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The inhomogeneous Cauchy-Riemann equation for weighted smooth vector-valued functions on strips with holes

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Abstract
This paper is dedicated to the question of surjectivity of the Cauchy-Riemann operator $\overline{\partial}$ on spaces $\mathcal{EV}(\Omega, E)$ of $\mathcal{C}^{\infty}$-smooth vector-valued functions whose growth on strips along the real axis with holes K is induced by a family of continuous weights $\mathcal{V}$. Vector-valued means that these functions have values in a locally convex Hausdorff space E over $\mathbb{C}$. We derive a counterpart of the Grothendieck-Köthe-Silva duality $\mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C}) \cong \mathscr{A}(K)$ with non-empty compact $K \subset \mathbb{R}$ for weighted holomorphic functions. We use this duality and splitting theory to prove the surjectivity of $\overline{\partial} : \mathcal{EV}(\Omega, E) \to \mathcal{EV}(\Omega, E)$ for certain E. This solves the smooth (holomorphic, distributional) parameter dependence problem for the Cauchy-Riemann operator on $\mathcal{EV}(\Omega, \mathbb{C})$.

Keywords Cauchy-Riemann · Parameter dependence · Weight · Smooth · Solvability · Vector-valued

Mathematics subject classification 35A01 · 35B30 · 32W05 · 46A63 · 46A32 · 46E40

1 Introduction

The smooth (holomorphic, distributional) parameter dependence problem for the Cauchy-Riemann operator $\overline{\partial} := (1/2)(\partial_1 + i\partial_2)$ on the space $\mathcal{C}^{\infty}(\Omega)$ of smooth complex-valued functions on an open set $\Omega \subset \mathbb{R}^2$ is whether for every family $(f_\lambda)_{\lambda \in U}$ in $\mathcal{C}^{\infty}(\Omega)$ depending smoothly (holomorphically, distributionally) on a parameter $\mathrm{Ext}^1(E, F)$ in an open set $U \subset \mathbb{R}^d$ there is a family $(u_\lambda)_{\lambda \in U}$ in $\mathcal{C}^{\infty}(\Omega)$ with the same kind of parameter dependence such that

$$\overline{\partial} u_\lambda = f_\lambda, \quad \lambda \in U.$$

Here, smooth (holomorphic, distributional) parameter dependence of $(f_\lambda)_{\lambda \in U}$ means that the map $\lambda \mapsto f_\lambda(x)$ is an element of $\mathcal{C}^{\infty}(U)$ (of the space of holomorphic functions $\mathcal{O}(U)$ on $U \subset \mathbb{C}$ open, the space of distributions $\mathcal{D}(V)'$ for open $V \subset \mathbb{R}^d$ where $U = \mathcal{D}(V)$) for each $x \in \Omega$.

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The parameter dependence problem for a variety of partial differential operators on several spaces of (generalised) differentiable functions has been extensively studied, see e.g. [4, 6, 7, 16, 31, 32] and the references and background in [3, 22]. The answer to this problem for the Cauchy-Riemann operator is affirmative since the Cauchy-Riemann operator

$$\overline{\partial}^E : \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\Omega, E) \tag{1}$$

on the space $\mathcal{C}^\infty(\Omega, E)$ of E-valued smooth functions is surjective if $E = \mathcal{C}^\infty(U)$ ($\mathcal{O}(U)$, $\mathcal{D}(V)'$) by [8, Corollary 3.9, p. 1112] which is a consequence of the splitting theory of Bonet and Domański for PLS-spaces [3, 4], the topological isomorphy of $\mathcal{C}^\infty(\Omega, E)$ to Schwartz' ε-product $\mathcal{C}^\infty(\Omega)\varepsilon E$ and the fact that $\overline{\partial} : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$ is surjective on the nuclear Fréchet space $\mathcal{C}^\infty(\Omega)$ (with its usual topology). More generally, the Cauchy-Riemann operator (1) is surjective if E is a Fréchet space by Grothendieck's classical theory of tensor products [12] or if $E := F_b'$ where F is a Fréchet space satisfying the condition (DN) by [31, Theorem 2.6, p. 174] or if E is an ultrabornological PLS-space having the property (PA) by [8, Corollary 3.9, p. 1112] since the space $\mathcal{O}(\Omega)$ of $\mathbb{C}$-valued holomorphic functions on Ω, i.e. the kernel of $\overline{\partial}$, has the property (Ω) by [31, Proposition 2.5 (b), p. 173]. The first and the last result cover the case that $E = \mathcal{C}^\infty(U)$ or $\mathcal{O}(U)$ whereas the last covers the case $E = \mathcal{D}(V)'$ as well. More examples of the second or third kind of such spaces E are arbitrary Fréchet–Schwartz spaces, the space $\mathcal{S}(\mathbb{R}^d)'$ of tempered distributions, the space $\mathcal{D}(V)'$ of distributions, the space $\mathcal{D}_{(w)}(V)'$ of ultradistributions of Beurling type and some more (see [4, 8, Corollary 4.8, p. 1116] and [22, Example 3, p. 7]).

In the present paper we consider the Cauchy-Riemann operator on weighted spaces $\mathcal{EV}(\Omega, E)$ of smooth E-valued functions where E is a locally convex Hausdorff space over $\mathbb{C}$ with a system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$ generating its topology. These spaces consist of functions $f \in \mathcal{C}^\infty(\Omega, E)$ fulfilling additional growth conditions induced by a family $\mathcal{V} := (v_n)_{n\in\mathbb{N}}$ of continuous functions $v_n : \Omega \to (0, \infty)$ on a sequence of open sets $(\Omega_n)_{n\in\mathbb{N}}$ with $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ given by the constraint

$$|f|_{n,m,\alpha} := \sup_{\substack{x\in\Omega_n \\ \gamma\in\mathbb{N}_0^2, |\gamma|\leq m}} p_\alpha\big((\partial^\gamma)^E f(x)\big) v_n(x) < \infty$$

for every $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$, where $(\partial^\gamma)^E f$ denotes the partial derivative of f w.r.t. the multi-index γ. Our main goal is to derive sufficient conditions on $\mathcal{V}$ and $(\Omega_n)_{n\in\mathbb{N}}$ such that

$$\overline{\partial}^E : \mathcal{EV}(\Omega, E) \to \mathcal{EV}(\Omega, E)$$

is surjective. We recall the main result of [22], which sets the course of the present paper.

Theorem 1 [22, **Theorem 5, p. 7-8**] *Let $\mathcal{EV}(\Omega)$ be a Schwartz space and $\mathcal{EV}_{\overline{\partial}}(\Omega)$ a nuclear subspace satisfying property (Ω). Assume that the $\mathbb{C}$-valued operator $\overline{\partial} : \mathcal{EV}(\Omega) \to \mathcal{EV}(\Omega)$ is surjective. Moreover, if*

(a) *$E := F_b'$ where F is a Fréchet space over $\mathbb{C}$ satisfying (DN), or*
(b) *E is an ultrabornological PLS-space over $\mathbb{C}$ satisfying (PA),*

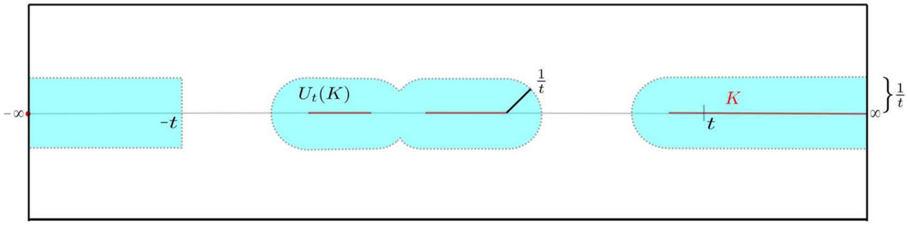

Fig. 1 $U_t(K)$ for $\pm\infty \in K$ (c.f. [19, Figure 3.1, p. 11])

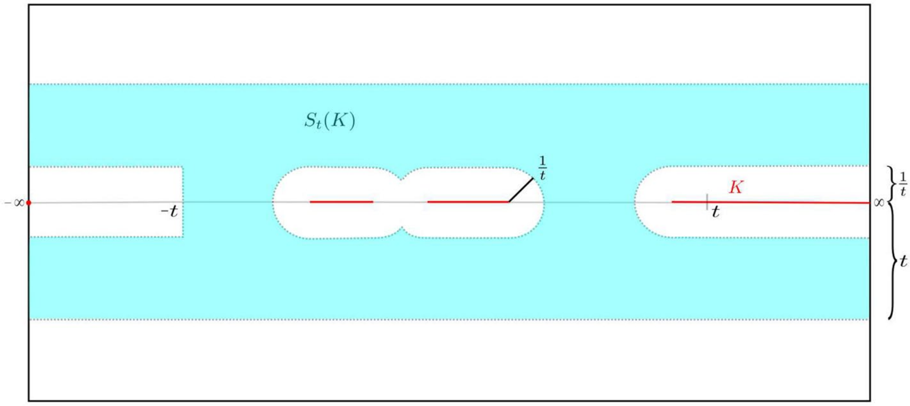

Fig. 2 $S_t(K)$ for $\pm\infty \in K$ (c.f. [19, Figure 3.2, p. 12])

then

$$\overline{\partial}^E : \mathcal{EV}(\Omega, E) \to \mathcal{EV}(\Omega, E)$$

is surjective.

Here $\mathcal{EV}(\Omega) := \mathcal{EV}(\Omega, \mathbb{C})$ and $\mathcal{EV}_{\overline{\partial}}(\Omega)$ is the kernel of $\overline{\partial}$ in $\mathcal{EV}(\Omega)$, i.e. its topological subspace

$$\mathcal{EV}_{\overline{\partial}}(\Omega) := \mathcal{O}(\Omega) \cap \mathcal{EV}(\Omega)$$

consisting of holomorphic functions.

We restrict to the case where the sequence $(\Omega_n)_{n\in\mathbb{N}}$ is given by strips $\Omega_n := S_n(K)$ along the real axis with holes around a compact set $K \subset [-\infty, \infty] =: \overline{\mathbb{R}}$, i.e. for $t \in \mathbb{R}$, $t \geq 1$, we define

$$S_t(K) := \left(\mathbb{C} \setminus \overline{U_t(K)}\right) \cap \{z \in \mathbb{C} \mid |\mathrm{Im}\,(z)| < t\}, \quad t > 1, \quad \text{and} \quad S_1(K) := S_{3/2}(K),$$

where the closure is taken in $\mathbb{C}$ and the open sets $U_t(K)$ are given

$$U_t(K) := \{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, K \cap \mathbb{C}) < 1/t\} \cup \begin{cases} \emptyset & , K \subset \mathbb{R}, \\ (t, \infty) + i(-1/t, 1/t) & , \infty \in K, \ -\infty \notin K, \\ (-\infty, -t) + i(-1/t, 1/t) & , \infty \notin K, \ -\infty \in K, \\ ((-\infty, -t) \cup (t, \infty)) + i(-1/t, 1/t) & , \pm\infty \in K, \end{cases}$$

where $\mathrm{d}(\{z\}, K \cap \mathbb{C})$ denotes the Euclidean distance of the sets $\{z\}$ and $K \cap \mathbb{C}$ (see Figs. 1, 2). We note that $\bigcup_{n\in\mathbb{N}} S_n(K) = \mathbb{C} \setminus K$ and the definition of $S_1(K)$ is motivated by $\left(\mathbb{C} \setminus \overline{U_1(\overline{\mathbb{R}})}\right) \cap \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < 1\} = \emptyset$. As a further simplification we only consider weights of the form $v_n(z) := \exp(a_n |\operatorname{Re}(z)|^\beta)$, $z \in \mathbb{C}$, for all $n \in \mathbb{N}$ for some $0 < \beta \leq 1$ and some strictly increasing real sequence $(a_n)_{n\in\mathbb{N}}$, and in combination with $\Omega_n := S_n(K)$, $n \in \mathbb{N}$, we fix the notation

$$\mathcal{E}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E) := \mathcal{EV}(\mathbb{C} \setminus K, E) \quad \text{and} \quad \mathcal{E}^{\beta}_{(a_n),\overline{\partial}}(\overline{\mathbb{C}} \setminus K) := \mathcal{EV}_{\overline{\partial}}(\mathbb{C} \setminus K)$$

with $\overline{\mathbb{C}} := \overline{\mathbb{R}} + i\mathbb{R}$. In the case $\beta = 1$ these spaces are of interest because they are the basic spaces for the theory of vector-valued Fourier hyperfunctions, see e.g. [13–15, 17, 19, 24]. Looking at Theorem 1, the main obstacle is to prove that $\mathcal{E}^{\beta}_{(a_n),\overline{\partial}}(\overline{\mathbb{C}} \setminus K)$ satisfies property (Ω). In [22, Corollary 14, p. 18] this is accomplished for $K = \emptyset$ and sequences $(a_n)_{n\in\mathbb{N}}$ such that $a_n \nearrow 0$ or $a_n \nearrow \infty$. We will use this result and extend it to the case that $K \subset \mathbb{R}$ is a non-empty compact set.

Let us summarise the content of our paper. In Sect. 2 we recall necessary definitions and preliminaries which are needed in the subsequent sections. Sect. 3 is dedicated to a counterpart for weighted holomorphic functions of the Silva-Köthe-Grothendieck duality

$$\mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C}) \cong \mathscr{A}(K)'_b$$

where $K \subset \mathbb{R}$ is a non-empty compact set and $\mathscr{A}(K)$ the space of germs of real analytic functions on K (see Theorem 11, Corollary 13, Corollary 15). In Sect. 4 we use this duality to show that $\mathcal{E}^{\beta}_{(a_n),\overline{\partial}}(\overline{\mathbb{C}} \setminus K)$ satisfies property (Ω) under some restrictions on K, or on $(a_n)_{n\in\mathbb{N}}$ and β (see Corollary 19). The preceding conditions on K, or on $(a_n)_{n\in\mathbb{N}}$ and β are used in Theorem 20 to show that $\overline{\partial}^E : \mathcal{E}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E) \to \mathcal{E}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)$ is surjective if $E := F'_b$ where F is a Fréchet space over $\mathbb{C}$ satisfying (DN), or if E is an ultrabornological PLS-space over $\mathbb{C}$ satisfying (PA).

2 Notation and preliminaries

The notation and preliminaries are essentially the same as in [22, 23, Sect. 2]. We define the distance of two subsets $M_0, M_1 \subset \mathbb{R}^2$ w.r.t. the Euclidean norm $|\cdot|$ on $\mathbb{R}^2$ via

$$\mathrm{d}(M_0, M_1) := \begin{cases} \inf_{x\in M_0, y\in M_1} |x - y| & , M_0, M_1 \neq \emptyset, \\ \infty & , M_0 = \emptyset \text{ or } M_1 = \emptyset. \end{cases}$$

Moreover, we denote by $\mathbb{B}_r(x) := \{w \in \mathbb{R}^2 \mid |w - x| < r\}$ the Euclidean ball around $x \in \mathbb{R}^2$ with radius $r > 0$ and identify $\mathbb{R}^2$ and $\mathbb{C}$ as (normed) vector spaces. We denote the closure of a subset $M \subset \mathbb{R}^2$ by $\overline{M}$, the boundary of M by ∂M and for a function $f : M \to \mathbb{C}$

and $K \subset M$ we denote by $f_{|K}$ the restriction of f to K. We write $\mathcal{C}(\Omega)$ for the space of continuous $\mathbb{C}$-valued functions on a set $\Omega \subset \mathbb{R}^2$ and $L^1(\Omega)$ for the space of (equivalence classes of) $\mathbb{C}$-valued Lebesgue integrable functions on a measurable set $\Omega \subset \mathbb{R}^2$.

