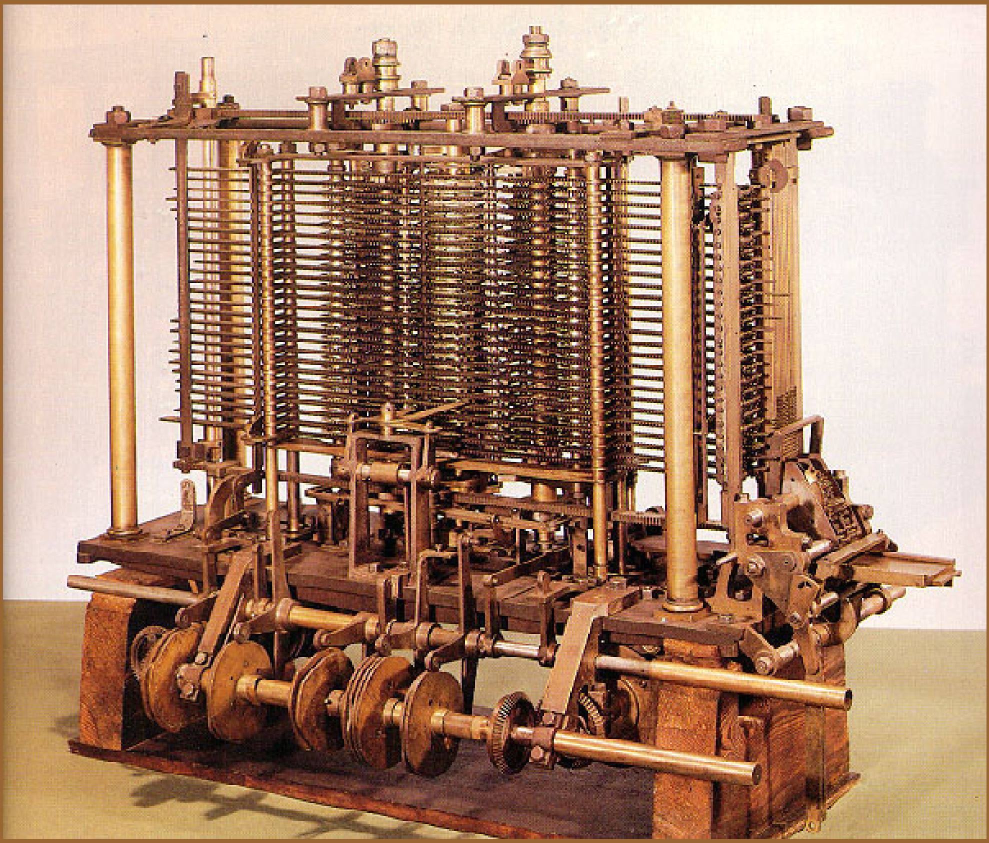

Computability Theory

Karl-Heinz Zimmermann

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For Gela and Eileen

Preface to the 2nd Edition

Why do we need a formalization of the notion of algorithm or effective computation? In order to show that a specific problem is algorithmically solvable, it is sufficient to provide an algorithm that solves it in a sufficiently precise manner. However, in order to prove that a problem is in principle not solvable by an algorithm, a rigorous formalism is necessary that allows mathematical proofs. The need for such a formalism became apparent in the studies of David Hilbert (1900) on the foundations of mathematics and Kurt Gödel (1931) on the incompleteness of elementary arithmetic.

The first investigations in the field were conducted by the logicians Alonzo Church, Stephen Kleene, Emil Post, and Alan Turing in the early 1930s. They have provided the foundation of computability theory as a branch of theoretical computer science. The fundamental results established Turing computability as the correct formalization of the informal idea of effective calculation. The results have led to Church's thesis stating that "everything computable is computable by a Turing machine". The theory of computability has grown rapidly from its beginning. Its questions and methods are penetrating many other mathematical disciplines. Today, computability theory provides an important theoretical background for logicians, pure mathematicians, and computer scientists. Many mathematical problems are known to be undecidable such as the word problem for groups, the halting problem, and Hilbert's tenth problem.

This book is a development of class notes for a two-hour lecture including a one-hour lab held for second-year Bachelor students of Computer Science at the Hamburg University of Technology during the last two years. The course aims to present the basic results of computability theory, including mathematical models of computability, primitive recursive and partial recursive functions, Ackermann's function, Gödel numbering, universal functions, smn theorem, Kleene's normal form, undecidable sets, theorems of Rice, and word problems. The manuscript has partly grown out of notes taken by the author during his studies at the University of Erlangen-Nuremberg. I would like to thank again my teachers Martin Becker[†] and Volker Strehl for giving inspiring lectures in this field.

The second edition contains minor changes. In particular, the section on Gödel numbering has been rewritten and a glossary of terms has been added.

Finally, I would like to express my thanks to Ralf Möller for valuable comments. I am also grateful to Mahwish Saleemi for conducting the lab and to Wolfgang Brandt for valuable technical support. Moreover, I would like to thank my students for their attention, their stimulating questions, and their dedicated work.

Hamburg, July 2012 *Karl-Heinz Zimmermann*

Mathematical Notation

General notation

$\mathbb{N}_0$	set of natural numbers
$\mathbb{N}$	set of natural number without 0
$\mathbb{Z}$	set of integers
Σ^*	set of words over Σ
Σ^+	set of non-empty words over Σ
$\mathcal{F}$	class of partial functions
$\mathcal{P}$	class of primitive recursive functions
$\mathcal{R}$	class of partial recursive functions
$\mathcal{T}$	class of recursive functions
$\mathcal{F}_{\mathrm{URM}}$	class of URM computable functions
$\mathcal{T}_{\mathrm{URM}}$	class of total URM computable functions
$\mathcal{F}_{\mathrm{LOOP}}$	class of LOOP computable functions
$\mathcal{P}_{\mathrm{LOOP}}$	class of LOOP programs
$\mathcal{F}_{\mathrm{LOOP}-n}$	class of LOOP-n computable functions
$\mathcal{P}_{\mathrm{LOOP}-n}$	class of LOOP-n programs
$\mathcal{F}_{\mathrm{GOTO}}$	class of GOTO computable functions
$\mathcal{P}_{\mathrm{GOTO}}$	class of GOTO programs
$\mathcal{P}_{\mathrm{SGOTO}}$	class of SGOTO programs
$\mathcal{T}_{\mathrm{Turing}}$	class of Turing computable functions

Chapter 1

Ω	state set of URM
$E(\Omega)$	set of state transitions
$\mathrm{dom} f$	domain of a function
$\mathrm{ran} f$	range of a function
$\dot{-}$	conditional decrement
sgn	sign function
csg	cosign function
f^{*n}	iteration of f w.r.t. nth register
$A\sigma$	increment
$S\sigma$	conditional decrement
$(P)\sigma$	iteration of a program
$P;Q$	composition of programs
$\lvert P\rvert$	state transition function of a program
$\lVert P\rVert_{k,m}$	function of a program
α_k	load function
β_m	result function

$R(i; j_1, \ldots, j_k)$	reload program
$C(i; j_1, \ldots, j_k)$	copy program

Chapter 2

ν	successor function
$c_0^{(n)}$	n-ary zero function
$\pi_k^{(n)}$	n-ary projection function
$\mathrm{pr}(g, h)$	primitive recursion
$g(h_1, \ldots, h_n)$	composition
Σf	bounded sum
Πf	bounded product
$\overline{\mu} f$	bounded minimalization
J_2	Cantor pairing function
K_2, L_2	inverse component functions of J_2
χ_S	characteristic function of a set S
p_i	ith prime
$(x)_i$	ith exponent in prime-power representation of x
$Z\sigma$	zero-setting of register σ
$\overline{C}(\sigma, \tau)$	copy program
$[P]\sigma$	iteration of a program

Chapter 3

μf	unbounded minimalization
$(l, x_i \leftarrow x_i + 1, m)$	GOTO increment
$(l, x_i \leftarrow x_i \dot{-} 1, m)$	GOTO conditional decrement
$(l, \texttt{if}\ x_i = 0, k, m)$	GOTO conditional jump
$V(P)$	set of GOTO variables
$L(P)$	set of GOTO labels
$\vdash$	one-step computation
G	encoding of URM state
G_i	URM state of ith register
$M(k)$	GOTO-2 multiplication
$D(k)$	GOTO-2 division
$T(k)$	GOTO-2 divisibility test

Chapter 4

B_n	small Ackermann function
A	Ackermann function
$\gamma(P)$	runtime of LOOP program
$\lambda(P)$	complexity of LOOP program
$f \leq g$	function bound

Chapter 5

ϵ	empty word
J	encoding of $\mathbb{N}_0^*$
K, L	inverse component functions of J
lg	length function
$I(s_l)$	encoding of SGOTO statement
$\Gamma(P)$	Gödel number of SGOTO program
P_e	SGOTO program with Gödel number e
$\phi_e^{(n)}$	n-ary computable function with index e
$s_{m,n}$	smn function
$\psi_{\text{univ}}^{(n)}$	n-ary universal function
E_A	unbounded existential quantification
U_A	unbounded universal quantification
μA	unbounded minimalization
S_n	Kleene set
T_n	Kleene predicate

Chapter 6

M	Turing machine
b	blank symbol
Σ	tape alphabet
Σ_I	input alphabet
Q	state set
T	state transition function
q_0	initial state
q_F	final state
L	left move
R	right move
Δ	no move
$\vdash$	one-step computation

Chapter 7

K	prototype of undecidable set
H	halting problem
$\mathcal{A}$	class of monadic partial recursive functions
$f_\uparrow$	nowhere defined function
prog($\mathcal{A}$)	set of indices of $\mathcal{A}$
r.e.	recursive enumerable
$f \subseteq g$	ordering relation
$p(X_1, \ldots, X_n)$	diophantine polynomial
$V(p)$	natural variety

Chapter 8

Σ	alphabet
R	rule set of TS
$\rightarrow$	rule
$\rightarrow_R$	one-step rewriting rule
$\rightarrow_R^*$	reflexive transitive closure
$R^{(s)}$	symmetric rule set of STS
$\rightarrow_{R^{(s)}}^*$	equivalence
$[s]$	equivalence class
Π	rule set of PCS
$\underline{\boldsymbol{i}}$	solution of PCS
$\alpha(\underline{\boldsymbol{i}})$	left string of solution
$\beta(\underline{\boldsymbol{i}})$	right string of solution

Contents

1

Unlimited Register Machine

The unlimited register machine (URM) introduced by Sheperdson and Sturgis (1963) is an abstract computing machine that allows to make precise the notion of computability. It consists of an infinite (unlimited) sequence of registers each capable of storing a natural number which can be arbitrarily large. The registers can be manipulated by using simple instructions. This chapter introduces the syntax and semantics of URMs and the class of URM computable functions.

1.1 States and State Transformations

An unlimited register machine (URM) contains an infinite number of registers named

$$R_0, R_1, R_2, R_3, \ldots. \tag{1.1}$$

The *state set* of an URM is given as

$$\Omega = \{\omega : \mathbb{N}_0 \to \mathbb{N}_0 \mid \omega \text{ is 0 almost everywhere}\}. \tag{1.2}$$

The elements of Ω are denoted as sequences

$$\omega = (\omega_0, \omega_1, \omega_2, \omega_3, \ldots), \tag{1.3}$$

where for each $n \in \mathbb{N}_0$, the component $\omega_n = \omega(n)$ denotes the content of the register R_n.

