Fredholm Theory with Applications to Random Operators

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1 Preliminaries

1.1 Notation and basic facts

1.1.1 Sets of numbers

We denote the natural, integer, rational, real and complex numbers by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. Moreover, we denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ by $\mathbb{T}$ and the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ by $\mathbb{D}$. By $\mathbb{I}$ we denote countable index sets, usually $\mathbb{N}$, $\mathbb{Z}^N$ or $\{1, \dots, n\}$ for some $N \in \mathbb{N}$ and $n \in \mathbb{N}$. The real and imaginary parts of a complex number $z \in \mathbb{C}$ are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively. We also use the subscript notations $\mathbb{N}_{\geq k} := \{n \in \mathbb{N} : n \geq k\}$ for $k \in \mathbb{N}$, $\mathbb{C}_{\operatorname{Re} \geq \lambda} := \{z \in \mathbb{C} : \operatorname{Re} z \geq \lambda\}$ for $\lambda \in \mathbb{R}$, etc. Real intervals are denoted by $(\cdot,\cdot)$, $(\cdot,\cdot]$, $[\cdot,\cdot)$ or $[\cdot,\cdot]$, respectively. A square bracket indicates that the endpoint is included whereas a parenthesis indicates that the endpoint is excluded. The distance $\inf\limits_{m \in M} |z - m|$ of a point $z \in \mathbb{C}$ to a set $M \subset \mathbb{C}$ is denoted by $\operatorname{dist}(z, M)$. The (topological) boundary of a set $M \subset \mathbb{C}$ will be denoted by ∂M.

1.1.2 Sequences

Sequences of objects are denoted with parentheses, i.e. $(z_n)_{n \in \mathbb{I}}$ for some countable index set $\mathbb{I}$ (usually $\mathbb{N}$ or $\mathbb{Z}$). For simplicity we use the notation $(z_n)_{n \in \mathbb{I}} \subset M$, which is shorthand for "the elements of the sequence $(z_n)_{n \in \mathbb{I}}$ are contained in the set M". If a sequence $(z_n)_{n \in \mathbb{N}} \subset M$ converges to some element $z \in M$, we write $z_n \to z$ as $n \to \infty$. In case the topology in unclear we add the supplements "strongly", "weakly" etc.

1.1.3 Banach spaces

Banach spaces are usually denoted by X (Y, Z in case we need more than one) and the corresponding norm is denoted by $\|\cdot\|_X$. We will only use complex Banach spaces and just use $\|\cdot\|$ as long as there is no ambiguity. If X is also a Hilbert space, we denote the corresponding inner product[1] by $\langle \cdot, \cdot \rangle_X$ or just $\langle \cdot, \cdot \rangle$. Typical examples of Banach spaces are the so-called ℓ^p-spaces or (generalized) sequence spaces. Let $p \in [1, \infty)$, $\mathbb{I}$ a countable set, e.g. $\{1, \dots, k\}$, $\mathbb{N}$ or $\mathbb{Z}^N$ for some $k, N \in \mathbb{N}$, and let X be an arbitrary Banach space. Then we define

$$\ell^p(\mathbb{I}, X) := \left\{ (x_n)_{n \in \mathbb{I}} \subset X : \sum_{n \in \mathbb{I}} \|x_n\|_X^p < \infty \right\},$$

$$\|(x_n)_{n \in \mathbb{I}}\|_{\ell^p(\mathbb{I}, X)} := \|(x_n)_{n \in \mathbb{I}}\|_p := \left(\sum_{n \in \mathbb{I}} \|x_n\|_X^p \right)^{1/p}.$$

Similarly we define

$$\ell^\infty(\mathbb{I}, X) := \left\{ (x_n)_{n \in \mathbb{I}} \subset X : \sup_{n \in \mathbb{I}} \|x_n\|_X < \infty \right\},$$

$$\|(x_n)_{n \in \mathbb{I}}\|_{\ell^\infty(\mathbb{I}, X)} := \|(x_n)_{n \in \mathbb{I}}\|_\infty := \sup_{n \in \mathbb{I}} \|x_n\|_X .$$

[1] We use the convention that inner products are linear in the first argument and conjugate linear in the second argument.

Elements $x \in \ell^p(\mathbb{I}, X)$ are sometimes called p-summable sequences whereas elements $x \in \ell^\infty(\mathbb{I}, X)$ are called bounded sequences. If $p = 2$ and X is a Hilbert space, then $\ell^2(\mathbb{I}, X)$ is also a Hilbert space with inner product

$$\langle (x_n)_{n\in\mathbb{I}}, (y_n)_{n\in\mathbb{I}} \rangle_{\ell^2(\mathbb{I},X)} = \sum_{n\in\mathbb{I}} \langle x_n, y_n \rangle_X .$$

Additionally, we define the normed vector space $c_{00}(\mathbb{I}, X)$:

$$\begin{aligned} c_{00}(\mathbb{I}, X) &:= \{(x_n)_{n\in\mathbb{I}} \subset X : (x_n)_{n\in\mathbb{I}} \text{ contains only finitely many non-zero elements}\} , \\ \|(x_n)_{n\in\mathbb{I}}\|_{c_{00}(\mathbb{I},X)} &:= \|(x_n)_{n\in\mathbb{I}}\|_\infty = \sup_{n\in\mathbb{I}} \|x_n\|_X . \end{aligned}$$

The closure of $c_{00}(\mathbb{I}, X)$ w.r.t. $\|\cdot\|_\infty$ is a Banach space and will be denoted by $\ell^0(\mathbb{I}, X)$. In case we have $X = \mathbb{C}$, we will drop the second entry and just write $\ell^p(\mathbb{I})$ for all $p \in \{0\} \cup [1, \infty]$. We further abbreviate $\mathbf{X} := \ell^p(\mathbb{I}, X)$ in case p, $\mathbb{I}$ and X are fixed. If the abbreviation is used, they are fixed at the beginning of the respective section. We will also use the letter $\mathbf{Y}$ for Banach spaces with additional structure like approximate projections etc. Informally speaking, every $\mathbf{X}$ is a $\mathbf{Y}$ but not vice versa.

1.1.4 Bounded linear operators

For two Banach spaces X and Y we write $\mathcal{L}(X, Y)$ for the space of bounded linear operators from X to Y. Equipped with the operator norm

$$\|A\|_{\mathcal{L}(X,Y)} := \sup_{\substack{x\in X \\ \|x\|_X \leq 1}} \|Ax\|_Y$$

and the usual pointwise addition and scalar multiplication, $\mathcal{L}(X, Y)$ is again a Banach space. In the case $X = Y$ we simply write $\mathcal{L}(X)$. With the usual composition of operators, $\mathcal{L}(X)$ forms a unital Banach algebra. The identity operator will be denoted by I. Sometimes we will also use a subscript if we want to emphasize the space it belongs to. The dual space $\mathcal{L}(X, \mathbb{C})$ of X will be denoted by X^*.

The Hahn-Banach theorem implies that X is canonically and isometrically embedded into its bidual $X^{**} := (X^*)^*$ via the evaluation map $J \colon x \mapsto (f \mapsto f(x))$ $(f \in X^*)$. If J is an isometric isomorphism (i.e. if J is surjective), X is called reflexive. All Hilbert spaces are reflexive by the Riesz representation theorem.

It holds $\ell^p(\mathbb{I}, X)^* \cong \ell^q(\mathbb{I}, X^*)$ for $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Also, $\ell^0(\mathbb{I}, X)^* \cong \ell^1(\mathbb{I}, X^*)$ and $\ell^1(\mathbb{I}, X)^* \cong \ell^\infty(\mathbb{I}, X^*)$. However, $\ell^\infty(\mathbb{I}, X)^*$ is usually strictly larger than $\ell^1(\mathbb{I}, X^*)$. For $p \in (1, \infty)$ and X reflexive, $\ell^p(\mathbb{I}, X)$ is reflexive, too.

A bounded linear operator $K \in \mathcal{L}(X, Y)$ is called compact if it maps bounded sets to relatively compact sets. The set of compact operators will be denoted by $\mathcal{K}(X, Y)$. It is easily seen that $\mathcal{K}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$, hence again a Banach space. Moreover, if $A \in \mathcal{L}(X, Y)$ and $K \in \mathcal{K}(Y, Z)$, then $KA \in \mathcal{K}(X, Z)$ and also if $K \in \mathcal{K}(X, Y)$ and $A \in \mathcal{L}(Y, Z)$, then $AK \in \mathcal{K}(X, Z)$. Hence $\mathcal{K}(X) := \mathcal{K}(X, X)$ is a two-sided ideal in $\mathcal{L}(X)$.

For two bounded linear operators $A, B \in \mathcal{L}(X)$ we define the commutator $[A, B] := AB - BA$.

1.1.5 Quotient spaces

For a Banach space X and a subspace $Y \subset X$ we denote the quotient space by X/Y. If additionally Y is closed, X/Y is again a Banach space equipped with the quotient norm

$$\|x+Y\|_{X/Y} := \inf_{y \in Y} \|x+y\| .$$

Clearly, if Y is finite-dimensional, the infimum is attained as a minimum by a compactness argument. In fact, this is still true if Y is merely assumed to be reflexive.

If X is a Banach algebra and Y is a closed two-sided ideal of X, then X/Y is again a Banach algebra. As an example we mention the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$.

1.1.6 Matrices

If $X = Y = \mathbb{C}^n$, we also use the matrix notation $\mathbb{C}^{n\times n} := \mathcal{L}(X,Y) = \mathcal{L}(X)$. By specifying an orthonormal basis $\{e_1, \dots, e_n\}$ w.r.t. the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$, we use the standard notation for matrix entries, namely $A_{i,j} = \langle Ae_j, e_i \rangle$ for $i,j \in \{1,\dots,n\}$. Furthermore, we use the notions i-th row for $(A_{i,j})_{1\le j\le n}$ and j-th column for $(A_{i,j})_{1\le i\le n}$. The sequence $(A_{i,i})_{1\le i\le n}$ is called the main diagonal of A. For $k \in \{0,\dots,n-1\}$ we use the notion k-th subdiagonal and k-th superdiagonal for $(A_{i+k,i})_{1\le i\le n-k}$ and $(A_{i-k,i})_{k+1\le i\le n}$, respectively. Often we skip the prefixes and just say k-th diagonal, where positive k refer to subdiagonals and negative k refer to superdiagonals. The determinant and the trace of a matrix A will be denoted by $\det(A)$ and $\operatorname{tr}(A)$, respectively.

For the sequence spaces $\ell^p(\mathbb{I}, X)$ defined above, we introduce the same notation as follows. We define the canonical projections

$$P_U \colon \ell^p(\mathbb{I},X) \to \ell^p(\mathbb{I},X), \quad (P_U x)_j = \begin{cases} x_j & \text{if } j \in U \\ 0 & \text{if } j \notin U \end{cases}$$

for $U \subset \mathbb{I}$. Then the matrix entries are defined as

$$A_{i,j} \colon \operatorname{im}(P_{\{j\}}) \to \operatorname{im}(P_{\{i\}}), \quad A_{i,j} := P_{\{i\}} A P_{\{j\}}|_{\operatorname{im} P_{\{j\}}} \tag{1}$$

and we identify both $\operatorname{im}(P_{\{i\}})$ and $\operatorname{im}(P_{\{j\}})$ with X so that $A_{i,j} \colon X \to X$. Now rows, columns and diagonals are defined exactly the same way as above, i.e. $(A_{i,j})_{j\in\mathbb{I}}$ is called the i-th row, $(A_{i,j})_{i\in\mathbb{I}}$ is called the j-th column and $(A_{i+k,i})_{i\in\mathbb{I}}$ is called the k-th diagonal. Also note that the entries $A_{i,j}$ are operators themselves if X is more than just one-dimensional. Of course, if $X = \mathbb{C}$, we identify $\mathcal{L}(\mathbb{C}) \cong \mathbb{C}$ and the entries are just numbers as expected. Even easier, if additionally $p = 2$, we can take the standard orthonormal basis $\{e_i\}_{i\in\mathbb{I}}$ and define $A_{i,j} = \langle Ae_j, e_i\rangle$ as above. This is of course in accordance with (1). The matrix $A := (A_{i,j})_{i,j\in\mathbb{I}}$ (we justify this ambiguous notation in the next sentence) then defines an operator on $\ell^p(\mathbb{I},X)$ by the usual matrix-vector multiplication

$$(Ax)_i = \sum_{j\in\mathbb{I}} A_{i,j} x_j .$$

By [57, Section 1.3.5], A coincides with its matrix representation for $p < \infty$. Thus it is reasonable to denote the matrix representation again by A. In the case $p = \infty$ things get a bit more complicated. However, throughout this thesis we will only work with operators where there is no problem about it.

1.1.7 Operator topologies

Besides the norm topology that is induced by $\|\cdot\|_{\mathcal{L}(X,Y)}$, we also use the strong operator topology. The strong operator topology is defined to be the weakest topology for which point evaluation $\mathcal{L}(X,Y) \to Y$, $A \mapsto Ax$ is continuous for all $x \in X$. A sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X,Y)$ converges strongly to $A \in \mathcal{L}(X,Y)$ if and only if $\|(A_n - A)x\|_Y \to 0$ as $n \to \infty$ for all $x \in X$. We also use the weak operator topology, which is defined to be the weakest topology for which dual point evaluation $\mathcal{L}(X,Y) \to \mathbb{C}$, $A \mapsto f(Ax)$ is continuous for all $x \in X$ and $f \in Y^*$. A sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X,Y)$ converges weakly to $A \in \mathcal{L}(X,Y)$ if and only if $f(A_n x) \to f(Ax)$ as $n \to \infty$ for all $x \in X$ and $f \in Y^*$. In the case where X and Y are Hilbert spaces, a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X,Y)$ converges weakly to $A \in \mathcal{L}(X,Y)$ if and only if $\langle A_n x, y\rangle_Y \to \langle Ax, y\rangle_Y$ as $n \to \infty$ for all $x \in X$ and $y \in Y$.

1.1.8 The spectrum

If $A \in \mathcal{L}(X,Y)$ is invertible, i.e. there exists a $B \in \mathcal{L}(Y,X)$ such that $BA = I_X$ and $AB = I_Y$, we denote the (unique) inverse by A^{-1}. It is well-known that A is invertible if and only if A is bijective (Bounded Inverse Theorem). The sets

$$\ker(A) := \{x \in X : Ax = 0\} \quad \text{and} \quad \operatorname{im}(A) := \{y \in Y : \exists x \in X \text{ such that } y = Ax\}$$

are called kernel and range (or image) of A, respectively. If $X = Y$, we define the spectrum of A:

$$\operatorname{sp}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}.$$

By the fact mentioned above, the spectrum divides into three (not necessarily disjoint) parts:

$$\begin{aligned}\operatorname{sp}(A) = &\{\lambda \in \mathbb{C} : A - \lambda I \text{ is not injective}\}\\ &\cup \{\lambda \in \mathbb{C} : \operatorname{im}(A - \lambda I) \text{ is not closed in } X\}\\ &\cup \{\lambda \in \mathbb{C} : \operatorname{im}(A - \lambda I) \text{ is closed but not dense in } X\}.\end{aligned}$$

This decomposition will be of great importance throughout this thesis.

Similarly, one can also define the spectrum for arbitrary Banach algebras $\mathcal{A}$ with unit e:

$$\operatorname{sp}(a) := \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible}\}$$

for $a \in \mathcal{A}$. The spectrum of an element $a \in \mathcal{A}$ is always non-empty and compact. Furthermore, the spectral radius formula

$$\rho(a) := \max\{|\lambda| : \lambda \in \operatorname{sp}(a)\} = \lim_{n\to\infty} \sqrt[n]{\|a^n\|} \leq \|a\|$$

holds.

In the case of a finite matrix $A \in \mathbb{C}^{n\times n}$ injectivity and surjectivity coincide. Furthermore, it holds

$$\operatorname{sp}(A) = \{\lambda \in \mathbb{C} : \det(A - \lambda I) = 0\}.$$

1.1.9 Convention on projections and strong convergence

Often we want to interpret expressions like $P_U A P_U$, where $U \subset \mathbb{I}$, as an operator $\ell^p(U,X) \to \ell^p(U,X)$. So technically we are considering $SP_U A P_U S^{-1}$, where $S\colon \operatorname{im}(P_U) \to \ell^p(U,X)$ is an isometric isomorphism. However, we do not want to write this down every time. Thus we just write "interpreted as a finite matrix" or whatever feels appropriate in the respective situation.

Conversely, we often want to write that a sequence of finite matrices $(A_n)_{n\in\mathbb{N}}$ converges strongly to a bounded linear operator $A \in \mathcal{L}(\ell^p(\mathbb{N}))$ (or $A \in \mathcal{L}(\ell^p(\mathbb{Z}))$). This does not quite make sense because the elements of the sequence $(A_n)_{n\in\mathbb{N}}$ and A do not belong to the same space. If such a case occurs, we want to understand the finite matrix A_n as a submatrix of an infinite matrix $\tilde{A}_n$ in the following way:

$$\tilde{A}_n := S^{-1}A_n SP_U + c(I - P_U) = \left(\begin{array}{c|c} \boxed{A_n} & \\ \hline & cI \end{array}\right) \in \ell^p(\mathbb{N})$$

(and similarly for $A \in \mathcal{L}(\ell^p(\mathbb{Z}))$), where I is of the appropriate size depending on the size of A_n and $c \in \mathbb{C}$ is chosen in such a way that the property we are interested in, does not change. For example, if we are interested in norms, we choose $c = 0$ to guarantee $\|\tilde{A}_n\| = \|A_n\|$ etc. Then $\tilde{A}_n \to A$ strongly makes sense at least formally. For convenience we still write $A_n \to A$.

1.1.10 The adjoint operator

For $A \in \mathcal{L}(X,Y)$ we define the adjoint operator $A^* \in \mathcal{L}(Y^*, X^*)$ by $(A^*f)(x) := f(Ax)$ for all $x \in X$ and $f \in Y^*$. In the case where X and Y are Hilbert spaces, one usually defines the adjoint slightly different, namely as the unique operator A^* that satisfies $\langle Ax, y\rangle_Y = \langle x, A^*y\rangle_X$ for all $x \in X$ and $y \in Y$. Unfortunately, this is not exactly the same as the definition for general Banach spaces. For example, if we consider $A \in \mathcal{L}(\ell^2(\mathbb{I}))$, then A^* in the Hilbert space definition is exactly the Hermitian adjoint of A (i.e. $A^*_{i,j} = \overline{A_{j,i}}$ for all $i,j \in \mathbb{I}$). On the other hand, in the Banach space definition A^* is rather the transpose of A (i.e. $A^*_{i,j} = A_{j,i}$ for all $i,j \in \mathbb{I}$). Therefore a different notation for the Banach space adjoint (e.g. A^T) could be beneficial. Nevertheless, we want to stick with A^* to be in accordance with the literature. We have to be careful in the Hilbert space case, though.

Now let X, Y and Z be Banach spaces. Then the following properties of the adjoint are immediate for all $A, B \in \mathcal{L}(X,Y)$, $C \in \mathcal{L}(Y,Z)$, $\lambda \in \mathbb{C}$:

(i) $(A+B)^* = A^* + B^*$,

(ii) $(\lambda A)^* = \lambda A^*$ ($(\lambda A)^* = \overline{\lambda} A^*$ in Hilbert space),

(iii) $(CA)^* = A^*C^*$,

(iv) $\|A\| = \|A^*\|$,

(v) $(A^*)^{-1} = (A^{-1})^*$ if A is invertible,

(vi) A is a restriction of $A^{**} := (A^*)^*$ under the canonical embeddings $X \to X^{**}$ and $Y \to Y^{**}$. If X is reflexive, then $A^{**} = A$ in the above sense.

If there is a Banach space Y such that $Y^* = X$ (a so-called predual space of X) and an operator $B \in \mathcal{L}(Y)$ such that $B^* = A \in \mathcal{L}(X)$, then B is called the preadjoint of A. If X is reflexive, then clearly $A^* \in \mathcal{L}(X^*)$ is the unique preadjoint of A. If X is not reflexive, the preadjoint may not be unique or not even exist. For example, $\ell^0(\mathbb{Z})$ does not have a predual space, hence no predual operators exist. On the other hand, $\ell^1(\mathbb{Z})$ has multiple predual spaces and hence an operator may have multiple predual operators.

If X is a Hilbert space, then $\|AA^*\| = \|A\|^2 = \|A^*A\|$ holds. Moreover, if $A^* = A$, then A is called self-adjoint. Similarly, if $AA^* = A^*A$, then A is called normal. Obviously every self-adjoint operator is normal, but not every normal operator is self-adjoint. Normal operators satisfy $\rho(A) = \|A\|$.

If X and Y are Hilbert spaces and $A\colon X \to Y$ satisfies $A^* = A^{-1}$, then A is called unitary. Equivalently, A is unitary if

$$\langle Ax, Ay\rangle_Y = \langle x, y\rangle_X$$

for all $x, y \in X$. Hence unitary operators are isometric isomorphisms (and vice versa).

1.2 Overview

1.2.1 Introduction and historical remarks

Fredholm theory has a fairly long history that started with a very famous paper by E.I. Fredholm [30] in 1903. Fredholm studied integral equations of the form

$$\varphi(x) + \int_a^b f(x,y)\varphi(y)\mathrm{d}y = \psi(x), \quad x \in [a,b]. \tag{2}$$

He observed that, under some integrability assumptions, Equation (2) either has a solution φ for all right-hand sides ψ or Equation (2) has a non-trivial solution φ for $\psi = 0$ (not both). In operator language, writing Equation (2) as $S_f\varphi = \psi$ as in [30], this means that S_f is either surjective or not injective (not both). In other words, S_f is either injective and surjective or none of the two. This is widely known as Fredholm's alternative and was later generalized by F.V. Atkinson [2]. Fredholm's work inspired D. Hilbert [45] to a spectral theorem for symmetric integral operators. This was later generalized by F. Riesz [77] to the now famous spectral theorem for compact operators. We will come back to the work of Hilbert below.

This thesis is mainly concerned with the study of Fredholm operators. An operator A is called Fredholm if both the kernel and the cokernel of A are finite-dimensional. The difference $\dim(\ker(A)) - \dim(\operatorname{coker}(A))$ is then called the Fredholm index of A. Since the identity operator is obviously Fredholm with index 0, Fredholm's theorem says that S_f, which can be seen as a pertubation of the identity by a compact operator, has the same index as the identity (if it is Fredholm). As Atkinson proved almost 50 years later [2], this is true in a much more general context. He showed that for operators on arbitrary Banach spaces the Fredholm index is invariant under compact perturbations. In particular, every compact perturbation of a Fredholm operator is again Fredholm.

Sometimes Fredholm operators are also called Noetherian, which refers to F. Noether who studied integral equations of the form

$$g(x)\varphi(x) + \int\limits_a^b f(x,y)\varphi(y)\mathrm{d}y = \psi(x), \quad x \in [a,b],$$

where $g\colon [a,b] \to \mathbb{R}$ is a continuous function (compare with (2)). Noether introduced the index of such an integral equation as the winding number of a certain function around 0. As it turned out, this index coincides with the Fredholm index defined above.

Alongside the Fredholm theory, we are also interested in the spectra of certain bounded linear operators. Combining these two notions, we consider the so-called essential spectrum

$$\mathrm{sp}_{\mathrm{ess}}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}$$

of an operator A. As it turns out, this can be viewed as the spectrum of the coset $A + \mathcal{K}(X)$ in the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$.

The first spectral theorem was given by R. Descartes [25] in 1637 and is known as the principal axes theorem for quadratic forms. It states that a quadratic form $Q\colon \mathbb{R}^2 \to \mathbb{R}$ given by $Q(x) := ax_1^2 + bx_1x_2 + cx_2^2$ for some real numbers a, b, c can be transformed to normal form $\lambda_1 y_1^2 + \lambda_2 y_2^2$ ($\lambda_1, \lambda_2 \in \mathbb{R}$), where the coordinate axes x_1 and x_2 are rotated to the principal axes y_1 and y_2. This can be viewed as an early form of the spectral theorem for symmetric 2×2 matrices. This was later generalized to quadratic forms $Q\colon \mathbb{R}^3 \to \mathbb{R}$ by J.L. Lagrange [51] in 1759. Lagrange also mentioned that his approach is not limited to three dimensions. However, his purely algebraic approach was not suited for a rigorous proof of the principal axes theorem in arbitrary dimension. It took another 70 years until A.L. Cauchy [13] was able to prove the general version of this theorem that is known as the spectral theorem for symmetric matrices nowadays. However, matrix language was not yet established for these kind of problems. Although matrices were used before by C.F. Gauss and other mathematicans at that time to solve systems of linear equations, it were J.J. Sylvester [88] and A. Cayley [14] in the 1850s who observed that the pricipal axes transformation is equivalent to the diagonalization of the symmetric matrix corresponding to the quadratic form and that the coefficients λ_i are exactly the roots of the characteristic polynomial of this matrix, the so-called eigenvalues.

In the 19th century J. Fourier [29] and others used the “method of infinitely many variables”, which is nothing else than our concept of sequence spaces and infinite matrices, to determine the solutions of differential equations. Fourier tried to write every function as an infinite linear combination of trigonometric functions, nowadays called the Fourier series. The coefficients of this linear combination then had to be determined by an infinite system of linear equations. Fourier’s attempt was to approximate the system by finite systems of increasing size. This method is still used today and called the finite section method (FSM). Fourier and other mathematicians at that time were not too worried about the convergence of infinite series and a rigorous definition of function spaces had yet to be established. In 1902, one year before the works of Fredholm, H. Lebesgue [54] introduced his theory of integrable functions. Riesz [76] extended the work of Lebesgue by introducing L^p-spaces a few years later. Around the same time the sequence spaces ℓ^p were extensively studied. Hilbert used the space of square-summable sequences ℓ^2, which was henceforth called Hilbert’s space, to prove his spectral theorem about symmetric integral operators [45]. Hilbert was also the first one to generalize eigenvalues to the concept of spectra in the case of bounded linear operators on ℓ^2. The set of eigenvalues he called point spectrum, whereas the residual part of the spectrum he called the continuous spectrum, notions that are still in use for self-adjoint bounded linear operators.

In the following, spectral theory was highly influenced by the development of quantum mechanics. W. Heisenberg observed that certain quantities, so-called observables, can not be measured simultaneously on the quantum level. If one considers the measurement of an observable as a projection onto an eigenspace, this means that the observables do not have the same eigenspaces and consequently, the observables do not commute. Clearly, this can not happen with the real numbers used in classical mechanics. Thus Heisenberg and his co-workers [10] used operators to describe physical quantities. At the same time E. Schrödinger [81] developed wave mechanics and deduced the exact same results as Heisenberg did before. The most influential part of his work definitely included the now famous Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi,$$

where $\hbar$ is the Planck constant, H denotes the Hamiltonian (an operator that describes the energy of the system, a concept used in almost every part of modern physics) and Ψ is a wave function. The spectrum of H determines the possible energy states of a system. As an example we mention the hydrogen atom, which consists of exactly one proton and one electron. Using the Schrödinger equation and computing the eigenvalues of the Hamiltonian, one can show that the electron can only attain a discrete number of energy states, which are represented by the eigenvalues of H.

Although we are not dealing with unbounded linear operators in this thesis, it should be mentioned that Hamiltonians in quantum mechanics are often unbounded. Thus Hilbert's approach of bounded linear operators on ℓ^2 was not quite sufficient to describe quantum systems. It was J. von Neumann [69] who introduced abstract Hilbert spaces and provided a rigorous framework for unbounded linear operators, which is still used to great success today.

In this thesis we are mainly going to consider the (Banach space valued) sequence spaces ℓ^p. It should be mentioned that the function spaces L^p can be considered as Banach space valued sequence spaces as well, thus including them in our considerations whenever we allow Banach space valued sequences. One of the main tools to study the Fredholm properties of bounded linear operators on ℓ^p are the so-called limit operators. Since the Fredholmness of an operator is invariant under finite-dimensional perturbations, the information has to be stored somewhere at infinity (in a way we will make precise below). In order to reach infinity in one way or another, we will use a certain limiting procedure, which is called the theory of limit operators. For a certain class of operators, the so-called band-dominated operators, these limit operators determine the essential spectrum entirely. As M. Lindner and M. Seidel [61] showed only recently, the essential spectrum of a band-dominated operator is equal to the union of the spectra of its limit operators. This result completed a long list of previous results in that direction that we mention below.

The first main ideas concerning limit operators go back to J. Favard [27], who studied systems of ordinary differential equations with almost periodic coefficients. About 50 years later, in the 1980s, E.M. Muhamadiev [68] was able to prove a similar result to that of Lindner and Seidel in the case of certain elliptic differential operators. Shortly after, B.V. Lange and V.S. Rabinovich [53] were able to apply the ideas of Muhamadiev to $\ell^p(\mathbb{Z}^N)$, where $p \in (1,\infty)$. They showed that a band-dominated operator on $\ell^p(\mathbb{Z}^N)$ is Fredholm if and only if all of its limit operators are invertible and their inverses are uniformly bounded. Unfortunately, this is not quite enough to formulate the analogous statement about spectra because of this "nasty" uniform boundedness condition. In the case of operators in the Wiener algebra, a particular subalgebra of the algebra of band-dominated operators, Lange and Rabinovich managed to show that the uniform boundedness condition is redundant. In particular, this implied the result about the spectra for these particular operators. In 1998 Rabinovich, S. Roch and B. Silbermann [73] were able to generalize these results to the case of vector valued ℓ^p-spaces, i.e. $\ell^p(\mathbb{Z}^N, \mathbb{C}^n)$, and provide an interesting connection to the finite section method we already mentioned above. The latter result was later generalized by Lindner [55], who showed that the finite section method is applicable to a band-dominated operator $A \in \ell^p(\mathbb{Z}^N, X)$, where $p \in [1, \infty]$ and X an arbitrary complex Banach space, if and only if A and certain submatrices of its limit operators are invertible.

It were again Rabinovich, Roch and Silbermann [74] who managed to generalize the Fredholm results even further. They considered the Hilbert space valued sequence space $\ell^2(\mathbb{Z}^N, H)$, where H was an arbitrary Hilbert space. However, this time one has to generalize notions like Fredholmness and compactness to what is called $\mathcal{P}$-Fredholmness and $\mathcal{P}$-compactness, respectively. In retrospect one may argue that these are exactly the right concepts to study ℓ^p-spaces even in the scalar case, in particular if $p \in \{1, \infty\}$. They provide a strong algebraic framework that we want to call the $\mathcal{P}$-framework. Using this new framework, Rabinovich, Roch and Silbermann [75] managed to further generalize the above results to $\ell^p(\mathbb{Z}^N, X)$ for $p \in (1, \infty)$ and X an arbitrary Banach space. The remaining cases $p \in \{1, \infty\}$ were solved by Lindner in [55]. It remained to show that the uniform boundedness assumption is actually redundant. This was achieved by Lindner and Seidel [61] in 2013, which completes the theory of limit operators in some sense. Efforts are being made to further generalize the limit operator theory from $\mathbb{Z}^N$ to certain discrete metric spaces, though (e.g. the recent paper by J. Špakula and R. Willett [87]).

As it turns out, limit operators are perfectly suited to study random operators on ℓ^p-spaces. Roughly speaking, we call an operator $A \in \mathcal{L}(\ell^p(\mathbb{Z}^N, X))$ random if its matrix entries are randomly distributed w.r.t. some probability measures. The first appearance of random operators was in a paper of E.P. Wigner [92] in 1955. This paper was motivated by an earlier paper by A.M. Lane, R.G. Thomas and Wigner [52] himself, where nucleon interactions were studied by statistical means. The idea of Wigner and his co-workers was to describe an overly complicated nucleon system containing many quantum particles that underlie the strong interaction (also called strong nuclear force) by a whole family of Hamiltonians equipped with a certain probability distribution. This allows to describe a whole system without knowing the exact interaction of every single particle. Similar ideas are also used in statistical approaches to thermodynamics and other fields where a high number of particles is involved. Wigner studied a particular class of symmetric random operators that he called bordered matrices. These were defined as the sum of an unbounded diagonal matrix and a symmetric random sign matrix with only finitely many non-zero diagonals, e.g.

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & -2 & v_{-2,-1} & 0 & 0 & 0 & \dots \\ \dots & v_{-2,-1} & -1 & v_{-1,0} & 0 & 0 & \dots \\ \dots & 0 & v_{-1,0} & 0 & v_{0,1} & 0 & \dots \\ \dots & 0 & 0 & v_{0,1} & 1 & v_{1,2} & \dots \\ \dots & 0 & 0 & 0 & v_{1,2} & 2 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3}$$

where $v_{j,j-1} \in \{\pm v\} \subset \mathbb{R}$ for all $j \in \mathbb{Z}$ and the signs are randomly distributed (see (4) in [92]). In fact, it is not hard to show that all possible samples are unitarily equivalent and thus share the same spectrum. This is no longer true if we drop the symmetry assumption. However, as we will see below, we still have that almost every sample has the same spectrum.

Three years later P.W. Anderson [1], who was awarded the Nobel Prize "for his fundamental theoretical investigations of the electronic structure of magnetic and disordered systems" in 1977, proposed a model that can be used to describe spin diffusion ("hopping quantum particles") on a lattice with randomly distributed potentials. In the one-dimensional case the corresponding Hamiltonian is given by

$$\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & 1 & & & & \\ & 1 & v_{-1} & 1 & & & \\ & & 1 & v_0 & 1 & & \\ & & & 1 & v_1 & 1 & \\ & & & & 1 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \in \mathcal{L}(\ell^2(\mathbb{Z})),$$

where the entries v_j are randomly distributed (with positive variance). In arbitrary dimension the Hamiltonian is given by the matrix $A \in \mathcal{L}(\ell^2(\mathbb{Z}^N))$, where

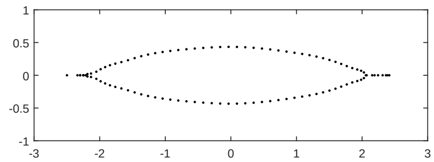
$$A_{i,j} = \begin{cases} v_i & \text{if } i = j, \\ 1 & \text{if } \|i-j\|_1 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Anderson conjectured that in dimension $N = 1$ and $N = 2$ the spectrum of the respective Hamiltonian only contains eigenvalues (with probability one) and that this is no longer true for $N \geq 3$ at least if the disorder, i.e. the variance of the distribution, is small. Physically this would mean that the system tends to insulate for $N \in \{1, 2\}$ and conducts for $N \geq 3$ at low disorder. The one-dimensional case was solved by H. Kunz and B. Souillard [50] in 1980 under the assumption that the underlying probability distribution is absolutely continuous w.r.t. the Lebesgue measure. This was generalized to arbitrary dimension by J. Fröhlich and T. Spencer [32] in the case of large disorder. The case of singular distributions, e.g. the Bernoulli distribution, was solved by R. Carmona, A. Klein and F. Martinelli [12] for $N = 1$. In the subsequent years the Anderson model was also discussed on different lattices like Cayley graphs (e.g. [49]).

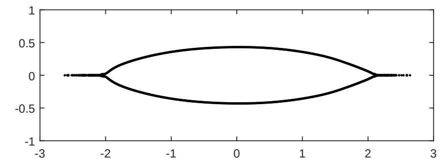
In 1996 N. Hatano and D.R. Nelson [41] started to consider non-self-adjoint random operators of the form

$$\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & e^{g} & & & & \\ & e^{-g} & v_{-1} & e^{g} & & & \\ & & e^{-g} & v_0 & e^{g} & & \\ & & & e^{-g} & v_1 & e^{g} & \\ & & & & e^{-g} & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \in \mathcal{L}(\ell^2(\mathbb{Z})),$$

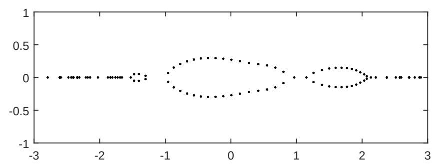
where the constant $g > 0$ is determined by the transverse magnetic field and the entries v_j are again randomly distributed, to study vortex line pinning in superconductors. Hatano and Nelson



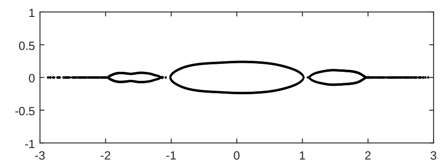
(a) Eigenvalues of a Hatano-Nelson sample with $g = \frac{1}{4}$ and $n = 100$ uniformly distributed random numbers in $[-1, 1]$.



(b) Eigenvalues of a Hatano-Nelson sample with $g = \frac{1}{4}$ and $n = 1000$ uniformly distributed random numbers in $[-1, 1]$.



(c) Eigenvalues of a Hatano-Nelson sample with $g = \frac{1}{4}$ and $n = 100$ uniformly distributed random numbers in $\{\pm 1\}$.



(d) Eigenvalues of a Hatano-Nelson sample with $g = \frac{1}{4}$ and $n = 1000$ uniformly distributed random numbers in $\{\pm 1\}$.

Figure 1

additionally assumed periodic boundary conditions which reduces the problem to the study of finite random matrices

$$\begin{pmatrix} v_1 & e^g & & & e^{-g} \\ e^{-g} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & e^g \\ e^g & & & e^{-g} & v_n \end{pmatrix}.$$

In contrast to the Anderson model (which can be interpreted as the case $g = 0$) the rightward hopping amplitude e^g is different from the leftward hopping amplitude e^{-g} here. The resulting complex eigenvalues can be interpreted in terms of certain properties of tilted vortex lines (see [42]). In Figure 1 we can see the eigenvalues of some Hatano-Nelson samples.

In 1999 J. Feinberg and A. Zee [28] proposed a different kind of non-self-adjoint random hopping model. In contrast to the Anderson or the Hatano-Nelson model, the randomness is now located on the first subdiagonal:

$$A := \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 0 & 1 & & & & \\ & v_{-1} & 0 & 1 & & & \\ & & v_0 & 0 & 1 & & \\ & & & v_1 & 0 & 1 & \\ & & & & v_2 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \in \mathcal{L}(\ell^2(\mathbb{Z})), \tag{4}$$

where the entries v_j are uniformly distributed random numbers in $\{\pm 1\}$. This matrix can be used to describe the system of a quantum particle that is hopping on a one-dimensional lattice and randomly changes its spin (where we consider only two spin states, i.e. "up" and "down") whenever it jumps to the right. The spectrum of A is completely unknown, lower and upper bounds can be found in Figure 2 below.

In 2001 E.B. Davies [23] introduced the concept of pseudo-ergodic operators and observed that, under some assumptions on the probability space, random operators are pseudo-ergodic almost surely. Altough this observation is somewhat obvious, it was the cornerstone of subsequent spectral studies because it eliminates all probabilistic arguments immediately. In this spirit several results for non-self-adjoint random operators were achieved, e.g. [15, 16, 17, 20, 59, 64, 65].

1.2.2 Main results

This thesis starts with an introduction to Fredholm theory. We present the standard results of classical Fredholm theory like Atkinson's theorem and perturbation results. We also show a result that generalizes Weyl's criterion and relates Fredholm operators to singular Weyl sequences. We proceed with an introduction to limit operators on $\ell^p(\mathbb{Z})$, where $p \in (1,\infty)$, and explain how they are used to get information about the essential spectrum.

In Section 3 we introduce the $\mathcal{P}$-Framework that enables us to treat the generalized sequences spaces $\mathbf{X} := \ell^p(\mathbb{Z}^N, X)$, where $N \in \mathbb{N}$, $p \in \{0\} \cup [1,\infty]$ and X an arbitrary Banach space, as well. It should be noted that, since the notions coincide, all subsequent results also apply without the $\mathcal{P}$ in the case of finite-dimensional X and $p \in (1,\infty)$. Thus, as immediate corollaries so to say, we obtain results that are interesting in the classical Fredholm theory as well. We take the theorem of Lindner and Seidel, that relates the $\mathcal{P}$-essential spectrum of a band-dominated operator with the spectra of its limit operators and was mentioned in the introduction, as a starting point and extend this result to other spectral quantities like the norm, the ε-pseudospectra, the numerical range and, to some extent, also the lower norm:

Result 1. *(Theorem 3.26, Theorem 3.35, Corollary 3.50, published in [39] and Theorem 3.76, partially published in [36])*
Let $A \in \mathrm{BDO}_\$(\mathbf{X})$, $\varepsilon > 0$ and denote the set of limit operators of A by $\sigma^{\mathrm{op}}(A)$. Then the following assertions holds:

(i) $\|A + \mathcal{K}(\mathbf{X},\mathcal{P})\| = \max\limits_{A_h \in \sigma^{\mathrm{op}}(A)} \|A_h\|,$

(ii) $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) := \mathrm{sp}_\varepsilon(A + \mathcal{K}(\mathbf{X},\mathcal{P})) = \bigcup\limits_{A_h \in \sigma^{\mathrm{op}}(A)} \mathrm{sp}_\varepsilon(A_h),$

(iii) $N_{\mathrm{ess}}(A) := \bigcap\limits_{K \in \mathcal{K}(\mathbf{X},\mathcal{P})} N(A+K) = \mathrm{conv}\left(\bigcup\limits_{B \in \sigma^{\mathrm{op}}(A)} N(B)\right).$

If in addition A is $\mathcal{P}$-Fredholm and $A + \mathcal{K}(\mathbf{X},\mathcal{P})$ contains an operator being bounded below, then

(iv) $\nu_{\mathrm{ess}}(A) := \sup\limits_{K \in \mathcal{K}(\mathbf{X},\mathcal{P})} \nu(A+K) = \min\limits_{A_h \in \sigma^{\mathrm{op}}(A)} \nu(A_h).$

The $\mathcal{P}$-essential pseudospectra are interesting because they approximate the $\mathcal{P}$-essential spectrum:

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcap_{\varepsilon>0} \mathrm{sp}_{\varepsilon,\mathrm{ess}}(A).$$

Also, they satisfy the following perturbation result, which is quite similar to the perturbation result for the usual pseudospectra:

Result 2. *(Theorem 3.38, published in [39])*
Let $\mathcal{C} \in \{\mathrm{BO}_\$(\mathbf{X}), \mathrm{BDO}_\$(\mathbf{X})\}$ and let $A \in \mathcal{C}$. Then for every $\varepsilon > 0$, we have

$$\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{\substack{\|T\|<\varepsilon,\\ T\in\mathcal{L}(\mathbf{X},\mathcal{P})}} \mathrm{sp}_{\mathrm{ess}}(A+T) = \bigcup_{\substack{\|T\|<\varepsilon,\\ T\in\mathcal{C}}} \mathrm{sp}_{\mathrm{ess}}(A+T).$$

In order to obtain more information about these $\mathcal{P}$-essential pseudospectra, we further investigate the inverse of the coset $A + \mathcal{K}(\mathbf{X},\mathcal{P})$ and obtain the following rather technical but useful result:

Result 3. *(Theorem 3.41, published in [39])*
Let $A \in \mathrm{BDO}_\$(\mathbf{X})$. Then

$$\left\|(A+\mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\right\|^{-1} = \lim_{m\to\infty} \min\left\{\nu(A|_{\mathrm{im}\,Q_m}), \nu(A^*|_{\mathrm{im}\,Q_m^*})\right\},$$

where $Q_m := P_{\mathbb{Z}^N\setminus\{-m,\ldots,m\}^N}$.

Using this, we deduce several new characterizations of the $\mathcal{P}$-essential pseudospectra. In the cases where $\mathbf{X}$ is a Hilbert space or X is finite-dimensional, we obtain particularly nice results. The results about the $\mathcal{P}$-essential pseudospectra are summarized in Theorem 3.69, which is also published in [39].

The results obtained in Section 3 are then applied to random operators on $\mathbf{X} := \ell^p(\mathbb{Z}, X)$. We thereby follow the pseudo-ergodic approach by Davies [23], which circumvents probabilistic arguments. In particular, we show that under some minor assumptions on the probability space ((C1) - (C4)), random operators are pseudo-ergodic almost surely. As a corollary of our results from Section 3 we thus get the following important result:

Result 4. *(Corollary 4.12)*
Let $(\Omega, \mathbb{P})$ be a probability space and let $A\colon \Omega \to \mathrm{BDO}(\mathbf{X})$ be a random operator that satisfies the conditions (C1) - (C4). Then

(i) $\mathrm{sp}(A(\omega)) = \mathrm{sp}_{\mathrm{ess}}(A(\omega)) = \bigcup\limits_{B\in A(\Omega)} \mathrm{sp}(B)$,

(ii) $\|A(\omega)\| = \|A(\omega) + \mathcal{K}(\mathbf{X},\mathcal{P})\| = \max\limits_{B\in A(\Omega)} \|B\|$,

(iii) $\mathrm{sp}_\varepsilon(A(\omega)) = \mathrm{sp}_{\varepsilon,\mathrm{ess}}(A(\omega)) = \bigcup\limits_{B\in A(\Omega)} \mathrm{sp}_\varepsilon(B)$ *for all $\varepsilon > 0$,*

(iv) $\nu(A(\omega)) = \nu_{\mathrm{ess}}(A(\omega)) = \min\limits_{B\in A(\Omega)} \nu(B)$

for almost every $\omega \in \Omega$. Additionally, if $\mathbf{X}$ is a Hilbert space, then

(v) $N(A(\omega)) = N_{\mathrm{ess}}(A(\omega)) = \bigcup\limits_{B\in A(\Omega)} N(B)$

for almost every $\omega \in \Omega$.

Thus by computing the spectral quantities of several operators on the right-hand side, we obtain information about the respective spectral quantities of the random operator. In the case of the norm, the lower norm, the pseudospectra and the numerical range we obtain approximation results in terms of periodic operators:

Result 5. *(Theorem 4.24, Theorem 4.27, Corollary 4.29, Theorem 4.30, partially published in [36])*
Let $(\Omega, \mathbb{P})$ be a probability space and let $A\colon \Omega \to \mathrm{BDO}(\mathbf{X})$ be a random operator that satisfies the conditions (C1) - (C4). Denote by $M_{per}(\Omega) \subset A(\Omega)$ the set of periodic operators contained in $A(\Omega)$. Then

(i) $\|A(\omega)\| = \sup_{B \in M_{per}(\Omega)} \|B\|$,

(ii) $\nu(A(\omega)) = \inf_{B \in M_{per}(\Omega)} \nu(B)$,

(iii) $\mathrm{sp}_\varepsilon(A(\omega)) = \bigcup_{B \in M_{per}(\Omega)} \mathrm{sp}_\varepsilon(B)$ *for all* $\varepsilon > 0$,

for almost every $\omega \in \Omega$. Additionally, if $\mathbf{X}$ is a Hilbert space, then

(iv) $N(A(\omega)) = \mathrm{clos}\left(\bigcup_{B \in M_{per}(\Omega)} N(B) \right)$

for almost every $\omega \in \Omega$.

As it turns out, one can not obtain a similar result for the spectrum. For example one can construct a random operator with two non-zero diagonals that satisfies $\mathrm{sp}(A(\omega)) = \mathbb{D}$ for almost every $\omega \in \Omega$ but $\bigcup_{B \in M_{per}(\Omega)} \mathrm{sp}(B) = \{0\} \cup \mathbb{T}$. However, one can obviously still get lower bounds by computing the spectra of periodic operators. To do so, we explain the symbol calculus, which can be viewed as a spectral theorem for periodic operators. In the case of tridiagonal periodic operators, the symbol calculus is particularly helpful and one can obtain very simple formulas that we carry out in detail. We then apply these results to tridiagonal random operators. A remarkable result for tridiagonal random operators is the following:

Result 6. *(Theorem 4.52, published in [36])*
Let $(\Omega, \mathbb{P})$ be a probability space and let $A\colon \Omega \to \mathrm{BDO}(\mathbf{X})$ be a tridiagonal random operator that satisfies the conditions (C1) - (C4). Denote by $L(\Omega) \subset A(\Omega)$ the set of Laurent operators contained in $A(\Omega)$. Then

$$N(A(\omega)) = \mathrm{conv}\left(\bigcup_{B \in L(\Omega)} \mathrm{sp}(B) \right)$$

for almost every $\omega \in \Omega$.

By the symbol calculus, the set on the right-hand side is just the convex hull of the union of certain ellipses and can thus be computed explicitly. Furthermore, this result implies the following surprising corollary:

Result 7. *(Corollary 4.53, published in [36])*
Let $(\Omega, \mathbb{P})$ be a probability space and let $A\colon \Omega \to \mathrm{BDO}(\mathbf{X})$ be a tridiagonal random operator that satisfies the conditions (C1) - (C4). Then

$$N(A(\omega)) = \mathrm{conv}(\mathrm{sp}(A(\omega)))$$

for almost every $\omega \in \Omega$.

In the last part of this thesis we consider the Feinberg-Zee random hopping matrix introduced above. Let π_∞ be the union of spectra of periodic operators with ones on the first superdiagonal, plus and minus ones on the first subdiagonal and zero everywhere else (cf. (4)). Similarly, let σ_∞ be the union of spectra of finite matrices of this kind. Then

Result 8. *(Theorem 4.58, published in [38])*
σ_∞ is a dense subset of π_∞.

Both π_∞ and σ_∞ are subsets of the spectrum of the Feinberg-Zee random hopping matrix that we want to call Σ. More precisely, a sample of the Feinberg-Zee random hopping matrix has spectrum Σ almost surely and $\sigma_\infty \subset \pi_\infty \subset \Sigma$. Therefore every subset of π_∞ or σ_∞ yields a subset of Σ and thus a better understanding of the spectrum of the Feinberg-Zee random hopping matrix. It is even conjectured (see [15]) that π_∞ (and hence also σ_∞) is a dense subset of Σ. For π_∞ we have the following symmetry result:

Result 9. *(Theorem 4.68, published in [37])*
There is an infinite set of polynomials S such that

$$p(\lambda) \in \pi_\infty \Longrightarrow \lambda \in \pi_\infty$$

for all $p \in S$.

These polynomials $p \in S$ are characteristic polynomials of certain periodic operators and can be computed explicitly. Since it is known that $\mathbb{D}$ is a subset of $\mathrm{clos}(\pi_\infty)$ (see [17]), we get $p^{-1}(\mathbb{D}) \subset \mathrm{clos}(\pi_\infty)$ for every $p \in S$. In this way we improve the lower bound to Σ by a considerable amount (see Figure 2). This contruction can also be iterated and yields the following result as a corollary:

Result 10. *(Remark 4.70, published in [37])*
Σ contains an infinite sequence of Julia sets.

After improving the lower bound to Σ by a sizeable amount, we try to give a new upper bound as well. To do so, we consider the numerical range of the square of the Feinberg-Zee random hopping matrix, which we compute explicitly using a method that is based on the Schur test. It is then easy to see that this set contains the spectrum and is actually a better upper bound than the numerical range without the square. The result reads as follows:

Result 11. *(Theorem 4.75, published in [36])*
Let $A\colon \Omega \to \mathrm{BDO}(\mathbf{X})$ be the Feinberg-Zee random hopping matrix and let $M_{per,4}(\Omega)$ denote the set of 4-periodic operators in $A(\Omega)$. Then

$$N(A(\omega)^2) = \mathrm{conv}\left(\bigcup_{B \in M_{per,4}(\Omega)} N(B^2)\right)$$

for almost every $\omega \in \Omega$.

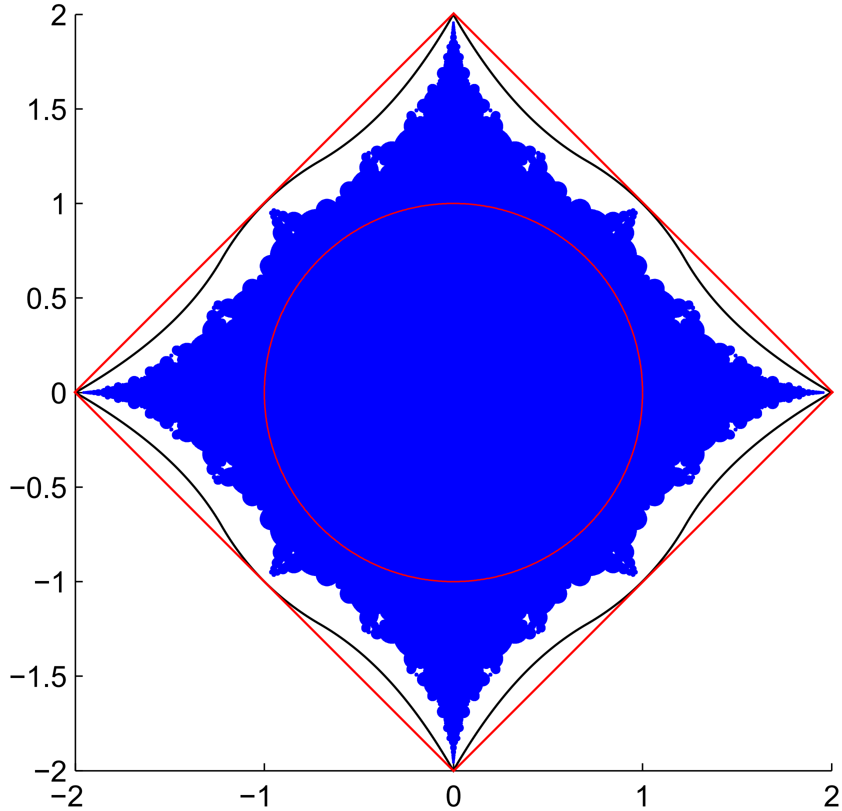


Figure 2: A picture of the lower bound (blue) and the upper bound (black) to Σ, the boundary of the numerical range of the Feinberg-Zee random hopping matrix (red square) and the unit circle as a reference.

In Figure 2 we see the upper and lower bound combined in one picture. This picture summarizes our results about the Feinberg-Zee random hopping matrix quite nicely.

2 Fredholm Theory of Band-Dominated Operators

2.1 Fredholm operators

Despite we are interested in the spectra of bounded linear operators, sometimes it does not suffice to know whether an operator is invertible or not. For example, if an operator is not invertible, how far is it from being invertible? Can we add a small perturbation (whatever that means) to make the operator invertible? Or at least invertible from one side? On the other hand, as we will see, it is sometimes easier to determine the part of the spectrum that is "far from invertible" because we have a big freedom there. Furthermore, most of the operators considered later in this thesis satisfy the property that they are either invertible or not even close to being invertible. Thus we start with the general theory of Fredholm operators.

2.1.1 Definition

Let X be a Banach space and $A \in \mathcal{L}(X)$. As mentioned in Section 1.1.8, A is invertible if and only if it is bijective. Thus invertibility can fail at two points. If A is not invertible, it is not injective or not surjective. To be more precise, we define the following two numbers:

$$\alpha(A) := \dim(\ker(A)), \quad \beta(A) := \dim(X/\operatorname{im}(A)).$$

In case both $\alpha(A)$ and $\beta(A)$ are finite we call A a Fredholm operator or just Fredholm. The difference

$$\operatorname{ind}(A) := \alpha(A) - \beta(A) \in \mathbb{Z}$$

is called the (Fredholm) index of A.

As a consequence of the open mapping theorem, $\operatorname{im}(A)$ is automatically closed if it is of finite codimension. In particular, every Fredholm operator has closed range and hence $X/\operatorname{im}(A)$ is a Banach space. Furthermore, if A is Fredholm, the following equalities hold:

$$\beta(A) = \alpha(A^*), \quad \beta(A^*) = \alpha(A).$$

In particular, A is Fredholm if and only if A^* is Fredholm and $\operatorname{ind}(A^*) = -\operatorname{ind}(A)$. This is an immediate consequence of Banach's closed range theorem [3]:

Theorem 2.1. *The following are equivalent:*

(i) $\operatorname{im}(A) \subset X$ *is closed,*

(ii) $\operatorname{im}(A^*) \subset X^*$ *is closed,*

(iii) $\operatorname{im}(A) = \{x \in X : f(x) = 0$ *for all* $f \in \ker(A^*)\}$*,*

(iv) $\operatorname{im}(A^*) = \{f \in X^* : f(x) = 0$ *for all* $x \in \ker(A)\}$*.*

Thus we can also write $\operatorname{ind}(A) = \alpha(A) - \alpha(A^*)$, which comes in handy because the dimension of the kernel is usually easier to compute. Also, often the adjoint is well-known. For example, for $\ell^p(\mathbb{Z})$ ($p \in \{0\} \cup [1, \infty)$) the adjoint is just the usual transpose of the (infinite) matrix. Here are some useful properties of the Fredholm index due to Atkinson:

Theorem 2.2. *([2, Theorems II-IV])*
Let $A, B \in \mathcal{L}(X)$ *be Fredholm and* $K \in \mathcal{K}(X)$*. Then the following assertions hold:*

(i) AB *and* BA *are Fredholm as well and* $\operatorname{ind}(AB) = \operatorname{ind}(BA) = \operatorname{ind}(A) + \operatorname{ind}(B)$.

(ii) $A + K$ *is Fredholm as well and* $\operatorname{ind}(A + K) = \operatorname{ind}(A)$.

(iii) The map $\operatorname{ind}\colon \{A \in \mathcal{L}(X) : A \text{ Fredholm}\} \to \mathbb{Z}$ *is continuous w.r.t. norm topology, hence constant on connected components.*

The statements (*ii*) and (*iii*) of Theorem 2.2 can be extended to so-called semi-Fredholm operators. An operator A is called semi-Fredholm if $\operatorname{im}(A)$ is closed and one of the numbers $\alpha(A)$ and $\beta(A)$ is finite (but not necessarily both). We will only need and state the following extension of Theorem 2.2(*ii*):

Theorem 2.3. *(e.g. [48, Section IV, Theorem 5.26])*
Let $A \in \mathcal{L}(X)$ *have closed range and let* $K \in \mathcal{K}(X)$. *If* $\alpha(A)$ *is finite, so is* $\alpha(A+K)$ *and* $\operatorname{im}(A+K)$ *is closed. If* $\beta(A)$ *is finite, so is* $\beta(A + K)$ *and* $\operatorname{im}(A + K)$ *is closed.*

In analogy to the spectrum we define the essential spectrum as

$$\operatorname{sp}_{\text{ess}}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}.$$

It is clear that $\operatorname{sp}_{\text{ess}}(A) \subset \operatorname{sp}(A)$ holds. Furthermore, as an application of Theorem 2.2(*iii*), we get that the Fredholm index is constant on connected components of $\operatorname{sp}(A) \setminus \operatorname{sp}_{\text{ess}}(A)$.

2.1.2 The Calkin algebra

There is also an algebraic approach to "almost invertibility" that is connected to Theorem 2.2(*ii*). Namely, we can ask whether the coset $A + \mathcal{K}(X)$ is invertible in the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$. In other words, are there an operator $B \in \mathcal{L}(X)$, a so-called Fredholm regularizer or just regularizer, and compact operators $K_1, K_2 \in \mathcal{K}(X)$ such that $AB = I + K_1$ and $BA = I + K_2$? As it turns out, this coincides with the question whether A is Fredholm.

Theorem 2.4. *([2, Theorem I])*
Let $A \in \mathcal{L}(X)$. *Then the following are equivalent:*

(i) A *is Fredholm,*

(ii) the coset $A + \mathcal{K}(X)$ *is invertible in the quotient algebra* $\in \mathcal{L}(X)/\mathcal{K}(X)$,

(iii) there exist $B \in \mathcal{L}(X)$ *and* $K_1, K_2 \in \mathcal{K}(X)$ *such that* $AB = I + K_1$ *and* $BA = I + K_2$.

As mentioned in Section 1.1.5, $\mathcal{L}(X)/\mathcal{K}(X)$ is again a unital Banach algebra, the so-called Calkin algebra. Thus we can also write

$$\operatorname{sp}_{\text{ess}}(A) = \operatorname{sp}(A + \mathcal{K}(X)).$$

This implies (cf. Section 1.1.8) that $\operatorname{sp}_{\text{ess}}(A)$ is always non-empty and compact.

2.1.3 Weyl sequences

There is a third characterization of Fredholm operators, which is in terms of singular Weyl sequences. For this we need some preparation.

Definition 2.5. Let $A \in \mathcal{L}(X)$. Then $(x_n)_{n\in\mathbb{N}} \subset X$ is called a Weyl sequence (for A) if $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $Ax_n \to 0$ as $n \to \infty$. If additionally $(x_n)_{n\in\mathbb{N}}$ possesses no convergent subsequence, we call $(x_n)_{n\in\mathbb{N}}$ a singular Weyl sequence.

Note that X needs to be infinite-dimensional for singular Weyl sequences to exist. Similarly, if X is finite-dimensional, all operators $A \in \mathcal{L}(X)$ are Fredholm. We will see that Fredholmness coincides with non-existence of singular Weyl sequences. In this sense the finite-dimensional case is a trivial case of this coincidence.

Next we define the lower norm of an operator $A \in \mathcal{L}(X,Y)$. For this section it would suffice to consider $X = Y$. But since we need it for operators on closed subspaces later on, we define the lower norm directly for arbitrary operators between two Banach spaces X and Y.

Definition 2.6. Let $A \in \mathcal{L}(X,Y)$. In analogy to the norm we define the lower norm of A by

$$\nu(A) := \inf_{\|x\|_X=1} \|Ax\|_Y .$$

If $\nu(A) > 0$, we say that A is bounded below.

Note that, despite its name, the lower norm is not a norm as it does not separate points and lacks subadditivity. The name solely refers to the similarly defined operator norm.

Here are some immediate properties of the lower norm. It is clear that the lower norm is supermultiplicative, i.e. $\nu(AB) \geq \nu(A)\nu(B)$ for all $A \in \mathcal{L}(Y,Z)$ and $B \in \mathcal{L}(X,Y)$ (compare with the submultiplicativity of the operator norm). It is also clear that $\nu(AB) \leq \|A\|\,\nu(B)$ holds. Furthermore, the lower norm is Lipschitz continuous.

Proposition 2.7. *(e.g. [57, Lemma 2.38])*
Let $A, B \in \mathcal{L}(X,Y)$. Then $|\nu(A) - \nu(B)| \leq \|A - B\|$.

Clearly, if an operator is bounded below, then it is injective. Moreover, its range is closed. In fact, also the converse is true. This is one of the main reasons we are interested in the lower norm.

Lemma 2.8. *(e.g. [57, Lemma 2.32])*
$A \in \mathcal{L}(X,Y)$ is bounded below if and only if A is injective and $\operatorname{im}(A)$ is closed.

The next result is an easy consequence of the previous lemma and Theorem 2.1. Nevertheless, it is an important observation that is helpful in many circumstances.

Corollary 2.9. *(e.g. [57, Lemma 2.35])*
$A \in \mathcal{L}(X,Y)$ is invertible if and only if A and A^* are bounded below. In that case, $\left\|A^{-1}\right\|^{-1} = \nu(A) = \nu(A^*)$.

Remark 2.10. Using the convention $\left\|A^{-1}\right\|^{-1} := 0$ for non-invertible operators $A \in \mathcal{L}(X,Y)$, we can rewrite the previous corollary in the form

$$\left\|A^{-1}\right\|^{-1} = \min\{\nu(A), \nu(A^*)\}$$

with $\nu(A) = \nu(A^*)$ if A is invertible.

It is clear by definition that A is bounded below if and only if A has no Weyl sequence. We conclude that A is invertible if and only if neither A nor A^* has a Weyl sequence. A similar characterization in terms of singular Weyl sequences is also possible for Fredholm operators.

Theorem 2.11. *Let $A \in \mathcal{L}(X)$. Then the following are equivalent:*

(i) A is Fredholm,

(ii) neither A nor A^* has a singular Weyl sequence.

In Hilbert space this result is due to Wolf ([93, Theorem 1.14]). The general Banach space case is very similar and only needs a few adjustments in the proof. For convenience we provide the full proof here.

Proof. Let A be Fredholm and assume that there exists a singular Weyl sequence $(x_n)_{n\in\mathbb{N}}$ for A. By Theorem 2.4, there exist $B \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $BA = I + K$. Thus we have

$$x_n + Kx_n = (I + K)x_n = BAx_n \to 0 \quad \text{as } n \to \infty.$$

Moreover, $K(\{x_n : n \in \mathbb{N}\})$ is relatively compact, hence the sequence $(Kx_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Denote this subsequence again by $(Kx_n)_{n\in\mathbb{N}}$ and denote its limit by y. It follows that $(x_n)_{n\in\mathbb{N}}$ converges to $-y$. But this is a contradiction to the assumption that $(x_n)_{n\in\mathbb{N}}$ is a singular Weyl sequence and hence does not have a convergent subsequence. Thus there exist no singular Weyl sequences for A. Similarly there exist no singular Weyl sequences for A^*.

Now assume that A is not Fredholm. This implies $\alpha(A) = \infty$ or $\beta(A) = \infty$. Using Theorem 2.1, we get three cases:

Case 1: $\alpha(A) = \infty$. In this case $\ker(A)$ is an infinite-dimensional closed subspace of X, hence an infinite-dimensional Banach space. The existence of a singular Weyl sequence for A is then a consequence of the fact that the closed unit balls of infinite-dimensional Banach spaces are non-compact.

Case 2: $\alpha(A^*) = \infty$. With the same argument there exists a singular Weyl sequence for A^*.

Case 3: $\operatorname{im}(A)$ is not closed. Let $\tilde{A}\colon X/\ker(A) \to X$, $x+\ker(A) \mapsto Ax$ be the induced operator on the coset $X/\ker(A)$. Then $\operatorname{im}(\tilde{A}) = \operatorname{im}(A)$ is not closed, hence $\tilde{A}$ is not bounded below by Lemma 2.8. Thus there exists a sequence $(x_n + \ker(A))_{n\in\mathbb{N}} \subset X/\ker(A)$ with $\|x_n + \ker(A)\|_{X/\ker(A)} = 1$ for all $n \in \mathbb{N}$ such that $\tilde{A}(x_n + \ker(A)) \to 0$ as $n \to \infty$. If this sequence possesses a convergent subsequence, converging to $x + \ker(A)$, say, then $\tilde{A}(x + \ker(A)) = 0$ by continuity of A. But this implies $x \in \ker(A)$, which is impossible since $\|x_n + \ker(A)\|_{X/\ker(A)} = 1$ for all $n \in \mathbb{N}$. Hence the sequence $(x_n + \ker(A))_{n\in\mathbb{N}}$ does not possess a convergent subsequence and $(x_n + \ker(A))_{n\in\mathbb{N}}$ is a singular Weyl sequence for $\tilde{A}$. W.l.o.g. we can assume that $\ker(A)$ is finite-dimensional (otherwise we are in case 1). Thus we can further assume that $\|x_n\| = 1$ holds for all $n \in \mathbb{N}$ (see Section 1.1.5). Also $Ax_n \to 0$ as $n \to \infty$ by construction and $(x_n)_{n\in\mathbb{N}}$ cannot have a convergent subsequence because $(x_n + \ker(A))_{n\in\mathbb{N}}$ does not possess one. Hence $(x_n)_{n\in\mathbb{N}}$ is a singular Weyl sequence for A.

Of course, the same argument can also be stated for A^*. This implies that if $\operatorname{im}(A)$ is not closed, there exist singular Weyl sequences for both A and A^*. □

2.1.4 Invertible perturbations

In Section 2.1.2 we observed that being Fredholm is equivalent to the existence of an inverse modulo compact operators, i.e. for every Fredholm operator $A \in \mathcal{L}(X)$ there exist $B \in \mathcal{L}(X)$ and

$K_1, K_2 \in \mathcal{K}(X)$ such that $AB = I + K_1$ and $BA = I + K_2$. A natural question is whether there is also a compact perturbation K such that $A + K$ is invertible. In other words, are there operators $B \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $(A+K)B = I$ and $B(A+K) = I$. Unfortunately, this is not possible in general. Indeed, consider an operator $A \in \mathcal{L}(X)$ that is Fredholm with $\operatorname{ind}(A) \neq 0$. By Theorem 2.2($ii$), we also have $\operatorname{ind}(A+K) \neq 0$ for all $K \in \mathcal{K}(X)$, which implies that there does not exist a compact operator K such that $A+K$ is invertible. However, we have the following:

Theorem 2.12. *(e.g. [33, Chapter XV, Corollary 2.4])*
Let $A \in \mathcal{L}(X)$ be Fredholm. Then the following is true:

(i) $\operatorname{ind}(A) = 0$ *if and only if there exists $K \in \mathcal{K}(X)$ such that $A + K$ is invertible,*

(ii) $\operatorname{ind}(A) \geq 0$ *if and only if there exists $K \in \mathcal{K}(X)$ such that $A + K$ is invertible from the right (i.e. there exists $B \in \mathcal{L}(X)$ such that $(A+K)B = I$),*

(iii) $\operatorname{ind}(A) \leq 0$ *if and only if there exists $K \in \mathcal{K}(X)$ such that $A + K$ is invertible from the left (i.e. there exists $B \in \mathcal{L}(X)$ such that $B(A+K) = I$).*

Furthermore, $\|K\|$ can be chosen arbitrarily small.

2.2 Limit operators on $\ell^p(\mathbb{Z},\mathbb{C})$, $p \in (1,\infty)$

In this section we focus on a special class of operators, the so-called band-dominated operators on $\mathbf{X} := \ell^p(\mathbb{I}, X)$. Unless stated otherwise, we restrict ourselves to $p \in (1,\infty)$, $\mathbb{I} = \mathbb{Z}$ and $X = \mathbb{C}$ in this section and come back to the general case in section 3.

2.2.1 Band-dominated operators

Definition 2.13. An operator $A \in \mathcal{L}(\mathbf{X})$ is called a band operator if A only has finitely many non-zero diagonals (cf. Section 1.1.6). The set of all band operators is denoted by $\mathrm{BO}(\mathbf{X})$. The closure of $\mathrm{BO}(\mathbf{X}) \subset \mathcal{L}(\mathbf{X})$ is denoted by $\mathrm{BDO}(\mathbf{X})$ and the elements of $\mathrm{BDO}(\mathbf{X})$ are called band-dominated operators.

In other words, a band-dominated operator is the (norm) limit of a sequence of band operators. If the only non-zero diagonal of an operator is the main diagonal, we call the operator diagonal. Often diagonal operators are also called (generalized) multiplication operators. If an operator has exactly two non-zero diagonals, namely the main diagonal and the first sub- or superdiagonal, it is called bidiagonal. If an operator has exactly three non-zero diagonals, namely the main diagonal and the first sub- and superdiagonal, it is called tridiagonal.

Example 2.14.

$$
\underset{\text{diagonal}}{\begin{pmatrix} \ddots &&&& \\ & v_{-1} &&& \\ && v_0 && \\ &&& v_1 & \\ &&&& \ddots \end{pmatrix}}
\qquad
\underset{\text{bidiagonal}}{\begin{pmatrix} \ddots & \ddots &&& \\ & v_{-1} & w_{-1} && \\ && v_0 & w_0 & \\ &&& v_1 & \ddots \\ &&&& \ddots \end{pmatrix}}
\qquad
\underset{\text{tridiagonal}}{\begin{pmatrix} \ddots & \ddots &&& \\ \ddots & v_{-1} & w_{-1} && \\ & u_0 & v_0 & w_0 & \\ && u_1 & v_1 & \ddots \\ &&& \ddots & \ddots \end{pmatrix}}
$$

The entries u_j, v_j, w_j ($j \in \mathbb{Z}$) are again bounded linear operators in general, i.e. $u_j, v_j, w_j \in \mathcal{L}(X)$ for all $j \in \mathbb{Z}$. Here, of course, $\mathcal{L}(\mathbb{C})$ is identified with $\mathbb{C}$ and hence $u_j, v_j, w_j \in \mathbb{C}$ for all $j \in \mathbb{Z}$.

Remember that if A is Fredholm, then there exist $B \in \mathcal{L}(\mathbf{X})$ and $K_1, K_2 \in \mathcal{K}(\mathbf{X})$ such that $AB = I + K_1$ and $BA = I + K_2$ (cf. Theorem 2.4). In case of a band-dominated operator A, a regularizer B has to be band-dominated as well:

Proposition 2.15. *(e.g. [57, Proposition 2.10])*
$\mathrm{BDO}(\mathbf{X})$ *is a unital, inverse closed Banach subalgebra of* $\mathcal{L}(\mathbf{X})$ *that contains* $\mathcal{K}(\mathbf{X})$ *as a two-sided ideal. Moreover,* $\mathrm{BDO}(\mathbf{X})/\mathcal{K}(\mathbf{X})$ *is also inverse closed in* $\mathcal{L}(\mathbf{X})/\mathcal{K}(\mathbf{X})$*, i.e. if A is Fredholm, then there exist* $B \in \mathrm{BDO}(\mathbf{X})$ *and* $K_1, K_2 \in \mathcal{K}(\mathbf{X})$ *such that* $AB = I+K_1$ *and* $BA = I+K_2$ *(cf. Theorem 2.4).*

Remark 2.16. Note that if we fix $\mathbb{I}$ and X, the set $\mathrm{BO}(\mathbf{X})$ does not depend on p. However, since we take the closure with respect to $\|\cdot\|_p$, $\mathrm{BDO}(\mathbf{X})$ may depend on p. Indeed, an example can be found in [57, Example 1.39].

For $n \in \mathbb{Z}$ let $V_n \in \mathrm{BO}(\mathbf{X})$ define the n-th shift operator, i.e. $(V_n x)_j = x_{j-n}$ for all $j \in \mathbb{Z}$ and $x \in \mathbf{X}$. Note that every V_n is an invertible isometry with $V_n^{-1} = V_{-n}$. It is clear that every band operator $A \in \mathrm{BO}(\mathbf{X})$ can be written as

$$A = \sum_{n=-\omega}^{\omega} D^{(n)} V_n, \tag{5}$$

where $D^{(n)}$ is the diagonal matrix associated to the n-th diagonal of A and ω is the so-called band-width.

Proposition 2.17. *Let A be a band operator of width ω and let $d_n := \sup_{j\in\mathbb{Z}} |D^{(n)}_{j,j}|$ be the supremum of the n-th diagonal for $n \in \{-\omega, \dots, \omega\}$. Then $\|A\| \leq \sum_{n=-\omega}^{\omega} d_n$.*

Proof. Since $\left\|D^{(n)}\right\| = d_n$ and V_n is an isometry, the assertion follows immediately from Equation (5). □

Note that the right-hand side of this estimate does not depend on p whereas the norm obviously does. In fact,

$$\|A\|_{\mathcal{W}} := \sum_{n=-\infty}^{\infty} d_n$$

defines a norm on $\mathrm{BO}(\mathbf{X})$ simultaneously for all $p \in \{0\} \cup [1,\infty]$. The closure of $\mathrm{BO}(\mathbf{X})$ with respect to $\|\cdot\|_{\mathcal{W}}$ is called the Wiener algebra and denoted by $\mathcal{W}(\mathbf{X})$, i.e.

$$\mathcal{W}(\mathbf{X}) := \left\{A \in \mathcal{L}(\mathbf{X}) : \|A\|_{\mathcal{W}} < \infty\right\}.$$

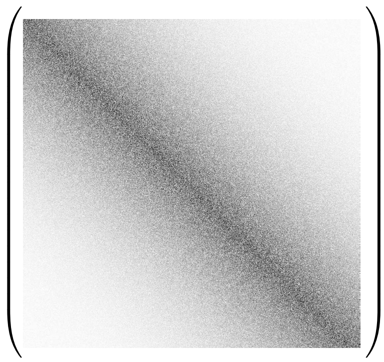
The good thing about the Wiener algebra is that it is independent of p but still a Banach algebra, in some sense a compromise between $\mathrm{BO}(\mathbf{X})$ and $\mathrm{BDO}(\mathbf{X})$. As a consequence, many properties of operators $A \in \mathcal{W}(\mathbf{X})$ do not depend on p, which means that we can often switch to one particular p (usually $p = 2$, the Hilbert space case). For example, the inverse closedness of $\mathcal{W}(\mathbf{X})$ (see [75, Theorem 2.5.2]) implies that the spectrum of an operator $A \in \mathcal{W}(\mathbf{X})$ is independent of p.

