in shallow water are obtained by solving n_p linear algebraic equations ($n_p \geq 1, \Delta p = 1/n_p$). Therefore, the simplest way to examine this new numerical method is to inspect whether or not the two boundary conditions (4.7) and (4.9) have been satisfied. For simplicity, let

$$(e_k)_{max} = max\left|\frac{\mathcal{R}\left[\phi(x,\zeta_f)\right]}{\sqrt{D\, g^3}}\right| \tag{4.94}$$

$$(e_d)_{max} = max\left|\frac{\zeta_f - \mathcal{Z}\left[\phi(x,\zeta_f)\right]}{D}\right| \tag{4.95}$$

denote the maximum non-dimentioned errors of two boundary conditions (4.7) and (4.9), where, ζ_f is the numerical result of wave-elevation.

Another way to test this numerical method is to examine whether or not the values of stream-function $\psi(x,y)$ on the wave-elevation are the same.

According to (4.73), the horizontal velocity $u(x,y)$ can be represented as follows :

$$u = \frac{\partial \phi}{\partial x} = \sum_{j=1}^{M} \frac{a_j(1)(jk)cosh(jky)cos(jkx)}{cosh(jkD)} \tag{4.96}$$

In the frame of reference moving in the same velocity C as the wave, the horizontal velocity u' is :

$$u' = u - C \quad .$$

Similar to the other authors', one can suppose

$$\psi(x,0) = 0$$

Thus, one can obtain the value $-Q$ of stream-function $\psi(x,y)$ on the wave-elevation $\zeta_f(x)$:

$$\begin{aligned} -Q_j &= \int_0^{\zeta_f(x_j)} \frac{\partial \psi}{\partial y}\, d\, y \\ &= \int_0^{\zeta_f(x_j)} u' dy \\ &= \int_0^{\zeta_f(x_j)} \left\{ \sum_{i=1}^{M} \frac{a_i(1)(ik)cosh(iky)cos(ikx_j)}{cosh(ikD)} - C \right\} dy \\ &= \sum_{i=1}^{M} \frac{a_i(1)sinh[ik\zeta_f(x_j)]cos(ikx_j)}{cosh(ikD)} - C\zeta_f(x_j) \end{aligned} \tag{4.97}$$

Define

$$-\overline{Q} = \frac{1}{M}\left\{\sum_{j=1}^{M} -Q_j\right\}$$

Table 4.1: convergence of Finite Process Method ($L_w/D = 9$)

Δp	1/10	1/20	1/50	1/100	1/200	1/500
H_w/D	0.4913933	0.4976689	0.5045102	0.5072117	0.5086361	0.5094969
$exp(-kQ/C)$	0.5108812	0.5104637	0.5102284	0.5101570	0.5101228	0.5101022
q_{max}	3.4803e-2	1.6840e-2	6.6904e-3	3.3450e-3	1.6768e-3	6.7507e-4
$(e_k)_{max}$	1.8268e-2	9.4735e-3	3.8581e-3	1.9396e-3	9.7242e-4	3.8592e-4
$(e_d)_{max}$	3.6731e-2	1.8009e-2	7.2612e-3	3.6448e-3	1.8296e-3	7.3320e-4

and

$$q_{max} = max\left|\frac{Q_j - \overline{Q}}{\overline{Q}}\right| \qquad \forall j \leq M$$

Similar to the method of Rienecker & Fenton [44] (1981), the $cosh(jkD)$ term is introduced in the expression (4.71) in order to avoid the undesirable numerical errors caused by the large value of j. When the number of points M is large, it is suggested to use double-precision data type in the computer program. The author used the VAX G-Floating data tpye in his program executing in computer VAX/VMS in institute of shipbuilding,university of Hamburg. It seems necessary to use G-Floating data tpye , specially for the very long waves near the highest .

4.4.2.1 convergence of Finite Process Method

As a new kind of numerical method, the convergence of Finite Process Method should be examined at first.

Select

$$\begin{aligned} f_1(p) &= p^{1.1} \\ f_2(p) &= (1-p)^{1.1} \end{aligned}$$

example 1

Select $L_w/D = 9.0; kC^2/g = 0.7; H_{w0}/D = 0.5; M = 32; \alpha = 0.90$.The numerical results are given in Table 4.1.

example 2

Select $L_w/D = 20.0; kC^2/g = 0.35; H_{w0}/D = 0.5; M = 32; \alpha = 0.65$.The numerical results are given in Table 4.2.

example 3

Select $L_w/D = 40.0; kC^2/g = 0.20; H_{w0}/D = 0.5; M = 32; \alpha = 0.90$.The numerical results are given in Table 4.3.

example 4

Select $L_w/D = 60.0; kC^2/g = 0.13; H_{w0}/D = 0.5; M = 64; \alpha = 0.95$.The numerical results are given in Table 4.4.

The wave with $L_w/D = 9$ is a moderately long wave. The waves corresponding to $L_w/D = 20$, $L_w/D = 40$ and specially to $L_w/D = 60$ are very long waves. From Table 4.1∼ 4.4, one can see that

Table 4.2: convergence of Finite Process Method ($L_w/D = 20$)

Δp	1/10	1/20	1/50	1/100	1/200	1/500
H_w/D	0.2970015	0.3122617	0.3215379	0.3238492	0.3262390	0.3271250
$exp(-kQ/C)$	0.7337657	0.7335614	0.7334444	0.7334083	0.7333886	0.7333791
q_{max}	2.0893e-2	1.0688e-2	4.3367e-3	2.1737e-3	1.0966e-3	4.4570e-4
$(e_k)_{max}$	1.3901e-3	7.4959e-4	3.1629e-4	1.6736e-4	8.4218e-5	3.3900e-5
$(e_d)_{max}$	2.4917e-2	1.2999e-2	5.1431e-3	2.5839e-3	1.2946e-3	5.1874e-4

Table 4.3: convergence of Finite Process Method ($L_w/D = 40$)

Δp	1/10	1/20	1/50	1/100	1/200	1/500
H_w/D	0.3723232	0.3911771	0.4029229	0.4069286	0.4089504	0.4101699
$exp(-kQ/C)$	0.8561234	0.8560977	0.8560824	0.8560773	0.8560748	0.8560732
q_{max}	3.2188e-2	1.6403e-2	6.6533e-3	3.3443e-3	1.6775e-3	6.7311e-4
$(e_k)_{max}$	3.6030e-3	1.9052e-3	7.8991e-4	3.9985e-4	2.0117e-4	8.0770e-5
$(e_d)_{max}$	3.5535e-2	1.8104e-2	7.3405e-3	3.6884e-3	1.8489e-3	7.4071e-4

Table 4.4: convergence of Finite Process Method ($L_w/D = 60$)

Δp	1/10	1/20	1/50	1/100	1/200	1/500
H_w/D	0.2927140	0.3074407	0.3165027	0.3195666	0.3211073	0.3220346
$exp(-kQ/C)$	0.9010935	0.9010919	0.9010918	0.9010918	0.9010919	0.9010920
q_{max}	2.5770e-2	1.3133e-2	5.5239e-3	2.6744e-3	1.3398e-3	5.3588e-4
$(e_k)_{max}$	2.4439e-3	1.3125e-3	5.5005e-4	2.7951e-4	1.4091e-4	5.6643e-5
$(e_d)_{max}$	2.7968e-2	1.4269e-2	5.7874e-3	2.9081e-3	1.4578e-3	5.8401e-4

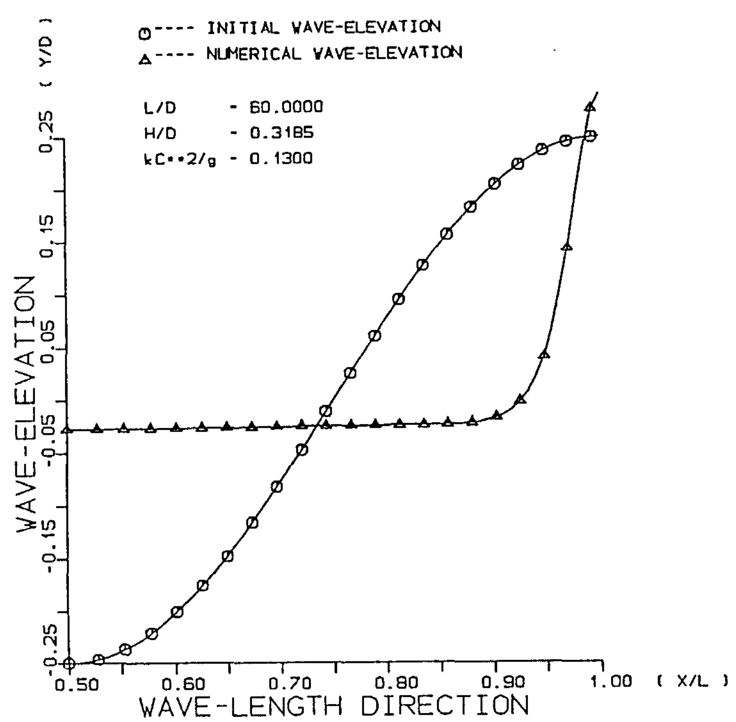

Figure 4.1: long wave with $L_w/D = 60$ in the case $\Delta p = 0.02$

there exists convergence for each case. Although the two boundary conditions (4.7) and (4.9) are not directly solved, the maximum non-dimensioned errors $(e_k)_{max}$ and $(e_k)_{max}$ of them will decrease when the Δp is decreasing. One would have reasons to say that $(e_k)_{max}$ and $(e_k)_{max}$ will tend to zero , if Δp tends to zero. In the same way, q_{max} decreases also, which means that the stream- function value $-Q$ on the wave-elevation tends to the same. It is interesting and should be also emphasized that this stream-function value $-Q$ is indirectly obtained.

Finite Process Method needs also an initial solution; but, it is insensitive to the initial values. For the case of example 4, $L_w/D = 60$, corresponding to a very long wave, its wave-elevation is very different from that of deep water wave which has been applied as the initial wave-elevation. Figure 4.1 shows the comparison of the initial deep water wave to the numerical long wave obtained by Finite Process Method in the case $\Delta p = 0.02$ (example 4). By means of iterative methods used by other authors, it was necessary to use the converged solutions obtained for lower waves as the initial approximation.

Let

$$e_{ks} = \sqrt{\frac{\int_0^{L_w} \left(\frac{\mathcal{R}(\phi)}{\sqrt{Dg^3}} \right)^2 dx}{L_w}}$$

$$e_{ds} = \sqrt{\frac{\int_0^{L_w} \left(\frac{\mathcal{Z}(\phi)-\zeta}{D} \right)^2 dx}{\mathrm{L}_w}}$$

denote respectically the statistical errors of the kinematic condition and dynamic condition of the free surface. Then, it is easy to see that smaller e_{ks} and e_{ds} are, more accurate the numerical solutions of

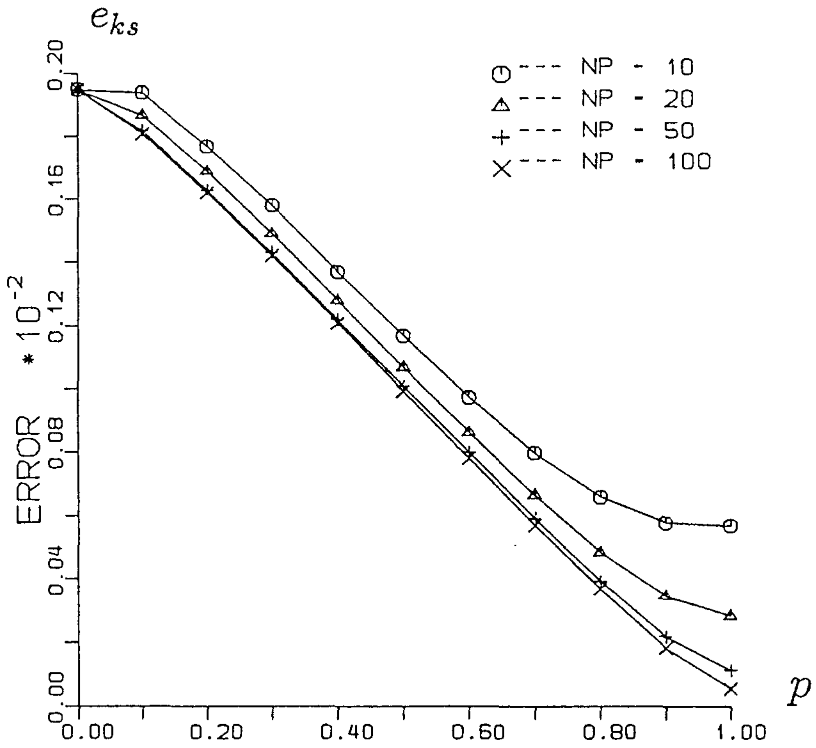

Figure 4.2: curve of error $e_{ks}(p)$ in $p \in [0,1]$ under different $\Delta p = 1/N_p$

$\phi(x,y)$ and $\zeta(x)$ are. Obviously, e_{ks} and e_{ds} are functions of $p \in [0,1]$; figure 4.2 and figure 4.3 depict the changes of them in process domain $p \in [0,1]$ under different value of $\Delta p = 1/N_p$. It is obvious from figure 4.2 and figure 4.3 that

$$\frac{d\, e_{ks}(p)}{dp} < 0 \qquad \text{in } p \in [0,1]$$

$$\frac{d\, e_{ds}(p)}{dp} < 0 \qquad \text{in } p \in [0,1]$$

it means that $e_{ds}(p)$ and $e_{ds}(p)$ will decrease when p increase from zero to unity. Smaller $\Delta p = 1/N_p$ is, i.e., finer the process domain $p \in [0,1]$ is discretized, then more accurately the kinematic and dynamic conditions of free surface is satesfied. If at $p = p_c < 1$ the numerical solutions are accurate enough, then the computation can be stoped. Figure 4.4 depicts the change of elevation from the initial solution ζ_0 to the approximate final solution ζ_f when p changes from zero to unity, while $N_p = 5$ ($\Delta p = 0.2$) is used in numerical computation. With comparison to the exacter numerical solutions given in figure 4.1, it is indeed true that the wave elevation is more and more closer to the numerical solution.

Using Finite Process Method, the iterative methods can be successfully avoided. Finite Process Method is insensitive not only to the selected initial solution but also to the number of unknowns. Therefore, the disadvantages and limitations of iterative methods can be overcomed. The disadvantage of Finite Process Method is that it needs more CPU time. This is mainly because the precision is approximately directly proportional to the CPU time by means of Finite Process Method. From Table 4.1~4.4, one can see that the errors $(e_k)_{max}$ and $(e_k)_{max}$ are approximately directly proportional to Δp.

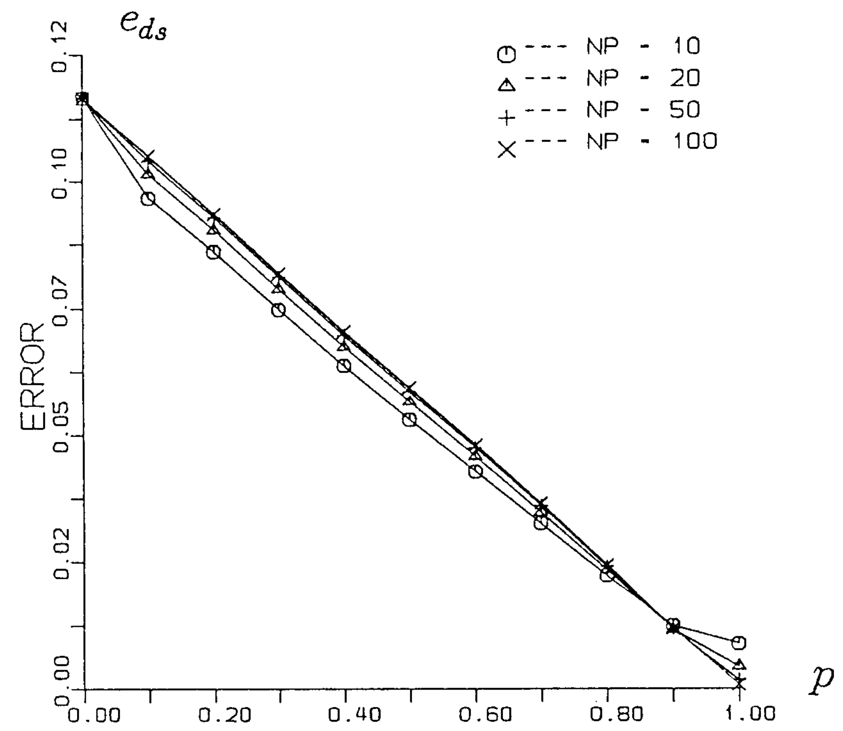

Figure 4.3: curve of error $e_{ds}(p)$ in $p \in [0,1]$ under different $\Delta p = 1/N_p$

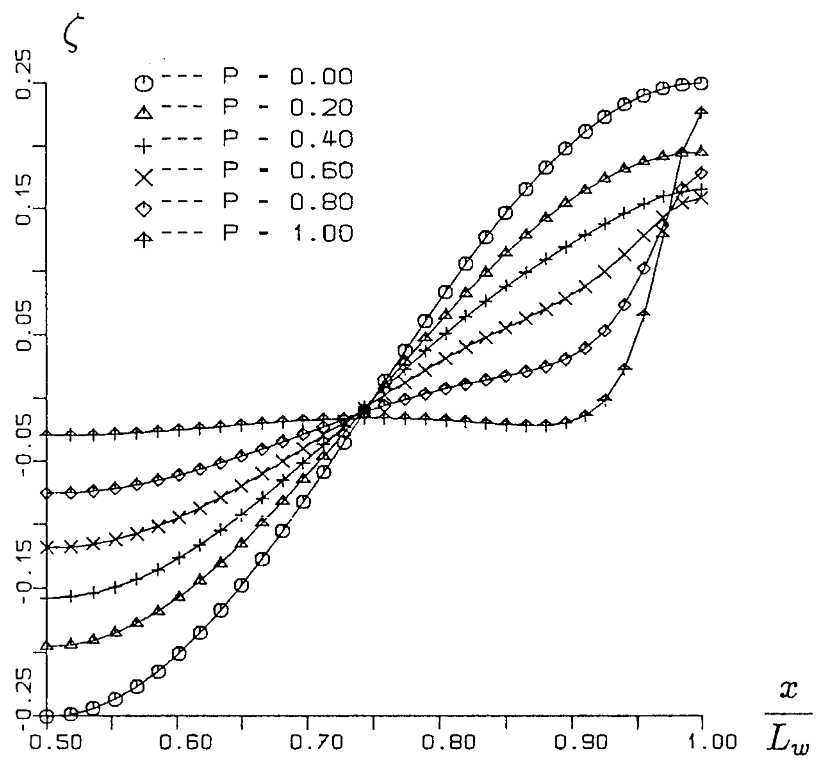

Figure 4.4: change of wave elevation in mapping domain $p \in [0,1]$

Table 4.5: comparisons to results of other authors

results of present method (FPM)				results of other authors $exp(-kQ/C) = 0.5$			
L_w/D	$exp(-kQ/C)$	kC^2/g	H_w/D	H_w/D	kC^2/g		
					Cokelet	Fenton	Schwartz
9.0158	0.5000197	0.615059	0.1720615	0.1729974	0.615059	0.615059	not presented
8.9776	0.5006100	0.631112	0.2528818	0.252630	0.631112	0.631112	not presented
8.8670	0.5004126	0.666501	0.3791625	0.3802645	0.666501	0.666501	0.666501
8.7634	0.5004327	0.706443	0.4898292	0.4944549	0.706443	0.706443	0.706443
8.6900	0.50076340	0.748230	0.5959385	0.602447	0.748230	0.748230	0.748230
8.6800	0.5097155	0.764403	0.6497747	0.651251	0.764403	0.764403	0.764403
8.6795	0.5008378	0.767748	0.6699720	0.672143	0.767748	0.767748	0.767748

4.4.2.2 comparison to te results of oter autors

The numerical results of Rienecker & Fenton [44] (1981) agreed very good with the results of Cokelet [8] (1977) and Vanden-Broeck & Schwartz [55] (1979) and even agreed with the experiment results given by Le Méhauté [29] (1968). It does not seem necessary to make again detailed comparison for all of these. This has two reasons. One is that the purpose of present work is mainly to introduce and examine a new kind of numerical method which can avoid using iterative methods and can be used to solve 3D nonlinear gravity wave problems, for example, the wave-resistance of a ship moving in a deep or shallow water. Another reason, which is perhaps the main , is that the method described in this chapter is based on velocity potential function and therefore it is very difficult to compare directly the results to those of other authors.

In spite of these, some approximate comparisons have been made, which are given in the Table 4.5.

For a selected value of kC^2/g , a large number of computation is needed in order to find such a value of L_w/D that $exp(-kQ/C) \approx 0.5$. This seems specially difficult for the waves near the highest. The corresponding values of L_w/D have been also given in Table 4.5. Although it is not possible to make directly the comparison, from the results of Table 4.5, one would have reasons to believe that the presented method can give results which agree good with those of Cokelet [8] (1977), Vanden-Broeck & Schwartz [55] (1979) and Rienecker & Fenton [44] (1981).

We can compare the present numerical scheme to the method of Rienecker & Fenton [44] (1981). Similar to other authors, their method were based on stream-function and Newton's iterative method were used to solve corresponding nonlinear algebraic equation. If M points on the wave-elevation were selected, a nonlinear algebraic euqation with $2M + 5$ unknowns should be solved; but by means of present method, n_p linear equations with M unknown quantities should be solved, because at each step $p_m = m\Delta p$, the wave-elevation $y = \zeta(x; p_m)$ is known. Different from methods based on iterative techniques, Finite Process Method is insensitive to the initial solution and the number of unknows. But, much more CPU time is needed. This is perhaps the greatest disadvantage of Finite Process Method.

4.5 conclusion

In this chapter, a simple but typical nonlinear problem in fluid mechanics, namely 2D steep gravity waves in shallow water, is used to introduce and examine a new kind of numerical method, called Finite Process Method.

Finite Process Method can avoid successfully the use of iterative methods and therefore can overcome the disadvantages and limitations of the iterative methods. In contrast to iterative methods, Finite Process Method is insensitive to initial solutions and the number of the unknowns. Even for the waves near the highest or for the very long waves, for example, waves with $L_w/D = 120$, a converged numerical results can be obtained by using the deep water wave as the initial solution. By means of the method of Rienecker & Fenton [44] (1981), it was necessary in this situation to extrapolate to the initial approximation from converged solutions for lower waves.

Although it is rather more difficult to compare directly the results of presented method to those of other authors, from the approximate comparisons given in Table 4.5, one could have reasons to believe that the Finite Process Method can be indeed used to solve numerically the 2D steep gravity waves in shallow water. It is interesting that, different from other authors, the presented method is based on velocity potential function and therefore can be applied to solve 3D nonlinear gravity waves by using Hess-Smith's method, e.g. , wave-resistance of a ship moving in a deep or shallow water, which will be discussed latter in this report.

In this chapter, the concept *process* and *process derivatives* are again used. *Process* is a kind of continuous mapping (homotopy) which connects the selected initial solution with the solutions of what one wants to solve. It is interesting that the equations of the first-order process derivatives, called first-order process equation, are linear. By means of this property of process, a nonlinear problem, e.g. , a nonlinear algebraic equation, a set of nonlinear algebraic equations, a differential or partial-differential equation and so on, can be transformed into a set of linear problems, because it is only needed for Finite Process Method to solve n_p linear equations ($n_p \geq 1, \Delta p = 1/n_p$). In another word, a nonlinear problem can be approximated by finite number of linear problems.Perhaps,one could say that a nonlinear problem would be discretized into n_p linear problems ($n_p \geq 1$). And finer the discretation is,i.e., smaller Δp is, exacter this kind of approximation. This is similar to the Finite Element Method and also to the Finite Difference Method, if we see each corresponding linear problem as an element of a nonlinear problem. It is well-known that a real function can be generally approximated by finite terms of Fourier series. One would see that Finite Process Method is also similar to it, if we would see a nonlinear problem as a function and linear problems as Fourier series.

The disadvantage of Finite Process Method is that more CPU time is needed in order to obtain a very accurate result. This is mainly bacause its precision of result is approximately directly proportional to the needed CPU time. Thus, exacter result means higher price.

It seems that Finite Process Method can avoid indeed the use of iterative techniques and is insensitive to the initial solutions. But, more CPU time is needed.

As last part of this chapter, we will compare simply Finite Process Method to Process Analysis Method. Clearly, both of them are based on the linearity- invariance under homtopy proved in appendix A. For Process Analysis Method, the homotopy is analysed in imbedding domain $p \in [0,1]$ by using kth-order process derivatives at $p = 0$ ($k = 0, 1, 2, \cdots$) and Taylor's expression; but for Finite Process Method, the imbedding domain $p \in [0,1]$ is discretized into finite number of small domains and in each such small domain a linear equation with respect to first-order process derivative should be solved. Clearly, Taylor's expression is not needed for Finite Process Method so that Finite Process

Method needs less suppositions than Process Analysis Method. In some cases, the first-order process derivative exists, but the corresponding higher-order process derivatives does not. In these cases, Finite Process Method can still give numerical results.

Chapter 5

Numerical Solution of Nonlinear Gravity Waves Part 2 : Steady Waves in Deep or Shallow Water

Summary

The steady nonlinear water wave is used as an example to reexamine a kind of numerical technique, namely Finite Process Method. On another side, the relationship between Finite Process Method and iteration is simply discussed. As simple examples, the folw past a submerged vortex and the flow past a submerged thin plain wing are solved.

5.1 introduction

The steady potential flow with free surface is a typical nonlinear problem. In 1977, Dawson [9] gave a numerical scheme of the ship-wave problems which was based on singularity-distribution method developed by Hess & Smith [49] in 1962. Dawson [9] distributed the simple singularities on the double-model ship surface and also on the limited region of the undisturbed free surface around the ship. The linear free surface conditions were used and a kind of four points upwind operator was applied to treat the radiation condition. In recent years, many researchers [5] [10] [19] [24] [25] [33] [42] expanded the basic idea of Dawson to study water wave problems with nonlinear free surface conditions. Some different iterative numerical schemes were given. But, it is well-known that nearly all iterative formulars are sensitive not only to initial solutions but also to number of unknowns.

In Chapter one, Chapter two and Chapter three, we have shown some applications of the linearity-invariance of continuous mapping (homotopy) in analysis. In chapter four, we have shown a kind of application of it in numerical computation. This chapter can be seen as the continue of chapter 4.

The main purpose of this chapter is to reexamine the new kind of numerical technique described in chapter 4, which can avoid the use of iteration to solve nonlinear problems.

5.2 mathematical description

The steady water wave problems can be generally described by following equations:

$$\nabla^2\phi(x,y,z) = 0 \qquad \text{in } \Omega \tag{5.1}$$

with the nonlinear free-surface conditions:

$$g\phi_z + \frac{1}{2}\nabla\phi\nabla(\nabla\phi\nabla\phi) = 0 \qquad \text{on } z = \zeta(x,y) \tag{5.2}$$

$$\zeta = \frac{1}{2g}\left(U^2 - \nabla\phi\nabla\phi\right) \qquad \text{on } z = \zeta(x,y) \tag{5.3}$$

and the boundary condition on body ∂B :

$$\left.\frac{\partial\phi}{\partial n}\right|_{\partial B} = 0 \tag{5.4}$$

where $\phi(x, y, z)$ is the velocity potential function, $\zeta(x, y)$ is the wave-elevation, g is the gravity acceleration, U is the velocity of the body. The coordinate system $o - xyz$ with z positive upwords is moving at the same velocity U with the body.