Proposition 1.1. *The set Ω is denumerable.*

Proof. Let $(p_0, p_1, p_2, \ldots)$ denote the sequence of prime numbers. Due to the unique factorization of each natural number into a product of prime powers, the mapping

$$\Omega \to \mathbb{N} : \omega \mapsto \prod_i p_i^{\omega_i}$$

is a bijection. □

Let $E(\Omega)$ denote the set of all partial functions from Ω to Ω. Here *partial* means that for each $f \in E(\Omega)$ and $\omega \in \Omega$ there exists not necessarily a value $f(\omega)$. Each partial function $f \in E(\Omega)$ has a *domain* given as

$$\mathrm{dom}(f) = \{\omega \in \Omega \mid f(\omega) \text{ is defined}\} \tag{1.4}$$

and a *range* defined by

$$\mathrm{ran}(f) = \{\omega' \in \Omega \mid \exists \omega \in \Omega : \ f(\omega) = \omega'\}. \tag{1.5}$$

Proposition 1.2. *The set of partial functions $E(\Omega)$ is non-denumerable.*

Two partial functions $f, g \in E(\Omega)$ are *equal*, written $f = g$, if they have the same domain, i.e., $\mathrm{dom}(f) = \mathrm{dom}(g)$, and for all arguments in the (common) domain, they coincide, i.e., for all $\omega \in \mathrm{dom}(f)$, $f(\omega) = g(\omega)$. A partial function $f \in E(\Omega)$ is called *total* if $\mathrm{dom}(f) = \Omega$. So a total function is a function in the usual sense.

Example 1.3. The *increment function* $a_k \in E(\Omega)$ with respect to the kth register is given by the assignment $a_k : \omega \mapsto \omega'$, where

$$\omega'_n = \begin{cases} \omega_n & \text{if } n \neq k, \\ \omega_k + 1 & \text{otherwise.} \end{cases} \tag{1.6}$$

The *decrement function* $s_k \in E(\Omega)$ w.r.t. the kth register is defined as $s_k : \omega \mapsto \omega'$, where

$$\omega'_n = \begin{cases} \omega_n & \text{if } n \neq k, \\ \omega_k \dot{-} 1 & \text{otherwise.} \end{cases} \tag{1.7}$$

The dyadic operator $\dot{-}$ on $\mathbb{N}_0$ denotes the *asymmetric difference* given as

$$x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

Both functions a_k and s_k are total. ♢

The *graph* of a partial function $f \in E(\Omega)$ is given by the relation

$$R_f = \{(\omega, f(\omega)) \mid \omega \in \mathrm{dom}(f)\}. \tag{1.9}$$

It is clear that two partial functions $f, g \in E(\Omega)$ are equal if and only if the corresponding graphs R_f and R_g are equal as sets.

The *composition* of two partial functions $f, g \in E(\Omega)$ is a partial function, denoted by $g \circ f$, defined as

$$(g \circ f)(\omega) = g(f(\omega)), \tag{1.10}$$

where ω belongs to the domain of $g \circ f$ given by

$$\mathrm{dom}(g \circ f) = \{\omega \in \Omega \mid \omega \in \mathrm{dom}(f) \wedge f(\omega) \in \mathrm{dom}(g)\}. \tag{1.11}$$

If f and g are total functions in $E(\Omega)$, the composition $g \circ f$ is also a total function.

Proposition 1.4. *The set $E(\Omega)$ together with the dyadic operation of composition is a semigroup.*

Proof. It is clear that the composition of partial functions is an associative operation. $\square$

In this way, the set $E(\Omega)$ is called the *semigroup of transformations* of Ω.

The *powers* of a partial function $f \in E(\Omega)$ are inductively defined as follows:

$$f^0 = \mathrm{id}_\Omega, \quad \text{and} \quad f^{n+1} = f \circ f^n,\ n \in \mathbb{N}_0. \tag{1.12}$$

In particular, $f^1 = f \circ \mathrm{id}_\Omega = f$.

Consider for each $f \in E(\Omega)$ and $\omega \in \Omega$ the following sequence of natural numbers:

$$\omega = f^0(\omega), f^1(\omega), f^2(\omega), \ldots. \tag{1.13}$$

This sequence is finite if $\omega \notin \mathrm{dom}(f^j)$ for some $j \in \mathbb{N}_0$. For this, put

$$\lambda(f,\omega) = \begin{cases} \min\{j \in \mathbb{N}_0 \mid \omega \notin \mathrm{dom}(f^j)\} & \text{if } \{\ldots\} \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases} \tag{1.14}$$

The *iteration* of $f \in E(\Omega)$ with respect to the nth register is the partial function $f^{*n} \in E(\Omega)$ defined as

$$f^{*n}(\omega) = f^k(\omega), \tag{1.15}$$

if there is an integer $k \geq 0$ with $k < \lambda(f,\omega)$ such that the content of the nth register is zero in $f^k(\omega)$, but non-zero in $f^j(\omega)$ for each $0 \leq j < k$. If no such integer k exists, the value of $f^{*n}(\omega)$ is taken to be undefined, written $\uparrow$. The computation of f^{*n} can be carried out by the `while` loop 1.1.

Algorithm 1.1 Computation of iteration f^{*n}.

Require: $\omega \in \Omega$
  while $\omega_n > 0$ **do**
    $w \leftarrow f(\omega)$
  end while

1.2 Syntax of URM Programs

The set of all (decimal) numbers over the alphabet of digits $\Sigma_{10} = \{0, 1, \ldots, 9\}$ is defined as

$$Z = (\Sigma_{10} \setminus \{0\})\Sigma_{10}^+ \cup \Sigma_{10}. \tag{1.16}$$

That is, a number is either the digit 0 or a non-empty word of digits that does not begin with 0.

The URM programs are words over the alphabet

$$\Sigma_{\mathrm{URM}} = \{A, S, (,), ;\} \cup Z. \tag{1.17}$$

Define the set of *URM programs* $\mathcal{P}_{\mathrm{URM}}$ inductively as follows:

1. $A\sigma \in \mathcal{P}_{\text{URM}}$ for each $\sigma \in Z$,
2. $S\sigma \in \mathcal{P}_{\text{URM}}$ for each $\sigma \in Z$,
3. if $P \in \mathcal{P}_{\text{URM}}$ and $\sigma \in Z$, then $(P)\sigma \in \mathcal{P}_{\text{URM}}$,
4. if $P, Q \in \mathcal{P}_{\text{URM}}$, then $P; Q \in \mathcal{P}_{\text{URM}}$.

The programs $A\sigma$ and $S\sigma$ are *atomic*, the program $(P)\sigma$ is the *iteration* of the program P with respect to the register R_σ, and the program $P; Q$ is the *composition* of the programs P and Q. For each program $P \in \mathcal{P}_{\text{URM}}$ and each integer $n \geq 0$, define the n-fold composition of P as

$$P^n = P; P; \ldots ; P \quad (n \text{ times}). \tag{1.18}$$

The atomic programs and the iterations are called *blocks*. The set of blocks in $\mathcal{P}_{\text{URM}}$ is denoted by $\mathcal{B}$.

Lemma 1.5. *For each program $P \in \mathcal{P}_{\text{URM}}$, there are uniquely determined blocks $P_1, \ldots, P_k \in \mathcal{B}$ such that*

$$P = P_1; P_2; \ldots ; P_k.$$

The separation symbol ";" can be removed although it eventually increases readability. In this way, we obtain the following result.

Proposition 1.6. *The set $\mathcal{P}_{\text{URM}}$ together with the operation of concatenation is a subsemigroup of Σ^+_{URM} which is freely generated by the set of blocks $\mathcal{B}$.*

Example 1.7. The URM program $P = (A3; A4; S1)1; ((A1; S3)3; S2; (A0; A3; S4)4; (A4; S0)0)2$ consists of the blocks $P_1 = (A3; A4; S1)1;$ and $P_2 = ((A1; S3)3; S2; (A0; A3; S4)4; (A4; S0)0)2$. $\diamondsuit$

1.3 Semantics of URM Programs

The URM programs can be interpreted by the semigroup of transformations $E(\Omega)$. The *semantics* of URM programs is a mapping $|\cdot| : \mathcal{P}_{\text{URM}} \to E(\Omega)$ defined inductively as follows:

1. $|A\sigma| = a_\sigma$ for each $\sigma \in Z$,
2. $|S\sigma| = s_\sigma$ for each $\sigma \in Z$,
3. if $P \in \mathcal{P}_{\text{URM}}$ and $\sigma \in Z$, then $|(P)\sigma| = |P|^{*\sigma}$,
4. if $P, Q \in \mathcal{P}_{\text{URM}}$, then $|P; Q| = |Q| \circ |P|$.

The semantics of blocks is defined by the first three items, and the last item indicates that the mapping $|\cdot|$ is a morphism of semigroups.

Proposition 1.8. *For each mapping $\psi : \mathcal{B} \to E(\Omega)$, there is a unique semigroup homomorphism $\phi : \mathcal{P}_{\text{URM}} \to E(\Omega)$ making the following diagram commutative:*

$$\begin{tikzcd}
\mathcal{P}_{\text{URM}} \arrow[r, "\phi"'] & E(\Omega) \\
\mathcal{B} \arrow[u, "\text{id}"'] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Proof. Given a mapping $\psi : \mathcal{B} \to E(\Omega)$. Since each URM program P is a composition of blocks, there are elements $B_0, \ldots, B_n$ of $\mathcal{B}$ such that $P = B_0; \ldots ; B_n$. Define $\phi(P) = \psi(B_0); \ldots ; \psi(B_n)$. This gives a semigroup homomorphism $\phi : \mathcal{P}_{\mathrm{URM}} \to E(\Omega)$ with the required property.

On the other hand, if $\phi' : \mathcal{P}_{\mathrm{URM}} \to E(\Omega)$ is a semigroup homomorphism with the property $\psi(B) = \phi'(B)$ for each block B. Then $\phi = \phi'$, since all URM programs are sequences of blocks. ☐

This algebraic statement asserts that the semantics on blocks can be uniquely extended to the full set of URM programs.

1.4 URM Computable Functions

A partial function $f \in E(\Omega)$ is *URM computable* if there is an URM program P such that $|P| = f$. Note that the class $\mathcal{P}_{\mathrm{URM}}$ is denumerable, while the set $E(\Omega)$ is not. It follows that there are partial functions in $E(\Omega)$ that are not URM computable. In the following, let $\mathcal{F}_{\mathrm{URM}}$ denote the class of all partial functions that are URM computable, and let $\mathcal{T}_{\mathrm{URM}}$ depict the class of all total functions which are URM computable. Clearly, we have $\mathcal{T}_{\mathrm{URM}} \subset \mathcal{F}_{\mathrm{URM}}$.

Functions like addition or multiplication of two natural numbers are URM computable. In general, the calculation of an URM-computable function $f : \mathbb{N}_0^k \to \mathbb{N}_0^m$ requires to load the registers with initial values and to read out the result. For this, define the total functions

$$\alpha_k : \mathbb{N}_0^k \to \Omega : (x_1, \ldots, x_k) \mapsto (0, x_1, \ldots, x_k, 0, 0, \ldots) \tag{1.19}$$

and

$$\beta_m : \Omega \to \mathbb{N}_0^m : (\omega_0, \omega_1, \omega_2, \ldots,) \mapsto (\omega_1, \omega_2, \ldots, \omega_m). \tag{1.20}$$

Given an URM program P and integers $k, m \in \mathbb{N}_0$, define the partial function $\|P\|_{k,m} : \mathbb{N}_0^k \to \mathbb{N}_0^m$ by the composition

$$\|P\|_{k,m} = \beta_m \circ |P| \circ \alpha_k. \tag{1.21}$$

Here the k-ary function $\|P\|_{k,m}$ is computed by loading the registers with an argument $x \in \mathbb{N}_0^k$, calculating the program P on the initial state $\alpha_k(x)$ and reading out the result using β_m.

A (partial) function $f : \mathbb{N}_0^k \to \mathbb{N}_0^m$ is called *URM computable* if there is an URM program P such that

$$f = \|P\|_{k,m}. \tag{1.22}$$

Examples 1.9.

