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# Stochastic averaging of roll-pitch and roll-heave motion in random seas

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**Abstract**

Multi-degree-of-freedom ship motion and ship stability in random seas are of major interest for the development of new advanced intact stability criteria. The purpose of this research is to improve the safety of new ship designs, but the results are relevant also for other engineering systems involving multiple scales. We focus on roll-pitch and roll-heave motion in random seas. The random wave excitation is modeled by a non-white stationary process. This process is derived from a spectral description of the random seaway using traveling effective wave.


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*Keywords:* Fluid-structure-interaction; random seas; stochastic averaging

## 1. Introduction

It is of major interest for the design process of ships to improve the calculation procedures for fluid-ship interaction in realistic sea states. Modeling of sea states by a random fluid field defined by measured spectral properties is more realistic then using deterministic fluid fields modeled by Airy or Stokes waves. However, numerical simulations are time consuming, if a random fluid field model is used, since simulations have to be repeated many times. In order to make the analysis of ship dynamics in random seas tractable in the early design process, a reduction of the multi-degree-of freedom state space is needed. This is possible by means of the stochastic averaging Method. Up to now, asymptotic methods were used for determining the averaged equations of roll motion subjected to real noise excitation [1,2,3,4]. Exact averaging of strongly nonlinear one-degree-of-freedom oscillator was done for the case of white noise excitation [5,6,7] and for the white noise excited Duffing

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oscillator with a two well potential [8]. Here, we use stochastic averaging as presented in [9,10] and include additional excitation due to heave or pitch motion of the ship. The purpose of this research is to improve the safety of new ship designs, but the results are relevant also for other engineering systems involving multiple scales.

We focus on coupled roll-pitch and roll-heave motion in random seas. The random wave excitation is modeled by a non-white stationary process. In Section 2 this process is derived from a spectral description of the random seaway using a traveling effective wave. Section 3 is devoted to roll-pitch and roll-heave equations of motion which are transformed later into a dynamical system with multiple time scales. The state space dimension of this system is reduced by means of stochastic averaging in Section 4, followed by possible application of the reduced equations in Section 5. Before we conclude our results, we calculate probabilistic measures for a real RoRo ship design.

## 2. Random Seas Waves

A well-known model for an irregular long crested wave surface is the superposition of infinitely many harmonic waves with wave numbers $k(\omega)$ and frequencies $\omega$, which correspond to a one-sided sea state spectral density $S(\omega)$. Such an irregular wave surface is too complex for further analysis, because there are infinitely many possibilities for the wave pressure field acting on the ship hull. Since high frequent pressure variations will have only small effect on the total fluid forces acting on the ship hull, we consider only an averaged incident wave with the same length as the ship length consisting of two harmonic components. Therefore, we approximate the irregular long crested wave surface by the following effective wave

$$Z_{eff}(x,t) = \eta_s(t)\sin\left(\frac{2\pi}{L}x\right) + \eta_c(t)\cos\left(\frac{2\pi}{L}x\right) = \eta(t)\cos\left(\frac{2\pi}{L}x + \psi(t)\right). \tag{1}$$

The Gaussian random processes $\eta_s(t)$ and $\eta_c(t)$ are determined by minimizing the error between the irregular wave surface and the effective wave. Then spectral densities of the above processes are

$$S_{\eta_s}(\omega) = 2 f_s(k(\omega))^2 S(\omega), \tag{2}$$

$$S_{\eta_c}(\omega) = 2 f_c(k(\omega))^2 S(\omega), \tag{3}$$

where the transfer functions $f_s$ and $f_c$ are given by

$$f_s(k(\omega)) = \frac{2\pi \sin\left(\frac{L}{2}k(\omega)\right)}{\pi^2 - \left(\frac{L}{2}k(\omega)\right)^2}, \qquad f_c(k(\omega)) = \frac{Lk(\omega)\sin\left(\frac{L}{2}k(\omega)\right)}{\pi^2 - \left(\frac{L}{2}k(\omega)\right)^2}. \tag{4}$$