For simplicity, define

$$\mathcal{R}(\phi) = g\phi_z + \frac{1}{2}\nabla\phi\nabla(\nabla\phi\nabla\phi) \tag{5.5}$$

$$\mathcal{Z}(\phi) = \frac{1}{2g}\left(U^2 - \nabla\phi\nabla\phi\right) \tag{5.6}$$

5.3 Finite Process Method

One can find two continuous functions $f_1(p)$ and $f_2(p)$ in $p \in [0, 1]$, called first-sort and second-sort of process function respectively, which satisfy

$$f_1(p) = \begin{cases} 0.0 & \text{when } p = 0.0 \\ 1.0 & \text{when } p = 1.0 \end{cases} \tag{5.7}$$

$$f_2(p) = \begin{cases} 1.0 & \text{when } p = 0.0 \\ 0.0 & \text{when } p = 1.0 \end{cases} \tag{5.8}$$

Then, a homotopy can be obtained as follows:

$$\nabla^2\phi(x, y, z; p) = 0.0 \qquad \text{in } \Omega(p) \tag{5.9}$$

with boundary conditions:

$$f_1(p)\mathcal{R}(\phi) + f_2(p)\left\{\mathcal{R}(\phi) - \mathcal{R}(\phi_0)\right\} = 0 \qquad \text{on } z = \zeta(x, y; p) \tag{5.10}$$

$$\zeta(x, y; p) = f_1(p)\mathcal{Z}(\phi) + f_2(p)\zeta_0(x, y) \qquad \text{on } z = \zeta(x, y; p) \tag{5.11}$$

and

$$\left.\frac{\partial\phi(x, y, z; p)}{\partial n}\right|_{\partial B} = 0 \tag{5.12}$$

where, $\zeta_0(x, y)$ is an initial wave-elevation and $\phi_0(x, y, z)$ is an initial velocity-potential function, which satisfies

$$\nabla^2\phi_0(x, y, z) = 0 \qquad \text{in } \Omega_0 \tag{5.13}$$

and the boundary condition:

$$\left.\frac{\partial\phi_0}{\partial n}\right|_{\partial B} = 0 \tag{5.14}$$

If $p = 0$, then $f_1(p) = 0$ and $f_2(p) = 1.0$. Hence, from (5.9) ~(5.12), one has the *initial equation*

$$\nabla^2\phi(x, y, z; 0) = 0.0 \qquad \text{in } \Omega(0) \tag{5.15}$$

with boundary conditions:

$$\mathcal{R}(\phi) = \mathcal{R}(\phi_0) \qquad \text{on } z = \zeta(x, y; 0) \tag{5.16}$$

$$\zeta(x,y;0) = \zeta_0(x,y) \qquad \text{on } z = \zeta(x,y;0) \tag{5.17}$$

and

$$\left.\frac{\partial\phi(x,y,z;0)}{\partial n}\right|_{\partial B} = 0 \tag{5.18}$$

Let $\zeta(x,y;0) = \zeta_0(x,y)$, then (5.17) is satisfied. It is interesting that $\phi_0(x,y,z)$ determined by (5.13)~(5.14) is just the solution of initial equation (5.15), (5.16) and (5.18); therefore

$$\phi(x,y,z;0) = \phi_0(x,y,z) \tag{5.19}$$

$$\zeta(x,y;0) = \zeta_0(x,y) \tag{5.20}$$

$$\Omega(0) = \Omega_0 \tag{5.21}$$

When $p = 1.0$, $f_1(p) = 1.0$ and $f_2(p) = 0.0$. Hence, from (5.9)~(5.12), one obtains the *final equation*

$$\nabla^2\phi(x,y,z;1.0) = 0 \qquad \text{in} \qquad \Omega(1.0) \tag{5.22}$$

with boundary conditions

$$\mathcal{R}(\phi) = 0 \qquad \text{on } z = \zeta(x,y;1) \tag{5.23}$$

$$\zeta(x,y;1.0) = \mathcal{Z}(\phi) \qquad \text{on } z = \zeta(x,y;1) \tag{5.24}$$

and

$$\left.\frac{\partial\phi(x,y,z;1)}{\partial n}\right|_{\partial B} = 0 \tag{5.25}$$

The final equations (5.22)~(5.25) are just the same as the original equations (5.1)~ (5.4), respectively. Suppose the solutions of the original equations (5.1)~(5.4) are $\phi_f(x,y,z)$ and $\zeta_f(x,y)$, called final solutions. Then, one has the relation

$$\phi_f(x,y,z) = \phi(x,y,z;1.0) \tag{5.26}$$

and

$$\zeta_f(x,y) = \zeta(x,y;1.0) \tag{5.27}$$

From above analysis, one can see that the homotopy determined by (5.9)~(5.12) gives a relation between the selected initial solutions $\phi_0(x,y,z), \zeta_0(x,y)$ and the final solutions $\phi_f(x,y,z)$, $\zeta_f(x,y)$ which can be described in the form of integral as follows :

$$\begin{aligned}\phi_f(x,y,z) &= \phi(x,y,z;1.0) \\ &= \phi(x,y,z;0.0) + \int_0^1 \phi^{[1]}(x,y,z;p)dp \\ &= \phi_0(x,y,z) + \int_0^1 \phi^{[1]}(x,y,z;p)dp\end{aligned} \tag{5.28}$$

and

$$\begin{aligned}\zeta_f(x,y) &= \zeta(x,y;1.0)\\ &= \zeta(x,y;0.0)+\int_0^1 \zeta^{[1]}(x,y;p)dp\\ &= \zeta_0(x,y)+\int_0^1 \zeta^{[1]}(x,y;p)dp\end{aligned}\tag{5.29}$$

where

$$\phi^{[1]}(x,y,z;p) = \frac{\partial\phi(x,y,z;p)}{\partial p}\tag{5.30}$$

$$\zeta^{[1]}(x,y;p) = \frac{\partial\zeta(x,y;p)}{\partial p}\tag{5.31}$$

are first-order partial-derivatives of $\phi(x,y,z;p)$ and $\zeta(x,y;p)$ with respect to p, respectively.

For simplicity, call the equations (5.9)~(5.12) zero-order process equation, call $\phi(x,y,z;p), \zeta(x,y;p)$ zero-order process of velocity-potential function and wave-elevation; $\phi^{[1]}(x,y,z;p)$ and $\zeta^{[1]}(x,y;p)$ the first-order process derivatives of $\phi(x,y,z;p)$ and $\zeta(x,y;p)$, respectively.

The first-order process derivatives $\phi^{[1]}(x,y,z;p)$ and $\zeta^{[1]}(x,y;p)$ can be obtained in the following way.

Deriving (5.9) and (5.12) with respect to p respectively, one has

$$\frac{\partial}{\partial p}\left(\nabla^2\phi\right) = \nabla^2\left(\frac{\partial\phi}{\partial p}\right) = \nabla^2\phi^{[1]} = 0 \qquad \text{in } \Omega(p)\tag{5.32}$$

and

$$\left.\frac{\partial\phi^{[1]}}{\partial n}\right|_{\partial B} = 0\tag{5.33}$$

Deriving equation (5.10) with respect to p, one has

$$f_1'(p)\mathcal{R}(\phi)+f_2'(p)\left\{\mathcal{R}(\phi)-\mathcal{R}(\phi_0)\right\}+\left\{f_1(p)+f_2(p)\right\}\frac{d\,\mathcal{R}(\phi)}{d\,p} = 0$$
$$\text{on } z=\zeta(x,y;p)\tag{5.34}$$

In the same way, deriving (5.11) with respect to p, one obtains

$$\zeta^{[1]}(x,y;p) = f_1'(p)\mathcal{Z}(\phi)+f_1(p)\frac{d\mathcal{Z}(\phi)}{dp}+f_2'(p)\zeta_0(x,y)$$
$$\text{on } z=\zeta(x,y;p)\tag{5.35}$$

Because wave-elevation $z=\zeta(x,y;p)$ is also the function of p, one has

$$\begin{aligned}\frac{d\,\mathcal{R}}{d\,p} &= \frac{\partial\mathcal{R}}{\partial p}+\frac{\partial\mathcal{R}}{\partial z}\frac{\partial z}{\partial p}\\ &= \frac{\partial\mathcal{R}}{\partial p}+\frac{\partial\mathcal{R}}{\partial z}\zeta^{[1]} \qquad \text{on } z=\zeta(x,y;p)\end{aligned}\tag{5.36}$$

$$\begin{aligned}\frac{d\,\mathcal{Z}}{d\,p} &= \frac{\partial\mathcal{Z}}{\partial p}+\frac{\partial\mathcal{Z}}{\partial z}\frac{\partial z}{\partial p}\\ &= \frac{\partial\mathcal{Z}}{\partial p}+\frac{\partial\mathcal{Z}}{\partial z}\zeta^{[1]} \qquad \text{on } z=\zeta(x,y;p)\end{aligned}\tag{5.37}$$

Substituting (5.37) into (5.35), one obtains

$$\zeta^{[1]}(x,y;p) = \frac{f_1'(p)\mathcal{Z}[\phi(x,y,z;p)] + f_2'(p)\zeta_0(x,y) + f_1(p)\frac{\partial \mathcal{Z}[\phi(x,y,z;p)]}{\partial p}}{1 - f_1(p)\frac{\partial \mathcal{Z}[\phi(x,y,z;p)]}{\partial z}} \quad \text{on } z = \zeta(x,y;p) \tag{5.38}$$

Substituting (5.36),(5.38) into (5.34), one obtains:

$$\frac{\partial \mathcal{R}}{\partial p} + f_1(p)\mathcal{S}(\phi;p)\frac{\partial \mathcal{Z}}{\partial p} = \mathcal{T}(\phi;p) \qquad \text{on } z = \zeta(x,y;p) \tag{5.39}$$

where

$$\mathcal{S}[\phi(x,y,z;p);p] = \frac{\frac{\partial \mathcal{R}[\phi(x,y,z;p)]}{\partial z}}{1 - f_1(p)\frac{\partial \mathcal{Z}[\phi(x,y,z;p)]}{\partial z}} \ , \ \text{on } z = \zeta(x,y;p) \tag{5.40}$$

and

$$\begin{aligned} &\mathcal{T}[\phi(x,y,z;p);p] \\ = \ &\frac{f_2'(p)\mathcal{R}[\phi_0(x,y;z)] - \{f_1'(p) + f_2'(p)\}\mathcal{R}[\phi(x,y,z;p)]}{f_1(p) + f_2(p)} \\ &-\mathcal{S}[\phi(x,y,z;p);p]\left\{f_1'(p)\mathcal{Z}[\phi(x,y,z;p)] + f_2'(p)\zeta_0(x,y)\right\} \\ &\text{on } z = \zeta(x,y;p) \end{aligned} \tag{5.41}$$

For simplicity, selecting $f_1(p) = p, f_2(p) = 1 - p$ and substituting

$$\frac{\partial \mathcal{R}}{\partial p} = g\phi_z^{[1]} + \frac{1}{2}\nabla\phi^{[1]}\nabla(\nabla\phi\nabla\phi) + \nabla\phi\nabla(\nabla\phi\nabla\phi^{[1]}) \tag{5.42}$$

and

$$\frac{\partial \mathcal{Z}}{\partial p} = -\frac{\nabla\phi\nabla\phi^{[1]}}{g} \tag{5.43}$$

$$\frac{\partial \mathcal{Z}}{\partial z} = -\frac{\nabla\phi\nabla\phi_z}{g} \tag{5.44}$$

into (5.38) and (5.39), one has the equations of $\phi^{[1]}(x,y,z;p)$ and $\zeta^{[1]}(x,y;p)$ as follows:

$$\begin{aligned} &g\phi_z^{[1]} + \frac{1}{2}\nabla\phi^{[1]}\nabla(\nabla\phi\nabla\phi) + \nabla\phi\nabla(\nabla\phi\nabla\phi^{[1]}) - \frac{p\,\mathcal{S}(\phi;p)\nabla\phi\nabla\phi^{[1]}}{g} \\ = \ &-\mathcal{R}(\phi_0) - \mathcal{S}(\phi;p)\{\mathcal{Z}(\phi) - \zeta_0(x,y)\} \qquad \text{on } z = \zeta(x,y;p) \end{aligned} \tag{5.45}$$

$$\zeta^{[1]}(x,y;p) = \frac{\mathcal{Z}(\phi) - \zeta_0(x,y) - \frac{p\nabla\phi\nabla\phi^{[1]}}{g}}{1 + \frac{p\,\nabla\phi\nabla\phi_z}{g}} \qquad \text{on } z = \zeta(x,y;p) \tag{5.46}$$

where

$$\mathcal{S}(\phi;p) = \frac{\frac{\partial \mathcal{R}}{\partial z}}{1 + \frac{p\,\nabla\phi\nabla\phi_z}{g}} \qquad \text{on } z = \zeta(x,y;p) \tag{5.47}$$

to $\phi^{[1]}(x, y, z; p)$ and $\zeta^{[1]}(x, y; p)$. If $\phi(x, y, z; p)$ and $\zeta(x, y; p)$ are known,then, $\phi^{[1]}(x, y, z; p)$ and $\zeta^{[1]}(x, y; p)$ can be obtained by solve this system of linear equations. The $\phi(x, y, z; p + \Delta p)$ and $\zeta(x, y; p+\Delta p)$ can be obtained by using Runge-Kutta's method in process domain $p \in [0, 1]$ as follows:

$$\begin{aligned}
\phi(x, y, z; p + \Delta p) &= \phi(x, y, z; p) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) && (5.48)\\
\zeta(x, y; p + \Delta p) &= \zeta(x, y; p) + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) && (5.49)
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= \Delta p \phi^{[1]}[\phi(x, y, z; p), \zeta(x, y; p); p] && (5.50)\\
m_1 &= \Delta p \zeta^{[1]}[\phi(x, y, z; p), \zeta(x, y; p); p] && (5.51)\\
k_2 &= \Delta p \phi^{[1]}[\phi(x, y, z; p) + k_1/2, \zeta(x, y; p) + m_1/2; p + \Delta p/2] && (5.52)\\
m_2 &= \Delta p \zeta^{[1]}[\phi(x, y, z; p) + k_1/2, \zeta(x, y; p) + m_1/2; p + \Delta p/2] && (5.53)\\
k_3 &= \Delta p \phi^{[1]}[\phi(x, y, z; p) + k_2/2, \zeta(x, y; p) + m_2/2; p + \Delta p/2] && (5.54)\\
m_3 &= \Delta p \zeta^{[1]}[\phi(x, y, z; p) + k_2/2, \zeta(x, y; p) + m_2/2; p + \Delta p/2] && (5.55)\\
k_4 &= \Delta p \phi^{[1]}[\phi(x, y, z; p) + k_3, \zeta(x, y; p) + m_3; p + \Delta p] && (5.56)\\
m_4 &= \Delta p \zeta^{[1]}[\phi(x, y, z; p) + k_3, \zeta(x, y; p) + m_3; p + \Delta p] && (5.57)
\end{aligned}$$

Note that the final results are obtained at $p = 1.0$.

5.4 simple numerical examples

The 2D potential flow past a submerged vortex is a simple but typical nonlinear problem [1]. Shown as figure 5.1, a vortex is submerged at $(0, -b)$ with circulation Γ. Upstream, there exists an uniform stream with velocity U. Downstream, there exists a steady wave with wave-length λ_w. In case of finite water depth, Salvesen and Kerczek [37] solved this problem by an iterative finite difference technique and compared their numerical results to their perturbation solutions in deep water. Similar to Salvesen & C. von Kerczek [37], we select uniform stream velocity $U = 10$ (fps), [2] vortex submergence $b = 4.5$ (ft), but the water depth is infinite so that more rigorous comparison to perturbation solutions in deep water can be made.

Similar as Jensen, Mi and Soeding [25], we use singularity distribution method to solve the corresponding equations of (5.32), (5.33), (5.45) and (5.46). The simple sources $\ln r$ are distributed continuously at a distance h_s above the undisturbed water surface in a limited region $\overline{AB} : x_a \leq x \leq x_b$, shown as figure 5.1.

[1] We have two reasons to select this problem as an example, although it seems too simple from the view point of engineering. At first, the main purpose of this paper is to examine the basic idea of a kind of numerical method. Secondly, there exist detailed numerical results of this problem given by other authors so that comparison can be made.

[2] For comparison to [37], we use British System. Where

1 feet (ft) = 0.3048 meter , 1 pound (bl.) = 0.454 kilogram

and (fps) means ' feet per second '

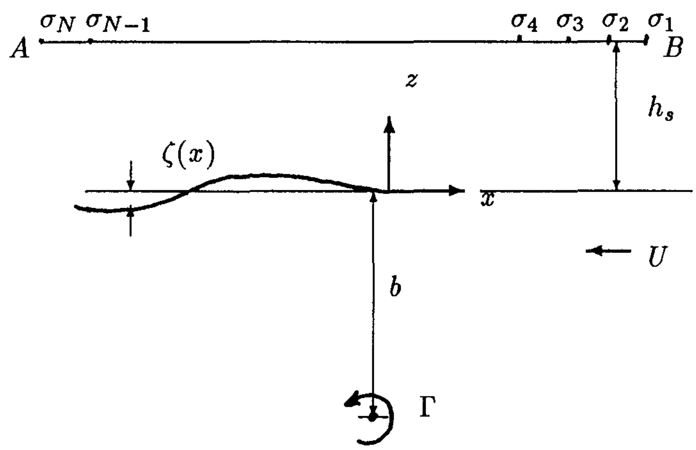

Figure 5.1: coordinate-system and grid for numerical computation of 2D flow past a submerged vortex

The treatment of radiation condition seems even more difficult than that of the nonlinear free surface conditions. A few numerical techniques, or more precisely, a few numerical arts, have been used to treat the radiation conditions. The method useed in [25] for treatment of radiation condition seems efficient and very simple, so, in this paper, we use the same technique as given in [25] to treat the radiation condition, i.e.,

$$\xi_i = x_i - \delta x \qquad (i = 1, 2, 3, \cdots, N)$$

where, (x_i, ζ_i) are coordinates of points on wave-elevation, (ξ_i, h_s) are coordinates of points on $\overline{AB}$, and $\delta x = |x_i - x_{i+1}|$ is a selected constant.

Use

$$\phi_0(x,z) = -Ux + \frac{\Gamma}{2\pi} arctg\left(\frac{z+b}{x}\right) - \frac{\Gamma}{2\pi} arctg\left(\frac{z-b}{x}\right) \tag{5.58}$$

as the initial velocity potential function and $\zeta_0(x) = 0$ as the initial wave-elevation. The corresponding continuous mapping $\phi(x, z; p)$ can be described as

$$\begin{aligned} \phi(x,z;p) &= \phi_0(x,z) + \int_{x_a}^{x_b} \sigma(\xi, h_s; p) \ln\sqrt{(x-\xi)^2 + (z-h_s)^2} d\xi \\ &= \phi_0(x,z) + \sum_{m=1}^{N} \varphi(x-\xi_m, z)\sigma_m(p) \end{aligned} \tag{5.59}$$

where, $\sigma_m(p) = \sigma(\xi_m, h_s; p)$ $(m = 1, 2, \cdots, N)$. Then, the first-order process derivatives $\phi^{[1]}(x, z; p)$ can be expressed as

$$\phi^{[1]}(x,z;p) = \int_{x_a}^{x_b} \sigma^{[1]}(\xi, h_s; p) \ln\sqrt{(x-\xi)^2 + (z-h_s)^2} d\xi$$

$$= \sum_{m=1}^{N} \varphi(x-\xi_m, z)\sigma_m^{[1]}(p) \tag{5.60}$$

Clearly, $\phi^{[1]}(x,z;p)$ described above satisfies (5.32).

From above expression, one obtains easily

$$\phi_x^{[1]}(x,z;p) = \sum_{m=1}^{N} \varphi_x(x-\xi_m, z)\sigma_m^{[1]}(p) \tag{5.61}$$

$$\phi_z^{[1]}(x,z;p) = \sum_{m=1}^{N} \varphi_z(x-\xi_m, z)\sigma_m^{[1]}(p) \tag{5.62}$$

$$\phi_{xx}^{[1]}(x,z;p) = \sum_{m=1}^{N} \varphi_{xx}(x-\xi_m, z)\sigma_m^{[1]}(p) \tag{5.63}$$

$$\phi_{xz}^{[1]}(x,z;p) = \sum_{m=1}^{N} \varphi_{xz}(x-\xi_m, z)\sigma_m^{[1]}(p) \tag{5.64}$$

Substituting (5.61)~(5.64) in (5.45), one obtains the set of linear algebraic equations of $\sigma_m^{[1]}$ $(m = 1, 2, \cdots, N)$ as follows:

$$\mathbf{E}(p)\sigma^{[1]}(p) = \mathbf{t}(p) \tag{5.65}$$

where

$$\begin{aligned} \mathbf{E}(p) &= \{e_{ij}(p)\} & (i,j = 1,2,\cdots,N) \\ \sigma^{[1]}(p) &= \left\{\sigma_m^{[1]}(p)\right\} & (m = 1,2,\cdots,N) \\ \mathbf{t}(p) &= \{\mathcal{T}_m(p)\} & (m = 1,2,\cdots,N) \end{aligned}$$

with

$$\begin{aligned} e_{ij}(p) = {} & b_1(x_i, z_i; p)\varphi_x(x_i - \xi_j, z_i) + b_2(x_i, z_i; p)\varphi_z(x_i - \xi_j, z_i) \\ & + b_3(x_i, z_i; p)\varphi_{xx}(x_i - \xi_j, z_i) + b_4(x_i, z_i; p)\varphi_{xz}(x_i - \xi_j, z_i) \\ & \text{on } z_i = \zeta(x_i; p) \end{aligned} \tag{5.66}$$

where

$$b_1(x,z;p) = 2(\phi_x\phi_{xx} + \phi_z\phi_{xz}) - p\mathcal{S}(\phi;p)\phi_x/g \quad \text{on } z = \zeta(x;p) \tag{5.67}$$

$$b_2(x,z;p) = g + 2(\phi_x\phi_{xz} - \phi_z\phi_{xx}) - p\mathcal{S}(\phi;p)\phi_z/g \quad \text{on } z = \zeta(x;p) \tag{5.68}$$

$$b_3(x,z;p) = \phi_x^2 - \phi_z^2 \qquad \text{on } z = \zeta(x;p) \tag{5.69}$$

$$b_4(x,z;p) = 2\phi_x\phi_z \qquad \text{on } z = \zeta(x;p) \tag{5.70}$$

and

$$\begin{aligned} \mathcal{T}_m(p) = {} & -\mathcal{R}[\phi_0(x_m, z_m)] - \mathcal{S}[\phi(x_m, z_m; p); p]\{\mathcal{Z}[\phi(x_m, z_m; p)] - \zeta_0(x_m)\} \\ & \text{on } z_m = \zeta(x_m; p) \qquad (m = 1, 2, \cdots, N) \end{aligned} \tag{5.71}$$

We solve the set of linear algebraic equations (5.65) by Gauss-Jordan elimination method. After $\phi^{[1]}(x,z;p)$ is known, $\zeta^{[1]}(x;p)$ can be easily obtained from expression (5.46). Note that $\phi(x,z;p+\Delta p)$ and $\zeta(x;p+\Delta p)$ are obtained by Runge-Kutta's method, shown as expressions (5.48)~(5.57).

The final solution $\phi_f(x,z)$ is

$$\begin{aligned}\phi_f(x,z) &= \phi_0(x,z)+\int_{x_a}^{x_b}\sigma_f(\xi,h_s)\ln\sqrt{(x-\xi)^2+(z-h_s)^2}d\xi \\ &= -Ux+\phi^*(x,z) \end{aligned} \qquad (5.72)$$

where

$$\sigma_f(\xi,h_s)=\int_0^1\sigma^{[1]}(\xi,h_s;p)dp \qquad (5.73)$$

or more precisely

$$\sigma_f(\xi_m,h_s)=\int_0^1\sigma_m^{[1]}(p)dp \qquad (m=1,2,\cdots,N) \qquad (5.74)$$

Note $\sigma_m(p+\Delta p)$ $(m=1,2,\cdots,N)$ are obtained also by Runge-kutta's method.

Let

$$(e_d)_{max} = \max_{1\le i\le N}\frac{g\ |\zeta_f(x_i)-\mathcal{Z}(\phi_f)|}{U^2} \qquad \text{on } z_i=\zeta_f(x_i) \qquad (5.75)$$

$$(e_k)_{max} = \max_{1\le i\le N}\frac{|\mathcal{R}(\phi_f)|}{g\ U} \qquad \text{on } z_i=\zeta_f(x_i) \qquad (5.76)$$

denote the maximum non-dimensional errors of nonlinear dynamic and kinematic conditions of free surface, respectively.

The exact wave-resistance formula for two-dimensional potential flow , neglecting surface tension, is from Wehausen [57]:

$$R_w=\frac{1}{2}\rho\int_{-\infty}^{\zeta(x_0)}\left[(\phi_z^*(x_0,z))^2-(\phi_x^*(x_0,z))^2\right]dz+\frac{1}{2}g\rho\zeta^2(x_0) \qquad (5.77)$$

where, the plane $x=x_0$ may be taken at any distance behind the submerged vortex, i.e., $x_0<0$. R_w can be obtained by numerical integral.

As a test computation, let us consider the flow past the submerged vortex $\Gamma/2\pi=2.20$ (ft^2/sec), corresponding to a strong nonlinear problem.

Select $h_s=1.6$ (ft), $N=80$, $x_a=-2.5\lambda_0, x_b=1.5\lambda_0$, where $\lambda_0=2\pi U^2/g=19.54$ (ft). The corresponding maximum initial errors are $(e_{k0})_{max}$ =5.1E-2, $(e_{d0})_{max}$=0.10. The results obtained in different value of Δp are given in Table 5.1.

From Table 5.1, clearly, smaller Δp is, more accurate the numerical results are. In case Δp=0.1, the numerical results are accurate enough. These mean that accurate enough results can be obtained by Finite Process Method, if small enough Δp is used. So, no iterative techniques are needed, although more CPU time will be used.

Using Finite Process Method, reasonable and accurate numerical results can be obtained in region $-6.49(ft^2/sec)\le\Gamma/2\pi\le 2.50(ft^2/sec)$.

Table 5.1: numerical results in different value of Δp

Δp	ζ_{max} (ft)	ζ_{min} (ft)	λ_w (ft)	R_w (bl/ft)	$(e_k)_{max}$	$(e_d)_{max}$
1.0	0.9342	-0.7201	18.3512	9.4341	$1.6E-2$	$8.1E-3$
1/2	0.9314	-0.7289	18.3613	9.5473	$2.5E-3$	$1.6E-3$
1/5	0.9329	-0.7301	18.3651	9.5787	$5.5E-4$	$1.5E-4$
1/10	0.9334	-0.7301	18.3644	9.5789	$7.5E-5$	$1.2E-5$
1/20	0.9334	-0.7301	18.3644	9.5778	$1.1E-5$	$7.8E-7$
1/50	0.9334	-0.7301	18.3644	9.5774	$4.8E-7$	$2.6E-8$
1/100	0.9334	-0.7301	18.3644	9.5774	$3.5E-8$	$7.7E-9$

For simplicity, we have selected in this paper $f_1(p) = p$ and $f_2(p) = 1 - p$. Clearly, there exist many other process functions, for example,

$$\begin{aligned} f_1(p) &= sin^m\left(\frac{p\pi}{2}\right) \qquad (m \geq 1) \\ f_2(p) &= cos^m\left(\frac{p\pi}{2}\right) \qquad (m \geq 1) \end{aligned}$$

or

$$\begin{aligned} f_1(p) &= p^m \\ f_2(p) &= (1-p)^m \end{aligned}$$

Using these process functions in our computer Program, we obtain the similar result: smaller Δp is, more accurate the numerical results are.