- The addition of natural numbers is URM computable. To see this, consider the URM program

$$P_+ = (A1; S2)2. \tag{1.23}$$

  This program transforms the initial state (ω_n) into the state $(\omega_0, \omega_1 + \omega_2, 0, \omega_3, \omega_4, \ldots)$ and thus realizes the function

$$\|P_+\|_{2,1}(x, y) = x + y, \quad x, y \in \mathbb{N}_0. \tag{1.24}$$

- The multiplication of natural number is URM computable. For this, take the URM program
$$P_\cdot = (A3; A4; S1)1; ((A1; S3)3; S2; (A0; A3; S4)4; (A4; S0)0)2. \tag{1.25}$$
The first block $(A3; A4; S1)1$ transforms the initial state $(0, x, y, 0, 0, \ldots)$ into $(0, 0, y, x, x, 0, 0, \ldots)$. Then the subprogram $(A1; S3)3; S_2; (A0; A3; S4)4; (A3; S0)0$ is carried out y times adding the content of R_3 to that of R_1 and copying the content of R_4 to R_3. This iteration provides the state $(0, xy, 0, x, x, 0, 0, \ldots)$. It follows that
$$\|P_\cdot\|_{2,1}(x, y) = xy, \quad x, y \in \mathbb{N}_0. \tag{1.26}$$
- The asymmetric difference is URM computable. For this, pick the URM program
$$P_{\dot{-}} = (S1; S2)2. \tag{1.27}$$
This program transforms the initial state (ω_n) into the state $(\omega_0, \omega_1 \dot{-} \omega_2, 0, \omega_3, \ldots)$ and thus yields the URM computable function
$$\|P_{\dot{-}}\|_{2,1}(x, y) = x \dot{-} y, \quad x, y \in \mathbb{N}_0. \tag{1.28}$$
- Consider the sign function $\mathrm{sgn} : \mathbb{N}_0 \to \mathbb{N}_0$ given by $\mathrm{sgn}(x) = 1$ if $x > 0$ and $\mathrm{sgn}(x) = 0$ if $x = 0$. This function is URM computable since it is calculated by the URM program
$$P_{\mathrm{sgn}} = (A2; S1)1; (A1; (S2)2)2. \tag{1.29}$$

$\diamondsuit$

Note that URM programs are *invariant of translation* in the sense that if an URM program P manipulates the registers $R_{i_1}, \ldots, R_{i_k}$, there is an URM program that manipulates the registers $R_{i_1+n}, \ldots, R_{i_k+n}$. This program will be denoted by $(P)[+n]$. For instance, if $P = (A1; S2)2$, $(P)[+5] = (A6; S7)7$.

Let $f : \mathbb{N}_0^k \to \mathbb{N}_0$ be an URM computable function. An URM program P with $\|P\|_{k,1} = f$ is *normal* if for all $(x_1, \ldots, x_k) \in \mathbb{N}_0^k$,
$$(P \circ \alpha_k)(x_1, \ldots, x_k) = \begin{cases} (0, f(x_1, \ldots, x_k), 0, 0, \ldots) & \text{if } (x_1, \ldots, x_k) \in \mathrm{dom}(f), \\ \uparrow & \text{otherwise.} \end{cases} \tag{1.30}$$
A normal URM-program computes a function in such a way that whenever the computation ends the register R_1 contains the result while all other registers are set to zero.

Proposition 1.10. *For each URM-computable function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ there is a normal URM-program P such that $\|P\|_{k,1} = f$.*

Proof. Let Q be an URM-program such that $\|Q\|_{k,1} = f$. Suppose σ is the largest number of a register that contains a non-zero value in the final state of computation. Then the corresponding normal URM-program is given by
$$P = Q; (S0)0; (S2)2; \ldots ; (S\sigma)\sigma. \tag{1.31}$$
Here the block $(Si)i$ sets the value of the ith register to zero. $\square$

Finally, we introduce two programs that are useful for the transport or distribution of the contents of registers. The first function *reloads* the content of register R_i into $k \geq 0$ registers $R_{j_1}, \ldots, R_{j_k}$ deleting the content of R_i. This is achieved by the URM program

$$R(i; j_1, j_2, \ldots, j_k) = (Aj_1; Aj_2; \ldots; Aj_k; Si)i. \tag{1.32}$$

Indeed, the program transforms the initial state (ω_n) into the state (ω'_n) where

$$\omega'_n = \begin{cases} \omega_n + \omega_i & \text{if } n \in \{j_1, j_2, \ldots, j_k\}, \\ 0 & \text{if } n = i, \\ \omega_n & \text{otherwise.} \end{cases} \tag{1.33}$$

The second function *copies* the content of register R_i, $i > 0$, into $k \geq 0$ registers $R_{j_1}, \ldots, R_{j_k}$ where the content of register R_i is retained. Here the register R_0 is used for distributing the value of R_i. This is achieved by the URM program

$$C(i; j_1, j_2, \ldots, j_k) = R(i; 0, j_1, j_2, \ldots, j_k); R(0; i). \tag{1.34}$$

In fact, the program transforms the initial state (ω_n) into the state (ω'_n) where

$$\omega'_n = \begin{cases} \omega_n + \omega_i & \text{if } n \in \{j_1, j_2, \ldots, j_k\}, \\ \omega_n + \omega_0 & \text{if } n = i, \\ 0 & \text{if } n = 0, \\ \omega_n & \text{otherwise.} \end{cases} \tag{1.35}$$

2

Primitive Recursive Functions

The primitive recursion functions form an important building block on the way to a full formalization of computability. They are formally defined using composition and primitive recursion as central operations. Most of the functions studied in arithmetics are primitive recursive such as the basic operations of addition and multiplication. Indeed, it is difficult to devise a function that is total but not primitive recursive. From the programming point of view, the primitive recursive functions can be implemented using `do`-loops only.

2.1 Peano Structures

We will use Peano structures to define the concept of primitive recursion, a major building block to introduce the class of primitive recursive functions.

A *semi-Peano structure* is a triple $\mathcal{A} = (A, \alpha, a)$ consisting of a non-empty set A, a monadic operation $\alpha : A \to A$, and an element $a \in A$. Let $\mathcal{A} = (A, \alpha, a)$ and $\mathcal{B} = (B, \beta, b)$ be semi-Peano structures. A mapping $\phi : A \to B$ is called a *morphism*, written $\phi : \mathcal{A} \to \mathcal{B}$, if ϕ commutes with the monadic operations, i.e., $\beta \circ \phi = \phi \circ \alpha$, and correspondingly assigns the distinguished elements, i.e., $\phi(a) = b$.

A *Peano structure* is a semi-Peano structure $\mathcal{A} = (A, \alpha, a)$ with the following properties:

- α is injective,
- $a \notin \text{ran}(\alpha)$, and
- A fulfills the *induction axiom*, i.e., if $T \subseteq A$ such that $a \in T$ and $\alpha(x) \in T$ whenever $x \in T$, then $T = A$.

Let $\nu : \mathbb{N}_0 \to \mathbb{N}_0 : n \mapsto n + 1$ be the *successor function*. The Peano structure given by the triple $(\mathbb{N}_0, \nu, 0)$ corresponds to the axioms postulated by the Italian mathematician Guiseppe Peano (1958-1932).

Lemma 2.1. *If $\mathcal{A} = (A, \alpha, a)$ is a Peano structure, then $A = \{\alpha^n(a) \mid n \in \mathbb{N}_0\}$.*

Proof. Let $T = \{\alpha^n(a) \mid n \in \mathbb{N}_0\}$. Then $a = \alpha^0(a) \in T$, and for each $\alpha^n(a) \in T$, $\alpha(\alpha^n(a)) = \alpha^{n+1}(a) \in T$. Hence by the induction axiom, $T = A$. □

Lemma 2.2. *If $\mathcal{A} = (A, \alpha, a)$ is a Peano structure, then for all $m, n \in \mathbb{N}_0$, $m \neq n$, we have $\alpha^m(a) \neq \alpha^n(a)$.*

Proof. Define T as the set of all elements $\alpha^m(a)$ such that $\alpha^n(a) \neq \alpha^m(a)$ for all $n \in \mathbb{N}_0$ with $n > m$.

First, suppose that $\alpha^n(a) = \alpha^0(a) = a$ for some $n > 0$. Then $a \in \text{ran}(\alpha)$ contradicting the definition. It follows that $a \in T$.

Second, let $x \in T$; that is, $x = \alpha^m(a)$ for some $m \geq 0$. Suppose that $\alpha(x) = \alpha^{m+1}(x) \notin T$. Then there is a number $n > m$ such that $\alpha^{m+1}(a) = \alpha^{n+1}(a)$. But α is injective and so $\alpha^m(a) = \alpha^n(a)$ contradicting the hypothesis. It follows that $\alpha(x) \in T$.

Thus the induction axiom implies that $T = A$ as required. $\square$

These assertions lead to the *Fundamental Lemma* for Peano structures.

Proposition 2.3. *If $\mathcal{A} = (A, \alpha, a)$ is a Peano structure and $\mathcal{B} = (B, \beta, b)$ is a semi-Peano structure, then there is a unique morphism $\phi : \mathcal{A} \to \mathcal{B}$.*

Proof. To prove existence, define $\phi(\alpha^n(a)) = \beta^n(b)$ for all $n \in \mathbb{N}_0$. The above assertions imply that ϕ is a mapping. Moreover, $\phi(a) = \phi(\alpha^0(a)) = \beta^0(b) = b$. Finally, let $x \in A$. Then $x = \alpha^m(a)$ for some $m \in \mathbb{N}_0$ and so $(\phi \circ \alpha)(x) = \phi(\alpha^{m+1}(a)) = \beta^{m+1}(b) = \beta(\beta^m(b)) = \beta(\phi(\alpha^m(a))) = (\beta \circ \phi)(x)$. Hence ϕ is a morphism.

To show uniqueness, suppose there is another morphism $\psi : \mathcal{A} \to \mathcal{B}$. Define $T = \{x \in A \mid \phi(x) = \psi(x)\}$. First, $\phi(a) = b = \psi(a)$ and so $a \in T$. Second, let $x \in T$. Then $\phi(\alpha(x)) = (\phi \circ \alpha)(x) = (\beta \circ \phi)(x) = (\beta \circ \psi)(x) = (\psi \circ \alpha)(x) = \psi(\alpha(x))$ and so $\alpha(x) \in T$. Hence by the induction axiom, $T = A$ and so $\phi = \psi$. $\square$

The Fundamental Lemma immediately leads to a result of Richard Dedekind (1931-1916).

Corollary 2.4. *Each Peano structure is isomorphic to $(\mathbb{N}_0, \nu, 0)$.*

Proof. We have already seen that $(\mathbb{N}_0, \nu, 0)$ is a Peano structure. Suppose there are Peano structures $\mathcal{A} = (A, \alpha, a)$ and $\mathcal{B} = (B, \beta, b)$. It is sufficient to show that there are morphisms $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{A}$ such that $\psi \circ \phi = \text{id}_A$ and $\phi \circ \psi = \text{id}_B$. For this, note that the composition of morphisms is also a morphism. Thus $\psi \circ \phi : \mathcal{A} \to \mathcal{A}$ is a morphism. On the other hand, the identity map $\text{id}_A : \mathcal{A} \to \mathcal{A} : x \mapsto x$ is a morphism. Hence, by the Fundamental Lemma, $\psi \circ \phi = \text{id}_A$. Similarly, it follows that $\phi \circ \psi = \text{id}_B$. $\square$

The Fundamental Lemma can be applied to the basic Peano structure $(\mathbb{N}_0, \nu, 0)$ in order to recursively define new functions.

Proposition 2.5. *If (A, α, a) is a semi-Peano structure, there is a unique total function $g : \mathbb{N}_0 \to A$ such that*

1. *$g(0) = a$,*
2. *$g(y + 1) = \alpha(g(y))$ for all $y \in \mathbb{N}_0$.*

Proof. By the Fundamental Lemma, there is a unique morphism $g : \mathbb{N}_0 \to A$ such that $g(0) = a$ and $g(y + 1) = g \circ \nu(y) = \alpha \circ g(y) = \alpha(g(y))$ for each $y \in \mathbb{N}_0$. $\square$

Example 2.6. There is a unique total function $f_+ : \mathbb{N}_0^2 \to \mathbb{N}_0$ such that

1. $f_+(x, 0) = x$ for all $x \in \mathbb{N}_0$,
2. $f_+(x, y+1) = f_+(x, y) + 1$ for all $x, y \in \mathbb{N}_0$.

To see this, consider the semi-Peano structure $(\mathbb{N}_0, \nu, x)$ for a fixed number $x \in \mathbb{N}_0$. By the Fundamental Lemma, there is a unique total function $f_x : \mathbb{N}_0 \to \mathbb{N}_0$ such that

1. $f_x(0) = x$,
2. $f_x(y+1) = f_x \circ \nu(y) = \nu \circ f_x(y) = f_x(y) + 1$ for all $y \in \mathbb{N}_0$.