By E we always denote a non-trivial locally convex Hausdorff space over the field $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$. If $E = \mathbb{C}$, then we set $(p_\alpha)_{\alpha\in\mathfrak{A}} := \{|\cdot|\}$. Further, we denote by $L(F, E)$ the space of continuous linear maps from a locally convex Hausdorff space F to E and sometimes use the notation $\langle T, f\rangle := T(f)$, $f \in F$, for $T \in L(F, E)$. If $E = \mathbb{C}$, we write $F' := L(F, \mathbb{C})$ for the dual space of F. If F and E are (linearly topologically) isomorphic, we write $F \cong E$. We denote by $L_b(F, E)$ the space $L(F, E)$ equipped with the locally convex topology of uniform convergence on the bounded subsets of F.

We recall that a function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{C}$ to E is called holomorphic if the limit

$$\left(\frac{\partial}{\partial z}\right)^E f(z_0) := \lim_{\substack{h\to 0\\ h\in\mathbb{C}, h\neq 0}} \frac{f(z_0+h)-f(z_0)}{h}$$

exists in E for every $z_0 \in \Omega$. The linear space of all functions $f : \Omega \to E$ which are holomorphic is denoted by $\mathcal{O}(\Omega, E)$. For a compact set $K \subset \overline{\mathbb{R}}$, $0 < \beta \le 1$ and a strictly increasing real sequence $(a_n)_{n\in\mathbb{N}}$ we set

$$\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E) := \{f \in \mathcal{O}(\mathbb{C} \setminus K, E) \mid \forall\, n \in \mathbb{N},\ \alpha \in \mathfrak{A} :\ |f|_{n,\alpha} < \infty\}$$

where

$$|f|_{n,\alpha} := \sup_{z\in S_n(K)} p_\alpha(f(z))e^{a_n|\operatorname{Re}(z)|^\beta}.$$

The subscript α in the notation of the seminorms is omitted in the $\mathbb{C}$-valued case and we write $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K) := \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, \mathbb{C})$.

Remark 2 We have $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K) = \mathcal{E}^{\beta}_{(a_n),\overline{\partial}}(\overline{\mathbb{C}} \setminus K)$ as Fréchet spaces by [22, Proposition 7 (b), p. 11] and [22, Example 6, p. 11].

Throughout the rest of the paper we make the following standing assumptions.

Assumption 3

- (i) E is sequentially complete,
- (ii) $K \subset \overline{\mathbb{R}}$ is a non-empty compact set,
- (iii) $0 < \beta \le 1$,
- (iv) $(a_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $a_n < 0$ for all $n \in \mathbb{N}$ or $a_n \ge 0$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = 0$ or $\lim_{n\to\infty} a_n = \infty$.

3 Duality

We recall the well-known topological Silva-Köthe-Grothendieck isomorphy

$$\mathcal{O}(\mathbb{C}\setminus K,E)/\mathcal{O}(\mathbb{C},E)\cong L_b(\mathscr{A}(K),E) \tag{2}$$

where E is a quasi-complete locally convex Hausdorff space, $\varnothing\neq K\subset\mathbb{R}$ is compact, $\mathcal{O}(\mathbb{C}\setminus K,E)$ is equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}\setminus K$, the quotient space with the induced quotient topology and $\mathscr{A}(K)$ is the space of germs of real analytic functions on K with its inductive limit topology (see e.g. [29, p. 6], [11, Proposition 1, p. 46], [18, §27.4, p. 375-378], [27, Theorem 2.1.3, p. 25]). The aim of this section is to prove a counterpart of this isomorphy for weighted vector-valued holomorphic functions and non-empty compact $K\subset\overline{\mathbb{R}}$.

The spaces $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)$ play the counterpart of $\mathcal{O}(\mathbb{C}\setminus K,E)$ for our version of the isomorphy (2). Next, we introduce the counterparts of $\mathscr{A}(K)$. Let $\Omega\subset\mathbb{C}$ be open and $f\in\mathcal{O}(\Omega)$. For $z\in\Omega$ and $n\in\mathbb{N}_0$ we denote the point evaluation of the nth complex derivative at z by $\delta^{(n)}_z f:=f^{(n)}(z)$.

Proposition 4 *For $n\in\mathbb{N}$ let*

$$\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right):=\{f\in\mathcal{O}(U_n(K))\cap\mathcal{C}\left(\overline{U_n(K)}\right)\mid \|f\|_{K,n}:=\|f\|_n<\infty\}$$

where

$$\|f\|_{K,n}:=\|f\|_n:=\sup_{z\in\overline{U_n(K)}}|f(z)|e^{-a_n|\operatorname{Re}(z)|^{\beta}}$$

and the spectral maps for $n,k\in\mathbb{N}$, $n\leq k$, are given by the restrictions

$$\pi_{n,k}:\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)\to\mathcal{O}^{-\beta}_{a_k}\left(\overline{U_k(K)}\right),\ \pi_{n,k}(f):=f_{|U_k(K)}.$$

Then the following assertions hold.

(a) *The inductive limit*

$$\mathcal{O}^{-\beta}_{(a_n)}(K):=\varinjlim_{n\in\mathbb{N}}\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$$

exists and is a DFS-space.

(b) *The span of the set of point evaluations of complex derivatives $\{\delta^{(n)}_{x_0}\mid x_0\in K\cap\mathbb{R},\ n\in\mathbb{N}_0\}$ is dense in $\mathcal{O}^{-\beta}_{(a_n)}(K)'_b$ if $K\subset\mathbb{R}$ or $K\cap\{\pm\infty\}$ contains no isolated points in K.*

Proof (*a*) It is easy to see that $\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$ is a Banach space for every $n\in\mathbb{N}$. Further, the maps $\pi_{n,m}:\mathcal{O}^{-\gamma}_{a_n}\left(\overline{U_n(K)}\right)\to\mathcal{O}^{-\beta}_{a_m}\left(\overline{U_m(K)}\right)$, $n\leq m$, are injective by virtue of the identity theorem and the definition of sets $U_n(K)$. Thus the considered spectrum is an embedding spectrum. For all $M\subset U_n(K)$ compact and $f\in B_n:=\{g\in\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)\mid\|g\|_n\leq 1\}$ we have

$$\|f\|_M:=\sup_{z\in M}|f(z)|=\sup_{z\in M}|f(z)|e^{-a_n|\operatorname{Re}(z)|^{\beta}}e^{a_n|\operatorname{Re}(z)|^{\beta}}\leq\sup_{z\in M}e^{a_n|\operatorname{Re}(z)|^{\beta}}\|f\|_n\leq\sup_{z\in M}e^{a_n|\operatorname{Re}(z)|^{\beta}}.$$

Thus B_n is bounded in $\mathcal{O}(U_n(K))$ w.r.t. the system of seminorms generated by $\|\cdot\|_M$ for compact $M \subset U_n(K)$. As this space is a Fréchet-Montel space, B_n is relatively compact and hence relatively sequentially compact in $\mathcal{O}(U_n(K))$.

What remains to be shown is that for all $n \in \mathbb{N}$ there exists $m > n$ such that $\pi_{n,m}$ is a compact map. Because the considered spaces are Banach spaces, it suffices to prove the existence of $m > n$ such that $(\pi_{n,m}(f_k))_{k\in\mathbb{N}}$ has a convergent subsequence in $\mathcal{O}^{-\beta}_{a_m}\left(\overline{U_m(K)}\right)$ for every sequence $(f_k)_{k\in\mathbb{N}}$ in B_n. We choose $m := 2n$. For $\varepsilon > 0$ we set

$$Q := \overline{U_{2n}(K)} \cap \{z \in \mathbb{C} \mid |\mathrm{Re}\,(z)| \leq \max(0, \ln(\varepsilon)/(a_n - a_{2n}))^{1/\beta} + n\},$$

and get

$$\sup_{z\in\overline{U_{2n}(K)}\setminus Q} \frac{e^{-a_{2n}|\mathrm{Re}\,(z)|^\beta}}{e^{-a_n|\mathrm{Re}\,(z)|^\beta}} = \sup_{z\in\overline{U_{2n}(K)}\setminus Q} e^{(a_n-a_{2n})|\mathrm{Re}\,(z)|^\beta} \leq \varepsilon. \tag{3}$$

Thus condition (RU) in [2, p. 67] is fufilled and it follows analogously to the proof of [2, Theorem (b), p. 67-68] that every sequence $(f_k)_{k\in\mathbb{N}}$ in B_n has a subsequence $(f_{k_l})_{l\in\mathbb{N}}$ such that $(\pi_{n,2n}(f_{k_l}))_{l\in\mathbb{N}}$ converges in $\mathcal{O}^{-\beta}_{a_{2n}}\left(\overline{U_{2n}(K)}\right)$, proving the compactness of $\pi_{n,2n}$. Hence the inductive limit $\mathcal{O}^{-\beta}_{(a_n)}(K)$ exists and is a DFS-space by [25, Proposition 25.20, p. 304].

(*b*) We set $F := \mathrm{span}\{\delta^{(n)}_{x_0} \mid x_0 \in K \cap \mathbb{R},\, n \in \mathbb{N}_0\}$. Let $x_0 \in K \cap \mathbb{R}$ and $n \in \mathbb{N}_0$. It follows from Cauchy's inequality that $\delta^{(n)}_{x_0}$ is continuous on $\mathcal{O}^{-\beta}_{a_k}(\overline{U_k(K)})$ for any $k \in \mathbb{N}$, implying $F \subset \mathcal{O}^{-\beta}_{(a_n)}(K)'$. As $\mathcal{O}^{-\beta}_{(a_n)}(K)$ is a DFS-space by part (*a*), it is reflexive by [25, Proposition 25.19, p. 303], which means that the canonical embedding $J : \mathcal{O}^{-\beta}_{(a_n)}(K) \to (\mathcal{O}^{-\beta}_{(a_n)}(K)'_b)'_b$ is a topological isomorphism. We consider the polar set of F, i.e.

$$F^\circ := \{y \in (\mathcal{O}^{-\beta}_{(a_n)}(K)'_b)'_b \mid \forall\, T \in F : \; y(T) = 0\}.$$

Let $y \in F^\circ$. Then there is $f \in \mathcal{O}^{-\beta}_{(a_n)}(K)$ such that $y = J(f)$. For $T := \delta^{(n)}_{x_0} \in F$

$$0 = y(T) = J(f)(T) = T(f) = f^{(n)}(x_0)$$

is valid for any $n \in \mathbb{N}_0$. Thus f is identical to zero on a neighbourhood of x_0 (by Taylor series expansion) since f is holomorphic near $x_0 \in U_n(K)$. Due to the assumptions every component of $U_n(K)$ contains a point $x_0 \in K \cap \mathbb{R}$ so f is identical to zero on $\overline{U_n(K)}$ by the identity theorem and continuity, yielding to $y = 0$. Therefore $F^\circ = \{0\}$ and thus F is dense in $\mathcal{O}^{-\beta}_{(a_n)}(K)'_b$ by the bipolar theorem. □

In the case $\beta := 1$ and $a_n := -1/n$ for all $n \in \mathbb{N}$ the spaces $\mathcal{O}^{-1}_{(a_n)}(K)$ play an essential role in the theory of Fourier hyperfunctions and it is already mentioned in [17, p. 469] resp. proved in [15, 1.11 Satz, p. 11] and [19, 3.5 Theorem, p. 17] that they are DFS-spaces.

Remark 5 If $K \subset \mathbb{R}$, then $\mathcal{O}^{-\beta}_{(a_n)}(K) \cong \mathscr{A}(K)$.

Now, we take a closer look at the sets $U_t(K)$ (c.f. [19, 3.3 Remark, p. 13]).

Remark 6 Let $t \in \mathbb{R}$, $t \geq 1$.

(a) The set $U_t(K)$ has finitely many components.
(b) Let Z be a component of $U_t(K)$. We define $a := \min(Z \cap K)$ and $b := \max(Z \cap K)$ if existing (in $\mathbb{R}$).

(i) If Z is bounded, there exists $0 < R \leq 1/t$ such that for all $0 < r \leq R$: $\{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, [a, b]) < r\} \subset Z$
(ii) If $Z \cap \mathbb{R}$ is bounded from below and unbounded from above and a exists, there exists $0 < R \leq 1/t$ such that for all $0 < r \leq R$: $\{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, [a, \infty)) < r\} \subset Z$
(iii) If $Z \cap \mathbb{R}$ is bounded from above and unbounded from below and b exists, there exists $0 < R \leq 1/t$ such that for all $0 < r \leq R$: $\{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, (-\infty, b]) < r\} \subset Z$
(iv) If $Z \cap \mathbb{R}$ is unbounded from below and above, there exists $0 < R \leq 1/t$ such that for all $0 < r \leq R$: $\{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, \mathbb{R}) < r\} \subset Z$
(v) If $Z \cap \mathbb{R}$ is bounded from below and unbounded from above and a does not exist, then $Z = (t, \infty) + i(-1/t, 1/t)$. If $Z \cap \mathbb{R}$ is bounded from above and unbounded from below and b does not exist, then $Z = (-\infty, -t) + i(-1/t, 1/t)$.

Proof (a) We only consider the case $\infty \in K$, $-\infty \notin K$. Let $(Z_j)_{j\in J}$ denote the (pairwise disjoint) components of $U_t(K)$. Then $U_t(K) = \bigcup_{j\in J} Z_j$ and by definition of a component there is $k \in J$ such that Z_k is the only component including $(t, \infty) + i(-1/t, 1/t)$. Furthermore there exists $m \in \mathbb{R}$ with $\bigcup_{j\in J\setminus\{k\}}(Z_j \cap \mathbb{R}) \subset [m, t]$ by assumption. For $j \neq k$ the length $\lambda(Z_j \cap \mathbb{R})$ of the interval $Z_j \cap \mathbb{R}$, where λ denotes the Lebesgue measure, is estimated from below by $\lambda(Z_j \cap \mathbb{R}) \geq 2/t$ by definition of $U_t(K)$. Since all Z_j are pairwise disjoint, this implies that J has to be finite. The others cases follow analogously.

(b)(i) Since $Z \cap K$ is closed in $\mathbb{R}$ and therefore compact, a and b exist. Hence $[a, b] \subset Z$ by the definition of $U_t(K)$ and as Z is connected. $[a, b]$ being a compact subset of the open set Z implies that there is $0 < R < 1/t$ such that $([a, b] + i(-R, R)) \subset Z$ by the tube lemma, which completes the proof.

(ii) If $Z \cap K \cap (-\infty, t] \neq \emptyset$, then a exists and analogously to (i) there exists $0 < R < 1/t$ such that for all $0 < r \leq R$

$$\{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, [a, t]) < r\} \subset Z.$$

By definition of $U_t(K)$ this brings forth $\{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, [a, \infty)) < r\} \subset Z$. If $Z \cap K \cap (-\infty, t] = \emptyset$ and a exists, the desired $0 < R < 1/t$ exists by the definition of $U_t(K)$ since $t \notin Z \cap K$ and $Z \cap K$ is closed in $\mathbb{R}$, which implies $\mathrm{d}(\{t\}, Z \cap K) > 0$.

(iii) Analogously to (ii).

(iv) By the assumptions $Z \cap K \cap [-t, t] \neq \emptyset$. Analogously to (i) there exists $0 < R < 1/t$ such that for all $0 < r \leq R$

$$\{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, [-t, t]) < r\} \subset Z.$$

Like in (ii) and (iii) this brings forth $\{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, \mathbb{R}) < r\} \subset Z$.