## 3. Model of Roll-pitch and Roll-heave coupling

Modeling of coupled 6-degrees of freedom ship motion can be done by many ways, depending on the necessary level of accuracy. Since our results have to be useable in the early design stage of a ship, we will include only leading nonlinearities and mode coupling. It follows from linear ship motion analysis, that for symmetric floating bodies with respect to the $x-z$ plane, the surge, heave, and pitch motions are decoupled from the sway, roll, and yaw motions. On the other hand, we include the coupling resulting from the influence of ship orientation in waves on the righting lever $GZ$ . This leads to multiplicative coupling in the righting lever curve approximation. From an order of magnitude analysis of heave-pitch-roll motions, it is common to neglect the influence of roll on heave and pitch. Then, the heave and pitch motion statistics can be computed beforehand and the roll motion is then obtained by including these forcing terms in the one-degree of freedom equation of motion for roll. We use a linear strip method to obtain the response amplitude operator (RAO) for heave and pitch motion. The spectral densities $S_{\xi_z\xi_z}$ and $S_{\xi_\theta\xi_\theta}$ for the heave and pitch motion with corresponding response amplitude operators $RAO_z$ and $RAO_\theta$ are given by

$$S_{\xi_z\xi_z}(\omega) = \left|RAO_z\right|^2 S(\omega), \qquad S_{\xi_\theta\xi_\theta}(\omega) = \left|RAO_\theta\right|^2 S(\omega). \tag{5}$$

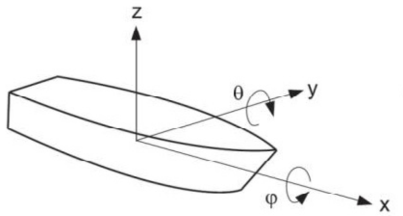


Figure 1. Definition of relevant ship motions

*3.1. Roll-pitch and Roll-heave coupling*

Heave and pitch motions change the position and orientation of the ship hull in incident waves. Therefore, these motions will have an influence on the ships restoring capability. The forces and moments on the hull are determined by integrating the fluid pressure over the wetted surface of the ship hull. This results in a moment $M_G$ with respect to the center of gravity $G$ of the ship, which is determined by $M_G = GZ \cdot \Delta \cdot g$ . Here, $g$ is acceleration due to gravity and $\Delta$ is the displacement of the ship. For each ship, the righting lever has to be computed numerically, taking the specific geometry of the ship hull into account. These numerical data are then fitted by analytical functions. For this, we consider the following approximation $GZ_{app}$ for the righting lever curve of coupled roll-pitch or roll-heave, where we denote the heave process by $\xi_1 := \xi_z$ and the pitch process by $\xi_2 := \xi_\theta$ . Then

$$GZ_{app}(\phi, \xi_i, \psi, \eta) = q_1\phi - q_2\phi^3 + q_3\eta\cos(\psi)\phi + q_{\phi i}\xi_i\phi, \quad i = 1, 2, \tag{6}$$

where all coefficients are positive. The coefficients $q_1, q_2$, and $q_3$ are obtained by least squares fitting of the righting lever curve data $GZ$ , cf. [11]. We have included a cubic nonlinearity in $\phi$ to account for the softening spring characteristic of roll restoring force. Moreover, the excitation process $\xi_\phi := \xi_{\eta_c} = \eta\cos(\psi)$ is given by the amplitude process $\eta_c$ defined by the spectral density from equation (3). With the above approximation of righting lever $GZ$, the roll dynamics of a ship in head or following long crested waves can be represented by

$$(I_{xx} + A_{xx})\ddot{\phi} + b_1\dot{\phi} + b_3\dot{\phi}^3 + \Delta g GZ_{app}(\phi, \xi_\phi, \xi_i) = 0, \qquad i = 1, 2. \tag{7}$$

Here, $I_{xx}$ is the roll moment of inertia, $A_{xx}$ is the hydrodynamic added mass evaluated at the natural roll eigenfrequency, $b_1$, and $b_3$ are linear and cubic damping coefficients.

### 3.2. Derivation of Multiple Scales Model

Using standard rescalings, equation (7) can be written as the following system of first order differential equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega_\phi^2 x + \alpha_3 x^3 - \varepsilon(2\beta_1 y + 2\beta_3 y^3) + \sqrt{\varepsilon}(\alpha_4 x \xi_\phi + k_{\phi i} x \xi_i), \qquad i = 1, 2. \end{aligned} \tag{8}$$

For the heave-roll and pitch-roll equations of motion we can assume, that the parameter ε in system (8) is much smaller than one, since hydrodynamic damping and excitation due to sea are small compared to the restoring forces. Then equations (8) form a weakly perturbed Hamiltonian system, with Hamiltonian

$$H(x, y) = \frac{y^2}{2} + \frac{\omega_\phi^2 x^2}{2} - \frac{\alpha_3 x^4}{4}. \tag{9}$$