The function f_+ is obtained by putting $f_+(x, y) = f_x(y)$ for all $x, y \in \mathbb{N}_0$. By induction, it follows that $f_+(x, y) = x + y$ for all $x, y \in \mathbb{N}_0$. Thus the addition of two numbers can be recursively defined. ♢

Proposition 2.7. *If $g : \mathbb{N}_0^k \to \mathbb{N}_0$ and $h : \mathbb{N}_0^{k+2} \to \mathbb{N}_0$ are total functions, there is a unique total function $f : \mathbb{N}_0^{k+1} \to \mathbb{N}_0$ such that*

$$f(\boldsymbol{x}, 0) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{N}_0^k, \tag{2.1}$$

and

$$f(\boldsymbol{x}, y+1) = h(\boldsymbol{x}, y, f(\boldsymbol{x}, y)), \quad \boldsymbol{x} \in \mathbb{N}_0^k, y \in \mathbb{N}_0. \tag{2.2}$$

Proof. For each $\boldsymbol{x} \in \mathbb{N}_0^k$, consider the semi-Peano structure $(\mathbb{N}_0^2, \alpha_{\boldsymbol{x}}, a_{\boldsymbol{x}})$, where $a_{\boldsymbol{x}} = (0, g(\boldsymbol{x}))$ and $\alpha_{\boldsymbol{x}} : \mathbb{N}_0^2 \to \mathbb{N}_0^2 : (y, z) \mapsto (y+1, h(\boldsymbol{x}, y, z))$. By Proposition 2.5, there is a unique total function $f_{\boldsymbol{x}} : \mathbb{N}_0 \to \mathbb{N}_0^2$ such that

1. $f_{\boldsymbol{x}}(0) = (0, g(\boldsymbol{x}))$,
2. $f_{\boldsymbol{x}}(y+1) = (f_{\boldsymbol{x}} \circ \nu)(y) = (\alpha_{\boldsymbol{x}} \circ f_{\boldsymbol{x}})(y) = \alpha_{\boldsymbol{x}}(y, f_{\boldsymbol{x}}(y)) = (y+1, h(\boldsymbol{x}, y, f_{\boldsymbol{x}}(y)))$ for all $y \in \mathbb{N}_0$.

The projection mapping $\pi_2^{(2)} : \mathbb{N}_0^2 \to \mathbb{N}_0 : (y, z) \mapsto z$ leads to the desired function $f(\boldsymbol{x}, y) = \pi_2^{(2)} \circ f_{\boldsymbol{x}}(y)$, $\boldsymbol{x} \in \mathbb{N}_0^k$ and $y \in \mathbb{N}_0$. □

The function f given in (2.1) and (2.2) is said to be defined by *primitive recursion* of the functions g and h. The above example shows that the addition of two numbers is defined by primitive recursion.

2.2 Primitive Recursive Functions

The class of primitive recursive functions is inductively defined. For this, the *basic functions* are the following:

1. The *0-ary constant function* $c_0^{(0)} : \to \mathbb{N}_0 : \mapsto 0$.
2. The *monadic constant function* $c_0^{(1)} : \mathbb{N}_0 \to \mathbb{N}_0 : x \mapsto 0$.
3. The *successor function* $\nu : \mathbb{N}_0 \to \mathbb{N}_0 : x \mapsto x + 1$.
4. The *projection function* $\pi_k^{(n)} : \mathbb{N}_0^n \to \mathbb{N}_0 : (x_1, \ldots, x_n) \mapsto x_k$, where $n \geq 1$ and $1 \leq k \leq n$.

Using these functions, more complex primitive recursive functions can be introduced.

1. If g is a k-ary total function and $h_1, \ldots, h_k$ are n-ary total functions, the *composition* of g along $(h_1, \ldots, h_k)$ is an n-ary function $f = g(h_1, \ldots, h_k)$ defined as

$$f(\boldsymbol{x}) = g(h_1(\boldsymbol{x}), \ldots, h_k(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathbb{N}_0^n. \tag{2.3}$$

2. If g is an n-ary total function and h is an $n+2$-ary total function, the *primitive recursion* of g along h is an $n+1$-ary function f given as

$$f(\boldsymbol{x}, 0) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{N}_0^n, \tag{2.4}$$

and

$$f(\boldsymbol{x}, y+1) = h(\boldsymbol{x}, y, f(\boldsymbol{x}, y)), \quad \boldsymbol{x} \in \mathbb{N}_0^n, y \in \mathbb{N}_0. \tag{2.5}$$

This function is denoted by $f = \mathrm{pr}(g, h)$.

The class of *primitive recursive functions* is given by the basic functions and those obtained from the basic functions by applying composition and primitive recursion a finite number of times. These functions were first studied by Richard Dedekind.

Proposition 2.8. *Each primitive recursive function is total.*

Proof. The basic functions are total. Let $f = g(h_1, \ldots, h_k)$ be the composition of g along $(h_1, \ldots, h_k)$. By induction, it can be assumed that the functions $g, h_1, \ldots, h_k$ are total. Then the function f is also total.

Let $f = \mathrm{pr}(g, h)$ be the primitive recursion of g along h. By induction, suppose that the functions g and h are total. Then the function f is total, too. $\square$

Examples 2.9. The dyadic functions of addition and multiplication are primitive recursive.

1. The function $f_+ : \mathbb{N}_0^2 \to \mathbb{N}_0 : (x, y) \mapsto x + y$ obeys the following scheme of primitive recursion:
   a) $f_+(x, 0) = x = \mathrm{id}_{\mathbb{N}_0}(x) = x$ for all $x \in \mathbb{N}_0$,
   b) $f_+(x, y+1) = f_+(x, y) + 1 = (\nu \circ \pi_3^{(3)})(x, y, f_+(x, y))$ for all $x, y \in \mathbb{N}_0$.
2. Define the function $f_\cdot : \mathbb{N}_0^2 \to \mathbb{N}_0 : (x, y) \mapsto xy$ inductively as follows:
   a) $f_\cdot(x, 0) = 0$ for all $x \in \mathbb{N}_0$,
   b) $f_\cdot(x, y+1) = f_\cdot(x, y) + x = f_+(x, f_\cdot(x, y))$ for all $x, y \in \mathbb{N}_0$.
   This leads to the following scheme of primitive recursion:
   a) $f_\cdot(x, 0) = c_0^{(1)}(x)$ for all $x \in \mathbb{N}_0$,
   b) $f_\cdot(x, y+1) = f_+(\pi_1^{(3)}, \pi_3^{(3)})(x, y, f_\cdot(x, y))$ for all $x, y \in \mathbb{N}_0$. $\diamondsuit$

Theorem 2.10. *Each primitive recursive function is URM computable.*

Proof. First, claim that the basic functions are URM computable.

1. 0-ary constant function: The URM program
$$P_0^{(0)} = A0; S0$$
   gives $\|P_0^{(0)}\|_{0,1} = c_0^{(0)}$.
2. Unary constant function: The URM program
$$P_0^{(1)} = (S1)1$$
   provides $\|P_0^{(1)}\|_{1,1} = c_0^{(1)}$.

3. Successor function: The URM program
$$P_{+1} = A1$$
yields $\|P_{+1}\|_{1,1} = \nu$.
4. Projection function $\pi_k^{(n)}$ with $n \geq 1$ and $1 \leq k \leq n$: The URM program
$$P_{p(n,k)} = R(k;0); (S1)1; R(0;1)$$
shows that $\|P_{n,k}\|_{n,1} = \pi_k^{(n)}$.

Second, consider the composition $f = g(h_1, \ldots, h_k)$. By induction, assume that there are normal URM programs P_g and $P_{h_1}, \ldots, P_{h_k}$ such that $\|P_g\|_{k,1} = g$ and $\|P_{h_i}\|_{n,1} = h_i$, $1 \leq i \leq k$. A normal URM program for the composite function f can be obtained as follows: For each $1 \leq i \leq k$,

- copy the values $x_1, \ldots, x_n$ into the registers $R_{n+k+2}, \ldots, R_{2n+k+1}$,
- compute the value $h_i(x_1, \ldots, x_n)$ by using the registers $R_{n+k+2}, \ldots, R_{2n+k+j}$, where $j \geq 1$,
- store the result $h_i(x_1, \ldots, x_n)$ in R_{n+i}.

Formally, this computation is carried out as follows:
$$Q_i = C(1; n+k+2); \ldots; C(n, 2n+k+1); (P_{h_i})[+n+k+1]; R(n+k+2; n+i) \tag{2.6}$$
Afterwards, the values in R_{n+i} are copied into R_i, $1 \leq i \leq k$, and the function $g(h_1(\boldsymbol{x}), \ldots, h_k(\boldsymbol{x}))$ is computed. Formally, the overall computation is achieved by the URM program
$$P_f = Q_1; \ldots; Q_k; (S1)1; \ldots; (Sn)n; R(n+1;1); \ldots; R(n+k;k); P_g \tag{2.7}$$
giving $\|P_f\|_{n,1} = f$.

Third, consider the primitive recursion $f = \mathrm{pr}(g, h)$. By induction, assume that there are normal URM programs P_g and P_h such that $\|P_g\|_{n,1} = g$ and $\|P_h\|_{n+2,1} = h$. As usual, the registers $R_1, \ldots, R_{n+1}$ contain the input values for the computation of $f(\boldsymbol{x}, y)$.

Note that the computation of the function value $f(\boldsymbol{x}, y)$ is accomplished in $y + 1$ steps:

- $f(\boldsymbol{x}, 0) = g(\boldsymbol{x})$, and
- for each $1 \leq i \leq y$, $f(\boldsymbol{x}, i) = h(\boldsymbol{x}, i-1, f(\boldsymbol{x}, i-1))$.

For this, the register R_{n+2} is used as a counter and the URM programs $(P_g)[+n+3]$ and $(P_h)[+n+3]$ make only use of the registers R_{n+3+j}, where $j \geq 0$. Formally, the overall computation is given as follows:
$$\begin{aligned} P_f = {} & R(n+1; n+2); \\ & C(1; n+4); \ldots; C(n, 2n+3); \\ & (P_g)[+n+3]; \\ & (R(n+4; 2n+5); C(1; n+4); \ldots; C(n+1; 2n+4); (P_h)[+n+3]; An+1; Sn+2)n+2; \\ & (S1)1; \ldots; (Sn+1)n+1; \\ & R(n+4; 1). \end{aligned} \tag{2.8}$$
First, the input value y is stored in R_{n+2} to serve as a counter, and the input values $x_1, \ldots, x_n$ are copied into $R_{n+4}, \ldots, R_{2n+3}$, respectively. Then $f(\boldsymbol{x}, 0) = g(\boldsymbol{x})$ is calculated. Afterwards, the following

iteration is performed while the value of R_{n+2} is non-zero: Copy $x_1, \ldots, x_n$ into $R_{n+4}, \ldots, R_{2n+3}$, respectively, copy the value of R_{n+1} into R_{2n+4}, which gives the ith iteration, and copy the result of the previous computation into R_{2n+5}. Then invoke the program P_h to obtain $f(\boldsymbol{x}, i) = h(\boldsymbol{x}, i-1, f(\boldsymbol{x}, i-1))$. At the end, the input arguments are set to zero and the result of the last iteration is copied into the first register. This provides the desired result: $\|P_f\|_{n+1,1} = f$. □

The URM programs for composition and primitive recursion also make sense if the URM subprograms used in the respective induction step are not primitive recursive. These ideas will be formalized in the remaining part of the section.

Let $g : \mathbb{N}_0^k \to \mathbb{N}_0$ and $h_i : \mathbb{N}_0^n \to \mathbb{N}_0$, $1 \leq i \leq k$, be partial functions. The *composition* of g along $(h_1, \ldots, h_k)$ is a partial function f, denoted by $f = g(h_1, \ldots, h_k)$, such that

$$\mathrm{dom}(f) = \{\boldsymbol{x} \in \mathbb{N}_0^n \mid \boldsymbol{x} \in \bigcap_{i=1}^{k} \mathrm{dom}(h_i) \wedge (h_1(\boldsymbol{x}), \ldots, h_k(\boldsymbol{x})) \in \mathrm{dom}(g)\} \tag{2.9}$$

and

$$f(\boldsymbol{x}) = g(h_1(\boldsymbol{x}), \ldots, h_k(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathrm{dom}(f). \tag{2.10}$$

The proof of the previous Theorem provides the following result.