It is pretty clear that $\mathcal{W}(\mathbf{X})$ is strictly larger than $\mathrm{BO}(\mathbf{X})$. That $\mathrm{BDO}(\mathbf{X})$ is strictly larger than $\mathcal{W}(\mathbf{X})$ for every $p \in \{0\} \cup [1,\infty]$ is not immediately obvious. For an example we refer to [57, Example 1.49 d)].

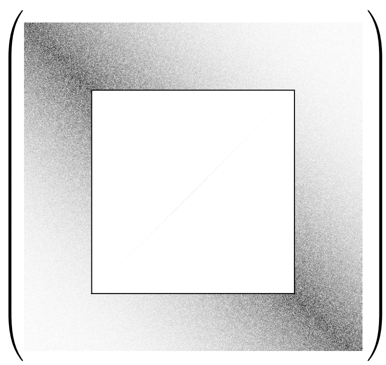
2.2.2 $\mathcal{P}$-compact operators

In order to apply the concepts from Section 2.1 to band-dominated operators, we have to identify the compact operators in $\mathcal{L}(\mathbf{X})$. For this purpose we introduce the notion of $\mathcal{P}$-compact operators. As it turns out in Section 3.1, this is not just an auxiliary notion but the main ingredient for the generalization to Banach space valued sequence spaces.

So assume that we are given some band-dominated operator

$A =$  $\in \mathcal{L}(\mathbf{X})$

and the task is to decide whether A is Fredholm or not. By Theorem 2.2(ii), it is clear that the Fredholmness of A is invariant under finite-dimensional perturbations, e.g.

$A - F :=$  $\in \mathcal{L}(\mathbf{X})$

is Fredholm if and only if A is Fredholm. In other words, changing finitely many entries does not change the Fredholmness of A. Let us make this statement precise. For $n \in \mathbb{N}$ we define the projections $P_n := P_{\{-n,\dots,n\}}$ and $Q_n := I - P_n$ (cf. Section 1.1.6). Note that $\|P_n\| = \|Q_n\| = 1$ for all $n \in \mathbb{N}$ and $P_n \to I$ strongly. The sequence of projections $(P_n)_{n\in\mathbb{N}}$ is denoted by $\mathcal{P}$. We further denote the set of operators with only finitely many non-zero entries

$$\{F \in \mathcal{L}(\mathbf{X}) : \exists\, n \in \mathbb{N} \text{ such that } P_n F P_n = F\}$$

by $\mathcal{K}_c(\mathbf{X}, \mathcal{P})$ and its closure in $\mathcal{L}(\mathbf{X})$ by $\mathcal{K}(\mathbf{X}, \mathcal{P})$. In analogy to compact operators, we call the operators in $\mathcal{K}(\mathbf{X}, \mathcal{P})$ $\mathcal{P}$-compact. It is clear that every $\mathcal{P}$-compact operator is band-dominated and compact because obviously $\mathcal{K}_c(\mathbf{X}, \mathcal{P}) \subset \mathrm{BO}(\mathbf{X}) \cap \mathcal{K}(\mathbf{X})$. In fact, compactness and $\mathcal{P}$-compactness coincide.

Proposition 2.18. *It holds* $\mathcal{K}(\mathbf{X}, \mathcal{P}) = \mathcal{K}(\mathbf{X})$.

To prove this, we need some auxiliary results on compact operators. The first one is Schauder's theorem [80] that relates compact operators with their adjoint.

Theorem 2.19. *Let X be an arbitrary Banach space and $A \in \mathcal{L}(X)$. Then A is compact if and only if A^* is compact. In particular,*

$$\mathcal{K}(X)^* \subset \mathcal{K}(X^*).$$

Note that strictly speaking, $\mathcal{K}(X)^*$ already has a meaning as the dual space of $\mathcal{K}(X)$ interpreted as a Banach space. However, we clearly mean the set $\{K^* : K \in \mathcal{K}(X)\}$. Equality holds if X (and hence X^*) is reflexive:

$$\mathcal{K}(X)^* \subset \mathcal{K}(X^*) = \mathcal{K}(X^*)^{**} \subset \mathcal{K}(X^{**})^* = \mathcal{K}(X)^*,$$

where we identified X with X^{**} in the usual way (cf. Section 1.1.10).

Another useful property of compact operators is that multiplication with a compact operator from the right maps strongly convergent to norm convergent sequences.

Theorem 2.20. *(e.g. [75, Theorem 1.1.3])*
Let X be an arbitrary Banach space, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ a sequence of bounded linear operators and $A \in \mathcal{L}(X)$. Then $A_n \to A$ strongly if and only if $\|A_nK - AK\| \to 0$ for every $K \in \mathcal{K}(X)$.

As an immediate corollary we have:

Corollary 2.21. *Let X be an arbitrary Banach space, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ a sequence of bounded linear operators and $A \in \mathcal{L}(X)$. If $A_n^* \to A^*$ strongly, then $\|KA_n - KA\| \to 0$ for every $K \in \mathcal{K}(X)$. If X is reflexive, then $\|KA_n - KA\| \to 0$ for every $K \in \mathcal{K}(X)$ also implies $A_n^* \to A^*$ strongly.*

Now we are ready to prove Proposition 2.18.

Proof of Proposition 2.18. $\mathcal{K}(\mathbf{X}, \mathcal{P}) \subset \mathcal{K}(\mathbf{X})$ is clear as mentioned above. So let $K \in \mathcal{K}(\mathbf{X})$. Since $P_n \to I$ strongly, we have $P_nK \to K$ in norm by Theorem 2.20. Furthermore, because of $\mathbf{X}^* = \ell^q(\mathbb{Z}, \mathbb{C})$, $q = \frac{p}{p-1} \in (1, \infty)$, we also get $P_n^* \to I^*$ strongly and hence $KP_n \to K$ in norm by Corollary 2.21. Let $F_n := P_nKP_n \in \mathcal{K}_c(\mathbf{X}, \mathcal{P})$. Then

$$\|K - F_n\| = \|K - P_nKP_n\| \leq \|K - P_nK\| + \|P_n(K - KP_n)\| \leq \|K - P_nK\| + \|K - KP_n\|,$$

hence $F_n \to K$ in norm, which implies $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$. $\square$

So Proposition 2.18 implies that Fredholmness equals "invertibility modulo finitely many entries". We conclude that Fredholmness has to be encoded somewhere at infinity. To make this precise, we introduce the concept of limit operators.

2.2.3 Limit operators

We say that a sequence $(A_n)_{n\in\mathbb{N}}$ of bounded linear operators converges $*$-strongly if both $(A_n)_{n\in\mathbb{N}}$ and $(A_n^*)_{n\in\mathbb{N}}$ converge strongly. In that case $\operatorname{s-lim}_{n\to\infty} A_n^* = (\operatorname{s-lim}_{n\to\infty} A_n)^*$, where s-lim denotes the limit w.r.t. the strong operator topology.

Definition 2.22. We say that a sequence of integers $h := (h_n)_{n\in\mathbb{N}}$ tends to infinity if $|h_n| \to \infty$ as $n \to \infty$. Let h be such a sequence and $A \in \mathcal{L}(\mathbf{X})$. If the $*$-strong limit $\underset{n\to\infty}{*\text{-}\lim} V_{-h_n} A V_{h_n}$ exists, we call it a limit operator of A and denote it by A_h. As an abbreviation we just say that A_h exists. The set of all limit operators is called the operator spectrum of A and denoted by $\sigma^{\mathrm{op}}(A)$.

Note that $(V_{-h_n} A V_{h_n})_{i,j} = A_{i+h_n,j+h_n}$ for all $i,j \in \mathbb{Z}$ and $n \in \mathbb{N}$. So taking limit operators corresponds to scrolling up/down the diagonals to $\pm\infty$ in some particular way (depending on h) and collecting what we get there. Hence as discussed in the previous section, Fredholmness of an operator $A \in \mathrm{BDO}(\mathbf{X})$ should correspond to invertibility of its limit operators in some way. In the more general case $A \in \mathcal{L}(\mathbf{X})$ one can only expect results in one direction. In general, bounded operators still control their limit operators, but the limit operators are usually not sufficient to determine the behavior of the operator. Roughly speaking, in the case of band-dominated operators most of the things are going on on the diagonals whereas for more general operators this might no longer be true. Since we are only taking two ends into account, we do not gather enough information in the general case. For example, if we consider the flip operator $J \in \mathcal{L}(\mathbf{X})$ defined by $(Jx)_j = x_{-j}$ for all $j \in \mathbb{Z}$, $x \in \mathbf{X}$, we do not get any information since the operator spectrum of J is empty.

Before we start to make this precise, let us observe that $*$-strong convergence, which is not that intuitive at first sight, is equivalent to different notions of convergence in many cases. For bounded sequences $(A_n)_{n\in\mathbb{N}}$ $*$-strong convergence is equivalent to norm convergence of $(P_m A_n)_{n\in\mathbb{N}}$ and $(A_n P_m)_{n\in\mathbb{N}}$ for all $m \in \mathbb{N}$.

Proposition 2.23. *Let $(A_n)_{n\in\mathbb{N}}$ be a bounded sequence of operators in $\mathcal{L}(\mathbf{X})$ and $A \in \mathcal{L}(\mathbf{X})$. Then the following are equivalent:*

(i) $\|P_m(A_n - A)\| \to 0$ *and* $\|(A_n - A)P_m\| \to 0$ *as* $n \to \infty$ *for all* $m \in \mathbb{N}$,

(ii) $\|K(A_n - A)\| \to 0$ *and* $\|(A_n - A)K\| \to 0$ *as* $n \to \infty$ *for all* $K \in \mathcal{K}(\mathbf{X})$,

(iii) $A_n \to A$ *and* $A_n^* \to A^*$ *strongly as* $n \to \infty$.

Proof. The equivalence of (ii) and (iii) was shown in Theorem 2.20 and Corollary 2.21. Moreover, "(ii) implies (i)" is trivial since $P_m \in \mathcal{K}(\mathbf{X})$ for all $m \in \mathbb{N}$. So assume that $\|(A_n - A)P_m\| \to 0$ holds for all $m \in \mathbb{N}$ and fix $K \in \mathcal{K}(\mathbf{X})$, $\varepsilon > 0$. Since K is compact and $Q_m \to 0$ strongly as $m \to \infty$, we can choose an $m \in \mathbb{N}$ such that $\|Q_m K\| < \varepsilon$ by Theorem 2.20. Furthermore, we choose n large enough such that $\|(A_n - A)P_m\| < \varepsilon$ as well. Then

$$\begin{aligned}\|(A_n - A)K\| &= \|(A_n - A)(P_m + Q_m)K\| \leq \|(A_n - A)P_m K\| + \|(A_n - A)Q_m K\| \\ &\leq \|(A_n - A)P_m\|\,\|K\| + (\|A_n\| + \|A\|)\,\|Q_m K\| < \varepsilon(\|K\| + \|A_n\| + \|A\|),\end{aligned}$$

hence $\|(A_n - A)K\| \to 0$ because $(A_n)_{n\in\mathbb{N}}$ was assumed to be bounded. The other assertion is of course exactly the same. □

If $(A_n)_{n\in\mathbb{N}}$ is a bounded sequence of band operators having a bounded band-width, then $*$-strong convergence is even equivalent to entrywise convergence. As a consequence, $*$-strong convergence coincides with strong convergence and the strong convergence of $(A_n)_{n\in\mathbb{N}}$ implies the strong convergence of $(A_n^*)_{n\in\mathbb{N}}$.

Proposition 2.24. *Let $(A_n)_{n\in\mathbb{N}}$ be a bounded sequence of band operators in $\mathcal{L}(\mathbf{X})$ and let $A \in \mathrm{BO}(\mathbf{X})$. Further assume that the band-widths of A and all A_n are bounded by ω. Then the following are equivalent:*

(i) $A_n \to A$ *entrywise, i.e.* $(A_n)_{i,j} \to A_{i,j}$ *as* $n \to \infty$*,*

(ii) $\|P_k(A_n - A)P_l\| \to 0$ *as* $n \to \infty$ *for all* $k, l \in \mathbb{N}$*,*

(iii) $A_n \to A$ *and* $A_n^* \to A^*$ *strongly as* $n \to \infty$*,*

(iv) $A_n \to A$ *strongly as* $n \to \infty$*.*

Proof. The assertions "*(iii)* implies *(iv)*", "*(iv)* implies *(i)*" and "*(i)* implies *(ii)*" are trivial. So let us assume that $\|P_k(A_n - A)P_l\| \to 0$ as $n \to \infty$ for all $k, l \in \mathbb{N}$. Since the band-width of $A_n - A$ is bounded by ω, we get $P_k(A_n - A)P_l = P_k(A_n - A)$ for l large enough. Similarly, $P_k(A_n - A)P_l = (A_n - A)P_l$ for k large enough. Hence *(iii)* follows by Proposition 2.23. □

So in the case of band operators, we have multiple viewpoints. The easiest is probably the entrywise convergence. Before we proceed with basic properties of limit operators, we have a look at some examples.

Example 2.25. (a) Let $\lambda \in \mathbb{C}$. Consider $A = \lambda I$. Then

$$V_{-n}AV_n = \lambda V_{-n}V_n = \lambda I = A$$

for all $n \in \mathbb{Z}$. Thus for every sequence $h = (h_n)_{n\in\mathbb{N}}$ of integers tending to infinity, the $*$-strong limit $\underset{n\to\infty}{*\text{-}\lim} V_{-h_n}AV_{h_n}$ exists and is equal to A. Hence $\sigma^{\mathrm{op}}(A) = \{A\}$.

(b) The previous example can be generalized to operators with constant diagonals, the so-called Laurent operators, e.g.

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & v & w & & \\ & u & v & w & \\ & & u & v & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

$u, v, w \in \mathbb{C}$. Then A is invariant under shifting, i.e. $V_{-n}AV_n = A$ for all $n \in \mathbb{Z}$. So again, $\sigma^{\mathrm{op}}(A) = \{A\}$. This is of course true for all Laurent operators, not only tridiagonal ones.

(c) Let $m \in \mathbb{N}$, $u, v, w \in \mathbb{C}^m$. Consider the operator

$$A := \begin{pmatrix} \ddots & \ddots & & & & & & \\ \ddots & v_m & w_m & & & & & \\ & u_1 & v_1 & w_1 & & & & \\ & & u_2 & \ddots & \ddots & & & \\ & & & \ddots & v_m & w_m & & \\ & & & & u_1 & v_1 & w_1 & \\ & & & & & u_2 & v_2 & \ddots \\ & & & & & & \ddots & \ddots \end{pmatrix}$$

with periodic diagonals, a so-called periodic operator. Then clearly, $V_{-(nm+k)}AV_{nm+k} = V_{-k}AV_k$ for all $n \in \mathbb{Z}$, $k \in \{0,\dots,m-1\}$ and hence $\{A, V_{-1}AV_1, \dots, V_{-(m-1)}AV_{m-1}\} \subset \sigma^{\mathrm{op}}(A)$. Considering entrywise convergence, it is easy to see that these are all limit operators and thus

$$\sigma^{\mathrm{op}}(A) = \left\{A, V_{-1}AV_1, \dots, V_{-(m-1)}AV_{m-1}\right\}.$$

So the operator spectrum consists of A and all of its shifts. This is again true for all periodic operators.

(d) Let $a, b \in \mathbb{C}$ and consider the diagonal operator with diagonal

$$(\dots, 1, \boxed{1}, a, b, b, a, a, a, b, b, b, b, a, a, a, a, a, b, b, b, b, b, b, \dots),$$

where the box indicates the 0-th entry. So first of all, any sequence $(h_n)_{n\in\mathbb{N}}$ tending to $-\infty$ yields the identity operator I as a limit operator. Choosing $h_1 = 1$, $h_2 = 6$, $h_3 = 15$ etc. (i.e. always the last a in a segment) we get

$$B := A_h = \begin{pmatrix} \ddots & & & & & \\ & a & & & & \\ & & \boxed{a} & & & \\ & & & b & & \\ & & & & b & \\ & & & & & \ddots \end{pmatrix}.$$

Choosing $h_1 = 3$, $h_2 = 10$, $h_3 = 21$ etc. (i.e. always the last b in a segment) we get

$$C := A_h = \begin{pmatrix} \ddots & & & & & \\ & b & & & & \\ & & \boxed{b} & & & \\ & & & a & & \\ & & & & a & \\ & & & & & \ddots \end{pmatrix}.$$

Similarly we can also get aI and bI if we choose each h_n in the middle of a segment. Furthermore, every shift of B and C is again a limit operator (it is readily seen that this statement is true in a more general context). As it turns out, these are all limit operators of A, hence

$$\sigma^{\mathrm{op}}(A) = \{I, aI, bI, V_{-k}BV_k, V_{-k}CV_k : k \in \mathbb{Z}\}.$$

This shows that there exist operators with infinitely many limit operators.

(e) Consider a diagonal operator A that carries a binary representation of the natural numbers on its main diagonal, e.g. the sequence

$$(\underbrace{\dots, 0,}_{\text{zeros}} \underbrace{0, 1}_{\text{1 digit}}, \underbrace{0, 0, 0, 1, 1, 0, 1, 1}_{\text{2 digits}}, \underbrace{0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1}_{\text{3 digits}}, \dots).$$

Then clearly, by choosing the right sequence $(h_n)_{n\in\mathbb{N}}$, one can find any (bi-infinite) sequence of zeros and ones on the main diagonal of a limit operator of A. Hence the operator spectrum of A consists of uncountably many operators.

Next we summarize the most important properties of limit operators. Most of them follow immediately from the definition. The first one is a consequence of the Banach-Steinhaus theorem [4] (uniform boundedness principle).

Proposition 2.26. *(e.g. [19, Theorem 5.12], [57, Proposition 3.4])*
Let $A, B \in \mathcal{L}(\mathbf{X})$ and let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(\mathbf{X})$. Furthermore, let $(h_n)_{n\in\mathbb{N}}$ be a sequence of integers tending to infinity. Then the following holds:

(i) if A_h exists, then $\|A_h\| \le \|A\|$,

(ii) if A_h exists, then $\nu(A_h) \ge \nu(A)$,

(iii) if A_h and B_h exist, so does $(A+B)_h$ and $(A+B)_h = A_h + B_h$,

(iv) if A_h and B_h exist, so does $(AB)_h$ and $(AB)_h = A_hB_h$,

(v) if A_h exists, so does $(A^*)_h$ and $(A^*)_h = (A_h)^*$,

(vi) if $A_n \to A$ in norm as $n \to \infty$ and $(A_n)_h$ exists for all $n \in \mathbb{N}$, so does A_h and $(A_n)_h \to A_h$ in norm as $n \to \infty$.

Proposition 2.27. *Let $A \in \mathrm{BDO}(\mathbf{X})$ and let $h = (h_n)_{n\in\mathbb{N}}$ be a sequence of integers tending to infinity. Then there exists a subsequence g of h such that A_g exists.*

Proof. First assume that $A \in \mathrm{BO}(\mathbf{X})$. The matrix entries $A_{i,j}$ $(i,j \in \mathbb{Z})$ are uniformly bounded by $\|A\|$. Thus every sequence $((V_{-h_n}AV_{h_n})_{i,j})_{n\in\mathbb{N}} = (A_{i+h_n,j+h_n})_{n\in\mathbb{N}} \subset \mathbb{C}$ is bounded and hence has a convergent subsequence. Taking subsequences of subsequences and using a diagonal argument, one obtains a subsequence g of h such that every sequence $((V_{-h_n}AV_{h_n})_{i,j})_{n\in\mathbb{N}}$ converges. We conclude that $(V_{-h_n}AV_{h_n})_{n\in\mathbb{N}}$ converges entrywise. Clearly, this limit is again a band operator of the same band-width. Proposition 2.24 now implies that A_g exists.

Now let $A \in \mathrm{BDO}(\mathbf{X})$ and choose a sequence of band operators $(A_m)_{m\in\mathbb{N}}$ such that $A_m \to A$ in norm as $m \to \infty$. As above we can find a subsequence g of h such that $(A_m)_g$ exists for every $m \in \mathbb{N}$. By Proposition 2.26(vi), this implies that A_g exists as well. □

Proposition 2.28. *Let $A \in \mathcal{L}(\mathbf{X})$ and let $(h_n)_{n\in\mathbb{N}}$ be a sequence of integers tending to infinity such that A_h exists. If A is a band operator, so is A_h. If A is band-dominated, so is A_h.*

Proof. Let $A \in \mathrm{BO}(\mathbf{X})$. Clearly, the sequence $(V_{-h_n}AV_{h_n})_{n\in\mathbb{N}}$ converges entrywise to a band operator B. Furthermore, the band-widths of $(V_{-h_n}AV_{h_n})_{n\in\mathbb{N}}$ and B are bounded by the band-width of A. Hence by Proposition 2.24, B coincides with A_h. This implies $A_h \in \mathrm{BO}(\mathbf{X})$.

If $A \in \mathrm{BDO}(\mathbf{X})$, the statement follows by approximation. Indeed, let $(A_m)_{m\in\mathbb{N}}$ be a sequence of band operators such that $\|A_m - A\| \to 0$ as $m \to \infty$. As in the proof of Proposition 2.27, we can find a subsequence g of h such that $(A_m)_g$ exists for every $m \in \mathbb{N}$. Applying the above to A_m yields $(A_m)_g \in \mathrm{BO}(\mathbf{X})$ for every $m \in \mathbb{N}$. Using Proposition 2.26(i) and (vi), we obtain

$$\|A_h - (A_m)_g\| = \|A_g - (A_m)_g\| = \|(A - A_m)_g\| \le \|A - A_m\| \to 0$$

as $m \to \infty$, which implies that A_h is band-dominated as well. □

2.2.4 The main theorem (spectrum)

The operator spectrum of a compact operator is trivial as we will show in the next proposition. This is somewhat clear since $\mathcal{K}(\mathbf{X})$ is the closure of $\mathcal{K}_c(\mathbf{X},\mathcal{P})$, the set of operators with only finitely many entries. This simple observation will allow us to understand that all limit operators of a Fredholm operator $A \in \mathcal{L}(\mathbf{X})$ are invertible and that their inverses are uniformly bounded.

Proposition 2.29. *For every sequence $(h_n)_{n\in\mathbb{N}}$ of integers tending to infinity and every $K \in \mathcal{K}(\mathbf{X})$, K_h exists and is equal to* 0.

Proof. Clearly $K_h = 0$ for every h and every $K \in \mathcal{K}_c(\mathbf{X},\mathcal{P})$. By Proposition 2.18, $\mathcal{K}(\mathbf{X})$ is the closure of $\mathcal{K}_c(\mathbf{X},\mathcal{P})$. Thus $K_h = 0$ for every $K \in \mathcal{K}(\mathbf{X})$ as well by Proposition 2.26(vi). □

Remark 2.30. For $A \in \mathrm{BDO}(\mathbf{X})$ also the converse holds. If A is band-dominated and $\sigma^{\mathrm{op}}(A) = \{0\}$, then A is compact. For band operators this is straightforward. To extend this to band-dominated operators, one has to argue that one can approximate A with band operators A_n that satisfy $\sigma^{\mathrm{op}}(A_n) = \{0\}$ for all $n \in \mathbb{N}$. One can also argue that the set

$$S := \{A \in \mathrm{BDO}(\mathbf{X}) : \sigma^{\mathrm{op}}(A) = \{0\}\}$$

is an ideal in $\mathcal{L}(\mathbf{X})$ containing $\mathcal{K}(\mathbf{X})$. But since $\mathcal{K}(\mathbf{X})$ is the largest proper ideal in $\mathcal{L}(\mathbf{X})$ (see [11, Theorem 1.4] for $p = 2$ and [34, Theorem 5.1] for $p = 0$ and $1 \leq p < \infty$), $S = \mathcal{K}(\mathbf{X})$ follows. However, this argument does not quite work in the more general context we want to consider later on and therefore we refer to Section 3, where this result emerges very naturally as a corollary of a much stronger statement.

Considering the previous proposition, it should not be too surprising that if A is band-dominated and Fredholm, then all of its limit operators are invertible and their inverses are uniformly bounded. Indeed, let $A, B \in \mathcal{L}(\mathbf{X})$ and $K_1, K_2 \in \mathcal{K}(\mathbf{X})$ such that $AB = I + K_1$ and $BA = I + K_2$ and let h be a sequence of integers tending to infinity such that A_h exists. By Proposition 2.15, B is band-dominated as well, so that we can apply Proposition 2.27 to get a subsequence g of h such that B_g exists. Clearly, A_g exists as well and is equal to A_h. Thus we have

$$I = I_g + (K_1)_g = (AB)_g = A_g B_g = A_h B_g$$

by Proposition 2.26. Similarly, we also have $B_g A_h = I$. Hence A_h is invertible and

$$\|(A_h)^{-1}\| = \|B_g\| \leq \|B\|\,.$$

This proves that all limit operators are invertible and their inverses are uniformly bounded if A is band-dominated and Fredholm. In fact, we do not have to require that A is band-dominated. This is because A_h exists if and only if B_h exists:

Proposition 2.31. *([84, Theorem 16])*
Let $A \in \mathcal{L}(\mathbf{X})$ be Fredholm, let B be a regularizer of A and let h be a sequence of integers tending to infinity such that A_h exists. Then B_h exists as well and is equal to $(A_h)^{-1}$. Thus all limit operators of A are invertible and their inverses are uniformly bounded.

The converse does not hold in general as already mentioned above. However, for band-dominated operators the converse is indeed true but harder to prove. We therefore refer to [57, Section 3.2.2] for the proof.

Theorem 2.32. *Let $A\in \mathrm{BDO}(\mathbf{X})$. Then A is Fredholm if and only if all of its limit operators are invertible and their inverses are uniformly bounded.*

The condition on the uniform boundedness of the inverse is a bit inconvenient because we want to write down

$$\mathrm{sp}_{\mathrm{ess}}(A)=\bigcup_{B\in\sigma^{\mathrm{op}}(A)}\mathrm{sp}(B),$$

which we cannot if the uniform boundedness condition is not redundant. Luckily, we have the following theorem by Rabinovich, Roch and Silbermann ([73, Theorem 8]) that was extended by Chandler-Wilde and Lindner ([18, Corollary 4.3],[19, Corollary 6.48]).

Theorem 2.33. *Let $A\in\mathcal{W}(\mathbf{X})$. Then the following are equivalent:*

(i) All limit operators of A are injective on $\ell^\infty(\mathbb{Z},\mathbb{C})$.

(ii) All limit operators of A are invertible on $\ell^p(\mathbb{Z},\mathbb{C})$ for some $p\in\{0\}\cup[1,\infty]$.

(iii) All limit operators of A are invertible on $\ell^p(\mathbb{Z},\mathbb{C})$ for all $p\in\{0\}\cup[1,\infty]$ and

$$\sup_{p\in[1,\infty]}\sup_{B\in\sigma^{\mathrm{op}}(A)}\|B^{-1}\|_p<\infty.$$

(iv) A is Fredholm on $\ell^p(\mathbb{Z},\mathbb{C})$ for some $p\in\{0\}\cup[1,\infty]$.

(v) A is Fredholm on $\ell^p(\mathbb{Z},\mathbb{C})$ for all $p\in\{0\}\cup[1,\infty]$.

Moreover,

$$\mathrm{sp}_{\mathrm{ess}}(A)=\bigcup_{B\in\sigma^{\mathrm{op}}(A)}\mathrm{sp}(B)=\bigcup_{B\in\sigma^{\mathrm{op}}(A)}\mathrm{sp}^\infty_{p.p.}(B),$$

where $\mathrm{sp}^\infty_{p.p.}(B)$ denotes the set of ℓ^∞-eigenvalues of B.

This theorem is useful in many ways. It does not only tell us that the uniform boundedness condition is redundant in the case $A\in\mathcal{W}(\mathbf{X})$, it moreover simplifies the computation of the essential spectrum quite a lot. Firstly, we only have to look for the eigenvalues of the limit operators, which is great because eigenvalues are usually easier to find than the whole spectrum. Secondly, we only have to consider ℓ^∞-eigenvalues. In many cases it is not hard to write down a formal eigenvector x of an operator $A\in\mathrm{BO}(\mathbf{X})$. It is then left to prove that x is bounded, the easiest condition to check under all ℓ^p-norms. We will make great use of this theorem later on.

In fact, a recent result of Lindner and Seidel shows that the uniform boundedness condition is redundant not only in $\mathcal{W}(\mathbf{X})$ but also in $\mathrm{BDO}(\mathbf{X})$.

Theorem 2.34. *([61, Theorem 11])*
Let $A\in\mathrm{BDO}(\mathbf{X})$. Then A is Fredholm if and only if all of its limit operators are invertible. Moreover,

$$\mathrm{sp}_{\mathrm{ess}}(A)=\bigcup_{B\in\sigma^{\mathrm{op}}(A)}\mathrm{sp}(B).$$

2.3 References

The general theory about Fredholm operators presented here is fairly standard and can probably be found in any book about operator theory, e.g. [24, 33, 48]. The results about limit operators, except some recent results as stated, can be found for example in [57] or [75]. Theorem 2.33, except for part (i), which was first proven in [18], is due to the authors of [73].

3 The $\mathcal{P}$-Framework

3.1 Limit operators in the general case

In this section we return to the general case $\mathbf{X} := \ell^p(\mathbb{Z}^N, X)$, where $N \in \mathbb{N}$, $p \in \{0\} \cup [1, \infty]$ and X is an arbitrary Banach space. Most of the things discussed in Section 2.2 are still valid under a small modification. Namely, we no longer consider operators modulo compact operators but operators modulo $\mathcal{P}$-compact operators as already mentioned in Section 2.2.2. We use the most general definition here that makes sense for arbitrary Banach spaces $\mathbf{Y}$.

3.1.1 $\mathcal{P}$-compactness revisited

Definition 3.1. Let $\mathbf{Y}$ be an arbitrary Banach space and let $\mathcal{P} := (P_n)_{n\in\mathbb{N}}$ be a sequence of projections P_n with the properties

($\mathcal{P}$1) $P_n = P_nP_{n+1} = P_{n+1}P_n$ and $\|P_n\| = \|Q_n\| = 1$ for all $n \in \mathbb{N}$, where $Q_n := I - P_n$;

($\mathcal{P}$2) $C_{\mathcal{P}} := \sup\limits_{U\subset\mathbb{N}} \|\sum\limits_{n\in U}(P_{n+1} - P_n)\| < \infty$ with the supremum taken over all finite sets $U \subset \mathbb{N}$.

Then $\mathcal{P}$ is called a uniform approximate projection in the sense of [75, 83, 86]. If additionally

($\mathcal{P}$3) $\sup\limits_{n\in\mathbb{N}} \|P_nx\| = \|x\|$ for all $x \in \mathbf{Y}$

then $\mathcal{P}$ is called a uniform approximate identity.

In what follows we always assume that $\mathcal{P}$ is a uniform approximate projection on $\mathbf{Y}$. In this case $\mathcal{P}^* = (P_n^*)_{n\in\mathbb{N}}$ defines a uniform approximate projection on $\mathbf{Y}^*$. The same statement is not necessarily true for uniform approximate identities.

Note that for $\mathbf{Y} = \mathbf{X}$ the standard projections $(P_n)_{n\in\mathbb{N}}$, $P_n := P_{\{-n,\dots,n\}^N}$ as defined in Section 1.1.6 and used in Section 2.2.2 define a uniform approximate identity on $\mathbf{X}$. Unless stated otherwise, we always use the standard projections $(P_n)_{n\in\mathbb{N}}$ on $\mathbf{X}$.

Definition 3.2. $K \in \mathcal{L}(\mathbf{Y})$ is called $\mathcal{P}$-compact if

$$\|(I - P_n)K\| + \|K(I - P_n)\| = \|Q_nK\| + \|KQ_n\| \to 0$$

as $n \to \infty$. The set of $\mathcal{P}$-compact operators is denoted by $\mathcal{K}(\mathbf{Y}, \mathcal{P})$.

This definition is in accordance with the definition given in Section 2.2.2 since

$$\|K - P_nKP_n\| \leq \|K - P_nK\| + \|P_n(K - KP_n)\| \leq \|(I - P_n)K\| + \|K(I - P_n)\|$$

and

$$\begin{aligned}\|Q_nK\| &= \|Q_n(K - P_nKP_n)\| \leq \|K - P_nKP_n\|, \\ \|KQ_n\| &= \|(K - P_nKP_n)Q_n\| \leq \|K - P_nKP_n\|.\end{aligned}$$

Thus these definitions are generalizations of the notions introduced in Section 2.2.2.

In Proposition 2.18 we showed that for $N = 1$, $p \in (1,\infty)$ and $X = \mathbb{C}$, it holds $\mathcal{K}(\mathbf{X},\mathcal{P}) = \mathcal{K}(\mathbf{X})$. This is no longer true if $p \in \{1,\infty\}$ or X is infinite-dimensional. Indeed, let $A \in \mathcal{L}(\ell^1(\mathbb{Z},\mathbb{C}))$ be given by

$$(Ax)_j = \begin{cases} \sum\limits_{i\in\mathbb{Z}} x_i & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}.$$

Then clearly, A is compact but $\|AQ_n\| = 1$ for all $n \in \mathbb{N}$. Similarly, $A^* \in \mathcal{L}(\ell^\infty(\mathbb{Z},\mathbb{C}))$ is compact but $\|Q_nA^*\| = 1$ for all $n \in \mathbb{N}$. Furthermore, if $\dim X = \infty$, it is easy to construct an operator which has only one entry but is not compact. However, we can recover the following from Section 2.2.2.

Proposition 3.3. *If $p \in \{0\} \cup (1,\infty)$, then $\mathcal{K}(\mathbf{X}) \subset \mathcal{K}(\mathbf{X},\mathcal{P})$. If $\dim(X) < \infty$, then $\mathcal{K}(\mathbf{X},\mathcal{P}) \subset \mathcal{K}(\mathbf{X})$.*

In general, $\mathcal{K}(\mathbf{Y},\mathcal{P})$ is not an ideal in $\mathcal{L}(\mathbf{Y})$ and hence $\mathcal{L}(\mathbf{Y})/\mathcal{K}(\mathbf{Y},\mathcal{P})$ is not an algebra. Therefore we replace $\mathcal{L}(\mathbf{Y})$ by the largest subalgebra $\mathcal{L}(\mathbf{Y},\mathcal{P})$ such that $\mathcal{K}(\mathbf{Y},\mathcal{P})$ is an ideal in $\mathcal{L}(\mathbf{Y},\mathcal{P})$:

$$\mathcal{L}(\mathbf{Y},\mathcal{P}) = \{A \in \mathcal{L}(\mathbf{Y}) : AK, KA \in \mathcal{K}(\mathbf{Y},\mathcal{P}) \text{ for all } K \in \mathcal{K}(\mathbf{Y},\mathcal{P})\}.$$

Now it is easy to see that $\mathcal{L}(\mathbf{Y},\mathcal{P})$ forms a closed subalgebra of $\mathcal{L}(\mathbf{Y})$ that contains $\mathcal{K}(\mathbf{Y},\mathcal{P})$ as a closed two-sided ideal. Furthermore, $\mathcal{L}(\mathbf{Y},\mathcal{P})$ is inverse closed by [75, Theorem 1.1.9]. If $\mathbf{Y} = \mathbf{X}$, $\mathcal{L}(\mathbf{X},\mathcal{P})$ consists exactly of those matrices for which every row and every column decays to 0 towards infinity. This can also be formulated in the general setting $(\mathbf{Y},\mathcal{P})$:

Proposition 3.4. *(e.g. [57, Proposition 1.20])*
It holds

$$\mathcal{L}(\mathbf{Y},\mathcal{P}) = \{A \in \mathcal{L}(\mathbf{Y}) : \|P_mAQ_n\| + \|Q_nAP_m\| \to 0 \text{ as } n \to \infty \text{ for every fixed } m \in \mathbb{N}\}.$$

Generalizing usual Fredholmness and the essential spectrum, we now study invertibility modulo $\mathcal{K}(\mathbf{Y},\mathcal{P})$. An operator $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$ is called $\mathcal{P}$-Fredholm if the coset $A + \mathcal{K}(\mathbf{Y},\mathcal{P})$ is invertible in the quotient algebra $\mathcal{L}(\mathbf{Y},\mathcal{P})/\mathcal{K}(\mathbf{Y},\mathcal{P})$. In other words, $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$ is $\mathcal{P}$-Fredholm if and only if there exists an operator $B \in \mathcal{L}(\mathbf{Y},\mathcal{P})$, a so-called $\mathcal{P}$-regularizer, and $K_1, K_2 \in \mathcal{K}(\mathbf{Y},\mathcal{P})$ such that $AB = I + K_1$ and $BA = I + K_2$. For $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$ the $\mathcal{P}$-essential spectrum, again denoted by $\operatorname{sp}_{\mathrm{ess}}(A)$, is then the set $\{\lambda \in \mathbb{C} : A - \lambda I \text{ is not } \mathcal{P}\text{-Fredholm}\}$. Clearly, if $\mathcal{K}(\mathbf{Y}) \subset \mathcal{K}(\mathbf{Y},\mathcal{P})$ holds, then Fredholmness implies $\mathcal{P}$-Fredholmness (cf. Proposition 3.3). Similarly, if $\mathcal{K}(\mathbf{Y},\mathcal{P}) \subset \mathcal{K}(\mathbf{Y})$ holds, then $\mathcal{P}$-Fredholmness implies Fredholmness. In the case $\mathbf{Y} = \mathbf{X}$ Fredholmness always implies $\mathcal{P}$-Fredholmness and if A is Fredholm of index 0, then the perturbation K such that $A + K$ is invertible (cf. Theorem 2.12) can be chosen in $\mathcal{K}(\mathbf{X},\mathcal{P})$:

Proposition 3.5. *([84, Corollary 12])*
Let $A \in \mathcal{L}(\mathbf{X},\mathcal{P})$ be Fredholm. Then A is $\mathcal{P}$-Fredholm and has a generalized inverse $B \in \mathcal{L}(\mathbf{X},\mathcal{P})$, i.e. $A = ABA$ and $B = BAB$. Moreover, A is Fredholm of index zero if and only if there exists an operator $K \in \mathcal{K}(\mathbf{X},\mathcal{P})$ of finite rank such that $A + K$ is invertible.

Finally, if $\mathcal{P}$ is a uniform approximate identity, we say that a sequence $(A_n)_{n\in\mathbb{N}}$ of operators in $\mathcal{L}(\mathbf{Y},\mathcal{P})$ converges $\mathcal{P}$-strongly to an operator $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$ if

$$\|K(A_n - A)\| + \|(A_n - A)K\| \to 0$$

as $n \to \infty$ for every $K \in \mathcal{K}(\mathbf{Y},\mathcal{P})$. As in Proposition 2.23, it suffices to consider the operators $P_m \in \mathcal{K}(\mathbf{Y},\mathcal{P})$ if the sequence $(A_n)_{n\in\mathbb{N}}$ is bounded:

Proposition 3.6. *Let $(A_n)_{n\in\mathbb{N}}$ be a bounded sequence of operators in $\mathcal{L}(\mathbf{Y},\mathcal{P})$ and $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$. Then the following are equivalent:*

(i) $\|P_m(A_n - A)\| \to 0$ *and* $\|(A_n - A)P_m\| \to 0$ *as* $n \to \infty$ *for all* $m \in \mathbb{N}$,

(ii) $\|K(A_n - A)\| \to 0$ *and* $\|(A_n - A)K\| \to 0$ *as* $n \to \infty$ *for all* $K \in \mathcal{K}(\mathbf{Y},\mathcal{P})$,

(iii) $A_n \to A$ *$\mathcal{P}$-strongly.*

Proof. The equivalence of (ii) and (iii) is by definition. The proof of the equivalence of (i) and (ii) is exactly the same as the one of Proposition 2.23 if we replace $\mathcal{K}(\mathbf{X})$ by $\mathcal{K}(\mathbf{Y},\mathcal{P})$ and strong convergence by $\mathcal{P}$-strong convergence. □

If we fix an approximate projection $\mathcal{P}$ and an operator $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$, we can always find an equivalent approximate projection that is tailored for A. This provides significant simplifications in many arguments.

Proposition 3.7. *(extension of [86, Theorem 1.15])*
Let $\mathcal{P}$ be a uniform approximate projection on $\mathbf{Y}$ and $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$. Then there exists a sequence $\hat{\mathcal{P}} = (F_n)_{n\in\mathbb{N}}$ of operators that satisfies $(\mathcal{P}1)$ and $(\mathcal{P}2)$ with $C_{\hat{\mathcal{P}}} \le C_{\mathcal{P}}$, and for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $F_nP_m = P_mF_n = F_n$ as well as $P_nF_m = F_mP_n = P_n$, and $\|[A,F_n]\| := \|AF_n - F_nA\| \to 0$ as $n \to \infty$. If $\mathcal{P}$ is a uniform approximate identity, then $\lim\limits_{n\to\infty} \|F_nx\| = \|x\|$ for every $x \in \mathbf{Y}$.

Proof. The existence of (F_n) with $F_nP_m = P_mF_n = F_n$ and $P_nF_m = F_mP_n = P_n$ as stated and $\|[A,F_n]\| = \|AF_n - F_nA\| \to 0$ as $n \to \infty$ was proven in [86, Theorem 1.15]. Actually, for every $n \in \mathbb{N}$, these F_n are of the form (see [86, Equation (1.4)] and the proof there)

$$F_n = \frac{1}{n}\sum_{k=1}^{n} kP_{U^n_{n-k}} = \frac{1}{n}\left(\sum_{k=1}^{n-1} k(P_{r^n_{n-k+1}} - P_{r^n_{n-k}}) + nP_{r^n_1}\right) = \frac{1}{n}\sum_{k=1}^{n} P_{r^n_k}$$

with certain integers $1 < r^n_1 < r^n_2 < \ldots < r^n_n$ and sets U^n_{n-k} as defined in the proof of [86, Theorem 1.15]. Thus,

$$1 = \|P_1\| = \|P_1F_n\| \le \|F_n\| \le \frac{1}{n}\sum_{k=1}^{n} \left\|P_{r^n_k}\right\| = \frac{n}{n} = 1.$$

Similarly, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$1 = \|I - P_m\| = \|(I - P_m)(I - F_n)\| \le \|I - F_n\| \le \frac{1}{n}\sum_{k=1}^{n} \left\|I - P_{r^n_k}\right\| = \frac{n}{n} = 1.$$

For $F_n = F_nF_{n+1} = F_{n+1}F_n$ and $(\mathcal{P}2)$ see again [86, Theorem 1.15]. Finally, since $\|F_nx\| = \|F_nP_mx\| \le \|P_mx\|$ and $\|P_nx\| = \|P_nF_mx\| \le \|F_mx\|$ for m large enough, we have

$$\sup_{n\in\mathbb{N}} \|F_nx\| = \lim_{n\to\infty} \|F_nx\| = \lim_{n\in\mathbb{N}} \|P_nx\| = \sup_{n\in\mathbb{N}} \|P_nx\|$$

for every $x \in \mathbf{Y}$. Hence if additionally $(\mathcal{P}3)$ is fulfilled, then $\lim\limits_{n\to\infty} \|F_nx\| = \|x\|$ for every $x \in \mathbf{Y}$. □

3.1.2 Limit operators revisited

As in Section 2.2.1 we define band and band-dominated operators.

Definition 3.8. An operator $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ is called a band operator if A only has finitely many non-zero diagonals (cf. Section 1.1.6). The set of all band operators is denoted by $\mathrm{BO}(\mathbf{X})$. The closure of $\mathrm{BO}(\mathbf{X}) \subset \mathcal{L}(\mathbf{X})$ is denoted by $\mathrm{BDO}(\mathbf{X})$ and the elements of $\mathrm{BDO}(\mathbf{X})$ are called band-dominated operators.

Again, $\mathrm{BDO}(\mathbf{X})$ defines a unital, inverse closed Banach algebra and also $\mathrm{BDO}(\mathbf{X})/\mathcal{K}(\mathbf{X}, \mathcal{P})$ is inverse closed in $\mathcal{L}(\mathbf{X}, \mathcal{P})/\mathcal{K}(\mathbf{X}, \mathcal{P})$:

Proposition 3.9. *(e.g. [57, Proposition 2.10])*
$\mathrm{BDO}(\mathbf{X})$ *is a unital, inverse closed Banach subalgebra of* $\mathcal{L}(\mathbf{X}, \mathcal{P})$*. Moreover,* $\mathrm{BDO}(\mathbf{X})/\mathcal{K}(\mathbf{X}, \mathcal{P})$ *is also inverse closed in* $\mathcal{L}(\mathbf{X}, \mathcal{P})/\mathcal{K}(\mathbf{X}, \mathcal{P})$*, i.e. if* $A \in \mathrm{BDO}(\mathbf{X})$ *is* $\mathcal{P}$*-Fredholm, then there exist* $B \in \mathrm{BDO}(\mathbf{X})$ *and* $K_1, K_2 \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ *such that* $AB = I + K_1$ *and* $BA = I + K_2$*.*

Also the decomposition of band operators is exactly the same as in Section 2.2.1. just note that the index set is now $\mathbb{Z}^N$, which makes everything a bit less intuitive. For $n \in \mathbb{Z}^N$ let $V_n \in \mathrm{BO}(\mathbf{X})$ define the n-th shift operator, i.e. $(V_n x)_j = x_{j-n}$ for all $j \in \mathbb{Z}^N$ and $x \in \mathbf{X}$. Note that every V_n is an invertible isometry with $V_n^{-1} = V_{-n}$. It is clear that every band operator $A \in \mathrm{BO}(\mathbf{X})$ can be written as

$$A = \sum_{\|n\|_\infty \leq \omega} D^{(n)} V_n,$$

where $D^{(n)}$ is the diagonal matrix associated with the n-th diagonal of A and ω is the band-width.

Replacing $*$-strong by $\mathcal{P}$-strong convergence, we can also define limit operators. Note that for $N > 1$ a sequence $h = (h_n)_{n \in \mathbb{N}}$ in $\mathbb{Z}^N$ can tend to infinity in many ways. Thus we just say that h tends to infinity if $(\|h_n\|_\infty)_{n \in \mathbb{N}}$ tends to infinity.

Definition 3.10. Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and let $h = (h_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{Z}^N$ tending to infinity. If the $\mathcal{P}$-strong limit $\mathcal{P}\text{-}\lim\limits_{n \to \infty} V_{-h_n} A V_{h_n}$ exists, we call it a limit operator of A and denote it by A_h. As a abbreviation we just say that A_h exists. The set of all limit operators is called the operator spectrum of A and denoted by $\sigma^{\mathrm{op}}(A)$.

As an immediate consequence of the definition, we have that $\sigma^{\mathrm{op}}(A)$ is shift-invariant:

Proposition 3.11. *(e.g. [57, Proposition 3.94])*
Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$*. If* $B \in \sigma^{\mathrm{op}}(A)$*, then also* $V_{-n} B V_n \in \sigma^{\mathrm{op}}(A)$ *for all* $n \in \mathbb{Z}^N$*.*

Also, $\mathcal{P}$-strong convergence is again equivalent to entrywise convergence in the case of a band operators with a bounded band-width. The proof is exactly the same as the one of Proposition 2.24.

Proposition 3.12. *Let* $(A_n)_{n \in \mathbb{N}}$ *be a bounded sequence of band operators in* $\mathcal{L}(\mathbf{X}, \mathcal{P})$ *and* $A \in \mathrm{BO}(\mathbf{X})$*. Further assume that the band-widths of* A *and every* A_n *are bounded by* ω*. Then the following are equivalent:*

(i) $A_n \to A$ *entrywise, i.e.* $\|(A_n)_{i,j} - A_{i,j}\|_{\mathcal{L}(X)} \to 0$ *as* $n \to \infty$ *for all* $i, j \in \mathbb{Z}^N$*,*

(ii) $\|P_k (A_n - A) P_l\| \to 0$ *as* $n \to \infty$ *for all* $k, l \in \mathbb{N}$*,*

(iii) $A_n \to A$ *$\mathcal{P}$-strongly as* $n \to \infty$.

We also have the same properties as in Proposition 2.26. However, for the statement about the adjoint (property (v) in Proposition 2.26/3.14), we need to restrict ourselves to $p \neq \infty$ because in the case $p = \infty$, the operator $A^* \in \mathcal{L}(\mathbf{X}^*)$ does no longer fit into the setting. Furthermore, we need (parts of) the following lemma to prove the statement about the lower norm (property (ii) in Proposition 2.26/3.14).

Lemma 3.13. *Let $\mathcal{P}$ be a uniform approximate identity on $\mathbf{Y}$ and $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Then*

(i) The set $\mathbf{Y}_0 := \{y \in \mathbf{Y} : \|Q_n y\| \to 0$ as $n \to \infty\}$ is a closed subspace of $\mathbf{Y}$. The restriction $A_0 := A|_{\mathbf{Y}_0}$ of A to $\mathbf{Y}_0$ belongs to $\mathcal{L}(\mathbf{Y}_0)$, $\|A_0\| = \|A\|$ and $\nu(A) = \nu(A_0)$.

(ii) The restriction $(A^*)_0 := A^*|_{(\mathbf{Y}^*)_0}$ of A^* to the (analogously defined) subspace $(\mathbf{Y}^*)_0$ belongs to $\mathcal{L}((\mathbf{Y}^*)_0)$ and $\|(A^*)_0\| = \|A^*\|$.

(iii) If A is invertible, then A_0 is invertible with $(A_0)^{-1} = (A^{-1})_0 \in \mathcal{L}(\mathbf{Y}_0)$ and $\left\|(A^{-1})_0\right\| = \left\|A^{-1}\right\|$. Moreover, $(A^*)_0$ is invertible in $\mathcal{L}((\mathbf{Y}^*)_0)$ with $((A^*)_0)^{-1} = ((A^*)^{-1})_0 = ((A^{-1})^*)_0$ and $\left\|((A^*)^{-1})_0\right\| = \left\|(A^*)^{-1}\right\| = \left\|A^{-1}\right\|$.

Proof. (i) It is easily checked that $\mathbf{Y}_0$ is a closed subspace of $\mathbf{Y}$. $A_0(\mathbf{Y}_0) \subset \mathbf{Y}_0$ is proven in [86, Proposition 1.18.1] and the formula on the norm can be found in [86, Proposition 1.18.2]. The inequality $\nu(A) \leq \nu(A_0)$ is trivial, so it remains to prove $\nu(A) \geq \nu(A_0)$. Let $(F_n)_{n\in\mathbb{N}}$ be the sequence given by Proposition 3.7 and observe that $F_n x \in \mathbf{Y}_0$ for all $n \in \mathbb{N}$. It follows

$$\|Ax\| = \|F_n\| \|Ax\| \geq \|F_n Ax\| \geq \|AF_n x\| - \|[A, F_n]\| \|x\| \geq \nu(A_0) \|F_n x\| - \|[A, F_n]\| \|x\|$$

for every $x \in \mathbf{Y}$ and every $n \in \mathbb{N}$. Taking the limit $n \to \infty$, we get $\|Ax\| \geq \nu(A_0) \|x\|$ for every $x \in \mathbf{Y}$, hence $\nu(A) \geq \nu(A_0)$.

(ii) The inclusion $(A^*)_0((\mathbf{Y}^*)_0) \subset (\mathbf{Y}^*)_0$ follows again from [86, Proposition 1.18.1]. However, [86, Proposition 1.18.2] may not be applicable anymore since $\mathcal{P}^*$ is not necessarily an approximate identity. Therefore we need another proof for the formula on the norms.

Let $\varepsilon > 0$ and choose $y \in \mathbf{Y}$ with $\|y\| = 1$ such that $\|A\| \leq \|Ay\| + \varepsilon$. Since $\mathcal{P}$ is an approximate identity, we find a k such that $\|Ay\| \leq \|P_k Ay\| + \varepsilon$. Now, by Hahn-Banach there is a functional g_0 on $\operatorname{im} P_k$ with $\|g_0\| = 1$ and $\|P_k Ay\| = |g_0(P_k Ay)|$. Thus, setting $g := g_0 \circ P_k$, we obtain a functional $g \in (\mathbf{Y}^*)_0$ with $\|g\| = 1$, hence

$$\|(A^*)_0\| \leq \|A^*\| = \|A\| \leq \|Ay\| + \varepsilon \leq \|P_k Ay\| + 2\varepsilon = |g(Ay)| + 2\varepsilon \leq \|A^* g\| + 2\varepsilon \leq \|(A^*)_0\| + 2\varepsilon.$$

Thus $\|(A^*)_0\| = \|A^*\|$.

(iii) Let A be invertible. Then A_0 is invertible by [86, Corollary 1.9, Corollary 1.19] and $(A_0)^{-1} = (A^{-1})_0$ is clear by (i). Since $\mathcal{L}(\mathbf{Y}, \mathcal{P})$ is inverse closed, $\left\|(A^{-1})_0\right\| = \left\|A^{-1}\right\|$ also follows by (i). Furthermore, A^* is invertible as well and $(A^*)^{-1} = (A^{-1})^*$. Thus the same formulas hold for A^*. For this, note that in [86, Corollary 1.19] only a uniform approximate projection is needed. □

Now we are able to state the mentioned properties. Lemma 3.13 is needed to deduce property (ii) from [19, Theorem 5.12 $(viii)$] in the case $p = \infty$. The other properties are more or less straightforward.

Proposition 3.14. *(e.g. [19, Theorem 5.12], [57, Proposition 3.4])*
Let $A, B \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(\mathbf{X}, \mathcal{P})$. Furthermore, let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}^N$ tending to infinity. then the following holds:

(i) if A_h exists, then $\|A_h\| \le \|A\|$,

(ii) if A_h exists, then $\nu(A_h) \ge \nu(A)$,

(iii) if A_h and B_h exist, so does $(A+B)_h$ and $(A+B)_h = A_h + B_h$,

(iv) if A_h and B_h exist, so does $(AB)_h$ and $(AB)_h = A_h B_h$,

(v) if A_h exists and $p \neq \infty$, so does $(A^*)_h$ and $(A^*)_h = (A_h)^*$,

(vi) if $A_n \to A$ in norm as $n \to \infty$ and $(A_n)_h$ exists for all $n \in \mathbb{N}$, so does A_h and $(A_n)_h \to A_h$ in norm as $n \to \infty$.

Proof. Properties (i), (iii), (iv), (v) and (vi) are straightforward (see e.g. [57, Proposition 3.4]). For (ii) we use assertion $(viii)$ in [19, Theorem 5.12] (again straightforward to prove):

$$\|A_h x\| \ge \nu(A)\,\|x\| \quad \text{for all } x \in \mathbf{X}_0.$$

This implies $\nu(A_h) = \nu((A_h)_0) \ge \nu(A)$ by Lemma 3.13(i). □

Clearly, Proposition 2.28 is still valid as well.

Proposition 3.15. *Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and let $(h_n)_{n\in\mathbb{N}}$ be a sequence of integers tending to infinity such that A_h exists. If A is a band operator, so is A_h. If A is band-dominated, so is A_h.*

So these were the things we were able to recover from Section 2.2. Now there are of course also assertions that are no longer true here. In particular, not every sequence h tending to infinity has a subsequence g such that A_g exists, not even for $A \in \mathrm{BO}(\mathbf{X})$. This leads to the unpleasant fact that there are band-dominated operators which do not have enough limit operators to work with (or even no limit operators at all!). To avoid these cases, we introduce the notion of rich operators.

Definition 3.16. An operator $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ is called rich if every sequence h in $\mathbb{Z}^N$ tending to infinity has a subsequence g such that the limit operator A_g exists. The set of rich operators in $\mathcal{L}(\mathbf{X}, \mathcal{P})$ is denoted by $\mathcal{L}_\$(\mathbf{X}, \mathcal{P})$, the set of rich band-dominated operators is denoted by $\mathrm{BDO}_\$(\mathbf{X})$, etc.

By Proposition 3.14, $\mathcal{L}_\$(\mathbf{X}, \mathcal{P})$, $\mathrm{BO}_\$(\mathbf{X})$ and $\mathrm{BDO}_\$(\mathbf{X})$ are again Banach algebras. Also, every operator $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ is rich and $\sigma^{\mathrm{op}}(K) = \{0\}$:

Proposition 3.17. *For every sequence h in $\mathbb{Z}^N$ tending to infinity and every $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ the limit operator K_h exists and is equal to 0.*

Proof. Let $h = (h_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}^N$ tending to infinity and let $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$. Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Choose $k \in \mathbb{N}$ large enough such that $\|KQ_k\| < \varepsilon$. Then

$$\|V_{-h_n} K V_{h_n} P_m\| = \|K V_{h_n} P_m\| \le \|K Q_k V_{h_n} P_m\| + \|K P_k V_{h_n} P_m\| \le \|KQ_k\| + \|K P_k V_{h_n} P_m\|.$$

The second term on the right-hand side vanishes for $\|h_n\|_\infty > m + k$ whereas the first term is bounded by ε. Hence $\|V_{-h_n} K V_{h_n} P_m\| \to 0$ as $n \to \infty$. Similarly, $\|P_m V_{-h_n} K V_{h_n}\| \to 0$ as $n \to \infty$. By Proposition 3.6, this implies that $V_{-h_n} K V_{h_n}$ converges $\mathcal{P}$-strongly to 0, thus $K_h = 0$. □

Richness also implies the following auxiliary result that we will make great use of in later sections.

Proposition 3.18. *(e.g. [57, Proposition 3.104])*
Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ be rich. Then $\sigma^{\mathrm{op}}(A)$ is $\mathcal{P}$-sequentially compact, i.e. every sequence of operators in $\sigma^{\mathrm{op}}(A)$ has a subsequence that converges $\mathcal{P}$-strongly in $\sigma^{\mathrm{op}}(A)$.

In fact, Proposition 3.18 is also one of the main ingredients of the proof of the main theorem that we will state next. Again, one direction is true for all $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ whereas the other direction needs $A \in \mathrm{BDO}_{\$}(\mathbf{X})$.

Proposition 3.19. *([84, Theorem 16])*
Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ be $\mathcal{P}$-Fredholm and let B be a $\mathcal{P}$-regularizer of A. Then all limit operators of A are invertible and their inverses are uniformly bounded. Moreover, for every sequence h in $\mathbb{Z}^N$ tending to infinity such that A_h exists B_h exists as well and is equal to $(A_h)^{-1}$. Hence

$$\sigma^{\mathrm{op}}(B) = \left\{(A_h)^{-1} : A_h \in \sigma^{\mathrm{op}}(A)\right\}.$$

Theorem 3.20. *([61, Theorem 11])*
Let $A \in \mathrm{BDO}(\mathbf{X})$ be rich. Then A is $\mathcal{P}$-Fredholm if and only if all of its limit operators are invertible. In this case the inverses of the limit operators are uniformly bounded. Moreover,

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{B \in \sigma^{\mathrm{op}}(A)} \mathrm{sp}(B).$$

We will see in the next section that this is not the only correspondence between an operator and its limit operators.

3.2 The $\mathcal{P}$-essential norm

Definition 3.21. The quotient norm on the quotient space $\mathcal{L}(\mathbf{Y}, \mathcal{P})/\mathcal{K}(\mathbf{Y}, \mathcal{P})$, given by

$$\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| := \inf_{K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})} \|A + K\|,$$

is called the $\mathcal{P}$-essential norm of $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. With this norm the space $\mathcal{L}(\mathbf{Y}, \mathcal{P})/\mathcal{K}(\mathbf{Y}, \mathcal{P})$ is again a Banach algebra.

By Proposition 3.14 and Proposition 3.17, we have $\|A + K\| \geq \|B\|$ for all $B \in \sigma^{\mathrm{op}}(A)$ and $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$, hence

$$\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| \geq \sup_{B \in \sigma^{\mathrm{op}}(A)} \|B\|$$

for all $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$. In this section we will prove that if $A \in \mathrm{BDO}_{\$}(\mathbf{X})$, then even equality holds and the supremum on the right-hand side is attained as a maximum. For this we need some auxiliary results. The first one relates the $\mathcal{P}$-essential norm to the limit of the norms $\|AQ_m\|$ and $\|Q_m A\|$ as $m \to \infty$. Remember that by definition $\|AQ_m\| + \|AQ_m\| \to 0$ if and only if $A \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$, which is of course equivalent to saying $\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| = 0$. In this context the following proposition should not be too surprising.

Proposition 3.22. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Then*

$$\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| = \|A^* + \mathcal{K}(\mathbf{Y}^*, \mathcal{P}^*)\| = \lim_{m \to \infty} \|AQ_m\| = \lim_{m \to \infty} \|Q_m A\|.$$

Proof. Let $\varepsilon > 0$ and choose $K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$ such that $\|A + K\| \le \|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| + \varepsilon$ and $m_0 \in \mathbb{N}$ such that $\|KQ_m\| \le \varepsilon$ for all $m \ge m_0$. It follows

$$\|AQ_m\| = \|A - AP_m\| \ge \|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| \ge \|A + K\| - \varepsilon \ge \|(A + K)Q_m\| - \varepsilon \ge \|AQ_m\| - 2\varepsilon$$

for all $m \ge m_0$ and therefore $\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| = \lim\limits_{m\to\infty} \|AQ_m\|$. The equality $\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\| = \lim\limits_{m\to\infty} \|Q_m A\|$ is similar. Finally, $\|A^* Q_m^*\| = \|Q_m A\|$ finishes the proof. □

3.2.1 Operator norm localization

Next we need a sequence of auxiliary norms for band-dominated operators that will allow us to localize the usual norm in $\mathcal{L}(\mathbf{X})$.

Definition 3.23. As usual, the support of an element $x = (x_n)_{n\in\mathbb{Z}^N} \in \mathbf{X}$ and the diameter of a set $M \subset \mathbb{Z}^N$ are defined as $\operatorname{supp} x := \left\{n \in \mathbb{Z}^N : x_n \neq 0\right\}$ and $\operatorname{diam} M := \sup\left\{\|n - m\|_\infty : n, m \in M\right\}$, respectively. Moreover, we write $|M|$ for the cardinality of a set $M \subset \mathbb{Z}^N$. For $D \in \mathbb{N}$ and $A \in \mathrm{BDO}(\mathbf{X})$ we now define

$$|||A|||_D := \sup\left\{\frac{\|Ax\|}{\|x\|} : x \in \mathbf{X} \setminus \{0\}, \operatorname{diam}\operatorname{supp} x \le D\right\}.$$

It is readily seen that this defines a norm on $\mathrm{BDO}(\mathbf{X})$ for every $D \in \mathbb{N}$. For operators $A \in \mathrm{BO}(\mathbf{X})$ with a fixed band-width ω this norm is equivalent to the operator norm in the following sense:

Proposition 3.24. *For every $\delta \in (0, 1)$ and $\omega \in \mathbb{N}$ there is a $D \in \mathbb{N}$ such that for all $A \in \mathrm{BO}(\mathbf{X})$ with band-width ω the following holds:*

$$(1 - \delta)\|A\| \le |||A|||_D \le \|A\|.$$

The important part is that the number $D \in \mathbb{N}$ can be chosen uniformly for all $A \in \mathrm{BO}(\mathbf{X})$ with a fixed band-width ω. There is a very similar statement in [61, Proposition 6] for the lower norm $\nu(A)$ and also the proof is very similar.

Proof. Clearly $|||A|||_D \le \|A\|$ for all $D \in \mathbb{N}$ and $A \in \mathrm{BO}(\mathbf{X})$. Now fix $\omega \in \mathbb{N}$ and let A be a band-operator with band-width ω. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}^N$ put

$$\begin{aligned}
C_n &:= \{-n, ..., n\}^N, \\
C_{n,k} &:= k + C_n, \\
D_n &:= C_{n+\omega} \setminus C_{n-\omega}, \\
D_{n,k} &:= k + D_n, \\
c_n &:= |C_n| = |C_{n,k}| = (2n + 1)^N, \\
d_n &:= |D_n| = |D_{n,k}| = c_{n+\omega} - c_{n-\omega} \sim n^{N-1}.
\end{aligned}$$

Let us abbreviate $P_{C_{n,k}}$ by $P_{n,k}$ and $P_{D_{n,k}} = P_{n+\omega,k} - P_{n-\omega,k}$ by $\Delta_{n,k}$. We start with the case $p \in [1, \infty)$, where we note the following facts:

(*i*) For all finite sets $S \subset \mathbb{Z}^N$ and all $x \in \mathbf{X}$, it holds

$$\sum_{k\in\mathbb{Z}^N} \|P_{k+S}\, x\|^p = |S| \cdot \|x\|^p.$$

(ii) By definition, remember the band-width ω, we have

$$P_{n,k}A\Delta_{n,k} = P_{n,k}A(P_{n+\omega,k} - P_{n-\omega,k}) = P_{n,k}A - AP_{n-\omega,k}$$

and

$$AP_{n,k}\Delta_{n,k} = AP_{n,k}(P_{n+\omega,k} - P_{n-\omega,k}) = AP_{n,k} - AP_{n-\omega,k}.$$

For the commutator $[P_{n,k}, A] = P_{n,k}A - AP_{n,k}$ it follows $[P_{n,k}, A] = [P_{n,k}, A]\Delta_{n,k}$ so that

$$\|[P_{n,k}, A]x\| = \|[P_{n,k}, A]\Delta_{n,k}x\| \leq \|[P_{n,k}, A]\| \, \|\Delta_{n,k}x\| \leq 2 \, \|A\| \, \|\Delta_{n,k}x\|$$

for all $x \in \mathbf{X}$ and hence

$$\sum_{k\in\mathbb{Z}^N} \|[P_{n,k}, A]x\|^p \; \leq \sum_{k\in\mathbb{Z}^N} 2^p \, \|A\|^p \, \|\Delta_{n,k}x\|^p \overset{(i)}{=} 2^p \, \|A\|^p \, d_n \, \|x\|^p \, .$$

(iii) Minkowski's inequality:

$$\left(\sum_{k\in\mathbb{Z}^N} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k\in\mathbb{Z}^N} |x_k|^p\right)^{1/p} + \left(\sum_{k\in\mathbb{Z}^N} |y_k|^p\right)^{1/p}$$

for all $(x_k)_{k\in\mathbb{Z}^N}, (y_k)_{k\in\mathbb{Z}^N} \in \ell^p(\mathbb{Z}^N)$, $p \in [1, \infty)$.

Now let $\delta \in (0, 1)$ and choose $n \in \mathbb{N}$ sufficiently large such that $\frac{d_n}{c_n} < (\frac{\delta}{4})^p$ and $x \in \mathbf{X}$ with $\|x\| = 1$ such that $(1 - \frac{\delta}{2}) \, \|A\| \leq \|Ax\|$. Then we conclude as follows:

$$\begin{aligned}
\left(1 - \frac{\delta}{2}\right) \|A\| \, c_n^{1/p} &\leq c_n^{1/p} \, \|Ax\| \\
&\overset{(i)}{=} \left(\sum_{k\in\mathbb{Z}^N} \|P_{n,k}Ax\|^p\right)^{1/p} \\
&\leq \left(\sum_{k\in\mathbb{Z}^N} (\|AP_{n,k}x\| + \|[P_{n,k}, A]x\|)^p\right)^{1/p} \\
&\overset{(iii)}{\leq} \left(\sum_{k\in\mathbb{Z}^N} \|AP_{n,k}x\|^p\right)^{1/p} + \left(\sum_{k\in\mathbb{Z}^N} \|[P_{n,k}, A]x\|^p\right)^{1/p} \\
&\overset{(ii)}{\leq} \left(\sum_{k\in\mathbb{Z}^N} \|AP_{n,k}x\|^p\right)^{1/p} + 2 \, \|A\| \, d_n^{1/p}
\end{aligned}$$

This means

$$\left(1 - \frac{\delta}{2} - 2\left(\frac{d_n}{c_n}\right)^{1/p}\right) \|A\| \, c_n^{1/p} \leq \left(\sum_{k\in\mathbb{Z}^N} \|AP_{n,k}x\|^p\right)^{1/p} .$$

Taking p-th powers and using $2\left(\frac{d_n}{c_n}\right)^{1/p} < \frac{\delta}{2}$ and $\sum\limits_{k\in\mathbb{Z}^N} \|P_{n,k}x\|^p = c_n$, we get

$$(1-\delta)^p \|A\|^p \sum_{k\in\mathbb{Z}^N} \|P_{n,k}x\|^p \le \left(1 - \frac{\delta}{2} - 2\left(\frac{d_n}{c_n}\right)^{1/p}\right)^p \|A\|^p c_n \le \sum_{k\in\mathbb{Z}^N} \|AP_{n,k}x\|^p .$$

The last inequality implies that there must be some $k \in \mathbb{Z}^N$ for which $P_{n,k}x \neq 0$ and

$$(1-\delta)^p \|A\|^p \|P_{n,k}x\|^p \le \|AP_{n,k}x\|^p .$$

Hence

$$(1-\delta)\|A\| \le \frac{\|AP_{n,k}x\|}{\|P_{n,k}x\|} \le |||A|||_{2n+1}.$$

This finishes the proof for $p \in [1,\infty)$.

Finally, let $p \in \{0,\infty\}$, $\varepsilon > 0$ and $x \in \mathbf{X}$. Then there is a $k \in \mathbb{Z}^N$ with

$$\|Ax\|_\infty - \varepsilon \le \left\|P_{\{k\}}Ax\right\|_\infty = \left\|P_{\{k\}}AP_{\omega,k}x\right\|_\infty \le |||A|||_{2\omega+1} \|x\|_\infty$$

and the assertion easily follows again. □

Since this localizability of the operator norm holds simultaneously for all $A \in \mathrm{BO}(\mathbf{X})$ of the same band-width ω, this is no longer a property of a particular operator but rather of the space $\mathbf{X}$, called the operator norm localization (ONL) property [21]. There is recent work by X. Chen, R. Tessera, X. Wang, G. Yu and H. Sako (see [79] and references therein) on metric spaces M with a certain measure such that $\mathbf{X} = \ell^2(M)$ has the (ONL) property. Sako proves in [79] that in case of a discrete metric space M with $\sup\limits_{m\in M} |\{n \in M : d(m,n) \le R\}| < \infty$ for all radii $R > 0$ (which clearly holds in our case, $M = \mathbb{Z}^N$) the ONL property is equivalent to the so-called Property A that was introduced by G. Yu ([94, Definition 2.1]) and is connected with amenability. Hence Proposition 3.24 also follows from [79, Proposition 3.1, Theorem 4.1] and [94, Example 2.3] in the case $p = 2$.

We also want to mention the very recent paper by Špakula and Willett [87] that generalizes the limit operator results from $\mathbb{Z}^N$ to certain discrete metric spaces. Based on the work of Roe [78], combined with ideas of [61], they prove a version of Theorem 3.20 under the sole assumption that these metric spaces have Yu's Property A.

Proposition 3.24 can be extended to band-dominated operators by approximation. However, since the approximation of band-dominated operators is not uniform, this cannot be done uniformly for all $A \in \mathrm{BDO}(\mathbf{X})$. Nevertheless, we have the following corollary.

Corollary 3.25. *Let $A \in \mathrm{BDO}(\mathbf{X})$ and $\delta > 0$. Then there is a $D \in \mathbb{N}$ such that for all $U \subset \mathbb{Z}^N$ and all operators $B \in \{A\} \cup \sigma^{\mathrm{op}}(A)$ the localization*

$$\|BP_U\| - \delta \le |||BP_U|||_D \le \|BP_U\|$$

holds.

Proof. Fix $\delta > 0$ and take a band operator C such that $\|A - C\| < \frac{\delta}{6}$. In particular, this implies

$$|||(A-C)P_U|||_D \le \|(A-C)P_U\| \le \|A-C\| < \frac{\delta}{6}.$$

Now choose D by applying Proposition 3.24 to C with $\frac{\delta}{6\|C\|}$ instead of δ. Then, for all $U \in \mathbb{Z}^N$,

$$\begin{aligned}\|AP_U\| &\geq |||AP_U|||_D \geq |||CP_U|||_D - |||(A-C)P_U|||_D \geq (1 - \frac{\delta}{6\|C\|})\|CP_U\| - \frac{\delta}{6} \\ &\geq \|CP_U\| - \frac{\delta}{3} \geq \|AP_U\| - \|(A-C)P_U\| - \frac{\delta}{3} \geq \|AP_U\| - \frac{\delta}{2}.\end{aligned}$$

Now let $A_h \in \sigma^{\mathrm{op}}(A)$. The estimate $|||A_hP_U|||_D \leq \|A_hP_U\|$ is clear. Furthermore, applying the computation above to $A_h \in \mathrm{BDO}(\mathbf{X})$, we get an m such that $\|A_hP_U\| \leq \|A_hP_UP_m\| + \frac{\delta}{2}$. For $\|A_hP_UP_m\|$ we have the estimate

$$\begin{aligned}\|A_hP_UP_m\| &\leq \|V_{-h_n}AV_{h_n}P_UP_m\| + \|(A_h - V_{-h_n}AV_{h_n})P_UP_m\| \\ &= \left\|AP_{U\cap\{-m,\dots,m\}^N+h_n}\right\| + \|(A_h - V_{-h_n}AV_{h_n})P_UP_m\| \\ &\leq \left|\left|\left|AP_{U\cap\{-m,\dots,m\}^N+h_n}\right|\right|\right|_D + \frac{\delta}{2} + \|(A_h - V_{-h_n}AV_{h_n})P_UP_m\| \\ &= |||V_{-h_n}AV_{h_n}P_UP_m|||_D + \frac{\delta}{2} + \|(A_h - V_{-h_n}AV_{h_n})P_UP_m\|.\end{aligned}$$

The last summand tends to zero as $n \to \infty$ since $V_{-h_n}AV_{h_n} \to A_h$ $\mathcal{P}$-strongly and $P_UP_m \in \mathcal{K}(\mathbf{X}, \mathcal{P})$. For the first one we have

$$\begin{aligned}\big||||V_{-h_n}AV_{h_n}P_UP_m|||_D - |||A_hP_UP_m|||_D\big| &\leq |||(V_{-h_n}AV_{h_n} - A_h)P_UP_m|||_D \\ &\leq \|(V_{-h_n}AV_{h_n} - A_h)P_UP_m\| \to 0\end{aligned}$$

as $m \to \infty$. Thus we obtain

$$\|A_hP_U\| - \delta \leq \|A_hP_UP_m\| - \frac{\delta}{2} \leq |||A_hP_UP_m|||_D \leq |||A_hP_U|||_D.$$

Hence the assertion follows. □

3.2.2 The main theorem (norm)

Now we are able to prove an analogue of Theorem 3.20 for the norm of rich band-dominated operators.