5.5 relationship between Finite Process Method and iteration

Consider the zero-order process equation (5.9)~(5.12). Clearly, the initial solution $\phi_0(x,z)$ and $\zeta_0(x)$ can be freely selected. If $\phi_0(x,z)$ and $\zeta_0(x)$ are just the true solution of original problems, then according to first-order process equation (5.32), (5.33), (5.45) and (5.46), $\phi^{[1]}(x,z;p) = 0$ and $\zeta^{[1]}(x;p) = 0$. But, at begining of computation, the solutions are unknown and we must select the initial solutions which may be far from the accurate solutions. The general form of initial solution $\phi_0(x,z)$ can be written as

$$\begin{aligned} \phi_0(x,z) &= -Ux + \frac{\Gamma}{2\pi}arctg\left(\frac{z+b}{x}\right) - \frac{\Gamma}{2\pi}arctg\left(\frac{z-b}{x}\right) \\ &+ \int_{x_a}^{x_b} \sigma_0(\xi,h_s)\ln\sqrt{(x-\xi)^2+(z-h_s)^2}d\xi \end{aligned} \tag{5.78}$$

The corresponding final solution $\phi_f(x,z)$ is now as:

$$\begin{aligned} \phi_f(x,z) &= -Ux + \frac{\Gamma}{2\pi}arctg\left(\frac{z+b}{x}\right) - \frac{\Gamma}{2\pi}arctg\left(\frac{z-b}{x}\right) \\ &+ \int_{x_a}^{x_b} \sigma_f(\xi,h_s)\ln\sqrt{(x-\xi)^2+(z-h_s)^2}d\xi \end{aligned} \tag{5.79}$$

Table 5.2: iterations in different value of Δp

Δp	iterations	ζ_{max} (ft)	ζ_{min} (ft)	λ_w (ft)	R_w (bl/ft)	$(e_k)_{max}$	$(e_d)_{max}$
1.0	1	0.9341	-0.7201	18.3512	9.4341	$1.6E-2$	$8.1E-3$
	2	0.9334	-0.7301	18.3644	9.5774	$3.6E-6$	$4.3E-6$
	3	0.9334	-0.7301	18.3644	9.5774	$2.7E-13$	$2.2E-13$
	4	0.9334	-0.7301	18.3644	9.5774	$8.1E-16$	$2.2E-16$
0.2	1	0.9329	-0.7301	18.3651	9.5787	$5.5E-4$	$1.5E-4$
	2	0.9334	-0.7301	18.3644	9.5774	$4.1E-12$	$9.9E-12$
	3	0.9334	-0.7301	18.3644	9.5774	$1.2E-15$	$4.7E-16$
	4	0.9334	-0.7301	18.3644	9.5774	$8.3E-16$	$2.9E-16$
0.1	1	0.9334	-0.7301	18.3644	9.5774	$7.5E-5$	$1.2E-5$
	2	0.9334	-0.7301	18.3644	9.5774	$1.1E-12$	$6.0E-13$
	3	0.9334	-0.7301	18.3644	9.5774	$9.3E-16$	$4.5E-16$
	4	0.9334	-0.7301	18.3644	9.5774	$9.3E-16$	$4.5E-16$

where

$$\sigma_f(\xi, h_s) = \sigma_0(\xi, h_s) + \int_0^1 \sigma^{[1]}(\xi, h_s; p)dp \tag{5.80}$$

or more clearly

$$\sigma_f(\xi_m, h_s) = \sigma_0(\xi_m, h_s) + \int_0^1 \sigma_m^{[1]}(p)dp \qquad (m = 1, 2, \cdots, N) \tag{5.81}$$

At begining of computation, select $\sigma_0(\xi, h_s) = 0$ and $\zeta_0(x) = 0$ to make a computation under a selected Δp. If Δp small enough, the results are accurate enough, which can be used as the solution of original problems. If Δp is not small enough, then crude results $\sigma_f^*(\xi, h_s)$ and $\zeta_f^*(x)$ are obtained. Although $\sigma_f^*(\xi, h_s)$ and $\zeta_f^*(x)$ are not accurate enough, they are clearly much better than the selected initial solutions $\sigma_0(\xi, h_s) = 0$ and $\zeta_0(x) = 0$. Clearly, a better numerical results will be obtained, if $\sigma_f^*(\xi, h_s)$ and $\zeta_f^*(x)$ are used as new initial solutions to make a new computation. This is in fact just the idea of iteration. From view point of this, expressions given by Finite Process Method in different value of Δp will give different iterative formulas. Smaller Δp is, more complex the corresponding iterative formulas are. Clearly, formulas in case $\Delta p = 1.0$ are the simplest of them.

We use the same example described in above section to show this point. At begining of computation, let $\sigma_0(\xi, h_s) = 0$ and $\zeta_0(x) = 0$. The new obtained results are used as initial soluions to make the next computation. Different value of Δp (Δp=1.0, Δp=0.2, Δp=0.1) are considered. The results are given in Table 5.2.

From Table 5.2, it seems that iterations in each case of Δp will converge. More complex the iterative formulas are, (i.e., smaller Δp is), faster the iteratation converges, but clearly more CPU time is needed.

It is interesting that if Δp is small enough, the numerical results will be accurate enough and no iteration is needed. So, we can see iterations as special cases of Finite Process Method in case of great Δp.

Table 5.3: convergence region of different iterative formulas

iterative model	convergence region (ft^2/sec)	ζ_{max} (ft)	max. slope (degree)
FPM ($\Delta p = 1$)	-6.49 $\leq \Gamma/2\pi \leq$ 2.45	1.206	23.0
Ref. [25]	-5.88 $\leq \Gamma/2\pi \leq$ 2.13	1.066	19.8

There exist some other iterative formulas of nonlinear water-wave problems. The formulas given by Jensen, Mi and Söding [25] are as follows:

$$\begin{aligned} & g\phi_z + \frac{1}{2}\nabla\phi\nabla(\nabla\Phi\nabla\Phi) + \nabla\Phi\nabla(\nabla\Phi\nabla\phi - \nabla\Phi\nabla\Phi) \\ + & \frac{\frac{\partial}{\partial z}\left\{g\Phi_z + \frac{1}{2}\nabla\Phi\nabla(\nabla\Phi\nabla\Phi)\right\}}{g + \nabla\Phi\nabla\Phi_z}\left\{\frac{1}{2}(U^2 - 2\nabla\Phi\nabla\phi + \nabla\Phi\nabla\Phi) - g\zeta_0\right\} \\ = & \ 0 \end{aligned} \tag{5.82}$$

and

$$\zeta = \zeta_0 + \frac{\frac{1}{2}(U^2 - 2\nabla\Phi\nabla\phi + \nabla\Phi\nabla\Phi) - g\zeta_0}{g + \nabla\Phi\nabla\Phi_z} \tag{5.83}$$

where, Φ, ζ_0 are old value and ϕ, ζ are new values.

Let us compare the above iterative formulas to the simplest one given by Finite Process Method in case $\Delta p = 1.0$. It is well known that iterations are generally sensitive to the initial solutions and will diverge in some case of strong nonlinearity. The convergece region and the numerical maximum wave-elevation (far downstream) and the maximum slope obtained by different formulas are given in Table 5.3.

According to Table 5.3, the iterative formulas given by FPM in case Δp =1 have a greater region of convergence than the iterative formulas (5.82) and (5.83). Note that $\Delta p = 1$ is corresponding to the simplest. In fact, smaller Δp is, more insensitive the corresponding iterative formulas are to the initial solutions.

Clearly, it is easy to use the basic ideas of Finite Process Method to derive a family of iterative formulas for any a reasonable nonlinear problem, among which the simplest is in case Δp =1, and more complex formulas are corresponding to smaller Δp. If Δp is small enough, then the numerical results are accurate enough and no iteration is needed, although more CPU time is needed.

In fact, the iterative formulas (5.82) and (5.83) can be also obtained from expressions (5.45) and (5.46).

In (5.45) and (5.46), let $p = 1.0$ and substitute ϕ_0 and ϕ by Φ, then, one has

$$\begin{aligned} & g\phi_z^{[1]} + \frac{1}{2}\nabla\phi^{[1]}\nabla(\nabla\Phi\nabla\Phi) + \nabla\Phi\nabla(\nabla\Phi\nabla\phi^{[1]}) - \frac{\nabla\Phi\nabla\phi^{[1]}\frac{\partial\mathcal{R}(\Phi)}{\partial z}}{g + \nabla\Phi\nabla\Phi_z} \\ = & \ -\mathcal{R}(\Phi) - \frac{\{\mathcal{Z}(\Phi) - \zeta_0(x,y)\}\frac{\partial\mathcal{R}(\Phi)}{\partial z}}{1 + \nabla\Phi\nabla\Phi_z/g} \end{aligned} \tag{5.84}$$

and

$$\zeta^{[1]} = \frac{\mathcal{Z}(\Phi) - \zeta_0(x,y) - \nabla\Phi\nabla\phi^{[1]}/g}{1 + \nabla\Phi\nabla\Phi_z/g} \tag{5.85}$$

In above expressions, substituting $\phi^{[1]}$ by $(\phi - \Phi)$ and $\zeta^{[1]}$ by $(\zeta - \zeta_0)$ respectively, one can obtain the same formulas as (5.82) and (5.83).

5.6 comparison to the results given by other others :

5.6.1 2D flow past a submerged vortex

In case $U = 10$ (fps) and $b = 4.5$ (ft), the 2D deep waves past a submerged vortex are researched by the numerical scheme described above. The used numerical parameters are $x_a/\lambda_0 = -3.5$, $x_b/\lambda_0 = 1.5$,$h_s = 1.6$ (ft) and $N = 200$.

The maximum and minimum wave-elevation, wave-length (downstream) and wave-resistance in different values of vortex circulation $\Gamma/2\pi$ are given respectively in Table 5.4, Table 5.5 and Table 5.6, whereas the perturbation solutions at third-order approximation for deep waves and the numerical results in finite water depth given in [37] are also listed in order to make a comparison. The converged results in cases of $\Gamma/2\pi < -3.20$ (ft^2/sec) and $\Gamma/2\pi > 2.20$ (ft^2/sec) are also given in Table 5.7.

From Table 5.4, Table 5.5,Table 5.6 and Table 5.7, it seems that, in cases of $|\Gamma/2\pi| < 1.7$ (ft^2/sec), the results given by Finite Process Method, specially the wave-length ,are in better agreement with the perturbation solutions at third-order of approximation than the numerical results given in [37]. In the cases of $|\Gamma/2\pi| > 1.70$ (ft^2/sec), we obtain the results which are considerablely different not only from the perturbation solutions at third-order of approximation but also from the numerical results given in [37]. It seems that perturbation solution at higher-order of approximation should be given for flow with stronger nonlinearity. And it seems also that water-depth has a great influence to 2D waves past a submerged vortex, specially to wave-length.

The maximum and minimum wave-elevation and the maximum slope of first crest are given in Table 5.8. In case of positive circulations, the height of first crest is nearly the same as those far downstream. But, in case of negative circulations, the first crest is always higher.

Figure 5.2 and figure 5.3 show the wave-elevation in case of different values of circulation. For positive circulations, the elevation height of first crest is nearly the same as those far downstream and all of them increase with the increment of vortex circulation until the limit status is reached. For the negative circulations, the wave-height of first crest is always greater than those far downstream and will increase with the decrement of vortex circulation, whereas, ζ_{max} far downstream has a crest-value in the near of $\Gamma/2\pi = -3.2$ (ft^2/sec) and a trough-value at $\Gamma/2\pi = -6.0$ (ft^2/sec), shown as figure 5.3. At $\Gamma/2\pi = -6.0$ (ft^2/sec), the height of wave elevation far downstream is so small that it is nearly an isolate wave. In the cases of negative circulation, the maximum wave-slope and wave-elevation far downstream are much less than the theoretical limits. It is clear from figure 5.3 that the wave-breaking will occur, when the first crest is too high. Accoding to the experiment of Salvensen [38], wave-breaking occurs at first crest. Our numerical results support his experiment.

Stokes [50] [51] had shown that the limiting form of steady irrotational gravity waves processes sharp crests containing an angel of 120 degree, that is, a maximum slope of 30 degree. And the limit of free-surface elevation is

$$\zeta_{lim} = \frac{U^2}{2g}$$

i.e., in case $U = 10.0$ (fps), $\zeta_{lim} = 1.54$ (ft).

The maximum ratio of wave height-to-length H_w/λ_w is given by Michell [9] as 0.142, by Thomas

Table 5.4: maximum and minimum wave-elevation (far downstream) in case of $U = 10$ (fps) and $b = 4.5$ (ft)

$\Gamma/2\pi$	3th-order perturbation [37]		numerical results			
			reference [37]		present method	
(ft^2/sec)	ζ_{max} (ft)	ζ_{min} (ft)	ζ_{max} (ft)	ζ_{min} (ft)	ζ_{max} (ft)	ζ_{min} (ft)
-3.20	0.544	-0.511	0.64	-0.58	0.558	-0.438
-2.70	0.495	-0.339	0.60	-0.54	0.553	-0.436
-2.10	0.440	-0.335	0.50	-0.47	0.497	-0.401
-1.70	0.390	-0.326	0.43	-0.40	0.466	-0.382
-1.40	0.341	-0.298	0.37	-0.35	0.369	-0.317
-1.15	0.293	-0.263	0.30	-0.29	0.312	-0.275
-0.90	0.239	-0.220	0.25	-0.24	0.250	-0.226
0.90	0.308	-0.281	0.29	-0.28	0.305	-0.280
1.15	0.409	-0.361	0.38	-0.36	0.406	-0.362
1.40	0.517	-0.440	0.47	-0.44	0.514	-0.445
1.70	0.655	-0.534	0.59	-0.54	0.659	-0.548
2.20	0.908	-0.685	0.84	-0.72	0.959	-0.729
2.70	1.187	-0.848	1.24	-0.93	‡	‡
3.20	1.495	-1.082	‡	‡	‡	‡

‡no convergence

[53] as 0.145 and by Schwartz [48] as 0.141∼0.142. The experiments by Salvensen [38] gives the maximum $H_w/\lambda_w = 0.11$ for the downstream waves, while the numerical maximum value given in [37] is $H_w/\lambda_w = 0.12$.

In the case of $U = 10$ (fps) and $b = 4.5$ (ft), the limit wave-elevation is obtained at $\Gamma/2\pi = 2.50$ (ft^2/sec) with $\zeta_{max} = 1.316$ (ft), maximum value of $H_w/\lambda_w = 0.125$ and maximum slope of 25.8 degree. With comparison to [37], we obtain a little steeper wave-elevation, shown as Table 5.9. The limiting form of wave-elevation for negative circulation is obtained in the case of $\Gamma/2\pi = -6.49$ (ft^2/sec) with the first crest $\zeta_{max} = 1.459$ (ft) and maximum slope 29.7 degree, while the wave-elevation far downstream is much evener. It is interesting that the limiting form of wave elevation in negative circulation is very different from that in positive circulation, shown as figure 5.2 and figure 5.3.

Figure 5.4 shows the wave-resistance in case $U = 10$ (fps) and $b = 4.5$ (ft). For $\Gamma > 0$, wave-resistance increases with the increment of vortex circulation; but, for $\Gamma < 0$, the curve of wave-resistance is very different : there exist a crest and also a trough in the region of negative circulation. At $\Gamma/2\pi = -6.0$ (ft^2/sec), the wave-resistance is nearly zero, while, the corresponding wave-elevation is nearly an isolate wave. For $\Gamma/2\pi < -6.0$ (ft^2/sec), the wave-resistance increases until the first crest of wave-elevation is too high and then the numerical wave-breaing occurs. These very interesting result have not been reported in [37].

Similar to wave-resistance, the wave-lengths in case of negative circulations are also considerably different from those in case of positive one, shown as figure 5.5. For positive circulation, the wave-length always decreases with the increment of vortex circulation. For negative circulation, the wave-length decrease at first but begin to *increase* later. This result is interesting.

Table 5.5: wave-length (ft) (far downstream) in case of $U = 10$ (fps) and $b = 4.5$ (ft)

$\Gamma/2\pi$ (ft^2/sec)	third-order perturbation method [37]	numerical results	
		reference [37]	present method
-3.20	17.73	19.2	18.82
-2.70	18.24	19.3	18.83
-2.10	18.75	19.4	18.96
-1.70	19.02	19.5	19.11
-1.40	19.18	19.5	19.23
-1.15	19.30	19.5	19.32
-0.90	19.39	19.6	19.40
0.90	19.39	19.6	19.40
1.15	19.30	19.5	19.29
1.40	19.18	19.5	19.14
1.70	19.02	19.1	18.89
2.20	18.67	18.8	18.26
2.70	18.24	18.0	‡
3.20	17.73	‡	‡

‡no convergence

Table 5.6: wave-resistance (lb/ft) of 2D waves past a submerged vortex in case of $U = 10$ (fps) and $b = 4.5$ (ft)

$\Gamma/2\pi$ (ft^2/sec)	perturbation results [37]			numerical results	
	first order	second order	third order	reference [37]	present method R_w
-3.20	13.92	10.40	4.04	5.53	3.616
-2.70	9.91	7.00	2.61	4.91	3.597
-2.10	5.99	4.13	2.26	3.54	3.011
-1.70	3.93	2.67	1.94	2.55	2.367
-1.40	2.66	1.94	1.55	1.91	1.809
-1.15	1.80	1.37	1.19	1.36	1.333
-0.90	1.10	0.88	0.81	0.88	0.884
0.90	1.10	1.41	1.33	1.22	1.321
1.15	1.80	2.48	2.23	2.08	2.251
1.40	2.66	3.95	3.40	3.15	3.476
1.70	3.93	6.34	5.10	4.84	5.351
1.90	4.91	8.38	6.40	6.19	6.894
2.20	6.58	12.23	8.57	8.73	9.642
2.50	8.49	17.17	10.96	11.76	‡
2.70	9.91	21.15	12.66	14.04	‡
3.20	13.92	33.97	16.97	‡	‡

‡no convergence

Table 5.7: 2D wave (far downstream) past a submerged vortex in case of U = 10.0 (fps) and b = 4.5 (ft)

$\Gamma/2\pi(ft^2/sec)$	ζ_{max} (ft)	ζ_{min} (ft)	max. slope (degree)	λ_w (ft)	R_w (bl/ft)
-6.49	0.199	-0.172	3.6	19.64	0.526
-6.4	0.163	-0.140	2.8	19.70	0.329
-6.0	0.004	-0.002	0.5	††	0.010
-5.0	0.268	-0.217	4.6	19.17	0.863
-4.7	0.346	-0.278	5.9	19.12	1.425
-3.7	0.523	-0.412	9.0	18.88	3.180
2.30	1.039	-0.767	19.2	18.07	10.690
2.40	1.144	-0.792	21.5	17.68	11.728
2.45	1.206	-0.825	23.0	17.64	12.388
2.50	1.316	-0.848	25.8	17.37	12.942

††no numerical value of wave-length because of the too small wave-height of downstream

Table 5.8: maximum and minimum wave-elevation and maximum slope of first crest

$\Gamma/2\pi$ (ft^2/sec)	-6.49	-6.4	-6.0	-5.0	-4.7	-3.7	-3.2	-2.7
ζ_{max} (ft)	1.459	1.441	1.372	1.230	1.184	1.006	0.901	0.786
ζ_{min} (ft)	-0.147	-0.115	0.000	-0.160	-0.229	-0.385	-0.419	-0.424
max. slope (degree)	29.7	25.8	21.7	17.5	16.6	14.3	13.1	11.7
$\Gamma/2\pi$ (ft^2/sec)	-2.1	-1.9	-1.7	-1.4	-1.15	-0.9	0.9	1.15
ζ_{max} (ft)	0.633	0.579	0.523	0.436	0.362	0.285	0.302	0.401
ζ_{min} (ft)	-0.394	-0.376	-0.353	-0.313	-0.272	-0.224	-0.299	-0.385
max. slope (degree)	9.8	8.9	8.1	6.9	5.8	4.6	5.5	7.2
$\Gamma/2\pi$ (ft^2/sec)	1.40	1.70	1.90	2.20	2.30	2.40	2.45	2.50
ζ_{max} (ft)	0.509	0.652	0.759	0.881	0.950	1.029	1.191	1.291
ζ_{min} (ft)	-0.472	-0.578	-0.650	-0.724	-0.762	-0.799	-0.857	-0.878
max. slope (degree)	9.2	11.7	13.7	16.1	17.4	19.17	22.7	25.1

Table 5.9: steepest elevation (far downstream) of 2D deep gravity waves

	theory	experiment [38]	numerical result	
			reference [37]	present method
ζ_{lim}	$U^2/2g$	0.82*$U^2/2g$	0.87*$U^2/2g$	0.85 * $U^2/2g$
maximum slope (degree)	30	25	24.4	25.8
$(H_w/\lambda_w)_{max}$	0.141~0.145	0.11	0.12	0.125

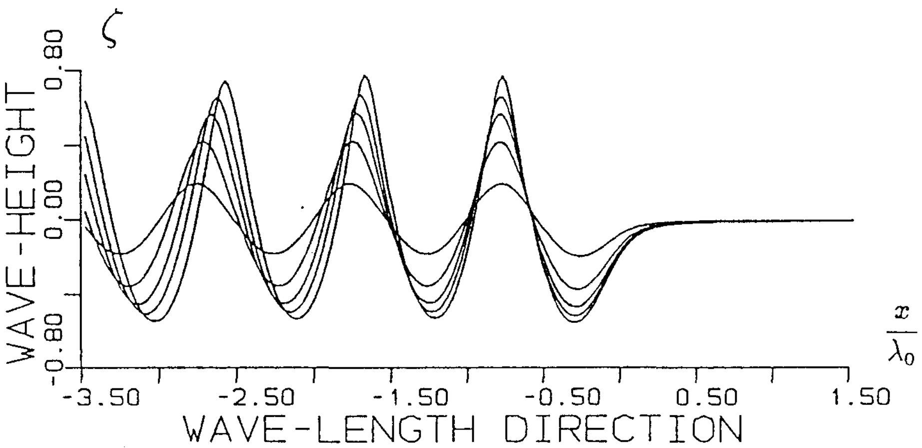

Figure 5.2: wave-elevation of 2D waves past positive circulations

U = 10 (fps) , b = 4.5 (ft), water depth is infinite.
ζ_{max} increase with the increament of $\Gamma/2\pi$ = 0.90, 1.70, 2.10, 2.30, 2.45 (ft^2/sec)

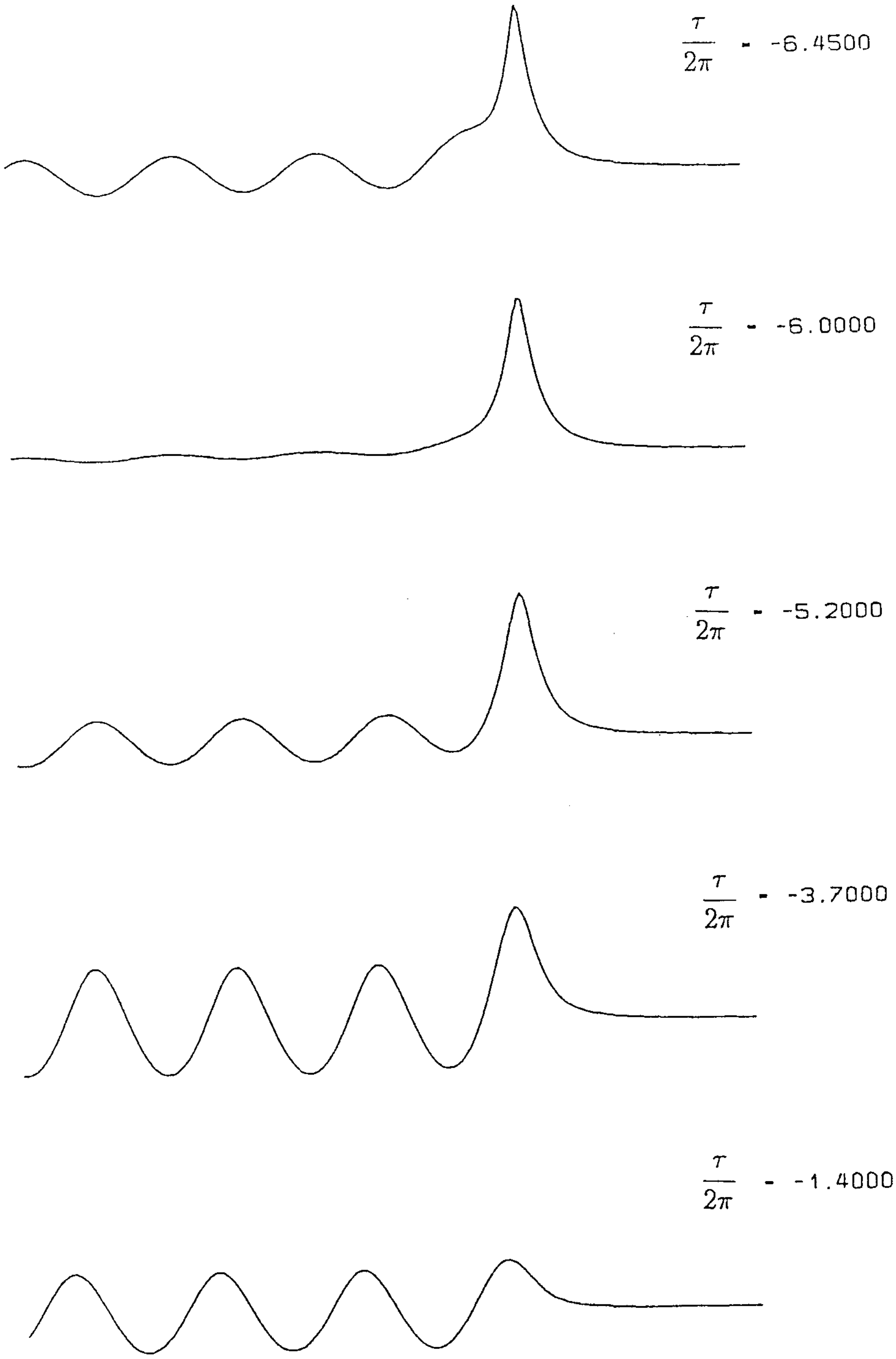

Figure 5.3: wave-elevation of 2D waves past negative circulations

U = 10 (fps), b = 4.5 (ft),water depth is infinite.
From bottom to top, the corresponding circulations are
$\Gamma/2\pi$ = -1.4, -3.7, -5.2, -6.0, -6.45 (ft^2/sec)

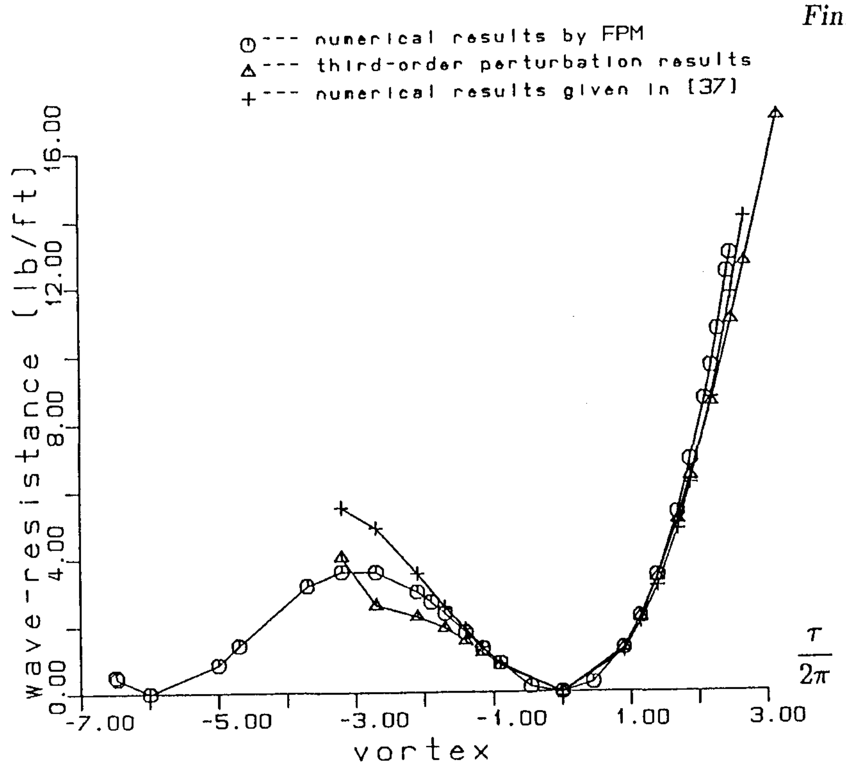

Figure 5.4: wave-resistance of 2D waves past a submerged vortex in case U = 10 (fps) and b = 4.5 (ft)