Proposition 2.11. *The class of URM computable functions is closed under composition; that is, if $g : \mathbb{N}_0^k \to \mathbb{N}_0$ and $h_i : \mathbb{N}_0^n \to \mathbb{N}_0$, $1 \leq i \leq k$, are URM computable, $f = g(h_1, \ldots, h_k)$ is URM computable.*

The situation is analogous for the primitive recursion.

Proposition 2.12. *Let $g : \mathbb{N}_0^n \to \mathbb{N}_0$ and $h : \mathbb{N}_0^{n+2} \to \mathbb{N}_0$ be partial functions. There is a unique function $f : \mathbb{N}_0^{n+1} \to \mathbb{N}_0$ such that*

1. *$(\boldsymbol{x}, 0) \in \mathrm{dom}(f)$ if and only if $\boldsymbol{x} \in \mathrm{dom}(g)$ for all $\boldsymbol{x} \in \mathbb{N}_0^n$,*
2. *$(\boldsymbol{x}, y+1) \in \mathrm{dom}(f)$ if and only if $(\boldsymbol{x}, y) \in \mathrm{dom}(f)$ and $(\boldsymbol{x}, y, f(\boldsymbol{x}, y)) \in \mathrm{dom}(h)$ for all $\boldsymbol{x} \in \mathbb{N}_0^n$, $y \in \mathbb{N}_0$,*
3. *$f(\boldsymbol{x}, 0) = g(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{N}_0^n$, and*
4. *$f(\boldsymbol{x}, y+1) = h(\boldsymbol{x}, y, f(\boldsymbol{x}, y))$ for all $\boldsymbol{x} \in \mathbb{N}_0^n$, $y \in \mathbb{N}_0$.*

The proof makes use of the Fundamental Lemma. The partial function f defined by g and h in this Proposition is denoted by $f = \mathrm{pr}(g, h)$ and said to be defined by *primitive recursion* of g and h.

Proposition 2.13. *The class of URM computable functions is closed under primitive recursion; that is, if $g : \mathbb{N}_0^n \to \mathbb{N}_0$ and $h : \mathbb{N}_0^{n+2} \to \mathbb{N}_0$ are URM computable, $f = \mathrm{pr}(g, h)$ is URM computable.*

Primitively Closed Function Classes

Let $\mathcal{F}$ be a class of functions, i.e., $\mathcal{F} \subseteq \bigcup_{k \geq 0} \mathbb{N}_0^{(\mathbb{N}_0^k)}$. The class $\mathcal{F}$ is called *primitively closed* if it contains the basic functions $c_0^{(0)}$, $c_0^{(1)}$, ν, $\pi_k^{(n)}$, $1 \leq k \leq n$, $n \geq 1$, and is closed under composition and primitive recursion.

Let $\mathcal{P}$ denote the class of all primitive recursive functions, $\mathcal{T}_{\mathrm{URM}}$ the class of all URM computable total functions, and $\mathcal{T}$ the class of all total functions.

Proposition 2.14. *The classes $\mathcal{P}$, $\mathcal{T}_{\mathrm{URM}}$, and $\mathcal{T}$ are primitively closed.*

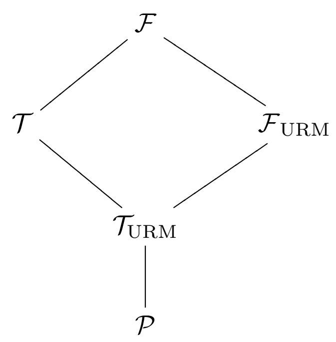
In particular, the class $\mathcal{P}$ of primitive recursive functions is the smallest class of functions which is primitively closed. Indeed, we have

$$\mathcal{P} = \bigcap\{\mathcal{F} \mid \mathcal{F} \subseteq \bigcup_{k\geq 0} \mathbb{N}_0^{(\mathbb{N}_0^k)}, \mathcal{F} \text{ primitively closed}\}. \tag{2.11}$$

The concept of primitive closure carries over to partial functions, since composition and primitive recursion have been defined for partial functions as well. Let $\mathcal{F}_{\mathrm{URM}}$ denote the class of URM computable functions and $\mathcal{F}$ the class of all functions.

Proposition 2.15. *The classes $\mathcal{F}_{\mathrm{URM}}$ and $\mathcal{F}$ are primitively closed.*

The lattice of the introduced classes (under inclusion) is the following:



All inclusions are strict. Indeed, the strict inclusions $\mathcal{T}_{\mathrm{URM}} \subset \mathcal{F}_{\mathrm{URM}}$ and $\mathcal{T} \subset \mathcal{F}$ are obvious, while the strict inclusions $\mathcal{T}_{\mathrm{URM}} \subset \mathcal{T}$ and $\mathcal{F}_{\mathrm{URM}} \subset \mathcal{F}$ follow by counting arguments. However, the strict inclusion $\mathcal{P} \subset \mathcal{T}_{\mathrm{URM}}$ is not so obvious. An example of a total URM computable function that is not primitive recursive will be given in Chapter 4.

2.3 Closure Properties

This section provides a small repository of algorithmic properties and constructions for later use.

Transformation of Variables and Parametrization

Given a function $f : \mathbb{N}_0^n \to \mathbb{N}_0$ and a mapping $\phi : [n] \to [m]$. The function f^ϕ obtained from f by *transformation of variables* with respect to ϕ is defined as

$$f^\phi : \mathbb{N}_0^m \to \mathbb{N}_0 : (x_1, \ldots, x_m) \mapsto f(x_{\phi(1)}, \ldots, x_{\phi(n)}). \tag{2.12}$$

Proposition 2.16. *If the function f is primitive recursive, the function f^ϕ is also primitive recursive.*

Proof. Transformation of variables can be described by the composition

$$f^\phi = f(\pi^{(m)}_{\phi(1)}, \ldots, \pi^{(m)}_{\phi(n)}). \tag{2.13}$$

□

Examples 2.17. Three important special cases for dyadic functions:

- permutation of variables: $f^\phi : (x, y) \mapsto f(y, x)$,
- adjunct of variables: $f^\phi : (x, y) \mapsto f(x)$,
- identification of variables: $f^\phi : x \mapsto f(x, x)$.

♢

Let $c^{(k)}_i$ denote the k-ary constant function with value $i \in \mathbb{N}_0$, i.e.,

$$c^{(k)}_i : \mathbb{N}_0^k \to \mathbb{N}_0 : (x_1, \ldots, x_k) \mapsto i. \tag{2.14}$$

Proposition 2.18. *The constant function $c^{(k)}_i$ is primitive recursive.*

Proof. If $k = 0$, $c^{(0)}_i = \nu^i \circ c^{(0)}_0$. Otherwise, $c^{(k)}_i = \nu^i \circ c^{(1)}_0 \circ \pi^{(k)}_1$. □

Let $f : \mathbb{N}_0^n \to \mathbb{N}_0$ be a function. Take a positive integer m with $m < n$ and $\boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbb{N}_0^m$. The function $f_{\boldsymbol{a}}$ obtained from f by *parametrization* with respect to $\boldsymbol{a}$ is defined as

$$f_{\boldsymbol{a}} : \mathbb{N}_0^{n-m} \to \mathbb{N}_0 : (x_1, \ldots, x_{n-m}) \mapsto f(x_1, \ldots, x_{n-m}, a_1, \ldots, a_m). \tag{2.15}$$

Proposition 2.19. *If the function f is primitive recursive, the function $f_{\boldsymbol{a}}$ is also primitive recursive.*

Proof. Parametrization can be described by the composition

$$f_{\boldsymbol{a}} = f(\pi^{(n-m)}_1, \ldots, \pi^{(n-m)}_{n-m}, c^{(n-m)}_{a_1}, \ldots, c^{(n-m)}_{a_m}). \tag{2.16}$$

□

Definition by Cases

Let $h_i : \mathbb{N}_0^k \to \mathbb{N}_0$, $1 \le i \le r$, be total functions with the property that for each $\boldsymbol{x} \in \mathbb{N}_0^k$ there is a unique index $i \in [r]$ such that $h_i(\boldsymbol{x}) = 0$. That is, the sets $H_i = \{\boldsymbol{x} \in \mathbb{N}_0^k \mid h_i(\boldsymbol{x}) = 0\}$ form a partition of the whole set $\mathbb{N}_0^k$. Moreover, let $g_i : \mathbb{N}_0^k \to \mathbb{N}_0$, $1 \le i \le r$, be arbitrary total functions. Define the function

$$f : \mathbb{N}_0^k \to \mathbb{N}_0 : \boldsymbol{x} \mapsto \begin{cases} g_1(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in H_1, \\ \vdots & \vdots \\ g_r(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in H_r. \end{cases} \tag{2.17}$$

The function f is clearly total and said to be *defined by cases.*

Proposition 2.20. *If the above functions g_i and h_i, $1 \le i \le r$, are primitive recursive, the function f is also primitive recursive.*

Proof. In case of $r = 2$, the function f is given in prefix notation as follows:

$$f = g_1 \cdot \mathrm{csg}(h_1) + g_2 \cdot \mathrm{csg}(h_2). \tag{2.18}$$

The general case follows by induction on r. $\square$

Example 2.21. Let csg denote the *cosign function*, i.e., $\mathrm{csg}(x) = 0$ if $x > 0$ and $\mathrm{csg}(x) = 1$ if $x = 0$. The mappings $h_1 : x \mapsto x \bmod 2$ and $h_2 : x \mapsto \mathrm{csg}(x \bmod 2)$ define a partition of the set $\mathbb{N}_0$ into the set of even natural numbers and the set of odd natural numbers. These mappings can be used to define a function defined by cases as follows:

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even}, \\ (x+1)/2 & \text{if } x \text{ is odd}. \end{cases} \tag{2.19}$$

$\diamondsuit$

Bounded Sum and Product

Let $f : \mathbb{N}_0^{k+1} \to \mathbb{N}_0$ be a total function. The *bounded sum* of f is the function

$$\Sigma f : \mathbb{N}_0^{k+1} \to \mathbb{N}_0 : (x_1, \ldots, x_k, y) \mapsto \sum_{i=0}^{y} f(x_1, \ldots, x_k, i) \tag{2.20}$$

and the *bounded product* of f is the function

$$\Pi f : \mathbb{N}_0^{k+1} \to \mathbb{N}_0 : (x_1, \ldots, x_k, y) \mapsto \prod_{i=0}^{y} f(x_1, \ldots, x_k, i). \tag{2.21}$$

Proposition 2.22. *If the function f is primitive recursive, the functions Σf and Πf are also primitive recursive.*

Proof. The function Σf is given as

$$\Sigma f(\boldsymbol{x}, 0) = f(\boldsymbol{x}, 0) \quad \text{and} \quad \Sigma f(\boldsymbol{x}, y+1) = \Sigma f(\boldsymbol{x}, y) + f(\boldsymbol{x}, y+1). \tag{2.22}$$

This corresponds to the primitive recursive scheme $\Sigma f = \mathrm{pr}(g, h)$, where $g(\boldsymbol{x}) = f(\boldsymbol{x}, 0)$ and $h(\boldsymbol{x}, y, z) = +(f(\boldsymbol{x}, \nu(y)), z)$. The function Πf can be similarly defined. $\square$

Example 2.23. Take the function $f : \mathbb{N}_0 \to \mathbb{N}_0$ defined by $f(x) = 1$ if $x = 0$ and $f(x) = x$ if $x > 0$. This function is primitive recursive and thus the bounded product, given as $\Pi f(x) = x!$ for all $x \in \mathbb{N}_0$, is primitive recursive. $\diamondsuit$

Bounded Minimalization

Let $f : \mathbb{N}_0^{k+1} \to \mathbb{N}_0$ be a total function. The *bounded minimalization* of f is the function

$$\overline{\mu} f : \mathbb{N}_0^{k+1} \to \mathbb{N}_0 : (\boldsymbol{x}, y) \mapsto \mu(i \le y)[f(\boldsymbol{x}, i) = 0], \tag{2.23}$$

where for each $(\boldsymbol{x}, y) \in \mathbb{N}_0^{k+1}$,

$$\mu(i \leq y)[f(\boldsymbol{x}, i) = 0] = \begin{cases} j & \text{if } j = \min\{i \mid i \leq y \wedge f(\boldsymbol{x}, i) = 0\} \text{ exists,} \\ y + 1 & \text{otherwise.} \end{cases} \tag{2.24}$$

That is, the value $\overline{\mu} f(\boldsymbol{x}, y)$ provides the smallest index j with $0 \leq j \leq y$ such that $f(\boldsymbol{x}, j) = 0$. If there is no such index, the value is $y + 1$.

