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Analytical and semi-analytical solutions of some fundamental nonlinear stochastic differential equations

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Abstract

We are interested in perturbed Hamiltonian systems in a plane, which are damped and excited by an absolutely regular non-white Gaussian process. We use two methods for the determination of analytical and semi-analytical solutions to such nonlinear stochastic differential equations (SDE). The first method is based on a limit theorem by Khashminskii, from which a class of methods was derived known as stochastic averaging. From the drift and diffusion of the resulting averaged process, probability density functions and mean exit times can be easily obtained. The second method enables the determination of a Gaussian mixture representation for probability density functions of SDE's. This method was proposed by Pradlwarter and is known as Local Statistical Linearization. The error evolution of such Gaussian mixture shows promising results for further research.

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Keywords: stochastic averaging; Gaussian mixture; Duffing oscillator; stochastic differential equations; Hamiltonian system

1. Introduction

In the presented work, we use two methods for the determination of analytical and semi-analytical solutions of nonlinear stochastic differential equations (SDE). The first method is based on a limit theorem by Khashminskii, which was rigorously proven in [1]. From this work, a class of methods was derived known as stochastic averaging. Depending on the analyzed nonlinear system and the type of stochastic processes involved, different stochastic averaging methods were developed [2,3,4]. The second method that we use was proposed by Pradlwarter [5] and is known as Local Statistical Linearization. This relatively new method is a semi-analytical approach, where the probability density function of the considered nonlinear SDE is approximated by a sum of Gaussian probability densities.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is the sample space, $\mathcal{F}$ is a σ - algebra over Ω and P is a probability measure. We are interested in perturbed Hamiltonian systems, which are damped and excited by an absolutely regular non-white Gaussian process $\boldsymbol{\xi}_t := \boldsymbol{\xi}(\omega, t) = (\xi_1(t), \ldots, \xi_k(t)) \in \mathbb{R}^k$, $\omega \in \Omega$, with sufficient mixing properties. Here, we

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consider a perturbed Hamiltonian system in a two dimensional state space $\mathbf{Z}^\varepsilon := (x, y) \in D \subset \mathbb{R}^2$ given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_\varepsilon} x &= \frac{\partial H(x,y)}{\partial y}, \\ \frac{\mathrm{d}}{\mathrm{d}t_\varepsilon} y &= -\frac{\partial H(x,y)}{\partial x} - \varepsilon f(y) + \sqrt{\varepsilon} g(x, \boldsymbol{\xi}_t), \end{aligned} \tag{1}$$

where the function $H(x, y)$ is the Hamiltonian, $f(y)$ is a damping function, $g(x, \boldsymbol{\xi}_t)$ is a function of random excitations and $\varepsilon > 0$.

2. Stochastic averaging of Hamiltonian

For the case of weakly perturbed systems of type (1) with small $\varepsilon \ll 1$, a stochastic averaging method is proposed in the following Theorem, using results by Khashminskii[2], Borodin[6] and Freidlin and Borodin[4]. With this method, the stochastic process of the Hamiltonian H can be obtained, which is the process of total energy of the corresponding nonlinear oscillator. Such a stochastic averaging procedure was used in[7] and can be generalized as follows.

Theorem 2.1. *Let $Z^\varepsilon = (X, Y) \in D \subseteq \mathbb{R}^2$ be the solution of the SDE*

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \varepsilon F_1(X, Y) + \sqrt{\varepsilon} F_0(X, Y)\, \boldsymbol{\xi}, \tag{2}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = G(X, Y), \qquad (X(0), Y(0)) = (x_0, y_0) \in D, \quad \varepsilon > 0, \tag{3}$$

and let the following conditions be fulfilled:

- *i) The stochastic process $\boldsymbol{\xi}_t := \boldsymbol{\xi}(\omega, t) = (\xi_1(t), \ldots, \xi_k(t))^{\mathrm{T}} \in \mathbb{R}^k$ is stationary, absolutely regular with sufficient mixing properties, and $E\{\xi_j(t)\} = 0$.*
- *ii) The functions F_0 and F_1 satisfy certain limits in X and in Y, which are specified in Borodin[6], in order to ensure uniqueness of the solution.*
- *iii) Without loss of generality, the functions $F_0 : D \mapsto \mathbb{R}^k$, $F_1 : D \mapsto \mathbb{R}$ and the solution of the equation $\dot{Y} = G(x, Y)$ are periodic with period $T(x)$ for fixed x.*
- *iv) Without loss of generality, the function $G(X, Y)$ has exactly one root at $(X, Y) = (0, 0)$ and has the form $G(X, Y) = \sqrt{Q}$ with $Q := 2X - 2U(Y)$ and the continuously differentiable function $U(Y) : \mathbb{R} \mapsto \mathbb{R}$. Furthermore, the Hessian matrix of $G(X, Y)$ evaluated at the root is positive definite.*

