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Fritz J. Ursell

VIII. Georg-Weinblum-Gedächtnis-Vorlesung
Mathematical Observations on the Methods of Multipoles

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VIII. Georg Weinblum Memorial Lecture

MATHEMATICAL OBSERVATIONS ON THE METHOD OF MULTIPOLES

F. Ursell

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Eighth Georg Weinblum Memorial Lecture

Mathematical Observations on the Method of Multipoles

by

Fritz J. Ursell

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1. Introduction

We are here today to commemorate a great naval architect, Georg Weinblum, whose life and work covered many countries and almost every aspect of naval architecture. He profoundly influenced ship hydrodynamics not only by his own scientific work but also by his attitudes and by his philosophy. Practice, experiment and theoretical analysis were all of them essential parts of his work. In particular he was not afraid of mathematics and gave every encouragement to mathematicians. I regard it as a great honour that I should have been invited to deliver a lecture which bears his name.

One of the problems discussed by Weinblum and St. Denis in 1950, in their great paper on the motion of ships at sea, is the determination of the virtual mass and damping of bodies heaving in the free surface of a fluid. This is the topic I have chosen for this lecture, and I shall suppose initially (in fact for most of the lecture) that the body is a horizontal half-immersed circular cylinder, about which I wrote a paper in 1949 (U1949a); my students and I are still working on various aspects of this problem nearly 40 years later. In the present lecture I shall be talking mainly about my own work but there have been many other contributors. Here I will only mention the distinguished work of Otto Grim (1953).

The following physical model will be adopted. It will be assumed that the fluid is frictionless, then the motion is irrotational and can be described by a velocity potential. It will also be assumed that the motion is so small that all the equations can be linearized, and that the motion is periodic in time. Our aim is to find the periodic vertical force on the cylinder. The component in phase with the acceleration is called the virtual-mass component and is workless, the component in quadrature is the wave-making or damping component and is related to the waves generated by the cylinder. The virtual-mass and damping coefficients have become important parameters in ship hydrodynamics, and we know now (but did not know then) that they are strongly frequency-dependent.

2. **Source representations.**

Let us begin by looking very briefly at early treatments of damping as described by Weinblum & St. Denis. These make use of distributions of pulsating wave sources. The velocity potential at the point (x,y) of a wave source located at the point (ξ,η) will be denoted by $G(x,y;\xi,\eta;K)e^{-i\omega t}$ and satisfies the following conditions:

$$\frac{\partial G}{\partial y} = -\frac{\omega^2}{g}G = -KG \text{ when } y = 0$$

$x \leftarrow$, $y \downarrow$

waves outgoing $\leftarrow$ at $x = +\infty$

$\odot$ (ξ,η)

$\rightarrow$ waves outgoing at $x = -\infty$

$e^{-i\omega t}$ omitted

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(x,y;\xi,\eta;K) = 0$$

except at $(x,y) = (\xi,\eta)$

Near (ξ,η) the potential behaves like a pulsating-source potential:

$$G(x,y;\xi,\eta;K)e^{-i\omega t} \sim \tfrac{1}{2}\log\left\{(x-\xi)^2 + (y-\eta)^2\right\}e^{-i\omega t}.$$

(The factor $e^{-i\omega t}$ will henceforth be omitted.)

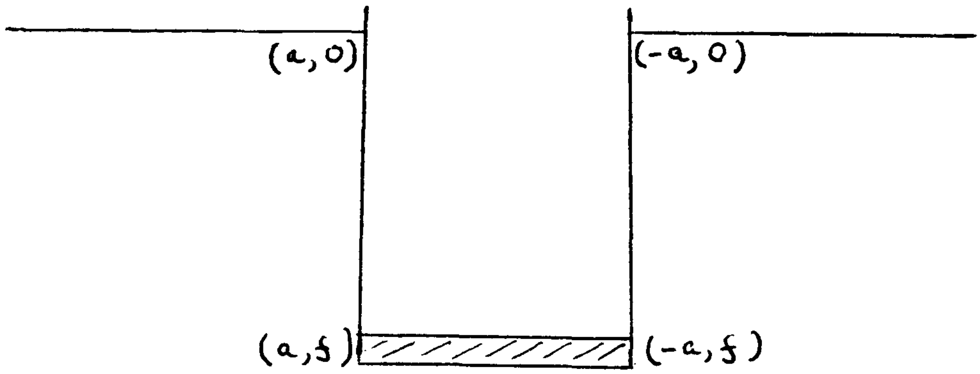
Then it can be verified that all conditions are satisfied by

$$G(x,y;\xi,\eta;K) = \tfrac{1}{2}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)} - 2\,⨍_0^\infty e^{-k(y+\eta)}\cos k(x-\xi)\frac{dk}{k-K}$$

$$\sim -2\pi i e^{-K(y+\eta)}\, e^{iK|x-\xi|} \quad \text{when} \quad |x-\xi| \to \infty.$$

We have arranged for G(   ) to behave like outgoing waves at infinity by choosing the path of integration to pass <u>under</u> the pole $k = K$.