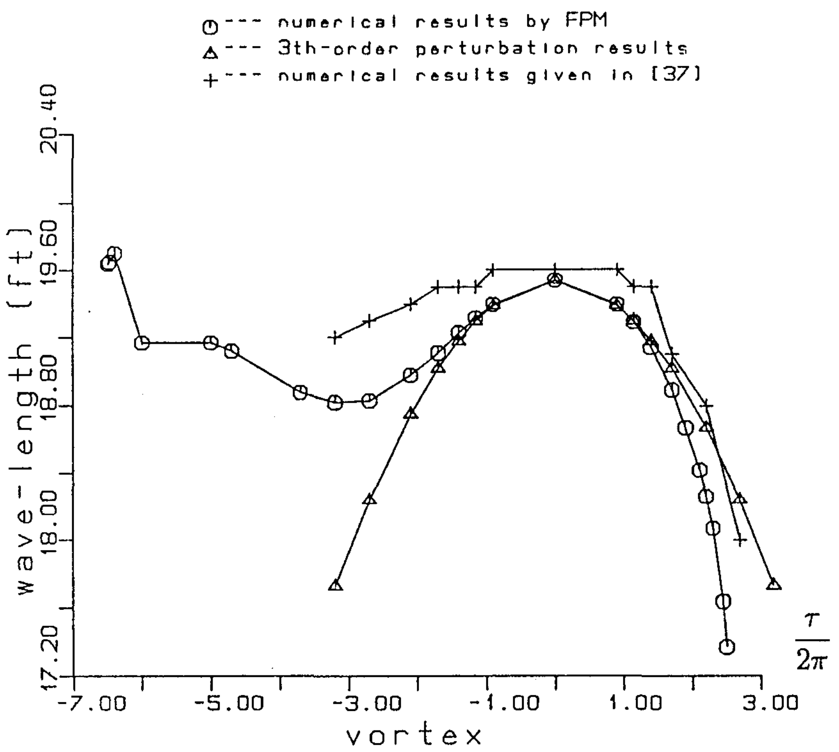

Figure 5.5: wave-length of 2D waves past a submerged vortex in case U = 10 (fps) and b = 4.5 (ft)

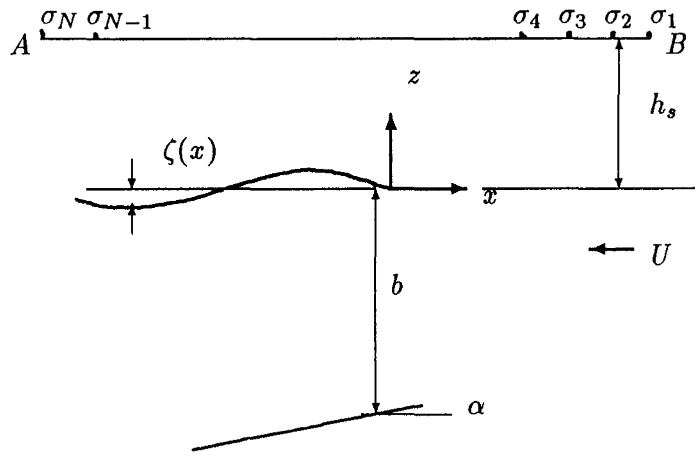

Figure 5.6: coordinate-system and grid for numerical computation of 2D flow past a submerged thin plain wing

5.6.2 flow past a 2D submerged thin plain wing

A submerged wing can be substituted simply by a submerged vortex, as shown in above sections. But, this model is perhaps too simple. So, in this subsection, we would use a little more complex model to research the folw past a 2D submerged thin plain wing.

Suppose that the fluid is inviscid, imcompressiable and without surface tensen. Consider a 2D wing with cord length c, angel of attack α, which is moving at velocity U (parallel to the undisterbed free surface) in a submerged distance b, where b is the perpendicular distance of the point on the wing quarter of the cord length c behind its nose to the free surface, shown as figure 5.6. Suppose that the 2D aerefoil is thin and plain so that it can be substituted by a plane plate with length c.[3]

The boundary conditions of free surface are still nonlinear and we treat them in the same way as mentioned above. Different from the former example, there exists now another boundary condition $\partial\phi/\partial n = 0$ which should be satisfied on the wing. In order to satisfy this linear boundary condition, we distribute continuous vortex with strength $\gamma(\xi)$ along the wing. So, the velocity-potential function should be described as follows:

$$\phi(x,z) \quad = \quad -Ux + \int_{x_a}^{x_b} \sigma(\xi, h_s) \ln \sqrt{(x-\xi)^2 + (z-h_s)^2} d\xi$$

[3]Note that this will bring singularity at the nose of wing, i.e., theoretically speaking, the velocity at nose is infinity.

$$+ \int_{-3c/4}^{c/4} \frac{\gamma(\xi)}{2\pi} arctg \left(\frac{z+b-\xi \sin\alpha}{x-\xi\cos\alpha} \right) d\xi \tag{5.86}$$

We discretize the surface of wing in NB elements and suppose that in each element $\gamma(\xi)$ is a linear function (i.e., we use linear boundary elements). At centre points of these elements, let boundary condition $\partial\phi/\partial n = 0$ be satisfied. At edge of the wing, set $\gamma(\xi) = 0$ so that the Joukowski's condition can be satisfied. These give NB algebraic equations which, combined with the N algebraic equations given by free surface conditions as mentioned above, construct the set of linear algebraic equations.

We use the linear boundary condition of free surface

$$g\phi_z + U^2\phi_{xx} = 0 \qquad \text{on } z = 0$$

to give the initial solution ϕ_0 and

$$\zeta_0 = \frac{1}{2g}(U^2 - \nabla\phi_0\nabla\phi_0).$$

Generally, we use the simple iterative formulas (5.82) and (5.83) given by Jensen, Mi and Söding. If they can't give converged results, then, we use more complex iterative formulas mentioned above in different value of Δp, for instance, $\Delta p = 0.5, \Delta p = 0.2$, $\Delta p = 0.1$ and so on.

Let

$$F_h = \frac{U}{\sqrt{gb}}$$

$$\delta = \frac{b}{c}$$

and define

$$C_k = \frac{R_w}{\rho U^2 b}$$

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 c}$$

as the coefficients of wave-resistance R_w and the lift-force L of the wing, respectively. It is easy to know that C_k and C_L are functions of F_h, δ and the angel of attack α.

Before discribing our numerical results, we would discuss simply some problems which should be specially payed attentions to in numerical computations.

We have

$$\frac{b}{\lambda_0} = \frac{1}{2\pi F_h^2} \tag{5.87}$$

where, $\lambda_0 = 2\pi U^2/g$ is the wave-length given by linear theory.

In case of great Froude number of depth, for instance, $F_h = 4.0$, we have

$$\frac{b}{\lambda_0} = \frac{1}{32\pi} \approx 0.01 \tag{5.88}$$

It menas that, if we set $\lambda_0 = 1.0$, then $b \approx 0.01$. In this case, the wing is very close to the free surface so that the change of flow between the submerged wing and the free surface is strong, specially in case of $b/c < 1$. In order to describe the flow in this region, very fine numerical model and numerical grid should be used. At first, we should use the continuously distributed vortex with strength $\gamma(\xi)$ on the wing, but not the discretized vortex Γ_i, although the latter model can give also accurate enough results in cases of small F_h (generally, $F_h < 2.5$) [4]. Secondly, if we use boundary elements with the same length on $\overline{AB}$, then much more boundary elements should be used on $\overline{AB}$, owing to the fact that the number of elements N should be approximately directly proportional to F_h^2 in case of great F_h ($F_h > 2.5$). [5] In order to avoid solving large algebraic equations, we use boundary elements with different length in case of great F_h: in a region of $2c$ above the wing, elements with length $\delta_0 = b/4$ or $b/3$ are used; then, the length of elements increase in form $(\Delta x)_k = q^k\delta_0$, where $q > 1$ should be given; and in the region far enough from the wing, elements with same length are used. Thirdly, more attentions should be payed, specially in case of great F_h, to the selection of h_s, the height of $\overline{AB}$ above the undisturbed free surface.

Note that the model of 2D plain plate will bring singularity at nose of it, where the velocity is infinite. The influence of this singularity is small to the far field but should be considered to the flow near the wing. In case of great values of F_h, for instance, $F_h > 4$, the free surface is very close to the submerged wing and the flow near free surface is just the flow near the wing. In this case, the influence of this kind of singularity to the free surface conditions is strong so that it should be considered whether or not the model of 2D plain plate can give accurate enough results. It seems that we had to use other better models to substitute a submerged wing or consider the viscid of the fluid in case of great F_h ($F_h > 4$).

Now, we would discuss simply our numerical results.

Figure 5.7 and figure 5.8 show respectively the curves of $dC_L/d\alpha$, C_k with respect to b/c in case of $F_h = 2.0, \alpha = 0.1$ (rad). It seems from Figure 5.7 and figure 5.8 that with increment of b/c, $dC_L/d\alpha$ will increase but C_k will decrease. It means that, longer the cord length of the wing is, greater the wave-resistance and the influence of free surface to the lift-force property of the wing are.

Figure 5.9 and figure 5.10 show respectively the curves of $dC_L/d\alpha$, C_k with respect to F_h in case of $b/c = 2.0, \alpha = 0.1$ (rad). In case of small F_h, C_k is very small and $dC_L/d\alpha > 2\pi$. In case of $F_h > 1$, $dC_L/d\alpha$ is generally smaller than 2π. Figure 5.9 and figure 5.10 show that the free surface has a great influence to wing in $0.8 < F_h < 2.0$ in this case.

In case of $b/c = 0.5, b/c = 0.25$ and $F_h > 2.5$, the nonlinearity is generally strong. Figure 5.11 shows the comparison of our numerical results of C_k to the perturbation results at second-order of

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[4] Our numerical experiences shows that in case $F_h > 2.5$, the model with discretized vortex Γ_i on the wing can't give reasonable results and in fact, in most cases, iteration will diverge. This is perhaps because the discretized vortex Γ_i can't describe finely the flow in the near of wing. But in case of small F_h, this model is simple and can give accurate enough results, because in this case, the flow near the free surface is the far-field with respect to the wing.

[5] We can show this as follows:

Suppose that the length of $\overline{AB}$ is $2\pi\Lambda U^2/g$ and the length of every boundary element on $\overline{AB}$ is the same, then the length of boundary element on $\overline{AB}$ is

$$\Delta x = \frac{2\pi\Lambda U^2}{gN}$$

In order to discribe finely the flow between the wing and the free surface, it is needed that $\Delta x \leq b$. (In practice, it seems better to let $\Delta x = b/4 \sim b/3$.) Without loss of generality, suppose $\Pi\Delta x = b$, where $\Pi \geq 1$, then

$$\frac{2\pi\Lambda\Pi U^2}{gN} = \frac{U^2}{gF_h^2}$$

so, $N = 2\pi\Lambda\Pi F_h^2$.

In the similar way, we can prove that, for 3D problems, N should be approximately directly proportional to F_h^4, in cases of great F_h, if we use the boundary elements with same length on $\overline{AB}$.

approximation given by Isay and Müscher [64] in case of $\alpha = 0.1$ (rad)and $b/c = 0.5, b/c = 0.25$. It seems that, in case of $b/c = 0.5$, the theoretical and the numerical results are in a good agreement, specially in case $F_h < 3.0$. In case of $F_h > 3.0$ or $b/c = 0.25$, the nonlinearity is strong and it seems that perturbations at higher-order of approximation should be needed.

Figure 5.12, figure 5.13 and figure 5.14 show respectively the wave-elevation, the distribution of singularity $\sigma(\xi, h_s)$ on $\overline{AB}$ and the distribution of continuous vortex $\gamma(\xi)$ on the wing in case $F_h = 4.0, b/c = 0.5, \alpha = 0.1$ (rad). Note that there exists a discontinuity in the distribution of singularity $\sigma(\xi, h_s)$. It seems that the wave in this case is near the state of breaking.

Detailed numerical results of the coefficients of wave-resistance and lift of the wing and the wave-elevations far downstream in cases of $\alpha = 0.1$ (rad) and $b/c = 0.50$ or $b/c = 0.25$ are given respectively in Table 5.10 and Table 5.11. Note that it should be considered whether or not these results have physical meanings in case of great F_h, because the singularity at nose of plain plate may be so great that the results can not be used in practice.

In above subsection, we have researched the flow fast a submerged vortex in case $U = 10$ (fps) and $b = 4.5$ (ft), corresponding to $F_h = 0.83136$. As mentioned in above subsection, the flow in case $\Gamma/2\pi = -6(ft^2/sec)$ is nearly an isolate wave and the corresponding wave-resistance is nearly zero. Clearly, a submerged vortex is a crude model for flow past a submerged wing. So, we use here the 2D plane plate as a better model to research the same problem in case $F_h = 0.83136$. Without loss of generality, we select $b/c = 1$. The results are shown as figure 5.15 and figure 5.16. Note that C_L is nearly the linear function of α, shown as figure 5.15. It is interesting that in case $\alpha = -0.210$ (rad), the wave-resistance coefficient C_k is also nearly zero and the corresponding wave-elevation is nearly an isolate wave, shown as figure 5.16 and figure 5.17. Note that in case $\alpha = -0.21$ (rad) the iterative formula (5.82) (5.83) will diverge so that the formulas given by Finite Process Method ($\Delta p = 1/3$) must be used in order to obtain these results. In case $\alpha = 0.13$ (rad), we obtain far downstream wave-elevation $\zeta_{max} = 0.832U^2/(2g)$, $\zeta_{min} = -0.501U^2/(2g)$, $\lambda_w = 0.872\lambda_0$ with maximum slope of wave-elevation 27.5 degree, corresponding to $(H_w/\lambda_w) = 0.122$ which is the maximum value we can obtained by Finite Process Method in case $F_h = 0.83136$ and $b/c = 1$.

All results given above are obtained in case of infinite water depth. In fact, it is very simple to consider the influence of water depth D. As an example, we compute the wave-resistance and lift force of the submerged wing in case of $F_h = 0.83136, b/c = 1$ in different water depth $D/b = 2, 3, 4$, shown as figure 5.15 and figure 5.16. The detailed numerical results are given in Table 5.12~5.15. It seems that when $D/b \geq 4$, the influence of water depth can be neglected. Just only in case $D/b \leq 2$, the influence of water depth must be considered.

In case $F_h = 0.83136, b/c = 1.0$, the above numerical computations show that in some case the wave-resistance of a 2D submerged thin plain wing would be nearly zero and the corresponding wave-elevation would be nearly an isolate wave. Is this interesting result a general property of 2D submerg thin plain wing? In order to have a more general understanding about this property of 2D submerged plain thin wing, we research the same problem also in case $b/c = 0.85$, $b/c = 1.0$ and $b/c = 2.0$ in cases of different Froude number F_h. The curve of wave-resistance coefficients are shown as figure 5.18, figure 5.19 and figure 5.20; the detailed results are given in table 5.16 ~ table 5.36. When $b/c = 0.85$, shown as figure 5.18, the wave-resistance is nearly zero near $\alpha = -0.15$ in cases of $F_h = 0.75, F_h = 0.80, F_h = 0.85, F_h = 0.90$. When $b/c = 1.00$, shown as figure 5.19, the wave-resistance is nearly zero near $\alpha = -0.20$ in cases of $F_h = 0.75, F_h = 0.80, F_h = 0.85, F_h = 0.90$. But when $b/c = 2.0$, shown as figure 5.20, the wave-resistance is only in case $F_h = 0.75$ of the researched cases nearly zero near $\alpha = -0.41$, however, the results might have no meaning because the numerical model used here is not suitful for the flow past a submerged wing in such a great angel of attact α.

Generally, the wave-resistance under positive value of the attack angel α will always increase with the increament of α and have a little difference in cases of different Froude F_h for a selected value of b/c. However, in case of small F_h, the wave-resistance will generally start to decrease when $-\alpha$ is greater than a value which seems dependent upon F_h and b/c. But in case of great F_h, for example, $F_h = 1.5$ in case $b/c = 1.0$, shown as figure 5.19, the wave-resistance will always increase with the value of $|\alpha|$. The numerical results show that, generally, the wave-resistance of a 2D submerged plain thin wing will be nearly zero near a angel of attack $\alpha < 0$ in case of small b/c and F_h. This results is interesting. If it is a general property of a submerged wing, perhaps, we can find some applications of it in decreasing the wave-resistance of ships. Naturally, the numerical model used in this paper is too crude for any practical applications. It seems valuable to apply 3D models under consideration of viscid to research the problem. It seems also valuable to do some experiments to examine this interesting result.

5.7 conclusion and discussion

The basic idea of Finite Process Method is to discretize an original nonlinear problem into a finite number of linear problems in continuous mapping domain $p \in [0, 1]$. Finer this discretization is, i.e., smaller Δp is, more accurate are the numerical results. If Δp is small enough, then the numerical results are accurate enough, but clearly more CPU time is needed. In this way, we can indeed avoid the use of iterative techniques to solve nonlinear problems.

On another side, the formulas of Finite Process Method at any definite values of $\Delta p = 1/n_p$ can be also used as iterative formulas. This gives a family of iterative models. Smaller Δp is, more complex but more insensitive to initial solutions the corresponding iterative formulas are. If Δp is small enough, then the results are accurate enough and no iteration is needed. So, we can see iterative formulas as special cases of Finite Process Method in great Δp.

Every thing has its light side and dark side. The simpler iterative formulars are more sensitive to initial solutions. On another side, the more complex formulas need more CPU time. For weak nonlinear problems, simpler formulas in case of great Δp, for example, $\Delta p = 1.0, \Delta p = 0.5$ and so on, can be used. But for stronger nonlinear problems, formulas in case of smaller Δp seem be needed in order to obtain converged results, and naturally more CPU time is needed in these cases. It seems that we had to make more efforts for a stronger nonlinear problem.

It is interesting that in cases of small F_h and b/c the wave-resistance of a submerged 2D wing would be nearly zero and the corresponding wave-elevation would be nearly an isolate wave. We have obtained this result by two different models, a submerged vortex and a submerged 2D plain plate. Both give the similar result. It should be emphasized that for both models the iterative formulas (5.82) (5.83) fail to obtain this result and more complex formulas given by Finite Process Method must be used. It means that this interesting phenomenon can be predicted by Finite Process Method but not by simple iterative techniques. Maybe, this is a good example to show the strong ability of Finite Process Method in solution of nonlinear problems. We are now trying to do some experiments to examine this interesting result. If it is true also for a 3D submerged wing, perhaps, some new ideas and methods of decreasing wave resistance of ships could be obtained. Clearly, the models used in this paper are too simple and more complex models close to the practical problems should be applied. At last, experiments should be done, and the feasibility in technology and the reasonablity in economy should be proved.

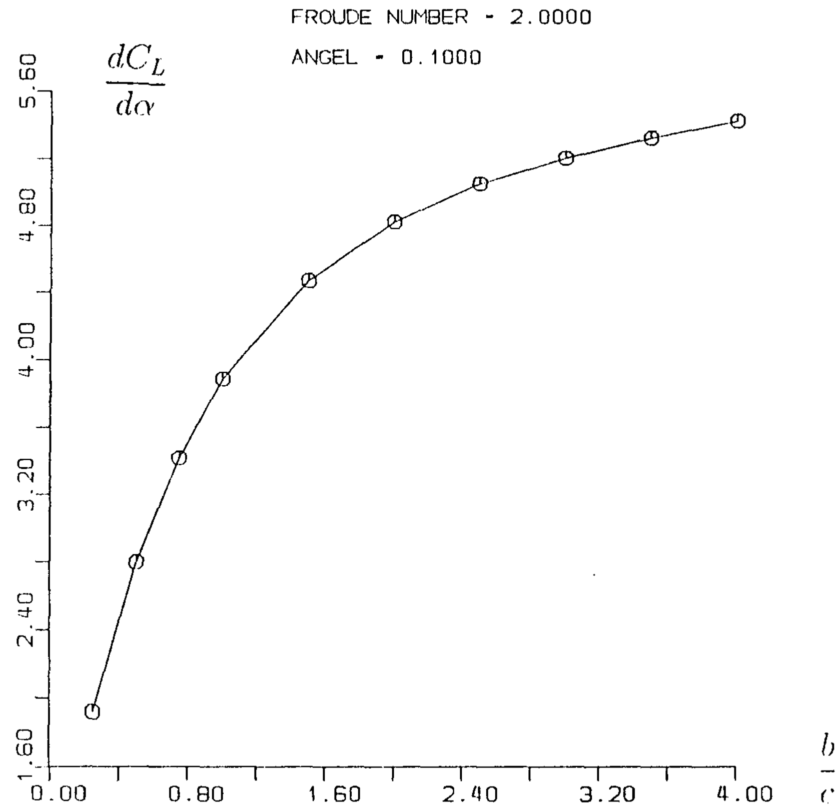

Figure 5.7: curve of $dC_L/d\alpha$ with respect to b/c in case $F_h = 2.0, \alpha = 0.1$ (rad)

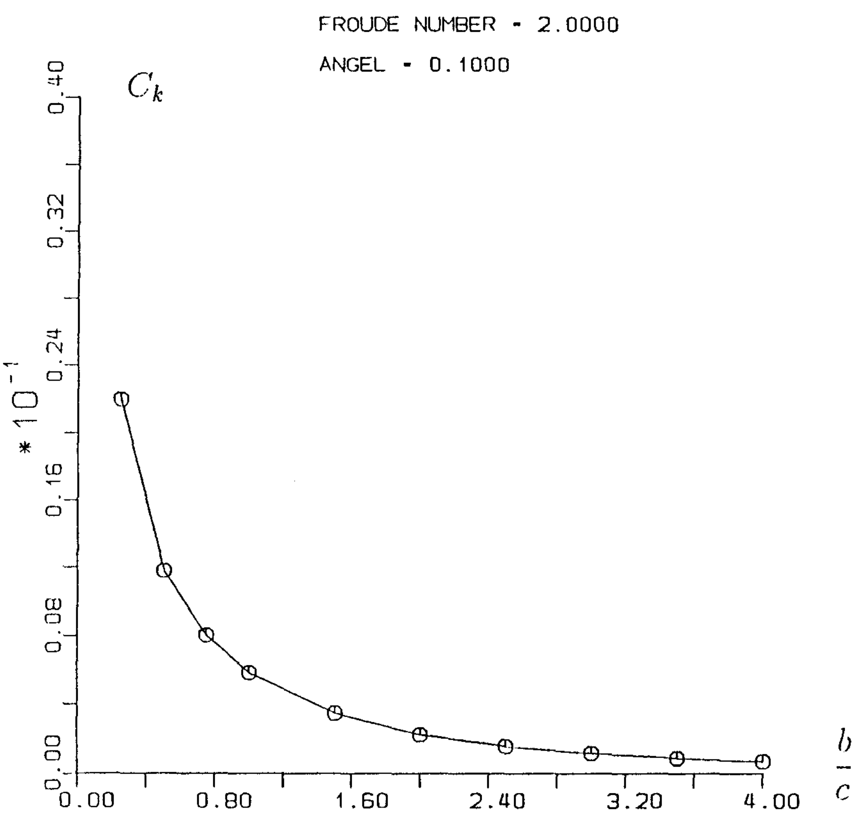

Figure 5.8: curve of C_k with respect to b/c in case $F_h = 2.0, \alpha = 0.1$ (rad)

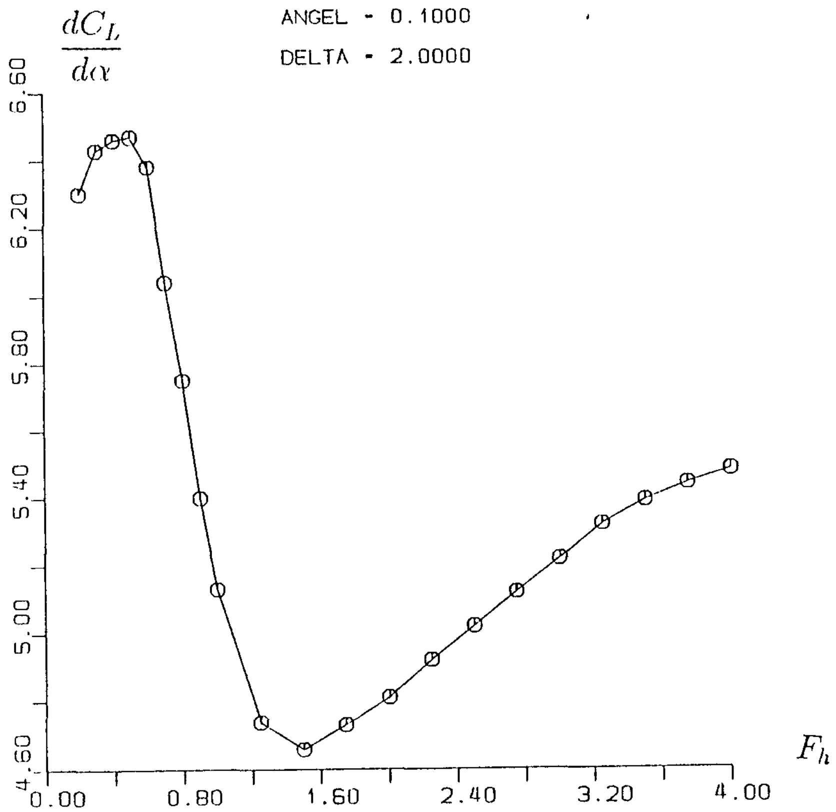

Figure 5.9: curve of $dC_L/d\alpha$ with respect to F_h in case $b/c = 2.0, \alpha = 0.1$ (rad)