Theorem 3.26. *Let $A \in \mathrm{BDO}(\mathbf{X})$ be rich. Then the following holds:*

$$\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = \max_{A_h \in \sigma^{\mathrm{op}}(A)} \|A_h\|.$$

Proof. As mentioned at the beginning of this section,

$$\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| \geq \sup_{A_h \in \sigma^{\mathrm{op}}(A)} \|A_h\| =: N_A$$

is clear by Proposition 3.14 and Proposition 3.17. So assume that $\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| > N_A$ holds. Then there is an $\varepsilon > 0$ such that $\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| > N_A + \varepsilon$. This implies that

$$\|AQ_m\| = \|A - AP_m\| \geq \|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| > N_A + \varepsilon$$

holds for every $m \in \mathbb{N}$. From Corollary 3.25 we get an $n \in \mathbb{N}$ such that $|||AQ_m|||_{2n+1} > N_A + \frac{\varepsilon}{2}$ for every $m \in \mathbb{N}$. In particular, we get a sequence $(k_m)_{m\in\mathbb{N}}$ in $\mathbb{Z}^N$ tending to infinity such that, in the notation $P_{n,k} = V_k P_n V_{-k}$ of the proof of Proposition 3.24, $N_A + \frac{\varepsilon}{2} < \|(AQ_m)P_{n,k_m}\| \leq \|AP_{n,k_m}\|$ for every $m \in \mathbb{N}$. Now pass to a subsequence $h = (h_m)_{m\in\mathbb{N}}$ of the sequence $(k_m)_{m\in\mathbb{N}}$ for which the limit operator A_h exists. Then

$$N_A + \frac{\varepsilon}{2} < \|AP_{n,h_m}\| = \|V_{-h_m} A V_{h_m} P_n\| \to \|A_h P_n\| \leq \|A_h\| \leq N_A$$

as $m \to \infty$ is a contradiction. Hence

$$\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = N_A = \sup_{A_h \in \sigma^{\mathrm{op}}(A)} \|A_h\| .$$

It remains to show that N_A exists as a maximum. The argument is similar to that in the proof of [61, Theorem 8]. Consider the numbers $\gamma_n := 2^{-n}$ and

$$r_l := \sum_{n=l}^{\infty} \gamma_n = 2^{-l+1}.$$

Then $(r_l)_{l\in\mathbb{N}}$ is a strictly decreasing sequence of positive numbers which tends to 0. From Corollary 3.25 we obtain a sequence $(D_l)_{l\in\mathbb{N}}$ of even numbers such that for every $l \in \mathbb{N}$, every $B \in \{A\} \cup \sigma^{\mathrm{op}}(A)$ and every $U \subset \mathbb{Z}^N$ the following holds:

$$D_{l+1} > 2D_l \quad \text{and} \quad |||BP_U|||_{D_l} > \|BP_U\| - \gamma_l.$$

Choose a sequence $(B_n)_{n\in\mathbb{N}} \subset \sigma^{\mathrm{op}}(A)$ such that $\|B_n\| \to N_A$ as $n \to \infty$. For $n \in \mathbb{N}$ we are going to construct a suitably shifted copy $C_n \in \sigma^{\mathrm{op}}(A)$ of B_n as follows:

We start with an $x_n^0 \in \mathbf{X}$ with $\|x_n^0\| = 1$ and $\operatorname{diam} \operatorname{supp} x_n^0 \leq D_n$ such that

$$\|B_n x_n^0\| \geq \|B_n\| - \gamma_n.$$

Now we choose a shift $j_n^0 \in \mathbb{Z}^N$ which centralizes $y_n^0 := V_{j_n^0} x_n^0$, i.e. $y_n^0 = P_{D_n/2} y_n^0$ and define $C_n^0 := V_{j_n^0} B_n V_{-j_n^0}$, which is contained in $\sigma^{\mathrm{op}}(A)$ by Proposition 3.11. Then we have

$$\|B_n\| \geq \|C_n^0 P_{D_n/2}\| \geq \|B_n\| - \gamma_n.$$

Now, for $k = 1, \ldots, n$, we gradually find a $x_n^k \in \operatorname{im} P_{D_{n-(k-1)}/2}$ with $\|x_n^k\| = 1$ and $\operatorname{diam} \operatorname{supp} x_n^k \leq D_{n-k}$ such that

$$\left\|C_n^{k-1} P_{D_{n-(k-1)}/2} x_n^k\right\| \geq \left\|C_n^{k-1} P_{D_{n-(k-1)}/2}\right\| - \gamma_{n-k},$$

pass to a centralized $y_n^k := V_{j_n^k} x_n^k$ via a shift $j_n^k \in \left\{-D_{n-(k-1)}/2, \ldots, D_{n-(k-1)}/2\right\}^N$ and define $C_n^k := V_{j_n^k} C_n^{k-1} V_{-j_n^k} \in \sigma^{\mathrm{op}}(A)$. For this we observe

$$\left\|C_n^k P_{D_{n-k}/2}\right\| \geq \left\|C_n^{k-1} P_{D_{n-(k-1)}/2}\right\| - \gamma_{n-k}.$$

In particular, for $n > l \geq 1$, the estimates

$$\left\|C_n^{n-l} P_{D_l/2}\right\| \geq \|B_n\| - \sum_{k=l}^{n} \gamma_k \geq \|B_n\| - r_l$$

hold. Finally define $C_n := C_n^n$ and notice that $C_n = V_{j_n^n+\ldots+j_n^{n-l+1}} C_n^{n-l} V_{-(j_n^n+\ldots+j_n^{n-l+1})}$, where $\left|j_n^n + \ldots + j_n^{n-l+1}\right| \leq D_l$ by construction, thus

$$\|C_n P_{2D_l}\| \geq \left\|C_n^{n-l} P_{D_l/2}\right\| \geq \|B_n\| - r_l.$$

By Proposition 3.18, $\sigma^{\mathrm{op}}(A)$ is $\mathcal{P}$-sequentially compact. Thus we can pass to a subsequence $(C_{h_n})_{n\in\mathbb{N}}$ of $(C_n)_{n\in\mathbb{N}}$ with $\mathcal{P}$-strong limit $C \in \sigma^{\mathrm{op}}(A)$. Then

$$\|C\| \geq \|CP_{2D_l}\| = \lim_{n\to\infty} \|C_{h_n} P_{2D_l}\| \geq \lim_{n\to\infty} \|B_{h_n}\| - r_l = N_A - r_l.$$

Since r_l tends to 0 as $l \to \infty$, the assertion follows. □

Note that both assumptions ($A \in \mathrm{BDO}(\mathbf{X})$ and A rich) are necessary as the following examples show.

Example 3.27. (a) Let $X := L^p([0,1])$ and consider the sequence of multiplication operators $(M_{a_k})_{k\in\mathbb{N}}$ in $\mathcal{L}(X)$ given by $a_k(x) := \sin(2\pi kx)$. Then the diagonal operator

$$A := \operatorname{diag}(\ldots, 0, 0, M_{a_1}, M_{a_2}, M_{a_3}, \ldots)$$

on $\mathbf{X} = \ell^p(\mathbb{Z}, X)$ has operator spectrum $\{0\}$ because the sequence $(M_{a_k})_{k\in\mathbb{N}}$ has no convergent subsequence. However, $\|M_{a_k}\| = \|a_k\|_\infty = 1$ for all $k \in \mathbb{N}$ and hence $\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = 1$ (cf. Proposition 3.22).

(b) Let $p \in (1, \infty)$ and $\mathbf{X} = \ell^p(\mathbb{Z})$. Consider the $n \times n$ matrices

$$B_n := \frac{1}{n} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = \frac{1}{n} v_n \otimes v_n,$$

where $\otimes$ denotes the Kronecker product and $v_n = (1, \ldots, 1) \in \mathbb{C}^n$. It is well-known that $\|v \otimes w\|_p = \|v\|_p \|w\|_q$ for $v \in \mathbb{C}^k$, $w \in \mathbb{C}^l$, $k, l \in \mathbb{N}$ and $\frac{1}{p} + \frac{1}{q} = 1$. In our case this implies

$$\|B_n\| = \frac{1}{n} \|v_n\|_p \|v_n\|_q = \frac{n^{1/p} n^{1/q}}{n} = 1$$

for all $n \in \mathbb{N}$. Now consider the block diagonal operator $A := \operatorname{diag}(\ldots, 0, 0, B_1, B_2, B_3, \ldots)$, where $A_{1,1} = B_1$. By the above, we get

$$\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = \lim_{m\to\infty} \|AQ_m\| = 1$$

by Proposition 3.22. Now fix $k \in \mathbb{N}$ and let $(h_m)_{m\in\mathbb{N}}$ be a sequence of integers tending to $+\infty$. It is not hard to check that

$$\|V_{-h_m} A V_{h_m} P_k\| \leq \frac{1}{n-1} \|v_n \otimes v_{2k+1}\| \leq \frac{n^{1/p}(2k+1)^{1/q}}{n-1}$$

for $\frac{(n-1)n}{2} - (k-1) \leq h_m < \frac{n(n+1)}{2} - k$ and $n > 2k+1$. Indeed, if h_m is in this interval, then all non-zero entries of $V_{-h_m} A V_{h_m} P_k$ are less or equal than $\frac{1}{n-1}$ and the largest block we can get is of size $n \times (2k+1)$. Therefore it follows $\|V_{-h_m} A V_{h_m} P_k\| \to 0$ as $m \to \infty$ for $p > 1$. Similarly, $\|P_k V_{-h_m} A V_{h_m}\| \to 0$ as $m \to \infty$ for $p < \infty$. Thus A_h exists and is equal to 0, hence $\sigma^{\mathrm{op}}(A) = \{0\}$.

The first example is banded but not rich whereas the second one is rich but not band-dominated. Note that in the cases $p \in \{0, 1, \infty\}$ the latter cannot happen since rich operators $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ are automatically band-dominated by [61, Theorem 15].

Remark 3.28. If $A \in \mathrm{BDO}_\$(\mathbf{X})$ satisfies $\sigma^{\mathrm{op}}(A) = \{0\}$, then by Theorem 3.26, $\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = 0$ and hence $A \in \mathcal{K}(\mathbf{X}, \mathcal{P})$. Together with Proposition 3.17 we thus have that $A \in \mathrm{BDO}_\$(\mathbf{X})$ is $\mathcal{P}$-compact if and only if $\sigma^{\mathrm{op}}(A) = \{0\}$ (cf. Remark 2.30). So in some sense $\mathcal{K}(\mathbf{X}, \mathcal{P})$ is the kernel of σ^{op}. This can be made precise by rearranging the operator spectrum (see e.g. [57, Section 3.5.2], [78] or [87]).

As a corollary of Theorem 3.26 we have the following result, which enables us to study pseudospectra as well.

Corollary 3.29. *Let $A \in \mathrm{BDO}_\$(\mathbf{X})$ be $\mathcal{P}$-Fredholm and let B be a $\mathcal{P}$-regularizer of A. Then*

$$\left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\| = \|B + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = \max_{B_h \in \sigma^{\mathrm{op}}(B)} \|B_h\| = \max_{A_h \in \sigma^{\mathrm{op}}(A)} \left\|A_h^{-1}\right\|.$$

Proof. The operator B is band-dominated by Proposition 3.9 and rich by Proposition 3.19. Hence we can apply Theorem 3.26 to B. The last equality follows again by Proposition 3.19. $\square$

What comes as a simple corollary here is in fact a cornerstone for large parts of the subsequent results. Remember Theorem 3.20 for $A \in \mathrm{BDO}_\$(\mathbf{X})$. Using the convention $\left\|a^{-1}\right\| = \infty$ for non-invertible elements a of a normed algebra, it says that

$$\left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\| < \infty \qquad \text{if and only if} \qquad \sup_{A_h \in \sigma^{\mathrm{op}}(A)} \left\|A_h^{-1}\right\| < \infty.$$

Now Corollary 3.29 goes far beyond. It shows that both quantities are always equal and that the supremum is actually attained as a maximum. This can be used to state a pseudospectral version of Theorem 3.20. We will do so in the next section.

3.3 The $\mathcal{P}$-essential pseudospectrum

We start with the definition and some properties of the (usual) pseudospectrum.

3.3.1 The pseudospectrum

Definition 3.30. Let $A \in \mathcal{L}(\mathbf{Y})$ and $\varepsilon > 0$. Then the ε-pseudospectrum of A is defined as

$$\mathrm{sp}_\varepsilon(A) := \left\{\lambda \in \mathbb{C} : \left\|(A - \lambda I)^{-1}\right\| > 1/\varepsilon\right\}.$$

Note that we used the convention $\left\|B^{-1}\right\| = \infty$ if B is not invertible. Thus it is immediate that $\mathrm{sp}(A) \subset \mathrm{sp}_\varepsilon(A)$ for every $\varepsilon > 0$. Moreover, by definition $\mathrm{sp}_\varepsilon(A) \subset \mathrm{sp}_\delta(A)$ for all $\delta > \varepsilon > 0$.

Remark 3.31. Sometimes pseudospectra are defined with "$\geq 1/\varepsilon$" instead of "$> 1/\varepsilon$", which leads to compact pseudospectra. However, we will find it benefical to work with open sets instead. A discussion on this particular question can be found in [91, Section 4].

From a computational perspective it is very natural to consider pseudospectra instead of the spectrum. This is because it is often hard to tell whether an operator (or a matrix) A is not invertible or the norm of the inverse is just very large, say $\|A^{-1}\| > 1/\varepsilon$. However, pseudospectra are only interesting in the case of non-normal operators. If $\mathbf{Y}$ is a Hilbert space and A is normal, then $\operatorname{sp}_\varepsilon(A) = \operatorname{sp}(A) + \varepsilon\mathbb{B}$, where $\mathbb{B}$ denotes the open unit ball in $\mathbb{C}$, i.e. $\mathbb{B} := \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{B}} = \mathbb{D}$. Indeed, if A is normal, $(A - \lambda I)^{-1}$ is normal as well, hence

$$\|(A-\lambda I)^{-1}\| = \rho((A-\lambda I)^{-1}) = \max_{z \in \operatorname{sp}((A-\lambda I)^{-1})} |z| = \left(\min_{z \in \operatorname{sp}(A-\lambda I)} |z|\right)^{-1} = \left(\min_{z \in \operatorname{sp}(A)} |z-\lambda|\right)^{-1}.$$

Normality is crucial in the first equality. In general, only $\|(A-\lambda I)^{-1}\| \geq \rho((A-\lambda I)^{-1})$ holds. Hence the inclusion $\operatorname{sp}_\varepsilon(A) \supset \operatorname{sp}(A) + \varepsilon\mathbb{B}$ is proper in general. Consider for example the matrix $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ with $a \geq 0$. It is not difficult to compute that the ε-pseudospectrum w.r.t. $\|\cdot\|_2$ is given by the open ball of radius $\varepsilon\sqrt{1 + a/\varepsilon}$ around 0. Observe that for large ε the radius of this ball is close to ε, which implies that $\operatorname{sp}_\varepsilon(A)$ is close to $\operatorname{sp}(A) + \varepsilon B$. This behavior is typical for pseudospectra: the interesting things happen for small ε (which justifies the choice of the letter ε). Note that although this was the case in the example, the pseudospectra need not be of the form $\operatorname{sp}(A) + r\mathbb{B}$ for some $r > 0$. Moreover, the pseudospectra depend on the chosen norm. In the example above the ε-pseudospectrum w.r.t. $\|\cdot\|_\infty$ is given by the open ball of radius $\varepsilon\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{a}{\varepsilon}}\right)$ around 0.

Still speaking from a computational perspective, there is another reason why pseudospectra are useful. If we compute the spectrum of an operator (or a matrix) numerically (and there is no way around it if $\mathbf{Y}$ is more than four-dimensional by the famous theorem of Abel-Ruffini), we are forced to use inexact methods (iterative methods, discretization, etc.). So in fact we are computing the spectrum of a slightly perturbed operator. Now a well-known result states that the pseudospectrum catches these errors, more precisely:

Proposition 3.32. *(e.g. [91, Theorem 4.2])*
Let $A \in \mathcal{L}(\mathbf{Y})$ and $\varepsilon > 0$. Then

$$\operatorname{sp}_\varepsilon(A) = \bigcup_{\|K\| < \varepsilon} \operatorname{sp}(A + K) = \bigcup_{\substack{\|K\| < \varepsilon \\ \operatorname{rank} K \leq 1}} \operatorname{sp}(A + K).$$

However, pseudospectra are also interesting from an analytic viewpoint. For example, the pseudospectrum has better continuity properties than the spectrum (see e.g. [35, Theorem 3.26]). Moreover, the spectrum is equal to the intersection of all pseudospectra.

Proposition 3.33. *(e.g. [91, Theorem 4.3])*
Let $A \in \mathcal{L}(\mathbf{Y})$. Then $\operatorname{sp}(A) = \bigcap_{\varepsilon > 0} \operatorname{sp}_\varepsilon(A)$.

3.3.2 The main theorem (pseudospectrum)

Pseudospectra can also be defined more generally for unital normed algebras. Here we define it for the quotient $\mathcal{L}(\mathbf{X}, \mathcal{P})/\mathcal{K}(\mathbf{X}, \mathcal{P})$.

Definition 3.34. For $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $\varepsilon > 0$, the $\mathcal{P}$-essential ε-pseudospectrum is defined as

$$\operatorname{sp}_{\varepsilon,\mathrm{ess}}(A) := \operatorname{sp}_\varepsilon(A + \mathcal{K}(\mathbf{X}, \mathcal{P})) := \left\{\lambda \in \mathbb{C} : \left\|(A - \lambda I + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\| > 1/\varepsilon\right\}.$$

From Corollary 3.29 we immediately get the following.

Theorem 3.35. *Let $A \in \mathrm{BDO}_\$(\mathbf{X})$ and $\varepsilon > 0$. Then*

$$\operatorname{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{A_h \in \sigma^{\mathrm{op}}(A)} \operatorname{sp}_\varepsilon(A_h).$$

Proof. Let $\lambda \in \mathbb{C}$. Corollary 3.29 implies

$$\left\|(A - \lambda I + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\| = \max_{A_h \in \sigma^{\mathrm{op}}(A)} \left\|(A_h - \lambda I)^{-1}\right\|$$

if $A - \lambda I$ is $\mathcal{P}$-Fredholm whereas Theorem 3.20 guarantees that both sides are infinite if $A - \lambda I$ is not $\mathcal{P}$-Fredholm, hence the assertion follows. ☐

3.3.3 Other properties of the $\mathcal{P}$-essential pseudospectrum

To motivate the $\mathcal{P}$-essential pseudospectrum, we show some useful properties. These are very similar to the properties of the (usual) pseudospectrum. So in some sense the $\mathcal{P}$-essential pseudospectrum is for the $\mathcal{P}$-essential spectrum what the pseudospectrum is for the spectrum. The first property is that again the intersection of all $\mathcal{P}$-essential pseudospectra equals the $\mathcal{P}$-essential spectrum. This follows immediately by definition and is true for every unital normed algebra:

$$\begin{aligned}\operatorname{sp}_{\varepsilon,\mathrm{ess}}(A) &= \left\{\lambda \in \mathbb{C} : \left\|(A - \lambda I + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\| > 1/\varepsilon\right\},\\ \operatorname{sp}_{\mathrm{ess}}(A) &= \left\{\lambda \in \mathbb{C} : \left\|(A - \lambda I + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\| = \infty\right\}.\end{aligned}$$

Proposition 3.36. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Then*

$$\operatorname{sp}_{\mathrm{ess}}(A) = \bigcap_{\varepsilon > 0} \operatorname{sp}_{\varepsilon,\mathrm{ess}}(A).$$

Combining this result with Theorem 3.35, we can write

$$\operatorname{sp}_{\mathrm{ess}}(A) = \bigcap_{\varepsilon > 0} \bigcup_{A_h \in \sigma^{\mathrm{op}}(A)} \operatorname{sp}_\varepsilon(A_h)$$

for $A \in \mathrm{BDO}_\$(\mathbf{X})$. Of course, this is just a weaker formulation of Theorem 3.20.

Next we want to prove a characterization similar to Proposition 3.32 in the case of rich band or band-dominated operators. For this we need the following extension of Proposition 3.32.

Proposition 3.37. *Let $\mathcal{C} \subset \mathcal{L}(\mathbf{X}, \mathcal{P})$ be a unital algebra that contains all rank-1-operators in $\mathcal{K}_c(\mathbf{X}, \mathcal{P})$, $A \in \mathcal{C}$ and $\varepsilon > 0$. Then*

$$\operatorname{sp}_\varepsilon(A) = \bigcup_{\|K\| < \varepsilon} \operatorname{sp}(A + K) = \bigcup_{\substack{\|K\| < \varepsilon,\\ K \in \mathcal{C}}} \operatorname{sp}(A + K) = \bigcup_{\substack{\|K\| < \varepsilon,\\ K \in \mathcal{K}(\mathbf{X}, \mathcal{P})}} \operatorname{sp}(A + K) = \bigcup_{\substack{\|K\| < \varepsilon,\\ K \in \mathcal{K}(\mathbf{X}, \mathcal{P}) \cap \mathcal{C},\\ \operatorname{rank} K \leq 1}} \operatorname{sp}(A + K).$$

Proof. Abbreviate the sets in this claim from left to right by $S_1, ..., S_5$. $S_1 = S_2$ holds by Proposition 3.32, $S_2 \supset S_3 \supset S_5$ and $S_2 \supset S_4 \supset S_5$ are obvious. Thus it remains to prove $S_5 \supset S_1$. So let $\lambda \in S_1$. Since the case $\lambda \in \mathrm{sp}(A)$ is clear, let $B := A - \lambda I$ be invertible with $\left\|B^{-1}\right\| > 1/\varepsilon$. By Lemma 3.13 c), also $B_0 = B|_{\mathbf{X}_0}$ is invertible and

$$\left\|(B_0)^{-1}\right\| = \left\|B^{-1}\right\| > 1/\varepsilon.$$

By Corollary 2.9, there exists an $x_0 \in \mathbf{X}_0$ with $\|x_0\| = 1$ such that $\|Bx_0\| = \|B_0 x_0\| < \varepsilon$. Now choose k large enough such that $x := \|P_k x_0\|^{-1} P_k x_0$ still satisfies $\|Bx\| < \varepsilon$. By the Hahn-Banach Theorem, there exists a functional φ with $\|\varphi\| = \varphi(x) = 1$ and $\varphi \circ P_k = \varphi$. Now let $K, \tilde{K} \in \mathcal{L}(\mathbf{X})$ be defined by

$$Ku := -\varphi(u)Bx \quad \text{and} \quad \tilde{K}u := -\varphi(u)x$$

for $u \in \mathbf{X}$. Clearly, both K and $\tilde{K}$ have rank 1. Moreover, we have $P_k \tilde{K} P_k = \tilde{K}$. This implies $\tilde{K} \in \mathcal{K}(\mathbf{X}, \mathcal{P}) \cap \mathcal{C}$ and thus also $K \in \mathcal{K}(\mathbf{X}, \mathcal{P}) \cap \mathcal{C}$ since $K = B\tilde{K}$ and $B \in \mathcal{L}(\mathbf{X}, \mathcal{P}) \cap \mathcal{C}$. We conclude with

$$(B + K)x = Bx - \varphi(x)Bx = Bx - Bx = 0$$

and $\|K\| = \|\varphi\| \|Bx\| < \varepsilon$. □

So here is the analogue of Proposition 3.32 for the $\mathcal{P}$-essential spectrum in the case of rich band(-dominated) operators. Once more, the proof is an application of limit operators, Theorem 3.35 to be precise.

Theorem 3.38. *Let $\mathcal{C} \in \{\mathrm{BO}_\$(\mathbf{X}), \mathrm{BDO}_\$(\mathbf{X})\}$ and let $A \in \mathcal{C}$. Then for every $\varepsilon > 0$, we have*

$$\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{\substack{\|T\| < \varepsilon, \\ T \in \mathcal{L}(\mathbf{X},\mathcal{P})}} \mathrm{sp}_{\mathrm{ess}}(A + T) = \bigcup_{\substack{\|T\| < \varepsilon, \\ T \in \mathcal{C}}} \mathrm{sp}_{\mathrm{ess}}(A + T).$$

Clearly, it suffices to consider only one representative per coset $T + \mathcal{K}(\mathbf{X}, \mathcal{P})$. Conversely, every coset $T + \mathcal{K}(\mathbf{X}, \mathcal{P})$ with $\|T + \mathcal{K}(\mathbf{X}, \mathcal{P})\| < \varepsilon$ contains an operator $\tilde{T}$ with $\|\tilde{T}\| < \varepsilon$. Thus an equivalent formulation in terms of cosets is also possible:

$$\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{\|T + \mathcal{K}(\mathbf{X},\mathcal{P})\| < \varepsilon} \mathrm{sp}(A + T + \mathcal{K}(\mathbf{X}, \mathcal{P})).$$

Proof. For every $L \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ we abbreviate the coset $L + \mathcal{K}(\mathbf{X}, \mathcal{P}) \in \mathcal{L}(\mathbf{X}, \mathcal{P})/\mathcal{K}(\mathbf{X}, \mathcal{P})$ by L°. Let $A \in \mathcal{C}$ and $\lambda \notin \mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$. With $B := A - \lambda I$, the coset B° is invertible and $\left\|(B^\circ)^{-1}\right\| \le 1/\varepsilon$. For arbitrary $T \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ with $\|T\| < \varepsilon$, one has

$$\left\|(B^\circ)^{-1} T^\circ\right\| \le \left\|(B^\circ)^{-1}\right\| \|T^\circ\| \le \left\|(B^\circ)^{-1}\right\| \|T\| < \frac{\varepsilon}{\varepsilon} = 1$$

so that $I^\circ + (B^\circ)^{-1} T^\circ$ is invertible by a Neumann series argument. Thus also

$$(B + T)^\circ = B^\circ (I^\circ + (B^\circ)^{-1} T^\circ)$$

is invertible and therefore $\lambda \notin \mathrm{sp}_{\mathrm{ess}}(A + T)$.

Together with Theorem 3.35 we obtain the following inclusions:

$$\bigcup_{\substack{\|T\|<\varepsilon,\\ T\in\mathcal{C}}} \mathrm{sp}_{\mathrm{ess}}(A+T) \subset \bigcup_{\substack{\|T\|<\varepsilon,\\ T\in\mathcal{L}(\mathbf{X},\mathcal{P})}} \mathrm{sp}_{\mathrm{ess}}(A+T) \subset \mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup_{A_h\in\sigma^{\mathrm{op}}(A)} \mathrm{sp}_{\varepsilon}(A_h).$$

Thus it remains to show that the right-most set is contained in the left-most. So let $A_h \in \sigma^{\mathrm{op}}(A)$ and $\lambda \in \mathrm{sp}_\varepsilon(A_h)$. By Proposition 3.37, λ is contained in $\mathrm{sp}(A_h+K)$ for some $K \in \mathcal{K}(\mathbf{X},\mathcal{P}) \cap \mathcal{C}$ with $\|K\| < \varepsilon$. Now choose a subsequence g of h such that all cubes $g_n + \{-n,\dots,n\}^N$ are pairwise disjoint and define

$$T := \sum_{n\in\mathbb{N}} V_{g_n} P_n K P_n V_{-g_n}.$$

T is a well-defined block-diagonal operator belonging to $\mathcal{C}$ with $\|T\| \leq \|K\| < \varepsilon$ and $T_g = K$. Since

$$(A - \lambda I + T)_g = A_g - \lambda I + T_g = A_h - \lambda I + K,$$

we find that $\lambda \in \mathrm{sp}(A_h+K) = \mathrm{sp}((A+T)_g)$. Thus $\lambda \in \mathrm{sp}_{\mathrm{ess}}(A+T)$ by Theorem 3.20. $\square$

3.4 The $\mathcal{P}$-essential lower norm

In this section we try to find an analogue of Theorem 3.20 and Theorem 3.26 in terms of lower norms. Combining Corollary 3.29 and Corollary 2.9, we get

$$\left\|(A+\mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\right\| = \max_{A_h\in\sigma^{\mathrm{op}}(A)} \left\|A_h^{-1}\right\| = \left(\min_{A_h\in\sigma^{\mathrm{op}}(A)} \nu(A_h)\right)^{-1}$$

if $A \in \mathrm{BDO}_\$(\mathbf{X})$ is $\mathcal{P}$-Fredholm. More generally, we can write

$$\left\|(A+\mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\right\|^{-1} = \min_{A_h\in\sigma^{\mathrm{op}}(A)} \min\left\{\nu(A_h), \nu(A_h^*)\right\}$$

for arbitrary $A \in \mathrm{BDO}_\$(\mathbf{X})$. In this section we prove that $\left\|(A+\mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\right\|^{-1}$ can be characterized in terms of lower norms as well.

3.4.1 Lower norms of asymptotic compressions

Definition 3.39. For $A \in \mathcal{L}(\mathbf{Y})$ set

$$\tilde{\mu}(A) := \lim_{m\to\infty} \nu(A|_{\operatorname{im} Q_m}) \quad \text{and} \quad \mu(A) := \min\left\{\tilde{\mu}(A),\, \tilde{\mu}(A^*)\right\},$$

where $A|_{\operatorname{im} Q_m} \colon \operatorname{im} Q_m \to \mathbf{Y}$.

Since $\nu(A|_{\operatorname{im} Q_m}) \leq \|AQ_m\| \leq \|A\|$ for all $m \in \mathbb{N}$, the sequence $(\nu(A|_{\operatorname{im} Q_m}))_{m\in\mathbb{N}}$ is bounded. Furthermore, this sequence is obviously monotonically increasing since $\operatorname{im} Q_{m+1} \subset \operatorname{im} Q_m$. Thus the limit always exists and is equal to $\sup\limits_{m\in\mathbb{N}} \nu(A|_{\operatorname{im} Q_m})$.

In Section 2.1.3 we studied the connection of Weyl sequences and invertibility/Fredholmness. We observed that an operator A is invertible if and only if neither A nor A^* has a Weyl sequence. Moreover, we showed in Theorem 2.11 that A is Fredholm if and only if neither A nor A^* has a singular Weyl sequence. The quantity $\mu(A)$ can also be interpreted in this setting. A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{Y}$ is called a $\mathcal{P}$-Weyl sequence (for A) if $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $Ax_n \to 0$ as $n \to \infty$ and $\|P_m x_n\| \to 0$ as $n \to \infty$ for every fixed $m \in \mathbb{N}$. We then have the following correspondence.

Lemma 3.40. *Let $A \in \mathcal{L}(\mathbf{Y})$. Then $\tilde{\mu}(A) = 0$ if and only if A has a $\mathcal{P}$-Weyl sequence.*

Proof. If $\tilde{\mu}(A) = 0$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements $x_n \in \mathbf{Y}$ with $\|x_n\| = 1$ such that $x_n \in \operatorname{im} Q_n$ and $Ax_n \to 0$ as $n \to \infty$. This obviously defines a $\mathcal{P}$-Weyl sequence.

Conversely, let $(x_n)_{n\in\mathbb{N}}$ be a $\mathcal{P}$-Weyl sequence of A. Then for every $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that $\|P_m x_n\| < \frac{1}{m}$ and $\|Ax_n\| < \frac{1}{m}$. This implies

$$\frac{\|AQ_m x_n\|}{\|Q_m x_n\|} = \frac{\|Ax_n - AP_m x_n\|}{\|x_n - P_m x_n\|} < \frac{\frac{1}{m} + \|A\| \frac{1}{m}}{1 - \frac{1}{m}} = \frac{1 + \|A\|}{m-1}.$$

Hence $\nu(A|_{\operatorname{im} Q_m}) \to 0$ as $m \to \infty$. □

Applying this lemma to A and A^*, we get $\mu(A) = 0$ if and only if A or A^* has a $\mathcal{P}$-Weyl sequence. This suggests that $\mu(A)$ should be connected to $\mathcal{P}$-Fredholmness in some way. Indeed, we have the following theorem.

Theorem 3.41. *Let $A \in \mathrm{BDO}_{\$}(\mathbf{X})$. Then*

$$\left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \mu(A).$$

The proof is divided into two lemmas. Both are presented in more generality than needed here. We will make use of this generality later on.

Lemma 3.42. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Then*

$$\left\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-1} \leq \mu(A).$$

If A is $\mathcal{P}$-Fredholm, then $\left\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-1} = \tilde{\mu}(A) = \tilde{\mu}(A^*) = \mu(A)$.

Proof. There is nothing to prove if A is not $\mathcal{P}$-Fredholm since $\left\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-1}$ vanishes by definition in this case. So assume that A is $\mathcal{P}$-Fredholm. Let $\varepsilon > 0$ be arbitrary and choose $B_0 \in (A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}$. Since $B_0 A - I =: K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$ we get

$$\|Q_m B_0 A Q_m - Q_m\| = \|Q_m K Q_m\| < \max\left\{\frac{\varepsilon}{\|B_0\| + \varepsilon}, 1\right\}$$

and $\|Q_m B_0\| \leq \left\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\| + \varepsilon$ for all m large enough. The first estimate is clear since $K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$ whereas the second estimate follows by Proposition 3.22. By a Neumann series argument, this implies that $Q_m B_0 A Q_m = Q_m + Q_m K Q_m$ is invertible in $\mathcal{L}(\operatorname{im} Q_m)$ and $\left\|(Q_m B_0 A Q_m)^{-1}\right\| \leq (1 - \|Q_m K Q_m\|)^{-1}$. Let $B_1 := Q_m (Q_m B_0 A Q_m)^{-1} Q_m B_0$. Then also

$$\begin{aligned}
\|Q_m B_0 - B_1\| &= \left\|(Q_m - Q_m (Q_m B_0 A Q_m)^{-1}) Q_m B_0\right\| \\
&= \left\|(Q_m B_0 A Q_m - Q_m)(Q_m B_0 A Q_m)^{-1} Q_m B_0\right\| \\
&\leq \|Q_m K Q_m\| \left(1 - \|Q_m K Q_m\|\right)^{-1} \|B_0\| \\
&< \frac{\varepsilon}{\|B_0\| + \varepsilon}\left(1 - \frac{\varepsilon}{\|B_0\| + \varepsilon}\right)^{-1} \|B_0\| \\
&= \varepsilon
\end{aligned}$$

Using

$$B_1AQ_m = Q_m(Q_mB_0AQ_m)^{-1}Q_mB_0AQ_m = Q_m,$$

we get that $\operatorname{im}(AQ_m)$ is closed by Lemma 2.8. Hence $A|_{\operatorname{im}Q_m}\colon \operatorname{im}Q_m \to \operatorname{im}(AQ_m)$ is invertible with inverse $B_1|_{\operatorname{im}(AQ_m)}\colon \operatorname{im}(AQ_m) \to \operatorname{im}Q_m$. By Corollary 2.9 and the estimates above, we get

$$\nu(A|_{\operatorname{im}Q_m})^{-1} = \left\|B_1|_{\operatorname{im}(AQ_m)}\right\| \le \|B_1\| \le \|Q_mB_0\| + \|B_1 - Q_mB_0\| \le \left\|(A+\mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\right\| + 2\varepsilon. \quad (6)$$

Since $AQ_m = A - AP_m$ is $\mathcal{P}$-Fredholm, there exists a $C \in \mathcal{L}(\mathbf{Y},\mathcal{P})$ such that $AQ_mC \in I + \mathcal{K}(\mathbf{Y},\mathcal{P})$. Thus $\|(AQ_mC - I)Q_k\| < \frac{\varepsilon}{\|B_1\|}$ for k large enough. Thus we get

$$\begin{aligned}
\|B_1Q_k\| &\le \|B_1AQ_mCQ_k\| + \|B_1(I - AQ_mC)Q_k\| \\
&\le \left\|B_1|_{\operatorname{im}(AQ_m)}\right\| \|AQ_mCQ_k\| + \|B_1\| \|(AQ_mC - I)Q_k\| \\
&< \left\|B_1|_{\operatorname{im}(AQ_m)}\right\| (1 + \frac{\varepsilon}{\|B_1\|}) + \|B_1\| \frac{\varepsilon}{\|B_1\|} \\
&\le \left\|B_1|_{\operatorname{im}(AQ_m)}\right\| + 2\varepsilon.
\end{aligned}$$

It follows

$$\begin{aligned}
\nu(A|_{\operatorname{im}Q_m})^{-1} &= \|B_1|_{\operatorname{im}AQ_m}\| \ge \|B_1Q_k\| - 2\varepsilon \ge \|Q_mB_0Q_k\| - \|(B_1 - Q_mB_0)Q_k\| - 2\varepsilon \\
&\ge \|Q_mB_0Q_k\| - \|B_1 - Q_mB_0\| - 2\varepsilon > \left\|(A+\mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\right\| - 3\varepsilon, \qquad (7)
\end{aligned}$$

where we used $Q_mB_0Q_k \in (A+\mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}$ in the last inequality.

Combining (6) and (7) and taking the limit $m \to \infty$, we obtain $\tilde{\mu}(A) = \left\|(A+\mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\right\|^{-1}$. By the same argument for $A^* \in \mathcal{L}(\mathbf{Y}^*,\mathcal{P}^*)$, we find

$$\tilde{\mu}(A^*) = \left\|(A^*+\mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*))^{-1}\right\|^{-1} = \left\|(A+\mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\right\|^{-1},$$

where we used Proposition 3.22 in the latter equality. We conclude

$$\left\|(A+\mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\right\|^{-1} = \tilde{\mu}(A) = \tilde{\mu}(A^*) = \mu(A). \qquad \square$$

Lemma 3.43. *Let $A \in \mathcal{L}(\mathbf{X},\mathcal{P})$. Then $\mu(A) \le \inf\left\{\left\|A_h^{-1}\right\|^{-1} : A_h \in \sigma^{\mathrm{op}}(A)\right\}$. For any invertible $A_h \in \sigma^{\mathrm{op}}(A)$ we even have both $\tilde{\mu}(A) \le \left\|A_h^{-1}\right\|^{-1}$ and $\tilde{\mu}(A^*) \le \left\|A_h^{-1}\right\|^{-1}$.*

Proof. Let $A_h \in \sigma^{\mathrm{op}}(A)$.

1st case: A_h is not invertible. For every $\varepsilon > 0$ there is a $\mathcal{P}$-compact operator T with $\|T\| = 1$ such that $\|A_hT\| < \varepsilon$ or $\|TA_h\| < \varepsilon$ (cf. [84, Theorem 11]). Since $(Q_m)_h = I$, $V_{-h_n}AQ_mV_{h_n}$ converges $\mathcal{P}$-strongly to A_h and it follows

$$\|V_{-h_n}AQ_mV_{h_n}T\| < 2\varepsilon \quad \text{or} \quad \|TV_{-h_n}Q_mAV_{h_n}\| < 2\varepsilon$$

for all $m \in \mathbb{N}$ and all $n \in \mathbb{N}$ large enough. Setting $T_n := V_{h_n}TV_{-h_n}$ we have $\|AQ_mT_n\| < 2\varepsilon$ or $\|T_nQ_mA\| < 2\varepsilon$. Furthermore,

$$\|Q_mT_n\| = \|Q_mV_{h_n}TV_{-h_n}\| = \|V_{-h_n}Q_mV_{h_n}T\| \to \|IT\| = 1$$

and similarly $\|T_nQ_m\| \to 1$ as $n \to \infty$. We conclude

$$\frac{\|AQ_mT_n\|}{\|Q_mT_n\|} < 3\varepsilon \quad \text{or} \quad \frac{\|T_nQ_mA\|}{\|T_nQ_m\|} < 3\varepsilon$$

for all $n \in \mathbb{N}$ large enough. This yields $\nu(A|_{\operatorname{im} Q_m}) < 3\varepsilon$ or $\nu(A^*|_{\operatorname{im} Q_m^*}) < 3\varepsilon$, and since ε and m were arbitrary, we conclude $\mu(A) = 0$.

2nd case: A_h is invertible. By Lemma 3.13 (*iii*), the compression $(A_h)_0$ is invertible with $((A_h)_0)^{-1} = (A_h^{-1})_0$ and $\left\|(A_h^{-1})_0\right\| = \left\|A_h^{-1}\right\|$. Let $\varepsilon > 0$. Then there exists an $x_0 \in \mathbf{X}_0$ with $\|x_0\| = 1$ such that

$$\|A_hx_0\| = \|(A_h)_0x_0\| < \nu((A_h)_0) + \varepsilon = \left\|((A_h)_0)^{-1}\right\|^{-1} + \varepsilon = \left\|A_h^{-1}\right\|^{-1} + \varepsilon.$$

For sufficiently large k also $x := \|P_kx_0\|^{-1} P_kx_0$ fulfills $\|A_hx\| < \left\|A_h^{-1}\right\|^{-1} + \varepsilon$. For sufficiently large n, $\|(V_{-h_n}AQ_mV_{h_n} - A_h)P_k\| \leq \varepsilon$ holds and we find

$$\|AQ_mV_{h_n}x\| = \|V_{-h_n}AQ_mV_{h_n}P_kx\| \leq \|A_hP_kx\| + \varepsilon = \|A_hx\| + \varepsilon \leq \left\|A_h^{-1}\right\|^{-1} + 2\varepsilon.$$

This implies $\nu(A|_{\operatorname{im} Q_m}) \leq \left\|A_h^{-1}\right\|^{-1} + 2\varepsilon$ for every $m \in \mathbb{N}$. We conclude $\mu(A) \leq \tilde{\mu}(A) \leq \left\|A_h^{-1}\right\|^{-1}$.

In the dual setting we proceed in exactly the same way to get

$$\tilde{\mu}(A^*) \leq \left\|(A_h^*)^{-1}\right\|^{-1} = \left\|(A_h)^{-1}\right\|^{-1}$$

by considering the compressions $(A_h^*)_0$. □

Proof of Theorem 3.41. Combining Lemma 3.42 and Lemma 3.43, we get

$$\left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} \leq \mu(A) \leq \inf\left\{\left\|(A_h)^{-1}\right\|^{-1} : A_h \in \sigma^{\mathrm{op}}(A)\right\}$$

for all $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$. If we restrict ourselves to rich band-dominated operators, we can use Corollary 3.29 to deduce

$$\left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} \leq \mu(A) \leq \min\left\{\left\|(A_h)^{-1}\right\|^{-1} : A_h \in \sigma^{\mathrm{op}}(A)\right\} = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1}$$

and hence $\left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \mu(A)$. □

3.4.2 Lower norms of $\mathcal{P}$-compact perturbations

Remember that we defined the $\mathcal{P}$-essential norm by $\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = \inf\limits_{K \in \mathcal{K}(\mathbf{Y}, \mathcal{P})} \|A + K\|$. In a similar way we now define the $\mathcal{P}$-essential lower norm.

Definition 3.44. For $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ we call

$$\nu_{\mathrm{ess}}(A) := \sup\{\nu(A + K) : K \in \mathcal{L}(\mathbf{Y}, \mathcal{P})\}$$

the $\mathcal{P}$-essential lower norm of A.

In order to write $\left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1}$ in terms of the $\mathcal{P}$-essential lower norm, we study the relations between $\nu_{\mathrm{ess}}(A)$ and $\tilde{\mu}(A)$.

Proposition 3.45. *Let $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$. Then $\nu_{\mathrm{ess}}(A) \le \tilde{\mu}(A)$ and $\nu_{\mathrm{ess}}(A^*) \le \tilde{\mu}(A^*)$. If $\nu(A) > 0$, then $\nu_{\mathrm{ess}}(A) = \tilde{\mu}(A)$. If $\nu(A^*) > 0$, then $\nu_{\mathrm{ess}}(A^*) = \tilde{\mu}(A^*)$.*

Proof. For $\varepsilon > 0$ choose $K \in \mathcal{K}(\mathbf{Y},\mathcal{P})$ such that $\nu(A+K) \ge \nu_{\mathrm{ess}}(A) - \varepsilon$ and $m \in \mathbb{N}$ such that $\|KQ_m\| \le \varepsilon$. Then

$$\tilde{\mu}(A) \ge \nu(A|_{\operatorname{im} Q_m}) \ge \nu((A+K)|_{\operatorname{im} Q_m}) - \|KQ_m\| \ge \nu(A+K) - \varepsilon \ge \nu_{\mathrm{ess}}(A) - 2\varepsilon$$

by Proposition 2.7. This implies $\tilde{\mu}(A) \ge \nu_{\mathrm{ess}}(A)$.

Now let A be bounded below and assume that there are constants $c, d \in \mathbb{R}$ such that

$$\nu_{\mathrm{ess}}(A) < c < d < \tilde{\mu}(A).$$

By definition, $\nu(Q_kA + \alpha P_kA) \le \nu_{\mathrm{ess}}(A)$ for all $k \in \mathbb{N}$, $\alpha > 0$. In particular, for every $k \in \mathbb{N}$ and $\alpha > 0$ there exists an $x_{k,\alpha}$ with $\|x_{k,\alpha}\| = 1$ such that $\|(Q_kA + \alpha P_kA)x_{k,\alpha}\| < c$. Furthermore,

$$\|Q_kAx_{k,\alpha}\| = \|Q_k(Q_kA + \alpha P_kA)x_{k,\alpha}\| < c$$

and similarly

$$\alpha \|P_kAx_{k,\alpha}\| = \|P_k(Q_kA + \alpha P_kA)x_{k,\alpha}\| < c.$$

Now choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{(d-c)\nu(A)}{\nu(A) + 2(\|A\| + d)},$$

which implies

$$\frac{c + \varepsilon + 2\varepsilon \|A\| / \nu(A)}{1 - 2\varepsilon/\nu(A)} < d,$$

and choose $\alpha > 1$ such that $c/\alpha < \varepsilon$.

Fix $n \in \mathbb{N}$, take the sequence $(F_m)_{m\in\mathbb{N}}$ from Proposition 3.7 and choose $m \in \mathbb{N}$ such that $P_nF_m = P_n$ and $\|[F_m, A]\| < \varepsilon$. Then choose $k \in \mathbb{N}$ such that $F_mP_k = F_m$. From $\alpha \|P_kAx_{k,\alpha}\| < c$ we get $\|F_mP_kAx_{k,\alpha}\| < c/\alpha$ and we conclude that

$$\|AF_mx_{k,\alpha}\| \le \|F_mAx_{k,\alpha}\| + \|[F_m, A]x_{k,\alpha}\| \le c/\alpha + \|[F_m, A]\| < 2\varepsilon.$$

Thus

$$\|P_nx_{k,\alpha}\| = \|P_nF_mx_{k,\alpha}\| \le \|F_mx_{k,\alpha}\| \le \|AF_mx_{k,\alpha}\| / \nu(A) < 2\varepsilon/\nu(A)$$

and

$$\|Q_nx_{k,\alpha}\| \ge \|x_{k,\alpha}\| - \|P_nx_{k,\alpha}\| > 1 - 2\varepsilon/\nu(A).$$

This implies

$$\begin{aligned}\nu(A|_{\operatorname{im} Q_n}) &\le \frac{\|AQ_nx_{k,\alpha}\|}{\|Q_nx_{k,\alpha}\|} \le \frac{\|Ax_{k,\alpha}\| + \|AP_nx_{k,\alpha}\|}{\|Q_nx_{k,\alpha}\|} \le \frac{\|Q_kAx_{k,\alpha}\| + \|P_kAx_{k,\alpha}\| + \|A\| \|P_nx_{k,\alpha}\|}{\|Q_nx_{k,\alpha}\|} \\ &< \frac{c + c/\alpha + 2\varepsilon \|A\| / \nu(A)}{1 - 2\varepsilon/\nu(A)} < d,\end{aligned}$$

hence $\tilde{\mu}(A) \le d < \tilde{\mu}(A)$, a contradiction. So $\nu_{\mathrm{ess}}(A) = \tilde{\mu}(A)$ in this case. Similarly we can prove $\nu_{\mathrm{ess}}(A^*) = \tilde{\mu}(A^*)$ if $\nu(A^*) > 0$. ☐

Observe that by Proposition 2.7

$$|\tilde{\mu}(A+K)-\tilde{\mu}(A)| = \lim_{m\to\infty} |\nu((A+K)|_{\operatorname{im} Q_m}) - \nu(A|_{\operatorname{im} Q_m})| \le \lim_{m\to\infty} \|KQ_m\| = 0$$

holds for all $K \in \mathcal{K}(\mathbf{Y},\mathcal{P})$ and $A \in \mathcal{L}(\mathbf{Y})$. This implies that both $\tilde{\mu}$ and ν_{ess} are invariant under $\mathcal{P}$-compact perturbations. Thus we have the following corollaries.

Corollary 3.46. *Let $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$. If $A+\mathcal{K}(\mathbf{Y},\mathcal{P})$ contains an operator being bounded below, then $\nu_{\text{ess}}(B) = \tilde{\mu}(B) > 0$ for all $B \in A+\mathcal{K}(\mathbf{Y},\mathcal{P})$. If $A^*+\mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)$ contains an operator being bounded below, then $\nu_{\text{ess}}(B) = \tilde{\mu}(B) > 0$ for all $B \in A^*+\mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)$.*

Corollary 3.47. *Let $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$. Then we either have $\nu_{\text{ess}}(A) = 0$ or $\nu_{\text{ess}}(A) = \tilde{\mu}(A) > 0$. Furthermore, we either have $\nu_{\text{ess}}(A^*) = 0$ or $\nu_{\text{ess}}(A^*) = \tilde{\mu}(A^*) > 0$.*

Note that Fredholm operators on $\mathbf{X} = \ell^p(\mathbb{Z})$ ($p \in (1,\infty)$) with positive index satisfy $0 = \nu_{\text{ess}}(A) < \tilde{\mu}(A)$ by Theorem 2.2(ii) (remember that $\mathcal{K}(\mathbf{X},\mathcal{P}) = \mathcal{K}(\mathbf{X})$ in this case, cf. Proposition 2.18). Similarly, Fredholm operators with negative index satisfy $0 = \nu_{\text{ess}}(A^*) < \tilde{\mu}(A^*)$.

For Fredholm operators we have another corollary:

Corollary 3.48. *Let $A \in \mathcal{L}(\mathbf{X},\mathcal{P})$ and let $A+\mathcal{K}(\mathbf{X},\mathcal{P})$ contain a Fredholm operator. Then*

$$\max\{\nu_{\text{ess}}(A), \nu_{\text{ess}}(A^*)\} \ge \mu(A)$$

If this Fredholm operator has index 0*, we additionally have*

$$\nu_{\text{ess}}(A) = \tilde{\mu}(A) = \tilde{\mu}(A^*) = \nu_{\text{ess}}(A^*).$$

Proof. Since both μ and ν_{ess} are invariant under $\mathcal{P}$-compact perturbations, we can w.l.o.g. assume that this Fredholm operator is A. Let us further assume that $\operatorname{ind}(A) \le 0$. Now choose Fredholm operators $S, T \in \mathcal{L}(\mathbf{X},\mathcal{P})$ such that $ST = I$ and $\operatorname{ind}(AS) = 0$ (cf. [84, Lemma 24], Theorem 2.2(i)). Applying Proposition 3.5 to AS we find an operator $K \in \mathcal{K}(\mathbf{X},\mathcal{P})$ such that $AS+K$ is invertible. This implies

$$\nu(A+KT) = \nu(AST+KT) = \nu((AS+K)T) \ge \nu(AS+K)\nu(T) > 0$$

because $1 = \nu(I) = \nu(ST) \le \|S\|\,\nu(T)$ and $AS+K$ is invertible. Corollary 3.46 now implies $\nu_{\text{ess}}(A) = \tilde{\mu}(A) \ge \mu(A)$.

Similarly, if $\operatorname{ind}(A) \ge 0$, we can choose Fredholm operators $S, T \in \mathcal{L}(\mathbf{X},\mathcal{P})$ such that $TS = I$ and $\operatorname{ind}(SA) = 0$. Applying Proposition 3.5 to SA we find an operator $K \in \mathcal{K}(\mathbf{X},\mathcal{P})$ such that $SA+K$ is invertible. This implies

$$\nu(A^*+K^*T^*) = \nu(A^*S^*T^*+K^*T^*) = \nu((A^*S^*+K^*)T^*) \ge \nu((SA+K)^*)\nu(T^*) > 0$$

and thus we get $\nu_{\text{ess}}(A^*) = \tilde{\mu}(A^*) \ge \mu(A)$.

In the case $\operatorname{ind}(A) = 0$, we obviously get both $\nu_{\text{ess}}(A) = \tilde{\mu}(A)$ and $\nu_{\text{ess}}(A^*) = \tilde{\mu}(A^*)$. Proposition 3.5 and Lemma 3.42 finish the proof. □

Let us summarize these results.

Theorem 3.49. *Let $A \in \mathcal{L}(\mathbf{X},\mathcal{P})$ be $\mathcal{P}$-Fredholm. Then the following holds:*

(i) If $A + \mathcal{K}(\mathbf{X}, \mathcal{P})$ contains an operator being bounded below, then

$$\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \nu_{\mathrm{ess}}(A) = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1}.$$

(ii) If $A^* + \mathcal{K}(\mathbf{X}^*, \mathcal{P}^*)$ contains an operator being bounded below, then

$$\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \nu_{\mathrm{ess}}(A^*) = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1}.$$

(iii) If $A + \mathcal{K}(\mathbf{X}, \mathcal{P})$ contains a Fredholm operator, then

$$\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1}.$$

Proof. (i) By Lemma 3.42, we have

$$\tilde{\mu}(A) = \tilde{\mu}(A^*) = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1},$$

Corollary 3.46 implies $\nu_{\mathrm{ess}}(A) = \tilde{\mu}(A)$ and Proposition 3.45 implies $\nu_{\mathrm{ess}}(A^*) \leq \tilde{\mu}(A^*)$. Hence the assertion follows.

(ii) Similar as (i).

(iii) Combining Lemma 3.42 with Corollary 3.48 and Proposition 3.45, we get

$$\left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \tilde{\mu}(A) = \tilde{\mu}(A^*) = \mu(A) \leq \max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} \leq \max\{\tilde{\mu}(A), \tilde{\mu}(A^*)\}.$$

Hence the assertion follows again. □

Combining Theorem 3.49 with Corollary 3.29, we get another analogue of Theorem 3.20. We have more restrictions here, though.

Corollary 3.50. *Let $A \in \mathrm{BDO}_{\$}(\mathbf{X})$ be $\mathcal{P}$-Fredholm. Then the following holds:*

(i) If $A + \mathcal{K}(\mathbf{X}, \mathcal{P})$ contains an operator being bounded below, then

$$\nu_{\mathrm{ess}}(A) = \min_{A_h \in \sigma^{\mathrm{op}}(A)} \nu(A_h).$$

(ii) If $A^* + \mathcal{K}(\mathbf{X}^*, \mathcal{P}^*)$ contains an operator being bounded below, then

$$\nu_{\mathrm{ess}}(A^*) = \min_{A_h \in \sigma^{\mathrm{op}}(A)} \nu(A_h^*).$$

Proof. Corollary 3.29 states

$$\left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \min_{A_h \in \sigma^{\mathrm{op}}(A)} \left\|A_h^{-1}\right\|^{-1}.$$

Moreover, by Theorem 3.20, all limit operators of A are invertible. This implies, using Corollary 2.9, that $\left\|A_h^{-1}\right\|^{-1} = \nu(A_h) = \nu(A_h^*)$ for every $A_h \in \sigma^{\mathrm{op}}(A)$. Thus the assertions follow immediately from Theorem 3.49. □

These results are a bit unsatisfactory because we do not have this full generality as in Theorem 3.20 or Theorem 3.26. Therefore we will try a different concept in the next section. In later sections we will also be able to improve the results for some special cases (Hilbert space and $\dim(X) < \infty$).

3.4.3 Symmetrization of the problem

In the two previous sections we looked at characteristics of both A and A^* and then combined them in order to get a complete (symmetric) picture. In this section we combine A and A^* first and then determine its essential lower norm. This will lead to similar results as obtained in the previous section. The benefits of this approach will only shine in the special cases discussed subsequently, though.

Given a Banach space $\mathbf{Y}$ with a uniform approximate projection $\mathcal{P}$, we write $\mathbf{Y} \oplus \mathbf{Y}^*$ for the Banach space of all pairs $(x, f) \in \mathbf{Y} \times \mathbf{Y}^*$ equipped with the norm $\|(x, f)\| := \max\{\|x\|, \|f\|\}$. For $A \in \mathcal{L}(\mathbf{Y})$, $B \in \mathcal{L}(\mathbf{Y}^*)$, we write $A \oplus B$ for the operator $(x, f) \mapsto (Ax, Bf)$ in $\mathcal{L}(\mathbf{Y} \oplus \mathbf{Y}^*)$. The following properties of $A \oplus B$ are easy to check:

$$\|A \oplus B\| = \max\{\|A\|, \|B\|\}, \quad \nu(A \oplus B) = \min\{\nu(A), \nu(B)\}.$$

To get a similar equality for the essential norm, we have to work a bit harder. Note that $\mathcal{P} \oplus \mathcal{P}^* = (P_n \oplus P_n^*)_{n\in\mathbb{N}}$ is again a uniform approximate projection on $\mathbf{Y} \oplus \mathbf{Y}^*$.

Proposition 3.51. *Let $A \oplus B \in \mathcal{L}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$. Then*

$$\|A \oplus B + \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)\| = \max\{\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\|, \|B + \mathcal{K}(\mathbf{Y}^*, \mathcal{P}^*)\|\}.$$

Proof. Observe that if $L \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$ and $M \in \mathcal{K}(\mathbf{Y}^*, \mathcal{P}^*)$, then $L \oplus M \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$. Hence

$$\begin{aligned}
\|A \oplus B + \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)\| &= \inf_{K \in \mathcal{K}(\mathbf{Y}\oplus\mathbf{Y}^*, \mathcal{P}\oplus\mathcal{P}^*)} \|A \oplus B + K\| \\
&\le \inf_{\substack{L \in \mathcal{K}(\mathbf{Y},\mathcal{P}) \\ M \in \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)}} \|A \oplus B + L \oplus M\| \\
&= \inf_{\substack{L \in \mathcal{K}(\mathbf{Y},\mathcal{P}) \\ M \in \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)}} \|(A + L) \oplus (B + M)\| \\
&= \inf_{\substack{L \in \mathcal{K}(\mathbf{Y},\mathcal{P}) \\ M \in \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)}} \max\{\|A + L\|, \|B + M\|\} \\
&= \max\left\{\inf_{L \in \mathcal{K}(\mathbf{Y},\mathcal{P})} \|A + L\|, \inf_{M \in \mathcal{K}(\mathbf{Y}^*,\mathcal{P}^*)} \|B + M\|\right\} \\
&= \max\{\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\|, \|B + \mathcal{K}(\mathbf{Y}^*, \mathcal{P}^*)\|\}
\end{aligned}$$

follows. This proves one direction.

Let $P_{(1)} \colon \mathbf{Y} \oplus \mathbf{Y}^* \to \mathbf{Y} \oplus \mathbf{Y}^*, (x, f) \mapsto (x, 0)$ and $P_{(2)} \colon \mathbf{Y} \oplus \mathbf{Y}^* \to \mathbf{Y} \oplus \mathbf{Y}^*, (x, f) \mapsto (0, f)$ be the canonical projections. Applying these to an operator $A \oplus B \in \mathbf{Y} \oplus \mathbf{Y}^*$ from both sides we get $P_{(1)}(A \oplus B)P_{(1)} = A \oplus 0$ and $P_{(2)}(A \oplus B)P_{(2)} = 0 \oplus B$, respectively. Furthermore, $\|P_{(1)}\| = \|P_{(2)}\| = 1$. It follows

$$\begin{aligned}
\|A \oplus B + K\| &\ge \max\left\{\left\|P_{(1)}(A \oplus B + K)P_{(1)}\right\|, \left\|P_{(2)}(A \oplus B + K)P_{(2)}\right\|\right\} \\
&= \max\left\{\left\|A \oplus 0 + P_{(1)}KP_{(1)}\right\|, \left\|0 \oplus B + P_{(2)}KP_{(2)}\right\|\right\} \\
&\ge \max\{\|A + \mathcal{K}(\mathbf{Y}, \mathcal{P})\|, \|B + \mathcal{K}(\mathbf{Y}^*, \mathcal{P}^*)\|\}.
\end{aligned}$$

for all $K \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$. Taking the infimum over all K, we get the other direction. □

At this point one is tempted to write down $\nu_{\mathrm{ess}}(A \oplus B) = \min\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(B)\}$. But this is wrong in general! Indeed, let $\mathbf{X} = \ell^p(\mathbb{Z})$, $p \in (1, \infty)$ (remember that $\mathcal{K}(\mathbf{X}, \mathcal{P}) = \mathcal{K}(\mathbf{X})$ in this case, cf. Proposition 2.18) and let $A \in \mathcal{L}(\mathbf{X})$ be Fredholm with positive index. Then $A \oplus A^*$ is Fredholm with index 0 and therefore there exists a compact operator $K \in \mathcal{L}(\mathbf{X} \oplus \mathbf{X}^*)$ such that $(A \oplus A^*) + K$ is invertible (see Theorem 2.12) and, in particular, bounded below. Therefore $\nu_{\mathrm{ess}}(A \oplus A^*) \geq \nu((A \oplus A^*) + K) > 0$. However, $\nu_{\mathrm{ess}}(A) = 0$ because $A + L$ has a positive index (see Theorem 2.2(ii)) and therefore a non-trivial kernel for all $L \in \mathcal{K}(\mathbf{X})$.

Nevertheless, the following is true: $\tilde{\mu}(A \oplus A^*) = \min\{\tilde{\mu}(A), \tilde{\mu}(A^*)\} = \mu(A)$. Furthermore, we can interpret Corollary 3.47 in this setting.

Corollary 3.52. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Then $\tilde{\mu}(A \oplus A^*) = \mu(A)$ and either $\nu_{\mathrm{ess}}(A \oplus A^*) = \mu(A) > 0$ or $\nu_{\mathrm{ess}}(A \oplus A^*) = 0$.*

Proof. Clearly,

$$\tilde{\mu}(A \oplus A^*) = \lim_{m \to \infty} \nu(A \oplus A^*|_{\mathrm{im}(Q_m \oplus Q_m^*)}) = \lim_{m \to \infty} \min\{\nu(A|_{\mathrm{im}\, Q_m}), \nu(A^*|_{\mathrm{im}\, Q_m^*})\} = \mu(A).$$

Corollary 3.47 applied to $A \oplus A^*$ then yields the assertion. □

If $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ is both Fredholm and $\mathcal{P}$-Fredholm, then $\mu(A)$ and $\nu_{\mathrm{ess}}(A \oplus A^*)$ actually coincide. This follows immediately from Corollary 3.52 if we can show that $\nu_{\mathrm{ess}}(A \oplus A^*) > 0$ in this case.

Theorem 3.53. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ be $\mathcal{P}$-Fredholm and $A + \mathcal{K}(\mathbf{Y}, \mathcal{P})$ contain a Fredholm operator. Then $\nu_{\mathrm{ess}}(A \oplus A^*) = \mu(A)$.*

Proof. W.l.o.g. A is already Fredholm. We show $\nu_{\mathrm{ess}}(A \oplus A^*) > 0$, which then implies the result by Corollary 3.52. In the case $\mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*) \subset \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$ this follows by the Fredholmness of $A \oplus A^*$ and Theorem 2.12. If $\mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*)$ is not a subset of $\mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$, we have to show that we can find a $K \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$ such that $A \oplus A^* + K$ is invertible or at least bounded below.

By [86, Corollary 1.9, Theorem 1.16] (cf. also the proof of [83, Corollary 12]), we have finite rank projections $P, P' \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$ with $\mathrm{im}(P) = \ker(A)$ and $\ker(P') = \mathrm{im}(A)$. Then $P^*, (P')^*$ are $\mathcal{P}^*$-compact projections with $\ker(P^*) = \mathrm{im}(A^*)$ and $\ker((P')^*) = \ker(A^*)$ by Theorem 2.1. Let $R := P \oplus (P')^* \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$ and $R' := P' \oplus P^* \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$. It follows $\mathrm{im}(R) = \ker(A \oplus A^*)$ and $\ker(R') = \mathrm{im}(A \oplus A^*)$. Since both projections are of the same finite rank there exists an isomorphism $C\colon \mathrm{im}(R) \to \mathrm{im}(R')$. Then $A \oplus A^* + R'CR = (I - R')(A \oplus A^*)(I - R) + R'CR$ is invertible and $R'CR \in \mathcal{K}(\mathbf{Y} \oplus \mathbf{Y}^*, \mathcal{P} \oplus \mathcal{P}^*)$ since

$$\begin{aligned}
\lim_{m \to \infty} \|R'CR(Q_m \oplus Q_m^*)\| &\leq \lim_{m \to \infty} \|R'C\| \, \|R(Q_m \oplus Q_m^*)\| = 0, \\
\lim_{m \to \infty} \|(Q_m \oplus Q_m^*)R'CR\| &\leq \lim_{m \to \infty} \|(Q_m \oplus Q_m^*)R'\| \, \|CR\| = 0.
\end{aligned}$$

In particular, $A \oplus A^* + R'CR$ is bounded below and hence Corollary 3.52 shows $\nu_{\mathrm{ess}}(A \oplus A^*) = \mu(A)$. □

By Proposition 3.5, all Fredholm operators $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ are also $\mathcal{P}$-Fredholm, so we can extend Theorem 3.49 by $\nu_{\mathrm{ess}}(A \oplus A^*)$.

Corollary 3.54. *Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $A + \mathcal{K}(\mathbf{X}, \mathcal{P})$ contain a Fredholm operator. Then*

$$\nu_{\mathrm{ess}}(A \oplus A^*) = \max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1}.$$

We still have this unpleasant condition that A must be Fredholm in order to get the equality of these properties. In the next section we will see that in case of a Hilbert space we can drop this assumption. Moreover, we can extend the result to arbitrary Hilbert spaces $\mathbf{Y}$.

3.4.4 The Hilbert space case

On a Hilbert space $\mathbf{Y}$ we consider a sequence of nested orthogonal projections $\mathcal{P} = (P_n)_{n\in\mathbb{N}}$, i.e. $P_n = P_n^* = P_n^2 = P_nP_{n+1} = P_{n+1}P_n$ for all $n \in \mathbb{N}$. With this condition $\mathcal{P}$ satisfies $(\mathcal{P}1)$ and $(\mathcal{P}2)$. If additionally $(\mathcal{P}3)$ is satisfied, we call $\mathcal{P}$ a Hermitian approximate identity (in short: happi) and the pairing $(\mathbf{Y}, \mathcal{P})$ a happi space. In this more particular case of $\mathbf{Y}$ being a Hilbert space and under these natural assumptions on $\mathcal{P}$ we will find that our results above apply to all operators $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. Note that $\mathbf{X} = \ell^2(\mathbb{Z}^N, X)$ together with the canonical projections $(P_n)_{n\in\mathbb{N}}$ defines a happi space if X is itself a Hilbert space.

Although this is somewhat obvious, we want to mention that we use the usual Hilbert space adjoint (cf. Section 1.1.10) in this section. In particular, A^* will be a bounded linear operator on Y rather than Y^*. This also means that we consider operators of the form $A \oplus B$ on $\mathbf{Y} \oplus \mathbf{Y}$, where $\mathbf{Y} \oplus \mathbf{Y}$ is defined as the Hilbert space of pairs $(x, y) \in \mathbf{Y} \times \mathbf{Y}$ equipped with the inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$ for $x_1, x_2, y_1, y_2 \in \mathbf{Y}$. Hence the norm on $\mathbf{Y} \oplus \mathbf{Y}$ is given by $\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}$ for all $x, y \in \mathbf{Y}$. Using $\mathcal{P} \oplus \mathcal{P} := (P_n \oplus P_n)_{n\in\mathbb{N}}$ as our Hermitian approximate identity, we observe that $(\mathbf{Y} \oplus \mathbf{Y}, \mathcal{P} \oplus \mathcal{P})$ is again a happi space.

We begin with the definition of the Moore-Penrose inverse. It is named after E.H. Moore [67] and R. Penrose [71].

Definition 3.55. Let $A \in \mathcal{L}(\mathbf{Y})$ have closed range. Then A^+ is called the Moore-Penrose inverse of A if the following conditions are satisfied:

- $AA^+A = A$,
- $A^+AA^+ = A^+$,
- $(AA^+)^* = AA^+$,
- $(A^+A)^* = A^+A$.

Note that these conditions imply that AA^+ is an orthogonal projection onto $\operatorname{im}(A)$. Indeed, $AA^+AA^+ = AA^+$ by the first property and the orthogonality follows by the third property. Moreover, for $Ax \in \operatorname{im}(A)$ we have $(AA^+)Ax = Ax$. Similarly, $I - A^+A$ is an orthogonal projection onto $\ker(A)$. Since orthogonal projections onto closed subspaces are unique, the Moore-Penrose inverse is unique, too. Indeed, suppose there is an operator $B \in \mathcal{L}(\mathbf{Y})$ that satisfies the above four conditions as well. Then $AB = AA^+$ and $BA = A^+A$, hence

$$B = BAB = BAA^+ = A^+AA^+ = A^+.$$

This justifies the notation A^+. Moreover, $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ has a Moore-Penrose inverse if and only if $\operatorname{im}(A)$ is closed. Clearly, if A^+ exists, then $\operatorname{im}(AA^+) = \operatorname{im}(A)$ has to be closed because AA^+ is an orthogonal projection. For the converse one has to consider the orthogonal projections onto $\ker(A)$ (closed by continuity) and $\operatorname{im}(A)$ (closed by assumption), then take the inverse of the "invertible part" of A and send the orthogonal complement to 0. For the details we refer to [35, Theorem 2.4].

Lemma 3.56. *(e.g. [35, Theorem 2.4])*
An operator $A \in \mathcal{L}(\mathbf{Y})$ is Moore-Penrose invertible if and only if $\operatorname{im}(A)$ *is closed.*

Here are some more properties of the Moore-Penrose inverse that are easy to check:

- if A is invertible, then $A^{-1} = A^+$,
- $(A^+)^+ = A$,
- $(\lambda A)^+ = \frac{1}{\lambda}A^+$,
- $(A^*)^+ = (A^+)^*$.

There are many more useful properties of the Moore-Penrose inverse not included here. The Moore-Penrose inverse has many important applications in operator theory and numerical analysis, e.g. least squares problems, minimum norm solutions, etc. Here we use the Moore-Penrose inverse for the projections it provides. But first we observe that the Moore-Penrose inverse of an operator $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ is again in $\mathcal{L}(\mathbf{Y}, \mathcal{P})$. For this we use a result from C^*-algebra theory. A C^*-algebra is a Banach algebra $\mathcal{A}$ together with an antilinear involution $*$ such that $(ab)^* = b^*a^*$ and $\|a^*a\| = \|a\|^2$ for all $a, b \in \mathcal{A}$. For an introduction to C^*-algebras we refer to [26] (general) and [35] (numerical analysis oriented).

Lemma 3.57. *If $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ has a Moore-Penrose inverse (i.e. closed range), then $A^+ \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$.*

Proof. Clearly, if $\mathbf{Y}$ is a Hilbert space, then $\mathcal{L}(\mathbf{Y})$ is a C^*-algebra and $\mathcal{L}(\mathbf{Y}, \mathcal{P})$ is a C^*-subalgebra of $\mathcal{L}(\mathbf{Y})$. Hence [35, Corollary 2.18] applies. $\square$

So here is our first result in the Hilbert space case. Observe that in contrast to Theorem 3.41 this theorem holds in much more generality, i.e. for all $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$.

Theorem 3.58. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ on a happi space $(\mathbf{Y}, \mathcal{P})$. Then*

$$\mu(A) = \left\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-1}.$$

Moreover, if $\mu(A) > 0$, then $\tilde{\mu}(A) = \tilde{\mu}(A^*)$.

Proof. By Lemma 3.42, it suffices to show that $\mu(A) > 0$ implies that A is $\mathcal{P}$-Fredholm. So assume that $\mu(A) > 0$. Let $m \in \mathbb{N}$ be large enough such that $\nu(A|_{\operatorname{im} Q_m}) > 0$. This implies that $\operatorname{im}(AQ_m)$ is closed by Lemma 2.8. From Lemma 3.56 and Lemma 3.57 we get that AQ_m is Moore-Penrose invertible with $(AQ_m)^+ \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. This implies that $I - (AQ_m)^+(AQ_m)$ is the orthogonal projection onto $\ker(AQ_m) = \operatorname{im}(P_m)$ because $\nu(A|_{\operatorname{im} Q_m}) > 0$. But the orthogonal projection onto $\operatorname{im}(P_m)$ is of course P_m itself, hence $I - (AQ_m)^+(AQ_m) = P_m \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$. Similarly we get $I - (A^*Q_m)^+(A^*Q_m) = P_m \in \mathcal{K}(\mathbf{Y}, \mathcal{P})$ for all sufficiently large $m \in \mathbb{N}$. Moreover, we have

$$(A^*Q_m)^+(A^*Q_m) = ((A^*Q_m)^+(A^*Q_m))^* = (Q_mA)(Q_mA)^+.$$

Summarizing, we have

$$(AQ_m)^+A = (AQ_m)^+AQ_m + (AQ_m)^+AP_m = I - P_m + (AQ_m)^+AP_m \in I + \mathcal{K}(\mathbf{Y}, \mathcal{P})$$

and

$$A(Q_mA)^+ = Q_mA(Q_mA)^+ + P_mA(Q_mA)^+ = I - P_m + P_mA(Q_mA)^+ \in I + \mathcal{K}(\mathbf{Y}, \mathcal{P}).$$

This implies that A is $\mathcal{P}$-Fredholm by a standard argument from group theory: If an element is one-sided invertible from both sides, then it is invertible and the one-sided inverses coincide, in our case $(AQ_m)^+ + \mathcal{K}(\mathbf{Y},\mathcal{P}) = (Q_mA)^+ + \mathcal{K}(\mathbf{Y},\mathcal{P})$. □

In addition to $\mathcal{P}$-Fredholmness we always had to assume that $A + \mathcal{K}(\mathbf{Y},\mathcal{P})$ contains a "good" operator (cf. Proposition 3.45, Theorem 3.49 et seq.). Notice that in the case $\mathbf{Y} = \ell^p(\mathbb{Z})$, $1 < p < \infty$ such a "good" operator always exists due to Theorem 2.12 and Proposition 2.18. In the case of a happi space $(\mathbf{Y},\mathcal{P})$ we can retain this property.