Let us now consider a heaving cylinder of box-like section. Then continuity considerations suggest that the



bottom is equivalent to a uniform distribution of wave sources at depth f while the side walls do not act as sources. The potential at (x,y) of this source distribution is proportional to

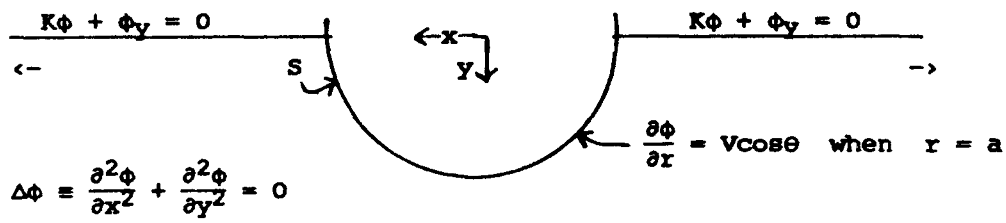
$$\int_{-a}^{a} G(x,y;\xi,f)d\xi,$$

and this is the expression that was used by Weinblum & St. Denis and others to calculate the wave damping, but it is obvious that it is not exact. In fact, the source distribution is easily seen to induce a velocity component normal to the side walls. We may also recall Green's representation (which is not difficult to prove, see U1954):

$$\phi(x,y) = -\frac{1}{2\pi}\int\left\{\phi(\xi,\eta)\frac{\partial G}{\partial n}(x,y;\xi,\eta) - G(x,y;\xi,\eta)\frac{\partial\phi}{\partial n}(\xi,\eta)\right\}ds(\xi,\eta);$$

when this is applied to our problem it states that the motion can be generated by a source distribution of strength $\partial\phi/\partial n$ over the bottom <u>together with a distribution of dipoles of strength ϕ over the bottom and side walls.</u> We note that the dipole strength ϕ cannot be found from continuity considerations, but that $\frac{\partial\phi}{\partial n}$ is prescribed on the boundary (as a constant on the bottom, as zero on the side walls).

From now on we shall concentrate on the half-immersed cylinder of circular section. To solve the heaving problem we must formulate it properly as a boundary-value problem: we require a velocity potential $\phi(x,y;K)$ which satisfies the following conditions:



Unless we are willing to consider a direct numerical solution, (which is not easy since the fluid is unbounded) we must represent the solution analytically in some appropriate way, either as an infinite series or as an integral. Suppose, for instance that, as before, we try to represent the motion by distributing wave sources over the semi-circle S; in mathematical terms, suppose that

$$\phi(x,y;K) = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \mu(\theta;K)G(x,y;a\sin\theta,a\cos\theta;K)a\,d\theta$$

where the source strength $\mu(\theta;K)$ is an unknown function which is to be determined. (Note that, when K is fixed μ is a function of only a single variable θ whereas $\phi(x,y;K)$ is a function of two variables x and y.) Are we justified in supposing that the required potential can indeed be represented in this form? Is it perhaps physically obvious? I do not think so. Let us examine the proposed representation. This satisfies all the conditions except the condition on S, and it can be shown that this is also satisfied if the source strength $\mu(\theta,K)$ satisfies the integral equation

$$\pi\mu(\alpha;K) + \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\mu(\theta;K)\left\langle\frac{\partial}{\partial r}G(r\sin\alpha,\ r\cos\alpha;\ a\sin\theta,\ a\cos\theta;\ K)\right\rangle a\,d\theta$$

$$= V\cos\alpha \quad \text{when} \quad -\tfrac{1}{2}\pi < \alpha < \tfrac{1}{2}\pi, \text{ and } r = a$$

where the right-hand side is the prescribed normal velocity. To solve our problem we must show that this integral equation has a solution. This can indeed be done but it requires some profound mathematics, Fredholm's famous theory of integral equations (1900, see e.g. U1974a) which applies to equations in which the unknown function occurs both outside as well as inside the integral sign. F. John (1950) used this theory to show that our problem does indeed have a unique solution of the proposed form, except at an infinite set of irregular frequencies where some alternative representation must be used. We note once again that the source strength $\mu(\theta;K)$ is not proportional to the prescribed normal velocity, mathematically because of the integral term, physically because a source at one boundary point induces a normal velocity at all boundary points. We also note again that the justification of the source representation is mathematical, not physical. When F. John's work appeared in 1950 there was no effective way of solving integral equations numerically.

3. Wavefree potentials and the method of multipoles.

I now want to talk about the method of multipoles. When I tried to solve this problem in the late 1940's I did not know about F. John's work (which was then unpublished) or about integral equations. I did know that the functions $r^{-m}\cos m\theta$ satisfy Laplace's equation $\Delta\phi = 0$ for any m. By trial and error I found (and this was the most important step) that for every positive integer m the combinations

$$\frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\,\frac{\cos(2m-1)\theta}{r^{2m-1}}, \quad m = 1,2,3,\ldots$$

also satisfy $K\phi + \phi_y = 0$ when $y = 0$ (i.e. when $\theta = \pm\tfrac{1}{2}\pi$). These combinations are now known as wavefree potentials.