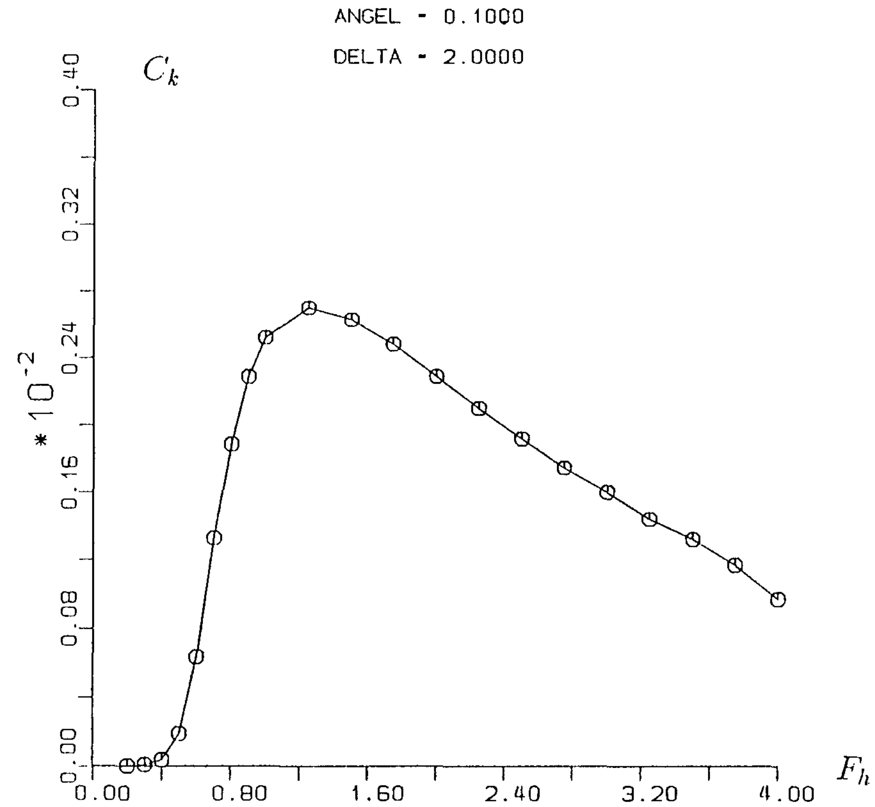

Figure 5.10: curve of C_k with respect to F_h in case $b/c = 2.0, \alpha = 0.1$ (rad)

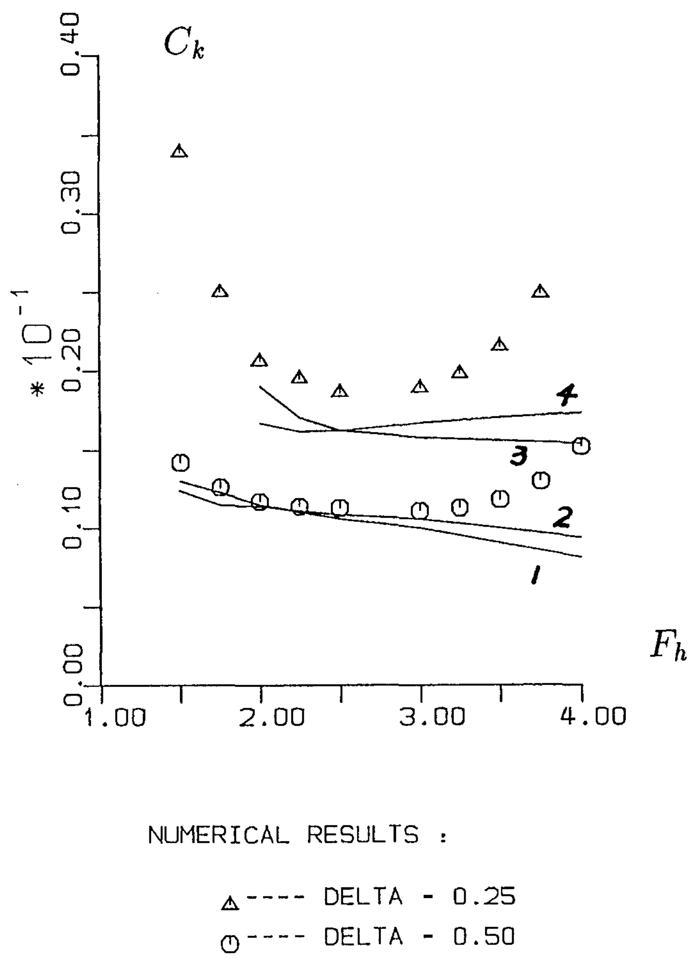

Figure 5.11: curve of C_k with respect to F_h in cases of $b/c = 0.25, b/c = 0.50$ and $\alpha = 0.1$ (rad) theoretical results given in [64]:

curve 1	:	$b/c = 0.5, \alpha = 0.1,$	(linear solution)
curve 2	:	$b/c = 0.5, \alpha = 0.1,$	(nonlinear solution)
curve 3	:	$b/c = 0.25, \alpha = 0.1,$	(linear solution)
curve 4	:	$b/c = 0.25, \alpha = 0.1,$	(nonlinear solution)

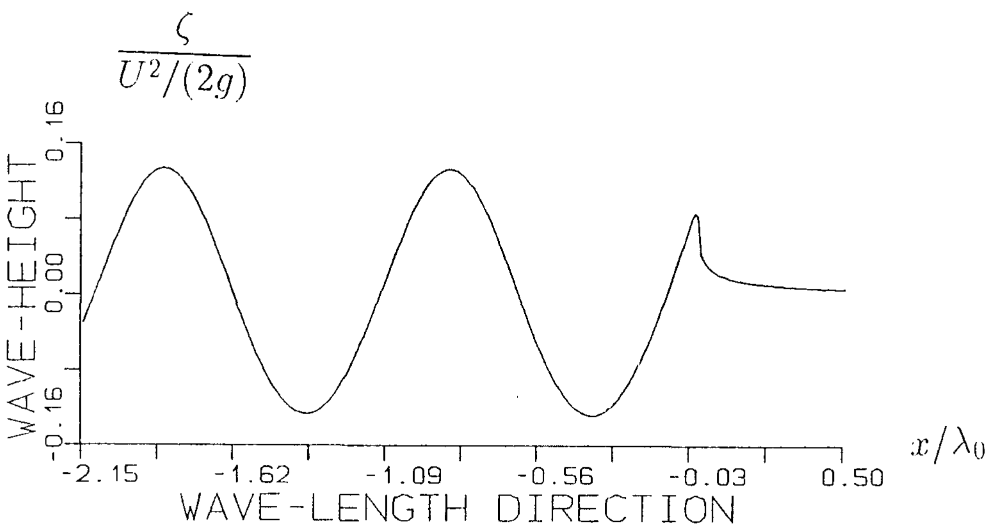

Figure 5.12: wave-elevation of flow past a submerged thin plain wing in case of $F_h = 4.0, b/c = 0.5, \alpha = 0.1$ (rad)

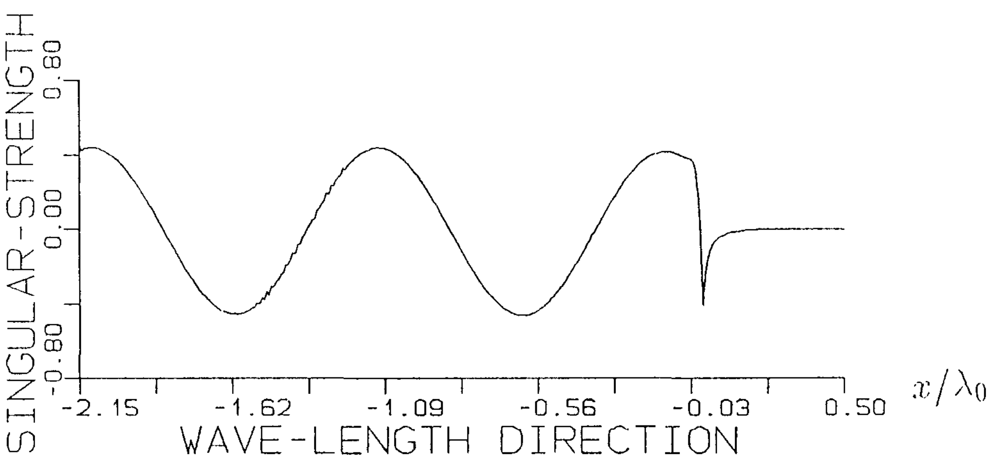

Figure 5.13: singularity-distribution on $\overline{AB}$ of flow past a submerged thin plain wing in case of $F_h = 4.0, b/c = 0.5, \alpha = 0.1$ (rad)

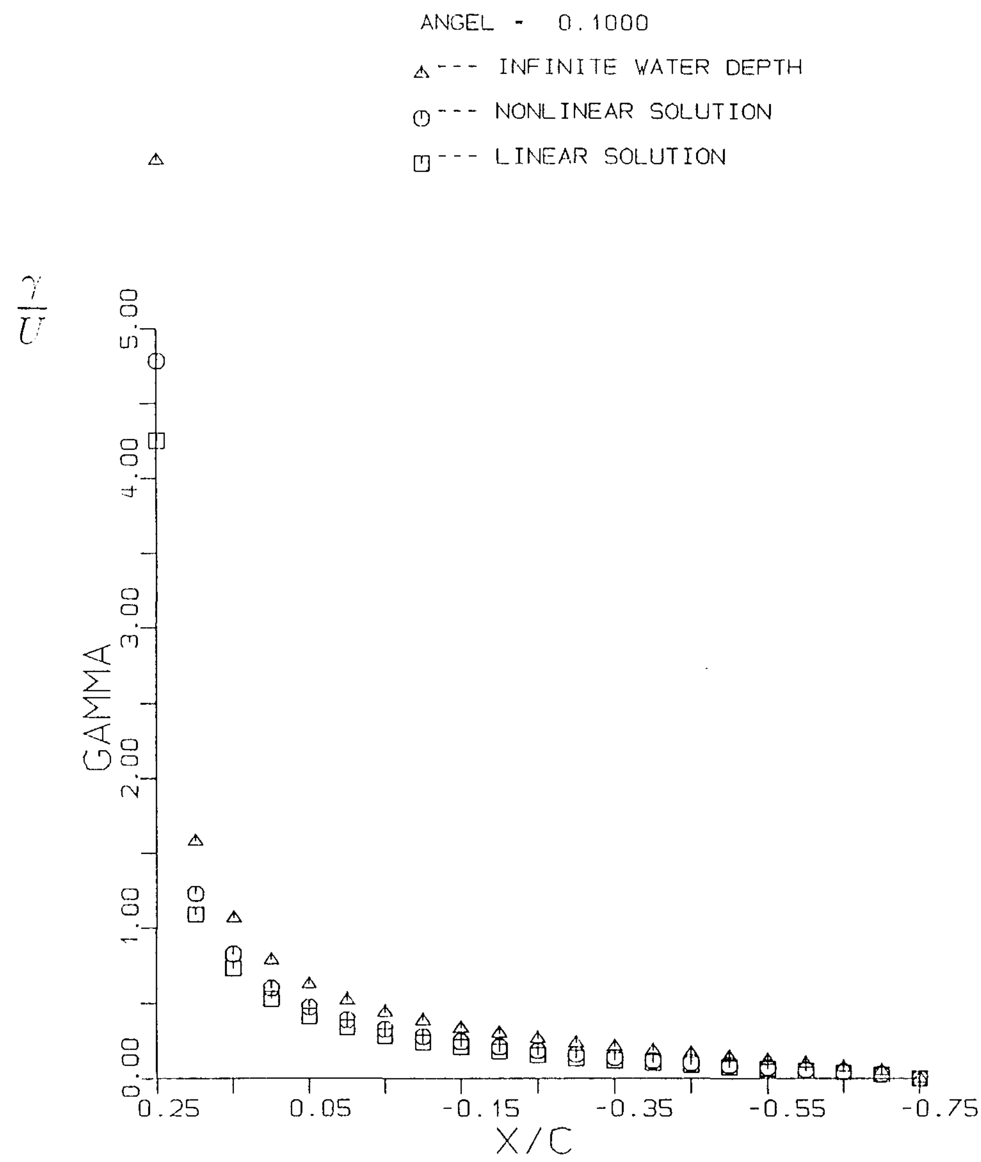

Figure 5.14: singularity-distribution on wing of flow past a submerged thin plain wing in case of $F_h = 4.0, b/c = 0.5, \alpha = 0.1$ (rad)

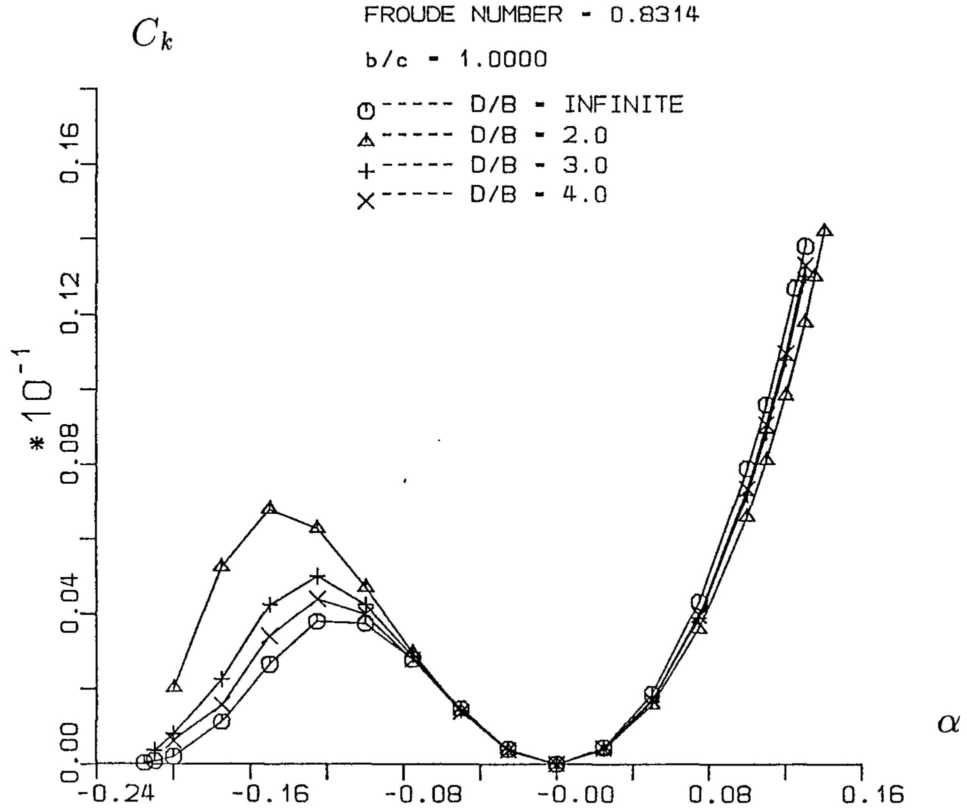

Figure 5.15: curve of C_k with respect to α (rad) in case $F_h = 0.83136, b/c = 1$ in case of different water depth.

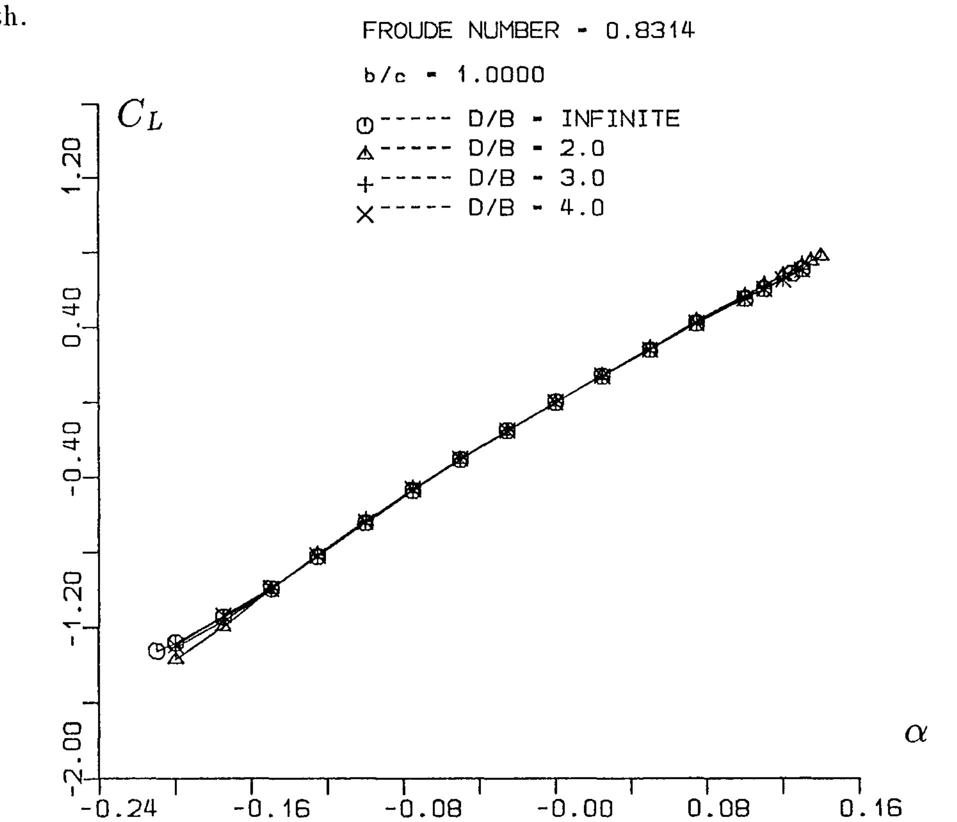

Figure 5.16: curve of C_L with respect to α (rad) in case $F_h = 0.83136, b/c = 1$ in case of different water depth.

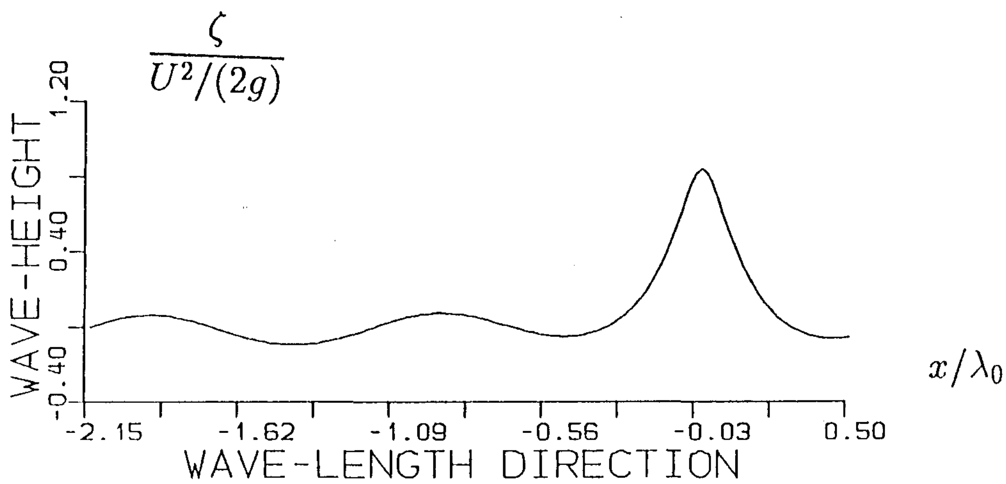

Figure 5.17: wave-elevation of flow past a submerged thin plain wing in case of $F_h = 0.83136, b/c = 1.0, \alpha = -0.21$ (rad)

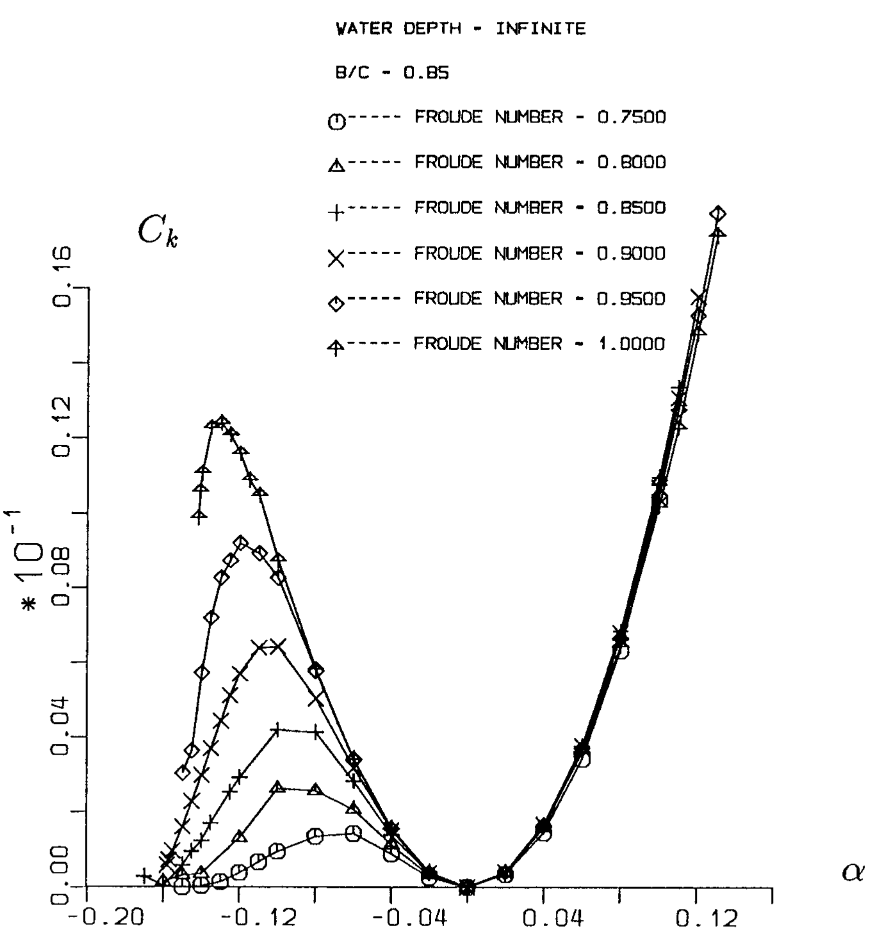

Figure 5.18: wave-resistance of a 2D submerged thin plain wing in case $b/c = 0.85$

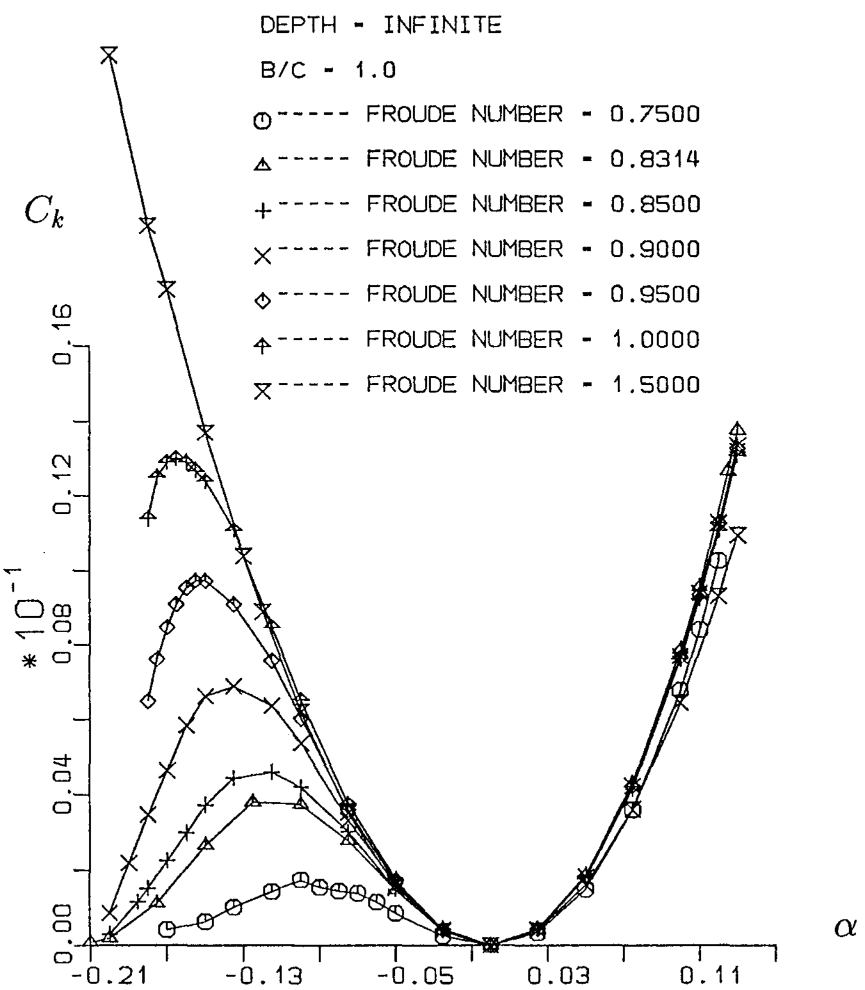

Figure 5.19: wave-resistance of a 2D submerged thin plain wing in case $b/c = 1.00$

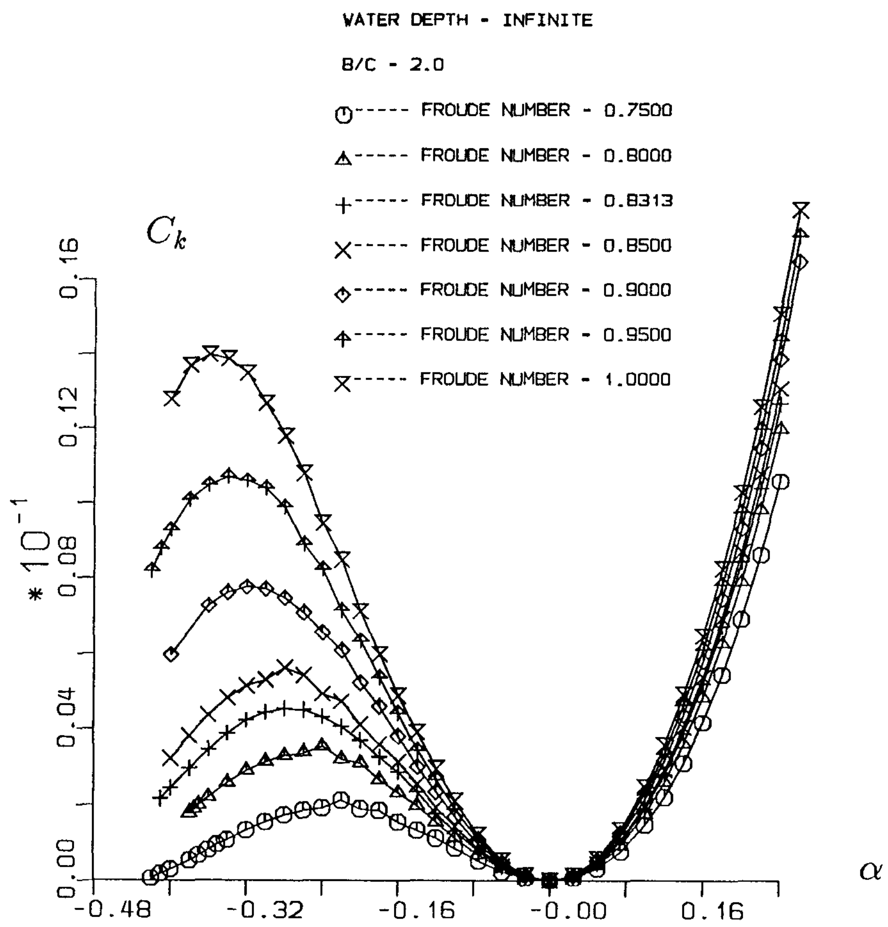

Figure 5.20: wave-resistance of a 2D submerged thin plain wing in case $b/c = 2.00$

Table 5.10: flow past a submerged wing in case $\alpha = 0.1$ (rad), $b/c = 0.5$

F_h	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
1.50	0.277	1.42e-2	0.368	-0.284	0.961
1.75	0.269	1.26e-2	0.290	-0.233	0.972
2.00	0.277	1.17e-2	0.243	-0.200	0.980
2.25	0.299	1.14e-2	0.205	-0.177	0.986
2.50	0.304	1.13e-2	0.184	-0.159	0.987
3.00	0.342	1.11e-2	0.156	-0.132	0.990
3.25	0.361	1.13e-2	0.143	-0.123	0.990
3.50	0.381	1.18e-2	0.135	-0.117	0.992
3.75	0.402	1.30e-2	0.131	-0.115	0.992
4.00	0.425	1.52e-2	0.132	-0.118	0.992

Table 5.11: flow past a submerged wing in case $\alpha = 0.1$ (rad), $b/c = 0.25$

F_h	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
1.50	0.236	3.39e-2	0.638	-0.421	0.903
1.75	0.198	2.50e-2	0.429	-0.320	0.949
2.00	0.187	2.06e-2	0.333	-0.262	0.965
2.25	0.190	1.95e-2	0.278	-0.231	0.975
2.50	0.195	1.86e-2	0.240	-0.203	0.983
3.00	0.222	1.89e-2	0.206	-0.171	0.984
3.25	0.238	1.98e-2	0.192	-0.162	0.988
3.50	0.256	2.15e-2	0.185	-0.157	0.988
3.75	0.275	2.49e-2	0.183	-0.159	0.989
4.00	0.301	3.45e-2	0.197	-0.178	0.989

Table 5.12: flow past a submerged 2D wing in case $F_h = 0.83136, b/c = 1.0$ with infinite water depth.