Proposition 3.59. *Let $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$ be $\mathcal{P}$-Fredholm on a happi space $(\mathbf{Y},\mathcal{P})$. Then there is a $K \in \mathcal{K}(\mathbf{Y},\mathcal{P})$ such that $A + K$ has a one-sided inverse in $\mathcal{L}(\mathbf{Y},\mathcal{P})$.*

Proof. Since A is $\mathcal{P}$-Fredholm, we get $\mu(A) > 0$ by Theorem 3.58 and thus $\operatorname{im}(AQ_m)$ is closed for sufficiently large $m \in \mathbb{N}$ by Lemma 2.8. In order to simplify notations, we may assume that $\operatorname{im}(A)$ is closed. Let $A_0 : \ker(A)^\perp \to \operatorname{im}(A)$ be defined by $A_0x = Ax$ for all $x \in \ker(A)^\perp$. Then A_0 is invertible by the first isomorphism theorem for Banach spaces. Now choose orthonormal bases $\{\beta_i\}_{i\in I}$ and $\{\gamma_j\}_{j\in J}$ of $\ker(A)$ and $\operatorname{im}(A)^\perp$, respectively. Depending on the cardinalities $|I|$ and $|J|$ there is an injection $\iota : I \to J$ or $\iota : J \to I$ (if $|I| = |J|$, there is even a bijection). Let us assume that $|I| \le |J|$. Then ι induces an isometry $\Phi : \ker(A) \to \operatorname{im}(A)^\perp$ by $\Phi(\beta_i) = \gamma_{\iota(i)}$ for all $i \in I$. Let $R_1 := I - A^+A$ and $R_2 := I - AA^+$ be the orthogonal projections onto $\ker(A)$ and $\operatorname{im}(A)^\perp$, respectively. Moreover, let B be a $\mathcal{P}$-regularizer of A. Then

$$BAA^+A \in A^+A + \mathcal{K}(\mathbf{Y},\mathcal{P}) \quad \text{and} \quad BAA^+A = BA \in I + \mathcal{K}(\mathbf{Y},\mathcal{P}),$$

hence $A^+A \in I + \mathcal{K}(\mathbf{Y},\mathcal{P})$ and therefore $R_1 \in \mathcal{K}(\mathbf{Y},\mathcal{P})$. Similarly, we get $R_2 \in \mathcal{K}(\mathbf{Y},\mathcal{P})$. Moreover, $\tilde{A} := A + R_2\Phi R_1 = A_0 \oplus \Phi$ is left invertible by construction. In particular, $\tilde{A}$ is injective and $\operatorname{im}(\tilde{A})$ is closed. This implies that $\tilde{A}$ is Moore-Penrose invertible and $\tilde{A}^+$ is a left inverse of $\tilde{A}$. It remains to show that $\tilde{A} \in A + \mathcal{K}(\mathbf{Y},\mathcal{P})$. But this is clear since

$$\|R_2\Phi R_1Q_m\| \le \|R_1Q_m\| \to 0 \quad \text{and} \quad \|Q_mR_2\Phi R_1\| \le \|Q_mR_2\| \to 0$$

as $m \to \infty$. This of course also implies $\tilde{A}^+ \in \mathcal{L}(\mathbf{Y},\mathcal{P})$ by Theorem 3.57. □

Combining Corollary 3.47, Theorem 3.58 and Proposition 3.59, this immediately yields

Corollary 3.60. *Let $A \in \mathcal{L}(\mathbf{Y},\mathcal{P})$ be $\mathcal{P}$-Fredholm on a happi space $(\mathbf{Y},\mathcal{P})$. Then*

$$\max\{\nu_{\text{ess}}(A), \nu_{\text{ess}}(A^*)\} = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\right\|^{-1}.$$

More precisely:

- *If $A + \mathcal{K}(\mathbf{Y},\mathcal{P})$ contains a left invertible operator, then*

$$\nu_{\text{ess}}(A) = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\right\|^{-1}.$$

  Otherwise, $\nu_{\text{ess}}(A) = 0$.

- *If $A + \mathcal{K}(\mathbf{Y},\mathcal{P})$ contains a right invertible operator, then*

$$\nu_{\text{ess}}(A^*) = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{Y},\mathcal{P}))^{-1}\right\|^{-1}.$$

  Otherwise, $\nu_{\text{ess}}(A^*) = 0$.

Similarly as in Corollary 3.54, we can extend this by considering $A \oplus A^* \in \mathcal{L}(Y \oplus Y, \mathcal{P} \oplus \mathcal{P})$.

Corollary 3.61. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ on a happi space $(\mathbf{Y}, \mathcal{P})$. Then*

$$\nu_{\mathrm{ess}}(A \oplus A^*) = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-1}.$$

Proof. If A is $\mathcal{P}$-Fredholm, we can apply Corollary 3.60 to $A \oplus A^*$. Using the observations $\tilde{\mu}(A \oplus A^*) = \tilde{\mu}(A^* \oplus A) = \mu(A \oplus A^*) = \mu(A)$ and $\nu_{\mathrm{ess}}(A \oplus A^*) = \nu_{\mathrm{ess}}(A^* \oplus A)$, Corollary 3.60 implies the assertion in this case.

If A is not $\mathcal{P}$-Fredholm, then the two rightmost terms are equal to zero by Theorem 3.58. So assume $\nu_{\mathrm{ess}}(A \oplus A^*) > 0$. Combining Corollary 3.47 and Theorem 3.58, we get that $A \oplus A^*$ is $\mathcal{P}$-Fredholm (w.r.t. $\mathcal{P} \oplus \mathcal{P}$ in $\mathbf{Y} \oplus \mathbf{Y}$). Restricting a $\mathcal{P}$-regularizer for $A \oplus A^*$ to the first component yields a $\mathcal{P}$-regularizer for A and thus A is $\mathcal{P}$-Fredholm, a contradiction. □

So in case of a happi space we got rid of all unpleasant conditions. Moreover, we were able to generalize our theorems to operators $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$. In fact, there is another way of doing the exact same thing. Namely, and in Hilbert space only, one can also consider products AA^* and A^*A instead of $A \oplus A^*$.

Corollary 3.62. *Let $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ on a happi space $(\mathbf{Y}, \mathcal{P})$. Then*

$$\min\left\{\sqrt{\nu_{\mathrm{ess}}(AA^*)}, \sqrt{\nu_{\mathrm{ess}}(A^*A)}\right\} = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-1}.$$

Proof. If A is $\mathcal{P}$-Fredholm, then

$$\left\|(AA^* + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-1} = \left\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-2} = \left\|(A^*A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-1}$$

since $\mathcal{L}(\mathbf{Y}, \mathcal{P})/\mathcal{K}(\mathbf{Y}, \mathcal{P})$ is a C^*-algebra. The assertion now follows from Corollary 3.60 applied to the self-adjoint operators AA^* and A^*A.

If A is not $\mathcal{P}$-Fredholm, then the two rightmost terms are equal to zero by Theorem 3.58. So assume that both $\nu_{\mathrm{ess}}(AA^*)$ and $\nu_{\mathrm{ess}}(A^*A)$ are greater than zero. Combining Proposition 3.45 and Theorem 3.58, we get that both AA^* and A^*A are $\mathcal{P}$-Fredholm. A simple algebra argument[2] shows that then both A and A^* must be $\mathcal{P}$-Fredholm, too. This is a contradiction. □

3.4.5 The finite-dimensional case

In this section we come back to the case $\mathbf{X} = \ell^p(\mathbb{Z}^N, X)$ with $N \in \mathbb{N}$, $p \in \{0\} \cup [1, \infty]$ and X a finite-dimensional Banach space. Remember that we considered the special case $N = 1$, $p \in (1, \infty)$, $X = \mathbb{C}$ already in Section 2.2. We will now extend the results obtained there by applying the notions of the previous sections to the finite-dimensional case. Due to the equivalence of Fredholmness and $\mathcal{P}$-Fredholmness, significant simplifications can be obtained here.

Proposition 3.63. *Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $\dim X < \infty$. Then A is Fredholm if and only if A is $\mathcal{P}$-Fredholm.*

This statement is of course obvious if $\mathcal{K}(\mathbf{X}, \mathcal{P}) = \mathcal{K}(\mathbf{X})$ like in Section 2.2. The interesting part is the case where $\mathcal{K}(\mathbf{X}, \mathcal{P}) \neq \mathcal{K}(\mathbf{X})$, i.e. $p \in \{1, \infty\}$.

[2]The following holds in every unital algebra: If ab and ba are invertible, then also both a and b are invertible with inverses $b(ab)^{-1} = (ba)^{-1}b$ and $a(ba)^{-1} = (ab)^{-1}a$, respectively.

Proof. By Proposition 3.5, Fredholmness implies $\mathcal{P}$-Fredholmness. Conversely, if $\dim X < \infty$, then $\mathcal{K}(\mathbf{X}, \mathcal{P}) \subset \mathcal{K}(\mathbf{X})$ by Proposition 3.3. Hence $\mathcal{P}$-Fredholmness implies Fredholmness by Theorem 2.4. □

We even have that the norms of the cosets $A + \mathcal{K}(\mathbf{X})$ and $A + \mathcal{K}(\mathbf{X}, \mathcal{P})$ coincide. This is really surprising because $\mathcal{K}(\mathbf{X})$ is strictly larger than $\mathcal{K}(\mathbf{X}, \mathcal{P})$ in the case $p \in \{1, \infty\}$.

Proposition 3.64. *Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $\dim X < \infty$. Then*

$$\|A + \mathcal{K}(\mathbf{X})\| = \|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = \|A^* + \mathcal{K}(\mathbf{X}^*, \mathcal{P}^*)\| = \|A^* + \mathcal{K}(\mathbf{X}^*)\|,$$
$$\left\|(A + \mathcal{K}(\mathbf{X}))^{-1}\right\|^{-1} = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \left\|(A^* + \mathcal{K}(\mathbf{X}^*, \mathcal{P}^*))^{-1}\right\|^{-1} = \left\|(A^* + \mathcal{K}(\mathbf{X}^*))^{-1}\right\|^{-1}.$$

Proof. By Proposition 3.63, A is Fredholm if and only if it is $\mathcal{P}$-Fredholm. Furthermore, A is Fredholm if and only if A^* is Fredholm (see Section 2.1). Also quite clearly, A^* is $\mathcal{P}^*$-Fredholm if A is $\mathcal{P}$-Fredholm. So it remains to show that A^* is Fredholm if it is $\mathcal{P}^*$-Fredholm to close the loop. For this it suffices to show that every $K \in \mathcal{K}(X^*, \mathcal{P}^*)$ is compact. By Theorem 2.19, $\mathcal{P}^*$ is a sequence of compact operators because $\mathcal{P}$ is. Hence $(KP_n^*)_{n \in \mathbb{N}}$ is a sequence of compact operators converging to K in norm. Thus K has to be compact as well. This implies that all terms in the second line are simultaneously zero or non-zero and hence the equalities follow directly from the first line. So let us prove the first line.

Proposition 3.22 implies $\|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\| = \|A^* + \mathcal{K}(\mathbf{X}^*, \mathcal{P}^*)\|$. Also, if $\mathcal{K}(\mathbf{X}, \mathcal{P}) = \mathcal{K}(\mathbf{X})$ and $\mathcal{K}(\mathbf{X}^*, \mathcal{P}^*) = \mathcal{K}(\mathbf{X}^*)$, then the first line is trivial. This is exactly the case for $p \in (1, \infty)$ (see Proposition 3.3 and observe that $\ell^p(\mathbb{Z}^N, X)^* \cong \ell^q(\mathbb{Z}^N, X^*)$ if $\frac{1}{p} + \frac{1}{q} = 1$, cf. Section 1.1.4). Moreover, $\ell^0(\mathbb{Z}^N, X)^* \cong \ell^1(\mathbb{Z}^N, X^*)$ and $\ell^1(\mathbb{Z}^N, X)^* \cong \ell^\infty(\mathbb{Z}^N, X^*)$. So it remains to show $\|A + \mathcal{K}(\mathbf{X})\| = \|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\|$ for $p \in \{1, \infty\}$ and $\|A + \mathcal{K}(\mathbf{X})\| = \|A^* + \mathcal{K}(\mathbf{X}^*)\|$ for $p = \infty$.

$p = 1$: Let $\varepsilon > 0$ and choose $K \in \mathcal{K}(\mathbf{X})$ such that $\|A + K\| \le \|A + \mathcal{K}(\mathbf{X})\| + \varepsilon$ and $m_0 \in \mathbb{N}$ such that $\|Q_m K\| \le \varepsilon$ for all $m \ge m_0$, which is possible because Q_m converges strongly to 0 as $m \to \infty$ and K is compact (see Theorem 2.20). Now we can proceed as in Proposition 3.22:

$$\|Q_m A\| = \|A - P_m A\| \ge \|A + \mathcal{K}(\mathbf{X})\| \ge \|A + K\| - \varepsilon \ge \|Q_m(A + K)\| - \varepsilon \ge \|Q_m A\| - 2\varepsilon$$

for all $m \ge m_0$ and therefore $\|A + \mathcal{K}(\mathbf{X})\| = \lim\limits_{m \to \infty} \|Q_m A\| = \|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\|$, again by Proposition 3.22.

$p = \infty$: Let $\varepsilon > 0$ and choose $K \in \mathcal{K}(\mathbf{X})$ such that $\|A + K\| \le \|A + \mathcal{K}(\mathbf{X})\| + \varepsilon$ and $m_0 \in \mathbb{N}$ such that $\|KQ_m|_{\mathbf{X}_0}\| \le \varepsilon$ for all $m \ge m_0$, which is possible because $(Q_m|_{\mathbf{X}_0})^* = Q_m|_{\ell^1(\mathbb{Z}^N, X^*)}$ converges strongly to 0 as $m \to \infty$ and $(K|_{\mathbf{X}_0})^*$ is compact (observe $\ell^\infty(\mathbb{Z}^N, X)_0 = \ell^0(\mathbb{Z}^N, X)$, cf. Section 1.1.4). Now we can proceed as before, using Lemma 3.13 i):

$$\begin{aligned}\|AQ_m\| &= \|A - AP_m\| \ge \|A + \mathcal{K}(\mathbf{X})\| \ge \|A + K\| - \varepsilon \ge \|(A + K)Q_m\| - \varepsilon \\ &\ge \|(A + K)Q_m|_{\mathbf{X}_0}\| - \varepsilon \ge \|AQ_m|_{\mathbf{X}_0}\| - \|KQ_m|_{\mathbf{X}_0}\| - \varepsilon \ge \|AQ_m\| - 2\varepsilon\end{aligned}$$

for all $m \ge m_0$ and therefore $\|A + \mathcal{K}(\mathbf{X})\| = \lim\limits_{m \to \infty} \|AQ_m\| = \|A + \mathcal{K}(\mathbf{X}, \mathcal{P})\|$. To prove the last assertion, observe that $\mathbf{X} = \ell^\infty(\mathbb{Z}^N, X)$ is the dual of $\mathbf{X}^\triangleleft := \ell^1(\mathbb{Z}^N, X^*)$ since $\dim X < \infty$ and hence $X^{**} = X$. Let $\iota_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{X}^{**}$ be the canonical embedding and consider the adjoint of the canonical embedding $\iota_{\mathbf{X}^\triangleleft} \colon \mathbf{X}^\triangleleft \to (\mathbf{X}^\triangleleft)^{**} = \mathbf{X}^*$. It has norm one and $(\iota_{\mathbf{X}^\triangleleft})^* A^{**} \iota_{\mathbf{X}} = A$ holds (cf. Section 1.1.10). It follows

$$\|A^{**} + K\| \ge \|(\iota_{\mathbf{X}^\triangleleft})^*(A^{**} + K)\iota_{\mathbf{X}}\| \ge \|A + (\iota_{\mathbf{X}^\triangleleft})^* K \iota_{\mathbf{X}}\| \ge \|A + \mathcal{K}(\mathbf{X})\|$$

for every $K \in \mathcal{K}(\mathbf{X}^{**})$ (see also Section 1.1.4). Taking the infimum over all $K \in \mathcal{K}(\mathbf{X}^{**})$, we get $\|A^{**} + \mathcal{K}(\mathbf{X}^{**})\| \geq \|A + \mathcal{K}(\mathbf{X})\|$. We conclude with an application of Theorem 2.19 (and the fact that $\|A+K\| = \|A^* + K^*\| = \|A^{**} + K^{**}\|$ for all $K \in \mathcal{K}(\mathbf{X})$):

$$\|A + \mathcal{K}(\mathbf{X})\| \geq \|A^* + \mathcal{K}(\mathbf{X}^*)\| \geq \|A^{**} + \mathcal{K}(\mathbf{X}^{**})\| \geq \|A + \mathcal{K}(\mathbf{X})\|. \qquad \square$$

It is natural to ask whether the same statement is true for the essential lower norm. For this (and to distinguish it from the $\mathcal{P}$-essential lower norm) we define

$$\nu_{\mathrm{ess,c}}(A) := \sup\{\nu(A+K) : K \in \mathcal{K}(\mathbf{Y})\}.$$

However,

Theorem 3.65. *Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $\dim X < \infty$. Then*

$$\nu_{\mathrm{ess,c}}(A \oplus A^*) = \nu_{\mathrm{ess}}(A \oplus A^*) = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \left\|(A + \mathcal{K}(\mathbf{X}))^{-1}\right\|^{-1}.$$

Moreover, if A is Fredholm of index zero, then

$$\nu_{\mathrm{ess,c}}(A) = \nu_{\mathrm{ess,c}}(A^*) = \nu_{\mathrm{ess}}(A) = \nu_{\mathrm{ess}}(A^*) = \mu(A) > 0.$$

Conversely, if $\nu_{\mathrm{ess,c}}(A) > 0$ and $\nu_{\mathrm{ess,c}}(A^*) > 0$, then A is Fredholm of index zero.

Proof. Let B be a Fredholm operator of index zero on a Banach space $\mathbf{Y}$. By Theorem 2.12(i), there exists an operator $K \in \mathcal{K}(\mathbf{Y})$ such that $B+K$ is invertible. W.l.o.g. we can assume that B is invertible itself. Moreover, we have $\operatorname{ind}(B+K) = 0$ for every $K \in \mathcal{K}(\mathbf{X})$ by Theorem 2.2(ii). So if $B+K$ is bounded below, it is already invertible. It therefore suffices to consider operators $B+K$ which are invertible in the definition of $\nu_{\mathrm{ess,c}}(B)$. It follows

$$\begin{aligned}\nu_{\mathrm{ess,c}}(B) &= \sup\{\nu(B+K) : K \in \mathcal{K}(\mathbf{Y}), B+K \text{ invertible}\}\\ &= \sup\left\{\left\|(B+K)^{-1}\right\|^{-1} : K \in \mathcal{K}(\mathbf{Y}), B+K \text{ invertible}\right\}\\ &= \left(\inf\left\{\left\|(B+K)^{-1}\right\| : K \in \mathcal{K}(\mathbf{Y}), B+K \text{ invertible}\right\}\right)^{-1}\end{aligned}$$

by Corollary 2.9. Now the inverse of every invertible $B+K \in B+\mathcal{K}(\mathbf{Y})$ can be written as $B^{-1}+L$ for some $L \in \mathcal{K}(\mathbf{Y})$ and vice versa. Indeed,

$$B^{-1} - (B+K)^{-1} = B^{-1}(B+K-B)(B+K)^{-1} = B^{-1}K(B+K)^{-1} \in \mathcal{K}(\mathbf{Y})$$

and conversely

$$(B^{-1}+L)^{-1} - B = (B^{-1}+L)^{-1} - (B^{-1})^{-1} = -(B^{-1}+L)^{-1}(B^{-1}+L-B^{-1})B = -(B^{-1}+L)^{-1}LB.$$

By the above formulas, we also see that $B+K$ is invertible if and only if $B^{-1}+L$ is. Hence,

$$\nu_{\mathrm{ess,c}}(B) = \left(\inf\left\{\left\|B^{-1}+L\right\| : L \in \mathcal{K}(\mathbf{Y}), B^{-1}+L \text{ invertible}\right\}\right)^{-1} = \left\|(B+\mathcal{K}(\mathbf{Y}))^{-1}\right\|^{-1} \qquad (8)$$

follows.

If A is Fredholm, then $A \oplus A^*$ is clearly Fredholm of index zero. Hence the above applies to $A \oplus A^* \in \mathcal{L}(\mathbf{X} \oplus \mathbf{X}^*, \mathcal{P} \oplus \mathcal{P}^*)$ and we get

$$\nu_{\mathrm{ess,c}}(A \oplus A^*) = \left\|(A \oplus A^* + \mathcal{K}(\mathbf{X} \oplus \mathbf{X}^*))^{-1}\right\|^{-1}.$$

Moreover,

$$\left\|(A \oplus A^* + \mathcal{K}(\mathbf{X} \oplus \mathbf{X}^*))^{-1}\right\| = \max\left\{\left\|(A + \mathcal{K}(\mathbf{X}))^{-1}\right\|, \left\|(A^* + \mathcal{K}(\mathbf{X}^*))^{-1}\right\|\right\}$$

by exactly the same arguments as in (the proof of) Proposition 3.51 applied to a regularizer $B \oplus B^*$ of $A \oplus A^*$. Hence,

$$\nu_{\mathrm{ess,c}}(A \oplus A^*) = \left\|(A \oplus A^* + \mathcal{K}(\mathbf{X} \oplus \mathbf{X}^*))^{-1}\right\|^{-1} = \left\|(A + \mathcal{K}(\mathbf{X}))^{-1}\right\|^{-1}$$

by Proposition 3.64 again. Using Theorem 3.53, Lemma 3.42 and Proposition 3.64 along with Proposition 3.63, we get

$$\nu_{\mathrm{ess}}(A \oplus A^*) = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \left\|(A + \mathcal{K}(\mathbf{X}))^{-1}\right\|^{-1}.$$

This proves the first part of the theorem if A is Fredholm.

If A is not Fredholm, then $\mathrm{im}(A)$ is not closed, $\ker(A)$ is infinite-dimensional or $\ker(A^*)$ is infinite-dimensional. In particular, $\mathrm{im}(A \oplus A^*)$ is not closed or $\ker(A \oplus A^*)$ is infinite-dimensional. In both cases $A \oplus A^* + K$ with $K \in \mathcal{K}(\mathbf{X} \oplus \mathbf{X}^*)$ can not be bounded below because of Theorem 2.3 and Lemma 2.8. Hence, $\nu_{\mathrm{ess,c}}(A \oplus A^*) = 0$ follows. Now assume that $\mu(A) > 0$. Then there exists an $m \in \mathbb{N}$ such that $\nu(A|_{\mathrm{im}\, Q_m}) > 0$ and $\nu(A^*|_{\mathrm{im}\, Q_m^*}) > 0$. In particular, $A|_{\mathrm{im}\, Q_m}$ is injective and has closed range by Lemma 2.8. This further implies that AQ_m has closed range as well and $\ker(AQ_m) = \mathrm{im}(P_m)$. Thus by Theorem 2.3, $A = AQ_m + AP_m \in AQ_m + \mathcal{K}(\mathbf{X})$ has closed range and finite-dimensional kernel, too. The same can of course been said about A^*. But this implies that A is Fredholm, a contradiction. Hence, $\mu(A) = 0$. Corollary 3.52 then implies $\nu_{\mathrm{ess}}(A \oplus A^*) = \mu(A) = 0$. This finishes the proof of the first part.

If A is Fredholm of index zero, then A^* is also Fredholm of index zero and we have

$$\nu_{\mathrm{ess,c}}(A) = \nu_{\mathrm{ess,c}}(A^*) = \left\|(A + \mathcal{K}(\mathbf{X}))^{-1}\right\|^{-1} = \mu(A) > 0$$

by (8), Proposition 3.64 and what we have just proven above. Combining this with Corollary 3.48, we get

$$\nu_{\mathrm{ess,c}}(A) = \nu_{\mathrm{ess,c}}(A^*) = \mu(A) = \nu_{\mathrm{ess}}(A) = \nu_{\mathrm{ess}}(A^*).$$

This proves the second claim.

If $\nu_{\mathrm{ess,c}}(A) > 0$, then there exists a $K \in \mathcal{K}(\mathbf{X})$ such that $\nu(A + K) > 0$. This implies that $A + K$ is injective and has closed range by Lemma 2.8. Theorem 2.3 now ensures that A has finite-dimensional kernel and closed range. If additionally A^* has finite-dimensional kernel, then A is Fredholm with $\mathrm{ind}(A) \leq 0$ because obviously $\mathrm{ind}(A + K) \leq 0$, see Theorem 2.2(*ii*).

Similarly, if $\nu_{\mathrm{ess,c}}(A^*) > 0$, then A^* has finite-dimensional kernel and closed range. In particular, A is Fredholm if both $\nu_{\mathrm{ess,c}}(A) > 0$ and $\nu_{\mathrm{ess,c}}(A^*) > 0$ hold. Moreover, we have both $\mathrm{ind}(A) \leq 0$ and $\mathrm{ind}(A^*) \leq 0$ and thus the index of A has to be zero. This proves the last part. $\square$

In the particular case $N = 1$, we also have the following Proposition for band-dominated operators:

Proposition 3.66. *Let* $\dim X < \infty$ *and* $A \in \mathrm{BDO}(\ell^p(\mathbb{Z}, X))$. *Then*

$$\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \mu(A) = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \left\|(A + \mathcal{K}(\mathbf{X}))^{-1}\right\|^{-1}.$$

Proof. The equalities

$$\mu(A) = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \left\|(A + \mathcal{K}(\mathbf{X}))^{-1}\right\|^{-1}$$

hold by Theorem 3.65. If A is Fredholm, then Theorem 3.49(*iii*) implies the remaining equality. If A is not Fredholm, it is not even semi-Fredholm by [85, Theorem 4.3], i.e. $\mathrm{im}(A)$ is not closed or both A and A^* have an infinite-dimensional kernel. This also remains true for all $B \in A + \mathcal{K}(\mathbf{X})$ by Theorem 2.3. It follows $\nu_{\mathrm{ess}}(A) \le \nu_{\mathrm{ess,c}}(A) = 0$ and $\nu_{\mathrm{ess}}(A^*) \le \nu_{\mathrm{ess,c}}(A^*) = 0$ by Lemma 2.8, hence $\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = 0$. □

Proposition 3.66 has a remarkable corollary:

Corollary 3.67. *Let* $\dim X < \infty$ *and* $A \in \mathrm{BDO}(\ell^p(\mathbb{Z}, X))$. *Then*

$$\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \min_{A_h \in \sigma^{\mathrm{op}}(A)} \min\{\nu(A_h), \nu(A_h^*)\}.$$

Proof. Proposition 3.66 implies

$$\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1}.$$

By Proposition 2.27 (at least for $p \in (1, \infty)$, but the proof is easily carried over to the remaining cases), every $A \in \mathrm{BDO}(\ell^p(\mathbb{Z}, X))$ is rich. Thus if A is $\mathcal{P}$-Fredholm, Corollary 3.29, Theorem 3.20 and Corollary 2.9 imply

$$\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\} = \min_{A_h \in \sigma^{\mathrm{op}}(A)} \left\|A_h^{-1}\right\|^{-1} = \min_{A_h \in \sigma^{\mathrm{op}}(A)} \nu(A_h) = \min_{A_h \in \sigma^{\mathrm{op}}(A)} \nu(A_h^*).$$

If A is not $\mathcal{P}$-Fredholm, then at least one $A_h \in \sigma^{\mathrm{op}}(A)$ is not invertible by Theorem 3.20. Corollary 2.9 then implies $\nu(A_h) = 0$ or $\nu(A_h^*) = 0$. Consequently,

$$\min_{A_h \in \sigma^{\mathrm{op}}(A)} \min\{\nu(A_h), \nu(A_h^*)\} = 0 = \left\|(A + \mathcal{K}(\mathbf{X}, \mathcal{P}))^{-1}\right\|^{-1} = \max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\}.$$

□

So far we mentioned quantities like $\nu_{\mathrm{ess,c}}$, μ or $\|\cdot + \mathcal{K}(\mathbf{X})\|$ that measure how "good" or how "bad" (w.r.t. the Fredholm property) an operator is and discussed the connections between them. There are even more of these quantities one can consider and we briefly want to mention two of them. For an operator $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $m \in \mathbb{N}$ we define the lower Bernstein numbers B_m by

$$B_m(A) := \sup\{\nu(A|_V) : V \subset \mathbf{X} \text{ is a closed subspace such that } \dim(\mathbf{X}/V) < m\}.$$

Moreover, we introduce the limit

$$B(A) := \lim_{m \to \infty} \min\{B_m(A), B_m(A^*)\},$$

which exists by monotonicity. Clearly, $\nu(A|_{\operatorname{im} Q_m}) \leq B_{\dim(\mathbf{X}/\operatorname{im} Q_m)+1}$ holds and hence $\mu(A) \leq B(A)$. So $B(A)$ looks more general in some sense because we consider more subspaces than in $\mu(A)$. However, we will show that actually $B(A) = \mu(A)$ holds. In other words, if we want to compute $B(A)$, it suffices to consider the subspaces $\operatorname{im} Q_m$, $m \in \mathbb{N}$. Another way of measuring how "good" an operator is, is to measure how good it can be approximated by certain operators. For this we define the approximation numbers

$$s_m^r(A) := \inf\left\{\|A - F\| : F \in \mathcal{L}(\mathbf{X}), \dim(\ker(F)) \geq m\right\},$$
$$s_m^l(A) := \inf\left\{\|A - F\| : F \in \mathcal{L}(\mathbf{X}), \dim(\mathbf{X}/\operatorname{im}(F)) \geq m\right\}$$

and the limit

$$S(A) := \lim_{m\to\infty} \min\left\{s_m^r(A), s_m^l(A)\right\}.$$

By [86, Corollary 2.11], we have $S(A) = 0$ if and only if A is Fredholm. The next theorem shows that $S(A) = \mu(A)$ holds more generally.

Theorem 3.68. *Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ and $\dim X < \infty$. Then*

$$\mu(A) = B(A) = S(A).$$

Proof. If A is not Fredholm, then $\mu(A) = 0$ by Theorem 3.65. Moreover, $S(A) = 0$ by [86, Corollary 2.11] as mentioned above. Assume that $B(A) > 0$. Then there exists a closed subspace $V \subset \mathbf{X}$ of finite co-dimension such that $\nu(A|_V) > 0$. Since V is a closed subspace of finite co-dimension, there exists a projection $P_V \in I + \mathcal{K}(\mathbf{X})$ onto V. Moreover, $A|_V$ is injective and has closed range by Lemma 2.8. This further implies that AP_V has closed range as well and $\ker(AP_V) = \operatorname{im}(I - P_V)$. Thus by Theorem 2.3, $A = AP_V + A(I - P_V) \in AP_V + \mathcal{K}(\mathbf{X})$ has closed range and finite-dimensional kernel, too. Similarly one finds a closed subspace $W \subset X^*$ for A^* and concludes that A^* has closed range and finite-dimensional kernel. Hence A is Fredholm, a contradiction. So $0 = \mu(A) = S(A) = B(A)$ holds in this case.

Now let A be Fredholm and let $\hat{\mathcal{P}} := (F_l)_{l\in\mathbb{N}}$ be an equivalent approximate identity provided by Proposition 3.7. The inequality $\mu(A) \leq B(A)$ is clear as mentioned above. Furthermore, $B(A) \leq S(A)$ holds by [86, Proposition 2.9]. Assume that there exist constants d, e such that $\tilde{\mu}(A) < d < e < S(A)$. In particular, for every $n \in \mathbb{N}$ there exists a $y_n \in \operatorname{im} Q_n$ such that $\|Ay_n\| < d\,\|y_n\|$. This also implies $\|AF_l y_n\| < d\|F_l y_n\|$ for sufficiently large l since $\|[A, F_l]\|$ tends to zero and $\|F_l y_n\|$ tends to $\|y_n\|$ as $l \to \infty$. Fix such an l (which depends on n) such that also $F_l P_n = P_n F_l = P_n$ holds and define $z_n := \|F_l y_n\|^{-1} F_l y_n$ for every $n \in \mathbb{N}$. Then $z_n \in \operatorname{im} Q_n$ is still true since $F_l y_n = F_l Q_n y_n = Q_n F_l y_n$. Next, we fix $m \in \mathbb{N}$ and choose numbers $n_1, \ldots, n_m$ as follows. Set $n_1 := 1$. Given n_i we choose l_i such that $P_{l_i} F_{n_i} = F_{n_i} P_{l_i} = F_{n_i}$. Then z_{n_i} is also in the range of P_{l_i}. Furthermore, we choose $k_i > l_i$ such that $\|Q_{k_i} A P_{l_i}\| < 2^{-i-1}(e - d)$ and $n_{i+1} > k_i$ such that $\left\|P_{k_i} A Q_{n_{i+1}}\right\| < 2^{-i-2}(e - d)$. For every $i \in \{1, \ldots, m\}$ let R_i be a projection of norm 1 onto $\operatorname{span}\{z_{n_i}\}$. Thus we have $R_i = P_{l_i} R_i = Q_{n_i} R_i$ for every $i \in \{1, \ldots, m\}$. Let $S_m := \sum\limits_{i=1}^{m} R_i$.

Then S_m is a projection of rank m and norm 1. Moreover,

$$\begin{aligned}
\|AS_m x\| &= \left\| \sum_{i=1}^m AR_i x \right\| \\
&= \left\| \sum_{i=1}^m P_{k_i} Q_{k_{i-1}} AR_i x + \sum_{i=1}^m P_{k_{i-1}} AR_i x + \sum_{i=1}^m Q_{k_i} AR_i x \right\| \\
&\leq \left\| \sum_{i=1}^m P_{k_i} Q_{k_{i-1}} AR_i x \right\| + \left\| \sum_{i=1}^m P_{k_{i-1}} AQ_{n_i} R_i x \right\| + \left\| \sum_{i=1}^m Q_{k_i} AP_{l_i} R_i x \right\| \\
&\leq \left\| \sum_{i=1}^m P_{k_i} Q_{k_{i-1}} AR_i x \right\| + \sum_{i=1}^m 2^{-i-1}(e-d)\,\|x\| + \sum_{i=1}^m 2^{-i-1}(e-d)\,\|x\| \\
&\leq \left\| \sum_{i=1}^m P_{k_i} Q_{k_{i-1}} AR_i x \right\| + (e-d)\,\|x\|
\end{aligned}$$

for all $x \in \mathbf{X}$. In the case $p \in [1, \infty)$ we can estimate the first term as follows:

$$\left\| \sum_{i=1}^m P_{k_i} Q_{k_{i-1}} AR_i x \right\|_p^p = \sum_{i=1}^m \left\| P_{k_i} Q_{k_{i-1}} AR_i x \right\|_p^p \leq d^p \sum_{i=1}^m \|R_i x\|_p^p = d^p \|S_m x\|_p^p \leq d^p \|x\|_p^p,$$

where we used $\|AR_i x\| \leq d\,\|R_i x\|$ and $R_i x \in \operatorname{span}\{z_{n_i}\}$. Similarly,

$$\left\| \sum_{i=1}^m P_{k_i} Q_{k_{i-1}} AR_i x \right\|_\infty = \max_{i \in \{1,\ldots,m\}} \left\| P_{k_i} Q_{k_{i-1}} AR_i x \right\|_\infty \leq d \max_{i \in \{1,\ldots,m\}} \|R_i x\|_\infty \leq d\,\|x\|_\infty.$$

Thus $\|AS_m x\| \leq e\,\|x\|$ for all $x \in \mathbf{X}$ and hence

$$s_m^r(A) = \inf\{\|A - F\| : \dim(\ker(F)) \geq m\} \leq \|A - A(I - S_m)\| = \|AS_m\| \leq e < S(A).$$

Sending $m \to \infty$ we arrive at a contradiction. Thus $\tilde{\mu}(A) \geq S(A)$.

Since A is $\mathcal{P}$-Fredholm by Proposition 3.63, we can apply the second part of Lemma 3.42 to obtain

$$S(A) \geq B(A) \geq \mu(A) = \tilde{\mu}(A) \geq S(A). \qquad \square$$

3.4.6 Summary of characterizations

In the previous sections we studied several equivalent descriptions of $\left\|(A + \mathcal{K}(\mathbf{Y}, \mathcal{P}))^{-1}\right\|^{-1}$ for the following classes of operators:

- $A \in \mathrm{BDO}_\$(\mathbf{X})$, where $\mathbf{Y} = \mathbf{X} = \ell^p(\mathbb{Z}^N, X)$,
- $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$, where $\mathbf{Y} = \mathbf{X} = \ell^p(\mathbb{Z}^N, X)$ and $\dim X < \infty$,
- $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$, where $(\mathbf{Y}, \mathcal{P})$ is a happi space.

These characterizations include (peudo)spectra of limit operators, $\mu(A)$, $\max\{\nu_{\mathrm{ess}}(A), \nu_{\mathrm{ess}}(A^*)\}$, $\min\left\{\sqrt{\nu_{\mathrm{ess}}(AA^*)}, \sqrt{\nu_{\mathrm{ess}}(A^*A)}\right\}$, $\nu_{\mathrm{ess}}(A \oplus A^*)$, $B(A)$ and $S(A)$. In this section we want to summarize these characterizations and write down (pseudo)spectral versions of them. The most complete picture we have in the happi space case because we only require $\mathbf{Y}$ being a Hilbert space and $\mathcal{P}$ being a Hermitian approximate identity. In the finite dimensional case ($\dim X < \infty$) we quantitatively have the most equivalent characterizations. In the general case we had to restrict ourselves to $\mathrm{BDO}_\$(\mathbf{X})$ in order to get the desired equalities. Moreover, we had to make some additional assumptions that do not allow us to write down (pseudo)spectral versions immediately. The results are thus not expected to be complete in the general Banach space case. Still, we can write down a lot of (pseudo)spectral versions in the special cases we considered.

Theorem 3.69. *a) Let $A \in \mathrm{BDO}_\$(\mathbf{X})$ and $\varepsilon > 0$. Then*

$$\begin{aligned}
\mathrm{sp}_{\mathrm{ess}}(A) &= \left\{\lambda \in \mathbb{C} : \left\|(A - \lambda I + \mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\right\|^{-1} = 0\right\} \\
&= \bigcup_{A_h \in \sigma^{\mathrm{op}}(A)} \mathrm{sp}(A_h) \\
&= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\}, \\
\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) &= \left\{\lambda \in \mathbb{C} : \left\|(A - \lambda I + \mathcal{K}(\mathbf{X},\mathcal{P}))^{-1}\right\|^{-1} < \varepsilon\right\} \\
&= \bigcup_{A_h \in \sigma^{\mathrm{op}}(A)} \mathrm{sp}_\varepsilon(A_h) \\
&= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) < \varepsilon\}.
\end{aligned}$$

If $\mathbf{X} = \ell^p(\mathbb{Z}, X)$ with $\dim X < \infty$, then

$$\begin{aligned}
\mathrm{sp}_{\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \max\{\nu_{\mathrm{ess}}(A - \lambda I), \nu_{\mathrm{ess}}((A - \lambda I)^*)\} = 0\}, \\
\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \max\{\nu_{\mathrm{ess}}(A - \lambda I), \nu_{\mathrm{ess}}((A - \lambda I)^*)\} < \varepsilon\}.
\end{aligned}$$

b) Let $(\mathbf{Y}, \mathcal{P})$ be a happi space, $A \in \mathcal{L}(\mathbf{Y}, \mathcal{P})$ and $\varepsilon > 0$. Then

$$\begin{aligned}
\mathrm{sp}_{\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\} \\
&= \{\lambda \in \mathbb{C} : \nu_{\mathrm{ess}}((A - \lambda I) \oplus (A - \lambda I)^*) = 0\} \\
&= \left\{\lambda \in \mathbb{C} : \min\left\{\sqrt{\nu_{\mathrm{ess}}((A - \lambda I)(A - \lambda I)^*)}, \sqrt{\nu_{\mathrm{ess}}((A - \lambda I)^*(A - \lambda I))}\right\} = 0\right\}, \\
\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) &= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : \nu_{\mathrm{ess}}((A - \lambda I) \oplus (A - \lambda I)^*) < \varepsilon\} \\
&= \left\{\lambda \in \mathbb{C} : \min\left\{\sqrt{\nu_{\mathrm{ess}}((A - \lambda I)(A - \lambda I)^*)}, \sqrt{\nu_{\mathrm{ess}}((A - \lambda I)^*(A - \lambda I))}\right\} < \varepsilon\right\} \\
&= \mathrm{sp}_{\mathrm{ess}}(A) \cup \{\lambda \in \mathbb{C} : \max\{\nu_{\mathrm{ess}}(A - \lambda I), \nu_{\mathrm{ess}}((A - \lambda I)^*)\} < \varepsilon\}.
\end{aligned}$$

c) Let $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$, $\dim X < \infty$ and $\varepsilon > 0$. Then

$$\begin{aligned}
\mathrm{sp}_{\mathrm{ess}}(A) &= \left\{\lambda \in \mathbb{C} : \left\|(A - \lambda I + \mathcal{K}(\mathbf{X}))^{-1}\right\|^{-1} = 0\right\} \\
&= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\} \\
&= \{\lambda \in \mathbb{C} : \nu_{\mathrm{ess}}((A - \lambda I) \oplus (A - \lambda I)^*) = 0\} \\
&= \{\lambda \in \mathbb{C} : B(A - \lambda I) = 0\} \\
&= \{\lambda \in \mathbb{C} : S(A - \lambda I) = 0\}, \\
\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) &= \left\{\lambda \in \mathbb{C} : \left\|(A - \lambda I + \mathcal{K}(\mathbf{X}))^{-1}\right\|^{-1} < \varepsilon\right\} \\
&= \{\lambda \in \mathbb{C} : \mu(A - \lambda I) < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : \nu_{\mathrm{ess}}((A - \lambda I) \oplus (A - \lambda I)^*) < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : B(A - \lambda I) < \varepsilon\} \\
&= \{\lambda \in \mathbb{C} : S(A - \lambda I) < \varepsilon\}, \\
&= \mathrm{sp}_{\mathrm{ess}}(A) \cup \{\lambda \in \mathbb{C} : \max\{\nu_{\mathrm{ess}}(A - \lambda I), \nu_{\mathrm{ess}}((A - \lambda I)^*)\} < \varepsilon\}
\end{aligned}$$

Proof. This is a collection of the following results:
a) Theorem 3.20, Theorem 3.41, Theorem 3.35, Proposition 3.66,
b) Theorem 3.58, Corollary 3.60, Corollary 3.61, Corollary 3.62,
c) Theorem 3.49(*iii*), Proposition 3.63, Proposition 3.64, Theorem 3.65, Theorem 3.68. □

3.5 The $\mathcal{P}$-essential numerical range

Returning to the Hilbert space case, we consider the numerical range of operators $A \in \mathcal{L}(\mathbf{Y})$ here. So let $\mathbf{Y}$ be a (non-trivial) Hilbert space. Then the numerical range of A is defined as

$$N(A) := \mathrm{clos}\,\{\langle Ax, x\rangle : x \in \mathbf{Y}, \|x\| = 1\}.$$

Furthermore, we define the numerical radius by

$$r(A) := \max\{|z| : z \in N(A)\}$$

and the (rotated) numerical abscissae

$$r_\varphi(A) := \max\left\{\mathrm{Re}(z) : z \in N(e^{i\varphi}A)\right\}$$

for $\varphi \in [0, 2\pi)$. Clearly, $r(A) = \max\limits_{\varphi \in [0,2\pi)} r_\varphi(A)$ holds.

Note that the numerical range is usually defined without the closure (and denoted by W(A)) and the closure appears when it comes to giving an upper bound to the spectrum. However, we will find it useful to take the closure right here in the definition in order to get a compact set. Also note that for finite matrices, the set $\{\langle Ax, x\rangle : x \in \mathbf{X}, \|x\| = 1\}$ is always closed, but for infinite matrices this is usually not the case.

The numerical range has many applications in operator theory. For example, the famous Lumer-Phillips theorem ([63, Theorem 2.1 (bounded case),Theorem 3.1 (unbounded case)]) can be formulated using the numerical range. More generally, the numerical range can be used to estimate the growth of the semigroup $(e^{tA})_{t \geq 0}$ (see [91, Section 17] for more information). However, we will mainly use the numerical range to estimate the spectrum here. For this, we will need the following well-known theorems.

Theorem 3.70. *(e.g. [40, Problem 214])*
Let $A \in \mathcal{L}(\mathbf{Y})$. Then $\mathrm{sp}(A) \subset N(A)$.

Theorem 3.71. *(Hausdorff-Toeplitz [43], see also [40, Problem 210])*
Let $A \in \mathcal{L}(\mathbf{Y})$. Then its numerical range $N(A)$ is convex.

Theorem 3.72. *(e.g. [40, Problems 214,216])*
Let $A \in \mathcal{L}(\mathbf{Y})$. Then

$$\rho(A) \leq r(A) \leq \|A\|,$$

where $\rho(A)$ denotes the spectral radius of A. If A is normal, then equality holds. Furthermore, we have

$$\mathrm{conv}(\mathrm{sp}(A)) \subset N(A),$$

where $\mathrm{conv}(M)$ denotes the convex hull of a set M. Again, equality holds if A is normal.

Furthermore, we will find it useful to talk about convergence of set sequences.

Definition 3.73. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{C}$. Then we define

$$\begin{aligned}\limsup_{n\to\infty} M_n &:= \{m \in \mathbb{C} : m \text{ is an accumulation point of a sequence } (m_n)_{n\in\mathbb{N}}, m_n \in M_n\},\\ \liminf_{n\to\infty} M_n &:= \{m \in \mathbb{C} : m \text{ is the limit of a sequence } (m_n)_{n\in\mathbb{N}}, m_n \in M_n\}.\end{aligned}$$

The *Hausdorff metric* for compact sets $A, B \subset \mathbb{C}$ is defined as

$$h(A,B) := \max\left\{\max_{a\in A}\min_{b\in B}|a-b|, \max_{b\in B}\min_{a\in A}|a-b|\right\}.$$

Moreover, we define $\lim\limits_{n\to\infty} M_n$ as the limit of the sequence $(M_n)_{n\in\mathbb{N}}$ w.r.t. the Hausdorff metric.

These notions are compatible with each other in the sense that they satisfy the same relations as they do for ordinary sequences:

Proposition 3.74. *([35, Proposition 3.6])*
Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{C}$. Then the limit $\lim\limits_{n\to\infty} M_n$ exists if and only if $\limsup\limits_{n\to\infty} M_n = \liminf\limits_{n\to\infty} M_n$ and in this case we have

$$\lim_{n\to\infty} M_n = \limsup_{n\to\infty} M_n = \liminf_{n\to\infty} M_n.$$

To get an analogue of Theorem 3.20 in terms of numerical ranges, we need the following lemma that we will then apply to sequences $(V_{-h_n}(A+K)V_{h_n})_{n\in\mathbb{N}}$, where $K \in \mathcal{K}(\mathbf{X},\mathcal{P})$ and $(h_n)_{n\in\mathbb{N}}$ is a sequence of integers tending to infinity.

Lemma 3.75. *Let X be a Hilbert space, $\mathbf{X} = \ell^2(\mathbb{Z}^N, X)$ and let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathbf{X})$ be a bounded sequence that converges entrywise to $A \in \mathcal{L}(\mathbf{X})$. Then $N(A) \subset \liminf\limits_{n\to\infty} N(A_n)$.*

Proof. Let $\varepsilon > 0$ and $x \in \mathbf{X}$ and choose m large enough such that $\|Q_m x\| < \varepsilon$. Then

$$\begin{aligned} |\langle (A_n - A)x, x\rangle| &= |\langle (A_n - A)x, x\rangle - \langle (A_n - A)x, P_m x\rangle + \langle (A_n - A)x, P_m x\rangle \\ &\quad - \langle (A_n - A)P_m x, P_m x\rangle + \langle (A_n - A)P_m x, P_m x\rangle| \\ &\le |\langle (A_n - A)x, Q_m x\rangle| + |\langle (A_n - A)Q_m x, P_m x\rangle| + |\langle (A_n - A)P_m x, P_m x\rangle| \\ &\le \|A_n - A\| \, \|x\| \, \|Q_m x\| + \|A_n - A\| \, \|Q_m x\| \, \|x\| + |\langle (A_n - A)P_m x, P_m x\rangle| \\ &\le 2\varepsilon \, \|A_n - A\| \, \|x\| + |\langle (A_n - A)P_m x, P_m x\rangle| \,. \end{aligned}$$

$A_n \to A$ entrywise implies $\langle (A_n - A)P_m x, P_m x\rangle \to 0$ as $n \to \infty$ for all $x \in \mathbf{X}$. Furthermore, we assumed that the sequence $(A_n)_{n\in\mathbb{N}}$ is bounded. Thus we get $\langle (A_n - A)x, x\rangle \to 0$ as $n \to \infty$ for all $x \in \mathbf{X}$.

Let $z \in N(A)$. Choose $x_1 \in \mathbf{X}$ with $\|x_1\| = 1$ such that $|z - \langle Ax_1, x_1\rangle| < 1$ and n_1 such that $|\langle (A_n - A)x_1, x_1\rangle| < 1$ for all $n \ge n_1$. For $j \in \mathbb{N}$, choose $x_{j+1} \in \mathbf{X}$ with $\|x_{j+1}\| = 1$ such that $|z - \langle Ax_{j+1}, x_{j+1}\rangle| < \frac{1}{j+1}$ and $n_{j+1} > n_j$ such that $|\langle (A_n - A)x_{j+1}, x_{j+1}\rangle| < \frac{1}{j+1}$ for all $n \ge n_{j+1}$. Of course this implies $|z - \langle A_n x_j, x_j\rangle| < \frac{2}{j}$ for all $n \ge n_j$. Now define a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ in the following way. For $n < n_1$, choose $z_n \in N(A_n)$ arbitrarily. For $j \in \mathbb{N}$ and $n_j \le n < n_{j+1}$, choose $z_n \in N(A_n)$ such that $|z - z_n| < \frac{2}{j}$. We get $|z - z_n| \to 0$ as $n \to \infty$. Thus $N(A) \subset \liminf\limits_{n\to\infty} N(A_n)$. □

These results allow us to prove one more analogue of Theorem 3.20.

Theorem 3.76. *Let X be a Hilbert space, $\mathbf{X} = \ell^2(\mathbb{Z}^N, X)$ and $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$. Then*

$$N_{\mathrm{ess}}(A) := \bigcap_{K \in \mathcal{K}(\mathbf{X},\mathcal{P})} N(A + K) \supset \operatorname{conv}\left(\bigcup_{B \in \sigma^{\mathrm{op}}(A)} N(B) \right). \tag{9}$$

If $A \in \mathrm{BDO}_\$(\mathbf{X})$, then equality holds.

Proof. Let $B \in \sigma^{\mathrm{op}}(A)$ and $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$. For the first part it suffices to show $N(B) \subset N(A+K)$ because the intersection of convex sets is again convex. So let h be a sequence of integers tending to infinity such that $A_h = B$. By Proposition 3.14(*iii*) and Proposition 3.17, B is also a limit operator of $A + K$:

$$(A + K)_h = A_h + K_h = A_h + 0 = A_h = B.$$

Applying Lemma 3.75 to the sequence $(V_{-h_n}(A + K)V_{h_n})_{n\in\mathbb{N}}$ and using that the numerical range is invariant under unitary transformations, we get

$$N(B) \subset \liminf_{n\to\infty} N(V_{-h_n}(A + K)V_{h_n}) = \liminf_{n\to\infty} N(A + K) = N(A + K).$$

Now let $A \in \mathrm{BDO}_\$(\mathbf{X})$ and set $z_0 := \|A\|$. The shift z_0 will ensure that $N(e^{i\varphi}B + z_0 I)$ is contained in the right half plane for every $B \in \sigma^{\mathrm{op}}(A)$ and $\varphi \in [0, 2\pi)$. We will find this convenient later on. By Theorem 3.71, both sets in (9) are convex. It thus suffices to show

$$\inf_{K \in \mathcal{K}(\mathbf{X},\mathcal{P})} r_\varphi(A + K) \le \max\left\{ r_\varphi(B) : B \in \sigma^{\mathrm{op}}(A) \right\}.$$

For $r_\varphi(A+K)$ we have

$$\begin{aligned}
r_\varphi(A+K) &= \sup_{\|x\|=1} \operatorname{Re}\left\langle e^{i\varphi}(A+K)x,x\right\rangle \\
&= \sup_{\|x\|=1} \operatorname{Re}\left\langle (e^{i\varphi}(A+K)+z_0I)x,x\right\rangle - z_0 \\
&\le \sup_{\|x\|=1} \left|\operatorname{Re}\left\langle (e^{i\varphi}(A+K)+z_0I)x,x\right\rangle\right| - z_0 \\
&= \sup_{\|x\|=1} \left|\frac{1}{2}\left\langle (e^{i\varphi}(A+K)+e^{-i\varphi}(A+K)^*+2z_0I)x,x\right\rangle\right| - z_0 \\
&= \frac{1}{2}\left\|e^{i\varphi}(A+K)+e^{-i\varphi}(A+K)^*+2z_0I\right\| - z_0,
\end{aligned}$$

where we applied Theorem 3.72 to the self-adjoint (hence normal) operator $e^{i\varphi}(A+K)+e^{-i\varphi}(A+K)^*+2z_0I$. Taking the infimum, we arrive at

$$\begin{aligned}
\inf_{K\in\mathcal{K}(\mathbf{X},\mathcal{P})} r_\varphi(A+K) &\le \frac{1}{2}\inf_{K\in\mathcal{K}(\mathbf{X},\mathcal{P})}\left\|e^{i\varphi}(A+K)+e^{-i\varphi}(A+K)^*+2z_0I\right\| - z_0 \\
&= \frac{1}{2}\inf_{\substack{K\in\mathcal{K}(\mathbf{X},\mathcal{P})\\ K=K^*}}\left\|e^{i\varphi}A+e^{-i\varphi}A^*+K+2z_0I\right\| - z_0.
\end{aligned}$$

For a self-adjoint operator $C\in\mathcal{L}(\mathbf{X},\mathcal{P})$, the norm $\|C+K\|$ is minimized by a self-adjoint operator $K\in\mathcal{K}(\mathbf{X},\mathcal{P})$. This can be seen as follows:

$$\begin{aligned}
\|C+K\| &\ge \sup_{\|x\|=1} |\langle (C+K)x,x\rangle| \\
&= \sup_{\|x\|=1} \left|\left\langle \left(C+\frac{K+K^*}{2}\right)x,x\right\rangle + \left\langle \left(\frac{K-K^*}{2}\right)x,x\right\rangle\right| \\
&\ge \sup_{\|x\|=1} \left|\left\langle \left(C+\frac{K+K^*}{2}\right)x,x\right\rangle\right| \\
&= \left\|C+\frac{K+K^*}{2}\right\|,
\end{aligned}$$

where we used Theorem 3.72 and the fact that $\left\langle \left(C+\frac{K+K^*}{2}\right)x,x\right\rangle \in \mathbb{R}$ and $\left\langle \left(\frac{K-K^*}{2}\right)x,x\right\rangle \in i\mathbb{R}$ for all $x\in\mathbf{X}$. Using this, Theorem 3.26 and Proposition 3.14, we get

$$\begin{aligned}
\inf_{K\in\mathcal{K}(\mathbf{X},\mathcal{P})} r_\varphi(A+K) &\le \frac{1}{2}\inf_{K\in\mathcal{K}(\mathbf{X},\mathcal{P})}\left\|e^{i\varphi}A+e^{-i\varphi}A^*+K+2z_0I\right\| - z_0 \\
&= \frac{1}{2}\max\left\{\|B\| : B\in\sigma^{\mathrm{op}}(e^{i\varphi}A+e^{-i\varphi}A^*+2z_0I)\right\} - z_0 \\
&= \frac{1}{2}\max\left\{\left\|e^{i\varphi}B+e^{-i\varphi}B^*+2z_0I\right\| : B\in\sigma^{\mathrm{op}}(A)\right\} - z_0.
\end{aligned}$$

Since $r_\varphi(B)\le r(B)\le\|B\|\le\|A\|$ by Proposition 3.14(i), $N(e^{i\varphi}B+z_0I)$ is contained in the right half plane for every $B \in \sigma^{\mathrm{op}}(A)$ and $\varphi \in [0, 2\pi)$. This implies

$$\begin{aligned} r_\varphi(B) &= \sup_{\|x\|=1} \operatorname{Re} \left\langle e^{i\varphi} Bx, x \right\rangle \\ &= \sup_{\|x\|=1} \operatorname{Re} \left\langle (e^{i\varphi} B + z_0 I)x, x \right\rangle - z_0 \\ &= \sup_{\|x\|=1} \left| \operatorname{Re} \left\langle (e^{i\varphi} B + z_0 I)x, x \right\rangle \right| - z_0 \\ &= \sup_{\|x\|=1} \left| \frac{1}{2} \left\langle (e^{i\varphi} B + e^{-i\varphi} B^* + 2z_0 I)x, x \right\rangle \right| - z_0 \\ &= \frac{1}{2} \left\| e^{i\varphi} B + e^{-i\varphi} B^* + 2z_0 I \right\| - z_0. \end{aligned}$$

We conclude

$$\inf_{K \in \mathcal{K}(\mathbf{X}, \mathcal{P})} r_\varphi(A + K) \leq \max \left\{ r_\varphi(B) : B \in \sigma^{\mathrm{op}}(A) \right\}.$$

□

3.6 References

The limit operator theory presented in Section 3.1 can be found in [57]. Good sources for results in $\mathcal{P}$-Fredholm theory are the works of Seidel and Silbermann [83, 84, 86]. Most of the non-standard results in Section 3.1 can be found there. Proposition 3.7 and Lemma 3.13 are extensions of results presented in [86]. Another excellent source is the Memoir of Chandler-Wilde and Lindner [19]. The main theorem of limit operator theory 3.20 was first proven in [61] following a long line of previous (less complete) versions including the results in [19], [53], [57] and [73]. Parts of Section 3.1 as well as almost all results in Sections 3.2, 3.3 and 3.4 are published by Lindner, Seidel and the author in [39], most of the results being probably new. The result about the $\mathcal{P}$-essential numerical range in Section 3.5 was published by the author in [36], altough the result is probably known in view of it being a Hilbert space result.

4 Application to Random Operators

In this section we apply the results obtained above to random operators. Let $(\Omega, \mathbb{P})$ be a probability space. Then a random operator A is a map $\Omega \to \mathcal{L}(\mathbf{X})$ such that $A(\cdot)_{i,j}\colon \Omega \to \mathcal{L}(X)$ is measurable for all $i, j \in \mathbb{Z}^N$, in short: an infinite matrix with random variables as entries. In the following we will consider a non-probabilistic approach by Davies [23]. The idea is that, under some reasonable assumptions on the probability space $(\Omega, \mathbb{P})$ and the map A, there is a particular subset $\Psi\mathrm{E}$ (to be defined below) of $A(\Omega) = \{A(\omega) : \omega \in \Omega\}$ such that $\mathbb{P}(\{\omega \in \Omega : A(\omega) \in \Psi\mathrm{E}\}) = 1$. In other words, the image $A(\omega)$ of the sample ω is contained in $\Psi\mathrm{E}$ with probability 1. The operators contained in the set $\Psi\mathrm{E}$ are called pseudo-ergodic operators. We will see that all pseudo-ergodic operators share the same spectrum, the same norm etc. This allows us to reduce the study of the spectral properties of a random operator A to the image of one arbitary sample in $\Psi\mathrm{E}$ that (under abuse of notation) will again be denoted by A. In particular, the operator A is purely deterministic and no further probabilistic arguments are needed, which is a great benefit of this approach. As a consequence we get that the corresponding random operator shares the properties of a pseudo-ergodic operator almost surely (i.e. with probability 1).

So here is our setting. Let $p \in \{0\} \cup [1, \infty]$, X a Banach space and $\mathbf{X} = \ell^p(\mathbb{Z}, X)$ as usual (for simplicity we restrict ourselves to the case $N = 1$ here). We consider random operators $A\colon \Omega \to \mathcal{L}(\mathbf{X})$ for which the sets $U_k := \{A(\omega)_{i,i+k} : \omega \in \Omega\}$ are compact for all $k \in \mathbb{Z}$.

Example 4.1. (a) Let $\mathbf{X} = \ell^p(\mathbb{Z})$ and $\Omega := \{1\}$. We equip Ω with the only probability measure $\mathbb{P}$ on $\{1\}$. Then the constant function $A\colon \{1\} \to \mathcal{L}(\mathbf{X})$, $A(1) = I$ is a random operator (although a boring one) with $U_0 = \{1\}$ and $U_k = \{0\}$ for all $k \neq 0$. Similarly, choosing the sets U_k appropriately, every Laurent operator (cf. Example 2.25) can be interpreted as the image of a random operator.

(b) Let $\mathbf{X} = \ell^p(\mathbb{Z})$, $\Omega_0 := \{0, 1\}$, $m \in \mathbb{N}$ and $\Omega := \{0, 1\}^m$. We equip Ω_0 with the probability measure defined by $\mathbb{P}_0(\{0\}) = \mathbb{P}_0(\{1\}) = \frac{1}{2}$. The set $\Omega = \Omega_0^m$ we equip with the product measure $\mathbb{P} := \bigotimes_{j=1}^{m} \mu_0$, i.e. the measure induced by

$$\mathbb{P}(W_1 \times \ldots \times W_m) = \mathbb{P}_0(W_1) \cdot \ldots \cdot \mathbb{P}_0(W_m)$$

for $W_1 \times \ldots \times W_m \subset \Omega$. The map $A\colon \Omega \to \mathcal{L}(\mathbf{X})$ defined by

$$A(\omega)_{i,j} := \begin{cases} \omega_{i \bmod m} & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for $\omega \in \Omega = \{0, 1\}^m$, $i, j \in \mathbb{Z}$ is a random operator with $U_0 = \{0, 1\}$ and $U_k = \{0\}$ for all $k \neq 0$. In this example, every operator $A(\omega)$ is m-periodic (cf. Example 2.25). Note that the entries of this random operator are not independent since $A(\omega)_{i+m,j+m} = A(\omega)_{i,j}$ for all $\omega \in \Omega$, $i, j \in \mathbb{Z}$, which is of course exactly the definition of an m-periodic operator. Again, this can be done with arbitrary sets U_k.

(c) Let $\mathbf{X} = \ell^p(\mathbb{Z})$ and $\Omega_0 := \{0, 1\}$. We equip Ω_0 with the probability measure $\mathbb{P}_0$ defined by $\mathbb{P}_0(\{0\}) = \mathbb{P}_0(\{1\}) = \frac{1}{2}$. We further define $(\Omega, \mathbb{P})$ as the product space $\{0, 1\}^{\mathbb{Z}}$ equipped with the usual product topology and product measure $\mathbb{P} := \bigotimes_{j \in \mathbb{Z}} \mathbb{P}_0$, i.e. the measure induced by

$$\mathbb{P}(\{\omega \in \Omega : \omega_{k_1} = x_1, \ldots, \omega_{k_n} = x_n\}) = \frac{1}{2^n}$$

for $k_1 < \ldots < k_n \in \mathbb{Z}$, $x_1, \ldots, x_n \in \{0,1\}$ and $n \in \mathbb{N}$. The map $A\colon \Omega \to \mathcal{L}(\mathbf{X})$ defined by

$$A(\omega)_{i,j} := \begin{cases} \omega_i & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for $\omega \in \Omega = \{0,1\}^{\mathbb{Z}}$, $i,j \in \mathbb{Z}$ is a random operator with $U_0 = \{0,1\}$ and $U_k = \{0\}$ for all $k \neq 0$. The image $A(\omega)$ of a sample ω is an infinite diagonal matrix with randomly distributed zeros and ones on its main diagonal. $A(\Omega)$ is the set of all images, i.e. the set of all diagonal binary matrices. In this example the entries on the main diagonal are independent and identically distributed (i.i.d.). Random operators that have i.i.d. entries along each diagonal will be our main topic for the rest of the thesis.

(d) Similarly, we can choose $\Omega_0 = \mathbb{D}$ and $\mathbb{P}_0$ the (scaled) Lebesgue measure on $\mathbb{D}$. Then again

$$A(\omega)_{i,j} := \begin{cases} \omega_i & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for $\omega \in \Omega = \mathbb{D}^{\mathbb{Z}}$, $i,j \in \mathbb{Z}$ defines a random operator with $U_0 = \mathbb{D}$ and $U_k = \{0\}$ for all $k \neq 0$.

(e) Let $\mathbf{X} = \ell^p(\mathbb{Z})$, $\Omega := \{0,1\}$ and $\mathbb{P}$ as $\mathbb{P}_0$ above. Then $A\colon \Omega \to \mathcal{L}(\mathbf{X})$ defined by

$$A(\omega)_{i,j} := \begin{cases} \omega & \text{if } i = j < 0, \\ 1-\omega & \text{if } i = j \geq 0, \\ 0 & \text{if } i \neq j \end{cases}$$

for $\omega \in \Omega = \{0,1\}$, $i,j \in \mathbb{Z}$ is another random operator with $U_0 = \{0,1\}$ and $U_k = \{0\}$ for all $k \neq 0$. As in (b), the entries are not independent here.

(f) Let $\mathbf{X} = \ell^p(\mathbb{Z})$, $\Omega_k := \{0,1\}$ and let $\mathbb{P}_k$ be defined by $\mathbb{P}_k(\{0\}) = 1 - \frac{1}{2^{|k|}}$ and $\mathbb{P}_k(\{1\}) = \frac{1}{2^{|k|}}$ for all $k \in \mathbb{Z}$. Then $\Omega := \prod\limits_{k \in \mathbb{Z}} \Omega_k^{\mathbb{Z}} = \{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$ equipped with the product measure $\mathbb{P} := \bigotimes\limits_{k\in\mathbb{Z}} \bigotimes\limits_{j\in\mathbb{Z}} \mathbb{P}_k$ as above is a probability space. However,

$$A(\omega)_{i,j} := (\omega_k)_j \quad \text{for } i-j=k,\ \omega_k \in \Omega_k \text{ and } i,j \in \mathbb{Z}$$

does not define a random operator. This is because $A(\omega)$ is not bounded for some $\omega \in \Omega$ despite the fact that $\mathbb{P}_k(\{1\})$ decays exponentially as $|k| \to \infty$. In fact, $\mathbb{P}(\{\omega \in \Omega : A(\omega) \text{ is bounded}\}) = 0$.

(g) Let $\mathbf{X} = \ell^p(\mathbb{Z})$, $\Omega_0 := \{0,1\}$ and let $\mathbb{P}_0$ be the probability measure defined by

$$\mathbb{P}_0(V) = \begin{cases} 1 & \text{if } 0 \in V, \\ 0 & \text{if } 0 \notin V \end{cases}$$

and $\mathbb{P} := \bigotimes\limits_{j\in\mathbb{Z}} \mathbb{P}_0$. Then

$$A(\omega)_{i,j} := \begin{cases} \omega_i & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for $\omega \in \Omega = \{0,1\}^{\mathbb{Z}}$, $i,j \in \mathbb{Z}$ defines another random operator with $U_0 = \{0,1\}$ and $U_k = \{0\}$ for all $k \neq 0$, although $\mathbb{P}(\{\omega \in \Omega : A(\omega) = I\}) = 1$.

Usually one defines a random operator by prescribing the sets U_k and then choosing appropriate probability measures (e.g. the uniform distribution). Pseudo-ergodic operator are constructed in a similar way as we will see below.

4.1 Pseudo-ergodic operators

4.1.1 Definition

As mentioned above we are looking for a subset ΨE of $A(\Omega) = \{A(\omega) : \omega \in \Omega\}$ with

$$\mathbb{P}(\{\omega : A(\omega) \in \Psi\mathrm{E}\}) = 1$$

such that all $B \in \Psi\mathrm{E}$ have the same spectral properties, e.g. spectrum, norm etc. For this we need some notation first.

Definition 4.2. Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$. For $n \in \mathbb{N}$ the set

$$\left\{B \in \mathcal{L}(X)^{n\times n} = \mathcal{L}(X^n) : B_{i,j} \in U_{i-j} \text{ for all } i,j \in \{1,\dots,n\}\right\}$$

will be denoted by $M_n((U_k)_{k\in\mathbb{Z}})$. Furthermore, we denote the union of all these sets by

$$M_{fin}((U_k)_{k\in\mathbb{Z}}) = \bigcup_{n\in\mathbb{N}} M_n((U_k)_{k\in\mathbb{Z}}).$$

$M_{fin}((U_k)_{k\in\mathbb{Z}})$ contains all finite matrices B with the property that the k-th diagonal of B only contains elements from U_k. If $U_k = \{0\}$ for some k, we will just drop this set from the list $(U_k)_{k\in\mathbb{Z}}$. So for example if $M_{fin}((U_k)_{k\in\mathbb{Z}})$ only consists of tridiagonal operators, we may just write $M_{fin}(U_{-1}, U_0, U_1)$ instead. Similarly, we define the set of all infinite matrices of this kind:

Definition 4.3. Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$. The set

$$\left\{B \in \mathcal{L}(X)^{\mathbb{Z}\times\mathbb{Z}} : B_{i,j} \in U_{i-j} \text{ for all } i,j \in \mathbb{Z}\right\}$$

will be denoted by $M((U_k)_{k\in\mathbb{Z}})$.

Note that this definition does not imply that $M((U_k)_{k\in\mathbb{Z}})$ contains any bounded linear operators at all (just infinite matrices). However, we will usually assume that the sets U_k are chosen in such a way that $M((U_k)_{k\in\mathbb{Z}}) \subset \mathcal{L}(\mathbf{X})$ (see Remark 4.5(a)).

Again, we will drop trivial sets from the list $(U_k)_{k\in\mathbb{Z}}$. Now we are able to define pseudo-ergodic operators.

Definition 4.4. Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$. A bounded linear operator $A \in M((U_k)_{k\in\mathbb{Z}})$ is called pseudo-ergodic if for all $\varepsilon > 0$, $n \in \mathbb{N}$ and every $B \in M_n((U_k)_{k\in\mathbb{Z}})$ there exists an $m \in \mathbb{Z}$ such that

$$\left\|P_{\{1,\dots,n\}}V_{-m}AV_mP_{\{1,\dots,n\}} - B\right\|_{\mathcal{L}(X^n)} \le \varepsilon,$$

where $P_{\{1,\dots,n\}}$ and V_m are defined as in Section 1.1.6 and 2.2.1, respectively. The set of all pseudo-ergodic operators in $M((U_k)_{k\in\mathbb{Z}})$ will be denoted by $\Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$.

The definition above should be taken with some care. The operator $P_{\{1,\dots,n\}}V_{-m}AV_mP_{\{1,\dots,n\}}$ has to be interpreted as an operator $X^n \to X^n$ so that the difference $P_{\{1,\dots,n\}}V_{-m}AV_mP_{\{1,\dots,n\}} - B$ makes sense. The norm $\|\cdot\|_{\mathcal{L}(X^n)}$ is defined by the restriction of $\|\cdot\|_{\mathbf{X}}$ to $\|\cdot\|_{X^n}$, i.e. just the respective p-norm on $\mathcal{L}(X^n)$. So an operator $A \in M((U_k)_{k\in\mathbb{Z}})$ is called pseudo-ergodic if every finite matrix of this kind can be found, up to ε, somewhere along the diagonals of A. In the case where all U_k are discrete (i.e. finite), we can just put ε equal to 0 such that the definition simplifies to

$$\forall n \in \mathbb{N}, \forall B \in M_n((U_k)_{k\in\mathbb{Z}})\, \exists m \in \mathbb{Z} \text{ such that } P_{\{1,\dots,n\}} V_{-m} A V_m P_{\{1,\dots,n\}} = B.$$

In other words: all of these finite matrices can be found somewhere along the diagonals of A. So for all $n \in \mathbb{N}$ and all $B \in M_n((U_k)_{k\in\mathbb{Z}})$ there exists an $m \in \mathbb{Z}$ such that for all $i, j \in \{1, \dots, n\}$ we have $A_{i+m,j+m} = B_{i,j}$. The sets $(U_k)_{k\in\mathbb{Z}}$ are often finite in applications. Therefore one should keep this simplification in mind.

Remark 4.5. (a) It should be noted that, by definition, every pseudo-ergodic operator is bounded. As a consequence, the sets U_k have to be chosen properly to make sure that pseudo-ergodic operators even exist. In regard to random operators this perfectly makes sense because we have to ensure $A(\omega) \in \mathcal{L}(\mathbf{X})$ for $\omega \in \Omega$ there as well. We will come back to this in Section 4.1.3.

(b) By definition, $A \in M((U_k)_{k\in\mathbb{Z}})$ is pseudo-ergodic if for all $\varepsilon > 0$, $n \in \mathbb{N}$ and every $B \in M_n((U_k)_{k\in\mathbb{Z}})$ there exists an $m \in \mathbb{Z}$ such that

$$\left\| P_{\{1,\dots,n\}} V_{-m} A V_m P_{\{1,\dots,n\}} - B \right\|_{\mathcal{L}(X^n)} \leq \varepsilon. \tag{10}$$

In fact, one can find infinitely many $m \in \mathbb{N}$ that satisfy this condition. If all the sets U_k are singletons, this is obvious. If not, then there are infinitely many matrices $C \in M_{fin}((U_k)_{k\in\mathbb{Z}})$ that have B as a submatrix. For all of them we can find an m such that Condition 10 is satisfied. By decreasing ε, we can make sure that we find infinitely many (different) m. Thus there are infinitely many m that satisfy Condition 10 for a given $B \in M_{fin}((U_k)_{k\in\mathbb{Z}})$.

The notion of pseudo-ergodic operators goes back to Davies [23]. The notations $\Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ and $M((U_k)_{k\in\mathbb{Z}})$ are in the style of [20] and [60], where the tridiagonal case is considered and the notations $\Psi\mathrm{E}(U, V, W)$ and $M(U, V, W)$ were used, respectively.

4.1.2 Reduction of random operators to pseudo-ergodic operators

In order to reduce a random operator to pseudo-ergodic operators, we want to impose further conditions. Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact sets. We consider the following conditions:

(C1) $A(\Omega) = M((U_k)_{k\in\mathbb{Z}})$.

(C2) For all $i, k \in \mathbb{Z}$ and all non-empty open subsets W_k of U_k we have

$$\mathbb{P}(\{\omega \in \Omega : A(\omega)_{i+k,i} \in W_k\}) > 0.$$

(C3) The diagonals of A are identically distributed, i.e. for all $i, j, k \in \mathbb{Z}$ and all (measureable) subsets W_k of U_k, we have

$$\mathbb{P}(\{\omega \in \Omega : A(\omega)_{i+k,i} \in W_k\}) = \mathbb{P}(\{\omega \in \Omega : A(\omega)_{j+k,j} \in W_k\}).$$

(C4) All entries $A(\cdot)_{i,j}$ are independent, i.e. for all $r \in \mathbb{N}$, all indices $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_r$ and all (measureable) subsets W_l of $U_{i_l - j_l}$, $l \in \{1, \dots, r\}$ we have

$$\mathbb{P}(\{\omega \in \Omega : A(\omega)_{i_1,j_1} \in W_1, \dots, A(\omega)_{i_r,j_r} \in W_r\}) = \prod_{l=1}^{r} \mathbb{P}(\{\omega \in \Omega : A(\omega)_{i_l,j_l} \in W_l\}).$$

In other words, every diagonal is a sequence of i.i.d. random variables and the diagonals are again independent from each other.

The first two conditions are rather mild in the sense that they are automatically satisfied if we properly define the probability spaces. Example 4.1(g) is an instance of a random operator that does not satisfy (C2). Similarly, one can easily construct random operators that do not satisfy (C1). However, these examples are rather artificial.

These latter two conditions are more strict as they exclude some reasonable examples we mentioned above. However, a lot of physical applications like the Anderson model [1], the non-periodic Hatano-Nelson model [41] or the Feinberg-Zee Hopping Sign model [28] satisfy these conditions. Clearly, the Examples 4.1(b) and (e) do not satisfy (C4). Example 4.1(a) trivially does, though.

With these conditions at hand we can now formulate the cornerstone of the subsequent results. It immediately eliminates all probabilistic arguments, which enables us to focus on the operator theoretical aspects. In fact, this is an instance of the so-called second Borel-Cantelli lemma ([9, Problème I]).

Theorem 4.6. *Let $(\Omega, \mathbb{P})$ be a probability space, let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A\colon \Omega \to \mathcal{L}(\mathbf{X})$ be a random operator that satisfies the conditions (C1) - (C4). Then*

$$\mathbb{P}(\{\omega \in \Omega : A(\omega) \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})\}) = 1,$$

i.e. $A(\omega)$ is pseudo-ergodic almost surely.