Can we express the required solution as a combination of just these wavefree potentials? Clearly not, for these all tend to 0 at infinity whereas the solution must have wavelike behaviour. We therefore need to add at least one wavelike function to our set, and I chose a wave source at the origin:

$$\Phi_0(Kr,\theta) = -\tfrac{1}{2}G(x,y;0,0;K) = \int_0^\infty e^{-ky}\cos kx \,\frac{dk}{k-K}$$

$$= \int_0^\infty e^{-kr\cos\theta}\cos(kr\sin\theta)\,\frac{dk}{k-K}$$

$$= -(\gamma + \log Kr)\sum_{s=0}^{\infty}\frac{(-Kr)^s}{s!}\cos s\theta$$

$$+ \theta\sum_{1}^{\infty}\frac{(-Kr)^s}{s!}\sin s\theta + \sum_{1}^{\infty}\frac{(-Kr)^s}{s!}\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{s}\right)\cos s\theta$$

$$+ \pi i\sum_{0}^{\infty}\frac{(-Kr)^s}{s!}\cos s\theta.$$

We now ask two questions. The first question: Is any solution of our problem necessarily of the form $\phi(x,y;K) = \phi(r\sin\theta, r\cos\theta;K)$

$$= C\left\{\Phi_0(Kr,\theta) + \sum_{m=1}^{\infty}p_{2m}(Ka)a^{2m}\left(\frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\,\frac{\cos(2m-1)\theta}{r^{2m-1}}\right)\right\}, \qquad \text{(A)}$$

or are there more terms that might be or ought to be included in the expansion? (The rather strange notation for the unknowns is taken from U1974b). The answer is, that no further term can be added, and the proof does not present any serious difficulty to a professional mathematician. Indeed, Green's representation shows that the potential $\phi(x,y;K)$ can be represented in this way provided that the source $G(x,y;\xi,\eta)$ and the dipole can be represented in this way, and this can be shown without much difficulty (U1981), alternative arguments are given in U1968b. The second question: Can the coefficients C, $p_2, p_4, \ldots$ be chosen so that the boundary condition on the semi-circle is also satisfied, and so that the three series for the velocity potential and the two velocity components are convergent outside and on the semi-circle? This can also be proved but the proof is more difficult. Let us look at this problem in more detail.

The boundary condition on the semi-circle is $\frac{\partial\phi}{\partial r} = V\cos\theta$ when $r = a$ and $0 < \theta < \tfrac{1}{2}\pi$. When the expression (A) is substituted we obtain

$$\left\langle a\,\frac{\partial\Phi_o}{\partial r}(Kr,\theta)\right\rangle_{r=a} - \sum_1^\infty 2mp_{2m}\left\{\cos 2m\theta + \frac{Ka}{2m}\cos(2m-1)\theta\right\}$$

$$= \frac{Va}{C}\cos\theta \quad \text{when} \quad 0 < \theta < \tfrac{1}{2}\pi,$$

or equivalently

$$\left\langle a\,\frac{\partial\Phi_o}{\partial r}(Kr,\theta)\right\rangle_{r=a} = F_1(\theta,Ka), \text{ say, a known function of } \theta \text{ and } Ka,$$

$$= \frac{Va}{C}\cos\theta + \sum_1^\infty 2mp_{2m}(Ka)\left\{\cos 2m\theta + \frac{Ka}{2m}\cos(2m-1)\theta\right\} \quad \text{when} \quad 0 < \theta < \tfrac{1}{2}\pi. \tag{B}$$

Thus we are asking whether the known function F_1 can be expressed as a combination of the known functions

$$\cos\theta, \text{ and } \cos 2m\theta + \frac{Ka}{2m}\cos(2m-1)\theta, \quad m = 1,2,3,\ldots .$$

We are reminded of Fourier series, indeed in the long-wave limit $(Ka = 0)$ we obtain the equation

$$-1 = \frac{Va}{C_o}\cos\theta + \sum_1^\infty 2mp_{2m}(0)\cos 2m\theta$$

which can indeed be solved by the usual Fourier method: To find C_o we integrate from 0 to $\tfrac{1}{2}\pi$; to find $2sp_{2s}(0)$ we multiply by $\cos 2s\theta$ and then integrate from 0 to $\tfrac{1}{2}\pi$. For general values of Ka we use the same procedure. Thus, on integrating (B) we find that

$$\int_0^{\frac{1}{2}\pi} F_1(\theta',Ka)d\theta' = \frac{Va}{C}\int_0^{\frac{1}{2}\pi}\cos\theta' d\theta'$$

$$+ \sum_1^\infty 2mp_{2m}(Ka).\frac{Ka}{2m}\int_0^{\frac{1}{2}\pi}\cos(2m-1)\theta' d\theta,$$

i.e., $$\frac{Va}{C} = \int_0^{\frac{1}{2}\pi} F_1(\theta',Ka)d\theta' - \sum_1^\infty 2mp_{2m}(Ka).\frac{(-1)^{m-1}Ka}{2m(2m-1)}.$$

When we substitute this into (B) we find that

$$F_1(\theta,Ka) - \cos\theta\int_0^{\frac{1}{2}\pi} F_1(\theta',Ka)d\theta'$$

$= F(\theta,Ka)$, say, a known function of θ and Ka,

$$= \sum_{m=1}^\infty 2mp_{2m}(Ka)\left\{\cos 2m\theta + \frac{Ka}{2m}\left(\cos(2m-1)\theta - \frac{(-1)^{m-1}}{2m-1}\cos\theta\right)\right\}.$$

Now we continue the Fourier procedure by multiplying by $\cos 2s\theta$, $s = 1,2,3,\ldots$ and integrating over $(0,\frac{1}{2}\pi)$. We find that

$$\frac{4}{\pi}\int_0^{\frac{1}{2}\pi} F(\theta,Ka)\cos 2s\theta d\theta, = c_{2s}(Ka), \text{ say}$$

$$= 2sp_{2s}(Ka) + \frac{4}{\pi}Ka\sum_{m=1}^{\infty} 2mp_{2m}(Ka).\frac{1}{2m}\int_0^{\frac{1}{2}\pi}\cos 2s\theta\left\{\cos(2m-1)\theta - \frac{(-1)^{m-1}}{2m-1}\cos\theta\right\}d\theta.$$