(v) This follows directly from the definition of $U_t(K)$ and as Z is a component of $U_t(K)$. □

Definition 7 Let $n \in \mathbb{N}$ and $(Z_j)_{j\in J}$ denote the components of $U_n(K)$. A component Z_j of $U_n(K)$ fulfils one of the cases of Remark 6 (b) and so for $a = a_j$, $b = b_j$ (in the cases (i)-(iii)), for $0 < r_j < R_j = R$ (in the cases (i)-(iv)) resp. $0 < r_j < 1/n =: R_j$ (in the case (v)) we define

$$V_{r_j}(Z_j) := \begin{cases} \{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, [a_j, b_j]) < r_j\} & , Z_j \text{ fulfils (i)}, \\ \{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, [a_j, \infty)) < r_j\} & , Z_j \text{ fulfils (ii)}, \\ \{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, (-\infty, b_j]) < r_j\} & , Z_j \text{ fulfils (iii)}, \\ \{z \in \mathbb{C} \mid \mathrm{d}(\{z\}, \mathbb{R}) < r_j\} & , Z_j \text{ fulfils (iv)}, \\ (1/r_j, \infty) + i(-r_j, r_j) & , Z_j = (n, \infty) + i(-1/n, 1/n), \\ (-\infty, -1/r_j) + i(-r_j, r_j) & , Z_j = (-\infty, -n) + i(-1/n, 1/n), \end{cases}$$

where Z_j fulfils (v) in the last two cases. By Remark 6 (a) there is w.l.o.g. $k \in \mathbb{N}$ with $U_n(K) = \bigcup_{j=1}^{k} Z_j$. We set $r := (r_j)_{1 \le j \le k}$ and the path

$$\gamma_{K,n,r} := \sum_{j=1}^{k} \gamma_j$$

where γ_j is the path along the boundary of $V_{r_j}(Z_j)$ in $\mathbb{C}$ in the positive sense (counterclockwise) (see Fig. 3).

Proposition 8 *Let $n \in \mathbb{N}$ and $\gamma_{K,n,r}$ be the path from Definition 7. Then the following assertions hold.*

(a) *$F \cdot \varphi$ is Pettis-integrable along $\gamma_{K,n,r}$ for all $F \in \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)$ and $\varphi \in \mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$.*

(b) *There are $m \in \mathbb{N}$ and $C > 0$ such that for all $\alpha \in \mathfrak{A}$, $F \in \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)$ and $\varphi \in \mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$*

$$p_\alpha\left(\int_{\gamma_{K,n,r}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta\right) \le C|F|_{m,\alpha}\|\varphi\|_n.$$

(c) *For all $F \in \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)$, $\varphi \in \mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$ and $\widetilde{r} := (\widetilde{r}_j)_{1 \le j \le k}$ with $0 < \widetilde{r}_j < R_j$ for all $1 \le j \le k$*

$$\int_{\gamma_{K,n,r}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta = \int_{\gamma_{K,n,\widetilde{r}}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta.$$

(d) *For all $F \in \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}, E)$ and $\varphi \in \mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$*

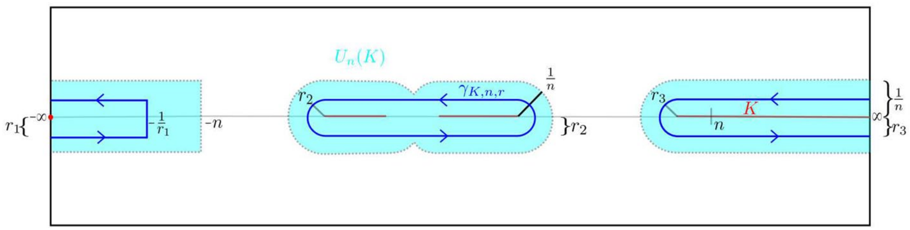

Fig. 3 Path $\gamma_{K,n,r}$ for $\pm\infty \in K$ (c.f. [19, Figure 4.1, p. 40])

$$\int_{\gamma_{K,n,r}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta = 0.$$

Proof $(a) + (b)$ We have to show that there is $e_{K,n,r} \in E$ such that

$$\langle e', e_{K,n,r}\rangle = \int_{\gamma_{K,n,r}} \langle e', F(\zeta)\varphi(\zeta)\rangle \mathrm{d}\zeta, \quad e' \in E',$$

which gives $\int_{\gamma_{K,n,r}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta = e_{K,n,r}$.

First, let $V_{r_j}(Z_j)$ be bounded for some $1 \leq j \leq k$. There is a parametrisation $\gamma_j : [0,1] \to \mathbb{C}$ which has a continuously differentiable extension $\widetilde{\gamma}_j$ on $(-1,2)$. As the map $(e' \circ (F \cdot \varphi) \circ \gamma_j) \cdot \gamma_j'$ is continuous on $[0,1]$ for every $e' \in E'$, it is an element of $L^1([0,1])$ for every $e' \in E'$. Thus the map

$$\mathfrak{I}_j : E' \to \mathbb{C},\ \mathfrak{I}_j(e') := \int_{\gamma_j} \langle e', F(\zeta)\varphi(\zeta)\rangle \mathrm{d}\zeta = \int_0^1 \langle e', (F \cdot \varphi)(\gamma_j(t))\rangle \gamma_j'(t)\mathrm{d}t,$$

is well-defined and linear. We estimate

$$|\mathfrak{I}_j(e')| \leq \underbrace{\int_0^1 |\gamma_j'(t)|\mathrm{d}t}_{=:\ell(\gamma_j)} \sup_{z \in (F\cdot\varphi)(\gamma_j([0,1]))} |e'(z)|, \quad e' \in E'.$$

Let us denote by $\overline{\mathrm{acx}}((F \cdot \varphi)(\gamma_j([0,1])))$ the closure of the absolutely convex hull of the set $(F \cdot \varphi)(\gamma_j([0,1]))$. Since $e' \circ (F \cdot \varphi) \circ \widetilde{\gamma}_j \in \mathcal{C}^1((-1,2))$ for every $e' \in E'$, the absolutely convex set $\overline{\mathrm{acx}}((F \cdot \varphi)(\gamma_j([0,1])))$ is compact in the sequentially complete space E by [5, Proposition 2, p. 354], yielding $\mathfrak{I}_j \in (E'_\kappa)' \cong E$ by the theorem of Mackey-Arens, i.e. there is $e_j \in E$ such that

$$\langle e', e_j\rangle = \mathfrak{I}_j(e') = \int_{\gamma_j} \langle e', F(\zeta)\varphi(\zeta)\rangle \mathrm{d}\zeta, \quad e' \in E'.$$

Therefore $F \cdot \varphi$ is Pettis-integrable along γ_j. Furthermore, we choose $m_j \in \mathbb{N}$ such that $(1/m_j) < r_j$ and for $\alpha \in \mathfrak{A}$ we set $B_\alpha := \{x \in E \mid p_\alpha(x) < 1\}$. We note that

$$\begin{aligned}
&p_\alpha\left(\int_{\gamma_j} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta\right) \\
&= \sup_{e' \in B_\alpha^\circ} \left|\langle e', \int_{\gamma_j} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta\rangle\right| \leq \ell(\gamma_j) \sup_{e' \in B_\alpha^\circ} \sup_{z \in \gamma_j([0,1])} |e'(F(z))\varphi(z)| \\
&\leq \ell(\gamma_j) \sup_{w \in \gamma_j([0,1])} e^{(a_n - a_{m_j})|\operatorname{Re}(w)|^\beta} \sup_{e' \in B_\alpha^\circ} \sup_{z \in S_{m_j}(K)} |e'(F(z)e^{a_{m_j}|\operatorname{Re}(z)|^\beta})| \|\varphi\|_n \\
&= \ell(\gamma_j) \sup_{w \in \gamma_j([0,1])} e^{(a_n - a_{m_j})|\operatorname{Re}(w)|^\beta} |F|_{m_j,\alpha} \|\varphi\|_n
\end{aligned}$$

where we used [25, Proposition 22.14, p. 256] in the first and the last equation to get from p_α to $\sup_{e' \in B_\alpha^\circ}$ and back. If $K \subset \mathbb{R}$, then all $V_{r_j}(Z_j)$, $1 \leq j \leq k$, are bounded and with the choice $e_{K,n,r} := \sum_{j=1}^{k} e_j$, $m := \max_{1\leq j\leq k} m_j$ and $C := k \max_{1\leq j\leq k} \ell(\gamma_j) \sup_{w\in\gamma_j([0,1])} e^{(a_n-a_{m_j})|\operatorname{Re}(w)|^\beta}$ we deduce our statement.

Second, let us consider the case $\infty \in K$, $-\infty \notin K$. Let Z_k be the unique unbounded component of $U_n(K)$. For $q \in \mathbb{N}$, $q > 1/r_k > n$, we denote by $\gamma_{k,q}$ the part of γ_k in $\{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq q\}$. Like in the first part the Pettis-integral

$$e_{k,q} := \int_{\gamma_{k,q}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta$$

exists (in E) and for $\alpha \in \mathfrak{A}$ and $m_k \in \mathbb{N}$, $(1/m_k) < r_k$, we have

$$p_\alpha\left(\int_{\gamma_{k,q}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta\right) \leq \ell(\gamma_{k,q}) \sup_{w\in\gamma_{k,q}([0,1])} e^{(a_n-a_{m_k})|\operatorname{Re}(w)|^\beta} |F|_{m_k,\alpha}\|\varphi\|_n.$$

Next, we prove that $(e_{k,q})_{q>1/r_k}$ is a Cauchy sequence in E. We choose $M := \max(m_k, 2n)$. For $q, p \in \mathbb{N}$, $q > p > 1/r_k > n$, we obtain

$$\begin{aligned}
&p_\alpha(e_{k,q} - e_{k,p}) \\
&= \sup_{e'\in B_\alpha^\circ} \left| \int_{\gamma_{k,q}-\gamma_{k,p}} e'(F(\zeta))\varphi(\zeta)\mathrm{d}\zeta \right| \\
&\leq \sup_{e'\in B_\alpha^\circ} \left(\int_p^q |e'(F(t-ir_k))\varphi(t-ir_k)|\mathrm{d}t + \int_p^q |e'(F(t+ir_k))\varphi(t+ir_k)|\mathrm{d}t \right) \\
&\leq 2 \sup_{e'\in B_\alpha^\circ} \int_p^q e^{(a_n-a_M)t^\beta}\mathrm{d}t |e' \circ F|_M \|\varphi\|_n \\
&= 2\int_p^q e^{(a_n-a_M)t^\beta}\mathrm{d}t|F|_{M,\alpha}\|\varphi\|_n \leq 2\int_p^q e^{(a_n-a_{2n})t^\beta}\mathrm{d}t|F|_{M,\alpha}\|\varphi\|_n
\end{aligned}$$

and observe that $(\int_0^q e^{(a_n-a_{2n})t^\beta}\mathrm{d}t)_q$ is a Cauchy sequence in $\mathbb{C}$ because

$$\int_0^\infty e^{(a_n-a_{2n})t^\beta}\mathrm{d}t = \frac{\Gamma(1/\beta)}{\beta|a_n - a_{2n}|^{1/\beta}}$$

where Γ is the gamma function. Therefore $(e_{k,q})_{q>1/r_k}$ is a Cauchy sequence in E, has a limit e_k in the sequentially complete space E and

$$e_k = \int_{\gamma_k} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta.$$

We fix $p \in \mathbb{N}$, $p > 1/r_k > n$, and conclude that

$$\begin{aligned}
p_\alpha\left(\int_{\gamma_k} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta\right) &\leq p_\alpha(e_k - e_{k,p}) + p_\alpha(e_{k,p}) \\
&\leq \left(\frac{2\Gamma(1/\beta)}{\beta|a_n - a_{2n}|^{1/\beta}} + \ell(\gamma_{k,p}) \sup_{w\in\gamma_{k,p}([0,1])} e^{(a_n-a_{m_k})|\operatorname{Re}(w)|^\beta} \right)|F|_{M,\alpha}\|\varphi\|_n
\end{aligned}$$

Consequently, our statement holds also in the case $\infty \in K$, $-\infty \notin K$ and in the remaining cases it follows analogously.

(*c*) We note that

$$\langle e', \int_{\gamma_{K,n,r}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta - \int_{\gamma_{K,n,\widetilde{r}}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta\rangle = \int_{\gamma_{K,n,r}-\gamma_{K,n,\widetilde{r}}} \langle e', F(\zeta)\varphi(\zeta)\rangle\mathrm{d}\zeta$$

for all $e' \in E'$. Thus statement (*c*) follows from Cauchy's integral theorem and the Hahn-Banach theorem if $K \subset \mathbb{R}$. Now, let us consider the case $\infty \in K, -\infty \notin K$. We denote by γ_k resp. $\widetilde{\gamma}_k$ the part of $\gamma_{K,n,r}$ resp. $\gamma_{K,n,\widetilde{r}}$ in the unbounded component of $U_n(K)$. It suffices to show that

$$\int_{\gamma_k-\widetilde{\gamma}_k} \langle e', F(\zeta)\varphi(\zeta)\rangle\mathrm{d}\zeta = 0, \quad e' \in E'. \tag{4}$$

Let $\varepsilon > 0$ and w.l.o.g. $r_k < \widetilde{r}_k$. We choose the compact set $Q \subset \overline{U_{2n}(K)}$ as in the proof of Proposition 4 (b). Further, we take $q \in \mathbb{R}$ such that $q > 1/r_k$ and $q \in \overline{U_{2n}(K)} \setminus Q$ and define the path $\gamma^+_{0,q} : [r_k, \widetilde{r}_k] \longrightarrow \mathbb{C}$, $\gamma^+_{0,q}(t) := q + it$. We deduce that for $m_k \in \mathbb{N}$, $(1/m_k) < \min(r_k, 1/(2n))$, and every $e' \in E'$

$$\begin{aligned}\left|\int_{\gamma^+_{0,q}} \langle e', F(\zeta)\varphi(\zeta)\rangle\mathrm{d}\zeta\right| &\leq \int_{r_k}^{\widetilde{r}_k} e^{(a_n-a_{m_k})|\operatorname{Re}(q+it)|^\beta}\mathrm{d}t\|\varphi\|_n|e'\circ F|_{m_k} \\ &= (\widetilde{r}_k - r_k)e^{(a_n-a_{m_k})q^\beta}\|\varphi\|_n|e'\circ F|_{m_k} \underset{(3)}{\leq} (\widetilde{r}_k - r_k)\|\varphi\|_n|e'\circ F|_{m_k}\varepsilon.\end{aligned}$$

In the same way we obtain with $\gamma^-_{0,q} : [-\widetilde{r}_k, -r_k] \longrightarrow \mathbb{C}$, $\gamma^-_{0,q}(t) := q + it$, that

$$\left|\int_{\gamma^-_{0,q}} \langle e', F(\zeta)\varphi(\zeta)\rangle\mathrm{d}\zeta\right| \leq (\widetilde{r}_k - r_k)\|\varphi\|_n|e'\circ F|_{m_k}\varepsilon.$$

Hence we get (4) by Cauchy's integral theorem and the Hahn-Banach theorem as well. The remaining cases follow similarly.