Let the averaging operator $\mathbb{M}$ for periodic functions $f : \mathbb{R}^+ \mapsto \mathbb{R}$ with period T be defined by

$$\mathbb{M}\{f\} = \frac{1}{T} \int_0^T f(t)\mathrm{d}t \tag{4}$$

and let Y^x be the solution of the ordinary differential equation

$$\mathrm{d}Y^x = G(X, Y^x)\, \mathrm{d}t, \; X = x, \;\; Y^x(0) = y.$$

If the limits

$$\begin{aligned} m(Z) &= \mathbb{M}\Big\{ F_1(x, Y^x(t)) + \int_{-\infty}^{0} \mathrm{cov}\left(\left\{ \frac{\partial F_0(X, Y^x(t))\, \boldsymbol{\xi}_t}{\partial X} \right\}_{X=x}, F_0(x, Y^x(t+s))\, \boldsymbol{\xi}_{t+s} \right) \mathrm{d}s \Big\}, \\ \sigma^2(Z) &= \mathbb{M}\Big\{ \int_{-\infty}^{\infty} \mathrm{cov}\left(F_0(x, Y^x(t))\, \boldsymbol{\xi}_t, F_0(x, Y^x(t+s))\, \boldsymbol{\xi}_{t+s} \right) \mathrm{d}s \Big\} \end{aligned}$$

exist, then the process $X(\tau)$, $\tau = \varepsilon t$, converges, as $\varepsilon \to 0$, weakly on the time interval of order $O(1/\varepsilon)$ to a diffusion Markov process Z satisfying the Itô stochastic differential equation

$$\mathrm{d}Z(\tau) = m(Z)\mathrm{d}\tau + \sigma(Z)\mathrm{d}W_\tau, \qquad Z(0) = x_0, \tag{5}$$

with the standard Wiener process W_τ.

For a proof, the deterministic solution Y^x of equation $\mathrm{d}Y^x = G(x, Y^x)\,\mathrm{d}t$ has to be determined for arbitrary but fixed x. Then, Y is replaced by Y^x in equation (2). The assertion follows by applying the Theorem from Borodin[6] for the resulting equation. If the functions F_0 and F_1 are not periodic, then the procedure as described in[1] has to be used. The essential result of Theorem 2.1 is, that the total energy $H(\mathbf{Z}^\varepsilon(t_\varepsilon))$ of system (1) converges in probability at a scale $O(t)$, $t = \varepsilon t_\varepsilon$, to the diffusion Markov process $\bar{H}(t)$ as $\epsilon \to 0$. The resulting stochastic process is given by the Itô equation

$$\mathrm{d}\bar{H}(t) = m(\bar{H})\mathrm{d}t + \sigma(\bar{H})\mathrm{d}W_t, \tag{6}$$

where W_t is the standard Wiener process.

In the following subsections we exemplarily demonstrate how simplified analytical expressions for the drift $m(Z)$ and diffusion $\sigma(Z)$ of the Itô equation (5) from Theorem 2.1 can be obtained for the Hamiltonians

$$H_4(x,y) = \frac{y^2}{2} + U_4(x), \qquad U_4(x) := \alpha_1\frac{x^2}{2} - \alpha_3\frac{x^4}{4}, \tag{7}$$

$$H_6(x,y) = \frac{y^2}{2} + U_6(x), \qquad U_6(x) := \alpha_1\frac{x^2}{2} + \alpha_3\frac{x^4}{4} - \alpha_5\frac{x^6}{6}, \tag{8}$$

where the coefficients α_i are positive. Here, U_4 is denoted as a quartic and U_6 as a sextic potential, respectively. If the drift $m(Z)$ and the diffusion $\sigma(Z)$ are known, then probability density functions and mean exit times of the averaged process Z can be easily obtained, cf. Dostal et al.[7].