The last integral is an elementary integral and can be readily found. We find in this way that the unknowns $p_2,p_4,\ldots$ satisfy the infinite system of equations

$$c_{2s}(Ka) = 2sp_{2s}(Ka) + Ka\sum_{m=1}^{\infty}(2mp_{2m}(Ka))b_{sm}, \qquad (S)$$

where $b_{sm} = (-1)^{m+s}\ \dfrac{32}{\pi}\ .\ \dfrac{s^2(m-1)}{(4s^2-1)(2m-1)(2s+2m-1)(2s-2m+1)}$

is known explicitly and is independent of Ka. The evaluation of the coefficients $c_{2s}(Ka)$ is slightly more difficult but can be carried out either by numerical integration or more simply by the summation of rapidly convergent power series in Ka where the coefficients can again be calculated explicitly without any numerical integration. At the time, in the late 1940's, this was the great advantage of the present method, because the equations which we have just obtained could easily be solved on a desk calculator whereas there was then no practical method of solving integral equations. The force coefficients, i.e. the virtual mass and damping, involves $\int_0^{\frac{1}{2}\pi}\Phi(a\sin\theta, a\cos\theta, K)\cos\theta d\theta$, and is thus known for a given value of Ka when we know C, $p_2,p_4,\ldots$.

We have not yet completed the solution of our problem for we must still show that the infinite system (S) has a solution and that the resulting series for the potential and for the velocities are convergent. For small Ka it was shown in the original paper (by iteration starting from $p_{2m}(0)$) that the infinite system has a solution and that for any fixed Ka we have $\qquad p_{2m}(Ka) = O(\frac{1}{m^3})$ as $m \to \infty$

This result can be extended to all values of Ka by using an analogue of Fredholm's theory of integral equations, but we shall not do so here (see U1974b).

4. **The method of multipoles, some criticisms.**

We have now seen that the multipole expansion

$$\phi = \text{wave source plus wavefree potentials}$$

can be justified mathematically but some ship hydrodynamicists have regretted that this expansion has no obvious physical interpretation. It is precisely for this reason that we need the mathematical justification which shows that the series is justified if and only if the original linear frictionless model is justified. There are some problems (e.g. the Kelvin-Neumann problem of wave resistance and the problems of slender-body theory), where either the physical or the mathematical arguments are unconvincing, and then there is indeed real doubt about the form of the proposed solutions. Another early comment was as follows: In the limit $Ka = \infty$ the heaving motion corresponds to the well-known motion in an unbounded fluid. It would therefore be more natural to start from the high-frequency rather than the low-frequency end, and to look for expansions in powers of $^1/Ka$ rather than in powers of Ka. Let us look at this suggestion.

We had the equation

$$F_1(\theta,Ka) = \frac{Va}{C}\cos\theta + \sum_1^\infty 2mp_{2m}(Ka)\left\{\cos 2m\theta + \frac{Ka}{2m}\cos(2m-1)\theta\right\}$$

which we can rewrite in the form

$$F_1(\theta,Ka) = \frac{Va}{C}\cos\theta + \sum_1^\infty \overset{*}{p}_{2m}(Ka)\left\{\cos(2m-1)\theta + \frac{2m}{Ka}\cos 2m\theta\right\}$$

where $\overset{*}{p}_{2m}(Ka) = Kap_{2m}(Ka)$. We want to find $\overset{*}{p}_2, \overset{*}{p}_4, \ldots$ for large Ka. In the limit the right-hand side becomes

$$\frac{Va}{C_\infty}\cos\theta + \sum_1^\infty \overset{*}{p}_{2m}(\infty)\cos(2m-1)\theta$$