(*d*) The proof is similar to (*c*). Let $F \in \mathcal{O}^\beta_{(a_n)}(\overline{\mathbb{C}}, E)$. Again, it suffices to prove that

$$\int_{\gamma_{K,n,r}} \langle e', F(\zeta)\varphi(\zeta)\rangle\mathrm{d}\zeta = 0, \quad e' \in E'.$$

This follows from Cauchy's integral theorem and the Hahn-Banach theorem if $K \subset \mathbb{R}$. Again, we only consider the case $\infty \in K, -\infty \notin K$ and only need to show that

$$\int_{\gamma_k} \langle e', F(\zeta)\varphi(\zeta)\rangle\mathrm{d}\zeta = 0, \quad e' \in E',$$

where γ_k is the part of $\gamma_{K,n,r}$ in the unbounded component of $U_n(K)$. Let $\varepsilon > 0$ and choose q as in (*c*). Then we have with $\gamma_{0,q} : [-r_k, r_k] \to \mathbb{C}$, $\gamma_{0,q}(t) := q + it$, that

$$\left|\int_{\gamma_{0,q}} \langle e', F(\zeta)\varphi(\zeta)\rangle\mathrm{d}\zeta\right| \leq 2r_k\|\varphi\|_n|e'\circ F|_{2n}\varepsilon$$

for every $e' \in E'$. Cauchy's integral theorem and the Hahn-Banach theorem imply our statement. □

An essential role in the proof of $\mathcal{O}(\mathbb{C}\setminus K,E)/\mathcal{O}(\mathbb{C},E)\cong L_b(\mathscr{A}(K),E)$ for non-empty compact $K\subset\mathbb{R}$ and quasi-complete E (see (2)) plays the fundamental solution $z\mapsto 1/(\pi z)$ of the Cauchy-Riemann operator. By the identity theorem we can consider $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$ as a subspace of $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)$ and we equip the quotient space $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$ with the induced locally convex quotient topology (which may not be Hausdorff, see Remark 14). We want to prove the isomorphy

$$\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)\cong L_b(\mathcal{O}^{-\beta}_{(a_n)}(K),E)$$

for non-empty compact $K\subset\overline{\mathbb{R}}$ under some assumptions on K, β and $(a_n)_{n\in\mathbb{N}}$. Since we have to deal with functions having some growth given by our exponential weights, we have to use the adapted fundamental solution $z\mapsto e^{-z^2}/(\pi z)$ of the Cauchy-Riemann operator.

Proposition 9 *Let $\gamma_{K,n,r}$ be the path from Definition 7. The map*

$$H_K:\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)\to L_b(\mathcal{O}^{-\beta}_{(a_n)}(K),E)$$

given by

$$H_K(f)(\varphi):=\int_{\gamma_{K,n,r}}F(\zeta)\varphi(\zeta)\mathrm{d}\zeta$$

for $f=[F]\in\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$ and $\varphi\in\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$, $n\in\mathbb{N}$, is well-defined, linear and continuous. For all non-empty compact sets $K_1\subset K$ it holds that

$$H_{K|\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K_1,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)}=H_{K_1} \tag{5}$$

on $\mathcal{O}^{-\beta}_{(a_n)}(K)$.

Proof In the following we omit the index K of H_K if no confusion seems to be likely. Let $f=[F]\in\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$ and $\varphi\in\mathcal{O}^{-\beta}_{(a_n)}(K)$. Then there is $n\in\mathbb{N}$ such that $\varphi\in\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$. Due to Proposition 8 (a) and (d) $H(f)(\varphi)\in E$ and $H(f)$ is independent of the representative F of f. From Proposition 8 (c) it follows that $H(f)$ is well-defined on $\mathcal{O}^{-\beta}_{(a_n)}(K)$, i.e. for all $k\in\mathbb{N}$, $k\geq n$, and $\varphi\in\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$ it holds that

$$H(f)(\varphi)=H(f)(\varphi_{|U_k(K)})=H(f)(\pi_{n,k}(\varphi)).$$

For all $n\in\mathbb{N}$ there are $m\in\mathbb{N}$ and $C>0$ such that

$$p_\alpha(H(f)(\varphi))\leq C|F|_{m,\alpha}\|\varphi\|_n \tag{6}$$

for all $f=[F]\in\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$, $\varphi\in\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$ and $\alpha\in\mathfrak{A}$ by Proposition 8 (b), which implies that $H(f)\in L\left(\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right),E\right)$ for every $n\in\mathbb{N}$. We deduce that $H(f)\in L(\mathcal{O}^{-\beta}_{(a_n)}(K),E)$ by [9, 3.6 Satz, p. 117]. Let

$$q:\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)\to\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E),\ q(F):=[F],$$

denote the quotient map. We equip the quotient space with its usual quotient topology generated by the system of quotient seminorms given by

$$|f|^{\wedge}_{l,\alpha} := \inf_{F\in q^{-1}(f)} |F|_{l,\alpha}, \quad f\in \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E),$$

for $l\in\mathbb{N}$ and $\alpha\in\mathfrak{A}$. Then the quotient space, equipped with these seminorms, is a locally convex space (but maybe not Hausdorff). Since (6) holds for every representative F of f, we obtain for every $f\in \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$, $\varphi\in\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$, $n\in\mathbb{N}$, and $\alpha\in\mathfrak{A}$ that

$$p_\alpha(H(f)(\varphi)) \leq C \inf_{F\in q^{-1}(f)} |F|_{m,\alpha}\|\varphi\|_n = C|f|^{\wedge}_{m,\alpha}\|\varphi\|_n. \tag{7}$$

Now, let $M\subset\mathcal{O}^{-\beta}_{(a_n)}(K)$ be a bounded set. Since the sequence $(B_n)_{n\in\mathbb{N}}$ of closed unit balls B_n of $\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$ is a fundamental system of bounded sets in $\mathcal{O}^{-\beta}_{(a_n)}(K)$ by [25, Proposition 25.19, p. 303], there exist $n\in\mathbb{N}$ and $\lambda>0$ with $M\subset\lambda B_n$. We derive from (7) that

$$\sup_{\varphi\in M} p_\alpha(H(f)(\varphi)) \leq |\lambda|C|f|^{\wedge}_{m,\alpha},$$

proving the continuity of H.

Moreover, let $K_1\subset\mathbb{R}$ be compact and $K_1\subset K$. We observe that for every $F\in\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K_1,E)$ and $\varphi\in\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_n(K)}\right)$, $n\in\mathbb{N}$, it holds that

$$H_K([F])(\varphi) = \int_{\gamma_{K,n,r}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta = \int_{\gamma_{K_1,n,r}} F(\zeta)\varphi(\zeta)\mathrm{d}\zeta = H_{K_1}([F])(\varphi)$$

by Cauchy's integral theorem and the Hahn-Banach theorem like in Proposition 8 (c) and (d). This yields to

$$H_{K|\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K_1,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)} = H_{K_1}$$

on $\mathcal{O}^{-\beta}_{(a_n)}(K)$. □

Now, we take a closer look at the potential inverse of H_K.

Proposition 10 *The map*

$$\Theta_K : L_b(\mathcal{O}^{-\beta}_{(a_n)}(K),E) \to \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$$

given by

$$\Theta_K(T) := \left[\mathbb{C}\setminus K \ni z \longmapsto \frac{1}{2\pi i}\left\langle T, \frac{e^{(z-\cdot)^2}}{z-\cdot}\right\rangle\right], \quad T\in L_b(\mathcal{O}^{-\beta}_{(a_n)}(K),E),$$

is well-defined, linear and continuous.

Proof We start with the proof that the map Θ_K is well-defined and take a closer look at its components. For $z,\zeta\in\mathbb{C}$ we set $G(z,\zeta) := e^{-(z-\zeta)^2}$ and note that $\frac{\partial}{\partial z}G(z,\zeta) = -2(z-\zeta)G(z,\zeta)$. We remark that for all $z=z_1+iz_2\in\mathbb{C}$ and all $n\in\mathbb{N}$

$$\begin{aligned}\|G(z,\cdot)\|_n &= \sup_{\zeta\in\overline{U_n(K)}} e^{-\operatorname{Re}((z-\zeta)^2)}e^{-a_n|\operatorname{Re}(\zeta)|^\beta}\\ &\le \sup_{\zeta_1+i\zeta_2\in\overline{U_n(K)}} e^{-(z_1-\zeta_1)^2+(z_2-\zeta_2)^2}e^{|a_n|(1+|\zeta_1|)}\\ &\le e^{(|z_2|+1/n)^2-z_1^2+|a_n|}\sup_{\zeta_1+i\zeta_2\in\overline{U_n(K)}} e^{-\zeta_1^2+(2|z_1|+|a_n|)|\zeta_1|}\\ &\le e^{(|z_2|+1/n)^2-z_1^2+|a_n|}e^{-(|z_1|+|a_n|/2)^2+(2|z_1|+|a_n|)(|z_1|+|a_n|/2)}\end{aligned}\tag{8}$$

and we deduce that $G(z,\cdot)\in\mathcal{O}_{a_n}^{-\beta}\left(\overline{U_n(K)}\right)$, in particular $G(\mathbb{B}_{1/n}(z),\cdot)\subset\mathcal{O}_{a_n}^{-\beta}\left(\overline{U_n(K)}\right)$. For $\zeta=\zeta_1+i\zeta_2\in\overline{U_n(K)}$ and $h\in\mathbb{C}$, $0<|h|\le 1$, we observe that

$$\begin{aligned}&\left|\frac{G(z+h,\zeta)-G(z,\zeta)}{h}-(-2(z-\zeta)G(z,\zeta))\right|e^{-a_n|\operatorname{Re}(\zeta)|^\beta}\\ &=\left|\left(\frac{e^{-2(z-\zeta)h-h^2}}{h}-\frac{1}{h}+2(z-\zeta)\right)G(z,\zeta)\right|e^{-a_n|\operatorname{Re}(\zeta)|^\beta}\\ &=\left|\left(-h+h\sum_{k=2}^\infty\frac{1}{k!}(-2(z-\zeta)-h)^k h^{k-2}\right)G(z,\zeta)\right|e^{-a_n|\operatorname{Re}(\zeta)|^\beta}\\ &\le|h|\left(1+\sum_{k=2}^\infty\frac{1}{k!}(2|z-\zeta|+1)^k\right)|G(z,\zeta)|e^{-a_n|\operatorname{Re}(\zeta)|^\beta}\\ &\le|h|e^{2|z-\zeta|+1}e^{-\operatorname{Re}((z-\zeta)^2)}e^{-a_n|\operatorname{Re}(\zeta)|^\beta}\\ &\le|h|e^{2|z|+2|\zeta_2|+2|\zeta_1|+1}e^{-(z_1-\zeta_1)^2+(z_2-\zeta_2)^2}e^{-a_n|\zeta_1|^\beta}\\ &\le|h|e^{2|z|+(2/n)+1-z_1^2+(|z_2|+1/n)^2+|a_n|}e^{-\zeta_1^2+(2|z_1|+2+|a_n|)|\zeta_1|}\\ &\le|h|e^{2|z|+(2/n)+1-z_1^2+(|z_2|+1/n)^2+|a_n|}e^{-(|z_1|+1+|a_n|/2)^2+(2|z_1|+2+|a_n|)(|z_1|+1+|a_n|/2)}=:|h|C_0,\end{aligned}$$

yielding to

$$\left\|\frac{G(z+h,\cdot)-G(z,\cdot)}{h}-(-2(z-\cdot))G(z,\cdot)\right\|_n\le|h|C_0\underset{h\to0}{\to}0.\tag{9}$$

We conclude that $\frac{\partial}{\partial z}G(z,\cdot)=-2(z-\cdot)G(z,\cdot)\in\mathcal{O}_{a_n}^{-\beta}\left(\overline{U_n(K)}\right)$ (inequality above and triangle inequality) holds.

For $z\in\mathbb{C}\setminus K$ and $\zeta\in\mathbb{C}\setminus\{z\}$ we define

$$g(z,\zeta):=\frac{G(z,\zeta)}{z-\zeta}=\frac{e^{-(z-\zeta)^2}}{z-\zeta}$$

and note that $g(z,\cdot)\in\mathcal{O}(\mathbb{C}\setminus\{z\})$. For $z\in\mathbb{C}\setminus K$ there is $k=k(z)\in\mathbb{N}$ such that

$$\mathrm{d}_k:=\mathrm{d}(\mathbb{B}_{1/k}(z),\overline{U_k(K)})>0$$

and we obtain

$$\|g(w,\cdot)\|_k=\sup_{\zeta\in\overline{U_k(K)}}\frac{1}{|w-\zeta|}|G(w,\zeta)|e^{-a_k|\operatorname{Re}(\zeta)|^\beta}\le\frac{1}{\mathrm{d}(\{w\},\overline{U_k(K)})}\|G(w,\cdot)\|_k\underset{(8)}{<}\infty$$

for all $w \in \mathbb{B}_{1/k}(z)$. We deduce that $g(w,\cdot) \in \mathcal{O}^{-\beta}_{a_k}\left(\overline{U_k(K)}\right)$ for all $w \in \mathbb{B}_{1/k}(z)$. Moreover, we observe that

$$\frac{\partial}{\partial z} g(z,\zeta) = \frac{\frac{\partial}{\partial z}G(z,\zeta)}{z-\zeta} - \frac{G(z,\zeta)}{(z-\zeta)^2} = -\left(2 + \frac{1}{(z-\zeta)^2}\right)G(z,\zeta)$$

for all $\zeta \in \overline{U_k(K)}$. Let $h \in \mathbb{C}$ with $0 < |h| < 1/k$. Then

$$\left|\frac{1}{z+h-\zeta} - \frac{1}{z-\zeta}\right| = \left|\frac{-h}{(z+h-\zeta)(z-\zeta)}\right| \leq \frac{|h|}{\mathrm{d}_k^2}$$

and

$$\left|\frac{1}{h}\left(\frac{1}{z+h-\zeta} - \frac{1}{z-\zeta}\right) + \frac{1}{(z-\zeta)^2}\right| = \left|\frac{h}{(z+h-\zeta)(z-\zeta)^2}\right| \leq \frac{|h|}{\mathrm{d}_k^3}$$

for all $\zeta \in \overline{U_k(K)}$. It follows that

$$\begin{aligned}
&\left|\frac{g(z+h,\zeta)-g(z,\zeta)}{h} - \frac{\partial}{\partial z}g(z,\zeta)\right| \\
&\quad \leq \frac{1}{|z+h-\zeta|}\left|\frac{G(z+h,\zeta)-G(z,\zeta)}{h} - \frac{\partial}{\partial z}G(z,\zeta)\right| + \left|\frac{\partial}{\partial z}G(z,\zeta)\right|\left|\frac{1}{z+h-\zeta} - \frac{1}{z-\zeta}\right| \\
&\qquad + |G(z,\zeta)|\left|\frac{1}{h}\left(\frac{1}{z+h-\zeta} - \frac{1}{z-\zeta}\right) + \frac{1}{(z-\zeta)^2}\right|
\end{aligned}$$

for all $\zeta \in \overline{U_k(K)}$, which implies

$$\begin{aligned}
&\left\|\frac{g(z+h,\cdot)-g(z,\cdot)}{h} - \frac{\partial}{\partial z}g(z,\cdot)\right\|_k \\
&\quad \leq \frac{1}{\mathrm{d}_k}\left\|\frac{G(z+h)-G(z)}{h} - \frac{\partial}{\partial z}G(z,\cdot)\right\|_k + \left\|\frac{\partial}{\partial z}G(z,\cdot)\right\|_k \frac{|h|}{\mathrm{d}_k^2} + \|G(z)\|_k \frac{|h|}{\mathrm{d}_k^3}.
\end{aligned}$$

We conclude that $\frac{\partial}{\partial z}g(z,\cdot) \in \mathcal{O}^{-\beta}_{a_k}\left(\overline{U_k(K)}\right)$ and $\frac{g(z+h,\cdot)-g(z,\cdot)}{h}$ converges to $\frac{\partial}{\partial z}g(z,\cdot)$ in the space $\mathcal{O}^{-\beta}_{a_k}\left(\overline{U_k(K)}\right)$ as $h \to 0$ by (9). Hence for all $T \in L(\mathcal{O}^{-\beta}_{(a_n)}(K), E)$ the limit

$$\lim_{\substack{h\to 0\\ h\in\mathbb{C},\, h\neq 0}} \frac{\langle T, g(z+h,\cdot)\rangle - \langle T, g(z,\cdot)\rangle}{h} = \left\langle \lim_{\substack{h\to 0\\ h\in\mathbb{C},\, h\neq 0}} \frac{g(z+h,\cdot)-g(z,\cdot)}{h}\right\rangle = \left\langle T, \frac{\partial}{\partial z}g(z,\cdot)\right\rangle$$

exists in E, meaning that $(z \mapsto \frac{1}{2\pi i}\langle T, \frac{e^{(z-\cdot)^2}}{z-\cdot}\rangle) \in \mathcal{O}(\mathbb{C}\setminus K, E)$.