Proposition 2.24. *If the function f is primitive recursive, the function $\overline{\mu} f$ is also primitive recursive.*

Proof. By definition,

$$\overline{\mu} f(\boldsymbol{x}, 0) = \operatorname{sgn}(f(\boldsymbol{x}, 0)) \tag{2.25}$$

and

$$\overline{\mu} f(\boldsymbol{x}, y + 1) = \begin{cases} \overline{\mu} f(\boldsymbol{x}, y) & \text{if } \overline{\mu} f(\boldsymbol{x}, y) \leq y \text{ or } f(\boldsymbol{x}, y + 1) = 0, \\ y + 2 & \text{otherwise.} \end{cases} \tag{2.26}$$

Define the $k + 2$-ary functions

$$\begin{aligned} g_1 &: (\boldsymbol{x}, y, z) \mapsto z, \\ g_2 &: (\boldsymbol{x}, y, z) \mapsto y + 2, \\ h_1 &: (\boldsymbol{x}, y, z) \mapsto (z \dot{-} y) \cdot \operatorname{sgn}(f(\boldsymbol{x}, y + 1)), \\ h_2 &: (\boldsymbol{x}, y, z) \mapsto \operatorname{csg}(h_1(\boldsymbol{x}, y, z)). \end{aligned} \tag{2.27}$$

These functions are primitive recursive. Moreover, the functions h_1 and h_2 provide a partition of $\mathbb{N}_0$. Thus the following function defined by cases is also primitive recursive:

$$g(\boldsymbol{x}, y, z) = \begin{cases} g_1(\boldsymbol{x}, y, z) & \text{if } h_1(\boldsymbol{x}, y, z) = 0, \\ g_2(\boldsymbol{x}, y, z) & \text{if } h_2(\boldsymbol{x}, y, z) = 0. \end{cases} \tag{2.28}$$

We have $h_1(\boldsymbol{x}, y, \overline{\mu} f(\boldsymbol{x}, y)) = 0$ if and only if $(\overline{\mu} f(\boldsymbol{x}, y)) \dot{-} y) \cdot \operatorname{sgn}(f(\boldsymbol{x}, y + 1)) = 0$, which is equivalent to $\overline{\mu} f(\boldsymbol{x}, y)) \leq y$ or $f(\boldsymbol{x}, y + 1) = 0$. In this case, $g_1(\boldsymbol{x}, y, \overline{\mu} f(\boldsymbol{x}, y)) = \overline{\mu} f(\boldsymbol{x}, y + 1)$. The other case can be similarly evaluated. It follows that the bounded minimalization $\overline{\mu} f$ corresponds to the primitive recursive scheme $\overline{\mu} f = \operatorname{pr}(s, g)$, where $s : \mathbb{N}_0^k \to \mathbb{N}_0$ is defined as $s(\boldsymbol{x}) = \operatorname{sgn}(f(\boldsymbol{x}, 0))$ and g is given as above. ☐

Example 2.25. Consider the integral division function

$$\div : \mathbb{N}_0^2 \to \mathbb{N}_0 : (x, y) \mapsto \begin{cases} \lfloor x/y \rfloor & \text{if } y > 0, \\ x & \text{if } y = 0, \end{cases} \tag{2.29}$$

where the expression $\lfloor x/y \rfloor$ means that $\lfloor x/y \rfloor = z$ if $y \cdot z \geq x$ and z is minimal with this property. Thus the value z can be provided by bounded minimalization. To this end, define the function $f : \mathbb{N}_0^3 \to \mathbb{N}_0 : (x, y, z) \mapsto \operatorname{csg}(y \cdot z \dot{-} x)$; that is,

$$f(x, y, z) = \begin{cases} 0 & \text{if } y \cdot z > x, \\ 1 & \text{otherwise.} \end{cases} \tag{2.30}$$

Applying bounded minimalization to f yields the primitive recursive function

$$\overline{\mu}f(x,y,z) = \begin{cases} \text{smallest } j \leq z \text{ with } y \cdot j > x & \text{if } j \text{ exists,} \\ z+1 & \text{otherwise.} \end{cases} \tag{2.31}$$

Identification of variables provides the primitive recursive function

$$(\overline{\mu}f)' : (x,y) \mapsto \overline{\mu}f(x,y,x), \tag{2.32}$$

which is given as

$$(\overline{\mu}f)'(x,y) = \begin{cases} \text{smallest } j \leq x \text{ with } y \cdot j > x & \text{if } y \geq 1, \\ x+1 & \text{if } y = 0. \end{cases} \tag{2.33}$$

It follows that $\div(x,y) = (\overline{\mu}f)'(x,y) \dot{-} 1$. Finally, the remainder of x modulo y is given by $\mathrm{rem}(x,y) = y \dot{-} x \cdot \div(x,y)$ and thus is also primitive recursive. ♢

Pairing Functions

A pairing function uniquely encodes pairs of natural numbers by single natural numbers. A primitive recursive bijection from $\mathbb{N}_0^2$ onto $\mathbb{N}_0$ is called a *pairing function*. In set theory, any pairing function can be used to prove that the rational numbers have the same cardinality as the natural numbers. For instance, the *Cantor function* $J_2 : \mathbb{N}_0^2 \to \mathbb{N}_0$ is defined as

$$J_2(m,n) = \binom{m+n+1}{2} + m. \tag{2.34}$$

This function will provide a proof that the cartesian product $\mathbb{N}_0^2$ is denumerable. To this end, write down the elements of $\mathbb{N}_0^2$ in a table as follows:

$$\begin{array}{l} (0,0)\ (0,1)\ (0,2)\ (0,3)\ (0,4)\ \ldots \\ (1,0)\ (1,1)\ (1,2)\ (1,3)\ \ \ldots \\ (2,0)\ (2,1)\ (2,2)\ \ \ldots \\ (3,0)\ (3,1)\ \ \ldots \\ (4,0)\ \ \ldots \\ \ \ldots \end{array} \tag{2.35}$$

Its *kth anti-diagonal* is given by the sequence

$$(0,k), (1,k-1), \ldots, (k,0). \tag{2.36}$$

Now generate a list of all elements of $\mathbb{N}_0^2$ by writing down the anti-diagonals in a consecutive manner starting from the 0-th anti-diagonal:

$$(0,0),(0,1),(1,0),(0,2),(1,1),(2,0),(0,3),(1,2),(2,1),(3,0),(0,4),\ldots. \tag{2.37}$$

Note the kth anti-diagonal consists of $k+1$ entries. Thus the pair (m,n) lies at position m in the $m+n$th anti-diagonal and hence occurs in the list at position

$$[1+2+\ldots+(m+n)]+m=\binom{m+n+1}{2}+m. \tag{2.38}$$

That is, $J_2(m,n)$ provides the position of the pair (m,n) in the above list. This shows that the function J_2 is bijective.

Proposition 2.26. *The Cantor function J_2 is primitive recursive.*

Proof. By using the integral division function $\div$, one obtains

$$J_2(m,n)=\div((m,n)\cdot(m+n+1),2)+m. \tag{2.39}$$

Thus J_2 is primitive recursive. $\square$

The Cantor function J_2 can be inverted by taking coordinate functions $K_2, L_2 : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$J_2^{-1}(n)=(K_2(n),L_2(n)),\quad n\in\mathbb{N}_0. \tag{2.40}$$

In order to define them, take $n \in \mathbb{N}_0$. Find a number $s \geq 0$ such that

$$\frac{1}{2}s(s+1)\leq n<\frac{1}{2}(s+1)(s+2) \tag{2.41}$$

and put

$$m=n-\frac{1}{2}s(s+1). \tag{2.42}$$

Then

$$m=n-\frac{1}{2}s(s+1)\leq[\frac{1}{2}(s+1)(s+2)-1]-[\frac{1}{2}s(s+1)]=s. \tag{2.43}$$

Finally, set

$$K_2(n)=m\quad\text{and}\quad L_2(n)=s-m. \tag{2.44}$$

Proposition 2.27. *The coordinate functions K_2 and L_2 are primitive recursive and the pair (K_2, L_2) is the inverse of J_2.*

Proof. Let $n \in \mathbb{N}_0$. The corresponding number s in (2.41) can be determined by bounded minimalization as follows:

$$\mu(i\leq n)[n\dot{-}\frac{1}{2}s(s+1)=0]=\begin{cases} j & \text{if } j=\min\{i\mid i\leq y\wedge n\dot{-}\frac{1}{2}s(s+1)=0\} \text{ exists,}\\ n+1 & \text{otherwise.}\end{cases} \tag{2.45}$$

The value j always exists and $s=j\dot{-}1$. Then $K_2(n)=n-\div(s(s+1),2)$ and $L_2(n)=s-K_2(n)$. Thus both, K_2 and L_2 are primitive recursive. Finally, by (2.42) and (2.44), $J_2(K_2(n),L_2(n))=J_2(m,s-m)=\frac{1}{2}s(s+1)+m=n$ and so (2.40) follows. $\square$

This assertion implies that the Cantor function J_2 is a pairing function.

Iteration

The *powers* of a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ are inductively defined as

$$f^0 = \mathrm{id}_{\mathbb{N}_0} \quad \text{and} \quad f^{n+1} = f \circ f^n,\ n \geq 0. \tag{2.46}$$

The *iteration* of a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ is given by the function

$$g : \mathbb{N}_0^2 \to \mathbb{N}_0 : (x, y) \mapsto f^y(x). \tag{2.47}$$

Example 2.28. Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0 : x \mapsto 2x$. The iteration of f is the function $g : \mathbb{N}_0^2 \to \mathbb{N}_0$ given by $g(x, y) = 2^y \cdot x$. $\diamondsuit$

Proposition 2.29. *If f is a monadic primitive recursive function, the iteration of f is also primitive recursive.*

Proof. The iteration g of f follows the primitive recursive scheme

$$g(x, 0) = x \tag{2.48}$$

and

$$g(x, y + 1) = f(g(x, y)) = f \circ \pi_3^{(3)}(x, y, g(x, y)),\ x, y \in \mathbb{N}_0. \tag{2.49}$$

$\square$

Iteration can also be defined for multivariate functions. For this, let $f : \mathbb{N}_0^k \to \mathbb{N}_0^k$ be a function defined by coordinate functions $f_i : \mathbb{N}_0^k \to \mathbb{N}_0$, $1 \leq i \leq k$, as follows:

$$f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathbb{N}_0^k. \tag{2.50}$$

Write $f = (f_1, \ldots, f_k)$ and define the *powers* of f inductively as follows:

$$f^0(\boldsymbol{x}) = \boldsymbol{x} \quad \text{and} \quad f^{n+1}(\boldsymbol{x}) = (f_1(f^n(\boldsymbol{x})), \ldots, f_k(f^n(\boldsymbol{x}))), \quad \boldsymbol{x} \in \mathbb{N}_0^k. \tag{2.51}$$

These definitions immediately give rise to the following result.