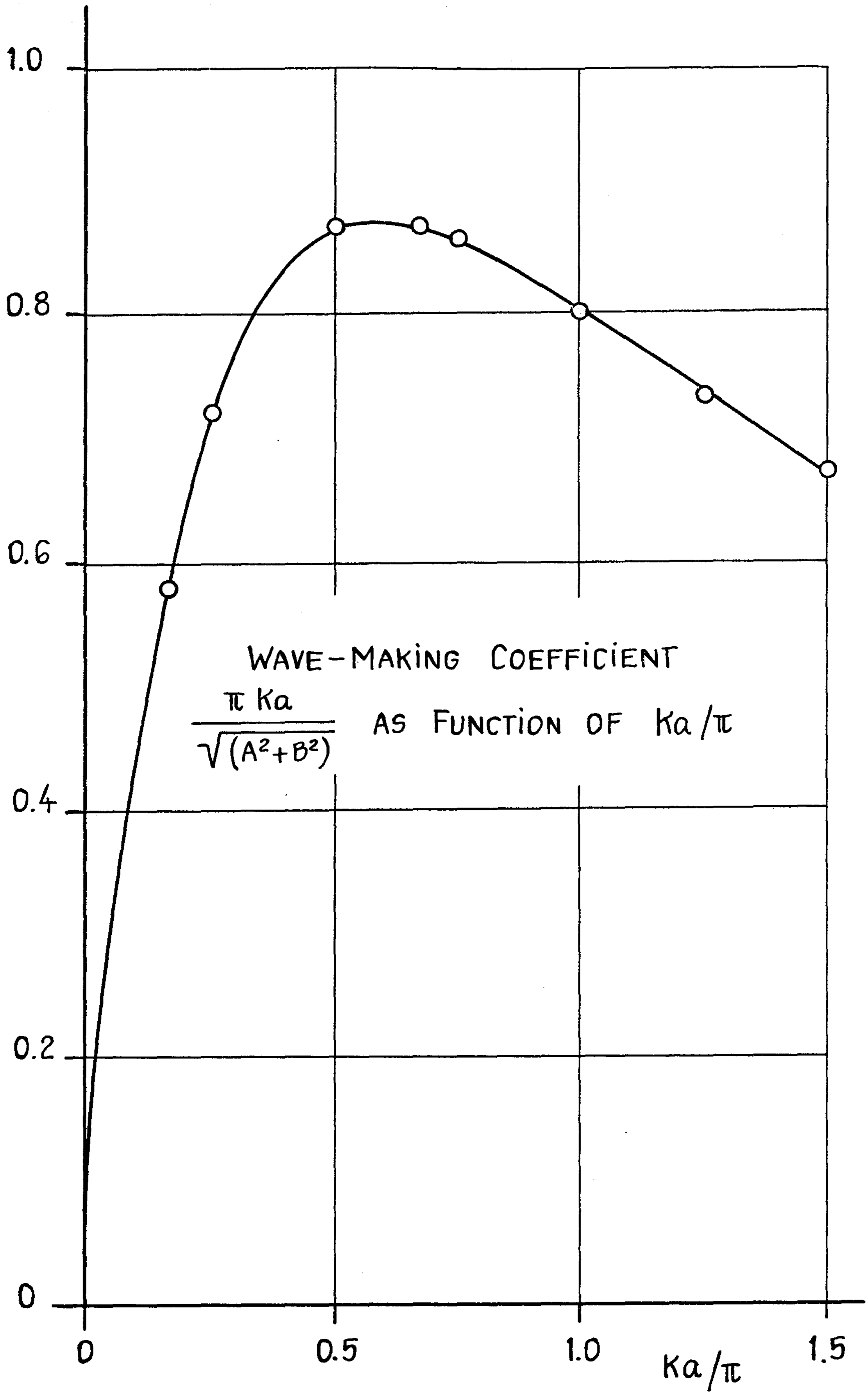


Fig. 3 Wave-making coefficient as function of Ka/π;
o points computed in Ursell 1949a .