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.210	-1.321	8.45e-5	0.030	-0.059	1.081
-0.200	-1.280	2.60e-4	0.069	-0.094	1.055
-0.175	-1.144	1.13e-3	0.182	-0.186	1.013
-0.150	-0.990	2.68e-3	0.272	-0.260	1.004
-0.125	-0.818	3.82e-3	0.322	-0.304	1.004
-0.100	-0.639	3.77e-3	0.314	-0.304	1.010
-0.075	-0.467	2.81e-3	0.261	-0.262	1.017
-0.050	-0.301	1.49e-3	0.186	-0.195	1.019
-0.025	-0.147	4.10e-4	0.095	-0.104	1.020
0.025	0.143	4.55e-4	0.115	-0.096	0.986
0.050	0.283	1.88e-3	0.242	-0.191	0.975
0.075	0.420	4.34e-3	0.381	-0.286	0.956
0.100	0.554	7.91e-3	0.539	-0.381	0.930
0.110	0.607	9.62e-3	0.608	-0.420	0.918
0.125	0.684	1.27e-2	0.749	-0.480	0.888
0.130	0.709	1.38e-2	0.832	-0.501	0.872

Table 5.13: flow past a submerged 2D wing in case $F_h = 0.83136, b/c = 1.0$ with water depth $D/b = 4.0$

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.200	-1.290	6.33e-4	0.086	-0.157	1.050
-0.175	-1.134	1.84e-3	0.216	-0.267	1.034
-0.150	-0.992	3.43e-3	0.294	-0.330	1.027
-0.125	-0.813	4.42e-3	0.336	-0.362	1.025
-0.100	-0.633	4.00e-3	0.316	-0.342	1.024
-0.075	-0.460	2.81e-3	0.260	-0.286	1.023
-0.050	-0.299	1.41e-3	0.179	-0.204	1.022
-0.025	-0.147	3.82e-4	0.091	-0.107	1.015
0.025	0.143	4.20e-4	0.116	-0.091	0.986
0.050	0.283	1.72e-3	0.241	-0.182	0.969
0.075	0.420	3.99e-3	0.376	-0.272	0.949
0.100	0.554	7.37e-3	0.525	-0.365	0.926
0.110	0.605	9.08e-3	0.593	-0.405	0.912
0.120	0.655	1.10e-2	0.675	-0.446	0.899
0.130	0.706	1.33e-2	0.788	-0.490	0.876

Table 5.14: flow past a submerged 2D wing in case $F_h = 0.83136, b/c = 1.0$ with water depth $D/b = 3.0$

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.200	-1.303	3.62e-4	0.105	-0.182	1.045
-0.175	-1.162	2.27e-3	0.227	-0.290	1.036
-0.150	-0.994	4.26e-3	0.318	-0.367	1.030
-0.125	-0.810	5.02e-3	0.349	-0.389	1.027
-0.100	-0.630	4.27e-3	0.319	-0.356	1.026
-0.075	-0.458	2.90e-3	0.257	-0.292	1.025
-0.050	-0.299	1.43e-3	0.177	-0.206	1.021
-0.025	-0.147	3.82e-4	0.090	-0.108	1.015
0.025	0.143	4.14e-4	0.116	-0.090	0.987
0.050	0.284	1.69e-3	0.240	-0.178	0.970
0.075	0.422	3.91e-3	0.372	-0.267	0.947
0.100	0.556	7.21e-3	0.517	-0.360	0.924
0.110	0.607	8.88e-3	0.583	-0.399	0.913
0.120	0.657	1.08e-2	0.660	-0.441	0.902
0.130	0.707	1.30e-2	0.763	-0.487	0.883

Table 5.15: flow past a submerged 2D wing in case $F_h = 0.83136, b/c = 1.0$ with water depth $D/b = 2.0$

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.200	-1.366	2.05e-3	0.203	-0.298	1.056
-0.175	-1.193	5.26e-3	0.342	-0.431	1.056
-0.150	-0.995	5.83e-3	0.401	-0.475	1.052
-0.125	-0.805	6.30e-3	0.384	-0.449	1.048
-0.100	-0.628	4.72e-3	0.327	-0.386	1.040
-0.075	-0.460	2.98e-3	0.255	-0.305	1.035
-0.050	-0.302	1.42e-3	0.173	-0.211	1.028
-0.025	-0.149	3.72e-4	0.087	-0.109	1.019
0.025	0.146	3.92e-4	0.116	-0.086	0.990
0.050	0.290	1.59e-3	0.237	-0.170	0.971
0.075	0.432	3.63e-3	0.365	-0.255	0.949
0.100	0.571	6.62e-3	0.501	-0.345	0.930
0.110	0.624	8.13e-3	0.562	-0.384	0.919
0.120	0.676	9.86e-3	0.628	-0.425	0.909
0.130	0.728	1.18e-2	0.710	-0.469	0.896
0.135	0.755	1.30e-2	0.765	-0.492	0.887
0.140	0.781	1.42e-2	0.850	-0.519	0.876

Table 5.16: flow past a 2D submerged thin plain wing in case b/c = 0.85 and F_h = 0.75 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.150	-1.018	1.70e-5	0.053	-0.078	/
-0.140	-0.967	4.09e-5	0.069	-0.096	/
-0.130	-0.913	1.42e-4	0.103	-0.120	1.044
-0.120	-0.857	3.92e-4	0.136	-0.147	1.033
-0.110	-0.797	6.73e-4	0.162	-0.173	1.029
-0.100	-0.733	9.56e-4	0.185	-0.195	1.027
-0.080	-0.594	1.34e-3	0.209	-0.221	1.026
-0.060	-0.444	1.43e-3	0.201	-0.214	1.028
-0.040	-0.291	8.86e-4	0.157	-0.170	1.028
-0.020	-0.143	2.76e-4	0.086	-0.096	/
0.020	0.137	3.44e-4	0.112	-0.092	0.981
0.040	0.269	1.45e-3	0.239	-0.186	0.970
0.060	0.397	3.42e-3	0.382	-0.280	0.952
0.080	0.519	6.33e-3	0.548	-0.376	0.923
0.100	0.634	1.04e-2	0.781	-0.479	0.883

Table 5.17: flow past a 2D submerged thin plain wing in case b/c = 0.85 and F_h = 0.80 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.160	-1.082	1.53e-4	0.048	-0.076	/
-0.150	-1.033	3.37e-4	0.077	-0.107	/
-0.140	-0.983	3.78e-4	0.122	-0.144	1.036
-0.120	-0.866	1.31e-3	0.202	-0.211	1.024
-0.100	-0.729	2.63e-3	0.255	-0.260	1.019
-0.080	-0.577	2.57e-3	0.268	-0.273	1.021
-0.060	-0.422	2.06e-3	0.235	-0.244	1.024
-0.040	-0.273	1.13e-3	0.170	-0.182	1.025
-0.020	-0.133	3.30e-4	0.088	-0.098	/
0.020	0.127	3.80e-4	0.110	-0.091	0.982
0.040	0.249	1.58e-3	0.233	-0.182	0.971
0.060	0.367	3.68e-3	0.369	-0.273	0.955
0.080	0.480	6.74e-3	0.523	-0.364	0.931
0.100	0.587	1.09e-2	0.723	-0.459	0.893

Table 5.18: flow past a 2D submerged thin plain wing in case b/c = 0.85 and F_h = 0.85 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.160	-1.085	5.73e-5	0.080	-0.107	1.041
-0.150	-1.040	5.91e-4	0.137	-0.153	1.027
-0.145	-1.013	9.56e-4	0.166	-0.177	1.023
-0.140	-0.987	1.22e-3	0.137	-0.153	1.027
-0.135	-0.953	1.71e-3	0.214	-0.219	1.015
-0.125	-0.888	2.52e-3	0.257	-0.256	1.013
-0.120	-0.856	2.93e-3	0.276	-0.274	1.014
-0.100	-0.700	4.21e-3	0.328	-0.321	1.014
-0.080	-0.538	4.15e-3	0.314	-0.311	1.018
-0.060	-0.388	2.82e-3	0.254	-0.260	1.023
-0.040	-0.250	1.39e-3	0.175	-0.186	1.023
-0.020	-0.122	3.73e-4	0.088	-0.098	/
0.020	0.117	4.03e-4	0.107	-0.088	0.986
0.040	0.230	1.65e-3	0.223	-0.175	0.974
0.060	0.339	3.79e-3	0.349	-0.261	0.958
0.080	0.444	6.86e-3	0.490	-0.347	0.935
0.100	0.545	1.10e-2	0.658	-0.434	0.905
0.110	0.594	1.34e-2	0.768	-0.480	0.882

Table 5.19: flow past a 2D submerged thin plain wing in case b/c = 0.85 and F_h = 0.90 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.158	-1.082	5.93e-4	0.126	-0.144	1.023
-0.157	-1.077	7.24e-4	0.136	-0.153	1.021
-0.155	-1.067	9.74e-4	0.155	-0.168	1.019
-0.150	-1.039	1.61e-3	0.195	-0.201	1.015
-0.145	-1.008	2.28e-3	0.230	-0.230	1.009
-0.140	-0.975	2.99e-3	0.263	-0.258	1.006
-0.135	-0.939	3.71e-3	0.294	-0.283	1.004
-0.130	-0.900	4.45e-3	0.322	-0.306	1.003
-0.125	-0.859	5.13e-3	0.346	-0.325	1.002
-0.120	-0.818	5.70e-3	0.365	-0.342	1.002
-0.110	-0.730	6.40e-3	0.388	-0.362	1.005
-0.100	-0.644	6.42e-3	0.386	-0.365	1.008
-0.080	-0.487	5.06e-3	0.335	-0.327	1.015
-0.060	-0.350	3.18e-3	0.258	-0.263	1.021
-0.040	-0.226	1.53e-3	0.173	-0.183	1.023
-0.020	-0.110	3.99e-4	0.087	-0.095	/
0.020	0.106	4.15e-4	0.101	-0.085	/
0.040	0.211	1.67e-3	0.210	-0.167	0.978
0.060	0.313	3.79e-3	0.327	-0.248	0.962
0.080	0.412	6.80e-3	0.454	-0.328	0.942
0.100	0.507	1.07e-2	0.598	-0.409	0.915
0.110	0.553	1.31e-2	0.688	-0.450	0.900
0.120	0.599	1.58e-2	0.798	-0.493	0.874

Table 5.20: flow past a 2D submerged thin plain wing in case $b/c = 0.85$ and $F_h = 0.95$ in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.150	-1.018	3.04e-3	0.253	-0.246	1.008
-0.145	-0.997	3.64e-3	0.274	-0.262	1.002
-0.140	-0.947	5.73e-3	0.347	-0.318	0.995
-0.135	-0.896	7.20e-3	0.393	-0.353	0.990
-0.130	-0.844	8.28e-3	0.425	-0.378	0.989
-0.120	-0.743	9.22e-3	0.451	-0.403	0.993
-0.110	-0.653	8.94e-3	0.441	-0.401	0.998
-0.100	-0.574	8.29e-3	0.413	-0.384	1.005
-0.080	-0.438	5.77e-3	0.335	-0.326	1.016
-0.060	-0.317	3.41e-3	0.251	-0.255	1.020
-0.040	-0.206	1.56e-3	0.166	-0.175	1.021
-0.020	-0.101	4.01e-4	0.082	-0.090	/
0.020	0.098	4.10e-4	0.095	-0.080	/
0.040	0.196	1.65e-3	0.196	-0.158	0.981
0.060	0.291	3.72e-3	0.304	-0.233	0.966
0.080	0.385	6.66e-3	0.420	-0.308	0.950
0.100	0.475	1.05e-2	0.548	-0.383	0.927
0.110	0.519	1.28e-2	0.620	-0.421	0.913
0.120	0.562	1.53e-2	0.709	-0.460	0.896
0.130	0.605	1.80e-2	0.820	-0.500	0.867

Table 5.21: flow past a 2D submerged thin plain wing in case $b/c = 0.85$ and $F_h = 1.00$ in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.142	-0.900	9.90e-3	0.448	-0.382	0.976
-0.141	-0.879	1.06e-2	0.468	-0.395	0.973
-0.140	-0.862	1.11e-2	0.481	-0.405	0.970
-0.135	-0.792	1.23e-2	0.510	-0.429	0.972
-0.130	-0.737	1.24e-2	0.511	-0.436	0.978
-0.125	-0.691	1.21e-2	0.502	-0.435	0.985
-0.120	-0.650	1.16e-2	0.583	-0.414	0.991
-0.115	-0.612	1.09e-2	0.468	-0.419	0.995
-0.110	-0.577	1.05e-2	0.448	-0.407	1.000
-0.100	-0.512	8.74e-3	0.407	-0.380	1.006
-0.080	-0.396	5.83e-3	0.322	-0.315	1.016
-0.060	-0.290	3.43e-3	0.238	-0.243	1.019
-0.040	-0.190	1.55e-3	0.157	-0.166	1.020
-0.020	-0.094	3.94e-4	0.077	-0.085	/
0.020	0.092	4.00e-4	0.089	-0.075	/
0.040	0.183	1.61e-3	0.183	-0.148	0.980
0.060	0.273	3.62e-3	0.283	-0.219	0.969
0.080	0.362	6.45e-3	0.389	-0.290	0.955
0.100	0.448	1.02e-2	0.505	-0.360	0.935
0.110	0.490	1.23e-2	0.566	-0.395	0.924
0.120	0.531	1.48e-2	0.637	-0.431	0.910
0.130	0.571	1.74e-2	0.721	-0.467	0.895

Table 5.22: flow past a 2D submerged thin plain wing in case b/c = 0.85 and F_h = 1.50 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.180	-0.607	2.37e-2	0.452	-0.407	1.000
-0.160	-0.543	1.87e-2	0.394	-0.368	1.006
-0.140	-0.478	1.46e-2	0.340	-0.326	1.013
-0.120	-0.412	1.08e-2	0.287	-0.284	1.018
-0.100	-0.345	7.58e-3	0.237	-0.240	1.019
-0.080	-0.277	4.89e-3	0.188	-0.194	1.020
-0.060	-0.208	2.77e-3	0.139	-0.148	1.020
-0.040	-0.139	1.24e-3	0.092	-0.100	/
-0.020	-0.070	3.13e-4	0.046	-0.050	/
0.020	0.070	3.15e-4	0.052	-0.045	/
0.040	0.141	1.26e-3	0.105	-0.089	0.991
0.060	0.211	2.84e-3	0.161	-0.132	0.984
0.080	0.282	5.06e-3	0.218	-0.175	0.980
0.100	0.352	7.91e-3	0.276	-0.217	0.971
0.120	0.422	1.14e-2	0.337	-0.259	0.964
0.140	0.491	1.56e-2	0.400	-0.300	0.955
0.160	0.558	2.04e-2	0.467	-0.342	0.944
0.180	0.624	2.59e-2	0.538	-0.383	0.930

Table 5.23: flow past a 2D submerged thin plain wing in case b/c = 1.00 and F_h = 0.75 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.170	-1.104	4.30e-4	0.099	-0.123	/
-0.150	-0.993	6.51e-4	0.159	-0.169	1.029
-0.135	-0.904	1.03e-3	0.190	-0.199	1.024
-0.115	-0.776	1.44e-3	0.218	-0.226	1.023
-0.100	-0.676	1.76e-3	0.225	-0.234	1.023
-0.090	-0.608	1.56e-3	0.222	-0.232	1.024
-0.080	-0.539	1.47e-3	0.213	-0.224	1.024
-0.070	-0.470	1.40e-3	0.200	-0.211	1.026
-0.060	-0.401	1.16e-3	0.181	-0.193	1.027
-0.050	-0.333	8.60e-4	0.158	-0.170	1.028
-0.025	-0.164	2.73e-4	0.086	-0.096	/
0.025	0.160	3.50e-4	0.114	-0.093	0.980
0.050	0.316	1.51e-3	0.244	-0.189	0.968
0.075	0.467	3.63e-3	0.394	-0.288	0.948
0.100	0.612	6.82e-3	0.571	-0.390	0.921
0.110	0.669	8.44e-3	0.658	-0.432	0.905
0.120	0.724	1.03e-2	0.770	-0.477	0.885

Table 5.24: flow past a 2D submerged thin plain wing in case b/c = 1.00 and F_h = 0.85 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.200	-1.259	3.07e-4	0.103	-0.127	1.036
-0.185	-1.182	1.16e-3	0.176	-0.187	1.022
-0.180	-1.153	1.53e-3	0.201	-0.207	1.018
-0.170	-1.093	1.27e-3	0.243	-0.243	1.015
-0.160	-1.030	3.01e-3	0.280	-0.274	1.011
-0.150	-0.964	3.75e-3	0.313	-0.303	1.010
-0.135	-0.860	4.47e-3	0.339	-0.326	1.008
-0.115	-0.716	4.62e-3	0.342	-0.331	1.012
-0.100	-0.611	4.23e-3	0.324	-0.319	1.016
-0.075	-0.444	3.05e-3	0.266	-0.270	1.022
-0.050	-0.290	1.50e-3	0.182	-0.192	1.022
-0.025	-0.142	4.07e-4	0.093	-0.102	/
0.025	0.140	4.37e-4	0.111	-0.092	0.985
0.050	0.276	1.81e-3	0.234	-0.183	0.973
0.075	0.409	4.19e-3	0.369	-0.274	0.956
0.100	0.539	7.65e-3	0.520	-0.366	0.930
0.110	0.589	9.36e-3	0.589	-0.403	0.920
0.120	0.638	1.13e-2	0.671	-0.442	0.902
0.130	0.686	1.34e-2	0.769	-0.481	0.884

Table 5.25: flow past a 2D submerged thin plain wing in case b/c = 1.00 and F_h = 0.90 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.200	-1.251	8.68e-4	0.143	-0.158	1.020
-0.190	-1.194	2.21e-3	0.225	-0.225	1.011
-0.180	-1.130	3.48e-3	0.284	-0.272	1.003
-0.170	-1.064	4.67e-3	0.330	-0.309	1.002
-0.160	-0.989	5.86e-3	0.371	-0.341	0.997
-0.150	-0.915	6.65e-3	0.397	-0.365	0.998
-0.135	-0.804	6.90e-3	0.404	-0.373	1.001
-0.115	-0.664	6.38e-3	0.378	-0.357	1.008
-0.100	-0.565	5.40e-3	0.343	-0.333	1.014
-0.075	-0.415	3.34e-3	0.263	-0.266	1.022
-0.050	-0.271	1.62e-3	0.178	-0.188	1.024
-0.025	-0.134	4.21e-4	0.089	-0.098	/
0.025	0.131	4.52e-4	0.106	-0.088	0.987
0.050	0.259	1.85e-3	0.222	-0.175	0.980
0.075	0.385	4.26e-3	0.347	-0.262	0.960
0.100	0.507	7.73e-3	0.486	-0.349	0.938
0.110	0.555	9.43e-3	0.548	-0.384	0.928
0.120	0.601	1.13e-2	0.618	-0.419	0.917
0.130	0.647	1.34e-2	0.695	-0.456	0.897

Table 5.26: flow past a 2D submerged thin plain wing in case b/c = 1.00 and F_h = 0.95 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.180	-1.092	6.50e-3	0.371	-0.332	0.987
-0.175	-1.048	7.63e-3	0.406	-0.358	0.984
-0.170	-1.005	8.49e-3	0.433	-0.377	0.982
-0.155	-0.882	9.76e-3	0.467	-0.408	0.982
-0.150	-0.843	9.75e-3	0.467	-0.410	0.989
-0.135	-0.737	9.12e-3	0.446	-0.403	0.997
-0.115	-0.610	7.59e-3	0.393	-0.369	1.007
-0.100	-0.522	6.06e-3	0.345	-0.333	1.012
-0.075	-0.384	3.63e-3	0.260	-0.263	1.022
-0.050	-0.253	1.69e-3	0.173	-0.182	1.024
-0.025	-0.125	4.37e-4	0.086	-0.095	/
0.025	0.123	4.59e-4	0.101	-0.084	/
0.050	0.244	1.87e-3	0.210	-0.167	0.978
0.075	0.363	4.27e-3	0.327	-0.249	0.963
0.100	0.480	7.71e-3	0.455	-0.331	0.944
0.110	0.525	9.39e-3	0.511	-0.364	0.935
0.120	0.570	1.13e-2	0.571	-0.397	0.923
0.130	0.614	1.33e-2	0.639	-0.430	0.910

Table 5.27: flow past a 2D submerged thin plain wing in case $b/c = 1.00$ and $F_h = 1.00$ in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.180	-1.013	1.14e-2	0.491	-0.404	0.964
-0.175	-0.960	1.25e-2	0.518	-0.424	0.964
-0.170	-0.914	1.29e-2	0.528	-0.435	0.964
-0.165	-0.873	1.30e-2	0.528	-0.440	0.970
-0.160	-0.835	1.29e-2	0.523	-0.441	0.976
-0.155	-0.800	1.25e-2	0.513	-0.439	0.982
-0.135	-0.667	1.11e-2	0.463	-0.417	0.997
-0.115	-0.557	8.49e-3	0.395	-0.371	1.007
-0.100	-0.481	6.54e-3	0.340	-0.329	1.014
-0.075	-0.359	3.78e-3	0.251	-0.255	1.019
-0.050	-0.235	1.76e-3	0.168	-0.176	1.022
-0.025	-0.116	4.52e-4	0.083	-0.091	/
0.025	0.116	4.61e-4	0.096	-0.081	/
0.050	0.231	1.87e-3	0.199	-0.159	0.980
0.075	0.344	4.25e-3	0.309	-0.237	0.966
0.100	0.456	7.64e-3	0.427	-0.314	0.949
0.110	0.500	9.29e-3	0.478	-0.345	0.939
0.120	0.543	1.11e-2	0.530	-0.376	0.931
0.130	0.586	1.31e-2	0.588	-0.407	0.921

Table 5.28: flow past a 2D submerged thin plain wing in case b/c = 1.00 and F_h = 1.50 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.240	-0.843	3.35e-2	0.568	-0.473	0.975
-0.200	-0.714	2.38e-2	0.453	-0.396	1.000
-0.180	-0.647	1.92e-2	0.402	-0.373	1.005
-0.170	-0.613	1.75e-2	0.377	-0.355	1.008
-0.150	-0.544	1.37e-2	0.328	-0.307	1.014
-0.130	-0.474	1.04e-2	0.281	-0.279	1.015
-0.120	-0.439	8.90e-3	0.258	-0.259	1.018
-0.100	-0.368	6.23e-3	0.213	-0.218	1.019
-0.075	-0.277	3.54e-3	0.158	-0.166	1.021
-0.050	-0.186	1.54e-3	0.104	-0.113	1.019
-0.025	-0.093	3.99e-4	0.052	-0.057	/
0.025	0.094	4.03e-4	0.059	-0.051	/
0.050	0.188	1.62e-3	0.120	-0.100	0.988
0.075	0.282	3.64e-3	0.183	-0.149	0.982
0.100	0.375	6.49e-3	0.248	-0.197	0.976
0.120	0.450	9.36e-3	0.303	-0.235	0.969
0.130	0.487	1.10e-2	0.317	-0.254	0.964
0.140	0.523	1.28e-2	0.358	-0.273	0.961
0.160	0.596	1.67e-2	0.416	-0.310	0.953
0.180	0.667	2.12e-2	0.478	-0.348	0.941
0.200	0.737	2.62e-2	0.543	-0.385	0.930

Table 5.29: flow past a 2D submerged thin plain wing in case b/c = 1.00 and F_h = 2.00 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.200	-0.763	2.26e-2	0.316	-0.302	1.008
-0.190	-0.726	2.05e-2	0.299	-0.288	1.009
-0.180	-0.689	1.84e-2	0.282	-0.274	1.012
-0.170	-0.652	1.65e-2	0.266	-0.261	1.013
-0.160	-0.615	1.46e-2	0.249	-0.246	1.014
-0.150	-0.577	1.29e-2	0.232	-0.232	1.015
-0.130	-0.502	9.75e-3	0.200	-0.203	1.016
-0.120	-0.464	8.33e-3	0.184	-0.188	1.016
-0.100	-0.388	5.80e-3	0.152	-0.158	1.017
-0.075	-0.291	3.28e-3	0.113	-0.119	1.019
-0.050	-0.195	1.46e-3	0.075	-0.081	/
-0.025	-0.098	3.66e-4	0.037	-0.041	/
0.025	0.098	3.67e-4	0.042	-0.037	/
0.050	0.195	1.47e-3	0.084	-0.073	/
0.075	0.293	3.30e-3	0.123	-0.108	0.988
0.100	0.390	5.86e-3	0.171	-0.143	0.988
0.120	0.467	8.44e-3	0.207	-0.171	0.982
0.140	0.544	1.15e-2	0.244	-0.198	0.979
0.160	0.619	1.50e-2	0.281	-0.226	0.975
0.180	0.694	1.91e-2	0.320	-0.253	0.970
0.200	0.768	2.37e-2	0.360	-0.280	0.964

Table 5.30: flow past a 2D submerged thin plain wing in case b/c = 2.00 and F_h = 0.75 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.420	-2.313	7.67e-5	0.095	-0.118	1.048
-0.410	-2.266	1.86e-4	0.111	-0.130	1.042
-0.400	-2.218	3.04e-4	0.127	-0.143	1.035
-0.390	-2.170	4.29e-4	0.141	-0.153	1.032
-0.380	-2.121	5.58e-4	0.155	-0.164	1.027
-0.370	-2.071	6.91e-4	0.169	-0.175	1.026
-0.360	-2.021	8.26e-4	0.181	-0.186	1.024
-0.350	-1.971	9.60e-4	0.192	-0.195	1.024
-0.340	-1.920	1.09e-3	0.201	-0.204	1.019
-0.330	-1.869	1.22e-3	0.211	-0.213	1.019
-0.320	-1.817	1.35e-3	0.219	-0.221	1.020
-0.310	-1.764	1.46e-3	0.227	-0.228	1.019
-0.300	-1.711	1.57e-3	0.233	-0.234	1.019
-0.280	-1.604	1.75e-3	0.244	-0.245	1.019
-0.260	-1.496	1.87e-3	0.250	-0.251	1.019
-0.240	-1.386	1.93e-3	0.253	-0.255	1.019
-0.220	-1.275	2.13e-3	0.250	-0.232	1.021
-0.180	-1.049	1.87e-3	0.234	-0.240	1.024
-0.160	-0.935	1.57e-3	0.219	-0.228	1.026
-0.140	-0.820	1.35e-3	0.201	-0.211	1.026
-0.120	-0.704	1.14e-3	0.180	-0.191	1.026
-0.100	-0.588	8.63e-4	0.155	-0.167	1.026
-0.075	-0.442	5.20e-4	0.121	-0.133	1.026
-0.050	-0.296	2.53e-4	0.083	-0.093	/
-0.025	-0.148	7.00e-5	0.043	-0.049	/
0.025	0.149	7.99e-5	0.053	-0.045	/
0.050	0.297	3.38e-4	0.111	-0.091	0.983
0.075	0.446	7.99e-4	0.174	-0.139	0.977
0.100	0.595	1.49e-3	0.241	-0.188	0.969
0.120	0.714	2.21e-3	0.299	-0.228	0.964
0.140	0.832	3.11e-3	0.360	-0.268	0.956
0.160	0.949	4.18e-3	0.425	-0.309	0.946
0.180	1.046	5.45e-3	0.496	-0.350	0.935
0.200	1.178	6.92e-3	0.576	-0.393	0.922
0.220	1.290	8.64e-3	0.666	-0.437	0.906
0.240	1.403	1.06e-2	0.792	-0.485	0.883