Let us turn to the continuity of Θ_K. Let $n \in \mathbb{N}$. We note that for $\zeta_1, z_1 \in \mathbb{R}$

$$-a_{2n}|\zeta_1|^\beta + a_n|z_1|^\beta \leq |a_{2n}||z_1-\zeta_1|^\beta \leq |a_{2n}|(1+|z_1-\zeta_1|).$$

It follows that

$$\begin{aligned}\sup_{z\in S_n(K)}\|G(z,\cdot)\|_{K,2n}e^{a_n|\operatorname{Re}(z)|^\beta} &= \sup_{z\in S_n(K)}\sup_{\zeta\in\overline{U_{2n}(K)}} e^{-\operatorname{Re}((z-\zeta)^2)}e^{-a_{2n}|\operatorname{Re}(\zeta)|^\beta}e^{a_n|\operatorname{Re}(z)|^\beta}\\ &\le e^{(n+1/(2n))^2}\sup_{z_1\in\mathbb{R}}\sup_{\zeta_1\in\mathbb{R}} e^{-(z_1-\zeta_1)^2-a_{2n}|\zeta_1|^\beta+a_n|z_1|^\beta}\\ &\le e^{(n+1/(2n))^2+|a_{2n}|}\sup_{z_1\in\mathbb{R}}\sup_{\zeta_1\in\mathbb{R}} e^{-(z_1-\zeta_1)^2+|a_{2n}||z_1-\zeta_1|} \qquad (10)\\ &\le e^{(n+1/(2n))^2+|a_{2n}|}\sup_{x\in\mathbb{R}} e^{-x^2+|a_{2n}|x}\\ &= e^{(n+1/(2n))^2+|a_{2n}|}e^{-(a_{2n}/2)^2+a_{2n}^2/2},\end{aligned}$$

which yields, in particular, that $G(S_n(K),\cdot)\subset\mathcal{O}^{-\beta}_{a_{2n}}\left(\overline{U_{2n}(K)}\right)$. Moreover, there is $k\in\mathbb{N}$ such that

$$\mathrm{D}_{n,k} := \mathrm{d}(S_n(K),\overline{U_k(K)})>0.$$

Again, it follows that $g(S_n(K),\cdot)\subset\mathcal{O}^{-\beta}_{a_m}\left(\overline{U_m(K)}\right)$ with $m:=\max(2n,k)$. Furthermore, we observe that $M:=\{g(z,\cdot)e^{a_n|\operatorname{Re}(z)|^\beta}\mid z\in S_n(K)\}\subset\mathcal{O}^{-\beta}_{a_m}\left(\overline{U_m(K)}\right)$ and

$$\sup_{\varphi\in M}\|\varphi\|_m = \sup_{z\in S_n(K)}\|g(z,\cdot)\|_{K,m}e^{a_n|\operatorname{Re}(z)|^\beta} \le \frac{1}{\mathrm{D}_{n,k}}\sup_{z\in S_n(K)}\|G(z,\cdot)\|_{K,2n}e^{a_n|\operatorname{Re}(z)|^\beta} \underset{(10)}{<} \infty,$$

showing that M is bounded in $\mathcal{O}^{-\beta}_{(a_n)}(K)$ by [25, Proposition 25.19, p. 303]. For every $\alpha\in\mathfrak{A}$ and $T\in L(\mathcal{O}^{-\beta}_{(a_n)}(K),E)$ we have

$$|\Theta_K(T)|^\wedge_{n,\alpha} \le \left| z\mapsto \frac{1}{2\pi i}\langle T, g(z,\cdot)\rangle\right|_{n,\alpha} = \frac{1}{2\pi}\sup_{\varphi\in M} p_\alpha(T(\varphi))$$

and therefore the map $\Theta_K : L_b(\mathcal{O}^{-\beta}_{(a_n)}(K),E)\to\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$ is well-defined, clearly linear and continuous. □

The map Θ_K is sometimes called (weighted) Cauchy transformation for obvious reasons (see [26, p. 84]).

Theorem 11 *If $K\subset\mathbb{R}$ or $K\cap\{\pm\infty\}$ has no isolated points in K, then the map*

$$H_K : \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)\to L_b(\mathcal{O}^{-\beta}_{(a_n)}(K),E)$$

is a topological isomorphism with inverse Θ_K.

Proof As before we omit the index K of H_K and Θ_K if it is not necessary. As a consequence of Proposition 9 and Proposition 10 the maps H and Θ are linear and continuous. First, we prove that $\Theta\circ H = \mathrm{id}$ on $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$, which implies the injectivity of H. Let $p\in\mathbb{N}$, $p\ge 2$. We choose $n\in\mathbb{N}$ such that $\mathrm{d}(S_p(K),\overline{U_n(K)})>0$. We define the path $\Gamma_p := \Gamma_- - \Gamma_+$ with

$$\Gamma_\pm : \mathbb{R}\to\mathbb{C},\ \Gamma_\pm(t) := t\pm ip,$$

Further, we choose $m \in \mathbb{N}$ such that $1/m < \min_{1\leq j\leq k} r_j < 1/n$ and $m > p$ where $r = (r_j)_{1\leq j\leq k}$ is from the path $\gamma_{K,n,r}$ in the definition of H. Due to this choice $\Gamma_{\pm}$ and $\gamma_{K,n,r}$ are within $S_m(K)$.

Let $f = [F] \in \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}, E)$ and $z = x + iy \in S_p(K)$. Let $u \in \mathbb{R}$, $u \neq x$, and $[t_0, t_1] \subset [-p, p]$ such that the path $\gamma_u : [t_0, t_1] \to \mathbb{C}$, $\gamma_u(t) := u + it$, is within $S_m(K)$. The map $\zeta \mapsto F(\zeta)\frac{e^{-(z-\zeta)^2}}{z-\zeta}$ is holomorphic on $\mathbb{C} \setminus \{z\}$ with values in E and like in Proposition 8 (a) and (b) we deduce that it is Pettis-integrable along γ_u and $\Gamma_{\pm|[s_0,s_1]}$ with $[s_0, s_1] \subset \mathbb{R}$ using [5, Proposition 2, p. 354] and the Mackey-Arens theorem. Then we have

$$\begin{aligned}
&\left|\left\langle e', \int_{\gamma_u} F(\zeta)\frac{e^{-(z-\zeta)^2}}{\zeta - z}\mathrm{d}\zeta \right\rangle\right| \\
&\quad\leq |e'\circ F|_m \int_{t_0}^{t_1} |e^{-(z-(u+it))^2}| e^{-a_m|\operatorname{Re}(u+it)|^{\beta}} \frac{1}{|z-u-it|}\mathrm{d}t \\
&\quad\leq |e'\circ F|_m (t_1 - t_0) e^{-(x-u)^2+(y-t)^2} e^{-a_m|u|^{\beta}} \frac{1}{|x-u|} \\
&\quad\leq \frac{1}{|x-u|}|e'\circ F|_m (t_1 - t_0) e^{(y-t)^2-x^2+|a_m|} \sup_{w\in\mathbb{R}} e^{-w^2+(2|x|+|a_m|)w} \\
&\quad= \frac{1}{|x-u|}|e'\circ F|_m (t_1 - t_0) e^{(y-t)^2-x^2+|a_m|} e^{-(|x|+|a_m|/2)^2+(2|x|+|a_m|)(|x|+|a_m|/2)} \underset{|u|\to\infty}{\to} 0
\end{aligned}$$

for all $e' \in E'$. Hence we derive from Cauchy's integral formula that

$$\langle e', F(z)\rangle = \frac{1}{2\pi i}\int_{\Gamma_p-\gamma_{K,n,r}} \left\langle e', F(\zeta)\frac{e^{-(z-\zeta)^2}}{\zeta - z}\right\rangle \mathrm{d}\zeta = -\frac{1}{2\pi i}\int_{\Gamma_p-\gamma_{K,n,r}} \left\langle e', F(\zeta)\frac{e^{-(z-\zeta)^2}}{z-\zeta}\right\rangle \mathrm{d}\zeta$$

for all $e' \in E'$ and $z \in S_p(K)$. Thus we have

$$F(z) = -\frac{1}{2\pi i}\int_{\Gamma_p-\gamma_{K,n,r}} F(\zeta)\frac{e^{-(z-\zeta)^2}}{z-\zeta}\mathrm{d}\zeta$$

for all $z \in S_p(K)$. By (the proof) of Proposition 10 the function $g(z, \cdot) = \frac{e^{-(z-\cdot)^2}}{z-\cdot} \in \mathcal{O}^{-\beta}_{(a_n)}(K)$ for all $z \in \mathbb{C} \setminus K$ and

$$W : \mathbb{C} \setminus K \to E,\ W(z) := \frac{1}{2\pi i}H([F])\left(\frac{e^{-(z-\cdot)^2}}{z-\cdot}\right) - F(z),$$

is an element of $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)$ since $T := H([F]) \in L(\mathcal{O}^{-\beta}_{(a_n)}(K), E)$ by Proposition 9. It follows that

$$\begin{aligned}
W(z) &= \frac{1}{2\pi i}\int_{\gamma_{K,n,r}} F(\zeta)\frac{e^{-(z-\zeta)^2}}{z-\zeta}\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{\Gamma_p-\gamma_{K,n,r}} F(\zeta)\frac{e^{-(z-\zeta)^2}}{z-\zeta}\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_{\Gamma_p} F(\zeta)\frac{e^{-(z-\zeta)^2}}{z-\zeta}\mathrm{d}\zeta =: W_p(z)
\end{aligned} \tag{11}$$

for all $z \in S_p(K)$. But the right-hand side W_p of (11), as a function in z, is weakly holomorphic on $S_p(\varnothing) = \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < p\}$, which follows from

$$\left\langle e', W_p(z)\right\rangle = \left\langle e', \frac{1}{2\pi i}\int_{\Gamma_p} F(\zeta)\frac{e^{-(z-\zeta)^2}}{z-\zeta}\mathrm{d}\zeta\right\rangle = \frac{1}{2\pi i}\int_{\Gamma_p} e'(F(\zeta))\frac{e^{-(z-\zeta)^2}}{z-\zeta}\mathrm{d}\zeta, \quad e'\in E',$$

and differentiation under the integral sign. The weak holomorphy and the sequential completeness of E imply that W_p is holomorphic on $S_p(\varnothing)$ by [10, Corollary 2, p. 404]. Thus W is extended by W_p to a function in $\mathcal{O}(\mathbb{C},E)$ and the extensions for each $p\in\mathbb{N}$ coincide because of the identity theorem. We denote this extension by W as well.

Then we have for $z=x+iy\in\overline{S_{1/n}} := \{w\in\mathbb{C} \mid |\operatorname{Im}(w)|\leq(1/n)\}\subset S_{2n}(\varnothing)$

$$\begin{aligned}
&2\pi|\langle e', W(z)\rangle|\\
&= 2\pi|\langle e', W_{2n}(z)\rangle| = \left|\int_{\Gamma_{2n}} e'(F(\zeta))\frac{e^{-(z-\zeta)^2}}{z-\zeta}\mathrm{d}\zeta\right|\\
&\leq \int_{-\infty}^{\infty}|e'(F(t-2ni))|\frac{|e^{-(z-(t-2ni))^2}|}{|z-(t-2ni)|}\mathrm{d}t + \int_{-\infty}^{\infty}|e'(F(t+2ni))|\frac{|e^{-(z-(t+2ni))^2}|}{|z-(t+2ni)|}\mathrm{d}t\\
&\leq \left(\frac{1}{|y+2n|}+\frac{1}{|y-2n|}\right)\max\int_{-\infty}^{\infty}|e^{-(z-(t\pm 2ni))^2}|e^{-a_{4n}|\operatorname{Re}(z-(t\pm 2ni))|^{\beta}}\mathrm{d}t\,|e'\circ F|_{K,4n}\\
&\leq 2\frac{1}{2n-\frac{1}{n}}\max\int_{-\infty}^{\infty}|e^{-(z-(t\pm 2ni))^2}|e^{-a_{4n}|\operatorname{Re}(z-(t\pm 2ni))|^{\beta}}\mathrm{d}t\,|e'\circ F|_{K,4n}.
\end{aligned}$$

Moreover, in combination with the estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty}|e^{-(z-(t\pm 2ni))^2}|e^{-a_{4n}|\operatorname{Re}(z-(t\pm 2ni))|^{\beta}}e^{a_n|\operatorname{Re}(z)|^{\beta}}\mathrm{d}t\\
&= \int_{-\infty}^{\infty}e^{-\operatorname{Re}((z-(t\pm 2ni))^2)}e^{a_n|x|^{\beta}-a_{4n}|t|^{\beta}}\mathrm{d}t \leq e^{(y\mp 2n)^2}\int_{-\infty}^{\infty}e^{-(x-t)^2}e^{|a_{4n}||x-t|^{\beta}}\mathrm{d}t\\
&\leq e^{((1/n)+2n)^2+|a_{4n}|}\int_{-\infty}^{\infty}e^{-(x-t)^2}e^{|a_{4n}||x-t|}\mathrm{d}t = 2e^{((1/n)+2n)^2+|a_{4n}|+a_{4n}^2/4}\int_{-|a_{4n}|/2}^{\infty}e^{-t^2}\mathrm{d}t\\
&\leq 2\sqrt{\pi}e^{((1/n)+2n)^2+|a_{4n}|+a_{4n}^2/4} =:\ C
\end{aligned}$$

we get for all $\alpha\in\mathfrak{A}$

$$\sup_{z\in\overline{S_{1/n}}}p_\alpha(W(z))e^{a_n|\operatorname{Re}(z)|^{\beta}} = \sup_{e'\in B_\alpha^\circ}\sup_{\substack{0\leq|y|\leq\frac{1}{n}\\ x\in\mathbb{R}}}|\langle e', W(x+iy)\rangle|e^{a_n|\operatorname{Re}(x+iy)|^{\beta}} \leq \frac{C|F|_{K,4n,\alpha}}{\pi\left(2n-\frac{1}{n}\right)},$$

yielding to

$$|W|_{\varnothing,n,\alpha} = \sup_{z\in S_n(\varnothing)}p_\alpha(W(z))e^{a_n|\operatorname{Re}(z)|^{\beta}} \leq \max\left(|W|_{K,n,\alpha}, \sup_{z\in\overline{S_{1/n}}}p_\alpha(W(z))e^{a_n|\operatorname{Re}(z)|^{\beta}}\right) < \infty.$$

Hence $W\in\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)$ and thus

$$(\Theta\circ H)(f) = \left[z \mapsto \frac{1}{2\pi i} H([F])\left(\frac{e^{-(z-\cdot)^2}}{z-\cdot}\right) - F(z)\right] + f = [W] + f = f,$$

i.e. H is injective.

Second, we prove that $H\circ\Theta =$ id on $L(\mathcal{O}^{-\beta}_{(a_n)}(K), E)$, which implies the surjectivity of H. Due to the Hahn-Banach theorem this is equivalent to the condition that

$$e'((H\circ\Theta)(T)(\varphi)) = e'(T(\varphi))$$

holds for all $T \in L(\mathcal{O}^{-\beta}_{(a_n)}(K), E)$, $\varphi \in \mathcal{O}^{-\beta}_{(a_n)}(K)$ and $e' \in E'$. As

$$e'((H\circ\Theta)(T)(\varphi)) = (H\circ\Theta)(e'\circ T)(\varphi)$$

for all $T \in L(\mathcal{O}^{-\beta}_{(a_n)}(K), E)$, $\varphi \in \mathcal{O}^{-\beta}_{(a_n)}(K)$ and $e' \in E'$, it suffices to show the result for $E = \mathbb{C}$.