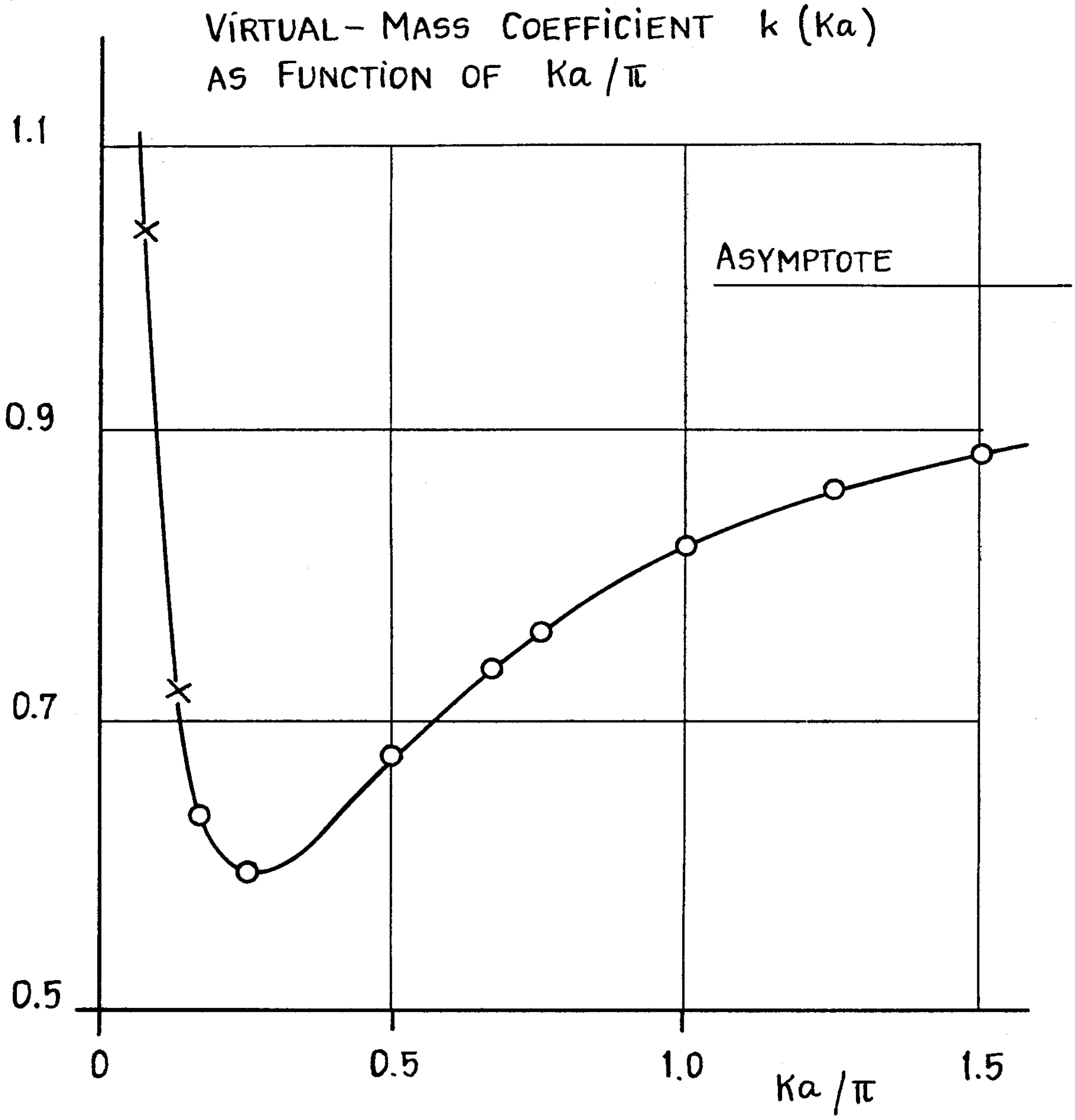


Fig. 5 Virtual-mass coefficient k as function of Ka/π ;
o points computed in Ursell 1949a
x points interpolated from Fig. 7 of Ursell 1959 .

which is again a Fourier series but now involving the complete set over $(0,\frac{1}{2}\pi)$ of functions $\cos\theta$, $\cos 3\theta$, $\cos 5\theta, \ldots$. The coefficient of $\cos\theta$ is

$$\frac{Va}{C_\infty} + \overset{*}{p}_2(\infty),$$

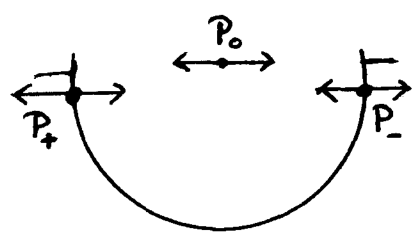
and we obtain only one equation for the two unknowns C_∞ and $\overset{*}{p}_2(m)$. And if we try to use this approximation as the first step in an iteration scheme we get divergence at the next step because of the factor $\frac{2m}{Ka}$ multiplying $\cos 2m\theta$. Thus the method of multipoles (although still theoretically valid) becomes useless for large Ka. The original computations were for the interval $0 < Ka < 5$ and are shown in Figs. 3 and 5 (from U1959).

5. **Short-wave asymptotics. An alternative representation.**

We still need results for larger values of Ka. What method can we use? We have just seen that the method of multipoles fails. F. John's integral equation also fails for large Ka because there are irregular values when Ka is near $(m+\frac{1}{2})\pi$. What is the reason for these and how can they be eliminated? We have seen that the source strength $\mu(\theta,Ka)$ on the semi-circle satisfies the integral equation

$$\pi\mu(\alpha,Ka) + \int_0^{\frac{1}{2}\pi}\mu(\theta,Ka)\left\langle a\frac{\partial}{\partial r}G(\alpha,\theta)+a\frac{\partial}{\partial r}G(\alpha,-\theta)\right\rangle d\theta = V\cos\alpha \text{ when } 0 < \alpha < \tfrac{1}{2}\pi.$$

where for large Ka the kernel is actually small except when θ and α are both near $\frac{1}{2}\pi$. The sources P_- near $\theta = -\frac{1}{2}\pi$ then induce large velocities near $\alpha = +\frac{1}{2}\pi$ by means of waves travelling along the horizontal diameter,