From the Hamiltonian we obtain

$$Q(x, H) := y^2 = 2H - \omega_\phi^2 x^2 + \frac{1}{2}\alpha_3 x^4. \tag{10}$$

It is clear, that the energy of system (8) changes due to the perturbations and damping. Therefore, the total derivative of the Hamiltonian (9) does not vanish and we have

$$\frac{dH}{dt} = -\varepsilon(2\beta_1 Q(x,H) + 2\beta_3 Q(x,H)^2) + \sqrt{\varepsilon}(\alpha_4 x \xi_\phi + k_{\phi i} x \xi_i)\sqrt{Q(x,H)}. \tag{11}$$

With this we state a system of differential equations for the energy H and the variable x

$$\begin{aligned} \dot{x} &= \sqrt{Q(x,H)}, \\ \dot{H} &= -\varepsilon(2\beta_1 Q(x,H) + 2\beta_3 Q(x,H)^2) + \sqrt{\varepsilon}(\alpha_4 x \xi_\phi + k_{\phi i} x \xi_i)\sqrt{Q(x,H)}. \end{aligned} \tag{12}$$

The state variable $x$ and $H$ in equation (12) exhibit different time scales, since the energy $H$ changes slowly with time compared to the change of the variable $x$. For fixed H a closed form solution of $x$ can be obtained in terms of Jacobian elliptic functions by

$$x(t) = b \; sn(qt, k), \tag{13}$$

where

$$b=\sqrt{-\frac{-\omega_\phi^2+\sqrt{\omega_\phi^4-4\alpha_3 H}}{\alpha_3}}, \tag{14}$$

and

$$a=\sqrt{\frac{4H}{b^2\alpha_3}}, \quad q=a\sqrt{\frac{\alpha_3}{2}}, \quad k=\frac{b}{a}.$$

With the equality $\dot{x}=\sqrt{Q(x,H)}$ we obtain from equation (13)

$$\sqrt{Q(x,H)}=bq\,cn(qt,k)\;dn\left(qt,k\right). \tag{15}$$

For a fixed energy $H$ in the unperturbed system with $\varepsilon=0$, the period of one oscillation of the fast variable $x$ is thus

$$T(H)=2dx\int_{-b}^{b}\frac{dx}{\sqrt{Q(x,H)}}. \tag{16}$$

## 4. Stochastic Averaging

A rigorous proof of a limit theorem concerning stochastic differential equations with multiple time scales was obtained in 1966 by Khashminskii [12]. This theorem was later extended by Papanicolao and Kohler [13]. We use the limit theorem from [13] to average the change of energy $H$ in system (12) during one oscillation period $T$ of the fast variable $x$. For this purpose, we define the averaging operator

$$\mathbb{M}\left\{f(t)\right\}=\frac{1}{T}\int_0^T f(t)dt. \tag{17}$$

Application of the limit theorem from [13] yields the following Itô stochastic differential equation

$$dH=m(H)dt+\sigma(H)dW, \tag{18}$$

which determines the energy variable $H$ of system (12). The equations for drift $m\left(H\right)$ and diffusion $\sigma(H)$ become

$$
\begin{aligned}
m(H) = {} & \mathbb{M}\{Q(x(t),H)(-2\beta_1 - 2\beta_3 Q(x(t),H))\} + \\
& + \alpha_4 k_{\phi i} b^2 \int_{-\infty}^{0} (R_{\xi_\phi \xi_i}(\tau) - R_{\xi_\phi \xi_i}(-\tau)) \mathbb{M}\left\{ sn_t sn_{t+\tau} \frac{cn_{t+\tau} dn_{t+\tau}}{cn_t dn_t} \right\} d\tau + \\
& + \alpha_4^2 b^2 \int_{-\infty}^{0} R_{\xi_\phi \xi_\phi}(\tau) \mathbb{M}\left\{ sn_t sn_{t+\tau} \frac{cn_{t+\tau} dn_{t+\tau}}{cn_t dn_t} \right\} d\tau + \\
& + k_{\phi i}^2 b^2 \int_{-\infty}^{0} R_{\xi_i \xi_i}(\tau) \mathbb{M}\left\{ sn_t sn_{t+\tau} \frac{cn_{t+\tau} dn_{t+\tau}}{cn_t dn_t} \right\} d\tau,
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\sigma^2(H) = {} & \alpha_4^2 b^4 q^2 \int_{-\infty}^{\infty} R_{\xi_\phi \xi_\phi}(\tau) \mathbb{M}\{ sn_t sn_{t+\tau} cn_t dn_t cn_{t+\tau} dn_{t+\tau} \} d\tau + \\
& + k_{\phi i}^2 b^4 q^2 \int_{-\infty}^{\infty} R_{\xi_i \xi_i}(\tau) \mathbb{M}\{ sn_t sn_{t+\tau} cn_t dn_t cn_{t+\tau} dn_{t+\tau} \} d\tau + \\
& + \alpha_4 k_{\phi i} b^4 q^2 \int_{-\infty}^{\infty} (R_{\xi_\phi \xi_i}(\tau) - R_{\xi_\phi \xi_i}(-\tau)) \mathbb{M}\{ sn_t sn_{t+\tau} cn_t dn_t cn_{t+\tau} dn_{t+\tau} \} d\tau.
\end{aligned} \tag{20}
$$