Since the span of the set of point evaluations of complex derivatives $\{\delta^{(n)}_{x_0} \mid x_0 \in K\cap\mathbb{R},\, n \in \mathbb{N}_0\}$ is dense in $\mathcal{O}^{-\beta}_{(a_n)}(K)'_b$ by virtue of Proposition 4 (b), we only need to show that $(H\circ\Theta)(\delta^{(n)}_{x_0})(\varphi) = \langle\delta^{(n)}_{x_0}, \varphi\rangle$ for all $x_0 \in K\cap\mathbb{R}$, $n \in \mathbb{N}_0$ and $\varphi \in \mathcal{O}^{-\beta}_{(a_n)}(K)$. Let $x_0 \in K\cap\mathbb{R}$ and $n \in \mathbb{N}_0$. Now, we have

$$(H\circ\Theta)(\delta^{(n)}_{x_0})(\varphi) = \frac{1}{2\pi i}\int_{\gamma_{K,k,r}} \left\langle \delta^{(n)}_{x_0}, \frac{e^{-(z-\cdot)^2}}{z-\cdot}\right\rangle \varphi(z)\mathrm{d}z \tag{12}$$

for all $\varphi \in \mathcal{O}^{-\beta}_{a_k}\left(\overline{U_k(K)}\right)$, $k \in \mathbb{N}$. Let us take a closer look at the integral on the right-hand side of (12). Let $m \in \mathbb{N}$, $m \geq 2$. Then $\frac{e^{-(z-\cdot)^2}}{z-\cdot} \in \mathcal{O}(\mathbb{B}_{1/m}(x_0))$ for every $z \in S_m(\{x_0\})$. We fix the notation $g_z(\zeta) := \frac{e^{-(z-\zeta)^2}}{z-\zeta}$ for $z \in S_m(\{x_0\})$ and $\zeta \in \mathbb{B}_{1/m}(x_0)$. Then we get by Cauchy's inequality

$$|g^{(n)}_z(x_0)| \leq n!(2m)^n \max_{\zeta\in\partial\mathbb{B}_{1/(2m)}(x_0)} \left|\frac{e^{-(z-\zeta)^2}}{z-\zeta}\right| \leq n!(2m)^{n-1} \max_{\zeta\in\partial\mathbb{B}_{1/(2m)}(x_0)} |e^{-(z-\zeta)^2}|$$

for every $z \in S_m(\{x_0\})$. We deduce that

$$\begin{aligned}
&\sup_{z\in S_m(\{x_0\})} |g^{(n)}_z(x_0)| e^{a_m|\operatorname{Re}(z)|^\beta} \\
&\leq n!(2m)^{n-1} \sup_{z\in S_m(\{x_0\})} \max_{\zeta\in\partial\mathbb{B}_{1/(2m)}(x_0)} |e^{-(z-\zeta)^2}| e^{a_m|\operatorname{Re}(z)|^\beta} \\
&\leq n!(2m)^{n-1} \sup_{z\in S_m(\{x_0\})} \max_{\zeta\in\partial\mathbb{B}_{1/(2m)}(x_0)} |e^{-(z-\zeta)^2}| e^{-a_{2m}|\operatorname{Re}(\zeta)|^\beta} e^{a_{2m}|\operatorname{Re}(\zeta)|^\beta} e^{a_m|\operatorname{Re}(z)|^\beta} \\
&\leq n!(2m)^{n-1} \sup_{\zeta\in\partial\mathbb{B}_{1/(2m)}(x_0)} e^{a_{2m}|\operatorname{Re}(\zeta)|^\beta} \sup_{z\in S_m(\{x_0\})} \|e^{-(z-\cdot)^2}\|_{\{x_0\},2m} e^{a_m|\operatorname{Re}(z)|^\beta} \underset{(10)}{<} \infty,
\end{aligned}$$

implying $(z \mapsto \langle\delta^{(n)}_{x_0}, \frac{e^{-(z-\cdot)^2}}{z-\cdot}\rangle) \in \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus \{x_0\})$. This means that the path of the integral on the right-hand side of (12) can be deformed using Cauchy's integral theorem (like in Proposition 8 (a) and (b)) and we get with $s := \min_j r_j > 0$ for $r = (r_j)$

$$\frac{1}{2\pi i}\int_{\gamma_{K,k,r}}\left\langle \delta^{(n)}_{x_0}, \frac{e^{-(z-\cdot)^2}}{z-\cdot}\right\rangle\varphi(z)\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\mathbb{B}_s(x_0)}\left\langle \delta^{(n)}_{x_0}, \frac{e^{-(z-\cdot)^2}}{z-\cdot}\right\rangle\varphi(z)\mathrm{d}z$$
$$= \frac{1}{2\pi i}\int_{\partial\mathbb{B}_s(x_0)} g^{(n)}_z(x_0)\varphi(z)\mathrm{d}z$$

for all $\varphi \in \mathcal{O}^{-\beta}_{a_k}\left(\overline{U_k(K)}\right)$, $k\in\mathbb{N}$. The Laurent series of $\frac{e^{-(z-\cdot)^2}}{z-\cdot}$ in $\zeta \neq z$ is

$$\frac{e^{-(z-\zeta)^2}}{z-\zeta} = \frac{1}{z-\zeta} + \sum_{j=1}^{\infty}\frac{(-1)^j}{j!}(z-\zeta)^{2j-1}$$

and so we have for the nth complex derivative of g_z at x_0

$$g^{(n)}_z(x_0) = \frac{n!}{(z-x_0)^{n+1}} + h(z,x_0)$$

with an entire function $h(\cdot,x_0)$. By Cauchy's integral theorem and Cauchy's integral formula for derivatives we have

$$\begin{aligned}(H\circ\Theta)(\delta^{(n)}_{x_0})(\varphi) &= \frac{1}{2\pi i}\int_{\partial\mathbb{B}_s(x_0)} g^{(n)}_z(x_0)\varphi(z)\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{\partial\mathbb{B}_s(x_0)}\left(\frac{n!}{(z-x_0)^{n+1}} + h(z,x_0)\right)\varphi(z)\mathrm{d}z\\ &= \frac{n!}{2\pi i}\int_{\partial\mathbb{B}_s(x_0)}\frac{\varphi(z)}{(z-x_0)^{n+1}}\mathrm{d}z = \varphi^{(n)}(x_0) = \langle \delta^{(n)}_{x_0},\varphi\rangle\end{aligned}$$

for all $\varphi \in \mathcal{O}^{-\beta}_{a_k}\left(\overline{U_k(K)}\right)$, $k\in\mathbb{N}$. □

Remark 12 If $K\subset\mathbb{R}$, then Theorem 11 is also valid for locally complete E because Proposition 8 still holds due to [5, Proposition 2, p. 354].

If $K\cap\{\pm\infty\}$ has isolated points in K, e.g. $K=\{+\infty\}$, then we cannot apply the preceding theorem directly since a counterpart for Proposition 4 (b) is missing. However, we can make use of the relation (5) if $\mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{R}})$ is dense in $\mathcal{O}^{-\beta}_{(a_n)}(K)$.

Corollary 13 *If $\mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{R}})$ dense in $\mathcal{O}^{-\beta}_{(a_n)}(K)$, then the map*

$$H_K : \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E) \to L_b(\mathcal{O}^{-\beta}_{(a_n)}(K),E)$$

is a topological isomorphism with inverse Θ_K and

$$\Theta_K(T) = \Theta_{\overline{\mathbb{R}}}(T), \quad T\in L(\mathcal{O}^{-\beta}_{(a_n)}(K),E). \tag{13}$$

Proof H_K and Θ_K are well-defined, linear and continuous maps by Proposition 9 and Proposition 10. $H_{\overline{\mathbb{R}}}$ is a topological isomorphism with inverse $\Theta_{\overline{\mathbb{R}}}$ by Theorem 11. The embedding of $\mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{R}})$ into $\mathcal{O}^{-\beta}_{(a_n)}(K)$ is continuous and dense, hence defines the embedding of $L(\mathcal{O}^{-\beta}_{(a_n)}(K),E)$ into $L(\mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{R}}),E)$ (the density of the first embedding implies the injectivity of the latter one) and we have

$$\Theta_K(T) = \Theta_{\overline{\mathbb{R}}}(T), \quad T \in L(\mathcal{O}^{-\beta}_{(a_n)}(K), E),$$

by the definition of Θ_K and $\Theta_{\overline{\mathbb{R}}}$. Furthermore, it follows from (5) that

$$H_{\overline{\mathbb{R}}|\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K,E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}},E)} = H_K$$

on $\mathcal{O}^{-\beta}_{(a_n)}(\overline{\mathbb{R}})$. We conclude for every $f \in \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}, E)$ that

$$(\Theta_K \circ H_K)(f) = \Theta_{\overline{\mathbb{R}}}(H_K(f)) = \Theta_{\overline{\mathbb{R}}}(H_{\overline{\mathbb{R}}}(f)) = f$$

and for every $T \in L(\mathcal{O}^{-\beta}_{(a_n)}(K), E)$ that

$$(H_K \circ \Theta_K)(T) = H_{\overline{\mathbb{R}}}(\Theta_K(T)) = H_{\overline{\mathbb{R}}}(\Theta_{\overline{\mathbb{R}}}(T)) = T$$

by Theorem 11. Thus H_K is bijective and Θ_K its inverse. □

Remark 14 Under the conditions of Theorem 11 resp. Corollary 13 it follows that $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}, E)$ is Hausdorff since E and thus $L_b(\mathcal{O}^{-\beta}_{(a_n)}(K), E)$ is Hausdorff. In particular, $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}, E)$ is closed in $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)$ by [25, Lemma 22.9, p. 254].

Corollary 15 *If $(a_n)_{n\in\mathbb{N}}$ is strictly increasing, $a_n < 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$, then the map*

$$H_K : \mathcal{O}^{1}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)/\mathcal{O}^{1}_{(a_n)}(\overline{\mathbb{C}}, E) \to L_b(\mathcal{O}^{-1}_{(a_n)}(K), E)$$

is a topological isomorphism with inverse Θ_K.

Proof We only need to prove that the condition of Corollary 13 is fulfilled. Due to [17, Theorem 2.2.1, p. 474] (and its correction in [28, Remark, p. 247-248]) the space $\mathcal{O}^{-1}_{(a_n)}(\overline{\mathbb{R}})$ is dense in $\mathcal{O}^{-1}_{(a_n)}(K)$ (where $\mathcal{O}^{-1}_{(a_n)}(\overline{\mathbb{R}})$ is called $\mathscr{P}_*$). □

The isomorphy $\mathcal{O}^{1}_{(a_n)}(\overline{\mathbb{C}} \setminus K, E)/\mathcal{O}^{1}_{(a_n)}(\overline{\mathbb{C}}, E) \cong L_b(\mathcal{O}^{-1}_{(a_n)}(K), E)$ in Corollary 15 is already known for special cases like $E = \mathbb{C}$ [17, Theorem 3.2.1, p. 480] and Fréchet spaces E [15, 3.9 Satz, p. 41] but the proof is of homological nature. In the special case $K = [a, \infty]$, $a \in \mathbb{R}$, and $E = \mathbb{C}$ the duality was proved in [26, Theorem 3.3, p. 85-86] and served as an initial point to prove Corollary 15 for complete E in [19, 4.1 Theorem, p. 41].

4 (Ω) for $\mathcal{O}^{\beta}_{(a_n)}$-spaces on strips with holes

In this section we derive sufficient conditions on K, $(a_n)_{n\in\mathbb{N}}$ and β such that $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)$ satisfies (Ω). The basic idea is to prove that the strong dual $\mathcal{O}^{-\beta}_{(a_n)}(K)'_b$ satisifies (Ω). Then we use the duality $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}) \cong \mathcal{O}^{-\beta}_{(a_n)}(K)'_b$ from the preceding section to obtain (Ω) for $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)$. Let us recall that a Fréchet space F with an increasing fundamental system of seminorms $(|||\cdot|||_k)_{k\in\mathbb{N}}$ satisfies (Ω) by [25, Chap. 29, Definition, p. 367] if

$$\forall\, p \in \mathbb{N}\ \exists\, q \in \mathbb{N}\ \forall\, k \in \mathbb{N}\ \exists\, n \in \mathbb{N},\ C > 0\ \forall\, r > 0 :\ U_q \subset Cr^n U_k + \frac{1}{r} U_p$$

where $U_k := \{x \in F \mid |||x|||_k \le 1\}$.

We start with a helpful observation concerning the inductive limit $\mathcal{O}^{-\beta}_{(a_n)}(K)$, namely, that the choice of the sequence $(1/n)_{n\in\mathbb{N}}$ for the neighbourhoods $U_n(K) = U_{1/(1/n)}(K)$ is irrelevant.

Remark 16 Let $(c_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence in $\mathbb{R}$ with $c_n \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = 0$. For $n \in \mathbb{N}$ let

$$\mathcal{O}^{-\beta}_{a_n}\left(\overline{U_{1/c_n}(K)}\right) := \{f \in \mathcal{O}(U_{1/c_n}(K)) \cap \mathcal{C}\left(\overline{U_{1/c_n}(K)}\right) \mid \|f\|_{n,1/c_n} < \infty\}$$

where

$$\|f\|_{n,1/c_n} := \sup_{z\in\overline{U_{1/c_n}(K)}} |f(z)|e^{-a_n|\operatorname{Re}(z)|^\beta}$$

and the spectral maps for $n, k \in \mathbb{N}$, $n \leq k$, be given by the restrictions

$$\widetilde{\pi}_{n,k} : \mathcal{O}^{-\beta}_{a_n}\left(\overline{U_{1/c_n}(K)}\right) \to \mathcal{O}^{-\beta}_{a_k}\left(\overline{U_{1/c_k}(K)}\right),\ \widetilde{\pi}_{n,k}(f) := f_{|U_{1/c_k}(K)}.$$

Then

$$\mathcal{O}^{-\beta}_{(a_n)}(K) \cong \varinjlim_{n\in\mathbb{N}} \mathcal{O}^{-\beta}_{a_n}\left(\overline{U_{1/c_n}(K)}\right).$$

Proof It follows directly from Proposition 4 (a) and [9, 4.2 Satz, p. 122]. □

We recall an equivalent description of the property (Ω). By [25, Lemma 29.13, p. 369] a Fréchet space F with an increasing fundamental system of seminorms $(|||\cdot|||_k)_{k\in\mathbb{N}}$ satisfies (Ω) if and only if

$$\forall\, p \in \mathbb{N}\ \exists\, q \in \mathbb{N}\ \forall\, k \in \mathbb{N}\ \exists\, 0 < \theta < 1,\ C > 0\ \forall\, y \in F' :\ \|y\|_q^* \leq C\|y\|_p^{*\,1-\theta}\|y\|_k^{*\theta} \tag{14}$$

holds where

$$\|y\|_k^* := \sup\{|y(x)| \mid |||x|||_k \leq 1\} \in \mathbb{R} \cup \{\infty\}$$

is the dual norm.

Lemma 17 *There is a strictly decreasing sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $c_n \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = 0$ such that*

$$\forall\, p, q, k \in \mathbb{N},\ p < q < k\ \exists\, C > 0\ \forall\, \zeta \in \mathbb{R},\ |\zeta| \geq 1 + c_k^{-1} :$$
$$\left(\sup_{z\in\mathbb{C},\,|z-\zeta|\leq c_k} e^{a_k|\operatorname{Re}(z)|^\beta}\right)^\theta \left(\sup_{z\in\mathbb{C},\,|z-\zeta|\leq c_p} e^{a_p|\operatorname{Re}(z)|^\beta}\right)^{1-\theta} \leq C \inf_{z\in\mathbb{C},\,|z-\zeta|\leq c_q} e^{a_q|\operatorname{Re}(z)|^\beta}$$

with $\theta := \frac{\ln(c_p/c_q)}{\ln(c_p/c_k)}$.