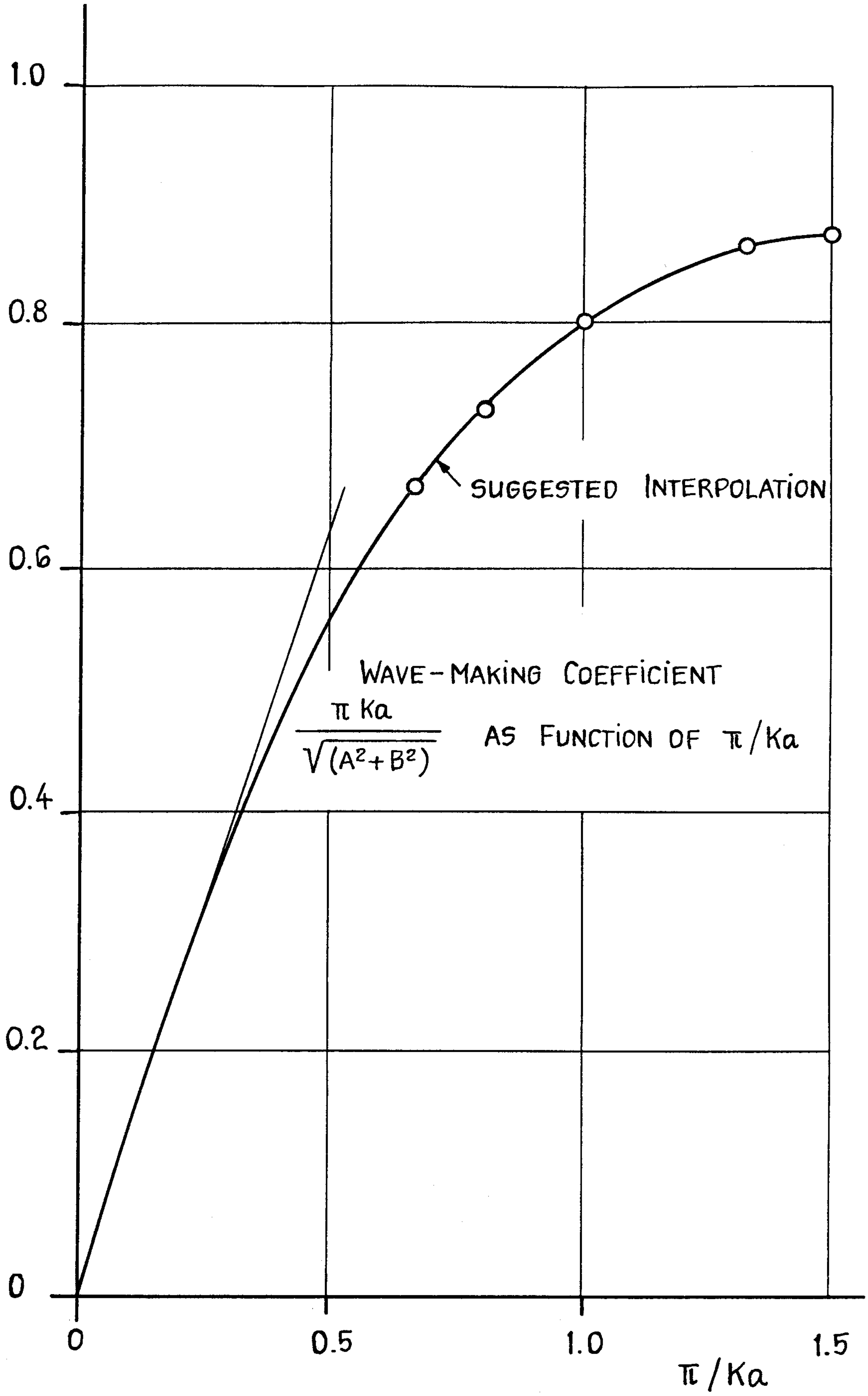


Fig. 4 Wave-making coefficient as function of π/Ka ;
o points computed in Ursell 1949a .

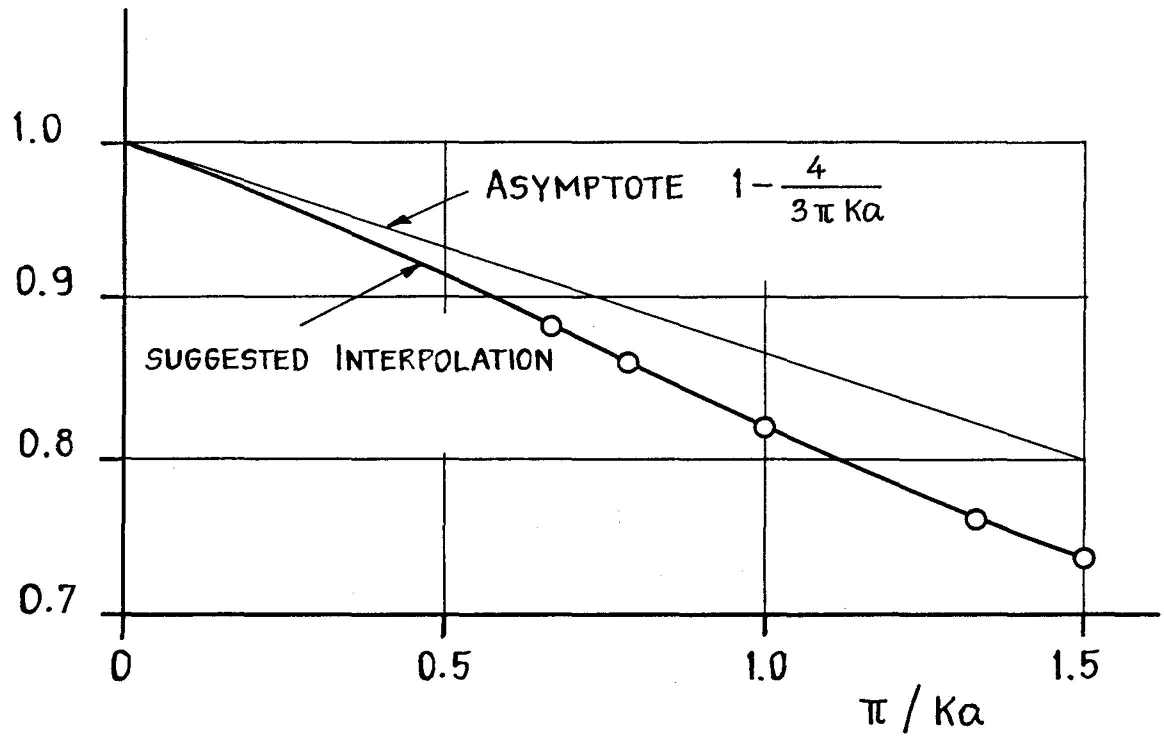


Fig. 6 Virtual-mass coefficient k(Ka) as function of π/Ka ;
o points computed in Ursell 1949a .

and these are responsible for the irregular values. (Similarly the sources near P_+ induce large velocities near P_-.) We can now see how this difficulty can be overcome. Let us try to represent the potential by triads of sources: to each pair P_+, P_- we add another source P_o at the centre, the source strength is chosen so that the waves generated by P_o towards P_+ just cancel out the waves generated by P_- towards P_+ (and, by symmetry, the waves generated by P_+ towards P_-). In other words, we write

$$\phi(x,y,K) = \int_0^{\frac{1}{2}\pi} \mu^*(\theta,K)\Big\{G(x,y,a\sin\theta,a\cos\theta)+G(x,y,-a\sin\theta,a\cos\theta) + A(\theta)G(x,y,0,0)\Big\}a\,d\theta$$

where $A(\theta)$ is known, and where $\mu^*(\quad)$ is the strength of the source triads. It turns out that the integral equation for $\mu^*(\quad)$ becomes comparatively simple for large Ka; in particular it has no irregular values. It can be shown by solving this integral equation that for large Ka the virtual-mass coefficient behaves like $1 - \frac{4}{3\pi Ka}$ and the wave-making coefficient like $\frac{4}{Ka}$. It was thus possible to extend our computations from $Ka = 5$ to $Ka = \infty$, see Figs. 4 & 6 (from U1959).