Table 5.31: flow past a 2D submerged thin plain wing in case b/c = 2.00 and F_h = 0.80 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.410	-2.255	1.04e-3	0.185	-0.191	1.020
-0.400	-2.206	1.26e-3	0.201	-0.204	1.016
-0.380	-2.083	1.80e-3	0.236	-0.233	1.013
-0.370	-2.033	2.01e-3	0.248	-0.244	1.011
-0.360	-1.982	2.21e-3	0.260	-0.253	1.011
-0.340	-1.880	2.58e-3	0.279	-0.271	1.009
-0.320	-1.776	2.90e-3	0.295	-0.285	1.009
-0.300	-1.671	3.15e-3	0.306	-0.295	1.011
-0.280	-1.565	3.31e-3	0.312	-0.301	1.012
-0.260	-1.457	3.38e-3	0.313	-0.304	1.013
-0.240	-1.349	3.54e-3	0.309	-0.302	1.014
-0.220	-1.239	3.20e-3	0.300	-0.296	1.016
-0.200	-1.129	3.10e-3	0.287	-0.286	1.019
-0.180	-1.019	2.67e-3	0.270	-0.272	1.020
-0.160	-0.907	2.32e-3	0.249	-0.254	1.021
-0.140	-0.796	1.99e-3	0.225	-0.233	1.023
-0.120	-0.683	1.57e-3	0.199	-0.209	1.023
-0.100	-0.570	1.14e-3	0.170	-0.181	1.024
-0.075	-0.429	7.05e-4	0.131	-0.142	1.024
-0.050	-0.287	3.35e-4	0.089	-0.099	/
-0.025	-0.144	8.88e-5	0.045	-0.051	/
0.025	0.144	9.82e-5	0.055	-0.047	/
0.050	0.288	4.10e-4	0.115	-0.094	0.984
0.075	0.433	9.57e-4	0.178	-0.143	0.977
0.100	0.577	1.76e-3	0.246	-0.192	0.970
0.120	0.692	2.60e-3	0.303	-0.231	0.964
0.140	0.806	3.63e-3	0.364	-0.271	0.957
0.160	0.919	4.85e-3	0.428	-0.312	0.947
0.180	1.031	6.28e-3	0.498	-0.353	0.936
0.200	1.141	7.93e-3	0.576	-0.395	0.922
0.220	1.250	9.84e-3	0.657	-0.438	0.907
0.240	1.369	1.20e-2	0.785	-0.484	0.883

Table 5.32: flow past a 2D submerged thin plain wing in case $b/c = 2.00$ and $F_h = 0.83136$ in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.410	-2.201	2.16e-3	0.248	-0.241	1.010
-0.400	-2.152	2.44e-3	0.263	-0.253	1.008
-0.380	-2.053	2.97e-3	0.289	-0.275	1.004
-0.360	-1.952	3.46e-3	0.312	-0.294	1.004
-0.340	-1.849	3.90e-3	0.329	-0.310	1.003
-0.320	-1.746	4.24e-3	0.342	-0.321	1.003
-0.300	-1.641	4.46e-3	0.350	-0.329	1.005
-0.280	-1.536	4.55e-3	0.352	-0.333	1.006
-0.260	-1.429	4.51e-3	0.348	-0.332	1.007
-0.240	-1.322	4.34e-3	0.340	-0.327	1.012
-0.220	-1.215	4.06e-3	0.327	-0.318	1.015
-0.200	-1.107	3.70e-3	0.309	-0.304	1.015
-0.180	-0.998	3.27e-3	0.288	-0.287	1.018
-0.160	-0.889	2.87e-3	0.264	-0.267	1.019
-0.140	-0.779	2.35e-3	0.237	-0.243	1.021
-0.120	-0.669	1.79e-3	0.207	-0.216	1.021
-0.100	-0.559	1.32e-3	0.176	-0.186	1.023
-0.075	-0.420	8.06e-4	0.135	-0.146	1.025
-0.050	-0.281	3.79e-4	0.091	-0.101	/
-0.025	-0.141	1.00e-4	0.046	-0.052	/
0.025	0.141	1.08e-4	0.056	-0.047	/
0.050	0.283	4.49e-4	0.115	-0.095	0.984
0.075	0.424	1.04e-3	0.179	-0.144	0.980
0.100	0.566	1.91e-3	0.246	-0.192	0.973
0.120	0.678	2.81e-3	0.303	-0.232	0.966
0.140	0.790	3.91e-3	0.363	-0.271	0.957
0.160	0.901	5.21e-3	0.426	-0.311	0.947
0.180	1.011	6.72e-3	0.494	-0.351	0.935
0.200	1.120	8.46e-3	0.570	-0.392	0.922
0.220	1.226	1.04e-2	0.648	-0.435	0.907
0.240	1.333	1.27e-2	0.768	-0.479	0.886

Table 5.33: flow past a 2D submerged thin plain wing in case b/c = 2.00 and F_h = 0.85 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.400	**-2.132**	**3.22e-3**	**0.294**	**-0.278**	**1.003**
-0.380	-2.032	3.83e-3	0.321	-0.299	1.002
-0.360	-1.932	4.38e-3	0.342	-0.317	0.997
-0.340	-1.829	4.83e-3	0.359	-0.331	0.998
-0.320	-1.726	5.15e-3	0.370	-0.342	1.001
-0.300	-1.622	5.31e-3	0.375	-0.348	1.002
-0.280	-1.517	5.62e-3	0.374	-0.349	1.005
-0.260	-1.412	5.43e-3	0.368	-0.347	1.006
-0.240	-1.306	4.94e-3	0.356	-0.339	1.011
-0.220	-1.200	4.74e-3	0.340	-0.328	1.011
-0.200	-1.093	4.13e-3	0.320	-0.313	1.014
-0.180	-0.986	3.61e-3	0.297	-0.295	1.017
-0.160	-0.878	3.14e-3	0.271	-0.273	1.019
-0.140	-0.770	2.56e-3	0.242	-0.248	1.021
-0.120	-0.661	1.94e-3	0.211	-0.220	1.021
-0.100	-0.552	1.45e-3	0.179	-0.189	1.023
-0.075	-0.415	8.63e-4	0.136	-0.147	1.023
-0.050	-0.277	4.04e-4	0.092	-0.102	/
-0.025	-0.139	1.06e-4	0.047	-0.053	/
0.025	0.140	1.14e-4	0.056	-0.047	/
0.050	0.279	4.71e-4	0.115	-0.095	0.985
0.075	0.419	1.09e-3	0.179	-0.144	0.979
0.100	0.559	1.99e-3	0.245	-0.192	0.972
0.120	0.671	2.92e-3	0.302	-0.231	0.965
0.140	0.781	4.06e-3	0.361	-0.270	0.956
0.160	0.891	5.39e-3	0.423	-0.310	0.946
0.180	1.000	6.95e-3	0.490	-0.350	0.938
0.200	1.107	8.73e-3	0.564	-0.390	0.923
0.220	1.212	1.08e-2	0.641	-0.432	0.909
0.240	1.318	1.31e-2	0.759	-0.476	0.886

Table 5.34: flow past a 2D submerged thin plain wing in case $b/c = 2.00$ and $F_h = 0.90$ in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.400	-2.073	5.97e-3	0.379	-0.337	0.988
-0.360	-1.872	7.28e-3	0.423	-0.371	0.985
-0.340	-1.771	7.64e-3	0.434	-0.382	0.988
-0.320	-1.669	7.78e-3	0.438	-0.388	0.989
-0.300	-1.568	7.73e-3	0.435	-0.389	0.991
-0.280	-1.466	7.49e-3	0.425	-0.386	0.997
-0.260	-1.364	7.09e-3	0.411	-0.378	1.000
-0.240	-1.262	6.56e-3	0.391	-0.366	1.004
-0.220	-1.159	6.10e-3	0.368	-0.350	1.011
-0.180	-0.953	4.60e-3	0.314	-0.308	1.016
-0.160	-0.849	3.83e-3	0.284	-0.283	1.018
-0.140	-0.745	3.01e-3	0.251	-0.255	1.020
-0.120	-0.640	2.35e-3	0.218	-0.225	1.020
-0.100	-0.535	1.70e-3	0.183	-0.193	1.024
-0.075	-0.402	9.99e-4	0.139	-0.149	1.023
-0.050	-0.269	4.62e-4	0.094	-0.102	/
-0.025	-0.135	1.20e-4	0.047	-0.053	/
0.025	0.135	1.27e-4	0.056	-0.047	/
0.050	0.271	5.21e-4	0.114	-0.095	0.988
0.075	0.407	1.20e-3	0.176	-0.142	0.982
0.100	0.543	2.17e-3	0.241	-0.190	0.973
0.120	0.651	3.18e-3	0.296	-0.228	0.966
0.140	0.759	4.39e-3	0.353	-0.266	0.960
0.160	0.865	5.81e-3	0.413	-0.304	0.950
0.180	0.971	7.46e-3	0.477	-0.343	0.939
0.200	1.075	9.34e-3	0.546	-0.382	0.929
0.240	1.280	1.39e-2	0.718	-0.462	0.895
0.260	1.383	1.65e-2	0.843	-0.505	0.860

Table 5.35: flow past a 2D submerged thin plain wing in case b/c = 2.00 and F_h = 0.95 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.420	-2.108	8.20e-3	0.430	-0.363	0.973
-0.410	-2.058	8.80e-3	0.448	-0.375	0.969
-0.400	-2.007	9.30e-3	0.463	-0.385	0.968
-0.380	-1.907	1.01e-2	0.485	-0.402	0.966
-0.360	-1.808	1.05e-2	0.497	-0.414	0.969
-0.340	-1.710	1.07e-2	0.499	-0.420	0.973
-0.320	-1.612	1.06e-2	0.493	-0.422	0.979
-0.300	-1.514	1.04e-2	0.480	-0.419	0.988
-0.280	-1.416	9.90e-3	0.462	-0.410	0.991
-0.260	-1.318	8.88e-3	0.440	-0.398	0.998
-0.240	-1.220	8.26e-3	0.414	-0.382	1.003
-0.220	-1.122	7.15e-3	0.385	-0.362	1.007
-0.200	-1.023	6.35e-3	0.355	-0.340	1.013
-0.180	-0.923	5.37e-3	0.323	-0.315	1.014
-0.160	-0.823	4.42e-3	0.289	-0.288	1.018
-0.140	-0.722	3.45e-3	0.255	-0.258	1.021
-0.120	-0.621	2.66e-3	0.220	-0.226	1.021
-0.100	-0.519	1.91e-3	0.184	-0.193	1.021
-0.075	-0.391	1.11e-3	0.139	-0.149	1.024
-0.050	-0.261	5.09e-4	0.093	-0.102	/
-0.025	-0.131	1.31e-4	0.046	-0.052	/
0.025	0.132	1.37e-4	0.055	-0.047	/
0.050	0.264	5.59e-4	0.112	-0.093	0.988
0.075	0.396	1.28e-3	0.172	-0.139	0.981
0.100	0.528	2.32e-3	0.235	-0.186	0.974
0.120	0.633	3.37e-3	0.288	-0.223	0.969
0.140	0.738	4.64e-3	0.343	-0.260	0.962
0.160	0.842	6.13e-3	0.400	-0.297	0.952
0.180	0.945	7.84e-3	0.460	-0.334	0.943
0.200	1.047	9.79e-3	0.525	-0.371	0.931
0.220	1.147	1.20e-2	0.594	-0.409	0.919
0.240	1.246	1.44e-2	0.676	-0.447	0.903
0.260	1.346	1.72e-2	0.779	-0.488	0.881

Table 5.36: flow past a 2D submerged thin plain wing in case b/c = 2.00 and F_h = 1.00 in infinite water depth

α (rad)	C_L	C_k	$\frac{\zeta_{max}}{(U^2/2g)}$	$\frac{\zeta_{min}}{(U^2/2g)}$	$\frac{\lambda_w}{(2\pi U^2/g)}$
-0.400	-1.961	1.28e-2	0.536	-0.418	0.946
-0.380	-1.857	1.37e-2	0.557	-0.437	0.949
-0.360	-1.760	1.40e-2	0.562	-0.448	0.956
-0.340	-1.664	1.39e-2	0.555	-0.453	0.964
-0.320	-1.569	1.35e-2	0.540	-0.451	0.972
-0.300	-1.475	1.27e-2	0.518	-0.443	0.980
-0.280	-1.380	1.18e-2	0.493	-0.433	0.988
-0.260	-1.285	1.08e-2	0.465	-0.417	0.997
-0.240	-1.197	9.47e-3	0.428	-0.393	1.000
-0.220	-1.101	8.49e-3	0.396	-0.370	1.007
-0.200	-1.004	7.13e-3	0.362	-0.345	1.011
-0.180	-0.907	5.99e-3	0.327	-0.318	1.014
-0.160	-0.809	4.89e-3	0.292	-0.290	1.018
-0.140	-0.710	3.92e-3	0.256	-0.259	1.019
-0.120	-0.611	2.96e-3	0.221	-0.227	1.021
-0.100	-0.511	2.11e-3	0.184	-0.193	1.021
-0.075	-0.384	1.22e-3	0.138	-0.148	1.021
-0.050	-0.257	5.55e-4	0.092	-0.101	/
-0.025	-0.129	1.42e-4	0.046	-0.052	/
0.025	0.130	1.48e-4	0.054	-0.046	/
0.050	0.260	5.99e-4	0.110	-0.092	0.988
0.075	0.390	1.37e-3	0.169	-0.137	0.982
0.100	0.520	2.46e-3	0.230	-0.182	0.976
0.120	0.624	3.58e-3	0.281	-0.218	0.970
0.140	0.727	4.92e-3	0.333	-0.254	0.962
0.160	0.831	6.48e-3	0.388	-0.290	0.955
0.180	0.934	8.27e-3	0.445	-0.326	0.947
0.200	1.047	1.03e-2	0.507	-0.363	0.937
0.220	1.122	1.26e-2	0.575	-0.398	0.925
0.240	1.230	1.51e-2	0.650	-0.435	0.907
0.260	1.327	1.79e-2	0.742	-0.473	0.887

Conclusion and Discussion

Based on a kind of invariance of homotopy which has been proved in Appendix A, the basic ideas of Process Analysis Method and Finite Process Method are described and examined by using some simple but typical nonlinear problems in mechanics. Clearly, more complex examples are needed in order to give a prudent conclusion, although the examples given in this paper show the success of both methods. In spite of this, I would try to give some discussions and conclusions in order to let this paper have a normal form.

In contrast with perturbation method, Process Analysis Method is *undependent* upon small or great parameters. Secondly, Process Analysis Method can give *infinite* number of solutions, among which there exist the best approximations to the considered problems, as shown in Chapter one; but, perturbation expansion method can give just only one or limited number of solutions. Generally, experiences seem important for perturbation methods, specially when singular perturbation methods, for example, multiplescale expansion method, matched asymptotic expansion method and so on, must be used; but, Process Analysis Method seems simple in logic. Therefore, it seems that Process Analysis Method would be more flexible and could be used to solve more nonlinear problems, specially those without small or great parameters.

Finite Process Method can avoid the use of iterative techniques to solve nonlinear problems. If Δp is small enough, accurate enough results can be obtained. Smaller Δp is, more accurate the results are. On another side, iterative formulas can be obtained in case of different Δp, for example, $\Delta p = 1.0, \Delta p = 0.5$ and so on. Smaller Δp is, more complex but more insensitive to initial solutions the corresponding iterative formulas are, but naturally more CPU time is needed. If Δp is small enough, then the numerical results are accurate enough and no iteration is needed – this is just Finite Process Method. It means that iterative methods are just only the special cases of Finite Process Method in great value of Δp. Note that in case the simple iterative formulas diverge, Finite Process Method can give converged results, as mentioned in chapter 4.

Clearly, every thing has its light side and dark side. The disadvantage of Process Analysis Method is that it is based on Taylor's serie which has generally finite radius of convergence. The disadvantage of Finite Process Method is that more CPU time is needed, specially when Δp is small.

We have discussed the limitations and disadvantages of perturbation expansion method and iterative techniques and have tried in this paper to overcome these limitations. As the last part of this paper, it seems better to reconsider the advantages of them. The evident advantages of perturbation methods and iterative techniques are that they are very simple in applications. If there exists indeed a small enough parameter, the considered problem can be greatly simplified and the accurate enough results can be obtained. The most "beautiful" application of perturbation expansion method would perhaps be that it prophesied the mass and orbit of Neptune. The important is whether or not there exists a parameter which is small enough in the considered case. Otherwise, we had to find another way. The simple iterative formulas can be easily obtained and in many cases they can give converged results. As mentioned in Chapter 4, in about 90% cases of computations, the simple iterative formulas

(4.82) (4.83) can give finely converged results; only in some cases of strong nonlinearity, they will diverge. Generally, if the nonlinearity of the considered problems is not strong, or you can find a good enough initial solution, then simple iterative formulas are generally good enough. Otherwise, the convergence of iteration is only dependent upon your luck.

So, it seems that Process Analysis Method and Finite Process Method should, if they could, be used as the assistant but not the substitute of perturbation expansion method and iterative techniques, in cases that there does not exist small enough parameters or the nonlinearity of the considered problems is strong.

Although only some simple examples have been used in this paper, it seems easy to apply the basic ideas of both methods to solve more complex problems in engineering. This is also needed in order to reexamine and improve both methods. On the other side, deeper mathematical research seems necessary, because hundreds of examples are not better than an abstract proof. But, it is a pity that this seems out of my ability.

It is ture that from the view points of Finite Process Method and Process Analsis Method nearly every nonlinear problem can be substituted by infinite number of linear problems . Therefore, theoretically speaking, if we have *large enough* computers, it is not difficult to find numerical solutions of any nonlinear problems. This means that the solution of nonlinear problems would be just dependent upon the ability of your computer. Unfortunately, the solution of linear algebraic equations with a large number of unknowns (for example, 5000 unknowns) is still not an easy thing today and need too much CPU time. Up to now, nearly all computers are based on the basic ideas of Turing and von Neumann. These Turing's machines can do just only one operation at its unit time while other parts of them are in waiting-state, although this unit time might be very small. This is a great disadvantage of Turing's machine. Some attempts, for example, the research of vector-computer, have been made to overcome the limitations of Turing's machine. On another side, it is exciting that the brain of humankind has been now also researched. It seems quite possible that this research might put forward some more effective models of automaton than the now widely used Turing's machine. It is real a luck that we are now in a world where science and technology are developing very quickly. I firmly believe in the ability of humankind and is looking for the new revolutions in computer science and industries.

Appendix A

A kind of linearity-invariance under homotopy[1]

Here, we will try to give a simple mathematical proof of a kind of interesting property of continuous mapping (homotopy) from view points of engineers.

Consider a general nonlinear problem,for example, a nonlinear algebraic equation, or a set of nonlinear algebraic equation, or a differential, partial-differential equation and so on. Suppose the considered nonlinear problem be constructed by one or more mathematical expressions and each of these mathematical expressions can be described as follows:

$$f(\mathbf{x},\mathbf{y}) = 0 \tag{A.1}$$

where

$$\begin{aligned} \mathbf{x} &= \{x_i\}, \qquad (i = 1, 2, \cdots, m) \\ \mathbf{y} &= \{y_j\}, \qquad (j = 1, 2, \cdots, l) \end{aligned}$$

are respectively the independent variable vector and dependent variable vector. Here,

$$f(\mathbf{x},\mathbf{y}) = 0$$

would be either an algebraic equation or a differential , partial-differential equation. It would be also either a governing equation or a boundary condition.

In order to give a general theorem and its proof, a concept, called *whole set*, need be given.

definition *a set of dependent variables* $\mathbf{y}$ *and their derivatives with respect to independent variable* $\mathbf{x}$ *appeared in a mathematical problem is called the whole set of it, denoted as*

$$\mathbf{w} = \{w_i\} \qquad (i = 1, 2, \cdots, n)$$

For example, the whole set of the following algebraic equation

$$y_1 - y_2^3 + \sin y_3 = 0$$

is $\mathbf{w} = \{y_1, y_2, y_3\}$; the following partial differential equation

$$\frac{\partial^2 y_1}{\partial x_1 \partial x_2} + \frac{\partial y_2}{\partial x_1} - \left(\frac{\partial y_1}{\partial x_2}\right)^3 + \cos(y_1 y_2) = 0$$

has the whole set

$$\mathbf{w} = \left\{ y_1, y_2, \frac{\partial y_1}{\partial x_2}, \frac{\partial y_2}{\partial x_1}, \frac{\partial^2 y_1}{\partial x_1 \partial x_2} \right\}$$

[1]Homotopy is an important part of topology. In order to have a basic idea of topology, we quote here simply the definition of topology given by Eric Temple Bell in his interesting book *Mathematics - Queen and Servant of Science*:

... topology .... division of modern geometry which deals with the properties of figures unchanged by continuous deformations. (pp.151)

... Topology is the study of those properties of spaces that are invariant under homomorphic transformation. (pp.157)

Generally, the whole set $\mathbf{w}$ of a set of nonlinear algebraic equation is the same as the dependent variable $\mathbf{y}$. For a differential or partial-differential equation, the dependent variable $\mathbf{y}$ is a subset of its whole set $\mathbf{w}$.

By means of concept whole set $\mathbf{w}$, (A.1) can be described as follows:

$$f(\mathbf{x},\mathbf{w}) = 0 \tag{A.2}$$

where,

$$\mathbf{w} = \{w_i\} \qquad , \qquad (i = 1, 2, \cdots, n). \tag{A.3}$$

In this part, the above more general expression of a nonlinear equation would be mainly used to describe and prove the linearity-invariance of the continuous mapping (homotopy).

Define two continuous real function $f_1(p) \in C^\infty$ and $f_2(p) \in C^\infty$, which satisfy:

$$f_1(p) = \begin{cases} 0.0 & \text{when } p = 0.0 \\ 1.0 & \text{when } p = 1.0 \end{cases} \tag{A.4}$$

and

$$f_2(p) = \begin{cases} 1.0 & \text{when } p = 0.0 \\ 0.0 & \text{when } p = 1.0 \end{cases} \tag{A.5}$$

$f_1(p)$ and $f_2(p)$ would be called respectively the *first-sort* and the *second-sort of process function*, p is the embedding variable.

Then, a continuous mapping (or more precisely speaking, homotopy) can be constructed as follows:

$$\mathcal{H}\{\mathbf{x},\mathbf{w}(\mathbf{x};p);p\} = f_1(p)f(\mathbf{x},\mathbf{w}) + f_2(p)\{f(\mathbf{x},\mathbf{w}) - f(\mathbf{x},\mathbf{w}_0)\} = 0 \tag{A.6}$$

where,$\mathbf{w}_0$ is known, corresponding to the freely selected initial solution $\mathbf{y}_0$.

For simplicity, call $\mathbf{y}(\mathbf{x};p)$ *process*, or more precisely, *zero-order process*. And then (A.6) is *zero-order process equation.*

Let $\mathbf{y}_f$ denote the solution of the original equation (A.2), and $\mathbf{w}_f$ denote the corresponding whole set at $\mathbf{y} = \mathbf{y}_f$. Let us research the relation between $\mathbf{y}_f$ and $\mathbf{y}_0$.

When $p = 0$, from (A.4),(A.5) and (A.6), one has:

$$f[\mathbf{x},\mathbf{w}(0)] = f[\mathbf{x},\mathbf{w}_0] \tag{A.7}$$

which has solution:

$$\mathbf{w}(0) = \mathbf{w}_0 \qquad \rightarrow \qquad \mathbf{y}(0) = \mathbf{y}_0 \tag{A.8}$$

When $p = 1.0$, from (A.4),(A.5) and (A.6), one has:

$$f[\mathbf{x},\mathbf{w}(\mathbf{x};1)] = 0 \tag{A.9}$$

which is the same as the original (A.2). Therefore, one has:

$$\mathbf{w}(\mathbf{x};1.0) = \mathbf{w}_f \qquad \rightarrow \qquad \mathbf{y}(\mathbf{x};1.0) = \mathbf{y}_f \tag{A.10}$$

One knows that $\mathbf{y}(\mathbf{x};0)$ or $\mathbf{w}(\mathbf{x};0)$ is the start-point of the zero-order process $\mathbf{y}(\mathbf{x};p)$ or $\mathbf{w}(\mathbf{x};p)$; $\mathbf{y}(\mathbf{x};1.0)$ or $\mathbf{w}(\mathbf{x};1.0)$ is the end-point of the zero-order process $\mathbf{y}(\mathbf{x};p)$ or $\mathbf{w}(\mathbf{x};p)$. Therefore, the zero-order process equation (A.6) connects the initial solution $\mathbf{y}_0(\mathbf{x})$ with the solution $\mathbf{y}_f(\mathbf{x})$ of the original equation (A.2). But, it is still a kind of nonlinear relation, because the zero-order process equation (A.6) is still a nonlinear one. In the following part, we would like to give a kind of linear relation between $\mathbf{y}_0$ and $\mathbf{y}_f$.

Define

$$\mathbf{y}^{[k]}(\mathbf{x};p) = \left\{ \frac{\partial^k y_j}{\partial p^k} \right\} \quad , \quad (j = 1, 2, \cdots, l) \tag{A.11}$$

as the *kth-order process derivatives* of $\mathbf{y}(\mathbf{x};p)$; and

$$\mathbf{w}^{[k]}(\mathbf{x};p) = \left\{ \frac{\partial^k w_i}{\partial p^k} \right\} \quad , \quad (i = 1, 2, \cdots, n) \tag{A.12}$$

as the *kth-order process derivatives* of $\mathbf{w}(\mathbf{x};p)$.

Suppose

1. there exists $\mathbf{y}(\mathbf{x};p), \mathbf{w}(\mathbf{x};p)$ in $p \in [0,1]$;
2. $\mathbf{y}^{[k]}(\mathbf{x};p), \mathbf{w}^{[k]}(\mathbf{x};p)$ $\forall k \in N$ have definition in the process region $p \in [0,1]$.

Then, according to (A.8), (A.10) and Taylor's theorem, one has the following *Taylor's serie relation* between $\mathbf{y}_0(\mathbf{x})$ and $\mathbf{y}_f(\mathbf{x})$ as follows:

$$\begin{aligned} \mathbf{y}_f(\mathbf{x}) &= \mathbf{y}(\mathbf{x};1.0) \\ &= \mathbf{y}(\mathbf{x};0.0) + \sum_{k=1}^{\infty} \left. \frac{\mathbf{y}^{[k]}(\mathbf{x};p)}{k\,!} \right|_{p=0} \\ &= \mathbf{y}_0(\mathbf{x}) + \sum_{k=1}^{\infty} \frac{\mathbf{y}_0^{[k]}(\mathbf{x})}{k\,!} \end{aligned} \tag{A.13}$$

and the following *integral relation:*

$$\mathbf{y}_f(\mathbf{x}) = \mathbf{y}_0(\mathbf{x}) + \int_0^1 \mathbf{y}^{[1]}(\mathbf{x};p)dp \tag{A.14}$$

where, $\mathbf{y}_0^{[k]}(\mathbf{x})$ denotes the value of $\mathbf{y}^{[k]}(\mathbf{x};p)$ at $p = 0$. $\mathbf{y}^{[1]}(\mathbf{x};p), \mathbf{y}^{[2]}(\mathbf{x};p), \cdots, \mathbf{y}^{[k]}(\mathbf{x};p)$ can be obtained as follows.