Here $R_{\xi_\phi \xi_\phi}$ and $R_{\xi_i \xi_i}$ are the autocorrelation functions of the processes $\xi_i$ and $\xi_\phi = \xi_{\eta_c}$. The cross correlation is denoted as $R_{\xi_\phi \xi_i}$. Equations (19) and (20) contain the Jacobian elliptic functions

$$sn_t := sn(qt,k),\ cn_t := cn(qt,k),\ dn_t := dn(qt,k),$$

and Jacobian elliptic functions with time shift

$$sn_{t+\tau} := sn(q(t+\tau),k),\ cn_{t+\tau} := cn(qt,k),\ dn_{t+\tau} = dn(q(t+\tau),k).$$

## 5. Stationary Density and Mean Exit Time

With the results from Section 4 for drift and diffusion of the averaged process for the energy $H$, we can calculate further relevant measures like probability density functions (pdf) or mean times until a specific energy level is reached. For this, we introduce the speed and scale measures, which transform the diffusion process (18) on its natural scale, where the drift is identically zero. This task is achieved via the transformation $\mathfrak{S}(x)$ given by

$$\mathfrak{S}(x) = \int^x s(x)dx, \quad s(y) = \exp\left( -2\int^y \frac{m(r)}{\sigma^2(r)} dr \right), \tag{21}$$

where $s(y)$ is called the scale density. Then, the speed density is defined by

$$\mu(y) = \frac{1}{\sigma^2(y)s(y)}. \tag{22}$$

The stationary pdf of energy $H$ can be stated in terms of

$$p_{st}(H) = \mu(H)\big(c_1 \mathfrak{S}(H) + c_2\big). \tag{23}$$

The coefficients $c_1$ and $c_2$ are determined by boundary and normality conditions. If we assume a reflecting boundary at $H = \omega_\phi^4 / (4\alpha_3)$ (i.e. no capsizing), which is approximately the case up to moderate noise intensities, then $c_1 = 0$. In this case a stationary solution exists and is given by

$$p_{st}(H) = \frac{c_2}{\sigma^2(H)} \exp\left( 2\int^H \frac{m(x)}{\sigma^2(x)} dx \right). \tag{24}$$

The mean first passage time $\mathbb{T}_e$ for a process on $x \in [x_e, x_c]$ with entrance boundary $x_e$ and exit boundary $x_c$ can be calculated by the following formula (cf. [14])

$$\mathbb{T}_e(x_0) = 2\int_{x_0}^{x_c} \left[ \int_z^{x_c} s(y)dy \right] \mu(z)dz, \tag{25}$$

where $x_0 \in [x_e, x_c]$ is the starting point.