Proof Let $c_n := \exp(1/a_n)$ for all $n \in \mathbb{N}$ if $a_n < 0$ for all $n \in \mathbb{N}$ and $c_n := \exp(-a_n)$ for all $n \in \mathbb{N}$ if $a_n \geq 0$ for all $n \in \mathbb{N}$. Then $(c_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence, $c_n \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = 0$. Let $p, q, k \in \mathbb{N}$ such that $p < q < k$ and $\theta := \frac{\ln(c_p/c_q)}{\ln(c_p/c_k)}$. Let $\zeta \in \mathbb{R}$ with $|\zeta| \geq 1 + c_k^{-1}$. For $z \in \mathbb{C}$ with $|z - \zeta| \leq c_n$, $n \in \{p, q, k\}$, we deduce from the inequalities

$$||\zeta| - |\operatorname{Re}(z) - \zeta||^\beta \leq |\operatorname{Re}(z)|^\beta \leq (|\operatorname{Re}(z) - \zeta| + |\zeta|)^\beta \leq (c_n + |\zeta|)^\beta$$

and

$$|\zeta| - |\operatorname{Re}(z) - \zeta| \geq |\zeta| - c_n \geq 1 + c_k^{-1} - c_n \geq c_k^{-1} > 0$$

that

$$\inf_{z\in\mathbb{C},\, |z-\zeta|\leq c_q} e^{a_q|\operatorname{Re}(z)|^\beta} \geq e^{a_q(c_q+|\zeta|)^\beta}$$

and

$$\sup_{z\in\mathbb{C},\, |z-\zeta|\leq c_k} e^{\theta a_k|\operatorname{Re}(z)|^\beta} \sup_{z\in\mathbb{C},\, |z-\zeta|\leq c_p} e^{(1-\theta)a_p|\operatorname{Re}(z)|^\beta} \leq e^{\theta a_k(|\zeta|-c_k)^\beta + (1-\theta)a_p(|\zeta|-c_p)^\beta},$$

if $a_n < 0$, as well as

$$\inf_{z\in\mathbb{C},\, |z-\zeta|\leq c_q} e^{a_q|\operatorname{Re}(z)|^\beta} \geq e^{a_q(|\zeta|-c_q)^\beta}$$

and

$$\sup_{z\in\mathbb{C},\, |z-\zeta|\leq c_k} e^{\theta a_k|\operatorname{Re}(z)|^\beta} \sup_{z\in\mathbb{C},\, |z-\zeta|\leq c_p} e^{(1-\theta)a_p|\operatorname{Re}(z)|^\beta} \leq e^{\theta a_k(c_k+|\zeta|)^\beta + (1-\theta)a_p(c_p+|\zeta|)^\beta},$$

if $a_n \geq 0$. Now, we only need to prove that there is $C > 0$ such that

$$e^{\theta a_k(|\zeta|-c_k)^\beta + (1-\theta)a_p(|\zeta|-c_p)^\beta} \leq C e^{a_q(c_q+|\zeta|)^\beta}, \quad a_n < 0,$$

resp.

$$e^{\theta a_k(c_k+|\zeta|)^\beta + (1-\theta)a_p(c_p+|\zeta|)^\beta} \leq C e^{a_q(|\zeta|-c_q)^\beta}, \quad a_n \geq 0.$$

If $a_n < 0$, we observe that

$$\begin{aligned}
&\theta a_k(|\zeta| - c_k)^\beta + (1-\theta)a_p(|\zeta| - c_p)^\beta - a_q(c_q + |\zeta|)^\beta \\
&\quad \leq \theta a_k(|\zeta| - c_p)^\beta + (1-\theta)a_p(|\zeta| - c_p)^\beta - a_q(|\zeta| - c_p)^\beta - a_q(c_p + c_q)^\beta \\
&\quad = (\theta a_k + (1-\theta)a_p - a_q)(|\zeta| - c_p)^\beta - a_q(c_p + c_q)^\beta
\end{aligned}$$

and, if $a_n \geq 0$, that

$$\begin{aligned}
&\theta a_k(c_k + |\zeta|)^\beta + (1-\theta)a_p(c_p + |\zeta|)^\beta - a_q(|\zeta| - c_q)^\beta \\
&\quad \leq \theta a_k(c_p + |\zeta|)^\beta + (1-\theta)a_p(c_p + |\zeta|)^\beta - a_q\big||\zeta| + c_p|^\beta - |c_p + c_q|^\beta\big| \\
&\quad \leq (\theta a_k + (1-\theta)a_p - a_q)(c_p + |\zeta|)^\beta + a_q(c_p + c_q)^\beta.
\end{aligned}$$

What remains to be shown is that

$$0 \geq \theta a_k + (1-\theta)a_p - a_q \tag{15}$$

because then we are done with $C := \exp(|a_q|(c_p + c_q)^{\beta})$. If $a_n < 0$, then

$$\theta = \frac{\ln(c_p/c_q)}{\ln(c_p/c_k)} = \frac{(1/a_p)-(1/a_q)}{(1/a_p)-(1/a_k)} = \frac{a_k(a_q-a_p)}{a_q(a_k-a_p)}$$

and (15) is equivalent to

$$0 \geq \frac{a_k^2(a_q-a_p)}{a_q(a_k-a_p)} + \left(1 - \frac{a_k(a_q-a_p)}{a_q(a_k-a_p)}\right)a_p - a_q,$$

which holds if and only if

$$\begin{aligned} 0 &\leq a_k^2(a_q-a_p) + \big(a_q(a_k-a_p) - a_k(a_q-a_p)\big)a_p - a_q^2(a_k-a_p) \\ &= a_k^2(a_q-a_p) + (a_k-a_q)a_p^2 - a_q^2(a_k-a_q+a_q-a_p) \\ &= (a_k^2-a_q^2)(a_q-a_p) + (a_k-a_q)(a_p^2-a_q^2) \\ &= (a_k-a_q)(a_k+a_q)(a_q-a_p) - (a_k-a_q)(a_q-a_p)(a_p+a_q) \end{aligned}$$

as $a_q(a_k-a_p) < 0$. Since $a_k - a_q > 0$ and $a_q - a_p > 0$, this is equivalent to

$$0 \leq (a_k+a_q) - (a_p+a_q) = a_k - a_p,$$

which is true. If $a_n \geq 0$, then

$$\theta = \frac{\ln(c_p/c_q)}{\ln(c_p/c_k)} = \frac{a_q-a_p}{a_k-a_p}$$

and (15) is equivalent to

$$0 \geq \frac{a_q-a_p}{a_k-a_p}a_k + \left(1-\frac{a_q-a_p}{a_k-a_p}\right)a_p - a_q,$$

which holds, as $a_k - a_p > 0$, if and only if

$$\begin{aligned} 0 &\geq (a_q-a_p)a_k + \big(a_k-a_p-(a_q-a_p)\big)a_p - (a_k-a_p)a_q \\ &= a_qa_k - a_pa_k + a_ka_p - a_qa_p - a_ka_q + a_pa_q = 0. \end{aligned}$$

□

We note that θ in the lemma above fulfils $0 < \theta < 1$ and state the following improvement of [19, 5.21 Lemma, p. 88].

Lemma 18 *The following assertions hold.*

(a) $\forall\, p,q,k \in \mathbb{N},\ p<q<k\ \exists\, 0<\theta<1,\ C>0\ \forall f \in \mathcal{O}_{a_p}^{-\beta}(\overline{U_{1/c_p}(K)})$:

$$\|f\|_{q,c_q} \leq C\|f\|_{p,c_p}^{1-\theta}\|f\|_{k,c_k}^{\theta}$$

with c_n from Lemma 17 if $K \cap \{\pm\infty\} \neq \varnothing$ resp. $c_n := 1/n$, $n \in \mathbb{N}$, if $K \subset \mathbb{R}$.

(b) $\mathcal{O}^{-\beta}_{(a_n)}(K)'_b$ *satisfies* (Ω).

Proof (a) Let $p, q, k \in \mathbb{N}$, $p < q < k$, and $f \in \mathcal{O}^{-\beta}_{a_p}\left(\overline{U_{1/c_p}(K)}\right)$. Considering the components of $U_{1/c_p}(K)$ we have to distinguish three different cases.

(i) Let Z_p be a bounded component of $U_{1/c_p}(K)$. By Remark 6 (a) there are only finitely many components Z_q of $U_{1/c_q}(K)$ with $Z_q \subset Z_p$. For every such component Z_q we choose $\zeta \in Z_q \cap K$, which exists since Z_q is bounded. Let Z_k be the (unique) component of $U_{1/c_k}(K)$ which contains ζ. Z_p is a proper simply connected subset of $\mathbb{C}$. Thus there exists a biholomorphic map $\widetilde{\psi} : Z_p \to \mathbb{B}_1(0)$ with $\widetilde{\psi}(\zeta) = 0$ due to the Riemann mapping theorem (and Möbius transformation). In addition, Z_p and $\mathbb{B}_1(0)$ are Jordan domains (for the definition see [1, 2.8.5 Lemma, p. 193, 1.8.5 Jordan Curve Theorem, p. 68]) and so there exists a homeomorphism $\psi : \overline{Z_p} \to \overline{\mathbb{B}_1(0)}$ such that $\psi_{|Z_p} = \widetilde{\psi}$ by [1, 2.8.8 Theorem (Caratheodory), p. 195]. Since $\psi(Z_q) \subset \psi(Z_p) = \mathbb{B}_1(0)$ and $\psi(\overline{Z_q})$ is compact, as $\overline{Z_q}$ is compact and ψ continuous, there is $0 < r_q < 1$ such that $\psi(\overline{Z_q}) \subset \mathbb{B}_{r_q}(0)$. Moreover, there exists $0 < r_k < r_q$ such that $\overline{\mathbb{B}_{r_k}(0)} \subset \psi(Z_k)$ since $0 \in \psi(Z_k)$, $\psi(Z_k)$ is open by the open mapping theorem (from complex analysis) and $\psi(Z_k) \subset \psi(Z_q)$. The function $u := f \circ (\psi^{-1})$ is holomorphic on $\mathbb{B}_1(0)$ and continuous on $\overline{\mathbb{B}_1(0)}$, in particular, $|u|$ is subharmonic on $\mathbb{B}_1(0)$ and continuous on $\overline{\mathbb{B}_1(0)}$. Setting

$$M(r) := \sup_{|z|=r} |u(z)|, \quad 0 < r \leq 1,$$

we obtain by virtue of [1, 4.4.32 Proposition (Hadamard's Three Circles Theorem), p. 338]

$$\ln(M(r_q)) \leq \frac{\ln(1/r_q)}{\ln(1/r_k)} \ln(M(r_k)) + \frac{\ln(r_q/r_k)}{\ln(1/r_k)} \ln(M(1))$$

and hence

$$M(r_q) \leq M(r_k)^{\theta} M(1)^{1-\theta}$$

with $\theta := \frac{\ln(1/r_q)}{\ln(1/r_k)}$. We note that $0 < \theta < 1$ because $0 < r_k < r_q < 1$. By the maximum principle we have

$$\begin{aligned} M(r_q) = \sup_{|z|\leq r_q} |u(z)| &\geq \inf_{|z|\leq r_q} e^{a_q |\operatorname{Re}(\psi^{-1}(z))|^{\beta}} \sup_{|z|\leq r_q} |f(\psi^{-1}(z))| e^{-a_q |\operatorname{Re}(\psi^{-1}(z))|^{\beta}} \\ &\underset{\psi(\overline{Z_q})\subset \mathbb{B}_{r_q}(0)}{\geq} \underbrace{\inf_{|z|\leq r_q} e^{a_q |\operatorname{Re}(\psi^{-1}(z))|^{\beta}}}_{=:C_0>0} \sup_{z\in \overline{Z_q}} |f(z)| e^{-a_q |\operatorname{Re}(z)|^{\beta}} \end{aligned}$$

as well as

$$
\begin{aligned}
&M(r_k)^{\theta}M(1)^{1-\theta}\\
&\quad= \sup_{|z|\le r_k}|u(z)|^{\theta}\sup_{|z|\le 1}|u(z)|^{1-\theta}\\
&\quad\le \left(\sup_{|z|\le r_k} e^{a_k|\operatorname{Re}(\psi^{-1}(z))|^{\beta}}\right)^{\theta}\left(\sup_{|z|\le r_k}|f(\psi^{-1}(z))|e^{-a_k|\operatorname{Re}(\psi^{-1}(z))|^{\beta}}\right)^{\theta}\\
&\qquad\cdot\left(\sup_{|z|\le 1} e^{a_p|\operatorname{Re}(\psi^{-1}(z))|^{\beta}}\right)^{1-\theta}\left(\sup_{|z|\le 1}|f(\psi^{-1}(z))|e^{-a_p|\operatorname{Re}(\psi^{-1}(z))|^{\beta}}\right)^{1-\theta}\\
&\quad\underset{\mathbb{B}_{r_k}(0)\subset\psi(\overline{Z_k})}{\le}\underbrace{\left(\sup_{|z|\le r_k} e^{a_k|\operatorname{Re}(\psi^{-1}(z))|^{\beta}}\right)^{\theta}\left(\sup_{|z|\le 1} e^{a_p|\operatorname{Re}(\psi^{-1}(z))|^{\beta}}\right)^{1-\theta}}_{=:C_1}\\
&\qquad\cdot\left(\sup_{z\in\overline{Z_k}}|f(z)|e^{-a_k|\operatorname{Re}(z)|^{\beta}}\right)^{\theta}\left(\sup_{z\in\overline{Z_p}}|f(z)|e^{-a_p|\operatorname{Re}(z)|^{\beta}}\right)^{1-\theta}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\sup_{z\in\overline{Z_q}}|f(z)|e^{-a_q|\operatorname{Re}(z)|^{\beta}} &\le \frac{C_1}{C_0}\left(\sup_{z\in\overline{Z_k}}|f(z)|e^{-a_k|\operatorname{Re}(z)|^{\beta}}\right)^{\theta}\left(\sup_{z\in\overline{Z_p}}|f(z)|e^{-a_p|\operatorname{Re}(z)|^{\beta}}\right)^{1-\theta}\\
&\le \frac{C_1}{C_0}\|f\|_{k,c_k}^{\theta}\|f\|_{p,c_p}^{1-\theta}.
\end{aligned}
\tag{16}
$$

(*ii*) Let $K\cap\{\pm\infty\}\neq\emptyset$. Let Z_p be an unbounded component of $U_{1/c_p}(K)$, w.l.o.g. the real part of Z_p is bounded from below and unbounded from above. Let $\zeta\in\mathbb{R}$ such that $\zeta\ge 1+c_k^{-1}$. Then we have $\overline{\mathbb{B}_{c_j}(\zeta)}\subset([c_j^{-1},\infty)+i[-c_j,c_j])$ for $j\in\{p,q,k\}$ since $c_p^{-1}<c_q^{-1}<c_k^{-1}$ and $c_j\le 1$. Applying Hadamard's Three Circles Theorem to $u:=|f|$, we get $M(c_q)\le M(c_k)^{\theta}M(c_p)^{1-\theta}$ with $\theta:=\frac{\ln(c_p/c_q)}{\ln(c_p/c_k)}$ fulfilling $0<\theta<1$. Like in (*i*) we obtain

$$
M(c_q)\ge \inf_{|z-\zeta|\le c_q} e^{a_q|\operatorname{Re}(z)|^{\beta}}\sup_{|z-\zeta|\le c_q}|f(z)|e^{-a_q|\operatorname{Re}(z)|^{\beta}}
$$

and

$$
\begin{aligned}
M(c_k)^{\theta}M(c_p)^{1-\theta}\le &\left(\sup_{|z-\zeta|\le c_k} e^{a_k|\operatorname{Re}(z)|^{\beta}}\right)^{\theta}\left(\sup_{|z-\zeta|\le c_p} e^{a_p|\operatorname{Re}(z)|^{\beta}}\right)^{1-\theta}\\
&\left(\sup_{|z-\zeta|\le c_k}|f(z)|e^{-a_k|\operatorname{Re}(z)|^{\beta}}\right)^{\theta}\left(\sup_{|z-\zeta|\le c_p}|f(z)|e^{-a_p|\operatorname{Re}(z)|^{\beta}}\right)^{1-\theta}.
\end{aligned}
$$