Deriving zero-order process equation (A.6) with respect to p, one has:

$$f_1'(p)f(\mathbf{x},\mathbf{w}) + f_2'(p)\left\{f(\mathbf{x},\mathbf{w}) - f(\mathbf{x},\mathbf{w}_0)\right\} + \left\{f_1(p) + f_2(p)\right\} \frac{df(\mathbf{x},\mathbf{w})}{dp} = 0 \tag{A.15}$$

One knows:

$$\begin{aligned} \frac{df(\mathbf{x},\mathbf{w})}{dp} &= \sum_{k=1}^{n} \frac{\partial f(\mathbf{x},\mathbf{w})}{\partial w_k} \frac{\partial w_k}{\partial p} \\ &= \sum_{k=1}^{n} \frac{\partial f(\mathbf{x},\mathbf{w})}{\partial w_k} w_k^{[1]}(\mathbf{x};p) \end{aligned} \tag{A.16}$$

Substituting (A.16) into (A.15), one has the *first-order process equation* as follows:

$$\{f_1(p)+f_2(p)\}\sum_{k=1}^{n}\frac{\partial f(\mathbf{x},\mathbf{w})}{\partial w_k}w_k^{[1]}(\mathbf{x};p) = -\left\{f_1'(p)f(\mathbf{x},\mathbf{w})+f_2'(p)\left[f(\mathbf{x},\mathbf{w})-f(\mathbf{x},\mathbf{w}_0)\right]\right\} \quad \text{(A.17)}$$

It is very interesting that this first-order process equation is linear with respect to $\mathbf{w}^{[1]}(\mathbf{x};p)\to \mathbf{y}^{[1]}(\mathbf{x};p)$. Generally, kth-order process equation, from which the kth-order process derivatives $\mathbf{y}^{[k]}(\mathbf{x};p)$ can be obtained, are linear with respect to $\mathbf{w}^{[k]}(\mathbf{x};p)\to \mathbf{y}^{[k]}(\mathbf{x};p)$. This can be described as follows:

theorem of linearity-invariance *for every zero-order process equation*

$$\mathcal{H}\{\mathbf{x},\mathbf{w}(\mathbf{x};p);p\} = f_1(p)f(\mathbf{x},\mathbf{w})+f_2(p)\{f(\mathbf{x},\mathbf{w})-f(\mathbf{x},\mathbf{w}_0)\}=0 \quad ,$$

its corresponding kth-order process equation about $\mathbf{w}^{[k]}(\mathbf{x};p)$ *, or more precisely, about* $\mathbf{y}^{[k]}(\mathbf{x};p)$*, can be described as follows:*

$$\{f_1(p)+f_2(p)\}\sum_{i=1}^{n}\frac{\partial f(\mathbf{x},\mathbf{w})}{\partial w_i}w_i^{[k]}(\mathbf{x};p) = R_k\left\{\mathbf{x},\mathbf{w}(\mathbf{x};p),\mathbf{w}^{[1]}(\mathbf{x};p),\cdots,\mathbf{w}^{[k-1]}(\mathbf{x};p);p\right\} \quad \text{(A.18)}$$

$$(k\geq 1)$$

where

$$\begin{aligned} & R_k\left\{\mathbf{x},\mathbf{w}(\mathbf{x};p),\mathbf{w}^{[1]}(\mathbf{x};p),\cdots,\mathbf{w}^{[k-1]}(\mathbf{x};p);p\right\} \\ = & \frac{dR_{k-1}}{dp}-\sum_{i=1}^{n}w_i^{[k-1]}(\mathbf{x};p)\frac{d}{dp}\left\{[f_1(p)+f_2(p)]\frac{\partial f(\mathbf{x},\mathbf{w})}{\partial w_i}\right\} \end{aligned} \quad \text{(A.19)}$$

with

$$R_1\{\mathbf{x},\mathbf{w}(\mathbf{x};p);p\} = -\left\{f_1'(p)f(\mathbf{x},\mathbf{w})+f_2'(p)\left[f(\mathbf{x},\mathbf{w})-f(\mathbf{x},\mathbf{w}_0)\right]\right\} \quad \text{(A.20)}$$

proof:

(I): when $k=1$, the expression (A.18) is just the same as the first-order process equation (A.17);

(II): suppose in case k=m

$$\{f_1(p)+f_2(p)\}\sum_{i=1}^{n}\frac{\partial f(\mathbf{x},\mathbf{w})}{\partial w_i}w_i^{[k]}(\mathbf{x};p) = R_k\left\{\mathbf{x},\mathbf{w}(\mathbf{x};p),\mathbf{w}^{[1]}(\mathbf{x};p),\cdots,\mathbf{w}^{[k-1]}(\mathbf{x};p);p\right\} \quad \text{(A.21)}$$

holds; then,deriving the above equation with respect to p, one has:

$$\{f_1(p)+f_2(p)\}\sum_{i=1}^{n}\frac{\partial f(\mathbf{x},\mathbf{w})}{\partial w_i}w_i^{[m+1]}(\mathbf{x};p)+\sum_{i=1}^{n}w_i^{[m]}(\mathbf{x};p)\frac{d}{dp}\left\{[f_1(p)+f_2(p)]\frac{\partial f(\mathbf{x};\mathbf{w})}{\partial w_i}\right\} = \frac{dR_m}{dp} \quad \text{(A.22)}$$

therefore,

$$\begin{aligned} & \{f_1(p)+f_2(p)\}\sum_{i=1}^{n}\frac{\partial f(\mathbf{x},\mathbf{w})}{\partial w_i}w_i^{[m+1]}(\mathbf{x};p) \\ = & \frac{dR_k}{dp}-\sum_{i=1}^{n}w_i^{[m]}(\mathbf{x};p)\frac{d}{dp}\left\{[f_1(p)+f_2(p)]\frac{\partial f(\mathbf{x};\mathbf{w})}{\partial w_i}\right\} \\ = & R_{m+1}\left\{\mathbf{x},\mathbf{w}(\mathbf{x};p),\mathbf{w}^{[1]}(\mathbf{x};p),\mathbf{w}^{[2]}(\mathbf{x};p),\cdots,\mathbf{w}^{[m]}(\mathbf{x};p);p\right\} \end{aligned} \quad \text{(A.23)}$$

so, (A.18) holds also in $k = m + 1$.

(III): according to (I),(II), (A.18) holds for every k. end of proof.

It means that every kth-order process equation[2] is always linear in kth-order process derivatives.[3] This is perhaps an important invariance under homotopy.

[2]perhaps, it would be better to call it kth-order deformation equation if we use the concept *deformation* in homotopy.

[3]It would be perhaps better to name it kth-order deformation derivatives.

Appendix B
Detailed deriving of formulas (2.14) and (2.15)

We know that

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \tag{B.1}$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \tag{B.2}$$

$$\cos^{2m+1} x = \frac{1}{2^{2m}} \sum_{k=0}^{\infty} C_{2m+1}^{k} \cos(2m-2k+1)x \tag{B.3}$$

$$cos^{2m} x = \frac{1}{2^{2m-1}} \left\{ \frac{C_{2m}^{m}}{2} + \sum_{k=0}^{m-1} C_{2m}^{k} \cos(2m-2k)x \right\} \tag{B.4}$$

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+m}}{k!(m+k)!} \tag{B.5}$$

where

$$C_m^n = \frac{m!}{n!(m-n)!} \tag{B.6}$$

and $J_m(z)$ is first-sort of Bessel function with positive integer m.

It is known that

$$\theta_0 = \beta \cos z$$

then, we have

$$\begin{aligned}
& sin(\theta_0) \\
= \; & sin(\beta \cos z) \\
= \; & \sum_{m=0}^{\infty} \frac{(-1)^m (\beta \cos z)^{2m+1}}{(2m+1)!} \\
= \; & \sum_{m=0}^{\infty} \frac{(-1)^m \beta^{2m+1}}{(2m+1)!} \left(\frac{1}{2^{2m}}\right) \sum_{k=0}^{m} C_{2m+1}^{k} \cos(2m-2k+1)z \\
= \; & 2 \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(-1)^m \left(\frac{\beta}{2}\right)^{2m+1}}{k!(2m-k+1)!} \cos(2m-2k+1)z \\
= \; & 2 \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^m \left(\frac{\beta}{2}\right)^{2m+1}}{(m-n)!(m+n+1)!} \cos(2n+1)z \\
= \; & 2 \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{(-1)^m \left(\frac{\beta}{2}\right)^{2m+1}}{(m-n)!(m+n+1)!} \cos(2n+1)z \\
= \; & 2 \sum_{n=0}^{\infty} \cos(2n+1)z \sum_{m=n}^{\infty} \frac{(-1)^m \left(\frac{\beta}{2}\right)^{2m+1}}{(m-n)!(m+n+1)!} \\
= \; & 2 \sum_{n=0}^{\infty} \cos(2n+1)z \sum_{k=0}^{\infty} \frac{(-1)^{k+n} \left(\frac{\beta}{2}\right)^{2k+2n+1}}{k!(k+2n+1)!}
\end{aligned}$$

$$= 2\sum_{n=0}^{\infty}(-1)^n \cos(2n+1)z \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\beta}{2}\right)^{2k+2n+1}}{k!(k+2n+1)!}$$

$$= 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(\beta)\cos(2n+1)z \tag{B.7}$$

Similarly, we have

$$\begin{aligned}
& \cos(\theta_0) \\
= \; & \cos(\beta \cos z) \\
= \; & \sum_{m=0}^{\infty} \frac{(-1)^m (\beta \cos z)^{2m}}{(2m)!} \\
= \; & 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \beta^{2m}}{(2m)!} \frac{1}{2^{2m-1}} \left\{ \frac{1}{2}\frac{(2m)!}{m!m!} + \sum_{k=0}^{m-1} \frac{(2m)!}{k!(2m-k)!} \cos(2m-2k)z \right\} \\
= \; & \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{\beta}{2}\right)^{2m}}{m!m!} + 2\sum_{m=1}^{\infty}(-1)^m \left(\frac{\beta}{2}\right)^{2m} \sum_{k=0}^{m-1} \frac{\cos(2m-2k)z}{k!(2m-k)!} \\
= \; & J_0(\beta) + 2\sum_{m=1}^{\infty}(-1)^m \left(\frac{\beta}{2}\right)^{2m} \sum_{n=1}^{m} \frac{cos(2nz)}{(m-n)!(m+n)!} \\
= \; & J_0(\beta) + 2\sum_{m=1}^{\infty}\sum_{n=1}^{m} \frac{(-1)^m \left(\frac{\beta}{2}\right)^{2m}}{(m-n)!(m+n)!} \cos(2nz) \\
= \; & J_0(\beta) + 2\sum_{n=1}^{\infty}\sum_{m=n}^{\infty} \frac{(-1)^m \left(\frac{\beta}{2}\right)^{2m}}{(m-n)!(m+n)!} \cos(2nz) \\
= \; & J_0(\beta) + 2\sum_{n=1}^{\infty} cos(2nz) \sum_{k=0}^{\infty} \frac{(-1)^{k+n} \left(\frac{\beta}{2}\right)^{2k+2n}}{k!(k+2n)!} \\
= \; & J_0(\beta) + 2\sum_{n=1}^{\infty}(-1)^n cos(2nz) \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\beta}{2}\right)^{2k+2n}}{k!(k+2n)!} \\
= \; & J_0(\beta) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(\beta)\cos(2nz)
\end{aligned} \tag{B.8}$$

Appendix C
Detailed deriving of formulas (3.2)

The nonlinear boundary conditions of 2D free surface of a kind of ideal fluid are described as follows:

$$\frac{\partial\phi(X,Y,t)}{\partial t}+\frac{1}{2}\left[grad\phi(X,Y,t)\right]^2+g\zeta(X,t)=0 \qquad \text{on } Y=\zeta(X,t) \tag{C.1}$$

$$\frac{\partial\zeta(X,t)}{\partial t}+\frac{\partial\phi(X,Y,t)}{\partial X}\frac{\partial\zeta(X,t)}{\partial X}=\frac{\partial\phi(X,Y,t)}{\partial Y} \qquad \text{on } Y=\zeta(X,t) \tag{C.2}$$

For the considered problem, there exists the relations

$$\phi(X,Y,t)=\phi(x+Ct,y)$$

and

$$\zeta(X,t)=\zeta(x+Ct)$$

where, C is phase velocity of 2D progressive wave. Then, using the coordinate transformation

$$X=x+Ct, \qquad Y=y \tag{C.3}$$

one has

$$\frac{\partial}{\partial t}=-C\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial X}=\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial Y}=\frac{\partial}{\partial y} \tag{C.4}$$

Substituting (C.3), (C.4) into (C.1) and (C.2), we have

$$\begin{aligned}\zeta(x) &= -\frac{1}{g}\left\{-C\frac{\partial\phi}{\partial x}+\frac{1}{2}\left(grad\phi\right)^2\right\}\\ &= -\frac{1}{g}\mathcal{H}(x,y) \qquad \text{on } y=\zeta(x)\end{aligned} \tag{C.5}$$

and

$$\left(-C+\frac{\partial\phi}{\partial x}\right)\frac{\partial\zeta}{\partial x}=\frac{\partial\phi}{\partial y} \qquad \text{on } y=\zeta(x) \tag{C.6}$$

where

$$\begin{aligned}\mathcal{H}(x,y) &= -C\frac{\partial\phi}{\partial x}+\frac{1}{2}\left(grad\phi\right)^2\\ &= -C\phi_x+\frac{1}{2}\nabla\phi\nabla\phi \qquad \text{on } y=\zeta(x)\end{aligned} \tag{C.7}$$

Note that in above expression ζ is now a function of x and ϕ is a function of $x, y=\zeta(x)$.

Because the independent variable of function ζ is only x, then, we can write

$$\frac{\partial\zeta}{\partial x}=\frac{d\zeta}{dx} \tag{C.8}$$

then, equation (C.6) can be written as

$$\left(-C+\frac{\partial\phi}{\partial x}\right)\frac{d\zeta}{dx}=\frac{\partial\phi}{\partial y} \qquad \text{on } y=\zeta(x) \tag{C.9}$$

Note that the left side of equation (C.5) is an expression only of x, but its right side is an expression not only of x but also of $y = \zeta(x)$, then from equation (C.5), we have

$$\begin{aligned} \frac{d\zeta}{dx} &= -\frac{1}{g}\left\{\frac{\partial \mathcal{H}}{\partial x} + \frac{\partial \mathcal{H}}{\partial y}\frac{dy}{dx}\right\} \\ &= -\frac{1}{g}\left\{\frac{\partial \mathcal{H}}{\partial x} + \frac{\partial \mathcal{H}}{\partial y}\frac{d\zeta}{dx}\right\} \qquad \text{on } y = \zeta(x) \end{aligned} \tag{C.10}$$

From above equation, we have

$$\frac{d\zeta}{dx} = \frac{-\frac{\partial \mathcal{H}}{\partial x}}{g + \frac{\partial \mathcal{H}}{\partial y}} \tag{C.11}$$

Substituting (C.11) into (C.9), one has

$$\left(-C + \frac{\partial \phi}{\partial x}\right)\frac{-\frac{\partial \mathcal{H}}{\partial x}}{g + \frac{\partial \mathcal{H}}{\partial y}} = \frac{\partial \phi}{\partial y} \qquad \text{on } y = \zeta(x) \tag{C.12}$$

From (C.12), we have

$$g\frac{\partial \phi}{\partial y} + \left(-C\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y}\right)\mathcal{H} = 0 \qquad \text{on } y = \zeta(x) \tag{C.13}$$

Substituting (C.7) into (C.12), we have

$$g\frac{\partial \phi}{\partial y} + \left(-C\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y}\right)\left\{-C\frac{\partial \phi}{\partial x} + \frac{1}{2}(grad\phi)^2\right\} = 0 \tag{C.14}$$

The above equation can be written as

$$g\phi_y + \left(-C\frac{\partial}{\partial x} + \nabla\phi\nabla\right)\left(-C\phi_x + \frac{1}{2}\nabla\phi\nabla\phi\right) = 0 \qquad \text{on } y = \zeta(x) \tag{C.15}$$

then, we have

$$g\phi_y + C^2\phi_{xx} - 2C\nabla\phi\nabla\phi_x + \frac{1}{2}\nabla\phi\nabla(\nabla\phi\nabla\phi) = 0 \qquad \text{on } y = \zeta(x) \tag{C.16}$$

Appendix D
Detailed deriving of $\phi_0^{[m]}(x,y), \zeta_0^{[m]}(x), C_0^{[m]}$ $(m = 1,2)$

For simplicity in following deriving, define

$$\mathcal{E}(\phi) = \nabla\phi\,\nabla\phi \tag{D.1}$$

D.1 initial solution:

The solution of the initial equation (3.23)~(3.26) can be easily obtained:

$$\phi_0(x,y) = AC_0 e^{ky} sin(kx) \tag{D.2}$$

$$C_0 = \sqrt{\frac{g}{k}} \tag{D.3}$$

where,A is the initial wave-amplitude , $k = \frac{2\pi}{\lambda}$ is the wave-number, λ is the wave-length.

The above initial solutions are the same as the well-known Aray's wave.

D.2 $\zeta_0^{[1]}(\mathbf{x})$:

According to (3.53), one has:

$$\begin{aligned}\zeta_0^{[1]}(x) &= \mathcal{Z}\left[\phi(x,y;p)\right]\big|_{p=0,y=0} \\ &= \mathcal{Z}\left[\phi_0(x,y)\right]\big|_{y=0} \\ &= A\left\{cos(kx) - \frac{1}{2}kA\right\}\end{aligned} \tag{D.4}$$

D.3 $\phi_0^{[1]}(x,y), C_0^{[1]}$:

According to (3.50),(3.51) and (3.52), one has the equation of $\phi_0^{[1]}(x,y)$ as follows:

$$\nabla^2\phi_0^{[1]}(x,y) = 0 \tag{D.5}$$

with boundary conditions:

$$\left.\frac{d\,\mathcal{L}}{d\,p}\right|_{p=0} + \mathcal{N}\left[\phi_0(x,y)\right] = 0 \quad , \quad \text{on } y = 0 \tag{D.6}$$

$$\lim_{y\to-\infty} \frac{\partial\phi_0^{[1]}(x,y)}{\partial y} = 0 \tag{D.7}$$

According to (3.48), one has:

$$\begin{aligned}\left.\frac{d\,\mathcal{L}}{d\,p}\right|_{p=0} &= \left.\frac{\partial\mathcal{L}}{\partial p}\right|_{p=0} + \zeta_0^{[1]}\frac{\partial\mathcal{L}\left[\phi_0(x,y)\right]}{\partial y} \\ &= C_0^2\left(\phi_0^{[1]}\right)_{xx} + g\left(\phi_0^{[1]}\right)_y + 2C_0C_0^{[1]}\left(\phi_0\right)_{xx} + \zeta_0^{[1]}\frac{\partial\mathcal{L}\left[\phi_0(x,y)\right]}{\partial y}\end{aligned} \tag{D.8}$$

Substituting (D.8) into (D.6), one has the boundary condition of $\phi_0^{[1]}(x,y)$ as follows:

$$\begin{aligned}&\left.\left\{C_0^2\left(\phi_0^{[1]}\right)_{xx} + g\left(\phi_0^{[1]}\right)_y\right\}\right|_{y=0} \\ &= 2C_0^2k^2A\left\{C_0^{[1]} - \frac{1}{2}C_0k^2A^2\right\}sin(kx)\end{aligned} \tag{D.9}$$

In order to let $\phi_0^{[1]}(x,y)$ have finite solution, it must be satisfied:

$$C_0^{[1]} = \frac{1}{2}C_0k^2A^2 \tag{D.10}$$

Then, according to (D.5),(D.7) and (D.9), the first-order process equation of velocity-potential at $p=0$ is as follows:

$$\nabla^2\phi_0^{[1]}(x,y) = 0 \tag{D.11}$$

with boundary conditions:

$$C_0^2\left(\phi_0^{[1]}\right)_{xx} + g\left(\phi_0^{[1]}\right)_y = 0 \quad , \qquad \text{on } y=0 \tag{D.12}$$

$$\lim_{y\to-\infty}\frac{\partial\phi_0^{[1]}(x,y)}{\partial y} = 0 \tag{D.13}$$

The above *linear* equations have solution:

$$\phi_0^{[1]}(x,y) = 0 \tag{D.14}$$

D.4 $\zeta_0^{[2]}(x)$:

According to (3.53) and (3.48), one has:

$$\begin{aligned}\zeta_0^{[2]}(x) &= 2\left.\frac{d\mathcal{Z}}{dp}\right|_{p=0,y=0} \\ &= 2\left.\left\{\left.\frac{\partial\mathcal{Z}}{\partial p}\right|_{p=0} + \zeta_0^{[1]}\frac{\partial\mathcal{Z}\left[\phi_0(x,y)\right]}{\partial y}\right\}\right|_{y=0}\end{aligned} \tag{D.15}$$

As a result of (3.18) and (D.14), one has:

$$\left.\frac{\partial \mathcal{Z}}{\partial p}\right|_{p=0} = \frac{1}{g}\left\{C_0^{[1]}(\phi_0)_x + C_0\left(\phi_0^{[1]}\right)_x - \nabla\phi_0\nabla\phi_0^{[1]}\right\}$$
$$= \frac{C_0^{[1]}(\phi_0)_x}{g}$$
$$= \frac{1}{2}k^2A^3e^{ky}cos(kx) \tag{D.16}$$

Substituting (D.4),(D.16) into (D.15), one obtains:

$$\zeta_0^{[2]}(x) = kA^2\left\{cos(2kx) - 2kAcos(kx) + 1 + k^2A^2\right\} \tag{D.17}$$

D.5 $\phi_0^{[2]}(x,y), C_0^{[2]}$:

According to (3.50), (3.51) and (3.52), the equations of $\phi_0^{[2]}(x,y)$ are as follows:

$$\nabla^2\phi^{[2]}(x,y) = 0 \tag{D.18}$$

with boundary conditions:

$$\left.\frac{d^2\mathcal{L}}{dp^2}\right|_{p=0} + 2\left.\frac{d\mathcal{N}}{dp}\right|_{p=0} = 0 \quad , \qquad \text{on } y = 0 \tag{D.19}$$

$$\lim_{y\to-\infty}\frac{\partial\phi_0^{[2]}(x,y)}{\partial y} = 0 \tag{D.20}$$

According to (3.49), one has:

$$\left.\frac{d^2\mathcal{L}}{dp^2}\right|_{p=0}$$
$$= \left.\frac{\partial^2\mathcal{L}}{\partial p^2}\right|_{p=0} + 2\zeta_0^{[1]}\left.\frac{\partial^2\mathcal{L}}{\partial y\partial p}\right|_{p=0} + \zeta_0^{[2]}\frac{\partial\mathcal{L}\left[\phi_0(x,y)\right]}{\partial y} + \left(\zeta_0^{[1]}\right)^2\frac{\partial^2\mathcal{L}\left[\phi_0(x,y)\right]}{\partial y^2} \tag{D.21}$$

As a result of (3.16) and (D.14), one has:

$$\left.\frac{\partial\mathcal{L}}{\partial p}\right|_{p=0} = 2C_0C_0^{[1]}(\phi_0)_{xx} + C_0^2\left(\phi_0^{[1]}\right)_{xx} + g\left(\phi_0^{[1]}\right)_y$$
$$= 2C_0C_0^{[1]}(\phi_0)_{xx} \tag{D.22}$$

Hence,

$$2\zeta_0^{[1]}\left.\frac{\partial^2\mathcal{L}}{\partial y\partial p}\right|_{p=0}$$
$$= 4C_0C_0^{[1]}\zeta_0^{[1]}(\phi_0)_{xxy} \tag{D.23}$$

According to (3.16), one has:

$$\left.\frac{\partial^2 \mathcal{L}}{\partial p^2}\right|_{p=0} = C_0^2\left(\phi_0^{[2]}\right)_{xx} + g\left(\phi_0^{[2]}\right)_g + 4C_0C_0^{[1]}\left(\phi_0^{[1]}\right)_{xx} + 2\left\{\left(C_0^{[1]}\right)^2 + C_0C_0^{[2]}\right\}(\phi_0)_{xx} \tag{D.24}$$

Substituting (D.17),(D.23) and (D.24) into (D.21), one obtains:

$$\begin{aligned}\left.\frac{d^2\mathcal{L}}{dp^2}\right|_{p=0,y=0} &= \left.\left\{C_0^2\left(\phi_0^{[2]}\right)_{xx} + g\left(\phi_0^{[2]}\right)_y\right\}\right|_{y=0} - C_0^3k^5A^4 sin(2kx) \\ &\quad -2C_0^2k^2A\left\{C_0^{[2]} - \frac{1}{4}C_0k^4A^4\right\} sin(kx)\end{aligned} \tag{D.25}$$

According to (3.48) and (3.17), one has:

$$\left.\frac{d\mathcal{N}}{dp}\right|_{p=0} = \left.\frac{\partial\mathcal{N}}{\partial p}\right|_{p=0} + \zeta_0^{[1]}\frac{\partial\mathcal{N}\left[\phi_0(x,y)\right]}{\partial y} \tag{D.26}$$

As the result of (D.14), one has:

$$\begin{aligned}\left.\frac{\partial\mathcal{N}}{\partial p}\right|_{p=0} &= -C_0^{[1]}\frac{\partial}{\partial x}\left(\nabla\phi_0\nabla\phi_0\right) \\ &= -C_0^{[1]}\frac{\partial\mathcal{E}\left(\phi_0\right)}{\partial x} \\ &= 0\end{aligned} \tag{D.27}$$

Substituting (D.27),(D.4) into (D.26), one obtains:

$$\left.\frac{d\mathcal{N}}{dp}\right|_{p=0,y=0} = \frac{3}{2}C_0^3k^5A^4\left\{sin(2kx) - kA sin(kx)\right\} \tag{D.28}$$

Substituting (D.25),(D.28) into (D.19), one has:

$$\left.\left\{C_0^2\left(\phi_0^{[2]}\right)_{xx} + g\left(\phi_0^{[2]}\right)_y\right\}\right|_{y=0} = 2C_0^2k^2A\left\{C_0^{[2]} + \frac{5}{4}C_0k^4A^4\right\} sin(kx) - 2C_0^3k^5A^4 sin(2kx) \tag{D.29}$$

In order to let $\phi_0^{[2]}(x,y)$ be finite, it must be satisfied :

$$C_0^{[2]} = -\frac{5}{4}C_0k^4A^4 \tag{D.30}$$

Thus, the equations of $\phi_0^{[2]}(x,y)$ are as follows:

$$\nabla^2\phi_0^{[2]}(x,y) = 0 \tag{D.31}$$

with boundary conditions:

$$C_0^2\left(\phi_0^{[2]}\right)_{xx} + g\left(\phi_0^{[2]}\right)_y = -2C_0^3k^5A^4 sin(2kx) \quad , \quad \text{on } y=0 \tag{D.32}$$

$$\lim_{y \to -\infty} \frac{\partial \phi_0^{[2]}(x,y)}{\partial y} = 0 \tag{D.33}$$

The above *linear* equations have following solution:

$$\phi_0^{[2]}(x,y) = C_0 k^3 A^4 e^{2ky} sin(2kx) \tag{D.34}$$

It is interesting and perhaps should be emphasized that the equations of $\phi_0^{[1]}(x,y)$,$\phi_0^{[2]}(x,y)$ are *linear*. It is easy to prove that this is true for any $\phi_0^{[m]}(x,y)$ $(m \geq 1)$. So, in the same way described above, one can obtain higher-order process derivatives of velocity-potential, wave-elevation and wave-velocity.

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