2.1. Duffing oscillator with negative cubic stiffness

Let the Hamiltonian H in system (1) be given by H_4 from equation (7) and let $\varepsilon \ll 1$. Using

$$Q(x,H) := y^2 = 2H - \alpha_1 x^2 + \alpha_3\frac{x^4}{2}, \tag{9}$$

and the derivative of the Hamiltonian with respect to time, and replacing $f(y)$ and $g(x,\xi_t)$ by specific functions, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t_\varepsilon}x = \sqrt{Q(x,H)},\\
&\frac{\mathrm{d}}{\mathrm{d}t_\varepsilon}H = \varepsilon Q(x,H)(-\beta_1 - \beta_2\sqrt{Q(x,H)} - \beta_3 Q(x,H)) + \sqrt{\varepsilon}\sqrt{Q(x,H)}(\nu_1\xi_1(t_\varepsilon) + \nu_2 x\,\xi_2(t_\varepsilon)).
\end{aligned} \tag{10}$$

The contour lines of $H_4(x,y)$ with the quartic potential U_4 from equation (7) are shown in Fig. 1. Theorem 2.1 can be applied to the resulting system of equations (10) and simplifies to

Theorem 2.2. *Let $\xi_1(t)$ and $\xi_2(t)$ in* (10) *be zero-mean stationary absolutely regular Gaussian process with sufficient mixing properties. Let further H be the solution of system* (10)*. Then for fixed $0 \le H < \frac{\alpha_1^2}{4\alpha_3}$ the integrals*

$$\begin{aligned}
m(H) =& \frac{4}{Tq}\int_{-\infty}^{0}\Big\{R_{\xi_1\xi_1}(\tau)\nu_1^2\int_0^{K(k)}\frac{\mathrm{cn}_{t+\tau}\mathrm{dn}_{t+\tau}}{\mathrm{cn}_t\mathrm{dn}_t}\mathrm{d}u +\\
&+ R_{\xi_2\xi_2}(\tau)b^2\nu_2^2\int_0^{K(k)}\mathrm{sn}\,\mathrm{sn}_{t+\tau}\frac{\mathrm{cn}_{t+\tau}\mathrm{dn}_{t+\tau}}{\mathrm{cn}_t\mathrm{dn}_t}\mathrm{d}u\Big\}\,\mathrm{d}\tau +\\
&+ \frac{1}{T}\int_0^T Q(x(t),H)G(x(t),H)\mathrm{d}t,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\sigma^2(H) =& \frac{4b^2q}{T}\int_{-\infty}^{\infty}\Big\{R_{\xi_1\xi_1}(\tau)\nu_1^2\int_0^{K(k)}\mathrm{cn}\,\mathrm{dn}\,\mathrm{cn}_{t+\tau}\,\mathrm{dn}_{t+\tau}\mathrm{d}u +\\
&+ R_{\xi_2\xi_2}(\tau)b^2\nu_2^2\int_0^{K(k)}\mathrm{sn}\,\mathrm{sn}_{t+\tau}\mathrm{cn}\,\mathrm{dn}\,\mathrm{cn}_{t+\tau}\,\mathrm{dn}_{t+\tau}\,\mathrm{d}u\Big\}\,\mathrm{d}\tau
\end{aligned} \tag{12}$$

exist, and the process H converges, as $\varepsilon \to 0$, weakly on the time interval of order $O(1/\varepsilon)$ to a diffusion Markov process $\bar{H}$ satisfying the Itô stochastic differential equation (6)*.*

The proof of this theorem follows the procedure derived in Dostal et al.[7,8]. The expressions in Theorem 2.2 contain the variables

$$b = \sqrt{-\frac{-\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_3 H}}{\alpha_3}}, \qquad a = \sqrt{\frac{4H}{b^2\alpha_3}}, \qquad q = a\sqrt{\frac{\alpha_3}{2}}, \qquad T(H) = \frac{4}{q}K(k), \tag{13}$$

the autocorrelation function $R_{\xi_t\xi_t}(\tau) = E\{\xi_{\tau+t}\xi_t\}$, the Jacobian elliptic functions $\mathrm{sn}(\cdot,k)$, $\mathrm{cn}(\cdot,k)$, $\mathrm{dn}(\cdot,k)$, and the complete elliptic integral of the first kind $K(k)$, see Byrd and Friedmann[9], where $k = b/a$ is the elliptic modulus. In addition, we use the following abbreviations

$$\mathrm{sn} := \mathrm{sn}(qt,k),\ \mathrm{cn} := \mathrm{cn}(qt,k),\ \mathrm{dn} := \mathrm{dn}(qt,k),\ u := qt. \tag{14}$$

If the subscript τ or $t+\tau$ is used, we refer to the argument $q\tau$ or $q(t+\tau)$, respectively.