Due to Lemma 17 there is $C_2>0$, independent of ζ, such that

$$
\sup_{|z-\zeta|\le c_q}|f(z)|e^{-a_q|\operatorname{Re}(z)|^{\beta}}\le C_2\left(\sup_{|z-\zeta|\le c_k}|f(z)|e^{-a_k|\operatorname{Re}(z)|^{\beta}}\right)^{\theta}\left(\sup_{|z-\zeta|\le c_p}|f(z)|e^{-a_p|\operatorname{Re}(z)|^{\beta}}\right)^{1-\theta}
$$

and thus

$$
\begin{aligned}
&\sup_{\substack{z\in\mathbb{C}\\ \mathrm{d}(\{z\},[1+c_k^{-1},\infty))\leq c_q}} |f(z)|e^{-a_q|\operatorname{Re}(z)|^\beta} = \sup_{\substack{\zeta\in\mathbb{R}\\ \zeta\geq 1+c_k^{-1}}} \sup_{|z-\zeta|\leq c_q} |f(z)|e^{-a_q|\operatorname{Re}(z)|^\beta}\\
&\leq C_2\left(\sup_{\substack{z\in\mathbb{C}\\ \mathrm{d}(\{z\},[1+c_k^{-1},\infty))\leq c_k}} |f(z)|e^{-a_k|\operatorname{Re}(z)|^\beta}\right)^{\theta}\left(\sup_{\substack{z\in\mathbb{C}\\ \mathrm{d}(\{z\},[1+c_k^{-1},\infty))\leq c_p}} |f(z)|e^{-a_p|\operatorname{Re}(z)|^\beta}\right)^{1-\theta}\\
&\leq C_2\|f\|_{k,c_k}^{\theta}\|f\|_{p,c_p}^{1-\theta}.
\end{aligned}
\tag{17}
$$

(iii) Let $K\cap\{\pm\infty\}\neq\varnothing$ and Z_p be w.l.o.g. like in (ii). We define $\widetilde{Z}_p := Z_p\cap((-\infty,1+c_k^{-1})+i\mathbb{R})$. By Remark 6 (a) there are only finitely many components $\widetilde{Z}_q$ of $U_{1/c_q}(K)\cap((-\infty,1+c_k^{-1})+i\mathbb{R})$ with $\widetilde{Z}_q\subset\widetilde{Z}_p$. For every such component $\widetilde{Z}_q$ we choose $\zeta\in\widetilde{Z}_q\cap(K\cup\{x\in\mathbb{R}\mid x>c_k^{-1}\})$. Let $\widetilde{Z}_k$ be the (unique) component of $U_{1/c_k}(K)\cap((-\infty,1+c_k^{-1})+i\mathbb{R})$ which contains ζ. The rest is analogous to (i) and thus there are $\widetilde{C}_0,\widetilde{C}_1>0$ and $0<\theta<1$ such that

$$
\sup_{z\in\overline{\widetilde{Z}_q}} |f(z)|e^{-a_q|\operatorname{Re}(z)|^\beta} \leq \frac{\widetilde{C}_1}{\widetilde{C}_0}\|f\|_{k,c_k}^{\theta}\|f\|_{p,c_p}^{1-\theta}. \tag{18}
$$

(iv) First, let us remark the following. Let B be a set, $B_0\subset B$, $0<\theta_0<\theta_1<1$, $h: B_0\to[0,\infty)$, $g: B\to[0,\infty)$ and $h\leq g$ on B_0. Then

$$
\left(\sup_{z\in B_0} h(z)\right)^{\theta_1}\left(\sup_{z\in B} g(z)\right)^{1-\theta_1} \leq \left(\sup_{z\in B_0} h(z)\right)^{\theta_0}\left(\sup_{z\in B} g(z)\right)^{1-\theta_0}.
$$

Now, we take the minimum of all the θs which appear in (i)-(iii). There are finitely many of them and denote their minimum again with θ. Take the maximum of the constants $\frac{C_1}{C_0}$, C_2 and $\frac{\widetilde{C}_1}{\widetilde{C}_0}$ which appear in (i)-(iii). There are again finitely many of them and denote their maximum with C. We apply the remark above to $B_0:=\overline{U_{1/c_k}(K)}$, $B:=\overline{U_{1/c_p}(K)}$, $h(z):=|f(z)|e^{-a_k|\operatorname{Re}(z)|^\beta}$ and $g(z):=|f(z)|e^{-a_p|\operatorname{Re}(z)|^\beta}$. Then we deduce from (16), (17) and (18) that

$$
\|f\|_{q,c_q}\leq C\|f\|_{k,c_k}^{\theta}\|f\|_{p,c_p}^{1-\theta}.
$$

(b) We recall Remark 16 and identify both inductive limits. Let $p\in\mathbb{N}$ and choose $q\in\mathbb{N}$, $q>p$. Let $k\in\mathbb{N}$. If $k\leq p$, then we get for any $0<\theta<1$ and all $y\in(\mathcal{O}^{-\beta}_{(a_n)}(K)'_b)'$ by definition of the dual norm

$$
\|y\|^*_{q,c_q} \underset{p<q}{\leq} \|y\|^*_{p,c_p} = {\|y\|^*_{p,c_p}}^{1-\theta}{\|y\|^*_{p,c_p}}^{\theta} \underset{k<p}{\leq} {\|y\|^*_{p,c_p}}^{1-\theta}{\|y\|^*_{k,c_k}}^{\theta}.
$$

Let $k>p$. If $k\leq q$, we have for any $0<\theta<1$ and all $y\in(\mathcal{O}^{-\beta}_{(a_n)}(K)'_b)'$ by definition of the dual norm

$$
\|y\|^*_{q,c_q} \underset{k\leq q}{\leq} \|y\|^*_{k,c_k} = {\|y\|^*_{k,c_k}}^{1-\theta}{\|y\|^*_{k,c_k}}^{\theta} \underset{p<k}{\leq} {\|y\|^*_{p,c_p}}^{1-\theta}{\|y\|^*_{k,c_k}}^{\theta}.
$$

Let $k > q$ and $y \in (\mathcal{O}^{-\beta}_{(a_n)}(K)'_b)'$. If $\|y\|^*_{p,c_p} = \infty$, then (14) is obviously fulfilled. Let $\|y\|^*_{p,c_p} < \infty$. As $\mathcal{O}^{-\beta}_{(a_n)}(K)$ is a DFS-space by Proposition 4 (a), the sets

$$B_n := \{f \in \mathcal{O}^{-\beta}_{a_n}(\overline{U_{1/c_n}(K)}) \mid \|f\|_{n,c_n} \leq 1\}, \quad n \in \mathbb{N},$$

are a fundamental system of bounded sets of $\mathcal{O}^{-\beta}_{(a_n)}(K)$ by [25, Proposition 25.19, p. 303] and hence the seminorms

$$|||x|||_n := \sup_{f \in B_n} |x(f)|, \quad x \in \mathcal{O}^{-\beta}_{(a_n)}(K)',$$

form a fundamental system of seminorms of $\mathcal{O}^{-\beta}_{(a_n)}(K)'_b$. Furthermore, $\mathcal{O}^{-\beta}_{(a_n)}(K)$ is reflexive and thus there is a unique $f \in \mathcal{O}^{-\beta}_{(a_n)}(K)$ such that $y(x) = x(f)$ for all $x \in \mathcal{O}^{-\beta}_{(a_n)}(K)'$. Then we obtain by [25, Proposition 22.14, p. 256] for all $n \in \mathbb{N}$, $n \geq p$,

$$\begin{aligned}\infty > \|y\|^*_{p,c_p} \underset{p \leq n}{\geq} \|y\|^*_{n,c_n} &= \sup\{|y(x)| \mid |||x|||_n \leq 1\} = \sup\{|x(f)| \mid x \in B_n^\circ\} \\ &= \inf\{t > 0 \mid f \in tB_n\}.\end{aligned}$$

In particular, this means that $\{t > 0 \mid f \in tB_n\} \neq \emptyset$ and thus we have $f \in \mathcal{O}^{-\beta}_{a_n}\left(\overline{U_{1/c_n}(K)}\right)$ as well as

$$\|y\|^*_{n,c_n} = \inf\{t > 0 \mid f \in tB_n\} = \|f\|_{n,c_n}$$

for all $n \geq p$. So by part (a), there are $C > 0$ and $0 < \theta < 1$, only depending on p, q and k, such that

$$\|y\|^*_{q,c_q} = \|f\|_{q,c_q} \leq C\|f\|^{1-\theta}_{p,c_p}\|f\|^{\theta}_{k,c_k} = C{\|y\|^*_{p,c_p}}^{1-\theta}{\|y\|^*_{k,c_k}}^{\theta}.$$

□

The idea to use Hadamard's Three Circles Theorem in the proof of Lemma 18 (a) is taken from the proof of [30, Lemma 5.2 (a)(3), p. 263-264]. If $K \subset \mathbb{R}$ is non-empty and compact, Lemma 18 (b) is already known. Indeed, the space $\mathcal{O}(\mathbb{C} \setminus K)$ satisfies (Ω) by [31, Proposition 2.5 (b), p. 173] and thus the quotient space $\mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C})$ as well by [25, Lemma 29.11 (2), p. 368]. Since (Ω) is a linear-topological invariant by [25, Lemma 29.11 (1), p. 368], it follows from $\mathcal{O}^{-\beta}_{(a_n)}(K)'_b \cong \mathscr{A}(K)'_b \cong \mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C})$ by (2) that $\mathcal{O}^{-\beta}_{(a_n)}(K)'_b$ also satisfies (Ω). Combining our duality result with the preceding lemma, we get a generalisation of [19, 5.22 Theorem, p. 92].

Corollary 19 *If*

(i) $K \subset \mathbb{R}$, *or* $K \cap \{\pm\infty\}$ *has no isolated points in* K, *or*
(ii) K *is arbitrary*, $a_n < 0$ *for all* $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n = 0$ *and* $\beta = 1$,

then $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)$ satisfies (Ω).

Proof The spaces $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)$ and $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}})$ are Fréchet spaces which is easily checked (similar to [20, 3.7 Proposition, p. 240]). By Theorem 11 in (i) resp. Corollary 15 in (ii) $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}} \setminus K)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}})$ is topologically isomorphic to $\mathcal{O}^{-\beta}_{(a_n)}(K)'_b$, in particular, the quotient is a Fréchet space as $\mathcal{O}^{-\beta}_{(a_n)}(K)$ is a DFS-space by Proposition 4 (a). Since (Ω) is a linear-topological invariant by [25, Lemma 29.11 (1), p. 368], $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}})$ satisfies (Ω) due to Lemma 18 (b). The sequence

$$0 \to \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}) \xrightarrow{i} \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K) \xrightarrow{q} \mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}) \to 0$$

is an exact sequence of Fréchet spaces where i means the inclusion and q the quotient map. $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}})$ satisfies (Ω) by [22, Corollary 14, p. 18] combined with Assumption 3 (iii)+(iv) and $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K)/\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}})$ as well, thus $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K)$ by [33, 1.7 Lemma, p. 230], too.

□

5 Surjectivity of the Cauchy-Riemann operator

In our last section we prove our main result on the surjectivity of the Cauchy-Riemann operator on $\mathcal{E}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K, E)$. This is done by using the results obtained so far and splitting theory. We recall that a Fréchet space $(F, (|||\cdot|||_k)_{k\in\mathbb{N}})$ satisfies (DN) by [25, Chap. 29, Definition, p. 359] if

$$\exists\, p\in\mathbb{N}\ \forall\, k\in\mathbb{N}\ \exists\, n\in\mathbb{N},\ C>0\ \forall\, x\in F:\ \ |||x|||_k^2 \leq C|||x|||_p|||x|||_n.$$

A *PLS-space* is a projective limit $X=\lim\limits_{\substack{\longleftarrow\\ N\in\mathbb{N}}} X_N$, where the $X_N=\lim\limits_{\substack{\longrightarrow\\ N\in\mathbb{N}}}\left(X_{N,n}, |||\cdot|||_{N,n}\right)$ are DFS-spaces, and it satisfies (PA) if

$$\forall\, N\ \exists\, M\ \forall\, K\ \exists\, n\ \forall\, m\ \forall\, \eta>0\ \exists\, k, C, r_0>0\ \forall\, r>r_0\ \forall\, x'\in X'_N:$$
$$\left|\left|\left|x'\circ i^M_N\right|\right|\right|^*_{M,m} \leq C\left(r^{\eta}\left|\left|\left|x'\circ i^K_N\right|\right|\right|^*_{K,k} + \frac{1}{r}|||x'|||^*_{N,n}\right)$$

where $|||\cdot|||^*$ denotes the dual norm of $|||\cdot|||$ and i^M_N, i^K_N the linking maps (see [4, Sect. 4, Eq. (24), p. 577]).

Theorem 20 *Let $(a_n)_{n\in\mathbb{N}}$ be strictly increasing, $a_n<0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$. If*

(i) $K\subset\mathbb{R}$, *or* $K\cap\{\pm\infty\}$ *has no isolated points in* K, *or*
(ii) K *is arbitrary and* $\beta=1$,

and

(a) $E := F'_b$ *where* F *is a Fréchet space over* $\mathbb{C}$ *satisfying* (DN), *or*
(b) E *is an ultrabornological PLS-space over* $\mathbb{C}$ *satisfying* (PA),

then

$$\overline{\partial}^E : \mathcal{E}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K, E) \to \mathcal{E}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K, E)$$

is surjective.

Proof We only need to check that the conditions of Theorem 1 are fulfilled. $\mathcal{E}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K)$ is nuclear, in particular a Schwartz space, and thus its subspace $\mathcal{E}^{\beta}_{(a_n),\overline{\partial}}(\overline{\mathbb{C}}\setminus K)$ as well by [21, Theorem 3.1, p. 188], [21, 2.8 Example (ii), p. 179], [21, Remark 2.7, p. 178-179] and [21, Remark 2.3 (b), p. 177]. Furthermore, $\mathcal{E}^{\beta}_{(a_n),\overline{\partial}}(\overline{\mathbb{C}}\setminus K)=\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K)$ by Remark 2. Due to Corollary 19 the space $\mathcal{O}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K)$ satisfies (Ω). The Cauchy-Riemann operator $\overline{\partial}:\mathcal{E}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K)\to\mathcal{E}^{\beta}_{(a_n)}(\overline{\mathbb{C}}\setminus K)$ in the $\mathbb{C}$-valued case is surjective by [23, Corollary 5.6, p. 27] which follows from [23, Example 5.7 (a), p. 27-28] in the case that $K\subset\mathbb{R}$ or $K\cap\{\pm\infty\}$ has no isolated points in K. If $K\cap\{\pm\infty\}$ has isolated points in K, then the proof that the conditions of [23, Corollary 5.6, p. 27] are fulfilled is verbatim as in [23, Example 5.7 (a), p. 27-28]. Hence all conditions of Theorem 1 are fulfilled. □

Theorem 20, together with [22, Corollary 18, p. 21] ($K=\varnothing$), generalises [19, 5.24 Theorem, p. 95] which is case (*ii*) above.

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