Proposition 2.30. *If the functions $f = (f_1, \ldots, f_k)$ are primitive recursive, the powers of f are also primitive recursive.*

The *iteration* of $f = (f_1, \ldots, f_k)$ is defined by the functions

$$g_i : \mathbb{N}_0^{k+1} \to \mathbb{N}_0 : (\boldsymbol{x}, y) \mapsto (\pi_i^{(k)} \circ f^y)(\boldsymbol{x}), \quad 1 \leq i \leq k. \tag{2.52}$$

Proposition 2.31. *If the functions $f = (f_1, \ldots, f_k)$ are primitive recursive, the iteration of f is also primitive recursive.*

Proof. The iteration of $f = (f_1, \ldots, f_k)$ follows the primitive recursive scheme

$$g_i(\boldsymbol{x}, 0) = x_i \tag{2.53}$$

and

$$g_i(\boldsymbol{x}, y + 1) = f_i(f^y(\boldsymbol{x})) = f_i \circ \pi_{k+2}^{(k+2)}(\boldsymbol{x}, y, g_i(\boldsymbol{x}, y)), \quad \boldsymbol{x} \in \mathbb{N}_0^k, y \in \mathbb{N}_0,\ 1 \leq i \leq k. \tag{2.54}$$

$\square$

2.4 Primitive Recursive Sets

The studies can be extended to relations given as subsets of $\mathbb{N}_0^k$ by taking their characteristic function. Let S be a subset of $\mathbb{N}_0^k$. The *characteristic function* of S is the function $\chi_S : \mathbb{N}_0^k \to \mathbb{N}_0$ defined by

$$\chi_S(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in S, \\ 0 \text{ otherwise.} \end{cases} \tag{2.55}$$

A subset S of $\mathbb{N}_0^k$ is called *primitive* if its characteristic function χ_S is primitive recursive.

Examples 2.32. Here are some primitive basic relations:

1. The equality relation $R_= = \{(x, y) \in \mathbb{N}_0^2 \mid x = y\}$ is primitive, since the corresponding characteristic function $\chi_{R_=}(x, y) = \mathrm{csg}(|x - y|)$ is primitive recursive.
2. The inequality relation $R_{\neq} = \{(x, y) \in \mathbb{N}_0^2 \mid x \neq y\}$ is primitive, since its characteristic function $\chi_{R_{\neq}}(x, y) = 1 - \mathrm{csg}(|x - y|)$ is primitive recursive.
3. The smaller relation $R_< = \{(x, y) \in \mathbb{N}_0^2 \mid x < y\}$ is primitive, since the associated characteristic function $\chi_{R_<}(x, y) = \mathrm{sgn}(y \dot{-} x)$ is primitive recursive.

♢

Proposition 2.33. *If S and T are primitive subsets of $\mathbb{N}_0^k$, $S \cup T$, $S \cap T$, and $\mathbb{N}_0^k \setminus S$ are also primitive.*

Proof. Clearly, $\chi_{S\cup T}(x) = \mathrm{sgn}(\chi_S(x) + \chi_T(x))$, $\chi_{S\cap T}(x) = \chi_S(x) \cdot \chi_T(x)$, and $\chi_{\mathbb{N}_0^k \setminus S}(x) = \mathrm{csg} \circ \chi_S(x)$. □

Let S be a subset of $\mathbb{N}_0^{k+1}$. The *bounded existential quantification* of S is a subset of $\mathbb{N}_0^{k+1}$ given as

$$\exists S = \{(\boldsymbol{x}, y) \mid (\boldsymbol{x}, i) \in S \text{ for some } 0 \leq i \leq y\}. \tag{2.56}$$

The *bounded universal quantification* of S is a subset of $\mathbb{N}_0^{k+1}$ defined by

$$\forall S = \{(\boldsymbol{x}, y) \mid (\boldsymbol{x}, i) \in S \text{ for all } 0 \leq i \leq y\}. \tag{2.57}$$

Proposition 2.34. *If S is a primitive subset of $\mathbb{N}_0^{k+1}$, the sets $\exists S$ and $\forall S$ are also primitive.*

Proof. Clearly, $\chi_{\exists S}(\boldsymbol{x}, y) = \mathrm{sgn}((\Sigma\chi_S)(\boldsymbol{x}, y))$ and $\chi_{\forall S}(\boldsymbol{x}, y) = (\Pi\chi_S)(\boldsymbol{x}, y)$. □

Consider the sequence of increasing primes $(p_0, p_1, p_2, p_3, p_4, \ldots) = (2, 3, 5, 7, 11, \ldots)$. By the fundamental theorem of arithmetics, each natural number $x \geq 1$ can be uniquely written as a product of prime powers, i.e.,

$$x = \prod_{i=0}^{r-1} p_i^{e_i}, \quad e_0, \ldots, e_{r-1} \in \mathbb{N}_0. \tag{2.58}$$

Write $(x)_i = e_i$ for each $i \in \mathbb{N}_0$, and put $(0)_i = 0$ for all $i \in \mathbb{N}_0$. For instance, $24 = 2^3 \cdot 3$ and so $(24)_0 = 3$, $(24)_1 = 1$, and $(24)_i = 0$ for all $i \geq 2$.

Proposition 2.35. *1. The divisibility relation $D = \{(x, y) \in \mathbb{N}_0^2 \mid x \text{ divides } y\}$ is primitive.*

2. The set of primes is primitive.
3. The function $p : \mathbb{N}_0 \to \mathbb{N}_0 : i \mapsto p_i$ is primitive recursive.
4. The function $\tilde{p} : \mathbb{N}_0^2 \to \mathbb{N}_0 : (x, i) \mapsto (x)_i$ is primitive recursive.

Proof. First, x divides y, written $x \mid y$, if and only if $x \cdot i = y$ for some $0 \leq i \leq y$. Thus the characteristic function of D can be written as

$$\chi_D(x, y) = \operatorname{sgn}[\chi_=(x \cdot 1, y) + \chi_=(x \cdot 2, y) + \ldots + \chi_=(x \cdot y, y)]. \tag{2.59}$$

Hence the relation D is primitive.

Second, a number x is prime if and only if $x \geq 2$ and i divides x implies $i = 1$ or $i = x$ for all $i \leq x$. Thus the characteristic function of the set P of primes is given as follows: $\chi_P(0) = \chi_P(1) = 0$, $\chi_P(2) = 1$, and

$$\chi_P(x) = \operatorname{csg}[\chi_D(2, x) + \chi_D(3, x) + \ldots + \chi_D(x - 1, x)], \quad x \geq 3. \tag{2.60}$$

Third, define the functions

$$g(z, x) = |\chi_{R_<}(z, x) \cdot \chi_P(x) - 1| = \begin{cases} 0 \text{ if } z < x \text{ and } x \text{ prime,} \\ 1 \text{ otherwise,} \end{cases} \tag{2.61}$$

and

$$h(z) = \overline{\mu} g(z, z! + 1) = \mu(y \leq z! + 1)[g(z, y) = 0]. \tag{2.62}$$

Both functions g and h are primitive recursive. By a theorem of Euclid, the $i + 1$th prime is bounded by the ith prime in a way that $p_{i+1} \leq p_i! + 1$ for all $i \geq 0$. Thus the value $h(p_i)$ provides the next prime p_{i+1}. That is, the sequence of prime numbers is given by the primitive recursive scheme

$$p_0 = 2 \quad \text{and} \quad p_{i+1} = h(p_i),\ i \geq 0. \tag{2.63}$$

Fourth, we have

$$(x)_i = \mu(y \leq x)[p_i^{y+1} \not| \, x] = \mu(y \leq x)[\chi_D(p_i^{y+1}, x) = 0]. \tag{2.64}$$

□

2.5 LOOP Programs

This section provides a mechanistic description of the class of primitive recursive functions. For this, a class of URM computable functions is introduced in which the use of loop variables is restricted. More specifically, the only loops or iterations allowed will be of the form $(M; S\sigma)\sigma$, where the variable σ does not appear in the program M. In this way, the program M cannot manipulate the register R_σ and thus it can be guaranteed that the program M will be carried out n times, where n is the content of the register R_σ at the start of the computation. Loops of this type allow an explicit control over the loop variable.

Two abbreviations will be used in the following: If P is an URM program and $\sigma \in Z$, write $[P]\sigma$ for the program $(P; S\sigma)\sigma$, and denote the URM program $(S\sigma)\sigma$ by $Z\sigma$.

The class $\mathcal{P}_{\text{LOOP}}$ of LOOP programs is inductively defined as follows:

1. Define the class of LOOP-0 programs $\mathcal{P}_{\mathrm{LOOP}(0)}$:
   a) For each $\sigma \in Z$, $A\sigma \in \mathcal{P}_{\mathrm{LOOP}(0)}$ and $Z\sigma \in \mathcal{P}_{\mathrm{LOOP}(0)}$.
   b) For each $\sigma, \tau \in Z$ with $\sigma \neq \tau$ and $\sigma \neq 0 \neq \tau$, $\bar{C}(\sigma, \tau) = Z\tau; Z0; C(\sigma; \tau) \in \mathcal{P}_{\mathrm{LOOP}(0)}$.
   c) If $P, Q \in \mathcal{P}_{\mathrm{LOOP}(0)}$, then $P; Q \in \mathcal{P}_{\mathrm{LOOP}(0)}$.
2. Suppose the class of LOOP-n programs $\mathcal{P}_{\mathrm{LOOP}(n)}$ has already been defined. Define the class of LOOP-$n+1$ programs $\mathcal{P}_{\mathrm{LOOP}(n+1)}$:
   a) Each $P \in \mathcal{P}_{\mathrm{LOOP}(n)}$ belongs to $\mathcal{P}_{\mathrm{LOOP}(n+1)}$.
   b) If $P, Q \in \mathcal{P}_{\mathrm{LOOP}(n+1)}$, then $P; Q \in \mathcal{P}_{\mathrm{LOOP}(n+1)}$.
   c) if $P \in \mathcal{P}_{\mathrm{LOOP}(n)}$ and $\sigma \in Z$ does not appear in P, then $[P]\sigma \in \mathcal{P}_{\mathrm{LOOP}(n+1)}$.

Note that for each $\omega \in \Omega$, $\bar{w} = \bar{C}(\sigma, \tau)(\omega)$ is given by

$$\bar{\omega}_n = \begin{cases} 0 & \text{if } n = 0, \\ \omega_\sigma & \text{if } n = \sigma \text{ or } n = \tau, \\ \omega_n & \text{otherwise.} \end{cases} \tag{2.65}$$

That is, the content of register R_σ is copied into register R_τ and the register R_0 is set to zero.

Note that the LOOP-n programs form as sets a proper hierarchy:

$$\mathcal{P}_{\mathrm{LOOP}(0)} \subset \mathcal{P}_{\mathrm{LOOP}(1)} \subset \mathcal{P}_{\mathrm{LOOP}(2)} \subset \ldots. \tag{2.66}$$

The class of *LOOP programs* is defined as as the union of LOOP-n programs for all $n \in \mathbb{N}_0$:

$$\mathcal{P}_{\mathrm{LOOP}} = \bigcup_{n \geq 0} \mathcal{P}_{\mathrm{LOOP}(n)}. \tag{2.67}$$

In particular, $\mathcal{P}_{\mathrm{LOOP}(n)}$ is called the class of LOOP programs *of depth* n, $n \in \mathbb{N}_0$.

Proposition 2.36. *For each LOOP program P, the function $\|P\|$ is total.*

Proof. For each LOOP-0 program P, it is clear that the function $\|P\|$ is total. Let P be a LOOP-n program and let $\sigma \in Z$ such that σ does not appear in P. By induction hypothesis, the function $\|P\|$ is total. Moreover, $\|[P]\sigma\| = \|P^k\|$, where k is the content of register R_σ at the beginning of the computation. Thus the function $\|[P]\sigma\|$ is also total. The remaining cases are clear. $\square$

A function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ is called *LOOP-n computable* if there is a LOOP-n program P such that $\|P\|_{k,1} = f$. Let $\mathcal{F}_{\mathrm{LOOP}(n)}$ denote the class of all LOOP-n computable functions and define the class of all LOOP computable functions $\mathcal{F}_{\mathrm{LOOP}}$ as the union of LOOP-n computable functions for all $n \geq 0$:

$$\mathcal{F}_{\mathrm{LOOP}} = \bigcup_{n \geq 0} \mathcal{F}_{\mathrm{LOOP}(n)}. \tag{2.68}$$

Note that if P is a LOOP-n program, $n \geq 1$, and P' is the normal program corresponding to P, then P' is also a LOOP-n program.