Proof. We first observe that $A(\omega) \in M((U_k)_{k\in\mathbb{Z}})$ for all $\omega \in \Omega$ by (C1). Now fix $\varepsilon > 0$, $B \in M_n((U_k)_{k\in\mathbb{Z}})$ and $i, j \in \{1, \ldots, n\}$. (C2) implies

$$p_{i,j} := \mathbb{P}\left(\left\{\omega \in \Omega : \|A(\omega)_{i,j} - B_{i,j}\|_{\mathcal{L}(X)} \leq \frac{\varepsilon}{n^2}\right\}\right) > 0$$

and thus

$$\begin{aligned}
p &:= \mathbb{P}\left(\left\{\omega \in \Omega : \left\|P_{\{1,\ldots,n\}}A(\omega)P_{\{1,\ldots,n\}} - B\right\|_{\mathcal{L}(X^n)} \leq \varepsilon\right\}\right) \\
&\geq \mathbb{P}\left(\left\{\omega \in \Omega : \sum_{i,j=1}^{n} \|A(\omega)_{i,j} - B_{i,j}\|_{\mathcal{L}(X)} \leq \varepsilon\right\}\right) \\
&\geq \mathbb{P}\left(\left\{\omega \in \Omega : \forall i, j \in \{1, \ldots, n\} : \|A(\omega)_{i,j} - B_{i,j}\|_{\mathcal{L}(X)} \leq \frac{\varepsilon}{n^2}\right\}\right) \\
&= \prod_{i,j=1}^{n} p_{i,j} \\
&> 0
\end{aligned}$$

by (C4). (C3) implies

$$\mathbb{P}\left(\left\{\omega \in \Omega : \left\|P_{\{1,\ldots,n\}}V_{-nm}A(\omega)V_{nm}P_{\{1,\ldots,n\}} - B\right\|_{\mathcal{L}(X^n)} \leq \varepsilon\right\}\right) = p$$

for all $m \in \mathbb{N}$. Fix $r \in \mathbb{N}$. Applying (C4) again, we get

$$\mathbb{P}\left(\left\{\omega \in \Omega : \exists m \in \{1, \ldots, r\} : \left\|P_{\{1,\ldots,n\}}V_{-nm}A(\omega)V_{nm}P_{\{1,\ldots,n\}} - B\right\|_{\mathcal{L}(X^n)} \leq \varepsilon\right\}\right) = 1 - (1-p)^r.$$

Since $p > 0$, we get

$$\mathbb{P}\left(\left\{\omega \in \Omega : \exists m \in \{1,\dots,r\} : \left\|P_{\{1,\dots,n\}}V_{-nm}A(\omega)V_{nm}P_{\{1,\dots,n\}} - B\right\|_{\mathcal{L}(X^n)} \le \varepsilon\right\}\right) \to 1$$

as $r \to \infty$. Thus for fixed $\varepsilon > 0$ and $B \in M_{fin}((U_k)_{k\in\mathbb{Z}})$ there exists $m \in \mathbb{N}$ such that

$$\left\|P_{\{1,\dots,n\}}V_{-nm}A(\omega)V_{nm}P_{\{1,\dots,n\}} - B\right\|_{\mathcal{L}(X^n)} \le \varepsilon \tag{11}$$

almost surely. Using the σ-additivity of $\mathbb{P}$ and the compactness of the sets U_k, we obtain that for fixed ε and all $B \in M_{fin}((U_k)_{k\in\mathbb{Z}})$ (11) holds almost surely. Similarly, choosing a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$, we obtain that for all ε and all $B \in M_{fin}((U_k)_{k\in\mathbb{Z}})$ (11) holds almost surely. In other words, $A(\omega) \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ almost surely. □

Note that we did not actually use that $A(\omega)$ is a two-sided infinite matrix. Hence, the theorem also applies to random operators $A\colon \Sigma \to \mathcal{L}(\ell^p(\mathbb{N},X))$ if we modify (C1) - (C4) appropriately. We will come back to this is Section 4.1.4.

4.1.3 Spectral properties of pseudo-ergodic operators

To apply the limit operator results obtained in Section 3, we restrict ourselves to rich band-dominated operators here. An immediate question is how to choose the sets U_k in such a way that $\Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}}) \subset \mathrm{BDO}_{\$}(\mathbf{X})$ holds. This is not the easiest question to answer in general. The easy part is the richness. If $A \in M((U_k)_{k\in\mathbb{Z}}) \cap \mathrm{BDO}(\mathbf{X})$, then A is automatically rich because we assumed that the sets U_k are compact (cf. proof of Proposition 2.27). So the hard part is to make sure that A is band-dominated (or even bounded). In some cases, however, it is easy to tell. For example, if only finitely many sets are different from $\{0\}$, then clearly $M((U_k)_{k\in\mathbb{Z}}) \subset \mathrm{BO}(\mathbf{X}) \subset \mathrm{BDO}(\mathbf{X})$ holds. This can of course be extended to sequences of sets that satisfy

$$\sum_{k=-\infty}^{\infty} \max_{u_k\in U_k} \|u_k\| < \infty. \tag{12}$$

In this case $M((U_k)_{k\in\mathbb{Z}}) \subset \mathcal{W}(\mathbf{X}) \subset \mathrm{BDO}(\mathbf{X})$. Recall that the Wiener algebra is defined by

$$\mathcal{W}(\mathbf{X}) = \left\{A \in \mathcal{L}(\mathbf{X}) : \sum_{k\in\mathbb{Z}} \sup_{i\in\mathbb{Z}} \|A_{i+k,i}\| < \infty\right\}.$$

Of course, starting with a Laurent operator that is not band-dominated, it is also not hard to construct more pseudo-ergodic operators that are not band-dominated. Having said that, a hand-made pseudo-ergodic operator is typically contained in $\mathcal{W}(\mathbf{X})$. Roughly speaking, this is because it is hard to ensure that A is bounded without having some particular operators like Laurent operators in mind. So the easiest thing one can do is to restrict the sets as in (12). If $\mathbf{X} = \ell^p(\mathbb{Z})$, $p \in \{0, 1, \infty\}$, then in fact all pseudo-ergodic operators are contained in the Wiener algebra. Similarly, if $\mathbf{X} = \ell^p(\mathbb{Z})$ and the maxima $\max\limits_{u_k\in U_k} |u_k|$ are contained in $U_k \subset \mathbb{C}$ for all $k \in \mathbb{Z}$, then $A \in \mathcal{W}(\mathbf{X})$. Moreover, random operators coming from physical models are usually banded, often even tridiagonal, so no problem there.

Proposition 4.7. *Let $X = \mathbb{C}$, $(U_k)_{k\in\mathbb{Z}}$ a sequence of non-empty compact subsets of $\mathbb{C}$ and $A \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$. If $p \in \{0, 1, \infty\}$ or if $\sup\limits_{i\in\mathbb{Z}} |A_{i+k,i}| \in U_k$ for all $k \in \mathbb{Z}$, then $A \in \mathcal{W}(\mathbf{X})$ and*

$$\|A\| = \|A\|_{\mathcal{W}} = \sum_{k=-\infty}^{\infty} \max_{v_k\in U_k} |v_k| \,. \tag{13}$$

In particular, Laurent operators are contained in $\mathcal{W}(\mathbf{X})$ if one of the above conditions is satisfied.

Proof. Since $\|A\| \leq \|A\|_{\mathcal{W}}$ always holds, we only have to show $\|A\| \geq \|A\|_{\mathcal{W}}$. This then implies equality and hence $A \in \mathcal{W}(\mathbf{X})$.

Let $p = 1$ and fix $m \in \mathbb{N}$, $\varepsilon > 0$. For every $k \in \{-m, \ldots, m\}$ choose $u_k \in U_k$ such that $|u_k| = \max\limits_{v_k\in U_k} |v_k|$. By the pseudo-ergodicity of A, we can find a column i such that $|A_{i+k,i} - u_k| < \frac{\varepsilon}{2^{|k|}}$ for all $k \in \{-m, \ldots, m\}$. Let $y \in \mathbf{X}$ be the i-th unit vector, i.e.

$$y_j = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{else.} \end{cases}$$

Then $\|y\| = 1$ and it follows

$$\|A\| \geq \|Ay\| \geq \sum_{k=-m}^{m} |A_{i+k,i}| > \sum_{k=-m}^{m} \left(|u_k| - \frac{\varepsilon}{2^{|k|}}\right) > \sum_{k=-m}^{m} |u_k| - 4\varepsilon.$$

Since m and ε were arbitrary, this implies $A \in \mathcal{W}(\mathbf{X})$ and Equation (13) holds.

Let $p \in \{0, \infty\}$ and fix $m \in \mathbb{N}$, $\varepsilon > 0$. For every $k \in \{-m, \ldots, m\}$ choose $u_k \in U_k$ such that $|u_k| = \max\limits_{v_k\in U_k} |v_k|$. By the pseudo-ergodicity of A, we can find a row i such that $|A_{i,i-k} - u_k| < \frac{\varepsilon}{2^{|k|}}$ for all $k \in \{-m, \ldots, m\}$. For every $k \in \{-m, \ldots, m\}$ choose $x_k \in X$ with $|x_k| = 1$ such that $|A_{i,i-k}| = A_{i,i-k}x_k$. Let $y \in \mathbf{X}$ be defined by

$$y_j = \begin{cases} x_{i-j} & \text{if } j \in \{i-m, \ldots, i+m\}, \\ 0 & \text{else.} \end{cases}$$

Then $\|y\| = 1$ and it follows

$$\|A\| \geq \|Ay\| \geq |(Ay)_i| = \left|\sum_{j=i-m}^{i+m} A_{i,j}x_{i-j}\right| = \sum_{k=-m}^{m} |A_{i,i-k}| > \sum_{k=-m}^{m} \left(|u_k| - \frac{\varepsilon}{2^{|k|}}\right) > \sum_{k=-m}^{m} |u_k| - 4\varepsilon.$$

Since m and ε were arbitrary, this implies $A \in \mathcal{W}(\mathbf{X})$ and Equation (13) holds.

The proof is very similar in the case $1 < p < \infty$. We just have to consider multiple rows. Let $u_k := \max\limits_{v_k\in U_k} |v_k| \in U_k$ for all $k \in \mathbb{Z}$. If $A = 0$, there is nothing to prove. Therefore we may assume that there exist $m_0, n_0 \in \mathbb{N}$ such that $\sum\limits_{k=-m}^{m} u_k \geq \frac{4}{n}$ for all $m \geq m_0$ and $n \geq n_0$. Fix $m \geq m_0$ and $n \geq n_0$. Then by pseudo-ergodicity, we can find $2n + 1$ consecutive rows $i - n, \ldots, i + n$ such that $u_k - \operatorname{Re}(A_{l,l-k}) < \frac{1}{2^{|k|}n}$ as well as $\operatorname{Re}(A_{l,l-k}) \geq 0$ for all $k \in \{-m-2n, \dots, m+2n\}$ and $l \in \{i-n, \dots, i+n\}$. Let $y \in \mathbf{X}$ be defined by

$$y_j = \begin{cases} 1 & \text{if } j \in \{i-m-n, \dots, i+m+n\}, \\ 0 & \text{else.} \end{cases}$$

Then $\|y\|^p = 2(m+n)+1$ and it follows

$$\begin{aligned}
(2(m+n)+1)\,\|A\|^p \geq \|Ay\|^p \geq \sum_{l=i-n}^{i+n} |(Ay)_l|^p &= \sum_{l=i-n}^{i+n} \left| \sum_{j=i-m-n}^{i+m+n} A_{l,j} \right|^p \\
&= \sum_{l=i-n}^{i+n} \left| \sum_{k=l-i-m-n}^{l-i+m+n} A_{l,l-k} \right|^p \\
&\geq \sum_{l=i-n}^{i+n} \left(\operatorname{Re}\left(\sum_{k=l-i-m-n}^{l-i+m+n} A_{l,l-k} \right) \right)^p \\
&> \sum_{l=i-n}^{i+n} \left(\sum_{k=l-i-m-n}^{l-i+m+n} \left(u_k - \frac{1}{2^{|k|}n} \right) \right)^p \\
&\geq \sum_{l=i-n}^{i+n} \left(-\frac{4}{n} + \sum_{k=-m}^{m} u_k \right)^p \\
&= (2n+1) \left(-\frac{4}{n} + \sum_{k=-m}^{m} u_k \right)^p .
\end{aligned}$$

We conclude

$$\|A\|^p > \frac{2n+1}{2(m+n)+1} \left(-\frac{4}{n} + \sum_{k=-m}^{m} u_k \right)^p .$$

Sending $n \to \infty$, we obtain

$$\|A\|^p \geq \left(\sum_{k=-m}^{m} u_k \right)^p .$$

Since m was arbitrary, this implies $A \in \mathcal{W}(\mathbf{X})$ and Equation (13) holds. $\square$

One could refine the last part of the proof to get even more cases where A is automatically contained in the Wiener algebra. But since we will not need this in the following, we just keep it as that. The intention of the previous proposition was to demonstrate what usually happens. It shows that the sets U_k are naturally restricted by the existence of pseudo-ergodic/random operators. The next result is another instance of this restriction.

Proposition 4.8. *Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ be band-dominated. Then $\sigma^{\mathrm{op}}(A) = M((U_k)_{k\in\mathbb{Z}})$. In particular, $M((U_k)_{k\in\mathbb{Z}}) \subset \mathrm{BDO}(\mathbf{X})$.*

Proof. Let h be a sequence of integers such that A_h exists. Then $V_{-h_n}AV_{h_n} \to A_h$ $\mathcal{P}$-strongly as $n \to \infty$. Since $V_{-h_n}AV_{h_n} \in M((U_k)_{k\in\mathbb{Z}})$ for all $n \in \mathbb{N}$ and we assumed that the sets U_k are compact, we conclude $A_h \in M((U_k)_{k\in\mathbb{Z}})$.

Conversely, let $B \in M((U_k)_{k\in\mathbb{Z}})$ and $B_n := P_nBP_n$ for $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we can find a $g_n \in \mathbb{Z}$ with $|g_n| > n^2$ such that

$$\left\|P_{\{1,\ldots,2n+1\}}V_{-g_n}AV_{g_n}P_{\{1,\ldots,2n+1\}} - V_{n+1}B_nV_{-(n+1)}\right\| \leq \frac{1}{n}$$

by pseudo-ergodicity (cf. Remark 4.5(b)). Choosing $h_n := g_n + (n+1)$ and observing that $P_{\{1,\ldots,2n+1\}} = V_{n+1}P_nV_{-(n+1)}$, we get

$$\|P_n(V_{-h_n}AV_{h_n} - B)P_n\| = \|P_nV_{-h_n}AV_{h_n}P_n - B_n\| \leq \frac{1}{n}.$$

This implies that $V_{-h_n}AV_{h_n}$ converges entrywise to B as $n \to \infty$. Now clearly, $h := (h_n)_{n\in\mathbb{N}}$ tends to infinity. As mentioned above, A is rich because the sets U_k are compact. Hence there exists a subsequence g of h such that A_g exists. But since $V_{-h_n}AV_{h_n}$ converges entrywise to B, A_g has to be equal to B. Hence, $B \in \sigma^{\mathrm{op}}(A)$.

The last assertion follows immediately from Proposition 3.15. □

Remark 4.9. The above proof also works if we assume $A \in \mathcal{L}_\$(\mathbf{X},\mathcal{P})$ instead of $A \in \mathrm{BDO}(\mathbf{X})$. The point is that entrywise convergence does not necessarily suffice to prove the existence of the corresponding limit operator. Thus we used richness to guarantee the existence and then of course the limit operator has to agree with the entrywise limit. It might very well be possible that one can circumvent this in a different way, though.

Proposition 4.8 can now be used to get the following nice results. All of them are in fact corollaries of the results in Section 3.

Theorem 4.10. *Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ be band-dominated. Then the following is true:*

(i) $\mathrm{sp}(A) = \mathrm{sp}_{\mathrm{ess}}(A) = \bigcup\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}(B)$,

(ii) $\|A\| = \|A + \mathcal{K}(\mathbf{X},\mathcal{P})\| = \max\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} \|B\|$,

(iii) $\mathrm{sp}_\varepsilon(A) = \mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}_\varepsilon(B)$ *for all* $\varepsilon > 0$,

(iv) $\nu(A) = \nu_{\mathrm{ess}}(A) = \min\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} \nu(B)$.

Additionally, if $\mathbf{X}$ is a Hilbert space, then

(v) $N(A) = N_{\mathrm{ess}}(A) = \bigcup\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} N(B)$.

Proof. By Proposition 4.8, we have $\sigma^{\mathrm{op}}(A) = M((U_k)_{k\in\mathbb{Z}})$. The assertions now easily follow from the following results in Section 3:

(*i*) Theorem 3.20 implies $\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{B \in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}(B)$. Since $\mathrm{sp}_{\mathrm{ess}}(A) \subset \mathrm{sp}(A)$ and $A \in M((U_k)_{k\in\mathbb{Z}})$, we get the other equality.

(*ii*), (*iii*) Similar, using Theorem 3.26 and Theorem 3.35.

(*iv*) We have $\sigma^{\mathrm{op}}(A+K) = \sigma^{\mathrm{op}}(A)$ for all $K \in \mathcal{K}(\mathbf{X},\mathcal{P})$. Thus

$$\nu_{\mathrm{ess}}(A) \leq \min_{B \in M((U_k)_{k\in\mathbb{Z}})} \nu(B) \leq \nu(A) \leq \nu_{\mathrm{ess}}(A)$$

follows by Proposition 3.14(*ii*).

(*v*) Again as above, we get $N(A) = N_{\mathrm{ess}}(A) = \mathrm{conv}\left(\bigcup_{B \in M((U_k)_{k\in\mathbb{Z}})} N(B)\right)$ by Theorem 3.76. But since

$$N(A) \subset \bigcup_{B \in M((U_k)_{k\in\mathbb{Z}})} N(B) \subset \mathrm{conv}\left(\bigcup_{B \in M((U_k)_{k\in\mathbb{Z}})} N(B)\right) = N(A),$$

taking the convex hull is clearly unnecessary. □

As a corollary we get the following result that we announced at the beginning of this section.

Corollary 4.11. *Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ be band-dominated. Then all $B \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ share the same spectrum, norm, pseudospectra and lower norm. In the case of a Hilbert space, they also share the same numerical range.*

Proof. By Proposition 4.8, all $B \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ are band-dominated. Thus Theorem 4.10 applies to all $B \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$. □

This leads to the surprising fact that a random operator that satisifies (C1) - (C4) has non-random spectral properties.

Corollary 4.12. *Let $(\Omega,\mathbb{P})$ be a probability space, let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A\colon \Omega \to \mathrm{BDO}(\mathbf{X})$ be a random operator that satisfies the conditions (C1) - (C4). Then*

(i) $\mathrm{sp}(A(\omega)) = \mathrm{sp}_{\mathrm{ess}}(A(\omega)) = \bigcup_{B \in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}(B)$,

(ii) $\|A(\omega)\| = \|A(\omega) + \mathcal{K}(\mathbf{X},\mathcal{P})\| = \max_{B \in M((U_k)_{k\in\mathbb{Z}})} \|B\|$,

(iii) $\mathrm{sp}_\varepsilon(A(\omega)) = \mathrm{sp}_{\varepsilon,\mathrm{ess}}(A(\omega)) = \bigcup_{B \in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}_\varepsilon(B)$ *for all* $\varepsilon > 0$,

(iv) $\nu(A(\omega)) = \nu_{\mathrm{ess}}(A(\omega)) = \min_{B \in M((U_k)_{k\in\mathbb{Z}})} \nu(B)$

for almost every $\omega \in \Omega$. Additionally, if $\mathbf{X}$ is a Hilbert space, then

(v) $N(A(\omega)) = N_{\mathrm{ess}}(A(\omega)) = \bigcup_{B \in M((U_k)_{k\in\mathbb{Z}})} N(B)$

for almost every $\omega \in \Omega$.

4.1.4 On one-sided pseudo-ergodic operators

In the previous sections we considered random operators $A\colon \Omega \to \mathcal{L}(\ell^p(\mathbb{Z}, X))$ and pseudo-ergodic operators $A \in \mathcal{L}(\ell^p(\mathbb{Z}, X))$. This is of course in accordance with the theory presented in Section 3. In applications, however, it is often more convenient to work with one-sided infinite matrices, i.e. $\mathbf{X} = \ell^p(\mathbb{N}, X)$. As it turns out, some properties match the two-sided case and some do not.

First we have to define limit operators for one-sided infinite matrices. The idea is the following. Let $\mathbf{X} = \ell^p(\mathbb{N}, X)$ and $A \in \mathcal{L}(\mathbf{X})$. For $c \in \mathbb{C}$ we consider the operators

$$A_c := \begin{pmatrix} cI & 0 \\ 0 & A \end{pmatrix} \in \mathcal{L}(\tilde{\mathbf{X}}),$$

where $\tilde{\mathbf{X}} = \ell^p(\mathbb{Z}, X)$. Roughly speaking, we identify $\ell^p(\mathbb{Z}, X)$ with $\ell^p(\mathbb{Z} \setminus \mathbb{N}, X) \oplus \ell^p(\mathbb{N}, X)$ (using the p-norm on $\ell^p(\mathbb{Z} \setminus \mathbb{N}, X) \oplus \ell^p(\mathbb{N}, X)$) and extend A to $\tilde{\mathbf{X}}$ by a multiple of the identity.

The definitions are then pretty obvious. We say $A \in \mathcal{L}(\mathbf{X}, \mathcal{P})$ if $A_c \in \mathcal{L}(\tilde{\mathbf{X}}, \mathcal{P})$ for some, hence all, $c \in \mathbb{C}$. Similarly, $A \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ if $A_c \in \mathcal{K}(\tilde{\mathbf{X}}, \mathcal{P})$ for $c = 0$. Limit operators are defined as follows. Let $h = (h_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{N}$ tending to infinity. If the $\mathcal{P}$-strong limit $\mathcal{P}\text{-}\lim\limits_{n \to \infty} V_{-h_n} A_c V_{h_n}$ exists, we call it a limit operator of A and denote it by A_h, i.e. $A_h = (A_c)_h$ if it exists. The set of all limit operators is again called the operator spectrum of A and denoted by $\sigma^{\mathrm{op}}(A)$. Note that A is an operator on $\mathbf{X}$ whereas a limit operator $B \in \sigma^{\mathrm{op}}(A)$ is an operator on $\tilde{\mathbf{X}}$. The notions rich, banded, band-dominated and Wiener algebra are also defined for A in the obvious way.

Remark 4.13. Unless we are talking about $\mathcal{P}$-compact operators, where we have to choose $c = 0$, the notions mentioned above do not depend on c. We want to have this extra freedom there to have an operator A_c that satisfies $\|A_c\| = \|A\|$ and another one that satisfies $\operatorname{sp}(A_c) = \operatorname{sp}(A)$ etc. For example, for the norm we usually choose $c = 0$ whereas for the specturm we want to choose some $c \in \operatorname{sp}(A)$. We will not further mention c as long as it is not important to take a particular one.

Similarly as in the two-sided case, we define the sets $M((U_k)_{k \in \mathbb{Z}})$ and $\Psi\mathrm{E}((U_k)_{k \in \mathbb{Z}})$. To distiguish them from the two-sided variants, we add a $+$ in the subscript.

Definition 4.14. Let $(U_k)_{k \in \mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$. The set

$$\left\{B \in \mathcal{L}(X)^{\mathbb{N} \times \mathbb{N}} : B_{i,j} \in U_{i-j} \text{ for all } i, j \in \mathbb{N}\right\}$$

will be denoted by $M_+((U_k)_{k \in \mathbb{Z}})$.

Definition 4.15. Let $(U_k)_{k \in \mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$. A bounded linear operator $A \in M_+((U_k)_{k \in \mathbb{Z}})$ is called pseudo-ergodic if for all $\varepsilon > 0, n \in \mathbb{N}$ and every $B \in M_n((U_k)_{k \in \mathbb{Z}})$ there exists $m \geq 0$ such that

$$\left\|P_{\{1,\ldots,n\}} V_{-m} A_c V_m P_{\{1,\ldots,n\}} - B\right\|_{\mathcal{L}(X^n)} \leq \varepsilon.$$

The set of all pseudo-ergodic operators in $M_+((U_k)_{k \in \mathbb{Z}})$ will be denoted by $\Psi\mathrm{E}_+((U_k)_{k \in \mathbb{Z}})$.

Note that a pseudo-ergodic operator $A \in \Psi\mathrm{E}_+((U_k)_{k \in \mathbb{Z}})$ can always be extended to a pseudo-ergodic operator $B \in \Psi\mathrm{E}((U_k)_{k \in \mathbb{Z}})$ in the sense that $A_c = P_{\mathbb{N}} B P_{\mathbb{N}} + c(I - P_{\mathbb{N}})$, which is equivalent to $A = P_{\mathbb{N}} B P_{\mathbb{N}}$ if interpreted correctly. This observation comes in handy when we compare the spectral properties of one-sided and two-sided pseudo-ergodic operators.

For $A \in \Psi\mathrm{E}_+((U_k)_{k \in \mathbb{Z}}) \cap \mathrm{BDO}(\mathbf{X})$ we again have that $\sigma^{\mathrm{op}}(A) = M((U_k)_{k \in \mathbb{Z}})$. This can be proven by adapting the proof of Proposition 4.8.

Proposition 4.16. *Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A \in \Psi\mathrm{E}_+((U_k)_{k\in\mathbb{Z}})$ be band-dominated. Then $\sigma^{\mathrm{op}}(A) = M((U_k)_{k\in\mathbb{Z}})$. In particular, $M((U_k)_{k\in\mathbb{Z}}) \subset \mathrm{BDO}(\mathbf{X})$.*

We have the following analogue of Theorem 4.10. Note that $\mathrm{sp}(A) \neq \mathrm{sp}_{\mathrm{ess}}(A)$, $\nu_{\mathrm{ess}}(A) \neq \min\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} \nu(B)$ and $\mathrm{sp}_\varepsilon(A) \neq \mathrm{sp}_{\varepsilon,\mathrm{ess}}(A)$ in general (see Example 4.18).

Theorem 4.17. *Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A \in \Psi\mathrm{E}_+((U_k)_{k\in\mathbb{Z}})$ be band-dominated. Then the following is true:*

(i) $\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}(B)$,

(ii) $\|A\| = \|A + \mathcal{K}(\mathbf{X},\mathcal{P})\| = \max\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} \|B\|$,

(iii) $\mathrm{sp}_{\varepsilon,\mathrm{ess}}(A) = \bigcup\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}_\varepsilon(B)$ *for all* $\varepsilon > 0$,

Additionally, if $\mathbf{X}$ *is a Hilbert space, then*

(iv) $N(A) = N_{\mathrm{ess}}(A) = \bigcup\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} N(B)$.

Proof. (i) Choose $c \in \mathbb{C}$ such that $c \notin \mathrm{sp}_{\mathrm{ess}}(A) \cup \bigcup\limits_{B\in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}(B)$. Then

$$\mathrm{sp}_{\mathrm{ess}}(A) \cup \{c\} = \mathrm{sp}_{\mathrm{ess}}(A_c) = \bigcup_{B\in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}(B) \cup \mathrm{sp}(cI) = \bigcup_{B\in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}(B) \cup \{c\}$$

by Theorem 3.20 and Proposition 4.16. This implies

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{B\in M((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}(B)$$

because c is not contained in one of the sets.

(ii) Let $B \in M((U_k)_{k\in\mathbb{Z}})$ be an extension of A, i.e. $A = P_\mathbb{N} B P_\mathbb{N}$ restricted to $\ell^2(\mathbb{N}, X)$. This implies $\|A\| \leq \|B\|$. Choosing $c = 0$, we get

$$\|A\| \leq \|B\| \leq \max_{B\in M((U_k)_{k\in\mathbb{Z}})} \|B\| = \|A_c + \mathcal{K}(\tilde{\mathbf{X}},\mathcal{P})\| = \|A + \mathcal{K}(\mathbf{X},\mathcal{P})\| \leq \|A\|$$

by Theorem 3.26 and Proposition 4.16 (cf. Proposition 3.51 for the equality of the essential norms).

(iii) Similar as (i), using Theorem 3.35 instead.

(iv) Choose $c \in N_{\mathrm{ess}}(A)$. It is not difficult to see that $N(C \oplus D) = \mathrm{conv}(N(C) \cup N(D))$ for $C \oplus D \in \mathcal{L}(\ell^2(\mathbb{Z} \setminus \mathbb{N}, X) \oplus \ell^2(\mathbb{N}, X)) = \mathcal{L}(\ell^2(\mathbb{Z}, X)) = \mathcal{L}(\tilde{\mathbf{X}})$. Indeed, let $P_\mathbb{N}$ be the orthogonal projection from $\tilde{\mathbf{X}}$ onto $\mathbf{X}$ and $Q_\mathbb{N} = I - P_\mathbb{N}$ as usual. Then

$$\begin{aligned}\langle (C_1 \oplus C_2)x, x\rangle &= \langle C_1 P_\mathbb{N} x, P_\mathbb{N} x\rangle + \langle C_2 Q_\mathbb{N} x, Q_\mathbb{N} x\rangle \\ &= \left\langle C_1 \frac{P_\mathbb{N} x}{\|P_\mathbb{N} x\|}, \frac{P_\mathbb{N} x}{\|P_\mathbb{N} x\|}\right\rangle \|P_\mathbb{N} x\|^2 + \left\langle C_1 \frac{Q_\mathbb{N} x}{\|Q_\mathbb{N} x\|}, \frac{Q_\mathbb{N} x}{\|Q_\mathbb{N} x\|}\right\rangle \|Q_\mathbb{N} x\|^2. \end{aligned} \tag{14}$$

In our case we have

$$N(cI \oplus A + 0 \oplus K) = \operatorname{conv}(N(cI) \cup N(A+K)) = \operatorname{conv}(\{c\} \cup N(A+K)) = N(A+K)$$

for $K \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ since $c \in N_{\text{ess}}(A)$. It follows $N_{\text{ess}}(A_c) \subset N_{\text{ess}}(A)$. Let $B \in M((U_k)_{k\in\mathbb{Z}})$ be an extension of A, i.e. $A = P_{\mathbb{N}} B P_{\mathbb{N}}$ restricted to $\ell^2(\mathbb{N}, X)$. Then

$$\begin{aligned} N(A) &= \operatorname{clos}\{\langle Ax, x\rangle : x \in \mathbf{X}, \|x\| = 1\} = \operatorname{clos}\left\{\langle P_{\mathbb{N}} B P_{\mathbb{N}} x, x\rangle : x \in \tilde{\mathbf{X}}, \|P_{\mathbb{N}} x\| = 1\right\} \\ &= \operatorname{clos}\left\{\langle B P_{\mathbb{N}} x, P_{\mathbb{N}} x\rangle : x \in \tilde{\mathbf{X}}, \|P_{\mathbb{N}} x\| = 1\right\} \subset N(B). \end{aligned}$$

Using again Proposition 4.16 and Theorem 3.76, we get

$$N(A) \subset N(B) \subset \bigcup_{B \in M((U_k)_{k\in\mathbb{Z}})} N(B) \subset N_{\text{ess}}(A_c) \subset N_{\text{ess}}(A) \subset N(A). \qquad \square$$

Example 4.18. In this example we show that $\operatorname{sp}(A) \neq \operatorname{sp}_{\text{ess}}(A)$, $\nu_{\text{ess}}(A) \neq \min\limits_{B \in M((U_k)_{k\in\mathbb{Z}})} \nu(B)$ and $\operatorname{sp}_\varepsilon(A) \neq \operatorname{sp}_{\varepsilon,\text{ess}}(A)$ in general. Let $\mathbf{X} = \ell^2(\mathbb{N})$ and let $A \in \mathcal{L}(\mathbf{X})$ be given by $(Ax)_i = x_{i+1}$ for all $i \in \mathbb{N}$ and $x \in \mathbf{X}$. Then trivially $A \in \Psi\mathrm{E}_+(U_{-1})$, where $U_{-1} = \{1\}$. Furthermore, $\sigma^{\text{op}}(A) = \{V_{-1}\}$.

It is well-known that $\operatorname{sp}(V_{-1}) = \mathbb{T}$ and $\operatorname{sp}(A) = \mathbb{D}$ (see e.g. Section 4.2.2 below). Thus by 4.17(i),

$$\operatorname{sp}(A) = \mathbb{D} \neq \mathbb{T} = \operatorname{sp}(V_{-1}) = \operatorname{sp}_{\text{ess}}(A).$$

Also, A is surjective and $\ker(A) = \operatorname{span}\{e_1\}$. This implies $\nu(A+K) = 0$ for all $K \in \mathcal{K}(\mathbf{X})$ by Theorem 2.2(ii), hence $\nu_{\text{ess}}(A) = 0$. But V_{-1} is an isometry, so $\nu(V_{-1}) = 1$.

To prove $\operatorname{sp}_\varepsilon(A) \neq \operatorname{sp}_{\varepsilon,\text{ess}}(A)$ for $\varepsilon < 1$, we observe that V_{-1} is unitary, i.e. $V_{-1}V_{-1}^* = V_{-1}^*V_{-1} = I$. This allows us to compute the resolvent norms:

$$\left\|(V_{-1} - \lambda I)^{-1}\right\| = \rho((V_{-1} - \lambda I)^{-1}) = \operatorname{dist}(0, \operatorname{sp}(V_{-1} - \lambda I))^{-1} = \operatorname{dist}(\lambda, \operatorname{sp}(V_{-1}))^{-1} = (1 - |\lambda|)^{-1}$$

for $|\lambda| < 1$, where we used Theorem 3.72 for the first equality. So for $\varepsilon < 1$ we have that $(1-\varepsilon)\mathbb{D}$ is not contained in $\operatorname{sp}_\varepsilon(V_{-1})$ whereas by definition, $\mathbb{D} = \operatorname{sp}(A) \subset \operatorname{sp}_\varepsilon(A)$. We conclude $\operatorname{sp}_\varepsilon(A) \neq \operatorname{sp}_{\varepsilon,\text{ess}}(A)$ for $\varepsilon < 1$.

In fact, the counterexample above (obviously) works for every operator $A \in \Psi\mathrm{E}_+((U_k)_{k\in\mathbb{Z}})$ that is Fredholm with $\operatorname{ind}(A) > 0$, although an explicit computation of the pseudospectra may be tedious to say the least.

The operator A used in the previous example is an example of a Toeplitz operator, which is just a one-sided Laurent operator. Toeplitz operators are extensively studied and have applications in various fields. Here we use them, along with Laurent operators, in examples and for explicit computations because they are the easiest pseudo-ergodic operators one can think of. So in particular, this is an extension of the theory of Toeplitz operators. This is the reason why pseudo-ergodic operators are sometimes called stochastic Toeplitz operators (e.g. in [90]). We refer to [8] for an introduction to Toeplitz operators.

As seen in the above example, we do not have $\operatorname{sp}_{\text{ess}}(A) = \operatorname{sp}(A)$ in general. So the question would be: "What has to be added to $\operatorname{sp}_{\text{ess}}(A)$ in order to get $\operatorname{sp}(A)$?". In the case of a band-dominated Toeplitz operator, the answer is that one has to fill one or the other hole in the essential spectrum (see e.g. [8, Theorem 2.47]). We cannot answer the question in general. However, we can at least restrict the numbers $\alpha(A)$ and $\beta(A)$ in the case of a pseudo-ergodic operator $A \in \mathrm{BO}(\ell^p(\mathbb{Z}))$. The main idea was discovered by Chandler-Wilde and Lindner in [20]. In the tridiagonal case discussed there, this result allows us to answer the question for many pseudo-ergodic operators.

Theorem 4.19. *(extension of [20, Theorem 2.2])*
Let $\mathbf{X} = \ell^p(\mathbb{N})$, *let* $n \leq m$ *be integers and let* $U_n, \ldots, U_m \subset \mathbb{C}$ *be non-empty compact sets. If* $A \in \Psi\mathrm{E}_+(U_n, \ldots, U_m)$ *is Fredholm, then all* $B \in M_+(U_n, \ldots, U_m)$ *are Fredholm with* $\alpha(B) + \beta(B) \leq \max\{|n|, |m|\}$. *In particular, if* A *is tridiagonal, all* $B \in M_+(U_n, \ldots, U_m)$ *are injective or surjective.*

Proof. Let $B \in M_+(U_n, \ldots, U_m)$. By Theorem 4.17($i$), all $C \in M(U_n, \ldots, U_m)$ are invertible, which in turn implies that B is Fredholm.

W.l.o.g. we can assume that $n = -m$. Let $B^T \colon \ell^q(\mathbb{N}) \to \ell^q(\mathbb{N})$, where $\frac{1}{p} + \frac{1}{q} = 1$, denote the transpose of B, i.e. $B^T_{i,j} = B_{j,i}$ for all $i, j \in \mathbb{N}$. It is not difficult to see that B^T is exactly the (Banach space) adjoint of B (pre-adjoint if $p = \infty$) and hence $\beta(B) = \alpha(B^T)$ (cf. Section 2.1). Let $J \colon \ell^q(\mathbb{N}) \to \ell^q(-\mathbb{N})$ denote the flip operator, i.e. $(Jx)_i = x_{-i}$ for every $i \in \mathbb{N}$ and $x \in \ell^q(\mathbb{N})$. Since B and JB^TJ^{-1} are band operators, we may consider them as operators on $\ell^\infty(\mathbb{N})$ and $\ell^\infty(-\mathbb{N})$, respectively. It thus makes sense to consider the following matrix:

$$C = \left(\begin{array}{ccc|ccc|ccc}
\multicolumn{3}{c|}{JB^TJ^{-1}} & * & & & & & \\
 & & & \vdots & \ddots & & & & \\
 & & & * & \cdots & * & & & \\
\hline
u_{n,1} & \cdots & u_{1,1} & * & \cdots & * & u_{-n,1} & & \\
 & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \\
 & & u_{n,n} & * & \cdots & * & u_{-1,n} & \cdots & u_{-n,n} \\
\hline
 & & & * & \cdots & * & & & \\
 & & & & \ddots & \vdots & & B & \\
 & & & & & * & & &
\end{array}\right) \in \mathrm{BO}(\ell^\infty(\mathbb{Z})),$$

where the stars can be chosen arbitrarily in their respective sets, i.e. elements from U_k in the k-th diagonal. Similarly, the elements $u_{i,j}$ can be chosen arbitrarily in U_i. The difference is that we need the entries $u_{i,j}$ for further reference. Let $x \in \ell^p(\mathbb{N})$ and $y \in \ell^q(-\mathbb{N})$ satisfy $Bx = 0$ and $JB^TJ^{-1}y = 0$, respectively. Then

$$\left(\begin{array}{ccc|ccc|ccc}
\multicolumn{3}{c|}{JB^TJ^{-1}} & * & & & & & \\
 & & & \vdots & \ddots & & & & \\
 & & & * & \cdots & * & & & \\
\hline
u_{n,1} & \cdots & u_{1,1} & * & \cdots & * & u_{-n,1} & & \\
 & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \\
 & & u_{n,n} & * & \cdots & * & u_{-1,n} & \cdots & u_{-n,n} \\
\hline
 & & & * & \cdots & * & & & \\
 & & & & \ddots & \vdots & & B & \\
 & & & & & * & & &
\end{array}\right)
\left(\begin{array}{c} y \\ \hline 0 \\ \hline x \end{array}\right)
=
\left(\begin{array}{c} 0 \\ \hline 0 \\ \hline 0 \end{array}\right)$$

is equivalent to the linear system

$$\begin{pmatrix} u_{n,1} & \dots & u_{1,1} & u_{-n,1} & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & u_{n,n} & u_{-1,n} & \dots & u_{-n,n} \end{pmatrix} \begin{pmatrix} y_n \\ \vdots \\ y_1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The latter is a linear system with n equations. It is thus not difficult to see that if $\dim(\ker(B)) + \dim(\ker(JB^TJ^{-1})) > n$, C cannot be injective (as an operator on $\ell^\infty(\mathbb{Z})$). Indeed, if both $\ker(B))$ and $\dim(\ker(JB^TJ^{-1}))$ are non-trivial, $\dim(\ker(B))+\dim(\ker(JB^TJ^{-1})) > n$ implies that we have a non-trivial solution. If either $\ker(B))$ or $\dim(\ker(JB^TJ^{-1}))$ is trivial, we can choose $x = 0$ and $y \in \ker(JBJ^{-1})$ such that $y_1 = \ldots = y_n = 0$ or vice versa. But as A is Fredholm, $C \in M(U_n, \ldots, U_m)$ has to be invertible (on all spaces $\ell^p(\mathbb{Z})$ by the inverse closedness of the Wiener algebra, see [75, Theorem 2.5.2]) by Theorem 4.17(i), a contradiction. We conclude $\alpha(B) + \alpha(JB^TJ^{-1}) \leq n$ and thus

$$\alpha(B) + \beta(B) = \alpha(B) + \alpha(B^T) = \alpha(B) + \alpha(JB^TJ^{-1}) \leq n$$

as claimed. □

So in the case of a tridiagonal operator, a Fredholm operator $A \in \Psi\mathrm{E}_+(U_{-1}, U_0, U_1)$ is at least injective or surjective. For Toeplitz operators this is known as Coburn's lemma [22] and holds for much more general Toeplitz operators. Although this comes along as an innocent result, Theorem 4.19 has huge consequences for one-sided tridiagonal pseudo-ergodic operators. Without going into too much detail, we just quote the corollaries from [20]. The first one uses [72, Theorem 1.2], an index result obtained via K-theory.

Corollary 4.20. *([20, Corollary 2.4])*
Let $\mathbf{X} = \ell^p(\mathbb{N})$ and let $U_{-1}, U_0, U_1 \subset \mathbb{C}$ be non-empty compact sets. If $A \in \Psi\mathrm{E}_+(U_{-1}, U_0, U_1)$ is Fredholm, then all $B \in M_+(U_{-1}, U_0, U_1)$ are Fredholm with the same index $\kappa(U_{-1}, U_0, U_1) :=$ $\operatorname{ind}(A) \in \{-1, 0, 1\}$.

Corollary 4.21. *([20, Corollary 2.5])*
Let $\mathbf{X} = \ell^p(\mathbb{N})$, let $U_{-1}, U_0, U_1 \subset \mathbb{C}$ be non-empty compact sets and let $A \in \Psi\mathrm{E}_+(U_{-1}, U_0, U_1)$. Then A is invertible if and only if all $B \in M_+(U_{-1}, U_0, U_1)$ are invertible. Furthermore,

$$\operatorname{sp}(A) = \bigcup_{B \in M_+(U_{-1}, U_0, U_1)} \operatorname{sp}(B).$$

So the latter corollary is an analogue of Theorem 4.10(i) in the tridiagonal case. The next corollary answers our question "What has to be added to $\operatorname{sp}_{\mathrm{ess}}(A)$ in order to get $\operatorname{sp}(A)$?" for $A \in \Psi\mathrm{E}_+(U_{-1}, U_0, U_1)$. Although this is already covered in [20], we add a proof for the reader's convenience.

Corollary 4.22. *([20, Theorem 2.7])*
Let $\mathbf{X} = \ell^p(\mathbb{N})$, let $U_{-1}, U_0, U_1 \subset \mathbb{C}$ be non-empty compact sets and let $A \in \Psi\mathrm{E}_+(U_{-1}, U_0, U_1)$.

Then A is invertible if and only if A is Fredholm and one $B \in M_+(U_{-1}, U_0, U_1)$ is invertible. Furthermore,

$$\operatorname{sp}(A) = \operatorname{sp}_{\mathrm{ess}}(A) \cup \bigcap_{B \in M_+(U_{-1},U_0,U_1)} \operatorname{sp}(B).$$

Moreover, if $|u_{-1}| \geq |u_1|$ and $|v_{-1}| \leq |v_1|$ for some $u_{-1}, v_{-1} \in U_{-1}$, $u_1, v_1 \in U_1$, then $\operatorname{sp}(A) = \operatorname{sp}_{\mathrm{ess}}(A)$.

Proof. If A is invertible, then obviously A is Fredholm and all $B \in M_+(U_{-1}, U_0, U_1)$ are invertible by Corollary 4.21. Conversely, if A is Fredholm and $B \in M_+(U_{-1}, U_0, U_1)$ is invertible, then $\operatorname{ind}(A) = \operatorname{ind}(B) = 0$ by Corollary 4.20, hence A is invertible by Theorem 4.19. The statement about the spectra follows by considering the operators $A - \lambda I$ and $B - \lambda I$.

To prove the last statement, take u_{-1}, u_1, v_{-1} and v_1 from the assumptions and arbitrary $u_0, v_0 \in U_0$. Assume that A is Fredholm and let $T_1 \in M_+(U_{-1}, U_0, U_1)$ be the Toeplitz operator with the entries u_{-1}, u_0 and u_1 on the respective diagonals. Similarly, let $T_2 \in M_+(U_{-1}, U_0, U_1)$ be the Toeplitz operator with the entries v_{-1}, v_0 and v_1 on the respective diagonals. By the index formula for Toeplitz operators (see e.g. [8, Theorem 2.47] or Section 4.2 below), T_1 and T_2 are Fredholm with $\operatorname{ind}(T_1) \in \{0, 1\}$ and $\operatorname{ind}(T_2) \in \{-1, 0\}$. It follows $\operatorname{ind}(A) = \operatorname{ind}(T_1) = \operatorname{ind}(T_2) = 0$ by Corollary 4.20, hence A is invertible by Theorem 4.19. Thus A is invertible if and only if A is Fredholm. The same argument applied to $A - \lambda I$, $T_1 - \lambda I$ and $T_2 - \lambda I$ implies $\operatorname{sp}(A) = \operatorname{sp}_{\mathrm{ess}}(A)$. □

4.2 Periodic operators

Theorem 4.10 provides a somewhat easy method to construct lower bounds for the spectral quantities of a pseudo-ergodic operator. Indeed, we can take the spectral quantities of any operator $B \in M((U_k)_{k\in\mathbb{Z}})$ and obtain a lower bound. It is therefore important to talk about operators for which we can actually (numerically) compute the spectral quantities. The operators we are going to talk about here are periodic operators, which we mentioned already a few times. Let us first make the definition precise here.

Definition 4.23. Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and $m \in \mathbb{N}$. A bounded linear operator $A \in M((U_k)_{k\in\mathbb{Z}})$ is called m-periodic if $A_{i,j} = A_{i+m,j+m}$ for all $i, j \in \mathbb{Z}$. The set of m-periodic operators is denoted by $M_{per,m}((U_k)_{k\in\mathbb{Z}})$. The set of all periodic operators is denoted by $M_{per}((U_k)_{k\in\mathbb{Z}})$. Similarly we define the one-sided m-periodic operators $M_{per,m,+}((U_k)_{k\in\mathbb{Z}})$ and the set of all one-sided periodic operators $M_{per,+}((U_k)_{k\in\mathbb{Z}})$.

Two-sided 1-periodic operators are called (block) Laurent operators and will be denoted by $L((U_k)_{k\in\mathbb{Z}})$. Similarly, one-sided 1-periodic operators are called (block) Toeplitz operators and denoted by $T((U_k)_{k\in\mathbb{Z}})$.

4.2.1 Approximation of spectral quantities

For $p \notin \{1, 2, \infty\}$ it is hard to compute the norm even in the case of a finite matrix (NP-hard to be precise, see [44]). So the attempt to compute norms (and lower norms, pseudospectra etc.) of infinite matrices is probably a bit too ambitious most of the time (see also [5]). In the Hilbert space case ($p = 2$) these tasks get a bit easier as we will see in Section 4.2.2. For the spectrum of an operator in the Wiener algebra the results carry over to the general case because the spectrum (unlike the norm for example) does not depend on p in this case (see [75, Theorem 2.5.2]).

Despite these computability problems, we still have some approximation results for band-dominated pseudo-ergodic operators.

Theorem 4.24. *Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ be band-dominated. Then we have the following improvement of Theorem 4.10(ii):*

$$\|A\| = \sup_{B\in M_{per}((U_k)_{k\in\mathbb{Z}})} \|B\| .$$

If $A \in \Psi\mathrm{E}_+((U_k)_{k\in\mathbb{Z}})$ is band-dominated, then

$$\|A\| = \sup_{C\in M_{per,+}((U_k)_{k\in\mathbb{Z}})} \|C\| = \sup_{B\in M_{per}((U_k)_{k\in\mathbb{Z}})} \|B\| .$$

To prove this, we need the following variant of the Banach-Steinhaus theorem.

Lemma 4.25. *([75, Proposition 1.1.17])*
Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(\mathbf{X},\mathcal{P})$ that converges $\mathcal{P}$-strongly to $A \in \mathcal{L}(\mathbf{X},\mathcal{P})$. Then

$$\|A\| \le \liminf_{n\to\infty} \|A_n\| .$$

Proof of Theorem 4.24. It is clear that all $B \in M_{per}((U_k)_{k\in\mathbb{Z}})$ are band-dominated by Proposition 4.8. It is also easy to find a sequence $(B_n)_{n\in\mathbb{N}}$ in $M_{per}((U_k)_{k\in\mathbb{Z}})$ that converges $\mathcal{P}$-strongly to A (just take an increasing number of columns of A and extend them periodically). Applying Lemma 4.25 and Theorem 4.10(ii) yields

$$\|A\| \le \liminf_{n\to\infty} \|B_n\| \le \sup_{B\in M_{per}((U_k)_{k\in\mathbb{Z}})} \|B\| \le \|A\| .$$

For the second assertion observe that for every $C \in M_{per,+}((U_k)_{k\in\mathbb{Z}})$ we can find a $B \in M_{per}((U_k)_{k\in\mathbb{Z}})$ such that $C = P_{\mathbb{N}}BP_{\mathbb{N}}$ (interpreted as an operator on $\ell^p(\mathbb{N},X)$). This implies $\|C\| \le \|B\|$ (in fact, $\|C\| = \|B\|$ because B is a limit operator of C). Thus we get

$$\|A\| \le \liminf_{n\to\infty} \|C_n\| \le \sup_{C\in M_{per,+}((U_k)_{k\in\mathbb{Z}})} \|C\| \le \sup_{B\in M_{per}((U_k)_{k\in\mathbb{Z}})} \|B\| \le \|A\|$$

as above by taking a suitable sequence $(C_n)_{n\in\mathbb{N}}$ in $M_{per,+}((U_k)_{k\in\mathbb{Z}})$ and applying Theorem 4.17(ii). $\square$

Remark 4.26. The same can be done with finite matrices. Choose a sequence $(D_n)_{n\in\mathbb{N}}$ in $M_{fin}((U_k)_{k\in\mathbb{Z}})$ that converges $\mathcal{P}$-strongly to A (for example $D_n = P_nAP_n$). For every $D \in M_{fin}((U_k)_{k\in\mathbb{Z}})$ we can find a $C \in M_{per,+}((U_k)_{k\in\mathbb{Z}})$ such that $D = P_nCP_n$ for some $n \in \mathbb{N}$, hence $\|D\| \le \|C\|$. This implies

$$\|A\| \le \liminf_{n\to\infty} \|D_n\| \le \sup_{D\in M_{fin}((U_k)_{k\in\mathbb{Z}})} \|D\| \le \sup_{C\in M_{per,+}((U_k)_{k\in\mathbb{Z}})} \|C\| = \|A\| .$$

In the two-sided case we can also state Theorem 4.24 for the lower norm. Unfortunatly we are unable to cover the one-sided case here. This is because we do not have Theorem 4.17 for the lower norm (cf. also Example 4.18).

Theorem 4.27. *Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ be band-dominated. Then we have the following improvement of Theorem 4.10(iv):*

$$\nu(A) = \inf_{B\in M_{per}((U_k)_{k\in\mathbb{Z}})} \nu(B).$$

For this we need an analogue of Lemma 4.25, of course.

Lemma 4.28. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(\mathbf{X},\mathcal{P})$ that converges $\mathcal{P}$-strongly to $A \in \mathcal{L}(\mathbf{X},\mathcal{P})$. Then*

$$\nu(A) \geq \limsup_{n\to\infty} \nu(A_n).$$

Proof. By Lemma 3.13(i), we have $\nu(A_0) = \nu(A)$. For every $\varepsilon \in (0,1)$ we can thus choose an $x \in \mathbf{X}_0 = \{x \in \mathbf{X} : \|Q_n x\| \to 0 \text{ as } n \to \infty\}$ with $\|x\| = 1$ such that $\|Ax\| - \nu(A) < \delta := \frac{\varepsilon}{1+\|A\|+\nu(A)+\varepsilon}$. Moreover, we can choose $m \in \mathbb{N}$ such that $\|Q_m x\| < \delta$. It follows

$$\frac{\|AP_m x\|}{\|P_m x\|} < \frac{\|Ax\| + \|AQ_m x\|}{1-\delta} < \frac{\nu(A) + \delta + \|A\|\,\delta}{1-\delta} = \nu(A) + \frac{\delta(1+\|A\|+\nu(A))}{1-\delta} = \nu(A) + \varepsilon.$$

Since $(A_n)_{n\in\mathbb{N}}$ converges $\mathcal{P}$-strongly to A, there exists $n_0 \in \mathbb{N}$ such that $\|(A_n - A)P_m\| < \varepsilon(1-\delta)$ for all $n \geq n_0$. It follows

$$\nu(A_n) \leq \frac{\|A_n P_m x\|}{\|P_m x\|} \leq \frac{\|(A_n - A)P_m x\|}{\|P_m x\|} + \frac{\|AP_m x\|}{\|P_m x\|} < \frac{\|(A_n - A)P_m\|}{1-\delta} + \nu(A) + \varepsilon < \nu(A) + 2\varepsilon$$

for all $n \geq n_0$. Thus $\limsup\limits_{n\to\infty} \nu(A_n) \leq \nu(A)$ follows. □

Proof of Theorem 4.27. Taking the same sequence as in the proof of Theorem 4.24, we get

$$\nu(A) \geq \limsup_{n\to\infty} \nu(B_n) \geq \inf_{B\in M_{per}((U_k)_{k\in\mathbb{Z}})} \nu(B) \geq \nu(A),$$

where we applied Lemma 4.28 and Theorem 4.10(iv). □

Of course, this implies the following corollary.

Corollary 4.29. *Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ be band-dominated. Then we have the following improvement of Theorem 4.10(iii):*

$$\mathrm{sp}_\varepsilon(A) = \bigcup_{B\in M_{per}((U_k)_{k\in\mathbb{Z}})} \mathrm{sp}_\varepsilon(B)$$

for every $\varepsilon > 0$.

Proof. We have

$$\mathrm{sp}_\varepsilon(A) = \left\{\lambda \in \mathbb{C} : \left\|(A-\lambda I)^{-1}\right\| > 1/\varepsilon\right\} = \left\{\lambda \in \mathbb{C} : \min\left\{\nu(A-\lambda I), \nu((A-\lambda I)^*)\right\} < \varepsilon\right\}$$

by Corollary 2.9. Now first assume that $p < \infty$. Applying Theorem 4.27 to $A - \lambda I$ and $(A - \lambda I)^*$, we get

$$\begin{aligned}
\operatorname{sp}_\varepsilon(A) &= \left\{\lambda \in \mathbb{C} : \min\left\{\inf_{B \in M_{per}((U_k)_{k\in\mathbb{Z}})} \nu(B - \lambda I), \inf_{B \in M_{per}((U_k)_{k\in\mathbb{Z}})} \nu((B - \lambda I)^*)\right\} < \varepsilon\right\} \\
&= \left\{\lambda \in \mathbb{C} : \inf_{B \in M_{per}((U_k)_{k\in\mathbb{Z}})} \min\left\{\nu(B - \lambda I), \nu((B - \lambda I)^*)\right\} < \varepsilon\right\} \\
&= \left\{\lambda \in \mathbb{C} : \sup_{B \in M_{per}((U_k)_{k\in\mathbb{Z}})} \left\|(B - \lambda I)^{-1}\right\| > 1/\varepsilon\right\} \\
&= \bigcup_{B \in M_{per}((U_k)_{k\in\mathbb{Z}})} \operatorname{sp}_\varepsilon(B).
\end{aligned}$$

If $p = \infty$, we use Lemma 3.13(iii) to get

$$\operatorname{sp}_\varepsilon(A) = \left\{\lambda \in \mathbb{C} : \left\|(A - \lambda I)^{-1}\right\| > 1/\varepsilon\right\} = \left\{\lambda \in \mathbb{C} : \left\|(A_0 - \lambda I_0)^{-1}\right\| > 1/\varepsilon\right\}.$$

Now we are in the case $p = 0$ and can apply the argument above. □

We also have a similar result for the numerical range in the case where $\mathbf{X}$ is a Hilbert space (i.e. $p = 2$ and X a Hilbert space):

Theorem 4.30. *Let $(U_k)_{k\in\mathbb{Z}}$ be a sequence of non-empty compact subsets of $\mathcal{L}(X)$ and let $A \in \Psi\mathrm{E}((U_k)_{k\in\mathbb{Z}})$ be band-dominated. Then we have the following improvement of Theorem 4.10(v):*

$$N(A) = \operatorname{clos}\left(\bigcup_{B \in M_{per}((U_k)_{k\in\mathbb{Z}})} N(B)\right).$$

If $A \in \Psi\mathrm{E}_+((U_k)_{k\in\mathbb{Z}})$ is band-dominated, then

$$N(A) = \operatorname{clos}\left(\bigcup_{C \in M_{per,+}((U_k)_{k\in\mathbb{Z}})} N(C)\right) = \operatorname{clos}\left(\bigcup_{B \in M_{per}((U_k)_{k\in\mathbb{Z}})} N(B)\right).$$

Proof. Again we can take the same sequence as in the proof of Theorem 4.24. Clearly, this sequence is bounded by $\|A\|$. Applying Lemma 3.75 and Theorem 4.10(v) yields

$$N(A) \subset \liminf_{n\to\infty} N(B_n) \subset \operatorname{clos}\left(\bigcup_{B \in M_{per}((U_k)_{k\in\mathbb{Z}})} N(B)\right) \subset N(A).$$

As in the proof of Theorem 4.24, every $C \in M_{per,+}((U_k)_{k\in\mathbb{Z}})$ can be written as $P_\mathbb{N} B P_\mathbb{N}$ for some $B \in M_{per}((U_k)_{k\in\mathbb{Z}})$. Since

$$\langle Cx, x\rangle = \langle P_\mathbb{N} B P_\mathbb{N} x, x\rangle = \langle B P_\mathbb{N} x, P_\mathbb{N} x\rangle,$$

for every $x \in \mathbf{X}$, we have $N(C) \subset N(B)$ (in fact, $N(C) = N(B)$ because B is a limit operator of C). Thus we get

$$N(A) \subset \liminf_{n\to\infty} N(C_n) \subset \operatorname{clos}\left(\bigcup_{C \in M_{per,+}((U_k)_{k\in\mathbb{Z}})} N(C)\right) \subset \operatorname{clos}\left(\bigcup_{B \in M_{per}((U_k)_{k\in\mathbb{Z}})} N(B)\right) \subset N(A)$$

as above by taking a suitable sequence $(C_n)_{n\in\mathbb{N}}$ in $M_{per,+}((U_k)_{k\in\mathbb{Z}})$ and applying Theorem 4.17(iv). □

Remark 4.31. Again, the same can be done with finite matrices. Choose a sequence $(D_n)_{n\in\mathbb{N}}$ in $M_{fin}((U_k)_{k\in\mathbb{Z}})$ that converges entrywise to A (for example $D_n = P_nAP_n$). Clearly, this sequence is again bounded by $\|A\|$. For every $D \in M_{fin}((U_k)_{k\in\mathbb{Z}})$ we can find a $C \in M_{per,+}((U_k)_{k\in\mathbb{Z}})$ such that $D = P_nCP_n$ for some $n \in \mathbb{N}$, hence $N(D) \subset N(C)$ as above. This implies

$$N(A) \subset \liminf_{n\to\infty} N(D_n) \subset \operatorname{clos}\left(\bigcup_{D\in M_{fin}((U_k)_{k\in\mathbb{Z}})} N(D)\right) \subset \operatorname{clos}\left(\bigcup_{C\in M_{per,+}((U_k)_{k\in\mathbb{Z}})} N(C)\right) = N(A).$$

So the only thing left is the spectrum, which is the main topic for the rest of this thesis. From many perspectives the spectrum is the most interesting spectral quantity, but, as it turns out, also the hardest to treat. Roughly speaking, this is because the spectrum has the worst continuity properties of all spectral quantities discussed here. Therefore we restrict ourselves to the simple case $\mathbf{X} = \ell^p(\mathbb{Z})$ here. In this case we can apply what is called the symbol calculus.

4.2.2 Symbols

It is well-known that in the case $\mathbf{X} = \ell^p(\mathbb{Z})$ the spectrum of a Laurent operator can be derived with the help of its symbol. The symbol of a Laurent operator $A \in \mathcal{L}(\mathbf{X})$ is defined by

$$a\colon [0,2\pi) \to \mathbb{C}, \quad \varphi \mapsto a(\varphi) := \sum_{k=-\infty}^{\infty} a_k e^{ik\varphi},$$

where a_k is the only element on the k-th diagonal of A. The symbol is well-defined by [8, Proposition 2.4] (note that the cases $p = 0, 1, \infty$ are already covered by Proposition 4.7). Furthermore, if $A \in \operatorname{BDO}(\mathbf{X})$, then a describes a closed curve, i.e. it is continuous and can be continuously extended to $a\colon [0,2\pi] \to \mathbb{C}$ with $\lim\limits_{\varphi\to 2\pi} a(\varphi) = a(0)$ (see [8, Section 2.5], cf. also [57, Remark 1.40]). Now A is invertible if and only if

$$b\colon [0,2\pi) \to \mathbb{C}, \quad \varphi \mapsto b(\varphi) := 1/a(\varphi)$$

is the symbol of a Laurent operator $B \in \mathcal{L}(\mathbf{X})$ (see e.g. [8, Proposition 2.28]). In this case, B is the inverse of A.

In the case $A \in \operatorname{BDO}(\mathbf{X})$ this statement can be simplified to

$$A \text{ invertible} \iff a(\varphi) \neq 0 \quad \forall \varphi \in [0,2\pi)$$

(see [8, Proposition 2.46] for $p \in [1,\infty)$, the remaining cases can be shown using a duality argument or Proposition 4.7). This also implies

$$\operatorname{sp}(A) = \operatorname{im}(a).$$

If $p = 2$, the Fourier transform

$$\mathcal{F}\colon L^2([0,2\pi)) \to \ell^2(\mathbb{Z}), \quad a \mapsto \mathcal{F}(a) := (a_k)_{k\in\mathbb{Z}},$$

where

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} a(\varphi) e^{-ik\varphi} \, \mathrm{d}\varphi,$$

is an isometric isomorphism (i.e. $\mathcal{F}^{-1} = \mathcal{F}^*$) and $\mathcal{F}^* A \mathcal{F} = M_a$, where

$$M_a \colon L^2([0,2\pi)) \to L^2([0,2\pi)), \quad f \mapsto af$$

denotes the multiplication operator corresponding to a. In other words, $\mathcal{F}$ diagonalizes A and it is thus clear that $\operatorname{sp}(A) = \operatorname{sp}(M_a)$. It follows that $\operatorname{sp}(A)$ is equal to the essential range of a. Moreover, this implies that Laurent operators are normal because multiplication operators commute. We also have $\|A\| = \|a\|_\infty$, the essential supremum of a. We will not go into detail here and just observe that the "essential" can be dropped in the case $A \in \mathrm{BDO}(\mathbf{X})$ (i.e. when a continuous).

Similarly, one defines the symbol of a Toeplitz operator $A \in \mathcal{L}(\ell^p(\mathbb{N}))$. If $A \in \mathrm{BDO}(\mathbf{X})$, we get the following (see e.g. [8, Theorem 2.47]):

$$\operatorname{sp}(A) = \operatorname{im}(a) \cup \{\lambda \in \mathbb{C} : \operatorname{wind}(a,\lambda) \neq 0\} = \operatorname{sp}_{\mathrm{ess}}(A) \cup \{\lambda \in \mathbb{C} : \operatorname{wind}(a,\lambda) \neq 0\}, \tag{15}$$

where $\operatorname{wind}(a,\lambda)$ denotes the winding number of the closed curve a around λ. Moreover, $\operatorname{ind}(A - \lambda I) = -\operatorname{wind}(a,\lambda)$ for all $\lambda \notin \operatorname{im}(a)$. So, as mentioned before, the difference between $\operatorname{sp}_{\mathrm{ess}}(A)$ and $\operatorname{sp}(A)$ is that we have to fill the holes with non-zero winding number.

Now the same can be done with periodic operators. The idea is the following. We identify periodic operators with block Laurent operators:

$$\begin{pmatrix}
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & A_{m,m} & A_{0,1} & \cdots & A_{0,m} & A_{-m,1} & \cdots & A_{-m,m} & A_{-2m,1} & \cdots \\
\cdots & A_{m+1,m} & A_{1,1} & \cdots & A_{1,m} & A_{-m+1,1} & \cdots & A_{-m+1,m} & A_{-2m+1,1} & \cdots \\
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & A_{2m,m} & A_{m,1} & \cdots & A_{m,m} & A_{0,1} & \cdots & A_{0,m} & A_{-m,1} & \cdots \\
\cdots & A_{2m+1,m} & A_{m+1,1} & \cdots & A_{m+1,m} & A_{1,1} & \cdots & A_{1,m} & A_{-m+1,1} & \cdots \\
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & A_{3m,m} & A_{2m,1} & \cdots & A_{2m,m} & A_{m,1} & \cdots & A_{m,m} & A_{0,1} & \cdots \\
\cdots & A_{3m+1,m} & A_{2m+1,1} & \cdots & A_{2m+1,m} & A_{m+1,1} & \cdots & A_{m+1,m} & A_{1,1} & \cdots \\
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots
\end{pmatrix}$$

$$\to \left(\begin{array}{cc|ccc|ccc|cc}
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & A_{m,m} & A_{0,1} & \cdots & A_{0,m} & A_{-m,1} & \cdots & A_{-m,m} & A_{-2m,1} & \cdots \\
\hline
\cdots & A_{m+1,m} & A_{1,1} & \cdots & A_{1,m} & A_{-m+1,1} & \cdots & A_{-m+1,m} & A_{-2m+1,1} & \cdots \\
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & A_{2m,m} & A_{m,1} & \cdots & A_{m,m} & A_{0,1} & \cdots & A_{0,m} & A_{-m,1} & \cdots \\
\hline
\cdots & A_{2m+1,m} & A_{m+1,1} & \cdots & A_{m+1,m} & A_{1,1} & \cdots & A_{1,m} & A_{-m+1,1} & \cdots \\
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\cdots & A_{3m,m} & A_{2m,1} & \cdots & A_{2m,m} & A_{m,1} & \cdots & A_{m,m} & A_{0,1} & \cdots \\
\hline
\cdots & A_{3m+1,m} & A_{2m+1,1} & \cdots & A_{2m+1,m} & A_{m+1,1} & \cdots & A_{m+1,m} & A_{1,1} & \cdots \\
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots
\end{array}\right).$$

Similarly, we identify one-sided periodic operators with block Toeplitz operators. This justifies the following definition:

Definition 4.32. Let A be an m-periodic operator on $\ell^p(\mathbb{Z})$. Then the matrix-valued function

$$a\colon [0,2\pi) \to \mathbb{C}^{m\times m}, \quad \varphi \mapsto a(\varphi) := \sum_{k=-\infty}^{\infty} a_k e^{ik\varphi},$$

where

$$a_k := \begin{pmatrix} A_{km+1,1} & \dots & A_{km+1,m} \\ \vdots & \ddots & \vdots \\ A_{(k+1)m,1} & \dots & A_{(k+1)m,m} \end{pmatrix}$$

is called the symbol of A.

As for Laurent operators, the spectrum of a periodic operator $A \in \mathcal{W}(\mathbf{X})$ can now be expressed in terms of the symbol a.

Theorem 4.33. *([8, Theorem 2.93 b)], [24, Theorem 4.4.9])*
Let $\mathbf{X} = \ell^p(\mathbb{Z})$, $m \in \mathbb{N}$ and let $A \in \mathcal{W}(\mathbf{X})$ be m-periodic. Then

$$\operatorname{sp}(A) = \bigcup_{\varphi\in[0,2\pi)} \operatorname{sp}(a(\varphi)) = \left\{\lambda \in \mathbb{C} : \det(a(\varphi) - \lambda I) = 0 \textit{ for some } \varphi \in [0,2\pi)\right\},$$

where a denotes the symbol of A.

If $p = 2$, the Fourier transform

$$\mathcal{F}\colon L^2([0,2\pi),\mathbb{C}^m) \to \ell^2(\mathbb{Z},\mathbb{C}^m), \quad f \mapsto \mathcal{F}(f) := (f_k)_{k\in\mathbb{Z}},$$

where

$$f_k := \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)e^{-ik\varphi}\,\mathrm{d}\varphi,$$

is an isometric isomorphism (i.e. $\mathcal{F}^{-1} = \mathcal{F}^*$) and $\mathcal{F}^*A\mathcal{F} = M_a$, where

$$M_a\colon L^2([0,2\pi),\mathbb{C}^m) \to L^2([0,2\pi),\mathbb{C}^m), \quad f \mapsto af$$

denotes the multiplication operator corresponding to a.

Note that we obviously cannot write $\operatorname{sp}(A) = \operatorname{im}(\det(a))$. In fact, the spectrum can be written as the union of up to m closed curves. However, it is usually not possible to label the eigenvalue curves individually. Consider for example the following 2-periodic tridiagonal operator A:

$$A = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 1 & -1 & & & \\ & 1 & -1 & 0 & & \\ & & 1 & 1 & -1 & \\ & & & 1 & -1 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}.$$

Its symbol is given by

$$a(\varphi) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e^{i\varphi} = \begin{pmatrix} 1 & -1+e^{i\varphi} \\ 1 & -1 \end{pmatrix}.$$

To be honest, there is some ambiguity here since we did not specify which entry is $A_{1,1}$ etc. So the symbol can also very well be

$$b(\varphi) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} e^{-i\varphi} + \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e^{i\varphi} = \begin{pmatrix} -1 & e^{i\varphi} \\ 1-e^{-i\varphi} & 1 \end{pmatrix}.$$

Of course, the spectrum does not depend on which representation we choose (or in which way we divide the matrix into blocks) because they are unitarily equivalent by a simple shift. For m-periodic operators there are m choices here, all of which are equivalent by a certain number of shifts. In this example we choose a as the symbol of A. Computing the characteristic polynomial, we get

$$\det(a(\varphi) - \lambda I) = (1-\lambda)(-1-\lambda) - (-1+e^{i\varphi}) = \lambda^2 - e^{i\varphi}.$$

Thus $\operatorname{sp}(A)$ is given by the unit circle. However, the eigenvalue curves

$$\lambda_1(\varphi) := e^{\frac{i\varphi}{2}} \quad \text{and} \quad \lambda_2(\varphi) := -e^{\frac{i\varphi}{2}}$$

are not closed on $[0, 2\pi)$. Hence we can not clearly assign one eigenvalue to this one closed curve. The eigenvalues form one closed curve together.

So in general we have to solve a φ-dependent $m \times m$ eigenvalue problem. For $m > 4$ we cannot do this analytically due to the famous Abel-Ruffini theorem. Therefore we often have to use numerical methods to get explicit results. This leads to an entirely different topic that is not covered in this thesis.

The determinant of the (analogously defined) symbol also plays an important role for one-sided periodic operators:

Theorem 4.34. *([8, Theorem 2.93 b), Theorem 2.94 b)])*
Let $\mathbf{X} = \ell^p(\mathbb{N})$ and let $A \in \mathcal{W}(\mathbf{X})$ be periodic. Then

$$\operatorname{sp_{ess}}(A) = \{\lambda \in \mathbb{C} : \det(a(\varphi) - \lambda I) = 0 \text{ for some } \varphi \in [0, 2\pi)\},$$

where a denotes the symbol of A. Moreover, $\operatorname{ind}(A - \lambda I) = -\operatorname{wind}(\det(a), \lambda)$ *for all $\lambda \notin \operatorname{sp_{ess}}(A)$.*

For one-sided periodic operators it is obvious where $A_{1,1}$ is located, i.e. there is only one reasonable choice for the symbol. Although the index formula looks very similar to the one for Toeplitz operators, it is not quite true that we just have to fill some holes. First of all, since $\operatorname{sp_{ess}}(A)$ does not coincide with the image of $\det(a)$, it is not immediately clear which holes should be filled. Moreover, we do not have Coburn's Lemma for periodic operators, i.e. index 0 does not imply invertibility. We will see in Section 4.2.3 that there are indeed periodic operators where Coburn's lemma fails. However, we will repair it with an extra condition in the case of tridiagonal periodic operators.

So now that we know how to compute the spectra of two-sided periodic operators, we want to know whether the spectra of the corresponding periodic operators approximate the spectrum of a given pseudo-ergodic operator. The answer is obviously YES for diagonal operators. For bidiagonal operators the answer is YES under the assumption that $0 \notin U_1$ or $0 \notin U_{-1}$, respectively, and NO if this assumption is dropped. Surprisingly, it holds

$$\operatorname{sp}(A) = \bigcup_{B \in T(U_0,U_1)} \operatorname{sp}(B)$$

for all $A \in \Psi\mathrm{E}_+(U_0, U_1)$ without restrictions on U_0 or U_1 (recall that $T(U_0, U_1)$ denotes the set of Toeplitz operators in $M_+(U_0, U_1)$). We will cover this in Section 4.3.1. For tridiagonal operators this is already a difficult question (and for more than three diagonals it gets even worse). We will treat the tridiagonal case in Section 4.3 and consider the special case of the Feinberg-Zee random hopping matrix in Section 4.4. As a preparation we deduce more explicit expressions for the spectrum of tridiagonal periodic operators.

4.2.3 Tridiagonal periodic operators

In the tridiagonal case the computation of the determinant is particularly easy. Therefore we can give a more explicit expression for the spectrum of a periodic operator. To simplify notation, we need the following definition:

Definition 4.35. Let $\mathbb{I} \in \{\mathbb{N}, \mathbb{Z}\}$ or $\mathbb{I} = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$, $\mathbf{X} = \ell^p(\mathbb{I})$ and $i, j \in \mathbb{I}$. For $A \in \mathcal{L}(\mathbf{X})$ we denote the finite matrix $(A_{k,l})_{i \le k,l \le j}$ by $A_{i;j}$. Furthermore, if $\mathbb{I} = \mathbb{Z}$, we denote the one-sided infinite matrix $(A_{k,l})_{1 \le k,l < \infty} \in \mathcal{L}(\ell^p(\mathbb{N}))$ by A_+.

We further define $\det A_{i;j} = 1$ if $i = j+1$ and $\det A_{i;j} = 0$ if $i \ge j+2$ so that we can apply Laplace's formula for determinants without worrying about the size of the (sub-)matrices.