6. Extensions of the method of multipoles.

I wish now to speak about various extensions of the method of multipoles. So far we have considered symmetric motions but it is not difficult to see that we can similarly construct anti-symmetric wavefree potentials for roll and sway. And we can generalize the method to boundaries other than the semi-circle by conformal transformation. Let us consider boundaries in the z-plane which are transformed into a semi-circle in the ζ-plane by means of a transformation

$$\frac{z}{\ell} = \zeta + \sum_{m=0}^{M} \frac{a_{2m+1}}{\zeta^{2m+1}}, \quad \text{(where } z = x+iy,\ \zeta = \xi+i\eta),$$

containing only a finite number of terms. When $M = 0$ we obtain ellipses;

when $M = 1$, Lewis forms; and so on. The free-surface condition $K\phi + \phi_y = 0$ when $y = 0$ is transformed into $K\phi\left(\frac{dz}{d\zeta}\right)_{\eta=0} + \phi_\eta = 0$ when $\eta = 0$, a different boundary condition, but it is remarkable that we can nevertheless still construct wavefree potentials in the ζ-plane, for ellipses each wavefree potential has 3 terms, for Lewis forms 4 terms, and so on. (For anti-symmetric motions these were given in U1949b.)

Another extension is to finite constant depth where an additional boundary condition

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{when} \quad y = h$$

must be imposed. The infinite-depth potentials can be modified so as to satisfy this, the wave source at the origin becomes

$$\Phi_o(Kr,\theta;Kh) = \int_0^\infty \frac{\cosh k(h-y)\cos kx \, dk}{k \sinh kh - K\cosh kh},$$

and the m^{th} wavefree potential becomes

$$\frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}$$

$$+ \frac{1}{(2m-1)!}\int_0^\infty \frac{e^{-kh}(k+K)(k\cosh ky - K\sinh ky)k^{2m-2}\cos kx \, dk}{k\sinh kh - K\cosh kh}$$

which however has wavelike behaviour at infinity. These potentials can be expressed in polar coordinates and have been used to calculate the force on a half-immersed heaving semi-circle (U1961, 1976). For other boundaries the boundary-value problem cannot be solved in this way but it can be solved by a distribution of wave sources; the wave-source potential for finite depth was given by Thorne (1953, p.714). For the half-immersed circle the method of multipoles and the integral-equation method have been compared by Sayer (1980a,b) who has found that numerically the method of multipoles is far more efficient.

For oblique seas the equation of continuity takes the form $(\Delta - k^2)\phi(x,y) = 0$ while the other boundary conditions are unaltered

(U1973, Green 1971). Wavefree potentials can still be constructed which now involve Bessel functions, and we can derive an expansion theorem analogous to our previous expansion theorem. In particular, we can consider head seas for which the equation of continuity is $(\Delta - K^2)\phi(x,y) = 0$ (U1968a) when it is assumed that head seas can travel along the cylinder without change of form. The boundary condition on the semi-circle is now $\partial\phi/\partial r = 0$, and there is no radiation condition at infinity. The expansion theorem for this case shows that we have one term too few and we therefore cannot satisfy the boundary condition. It follows that head seas cannot travel along the cylinder without change of form, an important physical result, and we can then go on to calculate the change of form.

There are many other problems relating to the half-immersed circular cylinder but I have time for only one more. Suppose that such a cylinder is given a vertical impulse from rest, then in the resulting motion waves radiate away from the cylinder, and it will be expected that the motion of the body is very nearly a damped harmonic motion. This is indeed found to be the case except in the initial and final stages (U1964, 1970). The vertical displacement of the cylinder at time $\tau(a/g)^{\frac{1}{2}}$ is described by the integral

$$h_1(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iu\tau}du}{1 - \frac{1}{4}\pi u^2(1 + \Lambda(u))}$$

where $\Lambda(u)$ is the force coefficient at frequency $u(g/a)^{\frac{1}{2}}$ (in suitable non-dimensional form). We see that $\Lambda(u)$ is needed at all frequencies, thus we must solve the infinite system of equations (S) for every frequency. The integrand decreases very slowly so that direct computation is not feasible but convergence can be accelerated by using the asymptotics of virtual mass and damping at high frequencies, and also by using complex rather than real frequencies in the integration.

7. Conclusion.

What use has been made of all this mathematical theory? The two-dimensional methods described here have found their widest use in the calculation of forces on oscillating ships, where they have been incorporated into a strip theory of three-dimensional ship motions. These were validated by Gerritsma and his collaborators at Delft who found good agreement between their measurements and the strip-theory calculations. This good agreement has never been explained by the theoreticians, and it remains a challenging problem for us. I think that Georg Weinblum would have been pleased if he could have known the contribution that mathematics has made to this problem, and also the highly original contribution that the experimenters have made by adapating this mathematics.

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