2.2. Oscillator with sextic potential

The accuracy of models of dynamical systems can be increased, if polynomial nonlinearities of higher order are used in system (1). Therefore, let the Hamiltonian H in system (1) be given by H_6 from equation (8), $\varepsilon \ll 1$ and let $f(y)$ and $g(x,\xi_t)$ be specified by the same functions as in equation (10). This yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_\varepsilon}x &= y, \\ \frac{\mathrm{d}}{\mathrm{d}t_\varepsilon}y &= -\frac{\partial}{\partial x}H_6(x,y) - \varepsilon\left(\beta_1 y + \beta_2\,|y|y + \beta_3 y^3\right) + \sqrt{\varepsilon}\left(\nu_1\,\xi_1(t_\varepsilon) + \nu_2\,x\,\xi_2(t_\varepsilon)\right). \end{aligned} \tag{15}$$

For $\varepsilon = 0$ equation (15) is reduced to the conservative system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_\varepsilon}x &= y, \\ \frac{\mathrm{d}}{\mathrm{d}t_\varepsilon}y &= -\frac{\partial}{\partial x}H_6(x,y). \end{aligned} \tag{16}$$

The total energy of the nonlinear oscillator corresponding to the system (15) is given by the Hamiltonian H_6 from equation (8). Solving this equation for y^2 we get

$$Q_6(x,H) := y^2 = 2H - 2U_6(x,H). \tag{17}$$

The fixed points of system (16) are

$$P_1 = (b_c, 0);\quad P_2 = (-b_c, 0);\quad S = (0,0). \tag{18}$$

Thereby, the critical oscillation amplitude b_c of the considered oscillator is given by

$$b_c = \sqrt{\frac{\alpha_3 + \sqrt{\alpha_3^2 + 4\alpha_1\alpha_5}}{2\alpha_5}}, \tag{19}$$

and the energy H_6^c at which the critical oscillation amplitude b_c is reached is

$$H_6^c = \frac{1}{3}\alpha_1 b_c^2 + \frac{1}{12}\alpha_3 b_c^4. \tag{20}$$

This defines the heteroclinic orbit

$$\gamma_6(x,y) = \left\{ x, y \in \mathbb{R},\ |x| < b_c :\ y^2 + \alpha_1 x^2 + \alpha_3\frac{x^4}{2} - \alpha_5\frac{x^6}{3} = 2H_6^c \right\} \tag{21}$$

which connects the saddle points P_1 and P_2. Only the trajectories inside the phase space domain

$$\mathbf{D}_{\gamma_6} := \left\{ x, y \in \mathbb{R},\, |x| < b_c \,:\, y^2 + \alpha_1 x^2 + \alpha_3 \frac{x^4}{2} - \alpha_5 \frac{x^6}{3} < 2H_6^c \right\}, \tag{22}$$

which is bounded by the heteroclinic orbit γ_6, lead to oscillations of the oscillator with the potential U_6. The contour lines of $H_6(x, y)$ with the sextic potential U_6 from equation (8) are shown in Fig. 2. Using $Q_6(x, H)$ and the derivative of the Hamiltonian with respect to time, the resulting system can be stated as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_\varepsilon} x &= \sqrt{Q_6(x, H)}, \\ \frac{\mathrm{d}}{\mathrm{d}t_\varepsilon} H &= \varepsilon Q_6(x, H)\, G_6(x, H) + \sqrt{\varepsilon}\, \sqrt{Q_6(x, H)}\, (\nu_1\, \xi_1(t_\varepsilon) + \nu_2\, x\, \xi_2(t_\varepsilon)), \end{aligned} \tag{23}$$

where

$$G_6(x, H) := -\beta_1 - \beta_2 \sqrt{Q_6(x(t), H)} - \beta_3 Q_6(x(t), H). \tag{24}$$

An analytical solution for the conservative system with the sextic potential from equation (8) in the region $\mathbf{D}_{\gamma_6}$ for the energy $H \in (0, H_6^c)$ can be written as

$$x(t) = \frac{\mathrm{sn}\,(qt, k)}{\sqrt{c - d\,\mathrm{sn}^2\,(qt, k)}}, \tag{25}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y(t) = \frac{c\, q\, \mathrm{cn}\,(qt, k)\, \mathrm{dn}\,(qt, k)}{(c - d\,\mathrm{sn}^2\,(qt, k))^{3/2}}. \tag{26}$$

The coefficients used in these equations for each energy level H are given by

$$c = \frac{6\,Hw - \alpha_1 + \sqrt{-12\,H^2w^2 + 4\,Hw\alpha_1 + {\alpha_1}^2 + 4\,H\alpha_3}}{4H}, \tag{27}$$

$$d = c - w, \qquad q = \sqrt{2cH}. \tag{28}$$

Here, w refers to the root of the function

$$W(z) = -\frac{1}{3}\,\alpha_5 + \frac{1}{2}\,\alpha_3 z + \alpha_1 z^2 - 2\,Hz^3, \tag{29}$$

which is given by

$$w = \frac{v}{12H} + \frac{3\,\alpha_3 H + {\alpha_1}^2}{3v} + \frac{\alpha_1}{6H}, \tag{30}$$

where

$$v = \left(36\,\alpha_3\alpha_1 H - 144\,\alpha_5 H^2 + 8\,{\alpha_1}^3 + 12\,H\,\sqrt{-12\,{\alpha_3}^3 H - 3\,{\alpha_3}^2{\alpha_1}^2 - 72\,\alpha_3\alpha_1 H\alpha_5 + 144\,{\alpha_5}^2 H^2 - 16\,\alpha_5{\alpha_1}^3}\right)^{\frac{1}{3}}. \tag{31}$$