The determinant of a tridiagonal matrix can be computed using so-called transfer matrices. Consider the following matrix

$$A = A_{1;m} = \begin{pmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & b_3 & \ddots & \\ & & \ddots & \ddots & c_{m-1} \\ & & & a_{m-1} & b_m \end{pmatrix} \in \mathbb{C}^{m \times m} \tag{16}$$

and the transfer matrices $M_j := \begin{pmatrix} 0 & 1 \\ -a_{j-1}c_{j-1} & b_j \end{pmatrix}$ for $j \in \{1, \ldots, m\}$, where a_0 and c_0 are some arbitrary constants. Furthermore, let $M := M_m \cdot \ldots \cdot M_1$. Then M is given by

$$M = \begin{pmatrix} -a_0c_0 \det A_{2;m-1} & \det A_{1;m-1} \\ -a_0c_0 \det A_{2;m} & \det A_{1;m} \end{pmatrix}. \tag{17}$$

This can be proven by induction using Laplace's formula. Thus $\det A_{1;m}$ can be read off. We will also find the following quantities useful:

$$\operatorname{tr}(M) = -a_0c_0 \det A_{2;m-1} + \det A_{1;m}, \tag{18}$$

$$\det(M) = \prod_{j=1}^{m} \det(M_j) = \prod_{j=0}^{m-1} a_jc_j. \tag{19}$$

So let us now consider a tridiagonal m-periodic operator

$$A := \begin{pmatrix} \ddots & \ddots & & & & & & & \\ \ddots & b_m & c_m & & & & & & \\ & a_m & b_1 & c_1 & & & & & \\ & & a_1 & \ddots & \ddots & & & & \\ & & & \ddots & b_m & c_m & & & \\ & & & & a_m & b_1 & c_1 & & \\ & & & & & a_1 & b_2 & \ddots & \\ & & & & & & \ddots & \ddots & \end{pmatrix} \in \mathcal{L}(\ell^2(\mathbb{Z})).$$

Then, by Theorem 4.33, A is invertible if and only if $\det(a(\varphi)) \neq 0$ for all $\varphi \in [0, 2\pi)$. The symbol a is given by

$$a(\varphi) = \begin{pmatrix} b_1 & c_1 & & & & a_m e^{i\varphi} \\ a_1 & b_2 & c_2 & & & \\ & a_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & c_{m-1} \\ c_m e^{-i\varphi} & & & & a_{m-1} & b_m \end{pmatrix}.$$

Using Laplace's formula twice, we get

$$\begin{aligned} \det(a(\varphi)) &= b_1 \det A_{2;m} - a_1 c_1 \det A_{3;m} + (-1)^{m+1} e^{i\varphi} \prod_{j=1}^{m} a_j - a_m c_m \det A_{2;m-1} \\ &\quad + (-1)^{m+1} e^{-i\varphi} \prod_{j=1}^{m} c_j \\ &= \det A_{1;m} - a_m c_m \det A_{2;m-1} + (-1)^{m+1} e^{i\varphi} \prod_{j=1}^{m} a_j + (-1)^{m+1} e^{-i\varphi} \prod_{j=1}^{m} c_j. \end{aligned} \tag{20}$$

Now let M be as in (17) above and choose $a_0 := a_m$ and $c_0 := c_m$ for consistency. Then we get

$$\det(a(\varphi)) = \operatorname{tr}(M) + (-1)^{m+1} e^{i\varphi} \prod_{j=1}^{m} a_j + (-1)^{m+1} e^{-i\varphi} \prod_{j=1}^{m} c_j. \tag{21}$$

Let us denote the two eigenvalues of M by λ_1 and λ_2, i.e.

$$\lambda_1 := \frac{\operatorname{tr}(M)}{2} + \frac{1}{2}\sqrt{\operatorname{tr}(M)^2 - 4\det(M)} \quad \text{and} \quad \lambda_2 := \frac{\operatorname{tr}(M)}{2} - \frac{1}{2}\sqrt{\operatorname{tr}(M)^2 - 4\det(M)}, \tag{22}$$

and assume that $\det(a(\varphi)) = 0$ for some $\varphi \in [0, 2\pi)$. Then, by Equation (21),

$$\begin{aligned}
\operatorname{sp}(M) &= \{\lambda_1, \lambda_2\} \\
&= \left\{ \frac{\operatorname{tr}(M)}{2} \pm \frac{1}{2}\sqrt{\operatorname{tr}(M)^2 - 4\det(M)} \right\} \\
&= \left\{ \frac{1}{2}(-1)^m e^{i\varphi} \prod_{j=1}^m a_j + \frac{1}{2}(-1)^m e^{-i\varphi} \prod_{j=1}^m c_j \pm \frac{1}{2}\sqrt{\left(e^{i\varphi} \prod_{j=1}^m a_j + e^{-i\varphi} \prod_{j=1}^m c_j \right)^2 - 4 \prod_{j=1}^m a_j c_j} \right\} \\
&= \left\{ \frac{1}{2}(-1)^m e^{i\varphi} \prod_{j=1}^m a_j + \frac{1}{2}(-1)^m e^{-i\varphi} \prod_{j=1}^m c_j \pm \frac{1}{2}\left(e^{i\varphi} \prod_{j=1}^m a_j - e^{-i\varphi} \prod_{j=1}^m c_j \right) \right\} \\
&\subset \left\{ \pm e^{i\varphi} \prod_{j=1}^m a_j, \pm e^{-i\varphi} \prod_{j=1}^m c_j \right\}. \qquad (23)
\end{aligned}$$

We were a bit sloppy in the last line, but we do not need the exact signs. First assume that $a_j, c_j \neq 0$ for all $j \in \{1, \dots, m\}$. Then we can observe that at least two out of $\left|\lambda_1 \prod_{j=1}^m \frac{1}{a_j}\right|$, $\left|\lambda_2 \prod_{j=1}^m \frac{1}{a_j}\right|$, $\left|\lambda_1 \prod_{j=1}^m \frac{1}{c_j}\right|$, $\left|\lambda_2 \prod_{j=1}^m \frac{1}{c_j}\right|$ have to be equal to 1. But since $\lambda_1\lambda_2 = \det(M) = \prod_{j=1}^m a_j c_j$ we get

$$\left|\lambda_1 \prod_{j=1}^m \frac{1}{a_j}\right| \left|\lambda_2 \prod_{j=1}^m \frac{1}{c_j}\right| = \left|\lambda_1 \prod_{j=1}^m \frac{1}{c_j}\right| \left|\lambda_2 \prod_{j=1}^m \frac{1}{a_j}\right| = 1.$$

This implies $\left|\lambda_1 \prod_{j=1}^m \frac{1}{a_j}\right| = 1$ or $\left|\lambda_2 \prod_{j=1}^m \frac{1}{a_j}\right| = 1$.

Now assume that $a_j = 0$ for at least one $j \in \{1, \dots, m\}$ and $c_k \neq 0$ for all $k \in \{1, \dots, m\}$. This implies $\det(M) = 0$ by Equation (19) and hence $\lambda_1 = \operatorname{tr}(M)$ and $\lambda_2 = 0$ or vice versa. Therefore $\left|\operatorname{tr}(M) \prod_{j=1}^m \frac{1}{c_j}\right| = 1$ by (23). Similarly, $\left|\operatorname{tr}(M) \prod_{j=1}^m \frac{1}{a_j}\right| = 1$ if $c_j = 0$ for at least one $j \in \{1, \dots, m\}$ and $a_k \neq 0$ for all $k \in \{1, \dots, m\}$. Finally, if $a_j = c_k = 0$ for some $j, k \in \{1, \dots, m\}$, then $\operatorname{tr}(M) = 0$ by (23) or directly by Equation (21).

So we just proved that if A is not invertible, we have certain conditions on the eigenvalues of M. In fact, also the converse is true as we will show now. First assume that $a_j, c_j \neq 0$ for all $j \in \{1, \dots, m\}$. Further assume that $\left|\lambda_1 \prod_{j=1}^m \frac{1}{a_j}\right| = 1$. Then there exists some $\psi \in [0, 2\pi)$ such that

$$\lambda_1 = \frac{\operatorname{tr}(M)}{2} + \frac{1}{2}\sqrt{\operatorname{tr}(M)^2 - 4\det(M)} = e^{i\psi} \prod_{j=1}^m a_j.$$

This implies

$$\operatorname{tr}(M)^2 - 4\det(M) = \left(2e^{i\psi}\prod_{j=1}^{m} a_j - \operatorname{tr}(M)\right)^2 = 4e^{2i\psi}\prod_{j=1}^{m} a_j^2 - 4e^{i\psi}\operatorname{tr}(M)\prod_{j=1}^{m} a_j + \operatorname{tr}(M)^2$$

or equivalently

$$\operatorname{tr}(M) = e^{i\psi}\prod_{j=1}^{m} a_j + e^{-i\psi}\det(M)\prod_{j=1}^{m}\frac{1}{a_j} = e^{i\psi}\prod_{j=1}^{m} a_j + e^{-i\psi}\prod_{j=1}^{m} c_j,$$

using Equation (19) and $a_0 = a_m$, $c_0 = c_m$. Comparing with Equation (21), we see that $\det(a(\varphi)) = 0$ for $\varphi = \psi$ or $\varphi = \psi + \pi$, depending on the sign of $(-1)^{m+1}$. This implies that A is not invertible by Theorem 4.33. The case $\left|\lambda_2 \prod\limits_{j=1}^{m} \frac{1}{a_j}\right| = 1$ is of course similar.

Now assume that $a_j = 0$ for at least one $j \in \{1, \dots, m\}$ and $c_k \neq 0$ for all $k \in \{1, \dots, m\}$. Further assume that $\left|\operatorname{tr}(M) \prod\limits_{j=1}^{m} \frac{1}{c_j}\right| = 1$. Then clearly, $\det(a(\varphi)) = 0$ for some $\varphi \in [0, 2\pi)$ by Equation (21). This implies that A is not invertible by Theorem 4.33. The case $c_j = 0$ for at least one $j \in \{1, \dots, m\}$ and $a_k \neq 0$ for all $k \in \{1, \dots, m\}$ is again similar. Finally, if $a_j = c_k = 0$ for some $j, k \in \{1, \dots, m\}$ and $\operatorname{tr}(M) = 0$, then $\det(a(\varphi)) = 0$ for all $\varphi \in [0, 2\pi)$ again by Equation (21). So A is not invertible in this case either.

We summarize:

Proposition 4.36. *Let $\mathbf{X} = \ell^p(\mathbb{Z})$, $m \in \mathbb{N}$ and let $A \in \mathcal{L}(\mathbf{X})$ be tridiagonal and m-periodic. Then the following is true:*

(i) Assume that $a_j, c_j \neq 0$ for all $j \in \{1, \dots, m\}$. Then A is invertible if and only if $\left|\lambda_1 \prod\limits_{j=1}^{m} \frac{1}{a_j}\right| \neq 1$ and $\left|\lambda_2 \prod\limits_{j=1}^{m} \frac{1}{a_j}\right| \neq 1$.

(ii) Assume that $a_j = 0$ for at least one $j \in \{1, \dots, m\}$ and $c_k \neq 0$ for all $k \in \{1, \dots, m\}$. Then A is invertible if and only if $\left|\operatorname{tr}(M) \prod\limits_{j=1}^{m} \frac{1}{c_j}\right| \neq 1$.

(iii) Assume that $c_j = 0$ for at least one $j \in \{1, \dots, m\}$ and $a_k \neq 0$ for all $k \in \{1, \dots, m\}$. Then A is invertible if and only if $\left|\operatorname{tr}(M) \prod\limits_{j=1}^{m} \frac{1}{a_j}\right| \neq 1$.

(iv) Assume that $a_j = c_k = 0$ for some $j, k \in \{1, \dots, m\}$. Then A is invertible if and only if $\operatorname{tr}(M) \neq 0$.

From Equation (21) we also get the following corollary:

Corollary 4.37. *Let $\mathbf{X} = \ell^p(\mathbb{Z})$, $m \in \mathbb{N}$ and let $A \in \mathcal{L}(\mathbf{X})$ be tridiagonal and m-periodic. Then the image of $\varphi \mapsto \det(a(\varphi))$ describes an ellipsis with half-axes $\left|\prod_{j=1}^m a_j\right| + \left|\prod_{j=1}^m c_j\right|$ and $\left|\left|\prod_{j=1}^m a_j\right| - \left|\prod_{j=1}^m c_j\right|\right|$ and center $\operatorname{tr}(M)$. In particular, if we consider $A - \lambda I$, only the center depends on λ.*

With the help of Proposition 4.36 we can prove the following variant of Coburn's lemma for tridiagonal periodic operators.

Theorem 4.38. *Let $\mathbf{X} = \ell^p(\mathbb{N})$, $m \in \mathbb{N}$ and let $A \in \mathcal{L}(\mathbf{X})$ be a tridiagonal, m-periodic Fredholm operator with $\det A_{1;m-1} \neq 0$. Then A is at least injective or surjective.*

For $m = 1$ the assumption $\det A_{1;m-1} \neq 0$ trivially holds and thus we get the usual Coburn lemma for tridiagonal Toeplitz operators.

Proof. Let $B \in \ell^p(\mathbb{Z})$ be tridiagonal and m-periodic such that

$$B_+ = A := \begin{pmatrix} b_1 & c_1 & & & & & \\ a_1 & b_2 & c_2 & & & & \\ & a_2 & \ddots & \ddots & & & \\ & & \ddots & b_m & c_m & & \\ & & & a_m & b_1 & c_1 & \\ & & & & a_1 & \ddots & \ddots \\ & & & & & \ddots & \end{pmatrix} \tag{24}$$

with $a_j, b_j, c_j \in \mathbb{C}$ for all $j \in \{1, \dots, m\}$. Since B is a limit operator of $B_+ = A$ (cf. Example 2.25(c)), B is invertible by Theorem 2.33.

Let us first assume that $a_j, c_j \neq 0$ for all $j \in \{1, \dots, m\}$. If A is not injective, there exists some $x \in \ell^p(\mathbb{N}) \setminus \{0\}$ such that $Ax = 0$. By our assumptions, it is clear that $x_1 \neq 0$ because $x_1 = 0$ would imply $x_2 = 0$ as well and we would get $x = 0$ by induction. Thus we can assume that $x_1 = 1$. Now the linear system $Ax = 0$ is easily solved recursively as follows:

$$\begin{pmatrix} x_j \\ x_{j+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{A_{j,j-1}}{A_{j,j+1}} & -\frac{A_{j,j}}{A_{j,j+1}} \end{pmatrix} \begin{pmatrix} x_{j-1} \\ x_j \end{pmatrix}$$

for all $j \in \mathbb{N}$ with $\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $A_{1,0} := a_m$. In order to determine the behavior of x towards infinity, we introduce the transfer matrices

$$\tilde{M}_j := \begin{pmatrix} 0 & 1 \\ -\frac{a_{j-1}}{c_j} & -\frac{b_j}{c_j} \end{pmatrix}$$

for $j \in \{1, \dots, m\}$ and their product $\tilde{M} := \tilde{M}_m \cdot \ldots \cdot \tilde{M}_1$. Since A is periodic, it holds

$$\begin{pmatrix} x_{km} \\ x_{km+1} \end{pmatrix} = \tilde{M}^k \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

for all $k \in \mathbb{N}$. Therefore the eigenvalues of $\tilde{M}$ determine the behavior of x towards infinity. Now the transfer matrices $\tilde{M}_j$ are very similar to the matrices M_j used above (cf. (17)). We only have to replace a_{j-1} by $-\frac{a_{j-1}}{c_j}$, b_j by $-\frac{b_j}{c_j}$ and c_{j-1} by -1 (of course, there are several choices here, but this one turns out to be the best one). Thus we get $\det(\tilde{M}) = \prod_{j=1}^{m} \frac{a_j}{c_j}$ and $\operatorname{tr}(\tilde{M}) = \det \tilde{A}_{1;m} - \frac{a_m}{c_1} \det \tilde{A}_{2;m-1}$ with

$$\tilde{A} := \begin{pmatrix} -\frac{b_1}{c_1} & -1 & & & & & \\ -\frac{a_1}{c_2} & -\frac{b_2}{c_2} & -1 & & & & \\ & -\frac{a_2}{c_3} & \ddots & \ddots & & & \\ & & \ddots & -\frac{b_m}{c_m} & -1 & & \\ & & & -\frac{a_m}{c_1} & -\frac{b_1}{c_1} & -1 & \\ & & & & -\frac{a_1}{c_2} & \ddots & \ddots \\ & & & & & \ddots & \end{pmatrix} \tag{25}$$

(see (18) and (19)). Using the multilinearity of determinants, we arrive at

$$\det \tilde{A}_{1;m} = \det A_{1;m}(-1)^m \prod_{j=1}^{m} \frac{1}{c_j} \quad \text{and} \quad \det \tilde{A}_{2;m-1} = \det A_{2;m-1}(-1)^{m-2} \prod_{j=2}^{m-1} \frac{1}{c_j}. \tag{26}$$

It follows

$$\det(\tilde{M}) = \det(M) \prod_{j=1}^{m} \frac{1}{c_j^2} \quad \text{and} \quad \operatorname{tr}(\tilde{M}) = \operatorname{tr}(M)(-1)^m \prod_{j=1}^{m} \frac{1}{c_j}.$$

Thus the eigenvalues of $\tilde{M}$ are given by $(-1)^m \prod_{j=1}^{m} \frac{1}{c_j}\lambda_1$ and $(-1)^m \prod_{j=1}^{m} \frac{1}{c_j}\lambda_2$, where λ_1 and λ_2 are the eigenvalues of M as in (22) above.

It is easily seen, using the spectral decomposition, that the sequence $(\tilde{M}^k)_{k\in\mathbb{N}}$ is unbounded if an eigenvalue of $\tilde{M}$ has an absolute value larger than 1. So since $x \in \ell^p(\mathbb{N})$ is bounded, we have that both eigenvalues of $\tilde{M}$ are contained in the closed unit disk or that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $\tilde{M}$ corresponding to an eigenvalue contained in the closed unit disk. The latter is impossible since this would imply

$$0 = \det \tilde{A}_{1;m-1} = \det A_{1;m-1}(-1)^{m-1} \prod_{j=1}^{m-1} \frac{1}{c_j}$$

by Equation (17) and hence $\det A_{1;m-1} = 0$, which was excluded by assumption. Thus we have

$$\max\left\{ \left| \prod_{j=1}^{m} \frac{1}{c_j}\lambda_1 \right|, \left| \prod_{j=1}^{m} \frac{1}{c_j}\lambda_2 \right| \right\} \leq 1. \tag{27}$$

Now assume that A^* (in the case $p = \infty$ just take the preadjoint/transpose) is not injective. Let $q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ if $p \in [1, \infty]$ and $q = 1$ if $p = 0$. Then there exists some $y \in \ell^q(\mathbb{N}) \setminus \{0\}$

such that $A^*y = 0$. Observe that A^* is just the transpose of A, hence a_j is interchanged with c_j for all $j \in \{1, \ldots, m\}$. Proceeding as above and using that trace and determinant are invariant under taking the transpose, we get

$$\max\left\{\left|\prod_{j=1}^{m} \frac{1}{a_j}\lambda_1\right|, \left|\prod_{j=1}^{m} \frac{1}{a_j}\lambda_2\right|\right\} \leq 1. \tag{28}$$

On the other hand we know that $\lambda_1\lambda_2 = \det(M) = \prod_{j=1}^{m} a_j c_j$. But this implies

$$\left|\prod_{j=1}^{m} \frac{1}{a_j}\lambda_1\right| = \left|\prod_{j=1}^{m} \frac{1}{c_j}\lambda_2\right| = \left|\prod_{j=1}^{m} \frac{1}{c_j}\lambda_1\right| = \left|\prod_{j=1}^{m} \frac{1}{a_j}\lambda_2\right| = 1$$

by (27) and (28). This is clearly a contradiction to Proposition 4.36 and the invertibility of B. Hence at least one of A and A^* is injective, which implies that A is injective or surjective (cf. Section 2.1).

Now let us consider the remaining cases:

- If $c_m = 0$, then $Ax = B_+x = 0$ implies $B\begin{pmatrix}0\\x\end{pmatrix} = 0$ because of the block structure that occurs in this case, a contradiction to the invertibility of B.
- If $a_m = 0$, then $A^*y = B_+^*y = 0$ implies $B^*\begin{pmatrix}0\\y\end{pmatrix} = 0$, again a contradiction to the invertibility of B.
- If $c_j = 0$ for some $j \in \{1, \ldots, m-1\}$, then consider the lowest index j for which $c_j = 0$. If $\det A_{1;j} \neq 0$, then $Ax = 0$ implies $x_1 = \ldots = x_j = 0$ because the linear system

$$\begin{pmatrix} b_1 & c_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & c_{j-1} \\ & & a_{j-1} & b_j \end{pmatrix}\begin{pmatrix}x_1\\ \vdots \\ \vdots \\ x_j\end{pmatrix} = \begin{pmatrix}0\\ \vdots \\ \vdots \\ 0\end{pmatrix}$$

  is uniquely solvable in this case. Thus again $B\begin{pmatrix}0\\x\end{pmatrix} = 0$, which is another contradiction to the invertibility of B. If $\det A_{1;j} = 0$, we get $\det A_{1;m-1} = \det A_{1;j} \det A_{j+1;m-1} = 0$ using the block structure of $A_{1;m-1}$. This is of course a contradiction to $\det A_{1;m-1} \neq 0$.
- If $a_j = 0$ for some $j \in \{1, \ldots, m-1\}$, we can proceed similarly and get again a contradiction.

□

Next we give an example to illustrate that the assumption $\det A_{1;m-1} \neq 0$ in Theorem 4.38, despite looking a bit odd, cannot be completely removed. However, it is also not completely necessary.

Example 4.39. Let $A \in \ell^p(\mathbb{N})$ be tridiagonal and 2-periodic with

$$A = \begin{pmatrix} 0 & 1 & & & \\ 1 & 1 & 2 & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & & \end{pmatrix}.$$

Using Theorem 4.34, it is readily seen that the essential spectrum of A is given by $[\frac{1}{2} - \frac{1}{2}\sqrt{37}, \frac{1}{2} - \frac{1}{2}\sqrt{5}] \cup [\frac{1}{2} + \frac{1}{2}\sqrt{5}, \frac{1}{2} + \frac{1}{2}\sqrt{37}]$, which implies that A is Fredholm. But since

$$(1, 0, -\frac{1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{8}, \ldots) \in \ker(A) \cap \ker(A^*),$$

A is neither injective nor surjective and hence the condition $\det A_{1;m-1} \neq 0$ cannot be removed. However, using the same procedure as in the proof of Theorem 4.38, we can also see that if we interchange 1 and 2 in the sub- and superdiagonal, A gets invertible. Hence the condition $\det A_{1;m-1} \neq 0$ is also not completely necessary. We will analyze this further below.

The previous example also shows that $\operatorname{sp}(A) \neq \operatorname{sp}_{\text{ess}}(A) \cup \{\text{some holes}\}$ in general. However, we can show that if there are holes in the essential spectrum of a tridiagonal periodic operator $A \in \mathcal{L}(\ell^p(\mathbb{N}))$, then these holes belong to the spectrum of A:

Theorem 4.40. *Let $\mathbf{X} = \ell^p(\mathbb{N})$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and let $A \in \mathcal{L}(\mathbf{X})$ be a tridiagonal, m-periodic operator with a_j, b_j and c_j as in* (24). *Then the following holds:*

- $\operatorname{ind}(A - \lambda I) = 1$ *if and only if* $\left|\prod_{j=1}^m c_j\right| > \left|\prod_{j=1}^m a_j\right|$ *and λ belongs to a bounded connected component of* $\mathbb{C} \setminus \operatorname{sp}_{\text{ess}}(A)$,
- $\operatorname{ind}(A - \lambda I) = -1$ *if and only if* $\left|\prod_{j=1}^m c_j\right| < \left|\prod_{j=1}^m a_j\right|$ *and λ belongs to a bounded connected component of* $\mathbb{C} \setminus \operatorname{sp}_{\text{ess}}(A)$,
- $\operatorname{ind}(A - \lambda I) = 0$ *if and only if λ belongs to the unbounded connected component of* $\mathbb{C} \setminus \operatorname{sp}_{\text{ess}}(A)$.

Proof. Let $\Psi\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \operatorname{tr}(M_{A-zI})$, where M_{A-zI} is defined as in (17) corresponding to the operator $A - zI$ in the obvious way. Denote the symbol of A by a. Then we have $z \in \operatorname{sp}_{\text{ess}}(A)$ if and only if $\det(a(\varphi) - zI) = 0$ for some $\varphi \in [0, 2\pi)$ by Theorem 4.34. From Equation (21) we know that $\det(a(\varphi) - zI)$ is given by

$$\det(a(\varphi) - zI) = \operatorname{tr}(M_{A-zI}) + (-1)^{m+1} e^{i\varphi} \prod_{j=1}^m a_j + (-1)^{m+1} e^{-i\varphi} \prod_{j=1}^m c_j.$$

Let $E := \left\{ e^{i\varphi} \prod_{j=1}^m a_j + e^{-i\varphi} \prod_{j=1}^m c_j : \varphi \in [0, 2\pi) \right\}$. So if $\Psi(z) = \operatorname{tr}(M_{A-zI}) \in E$, then $\det(a(\varphi) - zI) = 0$ for some $\varphi \in [0, 2\pi)$. Conversely, if $\det(a(\varphi) - zI) = 0$ for some $\varphi \in [0, 2\pi)$, then $\Psi(z) = \operatorname{tr}(M_{A-zI}) \in E$. Thus we have $\operatorname{sp}_{\text{ess}}(A) = \Psi^{-1}(E)$.

First assume that $\left|\prod_{j=1}^{m} a_j\right| = \left|\prod_{j=1}^{m} c_j\right|$ and let U be a bounded connected component of $\mathbb{C}\setminus\mathrm{sp}_{\mathrm{ess}}(A)$. It follows that $\Psi(U)$ is open and bounded since Ψ is a polynomial (and thus holomorphic) and not constant on U. Since E is a degenerated ellipse in this case (i.e. just a strip), it cannot be the boundary of a bounded open set. In particular, there exists some $x \in \partial\Psi(U) \setminus E$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\Psi(U) \setminus E$ with $x_n \to x$. Furthermore, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in U with $\Psi(y_n) = x_n$ for all $n \in \mathbb{N}$. Since U was supposed to be bounded, there exists a convergent subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ with $y_{n_k} \to y \in \overline{U}$ and $\Psi(y) = x$ by continuity. It follows $\Psi(y) = x \in \partial\Psi(U) \setminus E$, which is a contradiction because

$$\Psi(\overline{U}) = \Psi(U) \cup \Psi(\partial U) \subset \Psi(U) \cup \Psi(\mathrm{sp}_{\mathrm{ess}}(A)) = \Psi(U) \cup E$$

and $\Psi(U)$ is open. This implies that $\mathbb{C} \setminus \mathrm{sp}_{\mathrm{ess}}(A)$ does not have a bounded connected component in this case. From Theorem 4.34 we know that the index of $A - \lambda I$ is determined by the winding number of the closed curve $\varphi \mapsto \det(a(\varphi) - \lambda I)$:

$$\mathrm{ind}(A - \lambda I) = -\,\mathrm{wind}(\det(a - \lambda I), 0).$$

In the case at hand this curve (the ellipse) is degenerated and thus whenever $A - \lambda I$ is Fredholm, the index $\mathrm{ind}(A - \lambda I)$ vanishes. Thus we are done with this case.

Now let $\left|\prod_{j=1}^{m} a_j\right| \neq \left|\prod_{j=1}^{m} c_j\right|$ and let U be a bounded connected component of $\mathbb{C} \setminus \mathrm{sp}_{\mathrm{ess}}(A)$. We want to show that Ψ maps U to the bounded connected component of $\mathbb{C} \setminus E$. Without loss of generality we can assume that E is equal to the unit circle because they are biholomorphic to each other. This implies $|\Psi(z)| = 1$ for all $z \in \mathrm{sp}_{\mathrm{ess}}(A)$ and thus $|\Psi(z)| < 1$ for all $z \in U$ by the maximum principle. Let $z \in U$. Then we have

$$\{\det(a(\varphi) - zI) : \varphi \in [0, 2\pi)\} = \mathrm{tr}(M_{A-zI}) + E = \Psi(z) + E$$

and thus $\mathrm{wind}(\det(a - zI), 0) \in \{\pm 1\}$. The sign depends on the orientation of the ellipse. It is easy to check that the signs stated in the theorem are the correct ones. The last step is to check that the index $\mathrm{ind}(A - zI)$ vanishes on the unbounded component of $\mathbb{C}\setminus\mathrm{sp}_{\mathrm{ess}}(A)$. But this is obvious because the Fredholm index is constant on connected components by Theorem 2.2(iii) and $\mathrm{ind}(A - \lambda I) = 0$ whenever $A - \lambda I$ is invertible. □

As an immediate corollary of Theorem 4.40 and Theorem 4.38 we get the following. By "holes" we mean bounded connected components of the complement.

Corollary 4.41. *Let $\mathbf{X} = \ell^p(\mathbb{N})$, $m \in \mathbb{N}$ and let $A \in \mathcal{L}(\mathbf{X})$ be tridiagonal and m-periodic. Then we have the following inclusions:*

$$\mathrm{sp}_{\mathrm{ess}}(A) \cup \{\textit{holes of } \mathrm{sp}_{\mathrm{ess}}(A)\} \subset \mathrm{sp}(A) \subset \mathrm{sp}_{\mathrm{ess}}(A) \cup \{\textit{holes of } \mathrm{sp}_{\mathrm{ess}}(A)\} \cup \mathrm{sp}(A_{1;m-1}).$$

Proof. The first inclusion follows immediately from Theorem 4.40. To prove the second inclusion, we observe that the Fredholm index vanishes on the unbounded connected component of $\mathbb{C}\setminus\mathrm{sp}_{\mathrm{ess}}(A)$ as seen in Theorem 4.40. But this means, by Theorem 4.38, that we only have to care about the eigenvalues of $A_{1;m-1}$. □

So this already localizes the spectrum up to at most $m-1$ points. Analyzing the proof of Theorem 4.38 more carefully, we can actually make this a bit more precise. Note that the unions may not always be disjoint, though.

Theorem 4.42. *Let $\mathbf{X} = \ell^p(\mathbb{N})$, $m \in \mathbb{N}$ and let $A \in \mathcal{L}(\mathbf{X})$ be tridiagonal and m-periodic. Then we have:*

(i) If $a_j, c_j \neq 0$ for all $j \in \{1, \ldots, m\}$, then

$$\begin{aligned}\operatorname{sp}(A) &= \operatorname{sp}_{\mathrm{ess}}(A) \cup \{\textit{holes of } \operatorname{sp}_{\mathrm{ess}}(A)\} \\ &\quad \cup \left\{\lambda \in \operatorname{sp}(A_{1;m-1}) : |\det(A_{1;m} - \lambda I)| < \min\left\{\prod_{j=1}^{m}|a_j|, \prod_{j=1}^{m}|c_j|\right\}\right\} \\ &= \operatorname{sp}_{\mathrm{ess}}(A) \cup \{\textit{holes of } \operatorname{sp}_{\mathrm{ess}}(A)\} \\ &\quad \cup \left\{\lambda \in \operatorname{sp}(A_{1;m-1}) : |\det(A_{1;m} - \lambda I)| \leq \max\left\{\prod_{j=1}^{m}|a_j|, \prod_{j=1}^{m}|c_j|\right\}\right\}.\end{aligned}$$

(ii) If $a_m = 0$ or $c_m = 0$, then

$$\operatorname{sp}(A) = \operatorname{sp}_{\mathrm{ess}}(A) \cup \{\textit{holes of } \operatorname{sp}_{\mathrm{ess}}(A)\}.$$

(iii) If there is an index $k \in \{1, \ldots, m-1\}$ such that $a_k = 0$ and $a_j \neq 0$ for all $j \in \{1, \ldots, k-1\} \cup \{m\}$, then

$$\operatorname{sp}(A) = \operatorname{sp}_{\mathrm{ess}}(A) \cup \{\textit{holes of } \operatorname{sp}_{\mathrm{ess}}(A)\} \cup \operatorname{sp}(A_{1;k}).$$

iv) If there is an index $k \in \{1, \ldots, m-1\}$ such that $c_k = 0$ and $c_j \neq 0$ for all $j \in \{1, \ldots, k-1\} \cup \{m\}$, then

$$\operatorname{sp}(A) = \operatorname{sp}_{\mathrm{ess}}(A) \cup \{\textit{holes of } \operatorname{sp}_{\mathrm{ess}}(A)\} \cup \operatorname{sp}(A_{1;k}).$$

Proof. Let $B \in \mathcal{L}(\ell^p(\mathbb{Z}))$ be the m-periodic tridiagonal operator that satisfies $B_+ = A$. If A is Fredholm, then B is invertible by Theorem 2.33.

(i) If A is injective but not surjective or vice versa, then clearly $0 \in \{\text{holes of} \operatorname{sp}_{\mathrm{ess}}(A)\}$ by Theorem 4.40. So assume that A is Fredholm but neither injective nor surjective. Analyzing the proof of Theorem 4.38, we see that $\det A_{1;m-1}$ only comes into play if $\begin{pmatrix}0\\1\end{pmatrix}$ is an eigenvector of

$$\tilde{M} = \begin{pmatrix} -\frac{a_m}{c_1}\det \tilde{A}_{2;m-1} & \det \tilde{A}_{1;m-1} \\ -\frac{a_m}{c_1}\det \tilde{A}_{2;m} & \det \tilde{A}_{1;m} \end{pmatrix}$$

(cf. (17) and (25)). The corresponding eigenvalue is $\det \tilde{A}_{1;m} = \det A_{1;m}(-1)^m \prod\limits_{j=1}^{m} \frac{1}{c_j}$ (cf. Equation (26)), which implies $\left|\det A_{1;m} \prod\limits_{j=1}^{m} \frac{1}{c_j}\right| \leq 1$. Moreover, this implies

$$0 = \det \tilde{A}_{1;m-1} = \det A_{1;m-1}(-1)^{m-1} \prod_{j=1}^{m-1} \frac{1}{c_j}.$$

Consequently, $\det A_{1;m}$ is an eigenvalue of M (see Equation (17)). This implies that $\left|\det A_{1;m}\prod_{j=1}^{m}\frac{1}{c_j}\right|$ can actually not be equal to 1 because then B would not be invertible by Proposition 4.36 and hence A would not be Fredholm. This implies $\left|\det A_{1;m}\prod_{j=1}^{m}\frac{1}{c_j}\right| < 1$. Similarly, $\left|\det A_{1;m}\prod_{j=1}^{m}\frac{1}{a_j}\right| < 1$.

Conversely, if $\det A_{1;m-1} = 0$ and $\left|\det A_{1;m}\prod_{j=1}^{m}\frac{1}{c_j}\right| < 1$, then x, recursively defined by

$$\begin{pmatrix} x_{km+j} \\ x_{km+j+1} \end{pmatrix} = \left(\det A_{1;m}(-1)^m \prod_{j=1}^{m}\frac{1}{c_j}\right)^k \tilde{M}_j \cdot \ldots \cdot \tilde{M}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for $j \in \{1,\ldots,m\}$ and $k \in \mathbb{N}$, is contained in $\ell^p(\mathbb{N})$ by a geometric series argument in case $p \in [1,\infty)$ (for $p \in \{0,\infty\}$ it is obvious) and thus an eigenvector of A. Similarly, if $\left|\det A_{1;m}\prod_{j=1}^{m}\frac{1}{a_j}\right| < 1$, then we can construct an eigenvector of A^*.

We conclude: If A is not invertible, then A is either not Fredholm, $0 \in \{\text{holes of}\,\mathrm{sp}_{\mathrm{ess}}(A)\}$ or $\det A_{1;m-1} = 0$, $\left|\det A_{1;m}\prod_{j=1}^{m}\frac{1}{c_j}\right| < 1$ and $\left|\det A_{1;m}\prod_{j=1}^{m}\frac{1}{a_j}\right| < 1$. If A is invertible, then A is Fredholm, $0 \notin \{\text{holes of}\,\mathrm{sp}_{\mathrm{ess}}(A)\}$ and either $\det A_{1;m-1} \neq 0$ or $\left|\det A_{1;m}\prod_{j=1}^{m}\frac{1}{c_j}\right| > 1$ and $\left|\det A_{1;m}\prod_{j=1}^{m}\frac{1}{a_j}\right| > 1$. Applying this to $A - \lambda I$, we get

$$\begin{aligned}
\mathrm{sp}(A) &= \mathrm{sp}_{\mathrm{ess}}(A) \cup \{\text{holes of } \mathrm{sp}_{\mathrm{ess}}(A)\} \\
&\qquad \cup \left\{\lambda \in \mathrm{sp}(A_{1;m-1}) : |\det(A_{1;m} - \lambda I)| < \min\left\{\prod_{j=1}^{m}|a_j|, \prod_{j=1}^{m}|c_j|\right\}\right\} \\
&= \mathrm{sp}_{\mathrm{ess}}(A) \cup \{\text{holes of } \mathrm{sp}_{\mathrm{ess}}(A)\} \\
&\qquad \cup \left\{\lambda \in \mathrm{sp}(A_{1;m-1}) : |\det(A_{1;m} - \lambda I)| \le \max\left\{\prod_{j=1}^{m}|a_j|, \prod_{j=1}^{m}|c_j|\right\}\right\}.
\end{aligned}$$

(ii), *(iii)*, *(iv)* These follow immediately from Theorem 4.40 and the "remaining cases" in the proof of Theorem 4.38 □

Theorem 4.42 has the following two interesting corollaries. The first one was proven by slightly different means in [20, Theorem 3.3]. The second one is an (obvious) extension of [16, Theorem 4.1].

Corollary 4.43. *Let $U_{-1}, U_0, U_1 \subset \mathbb{C}$ be non-empty compact sets and let $B \in M_{fin}(U_{-1}, U_0, U_1)$. If B is not invertible, then there exists an $A \in M_{per,+}(U_{-1}, U_0, U_1)$ that is not invertible either. In* *particular,*

$$\bigcup_{B \in M_{fin}(U_{-1},U_0,U_1)} \operatorname{sp}(B) \subset \bigcup_{A \in M_{per,+}(U_{-1},U_0,U_1)} \operatorname{sp}(A) \subset \operatorname{sp}(C)$$

for all $C \in \Psi\mathrm{E}_+(U_{-1},U_0,U_1)$.

Proof. Let $B \in M_{m-1}(U_{-1},U_0,U_1)$ be given by

$$B = \begin{pmatrix} b_1 & c_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & c_{m-2} \\ & & a_{m-2} & b_{m-1} \end{pmatrix}.$$

If $0 \in U_1$, we can just choose

$$A := \begin{pmatrix} \boxed{B} & & \\ & c & \\ 0 & \boxed{B} & \\ & 0 & \ddots \end{pmatrix} \in M_{per,m-1,+}(U_{-1},U_0,U_1)$$

with $c \in U_{-1}$ arbitrary. If B is not invertible, then clearly A is not invertible either. If $0 \in U_{-1}$, we can proceed similarly. So let us now assume that $0 \notin U_{-1}, U_1$. Define

$$\hat{B} := \begin{pmatrix} b_{m-1} & c_{m-2} & & \\ a_{m-2} & b_{m-2} & \ddots & \\ & \ddots & \ddots & c_1 \\ & & a_1 & b_1 \end{pmatrix}.$$

Further define

$$A := \begin{pmatrix} \boxed{B} & & & & \\ & c & & & \\ a & b & c & & \\ & a & \boxed{B} & & \\ & & & c & \\ & & a & \ddots & \end{pmatrix} \in M_{per,m,+}(U_{-1},U_0,U_1)$$

and

$$\hat{A} := \begin{pmatrix} \boxed{\hat{B}} & & & & & \\ & c & & & & \\ a & b & c & & & \\ & a & \boxed{\hat{B}} & & & \\ & & & c & & \\ & & a & & \ddots & \end{pmatrix} \in M_{per,m,+}(U_{-1}, U_0, U_1),$$

where $a \in U_1$, $b \in U_0$ and $c \in U_{-1}$ can be chosen arbitrarily. Assume that B is not invertible. It is easy to check that then $\hat{B}$ is not invertible either. Clearly, $0 \in \mathrm{sp}(A_{1;m-1})$ and $0 \in \mathrm{sp}(\hat{A}_{1;m-1})$ because we have $A_{1;m-1} = B$ and $\hat{A}_{1;m-1} = \hat{B}$ by definition. Now the idea is to use Theorem 4.42(i) to prove that either A or $\hat{A}$ is not invertible, i.e. we have to check if

$$|\det(A_{1;m})| \leq \max\left\{ |a|^2 \prod_{j=1}^{m-2} |a_j|, |c|^2 \prod_{j=1}^{m-2} |c_j| \right\}$$

or

$$|\det(\hat{A}_{1;m})| \leq \max\left\{ |a|^2 \prod_{j=1}^{m-2} |a_j|, |c|^2 \prod_{j=1}^{m-2} |c_j| \right\}$$

holds. Using Laplace's formula, we get

$$\det(\hat{A}_{1;m}) = b \det(\hat{A}_{1;m-1}) - ac \det(\hat{A}_{1;m-2}) = -ac \det(\hat{A}_{1;m-2}).$$

The matrix $\hat{A}_{1;m-2}$ is just a flipped version of $A_{2;m-1}$ and therefore their determinants are equal. Since $\det A_{1;m-1} = 0$, we get

$$a^2 c^2 \prod_{j=1}^{m-2} a_j c_j = \det(M) = -ac \det A_{2;m-1} \det A_{1;m}$$

by Equations (17) and (19). Hence clearly, $|\det(A_{1;m})|$ and $|\det(\hat{A}_{1;m})| = |ac \det A_{2;m-1}|$ can not both be larger than $\max\left\{ |a|^2 \prod_{j=1}^{m-2} |a_j|, |c|^2 \prod_{j=1}^{m-2} |c_j| \right\}$. Hence either A or $\hat{A}$ is not invertible.

The second assertion follows by considering $B - \lambda I$ and taking Corollary 4.21 into account. $\square$

Corollary 4.44. *Let $U_{-1}, U_0, U_1 \subset \mathbb{C}$ be non-empty compact sets such that $U_{-1} = U_1$ and let $B \in M_{fin}(U_{-1}, U_0, U_1)$. If B is not invertible, then there exists an $A \in M_{per,+}(U_{-1}, U_0, U_1)$ that is not Fredholm. In particular,*

$$\bigcup_{B \in M_{fin}(U_{-1},U_0,U_1)} \mathrm{sp}(B) \subset \bigcup_{A \in M_{per,+}(U_{-1},U_0,U_1)} \mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{C \in M_{per}(U_{-1},U_0,U_1)} \mathrm{sp}(C).$$

Proof. Let B and $\hat{B}$ be defined as in the proof of Corollary 4.43. If $0 \in U_{-1} = U_1$, then we can choose

$$A := \begin{pmatrix} \boxed{B} & 0 & \\ 0 & \boxed{B} & 0 \\ & 0 & \ddots \end{pmatrix} \in M_{per,m-1,+}(U_{-1}, U_0, U_1)$$

and then clearly, B not invertible implies A not Fredholm. So let us now assume that $0 \notin U_{-1}$. This time we consider the operator

$$A := \begin{pmatrix} \boxed{B} & a & & & & \\ a & b & a & & & \\ & a & \boxed{\hat{B}^T} & a & & \\ & & a & b & a & \\ & & & a & \boxed{B} & a \\ & & & & a & \ddots \end{pmatrix} \in M_{per,2m,+}(U_{-1}, U_0, U_1),$$

where $a \in U_{-1}$ and $b \in U_0$ are arbitrary. Assume that B is not invertible. Then $\hat{B}^T$ is not invertible either. This implies that both $\det A_{1;m-1} = 0$ and $\det A_{m+1;2m-1} = 0$. Defining the transfer matrices M_j as in Equation (17), we get the following two (equal) products

$$M_{2m} \dots M_1 = \begin{pmatrix} -a^2 \det A_{2;2m-1} & \det A_{1;2m-1} \\ -a^2 \det A_{2;2m} & \det A_{1;2m} \end{pmatrix},$$

$$M_{2m} \dots M_{m+1} M_m \dots M_1 = \begin{pmatrix} -a^2 \det A_{m+2;2m-1} & 0 \\ -a^2 \det A_{m+2;2m} & \det A_{m+1;2m} \end{pmatrix} \begin{pmatrix} -a^2 \det A_{2;m-1} & 0 \\ -a^2 \det A_{2;m} & \det A_{1;m} \end{pmatrix},$$

hence $\det A_{1;2m} = \det A_{m+1;2m} \det A_{1;m}$ and $\det A_{1;2m-1} = 0$. Moreover,

$$\det A_{m+1;2m} = b \det A_{m+1;2m-1} - a^2 \det A_{m+1;2m-2} = -a^2 \det A_{2;m-1}.$$

It follows

$$\det A_{1;2m} = \det A_{m+1;2m} \det A_{1;m} = -a^2 \det A_{2;m-1} \det A_{1;m} = \det(M_m \dots M_1) = a^4 \prod_{j=1}^{m-2} a_j c_j,$$

where $a_j := B_{j+1,j}$ and $c_j := B_{j,j+1}$ as above. Since this is also equal to the product of the first $2m$ entries on the subdiagonal (or equally on the superdiagonal) of A and $\det A_{1;2m}$ is an eigenvalue of $M_{2m} \dots M_1$ by tridiagonality, we get that A is not Fredholm by Proposition 4.36 (i).

The second assertion follows by considering $B - \lambda I$ and taking Theorem 2.33 into account. □

4.3 Tridiagonal random operators

In this section we investigate tridiagonal random operators $A\colon \Omega \to \mathcal{L}(\ell^p(\mathbb{Z}))$ in more detail. In view of Theorem 4.6 we consider pseudo-ergodic operators $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$ for non-empty compact sets $U_{-1}, U_0, U_1 \subset \mathbb{C}$. As a warm-up we start with the bidiagonal case.

4.3.1 Bidiagonal random operators

We start with a simple lemma. For simplicity of notation we only formulate it for the lower bidiagonal case, i.e. $A_{i,j} = 0$ for $j \notin \{i-1, i\}$. In other words, we assume $U_{-1} = \{0\}$. The case $U_1 = \{0\}$ is of course exactly the same.

Lemma 4.45. *Let $\mathbf{X} = \ell^p(\mathbb{Z})$ and let $A \in \mathcal{L}(\mathbf{X})$ be lower bidiagonal. If*

(i) $\inf\limits_{i\in\mathbb{Z}} |A_{i,i}| > \sup\limits_{i\in\mathbb{Z}} |A_{i,i-1}|$ *or*

(ii) $\sup\limits_{i\in\mathbb{Z}} |A_{i,i}| < \inf\limits_{i\in\mathbb{Z}} |A_{i,i-1}|$

holds, then A is invertible.

Proof. (i) Let $A = D + T$, where D is diagonal and T only has non-zero entries on its first subdiagonal. D is invertible since $\inf\limits_{i\in\mathbb{Z}} |A_{i,i}| > \sup\limits_{i\in\mathbb{Z}} |A_{i,i-1}| \geq 0$. Thus $D^{-1}A = I + D^{-1}T$ and

$$\left\| D^{-1}T \right\| = \sup_{i\in\mathbb{Z}} \left| (D^{-1}T)_{i,i-1} \right| = \sup_{i\in\mathbb{Z}} \left| \frac{T_{i,i-1}}{D_{i,i}} \right| = \sup_{i\in\mathbb{Z}} \left| \frac{A_{i,i-1}}{A_{i,i}} \right| \leq \frac{\sup\limits_{i\in\mathbb{Z}} |A_{i,i-1}|}{\inf\limits_{i\in\mathbb{Z}} |A_{i,i}|} < 1.$$

Thus $D^{-1}A$ is invertible by a Neumann series argument. Of course, this implies that A is invertible as well.

(ii) Similar as (i). We only have to interchange the roles of D and T. $\square$

Remark 4.46. (a) The above actually works for any two diagonals j and k and not just $j = 0$, $k = 1$. Using an appropriate shift, we can move the diagonals such that one of the diagonals is the main diagonal. A decomposition $\ell^p(\mathbb{Z})$ into $\ell^p(|j-k|\,\mathbb{Z}) \oplus \ell^p(|j-k|\,\mathbb{Z}+1) \oplus \ldots \oplus \ell^p(|j-k|\,\mathbb{Z}+|j-k|-1)$ reduces the problem to the bidiagonal case (or just apply the same argument as above).

(b) One can weaken the assumptions a bit by requiring $\sup\limits_{i\in\mathbb{Z}} \left| \frac{A_{i,i-1}}{A_{i,i}} \right| < 1$, $\inf\limits_{i\in\mathbb{Z}} |A_{i,i}| > 0$ or $\sup\limits_{i\in\mathbb{Z}} \left| \frac{A_{i,i}}{A_{i,i-1}} \right| < 1$, $\inf\limits_{i\in\mathbb{Z}} |A_{i,i-1}| > 0$, respectively, instead.

We can now apply Lemma 4.45 to $A - \lambda I$ in order to get an upper bound to the spectrum. Again this is not limited to bidiagonal operators, but one of the two non-zero diagonals has to be the main diagonal so that $A - \lambda I$ still only has two non-zero diagonals.

Corollary 4.47. *Let $\mathbf{X} = \ell^p(\mathbb{Z})$ and let $A \in \mathcal{L}(\mathbf{X})$ be lower bidiagonal. Then*

$$\operatorname{sp}(A) \subset \left\{ \lambda \in \mathbb{C} : \inf_{i\in\mathbb{Z}} |A_{i,i} - \lambda| \leq \sup_{i\in\mathbb{Z}} |A_{i,i-1}| \right\} \setminus \left\{ \lambda \in \mathbb{C} : \sup_{i\in\mathbb{Z}} |A_{i,i} - \lambda| < \inf_{i\in\mathbb{Z}} |A_{i,i-1}| \right\}.$$

If the sets $\{A_{i,i} : i \in \mathbb{Z}\}$ *and* $\{A_{i,i-1} : i \in \mathbb{Z}\}$ *are compact, this is equivalent to*

$$\operatorname{sp}(A) \subset \left(\bigcup_{i\in\mathbb{Z}} \left\{ \lambda \in \mathbb{C} : |A_{i,i} - \lambda| \leq \sup_{i\in\mathbb{Z}} |A_{i,i-1}| \right\} \right) \setminus \left(\bigcap_{i\in\mathbb{Z}} \left\{ \lambda \in \mathbb{C} : |A_{i,i} - \lambda| < \inf_{i\in\mathbb{Z}} |A_{i,i-1}| \right\} \right).$$

Next we prove an approximation result for the spectrum of bidiagonal pseudo-ergodic operators. It is similar to the one for the norm (Theorem 4.24) and the numerical range (Theorem 4.30). The proof is an extension of the argument in [59, Section 3].

Theorem 4.48. *Let* $U_0, U_1 \subset \mathbb{C}$ *be non-empty compact sets with* $0 \notin U_1$ *and let* $A \in \Psi\mathrm{E}(U_0, U_1)$*. Then*

$$\operatorname{sp}(A) = \operatorname{clos} \left(\bigcup_{B\in M_{per}(U_0,U_1)} \operatorname{sp}(B) \right).$$

Proof. By Corollary 4.47, it is clear that

$$\operatorname{sp}(A) \subset \left(\bigcup_{u_0\in U_0} \left\{ \lambda \in \mathbb{C} : |u_0 - \lambda| \leq \max_{u_1\in U_1} |u_1| \right\} \right) \setminus \left(\bigcap_{v_0\in U_0} \left\{ \lambda \in \mathbb{C} : |v_0 - \lambda| < \min_{v_1\in U_1} |v_1| \right\} \right) =: \mathcal{M}.$$

Using Theorem 4.10(i), it remains to show

$$\mathcal{M} \subset \operatorname{clos} \left(\bigcup_{B\in M_{per}(U_0,U_1)} \operatorname{sp}(B) \right).$$

Let $m \in \mathbb{N}$ and fix $u_0, v_0 \in U_0$, $u_1, v_1 \in U_1$. Choose some $B \in M_{per,m}(\{u_0, v_0\}, \{u_1, v_1\}) \subset M_{per}(U_0, U_1)$ that satisfies

$$B_{j,j} = u_0 \Longleftrightarrow B_{j+1,j} = u_1 \qquad \text{and} \qquad B_{j,j} = v_0 \Longleftrightarrow B_{j+1,j} = v_1$$

for all $j \in \mathbb{Z}$. Theorem 4.33 and Equation (20) imply

$$\begin{aligned}
\lambda \in \operatorname{sp}(B) &\Longleftrightarrow \prod_{j=1}^{m} (B_{j,j} - \lambda) + (-1)^{m+1} e^{i\varphi} \prod_{j=1}^{m} B_{j+1,j} = 0 \quad \text{for some } \varphi \in [0, 2\pi) \\
&\Longleftrightarrow \prod_{j=1}^{m} |B_{j,j} - \lambda| = \prod_{j=1}^{m} |B_{j+1,j}| \\
&\Longleftrightarrow \prod_{j=1}^{m} \left| \frac{B_{j,j} - \lambda}{B_{j+1,j}} \right| = 1 \\
&\Longleftrightarrow \left| \frac{u_0 - \lambda}{u_1} \right|^{k} \left| \frac{v_0 - \lambda}{v_1} \right|^{m-k} = 1 \\
&\Longleftrightarrow \left| \frac{u_0 - \lambda}{u_1} \right|^{\frac{k}{m}} \left| \frac{v_0 - \lambda}{v_1} \right|^{1-\frac{k}{m}} = 1, \qquad (29)
\end{aligned}$$

where k denotes the cardinality of $\{j \in \{1, \ldots, m\} : B_{j,j} = u_0\}$.

Now if $|u_0 - \lambda| < |u_1|$ and $|v_0 - \lambda| > |v_1|$, there exists some $r(\lambda) \in (0,1)$ such that

$$\left|\frac{u_0 - \lambda}{u_1}\right|^{r(\lambda)} \left|\frac{v_0 - \lambda}{v_1}\right|^{1-r(\lambda)} = 1,$$

namely

$$r(\lambda) = \left(1 - \frac{\log\left|\frac{u_0-\lambda}{u_1}\right|}{\log\left|\frac{v_0-\lambda}{v_1}\right|}\right)^{-1}.$$

Define the set $S := \{\lambda \in \mathbb{C} : |u_0 - \lambda| < |u_1|, |v_0 - \lambda| > |v_1|\}$ and the function $R\colon S \to (0,1), \lambda \mapsto r(\lambda)$. Then S is open and R is continuous. Let $\lambda \in S$ and let $\varepsilon > 0$ be small enough such that $B_\varepsilon(\lambda) \subset S$. Since R is continuous and not constant on open sets, $R(B_\varepsilon(\lambda))$ is connected, hence $R(B_\varepsilon(\lambda)) \cap \mathbb{Q} \neq \emptyset$. This implies that

$$S_{\mathbb{Q}} := \left\{\lambda \in S : \left|\frac{u_0 - \lambda}{u_1}\right|^{r} \left|\frac{v_0 - \lambda}{v_1}\right|^{1-r} = 1 \text{ for some } r \in (0,1) \cap \mathbb{Q}\right\}$$

is dense in $\operatorname{clos}(S)$. Furthermore, $S_{\mathbb{Q}} \subset \bigcup_{B \in M_{per}(U_0,U_1)} \operatorname{sp}(B)$ by (29).

So let $\lambda \in \mathcal{M}$ and choose $u_0, v_0 \in U_0$, $u_1, v_1 \in U_1$ such that $|u_0 - \lambda| \leq |u_1|$ and $|v_0 - \lambda| \geq |v_1|$. Then

$$\lambda \in \operatorname{clos}(S) = \operatorname{clos}(S_{\mathbb{Q}}) \subset \operatorname{clos}\left(\bigcup_{B \in M_{per}(U_0,U_1)} \operatorname{sp}(B)\right).$$

This implies $\mathcal{M} \subset \operatorname{clos}\left(\bigcup_{B \in M_{per}(U_0,U_1)} \operatorname{sp}(B)\right)$, hence the Theorem follows. $\square$

In the proof of Theorem 4.48 we have seen that the spectrum of $A \in \Psi\mathrm{E}(U_0, U_1)$ is given by $\mathcal{M}$. This is actually true without the restriction $0 \notin U_1$. We formulate this in the next corollary.

Corollary 4.49. *Let $U_0, U_1 \subset \mathbb{C}$ be non-empty compact sets and let $A \in \Psi\mathrm{E}(U_0, U_1)$. Then*

$$\operatorname{sp}(A) = \left(\bigcup_{u_0 \in U_0} \left\{\lambda \in \mathbb{C} : |u_0 - \lambda| \leq \max_{u_1 \in U_1} |u_1|\right\}\right) \setminus \left(\bigcap_{v_0 \in U_0} \left\{\lambda \in \mathbb{C} : |v_0 - \lambda| < \min_{v_1 \in U_1} |v_1|\right\}\right).$$

Proof. For $0 \notin U_1$ this follows immediately from the proof of Theorem 4.48. For $0 \in U_1$ the right-hand side reduces to

$$\mathcal{M} := \bigcup_{u_0 \in U_0} \left\{\lambda \in \mathbb{C} : |u_0 - \lambda| \leq \max_{u_1 \in U_1} |u_1|\right\}$$

and $\operatorname{sp}(A) \subset \mathcal{M}$ is again clear by Corollary 4.47. Let $B \in \Psi\mathrm{E}_+(U_0, U_1)$. Then $\operatorname{sp}_{\mathrm{ess}}(B) = \operatorname{sp}(A)$ by Theorem 4.10 and Theorem 4.17. By Corollary 4.22 and Corollary 4.21, we get

$$\operatorname{sp}(A) = \operatorname{sp}_{\mathrm{ess}}(B) = \operatorname{sp}(B) = \bigcup_{C \in M_+(U_0,U_1)} \operatorname{sp}(C).$$

By Equation (15), the spectrum of the Toeplitz operator $T \in T(\{u_0\}, \{u_1\})$ is just a disk with radius $|u_1|$ and center u_0. This implies

$$\mathcal{M} = \bigcup_{T \in T(U_0,U_1)} \mathrm{sp}(T) \subset \bigcup_{C \in M_+(U_0,U_1)} \mathrm{sp}(C) = \mathrm{sp}(A).$$

Thus $\mathcal{M} = \mathrm{sp}(A)$. □

Remark 4.50. Note that if $0 \in U_1$, Theorem 4.48 is wrong in general. Consider for example $U_0 = \{0\}$ and $U_1 = \{0, 1\}$ and let $A \in \Psi\mathrm{E}(U_0, U_1)$. Then $\mathrm{sp}(A) = \mathbb{D}$ by Corollary 4.49. On the other hand we get

$$\mathrm{clos}\left(\bigcup_{B \in M_{per}(U_0,U_1)} \mathrm{sp}(B) \right) = \{0\} \cup \mathbb{T}$$

by Equation (20). Indeed, we have

$$\det(a(\varphi) - \lambda I) = (-\lambda)^m + (-1)^{m+1} e^{i\varphi} \prod_{j=1}^{m} a_j,$$

which implies, using Theorem 4.33, $\mathrm{sp}(A) = \mathbb{T}$ if $a_j \neq 0$ for all $j \in \{1, \ldots, m\}$ and $\mathrm{sp}(A) = \{0\}$ otherwise.

We also have a one-sided version of Theorem 4.48. Remarkably enough, the assumption $0 \notin U_1$ is again not needed here.

Theorem 4.51. *Let $U_0, U_1 \subset \mathbb{C}$ be non-empty compact sets and let $A \in \Psi\mathrm{E}_+(U_0, U_1)$. Then*

$$\mathrm{sp}(A) = \bigcup_{B \in M_{per,+}(U_0,U_1)} \mathrm{sp}(B) = \bigcup_{u_0 \in U_0} \left\{ \lambda \in \mathbb{C} : |u_0 - \lambda| \leq \max_{u_1 \in U_1} |u_1| \right\}.$$

Proof. By Theorem 4.17, we have

$$\mathrm{sp}(A) \supset \bigcup_{B \in M_{per,+}(U_0,U_1)} \mathrm{sp}(B)$$

and

$$\bigcup_{B \in M_{per,+}(U_0,U_1)} \mathrm{sp}(B) \supset \bigcup_{T \in T(U_0,U_1)} \mathrm{sp}(T) = \bigcup_{u_0 \in U_0} \left\{ \lambda \in \mathbb{C} : |u_0 - \lambda| \leq \max_{u_1 \in U_1} |u_1| \right\}$$

is clear. Let $\tilde{A} \in \Psi\mathrm{E}_+(U_0, U_1 \cup \{0\})$. Then $\mathrm{sp}(A) \subset \mathrm{sp}(\tilde{A})$ by Theorem 4.17. For $\tilde{A}$ we can proceed as in the proof of Corollary 4.49 to get

$$\mathrm{sp}(\tilde{A}) = \bigcup_{u_0 \in U_0} \left\{ \lambda \in \mathbb{C} : |u_0 - \lambda| \leq \max_{u_1 \in U_1} |u_1| \right\}.$$

This implies

$$\mathrm{sp}(A) \subset \mathrm{sp}(\tilde{A}) = \bigcup_{u_0 \in U_0} \left\{ \lambda \in \mathbb{C} : |u_0 - \lambda| \leq \max_{u_1 \in U_1} |u_1| \right\} \subset \bigcup_{B \in M_{per,+}(U_0,U_1)} \mathrm{sp}(B) \subset \mathrm{sp}(A).$$

This proves the assertion. □

4.3.2 The numerical range

As already mentioned a few times, the spectrum of a tridiagonal pseudo-ergodic operator is usually hard to compute. For the numerical range, however, we have the following remarkable result.

Theorem 4.52. *Let $\mathbf{X} = \ell^2(\mathbb{Z})$ and let $U_{-1}, U_0, U_1 \subset \mathbb{C}$ be non-empty compact sets. Then for all $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$ the following formula holds:*

$$N(A) = \operatorname{conv}\left(\bigcup_{B \in L(U_{-1},U_0,U_1)} \operatorname{sp}(B)\right) = \operatorname{conv}\left(\bigcup_{\substack{u_k \in U_k,\\ k=-1,0,1}} \left\{u_{-1}e^{-i\varphi} + u_0 + u_1 e^{i\varphi} : \varphi \in [0, 2\pi)\right\}\right).$$

In fact, pseudo-ergodicity is a bit stronger than needed in the proof. We only need that all Laurent operators are contained in the operator spectrum of A, i.e. $A \in M(U_{-1}, U_0, U_1)$ so that $L(U_{-1}, U_0, U_1) \subset \sigma^{\mathrm{op}}(A)$.

Proof. The second equality follows immediately from Theorem 4.33 applied to the case $m = 1$. Thus we focus on the proof of the first equality.

Let $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$. The idea is to compute the numerical abscissa $r_\varphi(A)$ for every angle $\varphi \in [0, 2\pi)$ and compare it with the abscissae of the Laurent operators. Since by Theorem 3.71 both sides of the first assertion are convex, proving the equality of all abscissae suffices to prove the theorem. But let us first observe that one direction follows immediately from Theorem 4.10, Theorem 3.72 and Theorem 3.71. Indeed, the spectrum of every operator in $L(U_{-1}, U_0, U_1)$ is contained in the spectrum of A by Theorem 4.10(i), which is of course contained in the numerical range of A by Theorem 3.72. Taking the convex hull on both sides yields

$$N(A) \supset \operatorname{conv}\left(\bigcup_{B \in L(U_{-1},U_0,U_1)} \operatorname{sp}(B)\right).$$

by Theorem 3.71.

Conversely, let $z_0 := \sup_{B \in M(U_{-1},U_0,U_1)} r(B)$. It is clear that z_0 is finite because

$$r(B) \leq \|B\| \leq \|A\|$$

holds for all $B \in M(U_{-1}, U_0, U_1)$ by Theorem 3.72 and Theorem 4.10(ii). As a consequence we have $N(e^{i\varphi}B + z_0 I) \subset \mathbb{C}_{\mathrm{Re} \geq 0}$ for all $B \in M(U_{-1}, U_0, U_1)$ and $\varphi \in [0, 2\pi)$. It follows

$$\begin{aligned}
r_\varphi(A) &= \sup_{\|x\|=1} \operatorname{Re}\left\langle e^{i\varphi}Ax, x\right\rangle \\
&= \sup_{\|x\|=1} \operatorname{Re}\left\langle (e^{i\varphi}A + z_0 I)x, x\right\rangle - z_0 \\
&= \sup_{\|x\|=1} \left|\operatorname{Re}\left\langle (e^{i\varphi}A + z_0 I)x, x\right\rangle\right| - z_0 \\
&= \frac{1}{2} \sup_{\|x\|=1} \left|\left\langle (e^{i\varphi}A + e^{-i\varphi}A^* + 2z_0 I)x, x\right\rangle\right| - z_0 \\
&= \frac{1}{2} \left\|e^{i\varphi}A + e^{-i\varphi}A^* + 2z_0 I\right\| - z_0,
\end{aligned}$$

where we used Theorem 3.72 again in the last line (observe that $e^{i\varphi}A+e^{-i\varphi}A^*+2z_0I$ is self-adjoint, hence normal). Using Proposition 2.17, we arrive at

$$\begin{aligned} r_\varphi(A) &= \frac{1}{2}\left\|e^{i\varphi}A+e^{-i\varphi}A^*+2z_0I\right\| - z_0 \\ &\leq \max_{\substack{u_{-1}\in U_{-1}\\ u_1\in U_1}} \left|e^{i\varphi}u_1+e^{-i\varphi}\overline{u_{-1}}\right| + \frac{1}{2}\max_{u_0\in U_0}\left|e^{i\varphi}u_0+e^{-i\varphi}\overline{u_0}+2z_0\right| - z_0. \end{aligned} \tag{30}$$

Choose $w_{-1}\in U_{-1}$, $w_0\in U_0$ and $w_1\in U_1$ such that the maxima in (30) are attained, i.e.

$$\max_{\substack{u_{-1}\in U_{-1}\\ u_1\in U_1}} \left|e^{i\varphi}u_1+e^{-i\varphi}\overline{u_{-1}}\right| = \left|e^{i\varphi}w_1+e^{-i\varphi}\overline{w_{-1}}\right|$$

and

$$\max_{u_0\in U_0}\left|e^{i\varphi}u_0+e^{-i\varphi}\overline{u_0}+2z_0\right| = \left|e^{i\varphi}w_0+e^{-i\varphi}\overline{w_0}+2z_0\right|.$$

Using Theorem 4.33, it is not hard to see that the spectrum of a (the) Laurent operator $C\in L(\{v_{-1}\},\{v_0\},\{v_1\})$ is given by an ellipse with center v_0 and half-axes $\left||v_{-1}|\pm|v_1|\right|$. If in addition C is self-adjoint, then clearly $\rho(C)=\|C\|=|v_0|+|v_{-1}|+|v_1|$. In our case, if we take $B\in L(\{w_{-1}\},\{w_0\},\{w_1\})$ and $C=e^{i\varphi}B+e^{-i\varphi}B^*+2z_0I$, we get

$$\left\|e^{i\varphi}B+e^{-i\varphi}B^*+2z_0I\right\| = 2\left|e^{i\varphi}w_1+e^{-i\varphi}\overline{w_{-1}}\right| + \left|e^{i\varphi}w_0+e^{-i\varphi}\overline{w_0}+2z_0\right|$$

and therefore

$$r_\varphi(A)\leq \frac{1}{2}\left\|e^{i\varphi}B+e^{-i\varphi}B^*+2z_0I\right\| - z_0.$$

From here, we can go the steps above backwards to finish the proof:

$$\begin{aligned} r_\varphi(A) &\leq \frac{1}{2}\left\|e^{i\varphi}B+e^{-i\varphi}B^*+2z_0I\right\| - z_0 \\ &= \frac{1}{2}\sup_{\|x\|=1}\left|\left\langle (e^{i\varphi}B+e^{-i\varphi}B^*+2z_0I)x,x\right\rangle\right| - z_0 \\ &= \sup_{\|x\|=1}\left|\operatorname{Re}\left\langle (e^{i\varphi}B+z_0I)x,x\right\rangle\right| - z_0 \\ &= \sup_{\|x\|=1}\operatorname{Re}\left\langle (e^{i\varphi}B+z_0I)x,x\right\rangle - z_0 \\ &= r_\varphi(B). \end{aligned}$$

Doing this for every $\varphi\in[0,2\pi)$ and using the observation at the beginning of the proof, we get

$$N(A)\subset\operatorname{conv}\left(\bigcup_{B\in L(U_{-1},U_0,U_1)} N(B)\right),$$

which is not yet what we wanted. But Laurent operators are normal (see Section 4.2.2) and thus

$$N(A)\subset\operatorname{conv}\left(\bigcup_{B\in L(U_{-1},U_0,U_1)}\operatorname{conv}(\operatorname{sp}(B))\right) = \operatorname{conv}\left(\bigcup_{B\in L(U_{-1},U_0,U_1)}\operatorname{sp}(B)\right)$$

by Theorem 3.72. □

So the numerical range of a tridiagonal pseudo-ergodic operator is easy to compute. We also know that the numerical range is an upper bound of the spectrum by Theorem 3.72. This means that we always have an easy upper bound in the tridiagonal case. If A is normal, then $\operatorname{conv}(\operatorname{sp}(A)) = N(A)$ and the upper bound is as tight as possible for a convex set. If not, this bound can be arbitrarily bad even for finite matrices as the simple example $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ with $a \geq 0$ shows. The spectrum of A is just $\{0\}$, whereas the numerical range is the closed ball of radius $a/2$ around 0. Surprisingly, tridiagonal pseudo-ergodic operators satisfy $\operatorname{conv}(\operatorname{sp}(A)) = N(A)$ just like normal operators:

Corollary 4.53. *Let U_{-1}, U_0 and U_1 be non-empty and compact and let $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$. Then*

$$N(A) = \operatorname{conv}(\operatorname{sp}(A)).$$

Proof. Using Theorem 4.52, Theorem 4.10(i) and Theorem 3.72, we get

$$N(A) = \operatorname{conv}\left(\bigcup_{B \in L(U_{-1},U_0,U_1)} \operatorname{sp}(B)\right) \subset \operatorname{conv}(\operatorname{sp}(A)) \subset N(A). \qquad \square$$

We do not know whether this is also true for more than three non-zero diagonals. However, we do know that Theorem 4.52 is wrong in general (see the example below). This implies that we can only use this upper bound in the tridiagonal case. This is somewhat in the line with other theorems we proved before. For example, Theorem 4.19 is so much more useful in the tridiagonal case (although we have to admit that we do not know whether one can improve Theorem 4.19 so that it is equally useful). For results like Theorem 4.40 we do not even know how a version for more than three diagonals should look like. Treating more than three non-zero diagonals seems to be much more difficult than treating only three of them and hence requires more sophisticated methods that have yet to be developed. The following example shows that Theorem 4.52 is wrong for five diagonals.

Example 4.54. Let $U_{-2} = \{i\}$, $U_{-1} = \{\pm i\}$, $U_0 = \{0\}$, $U_1 = \{i\}$ and $U_2 = \{i\}$ and let $A \in \Psi\mathrm{E}(U_{-2}, \ldots, U_2)$. In Figure 3 we can see the boundaries of the numerical ranges of all operators in $M_{per,3}(U_{-2}, \ldots, U_2)$ (red) and the spectra of both Laurent operators (blue). Thus clearly, $N(A) \neq \operatorname{conv}\left(\bigcup_{B \in L(U_{-2},\ldots,U_2)} \operatorname{sp}(B)\right)$ by Theorem 4.10(v). For an explicit computation see [36, Example 18].

4.3.3 The norm

As seen in the previous section, a tridiagonal pseudo-ergodic operator satisfies $\operatorname{conv}(\operatorname{sp}(A)) = N(A)$ just like normal operators. In view of Theorem 3.72 one may ask whether $\|A\| = \rho(A)$ is true as well. In some cases this is indeed true.

Theorem 4.55. *Let $\mathbf{X} = \ell^p(\mathbb{Z})$, let $U_n, \ldots, U_m \subset \mathbb{C}$ be non-empty compact sets and let $A \in \Psi\mathrm{E}(U_n, \ldots, U_m)$. If one of the following assumptions holds, then $\|A\| = \rho(A)$:*

(i) $\forall\, k \in \{n, \ldots, m\}\, \exists\, u_k \in U_k$ *with* $u_k \geq 0$ *and* $|u_k| = \max\limits_{v_k \in U_k} |v_k|$.

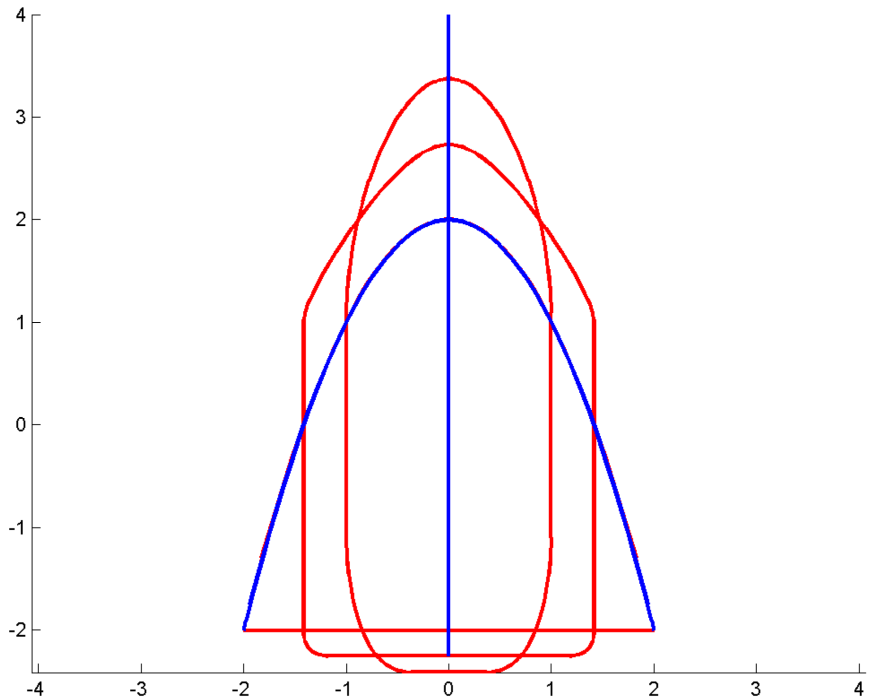


Figure 3: The boundaries of the numerical ranges of all 3-periodic operators (red) and the spectra of both Laurent operators (blue).

(ii) $\exists\, \varphi_0 \in [0, 2\pi)\ \forall\, k \in \{n, \dots, m\}\ \exists\, u_k \in U_k$ *with* $u_k e^{ik\varphi_0} \geq 0$ *and* $|u_k| = \max_{v_k \in U_k} |v_k|$.

(iii) $\exists\, \varphi_0, \psi_0 \in [0, 2\pi)\ \forall\, k \in \{n, \dots, m\}\ \exists\, u_k \in U_k$ *with* $u_k e^{i\psi_0} e^{ik\varphi_0} \geq 0$ *and* $|u_k| = \max_{v_k \in U_k} |v_k|$.

(iv) A *is tridiagonal and* $U_0 = \{0\}$.

If $\mathbf{X} = \ell^2(\mathbb{Z})$, *we can conclude* $\|A\| = r(A) = \rho(A)$ *also in these cases:*

(v) A *is tridiagonal and for* $k \in \{-1, 0, 1\}$ *there exist* $u_k \in U_k$ *and a* $\psi_0 \in [0, 2\pi)$ *such that* $|u_k| = \max_{v_k \in U_k} |v_k|$, $|u_{-1}| = |u_1|$,

$$\max_{j \in \mathbb{Z}} \left| A_{j,j-1} \overline{A_{j-1,j-1}} + A_{j,j} \overline{A_{j-1,j}} \right| \leq 2\, |u_1| \left| \mathrm{Re}(u_0 e^{i\psi_0}) \right|$$

and $u_{-1} e^{i\psi_0} = \overline{u_1} e^{-i\psi_0}$.

(vi) A *is tridiagonal,* $U_0 = \{u_0\}$ *is a singleton and* $|u_{-1}| = |u_1|$ *for all* $u_{-1} \in U_{-1}$, $u_1 \in U_1$.

Proof. Clearly, by Theorem 3.72, it suffices to prove $\|A\| \leq \rho(A)$. Also, (i) and (ii) are special cases of (iii).