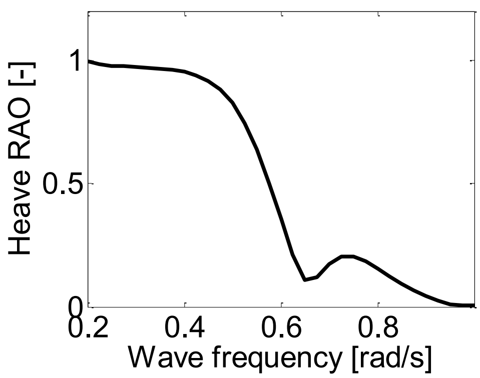


Figure 2. Heave RAO, U=15 kn

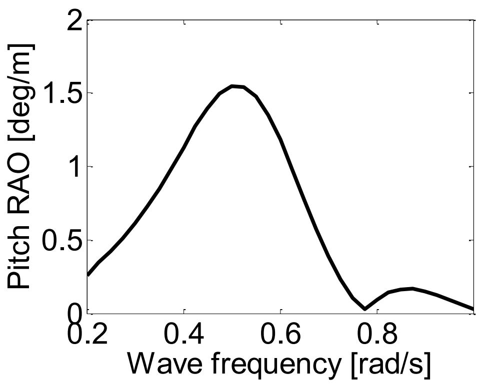


Figure 3. Pitch RAO, U=15 kn

**6. Results for a RoRo ship in random seas**

We now consider the case of a RoRo ship travelling in long crested following waves. The data of this ship can be obtained from [10,11,15]. At ship speed $U = 15kn$ in a sea state $S(\omega)$ defined by a Piersson-Moskowitz spectrum with modal frequency $\omega_m = 0.64 rad/s$ and significant wave height $H_s = 14m$, the RoRo ship is in the region of 2:1 parametric resonance, which leads to large amplitude roll motion. The heave and pitch RAO's for this scenario are plotted in Figures 2 and 3. For the case of roll-pitch coupling we calculate the drift $m(H)$ and diffusion $\sigma(H)$, which are respectively given in equation (19) and (20), by numerical integration. The numerical data are interpolated by cubic splines and shown in Figures 4 and 5.

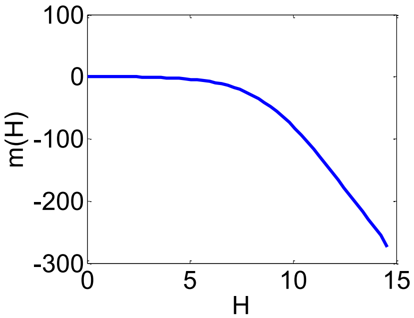


Figure 4. Drift *m(H)* for roll-pitch motion

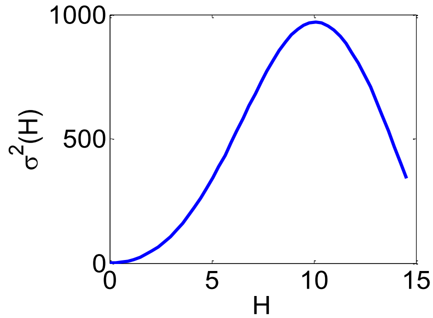


Figure 5. Diffusion $\sigma^2(H)$ for roll-pitch motion

We further can calculate an approximate probability density by means of equation (24). After the transformation from roll energy $H$ to maximal roll angle per roll period $b$, which is given in equation (14), we get the corresponding pdf as shown in Figure 6. Further calculations of mean first passage times by means of formula (25) are shown in Figure 7, where the scaled time for reaching the critical energy level for capsizing is computed, starting at the initial roll angle $\Phi_0$. Note, that the scaled time $t_\varepsilon = \varepsilon t$ is plotted, where we have used $\varepsilon = 0.1$ .

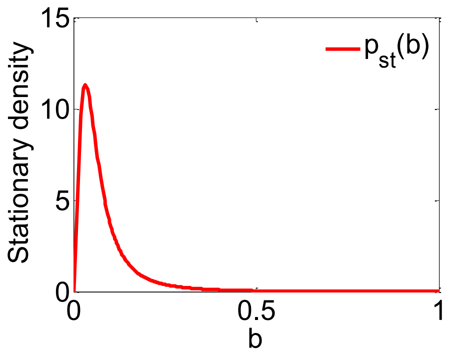


Figure 6. Pdf of maximal angle *b* for roll-pitch motion

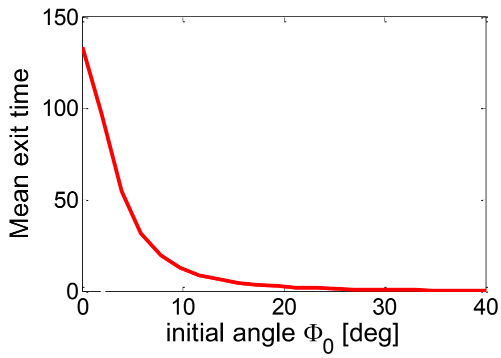


Figure 7. Mean time for capsizing for roll-pitch motion

## 7. Summary and Conclusions

In this paper we have obtained analytical results for the behavior of the roll-pitch and roll-heave motion of ships in random seas. It was shown, that the developed theory is applicable for the analysis of large real ships. An approximation for the probability density in a finite time interval was calculated for a RoRo ship, assuming reflecting boundary conditions for the corresponding Fokker-Planck equation. The approximate probability density can be used in the design process of a ship to optimize its stability. Because the roll dynamics are modeled by a softening spring type Duffing oscillator with additional cubic damping, which is excited by non-white stochastic processes, the presented results are also applicable to various engineering problems containing multiple scales.

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