The elliptic modulus k of the Jacobi elliptic functions is obtained from equation

$$k^2 = 1 + 4\,\frac{\left(4\,\alpha_1 w - 12\,Hw^2 + \alpha_3\right) H}{\left(6\,Hw - \alpha_1 + \sqrt{-12\,H^2w^2 + 4\,Hw\alpha_1 + {\alpha_1}^2 + 4\,H\alpha_3}\right)^2}. \tag{32}$$

With the solutions (25) and (26) of the conservative system of equations (16), we can state the following theorem.

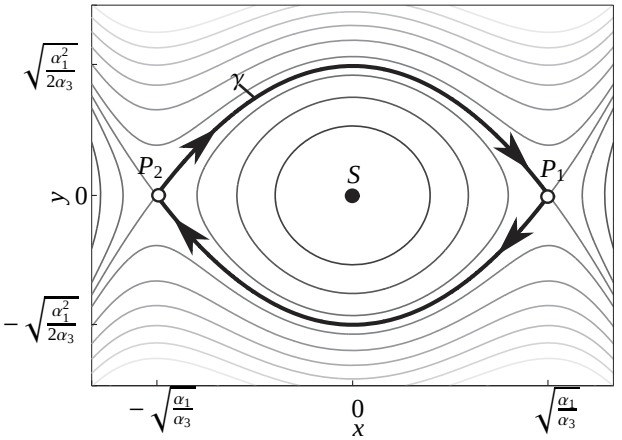

Figure 1. Contour lines of $H_4(x, y)$ from equation (7).

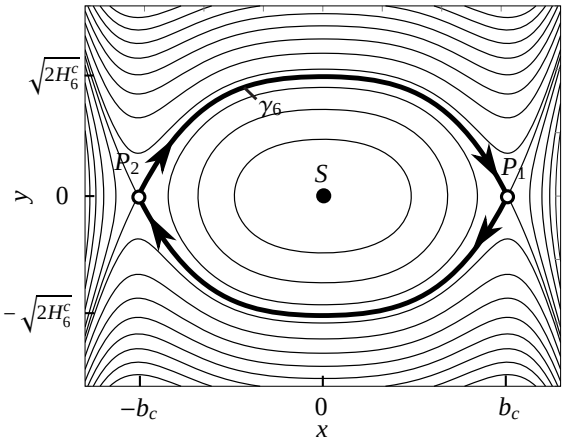

Figure 2. Contour lines of $H_6(x, y)$ with the sextic potential U_6 from equation (8).

Theorem 2.3. *Let $\xi_1(t)$ and $\xi_2(t)$ in* (23) *be zero-mean stationary absolutely regular Gaussian process with sufficient mixing properties. Let further H be the solution of system* (23)*. Then for fixed $0 \le H < H_6^c$ the integrals*

$$\begin{aligned} m(H) = \frac{4}{Tq}\int_{-\infty}^{0}\Big\{R_{\xi_1\xi_1}(\tau)\nu_1^2\int_0^{K(k)}\frac{\mathrm{cn}_{t+\tau}\mathrm{dn}_{t+\tau}\left(c-d\mathrm{sn}^2\right)^{3/2}}{\mathrm{cn}_t\mathrm{dn}_t\left(c-d\mathrm{sn}_{t+\tau}{}^2\right)^{3/2}}\mathrm{d}u+ \\ + R_{\xi_2\xi_2}(\tau)\nu_2^2\int_0^{K(k)}\mathrm{sn}\,\mathrm{sn}_{t+\tau}\frac{\mathrm{cn}_{t+\tau}\mathrm{dn}_{t+\tau}\left(c-d\mathrm{sn}^2\right)}{\mathrm{cn}_t\mathrm{dn}_t\left(c-d\mathrm{sn}_{t+\tau}{}^2\right)^2}\mathrm{d}u\Big\}\,\mathrm{d}\tau+ \\ + \frac{1}{T}\int_0^T Q_6(x(t),H)G_6(x(t),H)\mathrm{d}t, \end{aligned} \tag{33}$$