(*iii*) Let L be the Laurent operator contained in $L(\{u_n\},\ldots,\{u_m\})$. By Theorem 4.33, it holds

$$\operatorname{sp}(L) = \bigcup_{\varphi\in[0,2\pi)} \left\{ \sum_{k=n}^{m} u_k e^{ik\varphi} \right\}.$$

Using Theorem 4.10(*i*), Proposition 2.17 and assumption (*iii*), we get

$$\rho(A) \geq \rho(L) \geq \left| \sum_{k=n}^{m} u_k e^{ik\varphi_0} \right| = \left| \sum_{k=n}^{m} u_k e^{i\psi_0} e^{ik\varphi_0} \right| = \sum_{k=n}^{m} |u_k| \geq \|A\|.$$

(*iv*) For two complex numbers u_{-1} and u_1 there always exists a $\psi_0 \in [0,2\pi)$ such that

$$\frac{\overline{u_{-1}}}{|u_{-1}|} e^{-i\psi_0} = \frac{u_1}{|u_1|} e^{i\psi_0}.$$

Choosing $\varphi_0 \in [0,2\pi)$ such that $e^{i\varphi_0} = \frac{u_{-1}}{|u_{-1}|} e^{i\psi_0} = \frac{\overline{u_1}}{|u_1|} e^{-i\psi_0}$, we get

$$u_{-1} e^{i\psi_0} e^{-i\varphi_0} = u_{-1} \frac{\overline{u_{-1}}}{|u_{-1}|} = |u_{-1}| \geq 0 \quad \text{and} \quad u_1 e^{i\psi_0} e^{i\varphi_0} = u_1 \frac{\overline{u_1}}{|u_1|} = |u_1| \geq 0.$$

Thus we can apply (*iii*).

(*v*) Now let $\mathbf{X} = \ell^2(\mathbb{Z})$. The entries of AA^* are given by

$$\begin{aligned}
(AA^*)_{j,j-2} &= A_{j,j-1}A^*_{j-1,j-2} = A_{j,j-1}\overline{A_{j-2,j-1}}, \\
(AA^*)_{j,j-1} &= A_{j,j-1}A^*_{j-1,j-1} + A_{j,j}A^*_{j,j-1} = A_{j,j-1}\overline{A_{j-1,j-1}} + A_{j,j}\overline{A_{j-1,j}}, \\
(AA^*)_{j,j} &= A_{j,j-1}A^*_{j-1,j} + A_{j,j}A^*_{j,j} + A_{j,j+1}A^*_{j+1,j} = |A_{j,j-1}|^2 + |A_{j,j}|^2 + |A_{j,j+1}|^2, \\
(AA^*)_{j,j+1} &= A_{j,j+1}A^*_{j+1,j+1} + A_{j,j}A^*_{j,j+1} = A_{j,j+1}\overline{A_{j+1,j+1}} + A_{j,j}\overline{A_{j+1,j}}, \\
(AA^*)_{j,j+2} &= A_{j,j+1}A^*_{j+1,j+2} = A_{j,j+1}\overline{A_{j+2,j+1}}
\end{aligned}$$

for $j \in \mathbb{Z}$. By Proposition 2.17 and assumption (*v*), we have

$$\begin{aligned}
\|AA^*\| \leq{}& \left|A_{j,j-1}\overline{A_{j-2,j-1}}\right| + \left|A_{j,j-1}\overline{A_{j-1,j-1}} + A_{j,j}\overline{A_{j-1,j}}\right| + |A_{j,j-1}|^2 + |A_{j,j}|^2 + |A_{j,j+1}|^2 \\
&+ \left|A_{j,j+1}\overline{A_{j+1,j+1}} + A_{j,j}\overline{A_{j+1,j}}\right| + \left|A_{j,j+1}\overline{A_{j+2,j+1}}\right| \\
\leq{}& 4\,|u_1|^2 + 4\,|u_1| \left|\operatorname{Re}(u_0 e^{i\psi_0})\right| + |u_0|^2.
\end{aligned}$$

Let L be the Laurent operator contained in $L(\{u_{-1}\},\{u_0\},\{u_1\})$ and choose $\varphi_0 \in [0,2\pi)$ such that

$$e^{i\varphi_0} = \frac{u_{-1}}{|u_{-1}|} e^{i\psi_0} = \frac{\overline{u_1}}{|u_1|} e^{-i\psi_0}.$$

Then $LL^* \in \sigma^{\mathrm{op}}(AA^*)$ by Proposition 2.26 and we get

$$\begin{aligned}
\rho(AA^*) &\geq \rho(LL^*) \\
&\geq \left| \sum_{k=-2}^{2} (LL^*)_{0,-k} e^{ik\varphi_0} \right| \\
&= \Big| u_{-1}\overline{u_1} e^{-2i\varphi_0} + (u_{-1}\overline{u_0} + u_0\overline{u_1}) e^{-i\varphi_0} + |u_{-1}|^2 + |u_0|^2 + |u_1|^2 + (u_1\overline{u_0} + u_0\overline{u_{-1}}) e^{i\varphi_0} \\
&\quad + u_1\overline{u_{-1}} e^{2i\varphi_0} \Big| \\
&= \Big| |u_{-1}|\,|u_1| + |u_{-1}|\,\overline{u_0} e^{-i\psi_0} + u_0\,|u_1|\, e^{i\psi_0} + |u_{-1}|^2 + |u_0|^2 + |u_1|^2 + |u_1|\,\overline{u_0} e^{-i\psi_0} \\
&\quad + u_0\,|u_{-1}|\, e^{i\psi_0} + |u_1|\,|u_{-1}| \Big| \\
&= \Big| |u_1|^2 + 2\,|u_1| \operatorname{Re}(u_0 e^{i\psi_0}) + |u_1|^2 + |u_0|^2 + |u_1|^2 + 2\,|u_1| \operatorname{Re}(u_0 e^{i\psi_0}) + |u_1|^2 \Big|
\end{aligned}$$

as in (iii). If $\operatorname{Re}(u_0 e^{i\psi_0}) \geq 0$, then

$$\rho(AA^*) \geq \rho(LL^*) \geq 4\,|u_1|^2 + 4\,|u_1| \left|\operatorname{Re}(u_0 e^{i\psi_0})\right| + |u_0|^2 \geq \|AA^*\|$$

follows. If $\operatorname{Re}(u_0 e^{i\psi_0}) \leq 0$, replace ψ_0 by $\psi_0 + \pi$ and we end up with the same conclusion. Now AA^* is of course self-adjoint and thus $\rho(AA^*) = \|AA^*\|$ by Theorem 3.72, which means that we have equality everywhere. Recall that Laurent operators are normal (see Section 4.2.2). We conclude

$$\|A\|^2 = \|AA^*\| = \rho(AA^*) = \rho(LL^*) = \|LL^*\| = \|L\|^2 = \rho(L)^2 \leq \rho(A)^2 \leq \|A\|^2$$

by Theorem 3.72 and Theorem 4.10(i).

(vi) Let $u_{-1} \in U_{-1}$ and $u_1 \in U_1$. Choose $\psi_0 \in [0, 2\pi)$ so that $u_{-1} e^{i\psi_0} = \overline{u_1} e^{-i\psi_0}$. Then

$$\begin{aligned}
|u_1\overline{u_0} + u_0\overline{u_{-1}}|^2 &= 2\,|u_1|^2\,|u_0|^2 + 2\operatorname{Re}(\overline{u_1 u_{-1}} u_0^2) \\
&= 2\,|u_1|^2\,|u_0|^2 + 2\operatorname{Re}(|u_1|^2\, e^{2i\psi_0} u_0^2) \\
&= 2\,|u_1|^2\,|u_0|^2 + 2\,|u_1|^2 \operatorname{Re}((e^{i\psi_0} u_0)^2) \\
&= 2\,|u_1|^2 \left|u_0 e^{i\psi_0}\right|^2 + 2\,|u_1|^2 \left(\operatorname{Re}(u_0 e^{i\psi_0})^2 - \operatorname{Im}(u_0 e^{i\psi_0})^2\right) \\
&= 4\,|u_1|^2 \operatorname{Re}(u_0 e^{i\psi_0})^2.
\end{aligned}$$

Thus we can apply (v). □

Note that in the case $p \in \{0, 1, \infty\}$ the equality $\|A\|_p = \rho(A)$ implies

$$\rho(A) = \|A\|_p = \sum_{k=-\infty}^{\infty} \max_{v_k \in U_k} |v_k|$$

for all $p \in \{0\} \cup [1, \infty]$. This is because of Proposition 4.7 and the inverse closedness of $\mathcal{W}(\mathbf{X})$, which implies that the spectral radii coincide for all $p \in \{0\} \cup [1, \infty]$. Thus we have

$$\rho(A) = \sum_{k=-\infty}^{\infty} \max_{v_k \in U_k} |v_k|$$

in this case and it always holds

$$\rho(A) \leq \|A\|_p \leq \sum_{k=-\infty}^{\infty} \max_{v_k \in U_k} |v_k|$$

for all $p \in \{0\} \cup [1, \infty]$. It is thus not to be expected that $\|A\| = \rho(A)$ always holds for pseudo-ergodic operators. Indeed, we can even give a tridiagonal counterexample:

Example 4.56. Let $U_{-1} = \{1\}$, $U_0 = \{0\}$ and $U_1 = \{\pm 1\}$. Then $\|A - \lambda I\|_2 = r(A - \lambda I) = \rho(A - \lambda I)$ for all $\lambda \in \mathbb{C}$ by Theorem 4.55(vi). However,

$$N(A) = \{\lambda \in \mathbb{C} : |\mathrm{Re}\, \lambda| + |\mathrm{Im}\, \lambda| \leq 2\}$$

by Theorem 4.52. Thus, for example,

$$\rho(A - (1+i)I) = r(A - (1+i)I) = \sqrt{10} < 2 + \sqrt{2} = \|A - (1+i)I\|_1$$

by Proposition 4.7. Hence $\|A - (1+i)I\|_p = r(A - (1+i)I) = \rho(A - (1+i)I)$ does not hold for all $p \in \{0\} \cup [1, \infty]$ in this example.

In fact, not even $\|A\|_2 = r(A)$ holds in general as the next example shows.

Example 4.57. Let $U_{-1} = \{i\}$, $U_0 = \{\pm 1\}$, $U_1 = \{i\}$ and $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$. Consider the 2-periodic operator $B \in M_{per}(U_{-1}, U_0, U_1)$ given by the symbol

$$b(\varphi) = \begin{pmatrix} 1 & i + ie^{i\varphi} \\ i + ie^{-i\varphi} & -1 \end{pmatrix}.$$

By Theorem 4.33, $\|B\|_2 = \max_{\varphi \in [0, 2\pi)} \|b(\varphi)\|_2$ and it is easy to see that for $\varphi = 0$ we have $\|b(\varphi)\|_2 = 3$. Thus $\|A\|_2 \geq \|B\|_2 \geq 3$ by Theorem 4.10(ii). On the other hand, $\|A\|_2 \leq 3$ by Proposition 2.17. This implies $\|A\|_2 = 3$, but $r(A) = \sqrt{5}$ by Theorem 4.52. Thus $\|A\|_2 \neq r(A)$ in general.

We have seen that tridiagonal pseudo-ergodic operators share some spectral quantities with normal operators but not necessarily all of them. In case U_0 is a singleton and $|u_{-1}| = |u_1|$ for all $u_{-1} \in U_{-1}$ and $u_1 \in U_1$, both $N(A) = \mathrm{conv}(\mathrm{sp}(A))$ and $\|A - \lambda I\|_2 = \rho(A - \lambda I)$ hold for all $\lambda \in \mathbb{C}$ by Corollary 4.53 and Theorem 4.55(vi). The Feinberg-Zee random hopping matrix is a matrix of this kind. We will further investigate it in the next section.

4.4 The Feinberg-Zee random hopping matrix

The Feinberg-Zee random hopping matrix was first introduced by J. Feinberg and A. Zee in [28] (see also [46] for a physical interpretation). It is defined by

$$A := \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & b_0 & 0 & 1 & \\ & & b_1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \in \mathcal{L}(\ell^2(\mathbb{Z}))$$

for a random sequence $b \in \{\pm 1\}^{\mathbb{Z}}$. In the language we introduced in the previous sections this just means that we have $U_{-1} = \{1\}$, $U_0 = \{0\}$ and $U_1 = \{\pm 1\}$ and consider an operator $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$. Various efforts (e.g. [15], [16], [17]) have been made in recent years to determine the spectrum of A or at least describe it in terms of spectra of periodic or finite matrices. Nevertheless, a precise description has not been found yet. In this section we provide the best lower and upper bounds to the spectrum that are known so far. As the spectrum does not depend on p since $A \in \mathcal{W}(\mathbf{X})$ and $\mathcal{W}(\mathbf{X})$ is inverse closed, we will assume $p = 2$ in this section.

4.4.1 What is known and what is new

Let us quickly summarize what is known about the spectrum of A. For this we need some notation in order to be in accordance with [15] and [16]. The spectrum of A will be denoted by Σ. The unions of spectra of periodic and finite matrices are denoted by π_∞ and σ_∞, respectively:

$$\pi_\infty := \bigcup_{B \in M_{per}(U_{-1}, U_0, U_1)} \mathrm{sp}(B), \quad \sigma_\infty := \bigcup_{C \in M_{fin}(U_{-1}, U_0, U_1)} \mathrm{sp}(C).$$

By Theorem 4.10(i), we have

$$\Sigma = \bigcup_{B \in M(U_{-1}, U_0, U_1)} \mathrm{sp}(B),$$

so in particular $\pi_\infty \subset \Sigma$. Moreover, using the transformation TBT^* defined by the unitary diagonal operator $T \in \ell^2(\mathbb{Z})$ given by

$$T_{1,1} = 1 \quad \text{and} \quad T_{j+1,j+1} = B_{j,j+1} T_{j,j}$$

for $j \in \mathbb{Z}$, we can see that every operator in $M_{per}(U_1, U_0, U_1)$ is unitarily equivalent to an operator in $M_{per}(U_{-1}, U_0, U_1)$. This implies

$$\bigcup_{B \in M_{per}(U_1, U_0, U_1)} \mathrm{sp}(B) = \bigcup_{B \in M_{per}(U_{-1}, U_0, U_1)} \mathrm{sp}(B) = \pi_\infty$$

and therefore

$$\sigma_\infty \subset \bigcup_{C \in M_{fin}(U_1, U_0, U_1)} \mathrm{sp}(C) \subset \bigcup_{B \in M_{per}(U_1, U_0, U_1)} \mathrm{sp}(B) = \pi_\infty$$

by Corollary 4.44. Thus we have the inclusions

$$\sigma_\infty \subset \pi_\infty \subset \Sigma.$$

The first inclusion is obviously proper because σ_∞ is a countable set whereas π_∞ is a countable union of lines. These observations were first made by Chandler-Wilde, Chonchaiya and Lindner in [16]. That the second inclusion is also proper was proven by the same authors in [15]. They showed, using a beautiful connection to the Sierpinki triangle, that the unit disk $\mathbb{D}$ is contained in Σ, implying that Σ is two-dimensional while π_∞ is, as a countable union of lines, only one-dimensional. This also disproved earlier conjectures that Σ may have a fractal dimension. Shortly afterwards, Chandler-Wilde and Davies showed in [17] that $\mathbb{D}$ is already contained in $\mathrm{clos}(\pi_\infty)$ and that π_∞ and Σ share a square root symmetry, i.e.

$$\lambda^2 \in \pi_\infty \Longrightarrow \lambda \in \pi_\infty$$

and similarly for Σ. As noticed in [16] there are also the somewhat obvious rotational and reflection symmetries:

$$\lambda \in \pi_\infty \Longrightarrow i\lambda, \bar{\lambda} \in \pi_\infty$$

and similarly for σ_∞ and Σ.

The best known upper bound is the bound we get from Theorem 4.52:

$$\Sigma \subset N(A) = \{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| + |\mathrm{Im}\,\lambda| \leq 2\}\,.$$

The numerical range $N(A)$ was first derived in [16] by a direct computation. The authors of [16] also provided numerical evidence that Σ is a proper subset of $N(A)$.

In [16] it was conjectured that σ_∞ is dense in Σ and that Σ is a simply connected set which is the closure of its interior and which has a fractal boundary. We cannot confirm this conjecture yet, but we show that σ_∞ is at least dense in π_∞ in Section 4.4.2. This implies, in particular, that the unit disk is contained in the closure of σ_∞. Moreover, we extend the number of symmetries of π_∞ by an infinite amount in Section 4.4.3. These symmetries allow us to exploit a significantly larger part of Σ, which we will denote by Ξ. Furthermore, they imply that Σ contains an infinite sequence of Julia sets, which may explain the seemingly fractal structure of $\partial\Sigma$. Combining these two results, we get that Ξ is contained in $\mathrm{clos}(\sigma_\infty)$ as well. Last but not least, we are going to compute the numerical range of A^2 in order to obtain a new upper bound to Σ in Section 4.4.5.

4.4.2 σ_∞ is dense in π_∞

As announced, we are going to prove the following theorem:

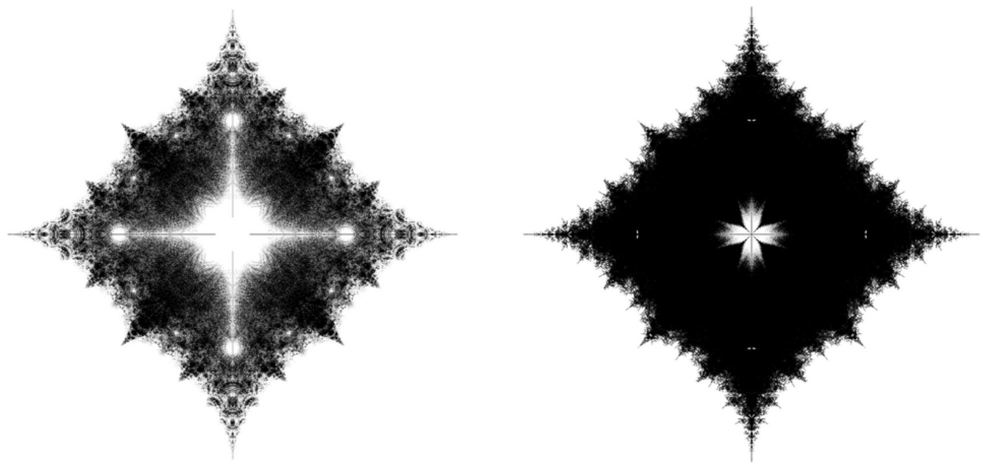
Theorem 4.58. *σ_∞ is a dense subset of π_∞.*

An approximation to σ_∞ and π_∞ can be found in Figure 4, which is due to the authors of [16]. We will use the notations

$$A^k_{per} := \begin{pmatrix} \ddots & \ddots & & & & & & \\ \ddots & 0 & 1 & & & & & \\ & k_m & 0 & 1 & & & & \\ & & k_1 & \ddots & \ddots & & & \\ & & & \ddots & \ddots & 1 & & \\ & & & & k_m & 0 & 1 & \\ & & & & & k_1 & 0 & \ddots \\ & & & & & & \ddots & \ddots \end{pmatrix} \in M_{per,m}(U_{-1}, U_0, U_1)$$

for $k \in \{\pm 1\}^m$ and

$$A^l_{fin} := \begin{pmatrix} 0 & 1 & & \\ l_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & l_n & 0 \end{pmatrix} \in M_{n+1}(U_{-1}, U_0, U_1)$$



(a) The eigenvalues of all 20×20 tridiagonal sign matrices.

(b) The spectra of all 20-periodic tridiagonal sign operators.

Figure 4

for $l \in \{\pm 1\}^n$ so that

$$\sigma_\infty = \bigcup_{n \in \mathbb{N}} \bigcup_{l \in \{\pm 1\}^n} \operatorname{sp}(A^l_{fin}), \qquad \pi_\infty = \bigcup_{m \in \mathbb{N}} \bigcup_{k \in \{\pm 1\}^m} \operatorname{sp}(A^k_{per}).$$

The symbol of A^k_{per} will be denoted by a^k.

Remark 4.59. Observe that k is not unique for a given periodic operator. For example, we can always use $(k_1, \dots, k_m, k_1, \dots, k_m) \in \{\pm 1\}^{2m}$ instead of $(k_1, \dots, k_m) \in \{\pm 1\}^m$ without changing the operator. We will make use of this freedom later on.

For the proof of Theorem 4.58 we need two auxiliary lemmas. In the first one we decompose the characteristic polynomial of the symbol a^k corresponding to A^k_{per}.

Lemma 4.60. *Let $m \in \mathbb{N}$ and $k \in \{\pm 1\}^m$. Then the only φ-dependent term in the characteristic polynomial of $a^k(\varphi)$ is the term of order zero. More precisely, there exists a polynomial $p_k \colon \mathbb{C} \to \mathbb{C}$ of degree m such that*

$$\det(a^k(\varphi) - \lambda I) = (-1)^m \left(p_k(\lambda) - e^{i\varphi} \prod_{j=1}^{m} k_j - e^{-i\varphi} \right) \tag{31}$$

for all $\varphi \in [0, 2\pi)$. The polynomial p_k is monic and given by

$$p_k(\lambda) = (-1)^m \left(\det \begin{pmatrix} -\lambda & 1 & & \\ k_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & k_{m-1} & -\lambda \end{pmatrix} - k_m \det \begin{pmatrix} -\lambda & 1 & & \\ k_2 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & k_{m-2} & -\lambda \end{pmatrix} \right).$$

Furthermore, p_k is an even (odd) function if m is even (odd).

Proof. The equation

$$\det(a^k(\varphi) - \lambda I) = (-1)^m \left(p_k(\lambda) - e^{i\varphi} \prod_{j=1}^{m} k_j - e^{-i\varphi} \right)$$

is a special case of Equation (20). That p_k is an even (odd) function if m is even (odd) follows easily by induction over m. □

The most important part of Lemma 4.60 is that there are no mixed terms of λ and φ in Equation (31). This leads to the fact (see Corollary 4.62 below) that the spectrum of every periodic operator A^k_{per} can be written as the preimage of the interval $[-2, 2]$ under some polynomial p_k. In this way the various parts of π_∞ are connected. We will make great use of this fact also in Section 4.4.3. But first observe the following. The term in Equation (31) involving φ can be simplified to $-2\cos(\varphi)$ or $2i\sin(\varphi)$ depending on the product $\prod\limits_{j=1}^{m} k_j$. To avoid unnecessary paperwork, we give the following definition.

Definition 4.61. Let $m \in \mathbb{N}$ and $k \in \{\pm 1\}^m$. Then we call k even if $\prod\limits_{j=1}^{m} k_j = 1$ and odd if $\prod\limits_{j=1}^{m} k_j = -1$.

Note that we can always assume that a periodic operator A^k_{per} has an even period k (cf. Remark 4.59).

Corollary 4.62. *Let $m \in \mathbb{N}$ and $k \in \{\pm 1\}^m$. Then $\mathrm{sp}(A^k_{per}) = p_k^{-1}([-2, 2])$ if k is even and $\mathrm{sp}(A^k_{per}) = p_k^{-1}(i[-2, 2])$ if k is odd.*

Proof. Let k be even first. By Theorem 4.33 and Lemma 4.60, we have

$$\begin{aligned} \mathrm{sp}(A^k_{per}) &= \{\lambda \in \mathbb{C} : \det(a(\varphi) - \lambda I) = 0 \text{ for some } \varphi \in [0, 2\pi)\} \\ &= \{\lambda \in \mathbb{C} : p_k(\lambda) - 2\cos(\varphi) = 0 \text{ for some } \varphi \in [0, 2\pi)\} \\ &= \{\lambda \in \mathbb{C} : p_k(\lambda) \in [-2, 2]\} \\ &= p_k^{-1}([-2, 2]). \end{aligned}$$

If k is odd, just replace $-2\cos(\varphi)$ by $2i\sin(\varphi)$. □

The second lemma is a discrete version of Theorem 4.33.

Lemma 4.63. *(e.g. [7, Section 2.1])*
Let $n, m \in \mathbb{N}$ and $A, B, C \in \mathbb{C}^{m\times m}$. Furthermore, denote by $e^{i\xi_1}, \ldots, e^{i\xi_n}$ the n-th roots of unity,

i.e. $\xi_j := \frac{2j}{n}\pi$ *for* $j \in \{1, \ldots, n\}$. *Then the following block matrices are unitarily equivalent:*

$$T_1 := \begin{pmatrix} B & C & & A \\ A & \ddots & \ddots & \\ & \ddots & \ddots & C \\ C & & A & B \end{pmatrix} \in \mathbb{C}^{nm \times nm},$$

$$T_2 := \begin{pmatrix} Ae^{i\xi_1} + B + Ce^{-i\xi_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & Ae^{i\xi_n} + B + Ce^{-i\xi_n} \end{pmatrix} \in \mathbb{C}^{nm \times nm}$$
$$= \operatorname{diag}(Ae^{i\xi_1} + B + Ce^{-i\xi_1}, \ldots, Ae^{i\xi_n} + B + Ce^{-i\xi_n}).$$

Proof. With the help of the Kronecker product $\otimes$, we can write $T_1 = P \otimes A + I_n \otimes B + P^* \otimes C$, where

$$P := \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

P is unitarily equivalent to the diagonal matrix

$$D := \begin{pmatrix} e^{i\xi_1} & & \\ & \ddots & \\ & & e^{i\xi_n} \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

Thus T_1 is unitarily equivalent to $D \otimes A + I_n \otimes B + D^* \otimes C$, which is exactly T_2. $\square$

Using these results, we are now able to prove Theorem 4.58.

Proof of Theorem 4.58. That $\sigma_\infty \subset \pi_\infty$ holds was proven in [16, Theorem 4.1] and sketched in Section 4.4.1. So let $m \in \mathbb{N}$, $k \in \{\pm 1\}^m$ and consider the operator A^k_{per}. W.l.o.g. we can assume that k is even (cf. Remark 4.59). This implies $\operatorname{sp}(A^k_{per}) = p_k^{-1}([-2, 2])$ by Corollary 4.62. In particular, the set

$$S^k_\infty := \left\{\lambda \in \mathbb{C} : \det(a^k(\varphi) - \lambda I_m) = 0 \text{ for some } \varphi \in \pi(\mathbb{Q} \setminus \mathbb{Z})\right\} = p_k^{-1}(2\cos(\pi(\mathbb{Q} \setminus \mathbb{Z})))$$

is dense in $\operatorname{sp}(A^k_{per})$, cf. Figure 5. Let $n \in \mathbb{N}$ and $\xi_j := \frac{2j}{n}\pi$ for $j \in \{1, \ldots, n\}$. We will show that for every $n \in \mathbb{N}$ there exists a finite matrix $A^l_{fin} \in \mathbb{C}^{(nm-1)\times(nm-1)}$ such that

$$S^k_n := \bigcup_{j \in \{1, \ldots, n-1\} \setminus \{\frac{n}{2}\}} \operatorname{sp}(a^k(\xi_j)) \subset \operatorname{sp}(A^l_{fin}) \subset \sigma_\infty. \tag{32}$$

Since

$$S^k_\infty = \bigcup_{n \in \mathbb{N}} S^k_n$$

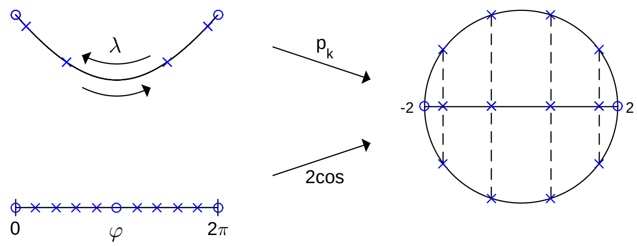


Figure 5: A sketch of the maps involved in the computation of the spectrum of a periodic operator A^k_{per}.

holds, this implies

$$S^k_\infty = \bigcup_{n\in\mathbb{N}} \bigcup_{j\in\{1,\dots,n-1\}\setminus\{\frac{n}{2}\}} \operatorname{sp}(a^k(\xi_j)) \subset \sigma_\infty.$$

Because $k \in \{\pm 1\}^m$ and $m \in \mathbb{N}$ were arbitrary, this implies that σ_∞ contains a dense subset of π_∞. Hence the assertion follows.

In order to construct this matrix A^l_{fin}, we arrange the matrices $a^k(\xi_j)$ in a block diagonal matrix and use Lemma 4.63:

$$\begin{pmatrix} a^k(\xi_1) & & & \\ & a^k(\xi_2) & & \\ & & \ddots & \\ & & & a^k(\xi_n) \end{pmatrix} \cong \begin{pmatrix} B & C & & A \\ A & \ddots & \ddots & \\ & \ddots & \ddots & C \\ C & & A & B \end{pmatrix} =: M \in \mathbb{C}^{nm\times nm},$$

where

$$A = \begin{pmatrix} & & k_m \\ & & \\ & & \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & & \\ k_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & k_{m-1} & 0 \end{pmatrix}, C = \begin{pmatrix} & & \\ & & \\ 1 & & \end{pmatrix}.$$

Now since $\det(a^k(\varphi) - \lambda I) = (-1)^m(p_k(\lambda) - 2\cos(\varphi))$ by Lemma 4.60, $a^k(\xi_j)$ and $a^k(\xi_{n-j})$ have the same spectrum for all $j \in \{1, \dots, n-1\}$. This implies that every eigenspace of M (excluding those coming from $a^k(\xi_{\frac{n}{2}})$ if n is even) corresponding to some $\lambda \in S^k_n$ is at least two-dimensional. Therefore we can choose two linearly independent eigenvectors $v, w \in \mathbb{C}^{nm}$ of M corresponding to λ and a non-trivial linear combination $x := \alpha v + \beta w \in \mathbb{C}^{nm}$ with $\alpha, \beta \in \mathbb{C}$ such that $x_1 = 0$. Of course, x is still an eigenvector of M corresponding to λ. Let $A^l_{fin} := M_{2;nm}$. Then we have

$$\begin{pmatrix} 0 & 1 & & k_m \\ k_1 & & & \\ & & \boxed{A^l_{fin}} & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 0 \\ \boxed{\begin{matrix} x_2 \\ \vdots \\ x_{nm} \end{matrix}} \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \boxed{\begin{matrix} x_2 \\ \vdots \\ x_{nm} \end{matrix}} \end{pmatrix}.$$

This implies that λ is also an eigenvalue of A^l_{fin}, hence (32) follows. $\square$

Theorem 4.58 heavily relies on the fact that all elements on the first sub- and superdiagonal have the same absolute value (and on tridiagonality of course). These cases are usually called critical or degenerated. Here degenerated refers to the fact that if all elements on the first sub- and superdiagonal have the same absolute value, the spectra of the periodic operators, which are loops in general, are degenerated to an arc. Changing the alphabet in the first subdiagonal to $\{\pm\sigma\}$, where $\sigma \in (0,1)$, as considered in [17] or [36] for example, the conclusion fails. This was of course to be expected because a finite matrix

$$\begin{pmatrix} 0 & l_1 & & \\ k_1 & \ddots & \ddots & \\ & \ddots & \ddots & l_n \\ & & k_n & 0 \end{pmatrix} \in \mathbb{C}^{(n+1)\times(n+1)}, k \in \{\pm\sigma\}^n, l \in \{\pm 1\}^n$$

is always similar to a matrix

$$\begin{pmatrix} 0 & \tilde{l}_1 & & \\ \tilde{k}_1 & \ddots & \ddots & \\ & \ddots & \ddots & \tilde{l}_n \\ & & \tilde{k}_n & 0 \end{pmatrix} \in \mathbb{C}^{(n+1)\times(n+1)}, \tilde{k} \in \left\{\pm\sqrt{\sigma}\right\}^n, \tilde{l} \in \left\{\pm\sqrt{\sigma}\right\}^n$$

(compare with the transformation used at the beginning of Section 4.4.1). This is no longer true for periodic operators. For tridiagonal operators on $\ell^2(\mathbb{Z})$ we can only shift phases to the other side. This remaining freedom, however, can be used to prove Theorem 4.58 for arbitrary alphabets as long as all elements share the same absolute value. This only needs Corollary 4.44 and a small modification of Lemma 4.63.

4.4.3 Symmetries

Next we show that π_∞ has an infinite number of symmetries. More precisely, we will show that there is an infinite set of polynomials S such that

$$p(\lambda) \in \pi_\infty \Longrightarrow \lambda \in \pi_\infty \tag{33}$$

for all $p \in S$. In [17] it was shown that the polynomial given by $p(z) = z^2$ is contained in S. Since $[-2,2] = \mathrm{sp}(L(1,0,1)) \subset \pi_\infty$, this implies $\mathbb{D} \subset \mathrm{clos}(\pi_\infty)$. Combining this with Theorem 4.58,

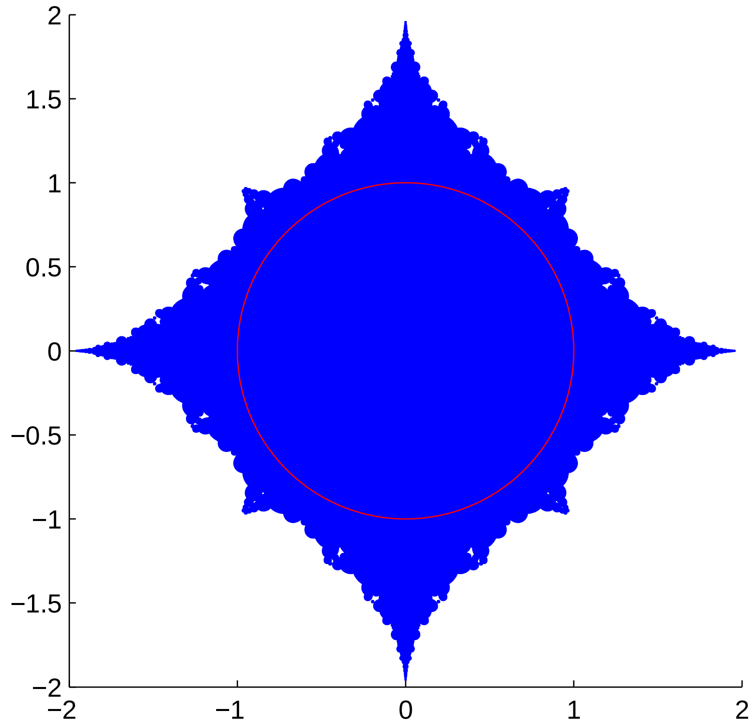


Figure 6: $\bigcup_{p\in S_{15}} p^{-1}(\mathbb{D})$, where $S_{15} := \{p \in S : \deg(p) \le 15\}$

Equation (33) and the fact that non-constant polynomials are open, we obtain new lower bounds to Σ:

$$p^{-1}(\mathbb{D}) \subset p^{-1}(\mathrm{clos}(\sigma_\infty)) = p^{-1}(\mathrm{clos}(\pi_\infty)) \subset \mathrm{clos}(\pi_\infty) \subset \Sigma$$

for every $p \in S$. Combining these lower bounds, we get

$$\bigcup_{p\in S} p^{-1}(\mathbb{D}) \subset \mathrm{clos}(\sigma_\infty) = \mathrm{clos}(\pi_\infty) \subset \Sigma. \tag{34}$$

The improvement is significant as Figure 6 shows.

Clearly, this construction can also be iterated, i.e.

$$p^{-n}(\mathbb{D}) \subset p^{-n}(\mathrm{clos}(\sigma_\infty)) = p^{-n}(\mathrm{clos}(\pi_\infty)) \subset \mathrm{clos}(\pi_\infty) \subset \Sigma$$

and hence

$$\bigcup_{n\in\mathbb{N}} p^{-n}(\mathbb{D}) \subset \bigcup_{n\in\mathbb{N}} p^{-n}(\mathrm{clos}(\sigma_\infty)) = \bigcup_{n\in\mathbb{N}} p^{-n}(\mathrm{clos}(\pi_\infty)) \subset \mathrm{clos}(\pi_\infty) \subset \Sigma.$$

This implies that Σ contains an infinite sequence of (presumably filled) Julia sets (see Remark 4.70 below), e.g. the set indicated in Figure 7.

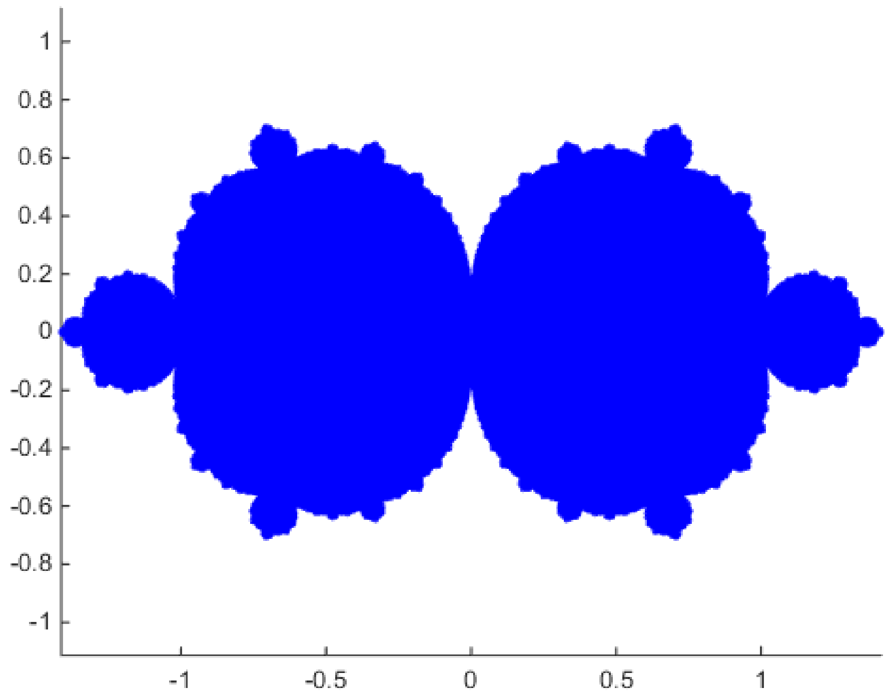


Figure 7: The filled Julia set corresponding to $p(\lambda) = \lambda^3 - \lambda$.

These two approaches can also be combined as follows. Let T be the closure of S with respect to composition, i.e.

$$T = \{q \colon \mathbb{C} \to \mathbb{C} : q = p_1 \circ \ldots \circ p_n \text{ for } p_1, \ldots, p_n \in S, n \in \mathbb{N}\}.$$

Then

$$\bigcup_{q \in T} q^{-1}(\mathbb{D}) \subset \operatorname{clos}(\sigma_\infty) = \operatorname{clos}(\pi_\infty) \subset \Sigma.$$

In this way one can construct even more Julia sets that are contained in Σ. This richness of symmetries might be a part of an explanation of the seemingly fractal boundary of Σ. Some more pictures of lower bounds to Σ can be found at the end of this section.

So let us now construct polynomials in S. We will need the following simple but beautiful proposition, which is in fact the cornerstone of the subsequent results.

Proposition 4.64. *Let $m \in \mathbb{N}$, $k \in \{\pm 1\}^m$ and let p_k be the corresponding polynomial given by Lemma 4.60. If k is even, then*

$$p_k(a^k(\varphi)) = 2\cos(\varphi) I$$

and $p_k(A^k_{per})$ is the Laurent operator with 1 on its m-th sub- and superdiagonal (and 0 everywhere *else):*

$$p_k(A^k_{per}) = \begin{pmatrix} & & \ddots & & & & \\ & & & 1 & & & \\ \ddots & & & & 1 & & \\ & 1 & & & & 1 & \\ & & 1 & & & & \ddots \\ & & & 1 & & & \\ & & & & \ddots & & \end{pmatrix}. \tag{35}$$

If k is odd, then

$$p_k(a^k(\varphi)) = -2i\sin(\varphi)I$$

and $p_k(A^k_{per})$ is the Laurent operator with -1 on its m-th subdiagonal and 1 on its m-th superdiagonal (and 0 everywhere else):

$$p_k(A^k_{per}) = \begin{pmatrix} & & \ddots & & & & \\ & & & 1 & & & \\ \ddots & & & & 1 & & \\ & -1 & & & & 1 & \\ & & -1 & & & & \ddots \\ & & & -1 & & & \\ & & & & \ddots & & \end{pmatrix}. \tag{36}$$

Proof. In both cases the first part follows immediately from Lemma 4.60 and the theorem of Cayley-Hamilton. For the second part observe that $2\cos(\varphi)I = (e^{i\varphi} + e^{-i\varphi})I$ is the symbol of (35) and $-2i\sin(\varphi)I_m = (-e^{i\varphi} + e^{-i\varphi})I_m$ is the symbol of (36). The assertion thus follows by Theorem 4.33. $\square$

Using this, we get the following theorem:

Theorem 4.65. *Let $m \geq 2$, $k := (k_1, \dots, k_{m-2}, -1, 1) \in \{\pm 1\}^m$ and $\hat{k} := (k_1, \dots, k_{m-2}, 1, -1) \in \{\pm 1\}^m$. Furthermore, let $b \in \{\pm 1\}^{\mathbb{Z}}$ and*

$$A^b := \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & b_0 & \boxed{0} & 1 & \\ & & b_1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

where the box indicates $A^b_{0,0}$. If the corresponding polynomials p_k and $p_{\hat{k}}$ are equal, then there exist $c \in \{\pm 1\}^{\mathbb{Z}}$ and $B \in \mathcal{L}(\ell^2(\mathbb{Z}))$ such that

$$p_k(A^c) \cong B \oplus A^b,$$

where we consider the following decomposition of the Hilbert space $\ell^2(\mathbb{Z})$*:*

$$\ell^2(\mathbb{Z}) \cong \ell^2(\mathbb{Z} \setminus m\mathbb{Z}) \oplus \ell^2(m\mathbb{Z}) \cong \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}).$$

In particular, $\mathrm{sp}(A^b) \subset \mathrm{sp}(p_k(A^c))$.

Proof. Let $c \in \{\pm 1\}^{\mathbb{Z}}$ be the sequence defined by:

- $c_0 = 1$, $c_1 = -1$,
- $c_{rm+j} = k_{j-1}$ for $j \in \{2, \ldots, m-1\}$, $r \in \mathbb{Z}$,
- $\prod_{j=0}^{m-1} c_{rm-j} = b_r$ for $r \in \mathbb{Z}$,
- $c_{rm+1} = -c_{rm}$ for $r \in \mathbb{Z}$.

Note that A^c is very similar to A^k_{per} and $A^{\hat{k}}_{per}$. The difference is that, depending on the sequence b, the entries c_{rm} and c_{rm+1} are swapped for some $r \in \mathbb{Z}$. For $m = 2$ this is exactly the same construction as in [17]. First we will prove the following claim by induction:

Claim 1: $((A^c)^s)_{i,i-s+2j}$ only depends on the coefficients $c_{i-s+j+1}, \ldots, c_{i+j}$ for $s \in \mathbb{N}$, $i \in \mathbb{Z}$ and $j \in \{0, \ldots, s-1\}$. Furthermore, $((A^c)^s)_{i,i+s} = 1$ for all $s \in \mathbb{N}$, $i \in \mathbb{Z}$ and all other entries are 0.

For $s = 1$ we have $(A^c)_{i,i-1+2j} = c_{i+j}$ for $j = 0$, $i \in \mathbb{Z}$, and $(A^c)_{i,i+1} = 1$ for $i \in \mathbb{Z}$. All other entries are 0. So assume that the claim holds for $s - 1$. Then

$$\begin{aligned} ((A^c)^s)_{i,i-s+2j} &= (A^c)_{i,i+1}((A^c)^{s-1})_{i+1,i-s+2j} + (A^c)_{i,i-1}((A^c)^{s-1})_{i-1,i-s+2j} \\ &= ((A^c)^{s-1})_{i+1,i-s+2j} + c_i((A^c)^{s-1})_{i-1,i-s+2j} \\ &= f(c_{i-s+j+2}, \ldots, c_{i+j}) + c_i g(c_{i-s+j+1}, \ldots, c_{i+j-1}), \end{aligned} \tag{37}$$

where f and g are some polynomials. Observe that c_i is contained in $\{c_{i-s+j+1}, \ldots, c_{i+j}\}$ for all $i \in \mathbb{Z}$, $j \in \{0, \ldots, s-1\}$. Thus $((A^c)^s)_{i,i-s+2j}$ only depends on the coefficients $c_{i-s+j+1}, \ldots, c_{i+j}$ for all $i \in \mathbb{Z}$, $j \in \{0, \ldots, s-1\}$. Plugging $j = s$ into (37) yields $((A^c)^s)_{i,i+s} = 1$ for all $i \in \mathbb{Z}$ by the same induction argument. Plugging in $j \in \mathbb{Z} + \frac{1}{2}$ and $j \in \mathbb{Z} \setminus \{0, \ldots, s\}$ shows that all other entries are 0. This finishes the proof of the claim.

Using Claim 1 with $s = m$ and $i = rm$ for $r \in \mathbb{Z}$, we get that $((A^c)^m)_{rm,(r-1)m+2j}$ only depends on the coefficients $c_{(r-1)m+j+1}, \ldots, c_{rm+j}$ for all $j \in \{0, \ldots, m-1\}$, $((A^c)^m)_{rm,rm+m} = 1$ and all other entries in row rm are 0. Similarly, $((A^c)^m)_{(r-1)m+2j,rm}$ only depends on the coeffcients $c_{(r-1)m+j+1}, \ldots, c_{rm+j}$ for all $j \in \{1, \ldots, m\}$, $((A^c)^m)_{rm-m,rm} = 1$ and all other entries in column rm are 0. Moreover, Claim 1 also implies that the same is true for $p_k(A^c)$ because p_k is an even/odd monic polynomial of degree m by Lemma 4.60.

Claim 2: $((A^c)^m)_{i,i-m} = \prod_{j=0}^{m-1} c_{i-j}$ for all $i \in \mathbb{Z}$.

This again follows easily by induction:

$$\begin{aligned} ((A^c)^m)_{i,i-m} &= (A^c)_{i,i-1}((A^c)^{m-1})_{i-1,i-m} + (A^c)_{i,i+1}((A^c)^{m-1})_{i+1,i-m} \\ &= c_i \cdot ((A^c)^{m-1})_{i-1,i-m} + 1 \cdot 0 \\ &= c_i \cdot \ldots \cdot c_{i-(m-1)}. \end{aligned}$$

Since $((A^c)^s)_{i,i-m} = 0$ for all $s < m$, also $(p_k(A^c))_{i,i-m} = \prod_{j=0}^{m-1} c_{i-j}$ for all $i \in \mathbb{Z}$. By definition of c, it thus follows

$$(p_k(A^c))_{rm,(r-1)m} = \prod_{j=0}^{m-1} c_{rm-j} = b_r$$

for all $r \in \mathbb{Z}$. Furthermore, $(p_k(A^c))_{rm,(r-1)m+2j}$ only depends on the coefficients $c_{(r-1)m+j+1}, \ldots,$ c_{rm+j} for $j \in \{1, \ldots, m-1\}$. In particular, these numbers all depend on c_{rm} and c_{rm+1} but not on $c_{(r-1)m}$, $c_{(r-1)m+1}$, $c_{(r+1)m}$ or $c_{(r+1)m+1}$. This implies

$$(p_k(A^c))_{rm,(r-1)m+2j} = (p_k(A^k_{per}))_{rm,(r-1)m+2j} = 0 \quad (\text{if } c_{rm} = -1, c_{rm+1} = 1)$$

or (using $p_k = p_{\hat{k}}$)

$$(p_k(A^c))_{rm,(r-1)m+2j} = (p_{\hat{k}}(A^{\hat{k}}_{per}))_{rm,(r-1)m+2j} = 0 \quad (\text{if } c_{rm} = 1, c_{rm+1} = -1)$$

for $j \in \{1, \ldots, m-1\}$ by Proposition 4.64. In other words, the entries $(p_k(A^c))_{rm,(r-1)m+2j}$ ($j \in \{1, \ldots, m-1\}$) can not "know" whether we swapped some of the entries c_{lm} and c_{lm+1} ($l \in \mathbb{Z}$) or not. Thus they have to remain zero. Similarly, the entries $(p_k(A^c))_{(r-1)m+2j,rm}$ ($j \in \{1, \ldots, m-1\}$) remain 0. Therefore $p_k(A^c)$ looks like this (where $*$ means "some unimportant entries"):

$$p_k(A^c) = \left(\begin{array}{ccccccccccccc}
 & \vdots & \vdots & \vdots & \ddots & & & & & & & & \\
\ldots & * & \vdots & * & \ldots & * & & & & & & & \\
\ldots & \ldots & 0 & \ldots & \ldots & 0 & 1 & & & & & & \\
\ldots & * & \vdots & * & \ldots & * & 0 & * & & & & & \\
\ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & & & & \\
 & * & 0 & * & \ldots & * & \vdots & * & \ldots & * & & & \\
 & & b_r & 0 & \ldots & \ldots & 0 & \ldots & \ldots & 0 & 1 & & \\
 & & & * & \ldots & * & \vdots & * & \ldots & * & 0 & * & \\
 & & & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \\
 & & & & & * & 0 & * & \ldots & * & \vdots & * & \ldots \\
 & & & & & & b_{r+1} & 0 & \ldots & \ldots & 0 & \ldots & \ldots \\
 & & & \multicolumn{3}{c}{\underbrace{\hphantom{xxxxxxxxxx}}_{m-1 \text{ columns}}} & & * & \ldots & * & \vdots & * & \ldots \\
 & & & & & & & & \ddots & \vdots & \vdots & \vdots &
\end{array}\right).$$

Decomposing our Hilbert space $\ell^2(\mathbb{Z}) \cong \ell^2(\mathbb{Z} \setminus m\mathbb{Z}) \oplus \ell^2(m\mathbb{Z}) \cong \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, we get the following decomposition of $p_k(A^c)$:

$$p_k(A^c) \cong B \oplus A^b$$

for some $B \in \mathcal{L}(\ell^2(\mathbb{Z}))$. In particular, $\operatorname{sp}(A^b) \subset \operatorname{sp}(p_k(A^c))$. □

By construction of the sequence c, we also have the following important corollary for periodic operators.

Corollary 4.66. *Under the same assumptions as in Theorem 4.65, we have that if $b \in \{\pm 1\}^{\mathbb{Z}}$ is an n-periodic sequence with even period $(b_1, \dots, b_n)$, then c, as defined in the proof of Theorem 4.65, is an nm-periodic sequence with even period $(c_1, \dots, c_{nm})$. $B \in \mathcal{L}(\ell^2(\mathbb{Z}))$ then is a periodic operator, too. Furthermore, if we denote the symbols of A^b, A^c and B by a^b, a^c and a^B, then also $p_k(a^c)$ can be decomposed as $p_k(a^c) \cong a^B \oplus a^b$. In particular, $\operatorname{sp}(a^b(\varphi)) \subset \operatorname{sp}(p_k(a^c(\varphi)))$ for every $\varphi \in [0, 2\pi)$.*

Proof. The first part follows by contruction of c. It remains to prove that the symbol $p_k(a^c)$ can be decomposed in a similar way. By Theorem 4.33, A^c is unitarily equivalent to the multiplication operator $M_{a^c} \in \mathcal{L}(L^2([0, 2\pi), \mathbb{C}^{nm}))$. Let us denote this equivalence by $\mathcal{F}_{nm}$, i.e. $A^c = \mathcal{F}_{nm}^* M_{a^c} \mathcal{F}_{nm}$. It follows

$$p_k(A^c) = p_k(\mathcal{F}_{nm}^* M_{a^c} \mathcal{F}_{nm}) = \mathcal{F}_{nm}^* p_k(M_{a^c}) \mathcal{F}_{nm} = \mathcal{F}_{nm}^* M_{p_k(a^c)} \mathcal{F}_{nm}.$$

Furthermore, A^b is unitarily equivalent to the multiplication operator $M_{a^b} \in \mathcal{L}(L^2([0, 2\pi), \mathbb{C}^m))$ and B is unitarily equivalent to the multiplication operator $M_{a^B} \in \mathcal{L}(L^2([0, 2\pi), \mathbb{C}^{(n-1)m}))$. Let us denote these equivalences by $\mathcal{F}_m$ and $\mathcal{F}_{(n-1)m}$. Furthermore, let us denote the decomposition $\ell^2(\mathbb{Z}) \cong \ell^2(\mathbb{Z} \setminus m\mathbb{Z}) \oplus \ell^2(m\mathbb{Z})$ by U and the decomposition $L^2([0, 2\pi), \mathbb{C}^{nm}) \cong L^2([0, 2\pi), \mathbb{C}^{(n-1)m}) \oplus L^2([0, 2\pi), \mathbb{C}^m)$ (in the obvious way) by V. It is not hard to see that

$$(\mathcal{F}_{(n-1)m} \oplus \mathcal{F}_m) U \mathcal{F}_{nm}^* = V$$

holds. Thus

$$\begin{aligned} M_{p_k(a^c)} &= \mathcal{F}_{nm} p_k(A^c) \mathcal{F}_{nm}^* \\ &= \mathcal{F}_{nm} U^* (B \oplus A^b) U \mathcal{F}_{nm}^* \\ &= \mathcal{F}_{nm} U^* (\mathcal{F}_{(n-1)m}^* M_{a^B} \mathcal{F}_{(n-1)m} \oplus \mathcal{F}_m^* M_{a^b} \mathcal{F}_m) U \mathcal{F}_{nm}^* \\ &= V^* (M_{a^B} \oplus M_{a^b}) V. \end{aligned}$$

This implies $p_k(a^c) \cong a^B \oplus a^b$ (as functions of φ). □

Definition 4.67. We define

$$S := \{p_k : p_k \text{ is a polynomial such that the assumptions of Theorem 4.65 are satisfied}\}$$

as our set of symmetries of π_∞.

In the case of periodic operators we can prove the following stronger version of Theorem 4.65 that justifies the definition of S.

Theorem 4.68. *Let $n \in \mathbb{N}$, let $b \in \{\pm 1\}^{\mathbb{Z}}$ be an n-periodic sequence and let $k \in \{\pm 1\}^m$ be such that $p := p_k \in S$. Moreover, let $c \in \{\pm 1\}^{\mathbb{Z}}$ be the sequence constructed in the proof of Theorem 4.65. Then the following assertion holds:*

$$p(\lambda) \in \operatorname{sp}(A^b) \iff \lambda \in \operatorname{sp}(A^c).$$

Proof. W.l.o.g. we can assume that $(b_1, \dots, b_n)$ is even (see Remark 4.59). By Corollary 4.66, $(c_1, \dots, c_{nm})$ is even, too. Let q and r be the polynomials given by Lemma 4.60 corresponding to b and c. Also denote by a^b and a^c the symbols of A^b and A^c. Fix some $\varphi \in [0, 2\pi)$ and let $\mu \in \mathbb{C}$ be an eigenvalue of $a^b(\varphi)$. Again by Corollary 4.66, there exists some $\lambda \in \operatorname{sp}(a^c(\varphi))$ such that

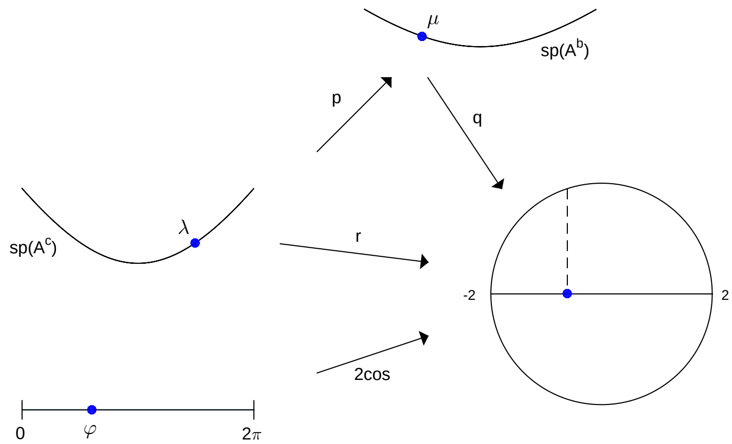


Figure 8: Schematic picture of the maps involved in the proof of Theorem 4.68.

$p(\lambda) = \mu$. Since $q(\mu) = 2\cos(\varphi)$ by Proposition 4.64, we have $q(p(\lambda)) = 2\cos(\varphi)$. On the other hand, also $r(\lambda) = 2\cos(\varphi)$ by Proposition 4.64. Thus

$$(q \circ p)(\lambda) = 2\cos(\varphi) = r(\lambda)$$

(cf. Figure 8). Since both $q \circ p$ and r are polynomials and the above argument is valid for every $\varphi \in [0, 2\pi)$, we conclude that $q \circ p$ and r are equal.

It follows

$$\mathrm{sp}(A^c) = r^{-1}([-2,2]) = p^{-1}(q^{-1}([-2,2])) = p^{-1}(\mathrm{sp}(A^b))$$

by Corollary 4.62. □

Theorem 4.68, combined with the result $\mathbb{D} \subset \mathrm{clos}(\pi_\infty)$ from [17], implies the following corollary.

Corollary 4.69. *Let $p \in S$. Then $p^{-1}(\pi_\infty) \subset \pi_\infty$ and hence $p^{-1}(\mathbb{D}) \subset p^{-1}(\mathrm{clos}(\pi_\infty)) \subset \mathrm{clos}(\pi_\infty)$.*

Recall that $k = (k_1, \ldots, k_m) \in \{\pm 1\}^m$ generates a polynomial $p = p_k \in S$ if $k_{m-1} \neq k_m$, so that $\hat{k} = (k_1, \ldots, k_{m-2}, k_m, k_{m-1}) \neq k$ but still $p_k = p_{\hat{k}}$. Thus it is immediate that all k of the form $(1, \ldots, 1, -1, 1)$ and $(-1, \ldots, -1, -1, 1)$ generate a polynomial $p_k \in S$. Indeed, if $k = (1, \ldots, 1, -1, 1)$ and $\hat{k} = (1, \ldots, 1, 1, -1)$, then A^k_{per} and $A^{\hat{k}}_{per}$ are unitarily equivalent by a simple shift. This implies that S contains a countable number of polynomials. However, there are a lot more than these trivial examples as the following table shows. We conjecture that there are approximately $2^{\lceil \frac{m}{2} \rceil - 1}$ polynomials of degree m in S.

No.	k	$p_k(\lambda)$
2.1	$(-1,1)$	λ^2
3.1	$(1,-1,1)$	$\lambda^3-\lambda$
3.2	$(-1,-1,1)$	$\lambda^3+\lambda$
4.1	$(1,1,-1,1)$	$\lambda^4-2\lambda^2$
4.2	$(-1,-1,-1,1)$	$\lambda^4+2\lambda^2$
5.1	$(1,1,1,-1,1)$	$\lambda^5-3\lambda^3+\lambda$
5.2	$(1,-1,1,-1,1)$	$\lambda^5-\lambda^3+\lambda$
5.3	$(-1,1,-1,-1,1)$	$\lambda^5+\lambda^3+\lambda$
5.4	$(-1,-1,-1,-1,1)$	$\lambda^5+3\lambda^3+\lambda$
6.1	$(1,1,1,1,-1,1)$	$\lambda^6-4\lambda^4+3\lambda^2$
6.2	$(1,-1,-1,1,-1,1)$	$\lambda^6-\lambda^2$
6.3	$(-1,-1,-1,-1,-1,1)$	$\lambda^6+4\lambda^4+3\lambda^2$
7.1	$(1,1,1,1,1,-1,1)$	$\lambda^7-5\lambda^5+6\lambda^3-\lambda$
7.2	$(1,1,-1,1,1,-1,1)$	$\lambda^7-3\lambda^5+2\lambda^3+\lambda$
7.3	$(1,-1,1,-1,1,-1,1)$	$\lambda^7-\lambda^5+2\lambda^3-\lambda$
7.4	$(1,-1,-1,-1,1,-1,1)$	$\lambda^7+\lambda^5-2\lambda^3+\lambda$
7.5	$(-1,1,1,1,-1,-1,1)$	$\lambda^7-\lambda^5-2\lambda^3-\lambda$
7.6	$(-1,1,-1,1,-1,-1,1)$	$\lambda^7+\lambda^5+2\lambda^3+\lambda$
7.7	$(-1,-1,1,-1,-1,-1,1)$	$\lambda^7+3\lambda^5+2\lambda^3-\lambda$
7.8	$(-1,-1,-1,-1,-1,-1,1)$	$\lambda^7+5\lambda^5+6\lambda^3+\lambda$

Table 1: Short list of elements in S.

Remark 4.70. As mentioned at the beginning of this section, we can iterate Corollary 4.69 to get

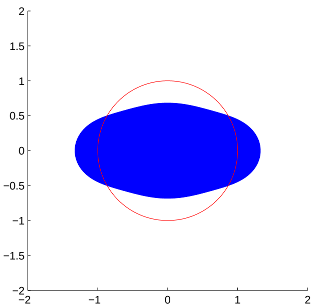
$$U := \bigcup_{n\in\mathbb{N}} p^{-n}(\mathbb{D}) \subset \bigcup_{n\in\mathbb{N}} p^{-n}(\operatorname{clos}(\pi_\infty)) \subset \operatorname{clos}(\pi_\infty)$$

for every $p \in S$. In other words, $z \in U$ if and only if $|p^n(z)| \le 1$ for some $n \in \mathbb{N}$. Thus there is clearly a connection to the filled Julia set corresponding to p, which is given by

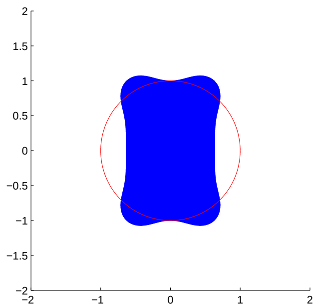
$$J_f(p) := \{z \in \mathbb{C} : (p^n(z))_{n\in\mathbb{N}} \text{ is bounded}\}.$$

(see [66, Lemma 17.1]). Indeed, the boundary $J(p) := \partial J_f(p)$ (which is usually just called the Julia set corresponding to p) is contained in the closure of $\bigcup_{n\in\mathbb{N}} p^{-n}(z)$ for every $z \in \mathbb{C}$ except for at most two points (see [6, Corollary 4.2,Corollary 4.7]). Hence $J(p) \subset \mathrm{clos}(U) \subset \mathrm{clos}(\pi_\infty)$. Considering the Figures $(viii)$ and (ix) below, it seems natural to conjecture that even the filled Julia set $J_f(p)$ is contained in $\mathrm{clos}(U)$.

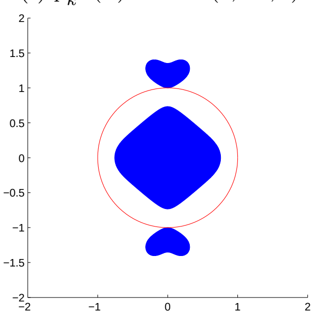
We conclude this section with some pictures of subsets of Σ. The red unit circle serves as a reference.



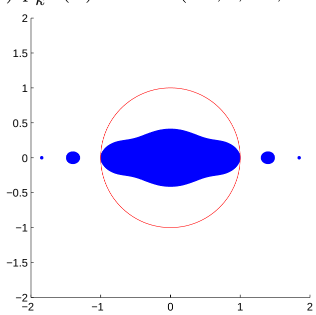
(i) $p_k^{-1}(\mathbb{D})$ for $k = (1, -1, 1)$



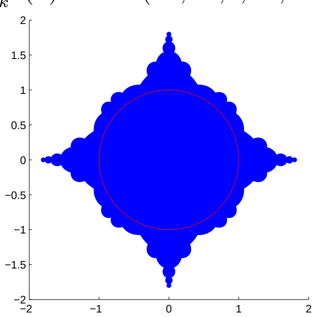
(ii) $p_k^{-1}(\mathbb{D})$ for $k = (-1, 1, -1, -1, 1)$



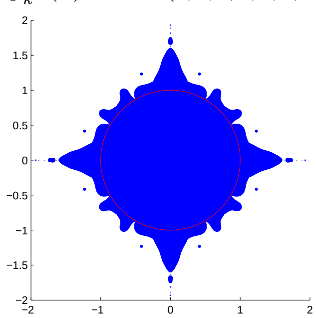
(iii) $p_k^{-1}(\mathbb{D})$ for $k = (-1, -1, 1, -1, -1, -1, 1)$



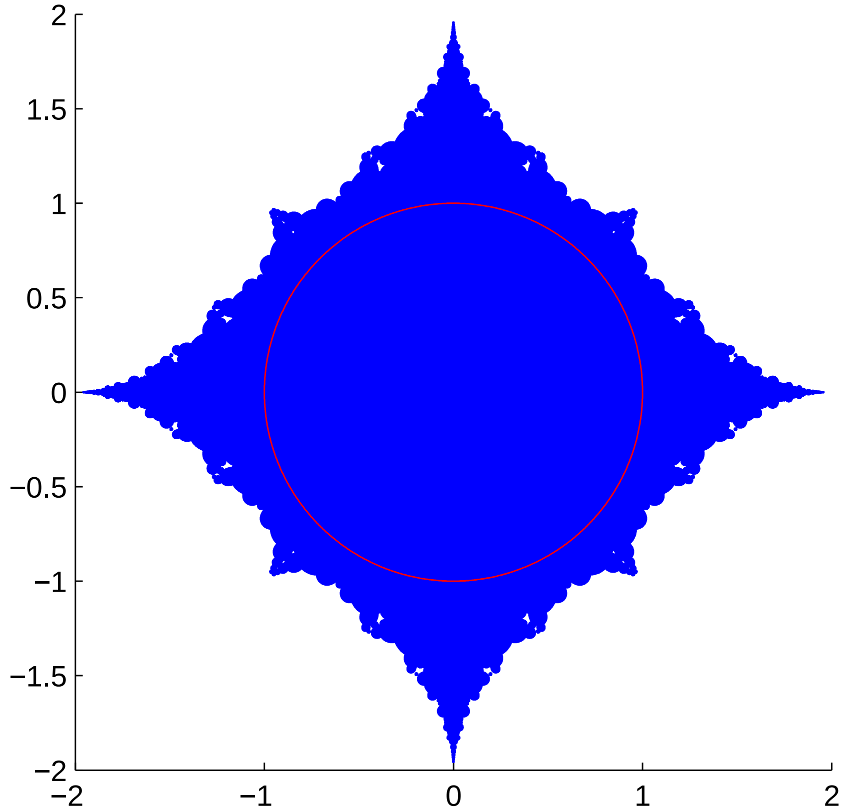
(iv) $p_k^{-1}(\mathbb{D})$ for $k = (1, 1, 1, 1, 1, 1, -1, 1)$



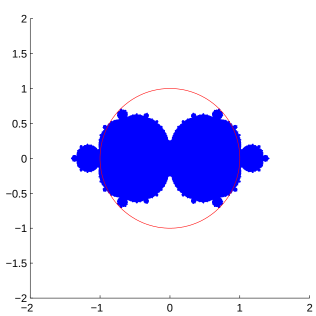
(v) $\bigcup_{\substack{p\in S \\ \deg(p)\le 7}} p^{-1}(\mathbb{D})$



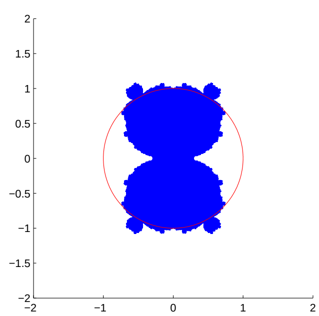
(vi) $\bigcup_{\substack{p\in S \\ \deg(p)=12}} p^{-1}(\mathbb{D})$



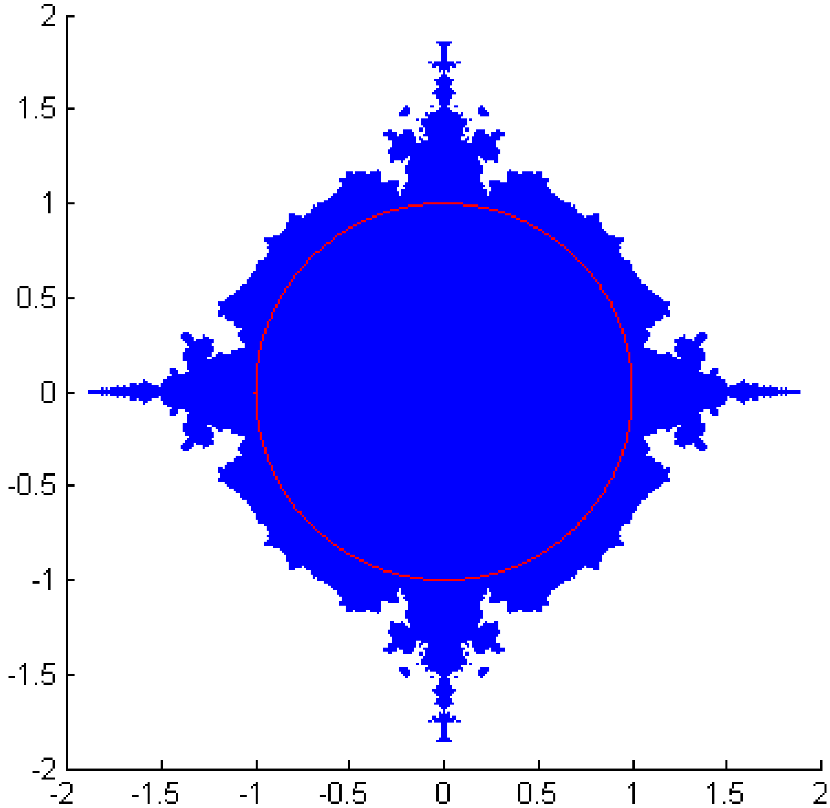
(vii) $\bigcup_{\substack{p\in S\\ \deg(p)\le 15}} p^{-1}(\mathbb{D})$



$(viii)$ $p_k^{-9}(\mathbb{D})$ for $k = (1, -1, 1)$



(ix) $p_k^{-6}(\mathbb{D})$ for $k = (-1, 1, -1, -1, 1)$



(x) $\bigcup_{\substack{p\in S\\ \deg(p)\le 8}} p^{-5}(\mathbb{D})$

Figure 11

4.4.4 Computing numerical ranges

In this section we present a method to compute numerical ranges of tridiagonal operators explicitly. Of course, there is no point in using this method for pseudo-ergodic operators because we already know their numerical range (Theorem 4.52). However, we will use this method to compute the numerical range of the square of the Feinberg-Zee random hopping matrix in Section 4.4.5. This will lead to an improved upper bound to the spectum. The method is based on the Schur test:

Theorem 4.71. *(Schur test [82])*
Let $\mathbb{I}$ be some countable index set, $\mathbf{X} = \ell^2(\mathbb{I})$ and let $A \in \mathcal{L}(\mathbf{X})$ have non-negative entries, i.e. $A_{i,j} \geq 0$ for all $i, j \in \mathbb{I}$. If there exist sequences $(x_i)_{i\in\mathbb{I}}$, $(y_i)_{i\in\mathbb{I}}$ in $\mathbb{R}_{>0}$ and numbers $a, b > 0$ such that

$$\sum_{j\in\mathbb{I}} A_{i,j}x_j \leq ay_i \quad \textit{and} \quad \sum_{j\in\mathbb{I}} A_{j,i}y_j \leq bx_i$$

for all $i \in \mathbb{I}$, *then* $\|A\| \leq \sqrt{ab}$.

The idea is the following ([47]): Since the numerical range is convex by Theorem 3.71, it suffices to compute the numerical abscissae $r_\varphi(A) = r_0(e^{i\varphi}A)$ for all angles $\varphi \in [0, 2\pi)$. Fix $\varphi \in [0, 2\pi)$ and let $B := \frac{1}{2}(e^{i\varphi}A + e^{-i\varphi}A^*)$. Then

$$r_\varphi(A) = \sup_{|x|=1} \operatorname{Re}\left\langle e^{i\varphi}Ax, x\right\rangle = \sup_{|x|=1} \frac{1}{2}\left\langle (e^{i\varphi}A + e^{-i\varphi}A^*)x, x\right\rangle = \sup_{|x|=1} \langle Bx, x\rangle = r_0(B) = \max_{\lambda \in \operatorname{sp}(B)} \lambda$$

since B is self-adjoint (see Theorem 3.72). In the case of a tridiagonal operator A the non-zero entries of B are given by

$$\begin{aligned} B_{j,j-1} &= \frac{1}{2}(e^{i\varphi}A_{j,j-1} + e^{-i\varphi}\overline{A_{j-1,j}}), \\ B_{j,j} &= \frac{1}{2}(e^{i\varphi}A_{j,j} + e^{-i\varphi}\overline{A_{j,j}}) = \operatorname{Re}(e^{i\varphi}A_{j,j}), \\ B_{j,j+1} &= \frac{1}{2}(e^{i\varphi}A_{j,j+1} + e^{-i\varphi}\overline{A_{j+1,j}}) \end{aligned}$$

for all $j \in \mathbb{I}$ that make sense. Every tridiagonal self-adjoint operator B is unitarily equivalent to a real symmetric operator C via a diagonal transformation that is defined as follows:

$$\begin{aligned} T_{1,1} &= 1, \\ T_{j+1,j+1} &= \operatorname{sign}(B_{j,j+1})T_{j,j} \end{aligned}$$

for all $j \in \mathbb{I}$ that make sense, where $\operatorname{sign}\colon \mathbb{C} \to \mathbb{T}$ is defined as

$$\operatorname{sign}(z) := \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

It is readily checked that $C = TBT^*$ is real and symmetric with $r_0(C) = r_0(B) = r_\varphi(A)$ and

$$\begin{aligned} C_{j,j} &= \operatorname{Re}(e^{i\varphi}A_{j,j}) \in \mathbb{R}, \\ C_{j,j+1} &= \frac{1}{2}\left|e^{i\varphi}A_{j,j+1} + e^{-i\varphi}\overline{A_{j+1,j}}\right| \geq 0. \end{aligned} \tag{38}$$

Thus the computation of $r_\varphi(A)$ is reduced to the computation of $r_0(C)$, which is equal to $\max\limits_{\lambda \in \operatorname{sp}(C)} \lambda$. In the following we can also assume that $C_{j,j+1} > 0$ for all j because if $C_{j,j+1} = 0$ for some j, then C can be divided into blocks and then the spectrum of C is given by the closure of the union of the spectra of these blocks. Moreover, shifting C by λI for some $\lambda \in \mathbb{R}$ only shifts the spectrum of C by λ. Therefore we can also assume that there are solely positive entries on the main diagonal of C. Applying Theorem 4.71 to C, we get the following lemma:

Lemma 4.72. *Let* $n \in \mathbb{N} \cup \{\infty\}$, $\mathbb{I} = \{1, \ldots, n\}$ *(i.e.* $\mathbb{I} = \mathbb{N}$ *if* $n = \infty$*),* $\mathbf{X} = \ell^2(\mathbb{I})$, *let* $C \in \mathcal{L}(\mathbf{X})$ *be real, symmetric and tridiagonal with* $C_{j,j}, C_{j,j+1} > 0$ *for all* $j \in \{1, \ldots, n-1\}$ *(*$j \in \mathbb{N}$ *if* $\mathbb{I} = \mathbb{N}$*) and* $N > \sup\limits_{j \in \mathbb{I}} C_{j,j}$. *If there is a sequence* $(g_j)_{j \in \mathbb{I}}$ *that satisfies* $0 \leq g_j \leq 1$ *and*

$$\frac{C_{j,j+1}^2}{(N - C_{j,j})(N - C_{j+1,j+1})} \leq g_{j+1}(1 - g_j) \tag{39}$$

for all $j \in \{1, \ldots, n-1\}$ *(*$j \in \mathbb{N}$ *if* $\mathbb{I} = \mathbb{N}$*), then* $r_0(C) \leq N$.