$$\begin{aligned} \sigma^2(H) = \frac{4c^2q}{T}\int_{-\infty}^{\infty}\Big\{R_{\xi_1\xi_1}(\tau)\nu_1^2\int_0^{K(k)}\frac{\mathrm{cn}\,\mathrm{dn}\,\mathrm{cn}_{t+\tau}\,\mathrm{dn}_{t+\tau}}{\left(c-d\mathrm{sn}^2\right)^{3/2}\left(c-d\mathrm{sn}_{t+\tau}{}^2\right)^{3/2}}\mathrm{d}u+ \\ + R_{\xi_2\xi_2}(\tau)\nu_2^2\int_0^{K(k)}\frac{\mathrm{sn}\,\mathrm{sn}_{t+\tau}\mathrm{cn}\,\mathrm{dn}\,\mathrm{cn}_{t+\tau}\,\mathrm{dn}_{t+\tau}}{\left(c-d\mathrm{sn}^2\right)^2\left(c-d\mathrm{sn}_{t+\tau}{}^2\right)^2}\,\mathrm{d}u\Big\}\,\mathrm{d}\tau \end{aligned} \tag{34}$$

exist, and the process H converges, as $\varepsilon \to 0$, weakly on the time interval of order $O(1/\varepsilon)$ to a diffusion Markov process $\bar{H}$ satisfying the Itô stochastic differential equation (6)*.*

The proof follows from Theorem 2.1 in analogy to the proof of Theorem 2.2. Contour lines of the Hamiltonian H_6 are shown in Fig. 2, where the separatrix γ_6 separates regions with different phase space structure.

2.3. Stochastic averaging of Hamiltonian for arbitrary polynomial potentials

If the nonlinearity of the weakly perturbed Hamiltonian system (1) is given by a polynomial, then an explicit solution of the corresponding conservative differential equation might be obtained by solving Jacobi's inversion problem. For cases with polynomials of high order, this requires in general explicit solutions to hyperelliptic integrals, cf. Baker[10].

3. Gaussian mixture by Local Statistical Linearization

For the general case with an arbitrary polynomial non-linearity and large ε a solution of the random systems (1) might be difficult to obtain. For such cases, we transform (1) into an Itô equation and approximate the non-white

Gaussian process ξ_t by an ARMA process in continuous time, cf. [11], such that the resulting system of equations has n state space variables $X_1, \ldots, X_n$. Then the probability density $p(\mathbf{X}, t)$, $\mathbf{X} = (X_1, \ldots, X_n)^T \in \mathbb{R}^n$, $t \in \mathbb{R}$, is approximated by a sum of Gaussian densities p_i which yields

$$p(\mathbf{X}, t) = \sum_i A_i p_i(\mathbf{X}, t), \qquad \sum_i A_i = 1, \qquad A_i \geq 0. \tag{35}$$

Such a Gaussian mixture for the probability density function can be determined by means of the Local Statistical Linearization method, which was proposed by Pradlwarter[5].

The time evolution of mean vectors $\boldsymbol{\mu}^i$ and covariance matrices $\mathbf{C}^i$ of each local density p_i, are determined by the derivatives of the first and second order moments of $\mathbf{X}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_j = \frac{\mathrm{d}}{\mathrm{d}t}E\{X_j\}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}C_{jk}(X_j, X_k) = \frac{\mathrm{d}}{\mathrm{d}t}E\{X_j X_k\} - E\{X_j\}\frac{\mathrm{d}}{\mathrm{d}t}E\{X_k\} - E\{X_k\}\frac{\mathrm{d}}{\mathrm{d}t}E\{X_j\}. \tag{36}$$

The original method uses Itô's differential rule and Gaussian closure for the calculation of the first and second order moments of $\mathbf{X}$, more details can be found in [12]. However, other well known methods can be used for the determination of the time evolution of local Gaussian densities. Such well known methods for calculation of the local density evolution are: Statistical linearization [13], equivalent linearization [13], closure schemes [14], generalized polynomial chaos expansion ? [15] (gPC). Indeed, we can use the well developed error estimations on the first and second order moments of local Gaussian densities from the gPC theory and obtain by this means a reliable iterative method for the determination of probability density functions for system (1).

3.1. Decomposition

Due to the diffusion of the stochastic differential equations, the local densities $p_i(\mathbf{X}, t)$ will spread. If their variance exceeds a certain level σ^2_{lim}, then they have to be decomposed into densities with a smaller variance in order to capture the local system dynamics. In this case some of the weighted densities $(A_j, p_j(\mathbf{X}, t))$ have to be decomposed into $2m + 1$ densities according to

$$A_j p_j(\mathbf{X}, t) = \sum_{k=-m}^{m} A_{jk} p_{jk}(\mathbf{X}, t), \qquad \sum_{k=-m}^{m} A_{jk} = 1, \qquad A_{jk} \geq 0. \tag{37}$$

If the covariance matrix is diagonalized, then the Gaussian density can be represented as a product of univariate Gaussian densities. It is sufficient to decompose only one univariate Gaussian density with the highest variance, say p_d, into $2m + 1$ densities with variance σ_0. After retransformation of densities to the original coordinates the decomposition is finished. This procedure results in the following equations for the decomposition. The density p_d is represented as

$$p_d(X_d) = \sum_{k=-m}^{m} B_{dk} p_{dk}(X_d), \tag{38}$$

where $X_d = \boldsymbol{\phi}_d^T \cdot (\mathbf{X} - \boldsymbol{\mu}_j)$ is the coordinate in direction of the highest variance and the coefficients B_{dk} are solutions to a least squares minimizing problem. The optimal solution is given by

$$\sum_{k=-m}^{m} \alpha_{jk} B_{dk} = b_j, \quad j = -m, \ldots, m, \qquad \alpha_{jk} = \frac{1}{2\sigma_0\sqrt{\pi}} \exp\left(-\frac{(j-k)^2}{4}\right), \qquad b_j = \frac{1}{\sqrt{2\pi(\sigma_d^2 + \sigma_0^2)}} \exp\left(-\frac{j^2\sigma_0^2}{2(\sigma_d^2 + \sigma_0^2)}\right). \tag{39}$$

This solution could result in negative amplitudes B_{dk} and thus negative densities, which are not defined. In order to circumvent a constrained minimizing problem, we introduce

$$\sigma_0^{max} = \max_{\sigma_0 \in \mathbb{R}^+} \min_k B_{dk}(\sigma_0) \tag{40}$$

and find the proportionality constant κ at which $\sigma_0^{max} = \sigma_d/\kappa$, where $\sigma_0^{max} > 0$. The retransformation of the decomposed densities to the original coordinates is done by

$$A_{jk} = A_j \cdot B_{jk}, \qquad \boldsymbol{\mu}_{jk} = \boldsymbol{\mu}_j + k\sigma_0 \cdot \boldsymbol{\phi}_{jk}, \qquad \mathbf{C}_{jk} = \sigma_0^2 \cdot \boldsymbol{\phi}_{ji} \cdot \boldsymbol{\phi}_{ji}^T + \sum_{s \neq i} \sigma_{js}^2 \boldsymbol{\phi}_{js} \cdot \boldsymbol{\phi}_{js}^T. \tag{41}$$

The eigenvectors $\boldsymbol{\phi}_{js}$ to the eigenvalues σ_{js}^2 have unit length and $\boldsymbol{\mu}_j$, $\mathbf{C}_j$ and $\boldsymbol{\mu}_{jk}$, $\mathbf{C}_{jk}$ denote mean vectors and covariance matrices to the corresponding densities p_j or p_{jk}.

3.2. Combination of Adjacent Densities

So far the algorithm would produce an exponential growth of densities because of the diffusion term in the stochastic differential equation. In order to obtain a feasible algorithm adjacent densities $p_i(\mathbf{X}, t)$, $p_j(\mathbf{X}, t)$ have to be combined to $p_c(\mathbf{X}, t)$. This is done by combining densities with a small difference between their mean vectors $\|\boldsymbol{\mu}_j - \boldsymbol{\mu}_i\| < TOL$, $\boldsymbol{\mu} \in \mathbb{R}^n$, using the following equations

$$A_c = A_i + A_j, \qquad \boldsymbol{\mu}_c = \frac{\boldsymbol{\mu}_i A_i + \boldsymbol{\mu}_j A_j}{A_c}, \qquad \mathbf{C}_c = \frac{A_i(\mathbf{C}_i + (\boldsymbol{\mu}_i - \boldsymbol{\mu}_c)(\boldsymbol{\mu}_i - \boldsymbol{\mu}_c)^T) + A_j(\mathbf{C}_j + (\boldsymbol{\mu}_j - \boldsymbol{\mu}_c)(\boldsymbol{\mu}_j - \boldsymbol{\mu}_c)^T)}{A_c}. \tag{42}$$

If the distances of all N densities have to be determined, then $N(N+1)/2$ distance evaluations are needed. This produces the highest computational cost in the Local Statistical Linearization routine, since adjacent densities have to be combined in each iteration step. An alternative approach is to cover the state space with boxes and limit the number of distributions within a box as described by Ellermann[16]. This approach reduces the computational cost for smaller system dimensions. Of course the number of boxes grows exponentially with the system dimension. Such a hybrid combination of the different methods is suggested to be computationally more efficient. More details on Gaussian mixture by local statistical linearization can be found in[12].