Proof. Let $(x_j)_{j\in\mathbb{I}}$ be the sequence given by

$$x_1 = 1 \qquad \text{and} \qquad x_{j+1} = \frac{C_{j,j+1}}{(N - C_{j+1,j+1})g_{j+1}} x_j$$

for $j \in \{1, \dots, n-1\}$ (observe that $g_{j+1} > 0$ by Equation (39)). Then $(x_j)_{j\in\mathbb{I}}$ is a positive sequence. To simplify notation we introduce the Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$$

and then $1 - \delta_{i,j}$ just means that the term it is attached to simply does not exist for $i = j$. Thus we have

$$\begin{aligned}
\sum_{k\in\mathbb{I}} C_{j,k}x_k &= C_{j,j-1}x_{j-1}(1-\delta_{j,1}) + C_{j,j}x_j + C_{j,j+1}x_{j+1}(1-\delta_{j,n}) \\
&= C_{j-1,j}\frac{(N - C_{j,j})g_j}{C_{j-1,j}} x_j(1-\delta_{j,1}) + C_{j,j}x_j + C_{j,j+1}\frac{C_{j,j+1}}{(N - C_{j+1,j+1})g_{j+1}} x_j(1-\delta_{j,n}) \\
&\leq ((N - C_{j,j})g_j + C_{j,j} + (N - C_{j,j})(1-g_j))x_j \\
&= Nx_j
\end{aligned}$$

for all $j \in \mathbb{I}$, where we used Condition (39) in the second line. Since C is symmetric and positive, Theorem 4.71 and Theorem 3.72 imply $N \geq \|C\| = r_0(C) = r_0(A)$. □

The next lemma is in some sense the converse of Lemma 4.72.

Lemma 4.73. *Let $n \in \mathbb{N} \cup \{\infty\}$, $\mathbb{I} = \{1, \dots, n\}$ ($\mathbb{I} = \mathbb{N}$ if $n = \infty$), $\mathbf{X} = \ell^2(\mathbb{I})$ and let $C \in \mathcal{L}(\mathbf{X})$ be real, symmetric and tridiagonal with $C_{j,j}, C_{j,j+1} > 0$ for all $j \in \{1, \dots, n-1\}$ ($j \in \mathbb{N}$ if $\mathbb{I} = \mathbb{N}$). Then there exists a sequence $(g_j)_{j\in\mathbb{I}}$ with the following properties:*

- *$0 \leq g_j \leq 1$ for all $j \in \mathbb{I}$,*
- *$g_j = 0$ if and only if $j = 1$,*
- *$g_j = 1$ only if $j = n$ ($g_j < 1$ for all $j \in \mathbb{I}$ if $\mathbb{I} = \mathbb{N}$),*
- *the following equality holds for all $1 \leq j \leq n-1$ ($j \in \mathbb{N}$ if $\mathbb{I} = \mathbb{N}$):*

$$\frac{C_{j,j+1}^2}{(r_0(C) - C_{j,j})(r_0(C) - C_{j+1,j+1})} = g_{j+1}(1 - g_j). \tag{40}$$

Proof. If $n = 1$, the theorem holds trivially. So let us assume $n \geq 2$ (including $\mathbb{I} = \mathbb{N}$).

We first observe that $r_0(C) > C_{j,j}$ for all $j \in \mathbb{I}$: Obviously we always have $r_0(C) \geq C_{j,j}$ since $\langle Ce_j, e_j\rangle = C_{j,j}$ for the j-th unit vector e_j. Let us assume $r_0(C) = C_{j,j}$ and choose $\lambda \geq 0$ sufficiently large so that $r_0(C + \lambda I) = \|C + \lambda I\|$ (cf. Theorem 3.72). Then

$$\begin{aligned}
r_0(C) &= \|C + \lambda I\| - \lambda \\
&\geq \|(C + \lambda I)e_j\| - \lambda \\
&= \sqrt{|C_{j-1,j}|^2(1-\delta_{j,1}) + |C_{j,j} + \lambda|^2 + |C_{j+1,j}|^2(1-\delta_{j,n})} - \lambda \\
&= \sqrt{|C_{j-1,j}|^2(1-\delta_{j,1}) + (r_0(C) + \lambda)^2 + |C_{j+1,j}|^2(1-\delta_{j,n})} - \lambda,
\end{aligned}$$

which implies $|C_{j-1,j}|^2\,(1-\delta_{j,1}) = |C_{j+1,j}|^2\,(1-\delta_{j,n}) = 0$, a contradiction since we assumed $n \geq 2$ and $C_{j,j+1} > 0$ for all $j \in \{1,\ldots,n-1\}$ ($j \in \mathbb{N}$ if $\mathbb{I} = \mathbb{N}$).

Let n be finite first. By the theorem of Perron-Frobenius [31], C has a strictly positive eigenvector x (i.e. $x_j > 0$ for all $j \in \{1,\ldots,n\}$) corresponding to the eigenvalue $r_0(C) = \|C\|$. Now choose the sequence $(g_j)_{j\in\mathbb{I}}$ as follows:

$$g_1 = 0 \qquad \text{and} \qquad g_{j+1} = \frac{C_{j+1,j}x_j}{(r_0(C) - C_{j+1,j+1})x_{j+1}} \tag{41}$$

for $j \in \{1,\ldots,n-1\}$. Thus we have $0 < g_j$ for all $1 < j \leq n$. Using

$$C_{j+1,j}x_j + C_{j+1,j+1}x_{j+1} < C_{j+1,j}x_j + C_{j+1,j+1}x_{j+1} + C_{j+1,j+2}x_{j+2} = r_0(C)x_{j+1},$$

we also get $g_j < 1$ for all $j \in \{1,\ldots,n-1\}$. Furthermore, we have the equality

$$C_{n,n-1}x_{n-1} + C_{n,n}x_n = r_0(C)x_n$$

and thus $g_n = 1$. So there is only property (40) left to prove:

$$\begin{aligned}
g_{j+1}(1-g_j) &= \frac{C_{j+1,j}x_j}{(r_0(C) - C_{j+1,j+1})x_{j+1}} \frac{(r_0(C) - C_{j,j})x_j - C_{j,j-1}x_{j-1}(1-\delta_{j,1})}{(r_0(C) - C_{j,j})x_j} \\
&= \frac{C_{j+1,j}}{(r_0(C) - C_{j+1,j+1})x_{j+1}} \frac{C_{j,j+1}x_{j+1}}{r_0(C) - C_{j,j}} \\
&= \frac{C^2_{j,j+1}}{(r_0(C) - C_{j,j})(r_0(C) - C_{j+1,j+1})}.
\end{aligned}$$

This proves the lemma in the case $n < \infty$. To prove the lemma also in the case $\mathbb{I} = \mathbb{N}$, we use a finite section approach. Let P_n be the projection onto span $\{e_1,\ldots,e_n\}$ and define $C^{(n)} := P_nCP_n$, interpreted as a finite matrix. Furthermore, let $x^{(n)}$ be a positive eigenvector of $C^{(n)}$ corresponding to the eigenvalue $r_0(C^{(n)})$ filled up with zeros such that $(x_j^{(n)})_{j\in\mathbb{N}}$ is an infinite sequence. W.l.o.g. we can assume $x_1^{(n)} = 1$ for all $n \in \mathbb{N}$. Using Remark 4.31, we get

$$r_0(C)x_1^{(n)} \geq r_0(C^{(n)})x_1^{(n)} = C_{1,1}^{(n)}x_1^{(n)} + C_{1,2}^{(n)}x_2^{(n)} = C_{1,1}x_1^{(n)} + C_{1,2}x_2^{(n)}$$

and

$$r_0(C)x_j^{(n)} \geq r_0(C^{(n)})x_j^{(n)} = C_{j,j-1}^{(n)}x_{j-1}^{(n)} + C_{j,j}^{(n)}x_j^{(n)} + C_{j,j+1}^{(n)}x_{j+1}^{(n)} = C_{j,j-1}x_{j-1}^{(n)} + C_{j,j}x_j^{(n)} + C_{j,j+1}x_{j+1}^{(n)}$$

for all $n \in \mathbb{N}$ and $j \in \{2,\ldots,n-1\}$. Thus every sequence $(x_j^{(n)})_{n\in\mathbb{N}}$ is bounded by induction (they are not necessarily uniformly bounded, of course). Choose a convergent subsequence $(x_1^{(n_{k_1})})_{k_1\in\mathbb{N}}$ of $(x_1^{(n)})_{n\in\mathbb{N}}$ and denote the limit by x_1 (of course $x_1 = 1$, but never mind). Now if x_j is defined, we choose a convergent subsequence $(x_{j+1}^{(n_{k_{j+1}})})_{k_{j+1}\in\mathbb{N}}$ of $(x_{j+1}^{(n_{k_j})})_{k_j\in\mathbb{N}}$ (we may always assume $n_{k_{j+1}} > j+1$, of course) and denote the limit by x_{j+1}. With these choices of sequences we have

$$r_0(C^{(n_{k_2})})x_1^{(n_{k_2})} = C_{1,1}^{(n_{k_2})}x_1^{(n_{k_2})} + C_{1,2}^{(n_{k_2})}x_2^{(n_{k_2})} = C_{1,1}x_1^{(n_{k_2})} + C_{1,2}x_2^{(n_{k_2})}$$

and

$$\begin{aligned} r_0(C^{(n_{k_{j+1}})})x_j^{(n_{k_{j+1}})} &= C_{j,j-1}^{(n_{k_{j+1}})}x_{j-1}^{(n_{k_{j+1}})} + C_{j,j}^{(n_{k_{j+1}})}x_j^{(n_{k_{j+1}})} + C_{j,j+1}^{(n_{k_{j+1}})}x_{j+1}^{(n_{k_{j+1}})} \\ &= C_{j,j-1}x_{j-1}^{(n_{k_{j+1}})} + C_{j,j}x_j^{(n_{k_{j+1}})} + C_{j,j+1}x_{j+1}^{(n_{k_{j+1}})} \end{aligned}$$

for fixed $j \geq 2$ and all $k_{j+1} \in \mathbb{N}$. Thus

$$r_0(C)x_1 = C_{1,1}x_1 + C_{1,2}x_2$$

and

$$r_0(C)x_j = C_{j,j-1}x_{j-1} + C_{j,j}x_j + C_{j,j+1}x_{j+1} \tag{42}$$

for all $j \geq 2$ since $\lim\limits_{k_{j+1}\to\infty} r_0(C^{(n_{k_{j+1}})}) = r_0(C)$ by Remark 4.31. Using $C_{j,j-1} > 0$, $C_{j,j+1} > 0$, $x_j \geq 0$ for all $j \in \mathbb{N}$ and $x_1 = 1$, we can conclude $x_j > 0$ for all $j \in \mathbb{N}$. Indeed, if $x_j = 0$, then Equation (42) implies $x_{j-1} = x_{j+1} = 0$. Thus $x_1 = 0$ by induction, a contradiction. Now we can define a sequence $(g_j)_{j\in\mathbb{N}}$ as in (41) and the desired properties follow similarly as in the finite-dimensional case. □

So if we are given a concrete tridiagonal $n \times n$ matrix A, we can deduce a real symmetric matrix C with $r_0(C) = r_\varphi(A)$ for a fixed angle $\varphi \in [0, 2\pi)$ as above. We can then make an educated guess $N > \max\limits_{j\in\{1,\dots,n\}} C_{j,j}$ for $r_0(C)$ and start computing the sequence recursively by the prescription

$$g_1 = 0 \qquad \text{and} \qquad g_{j+1} = \frac{C_{j,j+1}^2}{(1-g_j)(N-C_{j,j})(N-C_{j+1,j+1})}$$

for $j \in \{1,\dots,n-1\}$. Depending on the outcome of the sequence, we can decide whether $N < r_0(C)$, $N = r_0(C)$ or $N > r_0(C)$:

- if $g_j > 1$ (including ∞) for some $j \in \{1,\dots,n\}$, we have $N < r_0(C)$,
- if $g_j \leq 1$ for all $j \in \{1,\dots,n\}$, we have $N \geq r_0(C)$,
- if $g_j \leq 1$ for all $j \in \{1,\dots,n\}$ and $g_n = 1$, we have $N = r_0(C)$.

While the first two assertions are quite obvious from Lemma 4.72 and Lemma 4.73, the third assertion needs some clarification. So assume we got a sequence $(g_j)_{1\leq j\leq n}$ with $g_j \leq 1$ for all $j \in \{1,\dots,n\}$ and $g_n = 1$. From Lemma 4.72 we get $r_0(C) \leq N$. Assume $r_0(C) < N$. From Lemma 4.73 we then get another sequence $(h_j)_{1\leq j\leq n}$ with the properties $h_1 = 0$, $0 < h_j < 1$ for $1 < j < n$, $0 < h_n \leq 1$ and

$$\frac{C_{j,j+1}^2}{(r_0(C)-C_{j,j})(r_0(C)-C_{j+1,j+1})} = h_{j+1}(1-h_j).$$

But since

$$\frac{C_{j,j+1}^2}{(r_0(C)-C_{j,j})(r_0(C)-C_{j+1,j+1})} > \frac{C_{j,j+1}^2}{(N-C_{j,j})(N-C_{j+1,j+1})},$$

we have $h_j > g_j$ for all $j \in \{1,\dots,n\}$. Thus $h_n > 1$, a contradiction.

Using bisection, we could then narrow down $r_0(C) = r_\varphi(A)$ as precise as we want, although this may become a bit unstable because a small change in N can have huge effects on g_n. Then we could go on and compute the numerical abscissae $r_\varphi(A)$ for a bunch of angles and estimate the numerical range.

Although this algorithm actually works and is not the most inefficient one to compute a numerical range, it is not the best use of Lemma 4.72 and Lemma 4.73. What we are really interested in are infinite matrices. Of course, this algorithm cannot quite work for infinite matrices, not to mention the question on how to actually implement an infinite matrix... Therefore we have to be a bit smarter. The next proposition is a first example of how one can use Lemma 4.72 and Lemma 4.73 to compute numerical ranges of certain infinite matrices. In Section 4.4.5 we will use them to their full extent to compute the numerical range of the square of the Feinberg-Zee random hopping matrix. This will involve a somewhat combinatorial argument, which really should be viewed as the spirit of this method. Given the matrix entries, we have to combine a sequence such that Condition (39) is fulfilled.

Proposition 4.74. *Let $\mathbb{I} \in \{\mathbb{N}, \mathbb{Z}\}$, let $A \in \mathcal{L}(\ell^2(\mathbb{I}))$ be tridiagonal and 2-periodic and let $N > \sup\limits_{i\in\mathbb{I}} \operatorname{Re} A_{i,i}$. Further assume that $A + A^*$ is not diagonal. Define*

$$\eta_1(A) := \frac{\left|A_{1,2} + \overline{A_{2,1}}\right|^2}{4(N - \operatorname{Re} A_{1,1})(N - \operatorname{Re} A_{2,2})}, \qquad \eta_2(A) := \frac{\left|A_{2,3} + \overline{A_{3,2}}\right|^2}{4(N - \operatorname{Re} A_{2,2})(N - \operatorname{Re} A_{3,3})}.$$

Then we have $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ if and only if $N = r_0(A)$.

Proof. Let $A \in \mathcal{L}(\ell^2(\mathbb{Z}))$ be tridiagonal and 2-periodic and denote the corresponding one-sided periodic operator by A_+, i.e. $A_+ = P_\mathbb{N} A P_\mathbb{N}$ interpreted as operator on $\ell^2(\mathbb{N})$, where $P_\mathbb{N}$ denotes the orthogonal projection onto $\ell^2(\mathbb{N})$ as usual. Since $\langle A_+x, x\rangle = \langle Ax, x\rangle$ for all $x \in \operatorname{im} P_\mathbb{N}$, we have $N(A_+) \subset N(A)$.

Conversely, let $c \in N(A_+)$, $Q_\mathbb{N} := I - P_\mathbb{N}$ and consider $\tilde{A} := P_\mathbb{N} A P_\mathbb{N} + cQ_\mathbb{N}$. Then

$$\langle \tilde{A}x, x\rangle = \langle A_+P_\mathbb{N}x, P_\mathbb{N}x\rangle + c\,\langle Q_\mathbb{N}x, Q_\mathbb{N}x\rangle = \left\langle A_+\frac{P_\mathbb{N}x}{\|P_\mathbb{N}x\|}, \frac{P_\mathbb{N}x}{\|P_\mathbb{N}x\|}\right\rangle \|P_\mathbb{N}x\|^2 + c\,\|Q_\mathbb{N}x\|^2.$$

Since $\|P_\mathbb{N}x\|^2 + \|Q_\mathbb{N}x\|^2 = \|x\|^2$ and $N(A_+)$ is convex, we get $N(\tilde{A}) \subseteq N(A_+)$. Moreover, A is a limit operator of $\tilde{A}$ and thus $N(A) \subseteq N(\tilde{A})$ by Theorem 3.76. We conclude $N(A) = N(A_+)$. It therefore suffices to consider the case $\mathbb{I} = \mathbb{N}$ in the following.

Let C be as in (38) with $\varphi = 0$ so that $r_0(A) = r_0(C)$. We can assume that $C_{j,j} > 0$ for all $j \in \mathbb{N}$ (shifting by $\lambda \in \mathbb{R}$ does not change anything).

If $A_{1,2} + \overline{A_{2,1}} = 0$, then $\eta_1(A) = 0$ and an easy computation shows $\eta_2(A) = 1$ if and only if $N = r_0(A)$. The case $A_{2,3} + \overline{A_{3,2}} = 0$ is similar. So let us assume $A_{j,j+1} + \overline{A_{j+1,j}} \neq 0$ for all $j \in \mathbb{N}$ for the rest of the proof. Clearly, this implies $\eta_1(A), \eta_2(A) > 0$ and $C_{j,j+1} > 0$ for all $j \in \mathbb{N}$.

Let $N = r_0(A)$. Lemma 4.73 applied to C yields a sequence $(g_j)_{j\in\mathbb{N}}$ with the properties

- $0 \leq g_j < 1$ for all $j \in \mathbb{N}$
- $g_j = 0$ if and only if $j = 1$
- the following equality holds for all $j \in \mathbb{N}$:

$$\frac{\left|A_{j,j+1} + \overline{A_{j+1,j}}\right|^2}{4(r_0(A) - \operatorname{Re} A_{j,j})(r_0(A) - \operatorname{Re} A_{j+1,j+1})} = g_{j+1}(1 - g_j).$$

Since A is 2-periodic, we have

$$\begin{aligned}
\eta_1(A) &= \frac{\left|A_{1,2}+\overline{A_{2,1}}\right|^2}{4(r_0(A)-\text{Re } A_{1,1})(r_0(A)-\text{Re } A_{2,2})} = g_2 \\
\eta_2(A) &= \frac{\left|A_{2,3}+\overline{A_{3,2}}\right|^2}{4(r_0(A)-\text{Re } A_{2,2})(r_0(A)-\text{Re } A_{1,1})} = g_3(1-g_2) \\
\eta_1(A) &= \frac{\left|A_{1,2}+\overline{A_{2,1}}\right|^2}{4(r_0(A)-\text{Re } A_{1,1})(r_0(A)-\text{Re } A_{2,2})} = g_4(1-g_3) \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

We observe $\eta_1(A) = g_2 \in (0,1)$ and thus also $\eta_2(A) = g_3(1-g_2) \in (0,1)$. If j is odd, we deduce the following recursion:

$$g_{j+2} = \frac{\eta_2(A)}{1-g_{j+1}} = \frac{\eta_2(A)}{1-\frac{\eta_1(A)}{1-g_j}} = \frac{(1-g_j)\eta_2(A)}{1-g_j-\eta_1(A)}. \tag{43}$$

The corresponding iteration function

$$\begin{aligned}
&f : (0, 1-\eta_1(A)) \to \mathbb{R}, \\
&x \mapsto \frac{(1-x)\eta_2(A)}{1-x-\eta_1(A)}
\end{aligned} \tag{44}$$

has a positive derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{(1-x)\eta_2(A)}{1-x-\eta_1(A)} = \frac{\eta_1(A)\eta_2(A)}{(1-x-\eta_1(A))^2} > 0 \tag{45}$$

since $\eta_1(A), \eta_2(A) > 0$. Thus f is strictly increasing. Since $(g_j)_{j\in 2\mathbb{N}-1}$ is a sequence in $[0,1)$, it is in fact a sequence in $[0, 1-\eta_1(A))$. Indeed, if $g_j \geq 1-\eta_1(A)$, then by Equation (43), g_{j+2} is either not defined or negative, a contradiction. Moreover, we have

$$g_3 = \frac{\eta_2(A)}{1-\eta_1(A)} > 0 = g_1$$

since $\eta_1(A), \eta_2(A) \in (0,1)$. We conclude that $(g_j)_{j\in 2\mathbb{N}-1}$ is strictly increasing, hence convergent. Denote the limit of this sequence by x^*. By the fixed-point theorem, x^* has to be a fixed point of the iteration function f. After some rearranging, we get two possible candidates for a fixed point:

$$\begin{aligned}
\frac{(1-x^*)\eta_2(A)}{1-x^*-\eta_1(A)} = x^* \quad &\Leftrightarrow \quad (1-x^*)\eta_2(A) = x^*(1-x^*-\eta_1(A)) \\
&\Leftrightarrow \quad (x^*)^2 - (1+\eta_2(A)-\eta_1(A))x^* + \eta_2(A) = 0 \\
&\Leftrightarrow \quad x^* = \frac{1+\eta_2(A)-\eta_1(A) \pm \sqrt{(1+\eta_2(A)-\eta_1(A))^2 - 4\eta_2(A)}}{2}.
\end{aligned} \tag{46}$$

Of course the fixed point we are looking for has to be real and thus $(1+\eta_2(A)-\eta_1(A))^2 - 4\eta_2(A)$ has to be non-negative. It follows

$$\begin{aligned}
0 &\leq (1+\eta_2(A)-\eta_1(A))^2 - 4\eta_2(A) \\
&= 1 + \eta_2(A)^2 + \eta_1(A)^2 + 2\eta_2(A) - 2\eta_1(A) - 2\eta_1(A)\eta_2(A) - 4\eta_2(A) \\
&= \eta_2(A)^2 - 2(1+\eta_1(A))\eta_2(A) + (1-\eta_1(A))^2.
\end{aligned}$$

Solving for $\eta_2(A)$ yields

$$\eta_2(A) \leq 1 + \eta_1(A) - \sqrt{(1+\eta_1(A))^2 - (1-\eta_1(A))^2} = 1 + \eta_1(A) - 2\sqrt{\eta_1(A)} = (1 - \sqrt{\eta_1(A)})^2,$$

since $\eta_2(A) < 1$. This inequality now implies $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} \leq 1$. As we will prove later, this inequality is actually an equality.

Conversely, let $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$. Of course, we can again assume that $\mathbb{I} = \mathbb{N}$. Define the sequence $(g_j)_{j\in\mathbb{N}}$ as follows:

$$\begin{aligned} g_1 &:= 0, \\ g_{j+1} &:= \frac{\eta_1(A)}{1-g_j} \quad \text{if } j \text{ is odd}, \\ g_{j+1} &:= \frac{\eta_2(A)}{1-g_j} \quad \text{if } j \text{ is even}. \end{aligned}$$

In order to use Lemma 4.72, we have to check $g_j \in [0,1]$ for all $j \in \mathbb{N}$. Let us first consider $(g_j)_{j\in 2\mathbb{N}-1}$ and its iteration function (44). As seen in (46) the fixed points of f are given by

$$x^* = \frac{1 + \eta_2(A) - \eta_1(A) \pm \sqrt{(1+\eta_2(A) - \eta_1(A))^2 - 4\eta_2(A)}}{2}.$$

Plugging our assumption $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ into this equation, we get

$$\begin{aligned} x^* &= \frac{1 + \eta_2(A) - (1 - \sqrt{\eta_2(A)})^2 \pm \sqrt{(1 + \eta_2(A) - (1-\sqrt{\eta_2(A)})^2)^2 - 4\eta_2(A)}}{2} \\ &= \sqrt{\eta_2(A)} \pm \frac{\sqrt{4\eta_2(A) - 4\eta_2(A)}}{2} \\ &= \sqrt{\eta_2(A)}. \end{aligned}$$

Thus there is only one fixed point and $x^* < 1$. Furthermore, $g_1 = 0$ and thus $0 \leq g_j \leq x^* < 1$ for all $j \in 2\mathbb{N}-1$. We conclude $g_j \in [0,1]$ for odd j. Similarly (exchanging $\eta_1(A)$ and $\eta_2(A)$ and using the starting point $\eta_1(A) < \sqrt{\eta_1(A)} < 1$), we also get $g_j \in [0,1]$ for even j. Furthermore, Condition (39) is fulfilled by definition. Thus $(g_j)_{j\in\mathbb{N}}$ meets all the requirements and we can apply Lemma 4.72 to C, which implies $r_0(C) = r_0(A) \leq N$. So let us summarize what we have so far. We have

(*i*) $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} \leq 1$ if $N = r_0(A)$ and

(*ii*) $N \geq r_0(A)$ if $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$.

Now let $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ and assume $r_0(A) < N$. Then

$$\begin{aligned} \eta_1(A) &< \frac{\left|A_{1,2} + \overline{A_{2,1}}\right|^2}{4(r_0(A) - \text{Re } A_{1,1})(r_0(A) - \text{Re } A_{2,2})} =: \tilde{\eta}_1(A), \\ \eta_2(A) &< \frac{\left|A_{2,3} + \overline{A_{3,2}}\right|^2}{4(r_0(A) - \text{Re } A_{2,2})(r_0(A) - \text{Re } A_{3,3})} =: \tilde{\eta}_2(A) \end{aligned}$$

and thus $\sqrt{\tilde{\eta}_1(A)} + \sqrt{\tilde{\eta}_2(A)} > 1$. But this is a contradiction to (i). Thus $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ implies $N = r_0(A)$.

Conversely, let $N = r_0(A)$ and assume $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} < 1$. Then by continuity there exists an $\varepsilon > 0$ such that

$$\sqrt{\frac{\left|A_{1,2} + \overline{A_{2,1}}\right|^2}{4(N - \varepsilon - \operatorname{Re} A_{1,1})(N - \varepsilon - \operatorname{Re} A_{2,2})}} + \sqrt{\frac{\left|A_{2,3} + \overline{A_{3,2}}\right|^2}{4(N - \varepsilon - \operatorname{Re} A_{2,2})(N - \varepsilon - \operatorname{Re} A_{3,3})}} = 1.$$

This is a contradiction to (ii) since $N - \varepsilon < r_0(A)$. Thus $N = r_0(A)$ implies $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ and we proved all assertions. □

4.4.5 An improved upper bound to the spectrum

As seen in Theorem 4.52 and Corollary 4.53, the numerical range is a good upper bound to the spectrum of a tridiagonal pseudo-ergodic operator. However, since the numerical range is always convex by Theorem 3.71, we can only narrow the spectrum down to its convex hull. In order to get a bit further, we use the following idea inspired by [24, Section 9.4]. Since $A^2 - \lambda^2 I = (A - \lambda I)(A + \lambda I)$, it is clear that

$$\operatorname{sp}(A) \subset \left\{\lambda \in \mathbb{C} : \lambda^2 \in \operatorname{sp}(A^2)\right\} \subset \left\{\lambda \in \mathbb{C} : \lambda^2 \in N(A^2)\right\}.$$

Of course there is no guarantee that this indeed provides a better upper bound than $N(A)$. Fortunately, there is indeed an improvement for the Feinberg-Zee random hopping matrix. To compute $N(A^2)$, we will use the method introduced in 4.4.4. Here is the result:

Theorem 4.75. *Let $U_{-1} = \{1\}$, $U_0 = \{0\}$, $U_1 = \{\pm 1\}$ and $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$. Then*

$$N(A^2) = \operatorname{conv}\left(\bigcup_{B \in M_{per,4}(U_{-1},U_0,U_1)} N(B^2)\right).$$

That the right-hand side is a subset of the left-hand side is obvious by Theorem 4.10(v) and Proposition 2.26. To prove the other inclusion, we need to compute $N(B^2)$ for all $B \in M_{per,4}(U_{-1}, U_0, U_1)$ first. However, it turns out that it suffices to consider three of them (see Figure 12).

Proposition 4.76. *Let $U_{-1} = \{1\}$, $U_0 = \{0\}$ and $U_1 = \{\pm 1\}$ and let $B_1 \in M_{per,4}(U_{-1}, U_0, U_1)$ be an operator with period $(1, 1, 1, 1)$, $B_2 \in M_{per,4}(U_{-1}, U_0, U_1)$ an operator with period $(-1, -1, 1, 1)$ and $B_3 \in M_{per,4}(U_{-1}, U_0, U_1)$ an operator with period $(-1, -1, -1, -1)$. Then we have*

$$\begin{aligned} r_\varphi(B_1^2) &= 2\cos(\varphi) + 2\left|\cos(\varphi)\right|, \\ r_\varphi(B_2^2) &= 2, \\ r_\varphi(B_3^2) &= -2\cos(\varphi) + 2\left|\cos(\varphi)\right|, \end{aligned}$$

and the boundaries of $N(B_1^2)$, $N(B_2^2)$ and $N(B_3^2)$ are given by the following parametrizations:

$$\begin{aligned} \partial N(B_1^2) &: z(t) = 2 + 2\cos(t), \\ \partial N(B_2^2) &: z(t) = 2e^{it}, \\ \partial N(B_3^2) &: z(t) = -2 + 2\cos(t), \end{aligned}$$

$t \in [0, 2\pi)$.

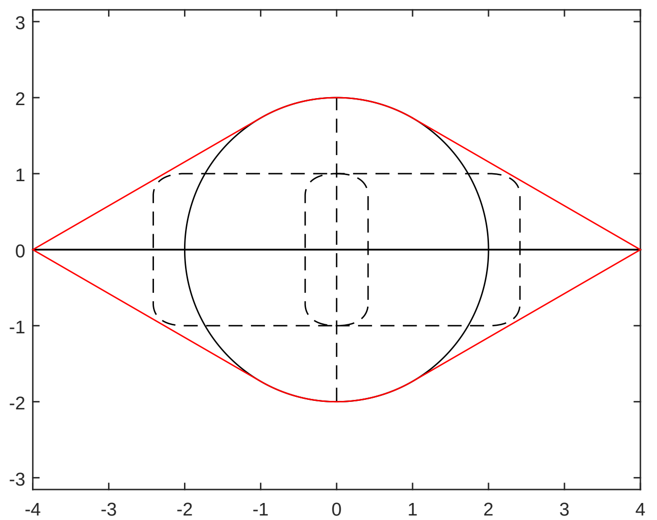


Figure 12: The boundaries of the numerical ranges of B_1^2, B_2^2 and B_3^2 (solid lines), the squares of the other operators in $M_{per,4}(U_{-1}, U_0, U_1)$ (dashed lines) and their convex hull (red).

Proof. B_1 is a Laurent operator with diagonals $(1)_{j\in\mathbb{Z}}$, $(0)_{j\in\mathbb{Z}}$ and $(1)_{j\in\mathbb{Z}}$ and therefore B_1^2 is a Laurent operator with diagonals $(1)_{j\in\mathbb{Z}}$, $(0)_{j\in\mathbb{Z}}$, $(2)_{j\in\mathbb{Z}}$, $(0)_{j\in\mathbb{Z}}$ and $(1)_{j\in\mathbb{Z}}$. Therefore the spectrum of B_1^2 is given by the degenerated ellipse $E := \{t \in [0, 2\pi) : 2 + 2\cos(t)\} = [0, 4]$ (see Theorem 4.33). Since Laurent operators are normal, E is also equal to the numerical range of B_1^2 (cf. Theorem 3.72). It is now easy to see that $r_\varphi(B_1^2) = 2\cos(\varphi) + 2\,|\cos(\varphi)|$.

Similarly, we get $r_\varphi(B_3^2) = -2\cos(\varphi) + 2\,|\cos(\varphi)|$.

B_2^2 is a 4-periodic operator that looks like this:

$$B_2^2 = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ & 1 & 0 & 0 & 0 & 1 & & & & \\ & & -1 & 0 & -2 & 0 & 1 & & & \\ & & & 1 & 0 & 0 & 0 & 1 & & \\ & & & & -1 & 0 & 2 & 0 & 1 & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Now we can split B_2^2 into an even (C_2) and an odd (D_2) part (use $\ell^2(\mathbb{Z}) \cong \ell^2(2\mathbb{Z}) \oplus \ell^2(2\mathbb{Z}+1)$) to get tridiagonal matrices:

$$C_2 = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad D_2 = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & -1 & -2 & 1 & & \\ & & -1 & 2 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

and $B_2 \cong C_2 \oplus D_2$. C_2 is again a Laurent operator. Thus we conclude that the boundary of the numerical range of C_2 is given by the degenerated ellipse $\{t \in [0, 2\pi) : 2\cos(t)\}$. D_2 is a 2-periodic operator, hence we can use Proposition 4.74. Let $D_{2,\varphi} := e^{i\varphi} D_2$, $N := 2$ and let us exclude the cases $\varphi \in \{0, \pi\}$ for the moment so that $\frac{1}{2}(D_{2,\varphi} + D_{2,\varphi}^*)$ is not diagonal. In the notation of Proposition 4.74 the numbers $\eta_1(D_{2,\varphi})$ and $\eta_2(D_{2,\varphi})$ are given by

$$\begin{aligned} \eta_1(D_{2,\varphi}) &= \frac{\left|e^{i\varphi} - e^{-i\varphi}\right|^2}{4(2 + 2\cos(\varphi))(2 - 2\cos(\varphi))} = \frac{\sin(\varphi)^2}{4 - 4\cos(\varphi)^2} = \frac{1}{4}, \\ \eta_2(D_{2,\varphi}) &= \eta_1(D_{2,\varphi}) = \frac{1}{4}. \end{aligned}$$

Thus $\sqrt{\eta_1(D_{2,\varphi})} + \sqrt{\eta_2(D_{2,\varphi})} = 1$. This implies $r_\varphi(D_2) = 2$ for all $\varphi \in (0, 2\pi) \setminus \{\pi\}$ by Proposition 4.74. In the remaining two cases $\frac{1}{2}(D_{2,\varphi} + D_{2,\varphi}^*)$ is a diagonal matrix and it is easily seen that $r_\varphi(D_2) = 2$ holds (one could also argue by continuity). Therefore we have $r_\varphi(D_2) = 2$ for all $\varphi \in [0, 2\pi)$. Now obviously $N(C_2) \subset N(D_2)$ holds and thus we get $r_\varphi(B_2^2) = 2$ for all $\varphi \in [0, 2\pi)$. A parametrization of $\partial N(B_2^2)$ is then of course given by $z(t) = 2e^{it}, t \in [0, 2\pi)$. $\square$

Next we have to compute

$$N(\varphi) := \max_{j \in \{1,2,3\}} r_\varphi(B_j^2) \tag{47}$$

for every $\varphi \in [0, 2\pi)$.

Proposition 4.77. *Let B_1, B_2 and B_3 be as in Proposition 4.76. Then*

$$N(\varphi) = \max_{j \in \{1,2,3\}} r_\varphi(B_j^2) = \begin{cases} 4\cos(\varphi) & \text{if } 0 \le \varphi \le \frac{\pi}{3}, \\ 2 & \text{if } \frac{\pi}{3} \le \varphi \le \frac{2\pi}{3}, \\ -4\cos(\varphi) & \text{if } \frac{2\pi}{3} \le \varphi \le \frac{4\pi}{3}, \\ 2 & \text{if } \frac{4\pi}{3} \le \varphi \le \frac{5\pi}{3}, \\ 4\cos(\varphi) & \text{if } \frac{5\pi}{3} \le \varphi < 2\pi. \end{cases}$$

Proof. By Proposition 4.76 and the continuity of all the functions involved, we only have to check where the graphs of $r_\varphi(B_1^2)$, $r_\varphi(B_2^2)$ and $r_\varphi(B_3^2)$ intersect. Let us have a look at $r_\varphi(B_1^2)$ and $r_\varphi(B_2^2)$ first:

$$\begin{aligned} r_\varphi(B_1^2) = r_\varphi(B_2^2) \quad &\Leftrightarrow \quad 2\cos(\varphi) + 2\left|\cos(\varphi)\right| = 2 \\ &\Leftrightarrow \quad 4\cos(\varphi)^2 = (2 - 2\cos(\varphi))^2 \\ &\Leftrightarrow \quad \cos(\varphi) = \frac{1}{2}. \end{aligned}$$

Thus the graphs of $r_\varphi(B_1^2)$ and $r_\varphi(B_2^2)$ only intersect at $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Similarly, the graphs of $r_\varphi(B_2^2)$ and $r_\varphi(B_3^2)$ only intersect at $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Finally, $r_\varphi(B_1^2)$ and $r_\varphi(B_3^2)$ obviously only intersect at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Plugging in some angles, one easily deduces the assertion. $\square$

Now let us focus on $N(A^2)$ and tabularize all possible combinations for (39) in Lemma 4.72. Let us denote the first subdiagonal of $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$ by $(h_j)_{j\in\mathbb{Z}}$, i.e. $h_j := A_{j+1,j}$ for all $j \in \mathbb{Z}$. Then A^2 has the following entries:

$$\begin{aligned}
(A^2)_{j,j+2} &= A_{j,j+1}A_{j+1,j+2} = 1, \\
(A^2)_{j,j+1} &= A_{j,j+1}A_{j+1,j+1} + A_{j,j}A_{j,j+1} = 0, \\
(A^2)_{j,j} &= A_{j,j+1}A_{j+1,j} + A_{j,j}A_{j,j} + A_{j,j-1}A_{j-1,j} = h_j + h_{j-1}, \\
(A^2)_{j,j-1} &= A_{j,j}A_{j,j-1} + A_{j,j-1}A_{j-1,j-1} = 0, \\
(A^2)_{j,j-2} &= A_{j,j-1}A_{j-1,j-2} = h_{j-1}h_{j-2}
\end{aligned}$$

and can be decomposed into an even and an odd part as follows. Let

$$\mathbf{X}_e := \{x \in \mathbf{X} : x_{2j+1} = 0 \text{ for all } j \in \mathbb{Z}\} \quad \text{and} \quad \mathbf{X}_o := \{x \in \mathbf{X} : x_{2j} = 0 \text{ for all } j \in \mathbb{Z}\}\,.$$

Then $A^2(\mathbf{X}_e) \subset \mathbf{X}_e$ and $A^2(\mathbf{X}_o) \subset \mathbf{X}_o$. Thus we can consider $C := A^2|_{\mathbf{X}_e}$ and $D := A^2|_{\mathbf{X}_o}$ and we get $A^2 \cong C \oplus D$ w.r.t. this decomposition of $\mathbf{X}$, where C and D are tridiagonal operators given by

$$\begin{aligned}
C_{j,j+1} &= 1, \\
C_{j,j} &= h_{2j} + h_{2j-1}, \\
C_{j,j-1} &= h_{2j-1}h_{2j-2}
\end{aligned}$$

and

$$\begin{aligned}
D_{j,j+1} &= 1, \\
D_{j,j} &= h_{2j+1} + h_{2j}, \\
D_{j,j-1} &= h_{2j}h_{2j-1}
\end{aligned}$$

for $j \in \mathbb{Z}$, respectively. We will focus on the computation of the numerical range of C. The computation of the numerical range of D is exactly the same so that we arrive at $N(A^2) = N(C) = N(D)$ (cf. Equation (14)).

By Theorem 4.10(v), the numerical range is the same for every operator $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$. Thus we may choose an operator $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$ such that $A_+ \in \Psi\mathrm{E}_+(U_{-1}, U_0, U_1)$. Since A_+ is pseudo-ergodic, we have $A \in \sigma^{\mathrm{op}}(A_+)$ by Proposition 4.16. Let g be a sequence of integers tending to $+\infty$ such that $(A_+)_g = A_g = A$. Then $(A^2)_g = (A_g)^2 = A^2 = C \oplus D$ by Proposition 2.26(iv). Observe that $V_{-g_n}(C \oplus D)V_{g_n} = V_{-g_n/2}CV_{g_n/2} \oplus V_{-g_n/2}DV_{g_n/2}$ if g_n is even and $V_{-g_n}(C \oplus D)V_{g_n} = V_{-(g_n-1)/2}DV_{(g_n-1)/2} \oplus V_{-(g_n+1)/2}CV_{(g_n+1)/2}$ if g_n is odd. Clearly, at least one of the two sets $\{n \in \mathbb{N} : g_n \text{ is even}\}$ and $\{n \in \mathbb{N} : g_n \text{ is odd}\}$ is infinite. Let us first assume that $\{n \in \mathbb{N} : g_n \text{ is even}\}$ is infinite and denote the sequence of even elements in g by g^e. Then by construction $V_{-g_n^e/2}CV_{g_n^e/2}$ converges $\mathcal{P}$-strongly to C and $V_{-g_n^e/2}DV_{g_n^e/2}$ converges $\mathcal{P}$-strongly to D as $n \to \infty$. Thus $C \in \sigma^{\mathrm{op}}(C)$ and $D \in \sigma^{\mathrm{op}}(D)$. Similarly, assume that $\{n \in \mathbb{N} : g_n \text{ is odd}\}$ is infinite and denote the sequence of odd elements in g by g^o. Then by construction $V_{-(g_n^o-1)/2}CV_{(g_n^o-1)/2}$ converges $\mathcal{P}$-strongly to D and $V_{-(g_n^o+1)/2}DV_{(g_n^o+1)/2}$ converges $\mathcal{P}$-strongly to C as $n \to \infty$. Thus $D \in \sigma^{\mathrm{op}}(C)$

and $C \in \sigma^{\mathrm{op}}(D)$ in this case. Since limit operators of limit operators are again limit operators of the original operator (see e.g. [57, Corollary 3.97]), we also get $C \in \sigma^{\mathrm{op}}(C)$ and $D \in \sigma^{\mathrm{op}}(D)$ in this case. Since g^e and g^o tend to $+\infty$, we get that C is also a limit operator of C_+ and thus $N(A^2) = N(C) = N(C_+)$ as in the beginning of the proof of Proposition 4.74.

Fix $\varphi \in [0, 2\pi)$ and let $E(\varphi)$ be the real symmetric tridiagonal operator that satisfies

$$\begin{aligned} E_{j,j}(\varphi) &= \mathrm{Re}(e^{i\varphi}(C_+)_{j,j}), \\ E_{j,j+1}(\varphi) &= \frac{1}{2}\left|e^{i\varphi}(C_+)_{j,j+1} + e^{-i\varphi}\overline{(C_+)_{j+1,j}}\right| \end{aligned} \tag{48}$$

and $r_\varphi(A^2) = r_\varphi(C_+) = r_0(E(\varphi))$ (cf. (38)). Now for every angle φ there are 16 different combinations for $(h_{2j-1}, h_{2j}, h_{2j+1}, h_{2j+2})$ in (39). Define

$$\eta_j(\varphi) := \frac{E_{j,j+1}(\varphi)^2}{(N(\varphi) - E_{j,j}(\varphi))(N(\varphi) - E_{j+1,j+1}(\varphi))} \tag{49}$$

for all $j \in \mathbb{N}$, where $N(\varphi)$ is given by Proposition 4.77. Let us consider $\varphi \in [\frac{\pi}{3}, \frac{\pi}{2}]$ first. For these angles, we have the following table. For later reference we numbered the 16 cases lexicographically.

t_j	$(h_{2j-1}, h_{2j}, h_{2j+1}, h_{2j+2})$	$\eta_j(\varphi)$
1	$(1,1,1,1)$	$\frac{\cos(\varphi)^2}{(2-2\cos(\varphi))^2}$
2	$(1,1,1,-1)$	$\frac{\cos(\varphi)^2}{2(2-2\cos(\varphi))}$
3	$(1,1,-1,1)$	$\frac{\sin(\varphi)^2}{2(2-2\cos(\varphi))} = \frac{1+\cos(\varphi)}{4}$
4	$(1,1,-1,-1)$	$\frac{\sin(\varphi)^2}{(2-2\cos(\varphi))(2+2\cos(\varphi))} = \frac{1}{4}$
5	$(1,-1,1,1)$	$\frac{\sin(\varphi)^2}{2(2-2\cos(\varphi))} = \frac{1+\cos(\varphi)}{4}$
6	$(1,-1,1,-1)$	$\frac{\sin(\varphi)^2}{4}$
7	$(1,-1,-1,1)$	$\frac{\cos(\varphi)^2}{4}$
8	$(1,-1,-1,-1)$	$\frac{\cos(\varphi)^2}{2(2+2\cos(\varphi))}$
9	$(-1,1,1,1)$	$\frac{\cos(\varphi)^2}{2(2-2\cos(\varphi))}$
10	$(-1,1,1,-1)$	$\frac{\cos(\varphi)^2}{4}$
11	$(-1,1,-1,1)$	$\frac{\sin(\varphi)^2}{4}$
12	$(-1,1,-1,-1)$	$\frac{\sin(\varphi)^2}{2(2+2\cos(\varphi))} = \frac{1-\cos(\varphi)}{4}$
13	$(-1,-1,1,1)$	$\frac{\sin(\varphi)^2}{(2-2\cos(\varphi))(2+2\cos(\varphi))} = \frac{1}{4}$
14	$(-1,-1,1,-1)$	$\frac{\sin(\varphi)^2}{2(2+2\cos(\varphi))} = \frac{1-\cos(\varphi)}{4}$
15	$(-1,-1,-1,1)$	$\frac{\cos(\varphi)^2}{2(2+2\cos(\varphi))}$
16	$(-1,-1,-1,-1)$	$\frac{\cos(\varphi)^2}{(2+2\cos(\varphi))^2}$

Table 2

This table has to be read as follows. The sequence $(h_j)_{j\in\mathbb{N}}$ induces a sequence $(t_j)_{j\in\mathbb{N}}$. For example if the sequence $(h_j)_{j\in\mathbb{N}}$ starts with $(1,-1,-1,1,1,1,1,-1,1,-1,\ldots)$, the sequence $(t_j)_{j\in\mathbb{N}}$ starts with $(7,9,2,6,\ldots)$. The numbers t_j are used to refer to the respective η_j, which are computed via Formula (49). So if, for example, $t_j = 6$, then $\eta_j(\varphi) = \frac{\sin(\varphi)^2}{4}$.

Using $\cos(\varphi) \in [0, \frac{1}{2}]$, it is not hard to see that $\eta_j(\varphi) \leq \frac{1}{2}$ for all $\varphi \in [\frac{\pi}{3}, \frac{\pi}{2}]$ and $j \in \mathbb{N}$ (i.e. for all possible values of $\eta_j(\varphi)$ in Table 2). We even have $\eta_j(\varphi) \leq \frac{1}{4}$ for all $\varphi \in [\frac{\pi}{3}, \frac{\pi}{2}]$ and $j \in \mathbb{N}$ with $t_j \notin \{3,5\}$. This observation is very useful to finally construct the sequence needed for Lemma 4.72.

Proposition 4.78. *Let $U_{-1} = \{1\}$, $U_0 = \{0\}$, $U_1 = \{\pm 1\}$ and let $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$. Fix $\varphi \in [\frac{\pi}{3}, \frac{\pi}{2}]$ and let $\eta_j := \eta_j(\varphi)$ and t_j for all $j \in \mathbb{N}$ be defined as above. Then the sequence $(g_j)_{j\in\mathbb{N}}$, defined by the following prescription, satisfies $g_j \in [0,1]$ and $\eta_j \leq g_{j+1}(1-g_j)$ for all $j \in \mathbb{N}$:*

- *If $t_1 = 5$, choose $g_1 = \frac{1-\cos(\varphi)}{2}$.*
- *If there is some $k \in \mathbb{N}$ such that $t_1 = \ldots = t_k = 6$ and $t_{k+1} = 5$, choose $g_1 = \frac{1-\cos(\varphi)}{2}$.*
- *If neither is true, choose $g_1 = \frac{1}{2}$.*
- *If $t_j \in \{2,6,10,14\}$ and $t_{j+1} = 5$, choose $g_{j+1} = \frac{1-\cos(\varphi)}{2}$.*
- *If $t_j \in \{2,6,10,14\}$, there is some $k > j$ such that $t_{j+1} = \ldots = t_k = 6$ and $t_{k+1} = 5$, choose $g_{j+1} = \frac{1-\cos(\varphi)}{2}$.*
- *If $t_j = 3$, choose $g_{j+1} = \frac{1+\cos(\varphi)}{2}$.*
- *If $t_j = 11$, there is some $k \leq j$ such that $t_k = \ldots = t_j = 11$ and $t_{k-1} = 3$, choose $g_{j+1} = \frac{1+\cos(\varphi)}{2}$.*
- *If none of the above is true, choose $g_{j+1} = \frac{1}{2}$.*

Proof. That $g_j \in [0,1]$ holds for all $j \in \mathbb{N}$ is clear. So it remains to prove $\eta_j \leq g_{j+1}(1-g_j)$. Above we observed that $\eta_j \leq \frac{1}{4}$ unless $t_j \in \{3,5\}$. So if $t_j \notin \{3,5\}$ for all $j \in \mathbb{N}$, then $\eta_j \leq g_{j+1}(1-g_j)$ is obviously satisfied. It remains to investigate what happens if $t_j \in \{3,5\}$ for some $j \in \mathbb{N}$. Roughly speaking, the idea is that the cases $t_j = 3$ and $t_j = 5$ affect the sequence $(g_k)_{k\in\mathbb{N}}$ only locally in the sense that $\left\{k \in \mathbb{N} : g_k = \frac{1}{2}\right\}$ is an infinite set. Thus if $t_j \in \{3,5\}$ occurs, we try to get back to $\frac{1}{2}$ as soon as possible as j increases. The argument can then be repeated by induction.

Let us consider the case $t_j = 3$ first and assume $g_j = \frac{1}{2}$. More precisely, we start our sequence with $g_1 = g_2 = \ldots = \frac{1}{2}$ until $t_j \in \{3,5\}$ occurs the first time and consider the case where $t_j = 3$ first. Then by definition $g_{j+1} = \frac{1+\cos(\varphi)}{2}$ and

$$g_{j+1}(1-g_j) = \frac{1+\cos(\varphi)}{4} = \eta_j.$$

We observe that η_j and η_{j+1} are not independent. Indeed, η_{j+1} depends on h_{2j+1}, h_{2j+2}, h_{2j+3} and h_{2j+4} whereas η_j depends on h_{2j-1}, h_{2j}, h_{2j+1} and h_{2j+2}. Thus if we fix η_j, there are only 4 possible combinations for η_{j+1}. In particular, if $t_j = 3$, then t_{j+1} has to be contained in $\{9, 10, 11, 12\}$. So there are four cases:

$$\begin{aligned}
\eta_{j+1} &= \frac{\cos(\varphi)^2}{2(2-2\cos(\varphi))} \quad (t_{j+1} = 9),\\
\eta_{j+1} &= \frac{\cos(\varphi)^2}{4} \quad (t_{j+1} = 10),\\
\eta_{j+1} &= \frac{\sin(\varphi)^2}{4} \quad (t_{j+1} = 11),\\
\eta_{j+1} &= \frac{\sin(\varphi)^2}{2(2+2\cos(\varphi))} \quad (t_{j+1} = 12).
\end{aligned}$$

In the first case we have $g_{j+2} = \frac{1}{2}$:

$$g_{j+2}(1-g_{j+1}) = \frac{1}{2} - \frac{1+\cos(\varphi)}{4} = \frac{1-\cos(\varphi)}{4} \geq \frac{1}{8} \geq \frac{\cos(\varphi)^2}{2(2-2\cos(\varphi))} = \eta_{j+1},$$

where we used $\cos(\varphi) \leq \frac{1}{2}$. In the second case we have $g_{j+2} = \frac{1-\cos(\varphi)}{2} \leq \frac{1}{2}$ if $t_{j+2} \in \{5, 6\}$ and $g_{j+2} = \frac{1}{2}$ if not:

$$g_{j+2}(1-g_{j+1}) \geq \frac{1-\cos(\varphi)}{2}\left(1 - \frac{1+\cos(\varphi)}{2}\right) = \frac{(1-\cos(\varphi))^2}{4} \geq \frac{\cos(\varphi)^2}{4} = \eta_{j+1},$$

where we used $1 - \cos(\varphi) \geq \cos(\varphi)$. In the third case we have $g_{j+2} = \frac{1+\cos(\varphi)}{2}$:

$$g_{j+2}(1-g_{j+1}) = \frac{1+\cos(\varphi)}{2}\left(1 - \frac{1+\cos(\varphi)}{2}\right) = \frac{1+\cos(\varphi)}{2}\,\frac{1-\cos(\varphi)}{2} = \frac{\sin(\varphi)^2}{4} = \eta_{j+1}.$$

In the fourth case we have $g_{j+2} = \frac{1}{2}$:

$$g_{j+2}(1-g_{j+1}) = \frac{1}{2} - \frac{1+\cos(\varphi)}{4} = \frac{1-\cos(\varphi)}{4} = \eta_{j+1}.$$

So either $g_{j+2} \leq \frac{1}{2}$ (and we included one special case that we need afterwards) or $g_{j+2} = g_{j+1}$. Thus either we are where we started with, namely $\frac{1}{2}$, or we are in the third case, where $t_{j+1} = 11$. But in this case we have $h_{2j+1} = h_{2j+3}$ and $h_{2j+2} = h_{2j+4}$ and thus we have again the same four cases for η_{j+2} and so on. So either we end up with an infinite sequence with $g_k = g_{j+1}$ for all $k > j$ (which is impossible by pseudo-ergodicity, but would still be just fine) or we eventually go out with $g_k \leq \frac{1}{2}$ for some $k \geq j+2$. Thus we are done by induction if we can control the case $t_j = 5$ as well.

The case $t_j = 5$ is very similar to the case $t_j = 3$, but we have to think backwards this time, which is a little bit more complicated. If we have a look at the generators (i.e. h_{2j-1}, h_{2j}, h_{2j+1} and h_{2j+2}) of the cases $t_j = 3$ and $t_j = 5$, it is intuitively clear, why this has to be the same but backwards. So assume that $t_j = 5$. Then $g_j = \frac{1-\cos(\varphi)}{2}$ and $g_{j+1} = \frac{1}{2}$ by definition and thus

$$g_{j+1}(1-g_j) = \frac{1}{2} - \frac{1-\cos(\varphi)}{4} = \frac{1+\cos(\varphi)}{4} = \eta_j.$$

As already mentioned, we have to look backwards here, i.e. we want to control g_{j-1}. Now there are five cases. The first case is $j=1$, which is trivial of course. The second case is where $t_{j-1}=2$. In this case we have $g_{j-1}=\frac{1}{2}$:

$$g_j(1-g_{j-1})=\frac{1-\cos(\varphi)}{4}\geq\frac{1}{8}\geq\frac{\cos(\varphi)^2}{2(2-2\cos(\varphi))}=\eta_{j-1},$$

where we used $\cos(\varphi)\leq\frac{1}{2}$. The third case is where $t_{j-1}=6$. In this case we have $g_{j-1}=\frac{1-\cos(\varphi)}{2}$:

$$g_j(1-g_{j-1})=\frac{1-\cos(\varphi)}{2}\left(1-\frac{1-\cos(\varphi)}{2}\right)=\frac{1-\cos(\varphi)}{2}\frac{1+\cos(\varphi)}{2}=\frac{\sin(\varphi)^2}{4}=\eta_{j-1}.$$

The fourth case is where $t_{j-1}=10$. In this case we either have $g_{j-1}=\frac{1+\cos(\varphi)}{2}\geq\frac{1}{2}$ if $t_{j-2}\in\{3,11\}$ or $g_{j-1}=\frac{1}{2}$ if not:

$$g_j(1-g_{j-1})\geq\frac{1-\cos(\varphi)}{2}\left(1-\frac{1+\cos(\varphi)}{2}\right)=\frac{(1-\cos(\varphi))^2}{4}\geq\frac{\cos(\varphi)^2}{4}=\eta_{j-1},$$

where we used $1-\cos(\varphi)\geq\cos(\varphi)$. Note that this case matches perfectly with the second case above. The fifth case is where $t_{j-1}=14$. In this case we have $g_{j-1}=\frac{1}{2}$:

$$g_j(1-g_{j-1})=\frac{1-\cos(\varphi)}{4}=\eta_{j-1}.$$

Again we conclude that either $g_{j-1}\geq\frac{1}{2}$ (note that the inequality is in the other direction this time, which is good!) or $g_{j-1}=g_j$. Thus either we started where we ended, namely $\frac{1}{2}$ (or even better, we started with something $\geq\frac{1}{2}$ and the sequence reduced to $\frac{1}{2}$, compare with the mentioned special case above), or we are in the third case, where $t_{j-1}=6$. But in this case we have $h_{2j-1}=h_{2j-3}$ and $h_{2j-2}=h_{2j-4}$ and thus we again have the same four cases for η_{j-2} and so on. Thus we either end up at g_1, which is fine or we eventually have $g_k\geq\frac{1}{2}$ for some $k\leq j-1$. In either case we are done by induction. □

Using this sequence, we can apply Lemma 4.72 to obtain $r_\varphi(A)=r_0(E(\varphi))\leq N(\varphi)=2$ for $\varphi\in[\frac{\pi}{3},\frac{\pi}{2}]$. So let us now consider the case $\varphi\in[0,\frac{\pi}{3}]$. This will be the last case since the other cases will follow by symmetry. For $\varphi\in[0,\frac{\pi}{3}]$ we have $N(\varphi)=4\cos(\varphi)$. This implies the following table:

t_j	$(h_{2j-1},h_{2j},h_{2j+1},h_{2j+2})$	$\eta_j(\varphi)$
1	$(1,1,1,1)$	$\frac{\cos(\varphi)^2}{4\cos(\varphi)^2}=\frac{1}{4}$
2	$(1,1,1,-1)$	$\frac{\cos(\varphi)^2}{8\cos(\varphi)^2}=\frac{1}{8}$
3	$(1,1,-1,1)$	$\frac{\sin(\varphi)^2}{8\cos(\varphi)^2}=\frac{\tan(\varphi)^2}{8}$
4	$(1,1,-1,-1)$	$\frac{\sin(\varphi)^2}{12\cos(\varphi)^2}=\frac{\tan(\varphi)^2}{12}$
5	$(1,-1,1,1)$	$\frac{\sin(\varphi)^2}{8\cos(\varphi)^2}=\frac{\tan(\varphi)^2}{8}$
6	$(1,-1,1,-1)$	$\frac{\sin(\varphi)^2}{16\cos(\varphi)^2}=\frac{\tan(\varphi)^2}{16}$
7	$(1,-1,-1,1)$	$\frac{\cos(\varphi)^2}{16\cos(\varphi)^2}=\frac{1}{16}$
8	$(1,-1,-1,-1)$	$\frac{\cos(\varphi)^2}{24\cos(\varphi)^2}=\frac{1}{24}$

t_j	$(h_{2j-1}, h_{2j}, h_{2j+1}, h_{2j+2})$	$\eta_j(\varphi)$
9	$(-1, 1, 1, 1)$	$\frac{\cos(\varphi)^2}{8\cos(\varphi)^2} = \frac{1}{8}$
10	$(-1, 1, 1, -1)$	$\frac{\cos(\varphi)^2}{16\cos(\varphi)^2} = \frac{1}{16}$
11	$(-1, 1, -1, 1)$	$\frac{\sin(\varphi)^2}{16\cos(\varphi)^2} = \frac{\tan(\varphi)^2}{16}$
12	$(-1, 1, -1, -1)$	$\frac{\sin(\varphi)^2}{24\cos(\varphi)^2} = \frac{\tan(\varphi)^2}{24}$
13	$(-1, -1, 1, 1)$	$\frac{\sin(\varphi)^2}{12\cos(\varphi)^2} = \frac{\tan(\varphi)^2}{12}$
14	$(-1, -1, 1, -1)$	$\frac{\sin(\varphi)^2}{24\cos(\varphi)^2} = \frac{\tan(\varphi)^2}{24}$
15	$(-1, -1, -1, 1)$	$\frac{\cos(\varphi)^2}{24\cos(\varphi)^2} = \frac{1}{24}$
16	$(-1, -1, -1, -1)$	$\frac{\cos(\varphi)^2}{36\cos(\varphi)^2} = \frac{1}{36}$

Table 3

Using $\tan(\varphi) \in [0, \sqrt{3}]$, we observe again that $\eta_j(\varphi) \leq \frac{1}{2}$ for all $\varphi \in [0, \frac{\pi}{3}]$ and $j \in \mathbb{N}$. Also, $\eta_j(\varphi) \leq \frac{1}{4}$ for all $\varphi \in [0, \frac{\pi}{3}]$ and $j \in \mathbb{N}$ with $t_j \notin \{3, 5\}$ as before. If $\tan(\varphi)^2 \leq 2$, then even $\eta_j(\varphi) \leq \frac{1}{4}$ for all $\varphi \in [0, \frac{\pi}{3}]$ and $j \in \mathbb{N}$. In this case we can choose $g_j = \frac{1}{2}$ for all $j \in \mathbb{N}$ and we are done. Thus we only have to consider the angles where $\tan(\varphi)^2 > 2$. The argument is exactly the same as in the proof of Proposition 4.78.

Proposition 4.79. *Let $U_{-1} = \{1\}$, $U_0 = \{0\}$, $U_1 = \{\pm 1\}$ and let $A \in \Psi\mathrm{E}(U_{-1}, U_0, U_1)$. Let $\varphi \in [0, \frac{\pi}{3}]$ be such that $\tan(\varphi)^2 \geq 2$ and let $\eta_j := \eta_j(\varphi)$ for all $j \in \mathbb{N}$ be defined as above. Then the sequence $(g_j)_{j\in\mathbb{N}}$, defined by the following prescription, satisfies $g_j \in [0, 1]$ and $\eta_j \leq g_{j+1}(1 - g_j)$ for all $j \in \mathbb{N}$:*

- *If $t_1 = 5$, choose $g_1 = 1 - \frac{1}{4}\tan(\varphi)^2$.*
- *If there is some $k \in \mathbb{N}$ such that $t_1 = \ldots = t_k = 6$ and $t_{k+1} = 5$, choose $g_1 = 1 - \frac{1}{4}\tan(\varphi)^2$.*
- *If neither is true, choose $g_1 = \frac{1}{2}$.*
- *If $t_j \in \{2, 6, 10, 14\}$ and $t_{j+1} = 5$, choose $g_{j+1} = 1 - \frac{1}{4}\tan(\varphi)^2$.*
- *If $t_j \in \{2, 6, 10, 14\}$, there is some $k > j$ such that $t_{j+1} = \ldots = t_k = 6$ and $t_{k+1} = 5$, choose $g_{j+1} = 1 - \frac{1}{4}\tan(\varphi)^2$.*
- *If $t_j = 3$, choose $g_{j+1} = \frac{1}{4}\tan(\varphi)^2$.*
- *If $t_j = 11$, there is some $k \leq j$ such that $t_k = \ldots = t_j = 11$ and $t_{k-1} = 3$, choose $g_{j+1} = \frac{1}{4}\tan(\varphi)^2$.*
- *If none of the above is true, choose $g_{j+1} = \frac{1}{2}$.*

Proof. The proof is exactly the same as the proof of Proposition 4.78. We only have to change the numbers. That $g_j \in [0, 1]$ holds for all $j \in \mathbb{N}$ is clear since $\tan(\varphi) \in [0, \sqrt{3}]$ for $\varphi \in [0, \frac{\pi}{3}]$. So it remains to prove $\eta_j \leq g_{j+1}(1 - g_j)$. Above we observed that $\eta_j \leq \frac{1}{4}$ unless $t_j \in \{3, 5\}$. Thus if the cases $t_j = 3$ and $t_j = 5$ do not occur, then $\eta_j \leq g_{j+1}(1 - g_j)$ is obviously satisfied. So we are left with the cases $t_j = 3$ and $t_j = 5$ again.

Let us consider the case $t_j = 3$ first and assume $g_j = \frac{1}{2}$. Then by definition $g_{j+1} = \frac{1}{4}\tan(\varphi)^2$ and

$$g_{j+1}(1-g_j) = \frac{1}{8}\tan(\varphi)^2 = \eta_j.$$

Now there are again four possible cases for η_{j+1}:

$$\begin{aligned}
\eta_{j+1} &= \frac{1}{8} \quad (t_{j+1} = 9),\\
\eta_{j+1} &= \frac{1}{16} \quad (t_{j+1} = 10),\\
\eta_{j+1} &= \frac{\tan(\varphi)^2}{16} \quad (t_{j+1} = 11),\\
\eta_{j+1} &= \frac{\tan(\varphi)^2}{24} \quad (t_{j+1} = 12).
\end{aligned}$$

In the first case we have $g_{j+2} = \frac{1}{2}$:

$$g_{j+2}(1-g_{j+1}) = \frac{1}{2} - \frac{1}{8}\tan(\varphi)^2 \geq \frac{1}{8} = \eta_{j+1},$$

where we used $\tan(\varphi) \leq \sqrt{3}$. In the second case we have $g_{j+2} = 1 - \frac{1}{4}\tan(\varphi)^2 \leq \frac{1}{2}$ if $t_{j+2} \in \{5,6\}$ and $g_{j+2} = \frac{1}{2}$ if not:

$$g_{j+2}(1-g_{j+1}) \geq \left(1 - \frac{1}{4}\tan(\varphi)^2\right)^2 \geq \frac{1}{16} = \eta_{j+1},$$

where we used $\tan(\varphi) \leq \sqrt{3}$ again. In the third case we have $g_{j+2} = \frac{1}{4}\tan(\varphi)^2$:

$$g_{j+2}(1-g_{j+1}) = \frac{1}{4}\tan(\varphi)^2\left(1 - \frac{1}{4}\tan(\varphi)^2\right) = \frac{1}{16}\tan(\varphi)^2(4-\tan(\varphi)^2) \geq \frac{1}{16}\tan(\varphi)^2 = \eta_{j+1}.$$

In the fourth case we have $g_{j+2} = \frac{1}{2}$:

$$g_{j+2}(1-g_{j+1}) = \frac{1}{2} - \frac{1}{8}\tan(\varphi)^2 \geq \frac{1}{6}\tan(\varphi)^2 - \frac{1}{8}\tan(\varphi)^2 = \frac{\tan(\varphi)^2}{24} = \eta_{j+1}.$$

So either $g_{j+2} \leq \frac{1}{2}$ or $g_{j+2} = g_{j+1}$. As in the proof of Proposition 4.78 we conclude that we eventually go out with $g_k \leq \frac{1}{2}$ for some $k \geq j+2$. Thus we are done by induction if we can control the case $t_j = 5$ as well.

So assume that $t_j = 5$. Then $g_j = 1 - \frac{1}{4}\tan(\varphi)^2$ and $g_{j+1} = \frac{1}{2}$ by definition and thus

$$g_{j+1}(1-g_j) = \frac{1}{8}\tan(\varphi)^2 = \eta_j.$$

Again there are five cases here. The first case is $j = 1$, which is again trivial. The second case is where $t_{j-1} = 2$. In this case we have $g_{j-1} = \frac{1}{2}$:

$$g_j(1-g_{j-1}) = \frac{1}{2} - \frac{1}{8}\tan(\varphi)^2 \geq \frac{1}{8} = \eta_{j-1}.$$

The third case is where $t_{j-1} = 6$. In this case we have $g_{j-1} = 1 - \frac{1}{4}\tan(\varphi)^2$:

$$g_j(1-g_{j-1}) = \left(1 - \frac{1}{4}\tan(\varphi)^2\right)\frac{1}{4}\tan(\varphi)^2 = \frac{1}{16}\tan(\varphi)^2(4-\tan(\varphi)^2) \geq \frac{1}{16}\tan(\varphi)^2 = \eta_{j-1}.$$

The fourth case is where $t_{j-1} = 10$. In this case we either have $g_{j-1} = \frac{1}{4}\tan(\varphi)^2 \geq \frac{1}{2}$ if $t_{j-2} \in \{3, 11\}$ or $g_{j-1} = \frac{1}{2}$ if not:

$$g_j(1-g_{j-1}) \geq \left(1 - \frac{1}{4}\tan(\varphi)^2\right)^2 \geq \frac{1}{16} = \eta_{j-1}.$$

The fifth case is where $t_{j-1} = 14$. In this case we have $g_{j-1} = \frac{1}{2}$:

$$g_j(1-g_{j-1}) = \frac{1}{2} - \frac{1}{8}\tan(\varphi)^2 \geq \frac{1}{6}\tan(\varphi)^2 - \frac{1}{8}\tan(\varphi)^2 = \frac{1}{24}\tan(\varphi)^2 = \eta_{j-1}.$$

Thus we either have $g_{j-1} \geq \frac{1}{2}$ or $g_{j-1} = g_j$ again. As in the proof of Proposition 4.78 we conclude that we either end up at g_1, which is fine or we eventually have $g_k \geq \frac{1}{2}$ for some $k \leq j-1$. In either case we are done by induction. □

Using the sequences obtained by Proposition 4.78 and Proposition 4.79, we can now apply Lemma 4.72 to prove Theorem 4.75:

Proof of Theorem 4.75.

$$N(A^2) \supset \operatorname{conv}\left(\bigcup_{B \in M_{per,4}(U_{-1},U_0,U_1)} N(B^2)\right)$$

is clear by Theorem 4.10(v) and Proposition 2.26. To prove the other direction, we have to show $r_\varphi(A^2) \leq N(\varphi)$ for all $\varphi \in [0, 2\pi)$, where $N(\varphi)$ is given by Proposition 4.77. Using the transformations $\varphi \mapsto \pi - \varphi$ and $\varphi \mapsto \varphi + \pi$, it is clear that is suffices to consider $\varphi \in [0, \frac{\pi}{2}]$. Indeed, $N(\varphi)$ is invariant under these transformations and in the Tables 2 and 3 only the roles of $+1$ and -1 are interchanged.

To apply Lemma 4.72 to $E(\varphi)$ (as defined in (48)), we have to assure

$$E_{j,j+1}(\varphi) = \frac{1}{2}\left|e^{i\varphi}(C_+)_{j,j+1} + e^{-i\varphi}\overline{(C_+)_{j+1,j}}\right| > 0$$

and $E_{j,j}(\varphi) > 0$ for all $j \in \mathbb{N}$. The latter can be achieved by shifting and the former can only fail if $\varphi = 0$ or $\varphi = \frac{\pi}{2}$. If $\varphi = 0$, then

$$r_0(E(\varphi)) \leq \|E(\varphi)\| \leq 4 = N(\varphi)$$

by Proposition 2.17. Similarly, if $\varphi = \frac{\pi}{2}$, then

$$r_0(E(\varphi)) \leq \|E(\varphi)\| \leq 2 = N(\varphi)$$

again by Proposition 2.17 (observe that $E_{j,j}(\frac{\pi}{2}) = 0$ for all $j \in \mathbb{N}$). In the remaining cases we clearly have $N(\varphi) > \sup\limits_{j \in \mathbb{N}} E_{j,j}(\varphi)$ as $E_{j,j}(\varphi) \in \{-2\cos(\varphi), 0, 2\cos(\varphi)\}$ for all $j \in \mathbb{N}$ (cf. Proposition 4.77). We can thus apply Lemma 4.72, using the sequences from Proposition 4.78 and Proposition 4.79 (including the trivial case $\tan(\varphi)^2 \leq 2$), to obtain $r_\varphi(A^2) = r_0(E(\varphi)) \leq N(\varphi)$ for all $\varphi \in [0, \frac{\pi}{2}]$ and hence for all $\varphi \in [0, 2\pi)$. □

In Theorem 4.75 we gave a viable formula for the numerical range of A^2, where A is a very specific operator, i.e. we have

$$N(A^2) = \operatorname{conv}\left(\bigcup_{B \in M_{per,4}(U_{-1},U_0,U_1)} N(B^2)\right) \tag{50}$$

for $A \in \Psi\mathrm{E}(\{1\},\{0\},\{\pm 1\})$. This result was generalized to $A \in \Psi\mathrm{E}(\{1\},\{0\},\{\pm\sigma\})$, $\sigma \in \mathbb{C}$ by the author in [36]. Formula (50) still holds if we assume $U_1 = \left\{\sigma e^{\frac{2\pi i k}{n}} : 0 \leq k \leq n-1\right\}$ for some $n \in \mathbb{N}$ instead. However, the calculations get much more tedious than in the case $n = 2$ discussed here. Moreover, a numerical analysis of several operators in $\Psi\mathrm{E}(U_{-1},\{0\},U_1)$, where U_{-1} and U_1 are arbitrary but compact, suggests that Formula (50) holds in much more generality. In fact, we have not found any counterexamples yet and therefore we conjecture that Theorem 4.75 holds for all $A \in \Psi\mathrm{E}(U_{-1},\{0\},U_1)$, where U_{-1} and U_1 are arbitrary but compact. However, the computational method introduced in Section 4.4.4 is a bit too situational and probably not suited to prove this theorem in full generality because the computations get more and more tedious the more parameters we introduce.

Beside these generalizations a natural question is whether we can get similar results for A^4, A^8 and so on. This is a difficult question to answer. On the one hand, numerical considerations suggest that the numerical ranges of A^4, A^8 and so on yield better and better upper bounds to the spectrum of $A \in \Psi\mathrm{E}(\{1\},\{0\},\{\pm 1\})$ and this might also explain the somewhat fractal structure of $\partial\Sigma$. This is also supported by [24, Section 9.4]. On the other hand, we do not know how to actually compute $N(A^4)$ because A^4 can not be decomposed into tridiagonal operators (at least not in an obvious way) as this was the case for A^2.

We conclude with a picture of $\left\{\lambda \in \mathbb{C} : \lambda^2 \in N(A^2)\right\}$, $N(A)$ and the lower bound obtained in Section 4.4.3. It shows that $\left\{\lambda \in \mathbb{C} : \lambda^2 \in N(A^2)\right\}$ improves the upper bound to $\mathrm{sp}(A)$ by a decent amount.

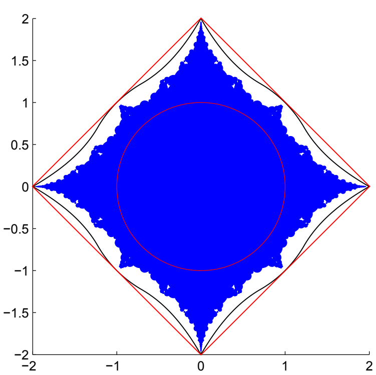


Figure 13: The boundaries of $N(A)$ and $\left\{\lambda \in \mathbb{C} : \lambda^2 \in N(A^2)\right\}$, the lower bound computed in Section 4.4.3 and the unit circle as a reference.

4.5 References

Section 4.1 follows the ideas of Davies [23]. The notation is an adaption of the notation in [18] and [60]. That random operators are pseudo-ergodic almost surely (Theorem 4.6) is well-known and basically a direct implication of the second Borel-Cantelli lemma. An excellent source for the theory of Laurent and Toeplitz operators is the book by Böttcher and Silbermann [8]. Also periodic operators (called block Toeplitz operators there) are covered, for example Theorem 4.33 and Theorem 4.34 can be found there. The approximation results of Section 4.2.1 are published by the author in [36]. The ideas in Section 4.3.1 are due to Lindner [59], who extended a result in [90]. The results in Section 4.3.2 are again published by the author in [36]. Theorem 4.55 was not published in [36] and is probably new. Except for Section 4.4.1, which is a summary of known results, and Section 4.4.4, which is a modification of an observation by Szwarc [89], the results of Section 4.4 are all new and published by the author in [36], [37] and [38], respectively. Proposition 4.74 was also published in [36].

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