3.3. Result for Duffing oscillator

For validation, the invariant probability density of the Duffing oscillator with additive excitation by white noise was calculated with the Local Statistical Linearization method. Such stochastic Duffing oscillator can be written as

$$\mathrm{d}x = y\,\mathrm{d}t, \qquad \mathrm{d}y = (\alpha_1 x + \alpha_2 x + \alpha_3 x^3)\,\mathrm{d}t + \nu_1\,\mathrm{d}W_t, \tag{43}$$

where $\alpha_i, \nu_1 \in \mathbb{R}$. If the parameters $\alpha_1 = 1$, $\alpha_2 = -0.2$, $\alpha_3 = -1$, $\nu_1 = \sqrt{0.025}$ are chosen, then the exact stationary probability density function for this Duffing oscillator is given by $p(x, y) = c\,\exp(4x^2 - 2x^4 - 4y^2)$, with the normalization constant $c = 0.108234406$, cf. Pradlwarter[5]. At time $t = 0$ the initial probability density is a Gaussian density with mean μ and covariance matrix $\mathbf{C}$

$$\boldsymbol{\mu}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \mathbf{C}(0) = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}. \tag{44}$$

At $t = 40$, the unsteady probability density of the considered Duffing oscillator consists of 507 local Gaussian densities. At $t = 200$, the probability density converged to a stationary state and the Gaussian mixture consists of 907 Gaussian densities. The error evolution is measured by the Kullback-Leibler divergence in Fig. 3 and in Fig. 4 by the least squares error between the exact probability density function and the Gaussian mixture obtained by Local Statistical Linearization. In Fig. 3b) and in Fig. 4b) the error evolution can be seen in more detail. Further refinement of the Gaussian mixture leads to 1199 Gaussian densities at $t = 320$, which decreases the error slightly.

4. Conclusions

New procedures for the analysis of nonlinear dynamics of randomly perturbed Hamiltonian systems are presented. Thereby, the method of stochastic averaging of energy is developed further, and error evolution of Gaussian mixture by Local Statistical Linearization shows promising results for further research. The results enable the determination of functional relationships between the parameters of the considered Hamiltonian system and probability measures such as probability density functions and mean first passage times. Our results are applicable to various engineering problems, such as random vibrations of mechanical systems with nonlinear springs and dampers. Since the roll motion of a ship in random seas can be modelled as a nonlinear random dynamical system including nonlinear damping and a nonlinear softening spring, the results are also significant for the determination of new criteria for ship stability.

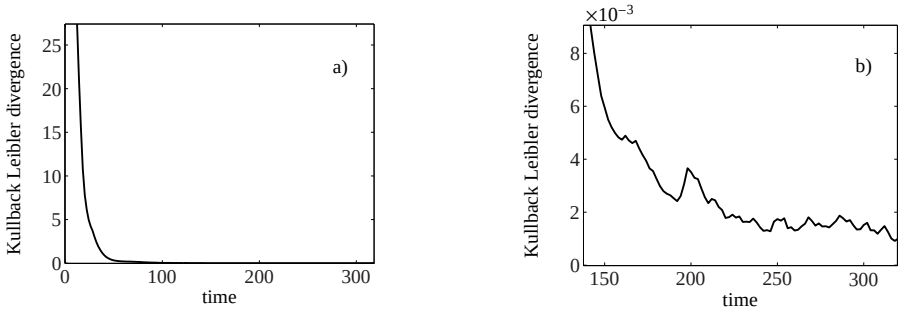

Figure 3. (a) Kullback-Leibler divergence; (b) zoom into figure a).

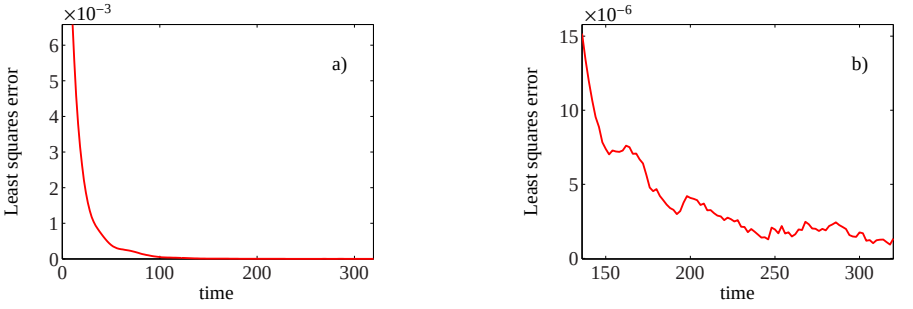

Figure 4. (a) Least squares error ; (b) zoom into figure a).

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