Example 2.37. The program $S\sigma$ does not belong to the basic LOOP programs. But it can be described by a LOOP-1 program. Indeed, put

$$P_{\dot{-}1} = \bar{C}(1; 3); [\bar{C}(2; 1); A2]3. \tag{2.69}$$

Then we have for input $x = 0$,

0	1	2	3	4 ...	registers
0	0	0	0	0 ...	initially
0	0	0	0	0 ...	$\bar{C}(1;3)$
0	0	0	0	0 ...	finally

and for input $x \geq 1$,

0	1	2	3	4 ...	registers
0	x	0	0	0 ...	initially
0	x	0	x	0 ...	$\bar{C}(1;3)$
0	0	0	x	0 ...	$\bar{C}(2;1)$
0	0	1	x	0 ...	$A2$
0	0	1	$x-1$	0 ...	$S3$
...					
0	1	2	$x-2$	0 ...	
...					
0	$x-1$	x	0	0 ...	finally

It follows that $\|P_{\dot{-}1}\|_{1,1} = \|S1\|_{1,1}$. ♢

The LOOP-n computable functions form a hierarchy but at this stage it is not clear whether it is proper or not:

$$\mathcal{T}_{\text{LOOP}(0)} \subseteq \mathcal{T}_{\text{LOOP}(1)} \subseteq \mathcal{T}_{\text{LOOP}(2)} \subseteq \ldots. \tag{2.70}$$

Theorem 2.38. *The class of LOOP computable functions equals the class of primitive recursive functions.*

Proof. First, claim that each primitive recursive function is LOOP computable. Indeed, the basic primitive recursive functions are LOOP computable:

1. 0-ary constant function : $\|Z0\|_{0,1} = c_0^{(0)}$,
2. monadic constant function : $\|Z1\|_{1,1} = c_0^{(1)}$,
3. successor function: $\|A1\|_{1,1} = \nu$,
4. projection function : $\|Z0\|_{k,1} = \pi_1^{(k)}$ and $\|\bar{C}(\sigma;1)\|_{k,1} = \pi_\sigma^{(k)}$, $\sigma \neq 1$.

Moreover, the class of LOOP computable functions is closed under composition and primitive recursion. This can be shown as in the proof of Theorem 2.10, where subtraction is replaced by the program in 2.37. But the class of primitive recursive functions is the smallest class of functions that is primitively closed. Hence, all primitive recursive functions are LOOP computable. This proves the claim.

Second, claim that each LOOP computable function is primitive recursive. Indeed, for each LOOP program P, let $n(P)$ denote the largest address (or register number) used in P. For integers $m \geq n(P)$ and $0 \leq j \leq m$, consider the functions

$$k_j^{(m+1)}(P) : \mathbb{N}_0^{m+1} \to \mathbb{N}_0 : (x_0, x_1, \ldots, x_m) \mapsto (\pi_j \circ |P|)(x_0, x_1, \ldots, x_m, , 0, 0, \ldots), \tag{2.71}$$

where for each $j \in \mathbb{N}_0$,

$$\pi_j : \Omega \to \mathbb{N}_0 : (\omega_0, \omega_1, \omega_2, \ldots) \mapsto \omega_j. \tag{2.72}$$

The assertion to be shown is a special case of the following assertion: For all LOOP programs P, for all integers $m \geq n(P)$ and $0 \leq j \leq m$, the function $k_j^{(m+1)}(P)$ is primitive recursive. The proof makes use of the inductive definition of LOOP programs.

First, let $P = A\sigma$, $m \geq \sigma$ and $0 \leq j \leq m$. Then

$$k_j^{(m+1)}(P) : \boldsymbol{x} \mapsto \begin{cases} (\nu \circ \pi_j^{(m+1)})(\boldsymbol{x}) & \text{if } j = \sigma, \\ \pi_j^{(m+1)}(\boldsymbol{x}) & \text{otherwise.} \end{cases} \tag{2.73}$$

Clearly, this function is primitive recursive.

Second, let $P = Z\sigma$, $m \geq \sigma$ and $0 \leq j \leq m$. We have

$$k_j^{(m+1)}(P) : \boldsymbol{x} \mapsto \begin{cases} (c_0^{(1)} \circ \pi_j^{(m+1)})(\boldsymbol{x}) & \text{if } j = \sigma, \\ \pi_j^{(m+1)}(\boldsymbol{x}) & \text{otherwise.} \end{cases} \tag{2.74}$$

This function is also primitive recursive.

Third, let $P = \bar{C}(\sigma, \tau)$, where $\sigma \neq \tau$ and $\sigma \neq 0 \neq \tau$, $m \geq n(P) = \max\{\sigma, \tau\}$ and $0 \leq j \leq m$. Then

$$k_j^{(m+1)}(P) : \boldsymbol{x} \mapsto \begin{cases} (c_0^{(1)} \circ \pi_j^{(m+1)})(\boldsymbol{x}) & \text{if } j = 0, \\ \pi_\sigma^{(m+1)}(\boldsymbol{x}) & \text{if } j = \tau, \\ \pi_j^{(m+1)}(\boldsymbol{x}) & \text{otherwise.} \end{cases} \tag{2.75}$$

This function is clearly primitive recursive.

Fourth, let $P = Q; R \in \mathcal{P}_{\text{LOOP}}$. By induction, assume that the assertion holds for Q and R. Let $m \geq n(Q; R) = \max\{n(Q), n(R)\}$ and $0 \leq j \leq m$. Then

$$\begin{aligned} k_j^{(m+1)}(P)(\boldsymbol{x}) &= (\pi_j \circ P)(\boldsymbol{x}, 0, 0, \ldots) \\ &= (\pi_j \circ |R| \circ |Q|)(\boldsymbol{x}, 0, 0, \ldots) \\ &= k_j^{(m+1)}(R)(k_0^{(m+1)}(Q)(\boldsymbol{x}), \ldots, k_m^{(m+1)}(Q)(\boldsymbol{x})), \\ &= k_j^{(m+1)}(R)(k_0^{(m+1)}(Q), \ldots, k_m^{(m+1)}(Q))(\boldsymbol{x}). \end{aligned} \tag{2.76}$$

Thus $k_j^{(m+1)}(P)$ is a composition of primitive recursive functions and hence also primitive recursive.

Finally, let $P = [Q]\sigma \in \mathcal{P}_{\text{LOOP}}$, where Q is a LOOP program in which the address σ is not involved. By induction, assume that the assertion holds for Q. Let $m \geq n([Q]\sigma) = \max\{n(Q), \sigma\}$ and $0 \leq j \leq m$.

First the program $Q; S\sigma$ yields

$$k_j^{(m+1)}(Q; S\sigma) : \boldsymbol{x} \mapsto \begin{cases} k_j^{(m+1)}(Q)(\boldsymbol{x}) & \text{if } j \neq \sigma, \\ (f_{\dot{-}1} \circ \pi_j^{(m+1)})(\boldsymbol{x}) & \text{otherwise.} \end{cases} \tag{2.77}$$

Let $k^{(m+1)}(Q; S\sigma) : \mathbb{N}_0^{m+1} \to \mathbb{N}_0^{m+1}$ denote the product of the $m+1$ functions $k_j^{(m+1)}(Q; S\sigma)$ for $0 \leq j \leq m$. That is,

$$k^{(m+1)}(Q; S\sigma)(\boldsymbol{x}) = (k_0^{(m+1)}(Q; S\sigma)(\boldsymbol{x}), \ldots, k_m^{(m+1)}(Q; S\sigma)(\boldsymbol{x})). \tag{2.78}$$

Let $g : \mathbb{N}_0^{m+2} \to \mathbb{N}_0^{m+1}$ denote the iteration of $k^{(m+1)}(Q; S\sigma)$; that is,

$$g(\boldsymbol{x}, 0) = \boldsymbol{x} \quad \text{and} \quad g(\boldsymbol{x}, y+1) = k^{(m+1)}(Q; S\sigma)(g(\boldsymbol{x}, y)). \tag{2.79}$$

For each index j, $0 \leq j \leq m$, the composition $\pi_j^{(m+1)} \circ g$ is also primitive recursive giving

$$(\pi_j^{(m+1)} \circ g)(\boldsymbol{x}, y) = k_j^{(m+1)}((Q; S\sigma)^y)(\boldsymbol{x}). \tag{2.80}$$

But the register R_σ is never used by the program Q and thus

$$|P|(\omega) = |(Q; S\sigma)\sigma|(\omega) = |(Q; S\sigma)^{\omega_\sigma}|(\omega), \ \omega \in \Omega. \tag{2.81}$$

It follows that the function $k_j^{(m+1)}(P)$ can be obtained from the primitive recursive function $\pi_j^{(m+1)} \circ g$ by transformation of variables, i.e.,

$$k_j^{(m+1)}(P)(\boldsymbol{x}) = (\pi_j^{(m+1)} \circ g)(\boldsymbol{x}, \pi_\sigma^{(m+1)}(\boldsymbol{x})), \ \boldsymbol{x} \in \mathbb{N}_0^{m+1}. \tag{2.82}$$

Thus $k_j^{(m+1)}(P)$ is also primitive recursive. $\square$

3

Partial Recursive Functions

The partial recursive functions form a class of partial functions that are computable in an intuitive sense. In fact, the partial recursive functions are precisely the functions that can be computed by Turing machines or unlimited register machines. By Church's thesis, the partial recursive functions provide a formalization of the notion of computability. The partial recursive functions are closely related to the primitive recursive functions and their inductive definition builds upon them.

3.1 Partial Recursive Functions

The class of partial recursive functions is the basic object of study in computability theory. This class was first investigated by Stephen Cole Kleene (1909-1994) in the 1930s and provides a formalized analogue of the intuitive notion of computability. To this end, a formal analogon of the `while` loop is required. For this, each partial function $f : \mathbb{N}^{k+1} \to \mathbb{N}_0$ is associated with a partial function

$$\mu f : \mathbb{N}_0^k \to \mathbb{N}_0 : \boldsymbol{x} \mapsto \begin{cases} y \text{ if } f(\boldsymbol{x}, y) = 0 \text{ and } f(\boldsymbol{x}, i) \neq 0 \text{ for } 0 \leq i < y, \\ \uparrow \text{ otherwise.} \end{cases} \tag{3.1}$$

The function μf is said to be defined by *(unbounded) minimalization* of f. The domain of the function μf is given by all elements $\boldsymbol{x} \in \mathbb{N}_0^k$ with the property that $f(\boldsymbol{x}, y) = 0$ and $(\boldsymbol{x}, i) \in \mathrm{dom}(f)$ for all $0 \leq i \leq y$. It is clear that in the context of programming, unbounded minimalization corresponds to a `while` loop (Algorithm 3.1).

Algorithm 3.1 Minimalization of f.

Require: $\boldsymbol{x} \in \mathbb{N}_0^n$
  $y \leftarrow -1$
  repeat
    $y \leftarrow y + 1$
    $z \leftarrow f(\boldsymbol{x}, y)$
  until $z = 0$
  return y

Examples 3.1.

- The minimalization function μf may be partial even if f is total: The function $f(x,y) = (x+y) \dot{-} 3$ is total, while its minimalization μf is partial with $\mathrm{dom}(\mu f) = \{0,1,2,3\}$ and $\mu f(0) = \mu f(1) = \mu f(2) = \mu f(3) = 0$.
- The minimalization function μf may be total even if f is partial: Take the partial function $f(x,y) = x - y$ if $y \leq x$ and $f(x,y) =\uparrow$ if $y > x$. The corresponding minimalization $\mu f(x) = x$ is total with $\mathrm{dom}(\mu f) = \mathbb{N}_0$. ♢

The class $\mathcal{R}$ of *partial recursive functions* over $\mathbb{N}_0$ is inductively defined:

- $\mathcal{R}$ contains all the base functions.
- If partial functions $g : \mathbb{N}_0^k \to \mathbb{N}_0$ and $h_i : \mathbb{N}_0^n \to \mathbb{N}_0$, $1 \leq i \leq k$, belong to $\mathcal{R}$, the composite function $f = g(h_1, \ldots, h_k) : \mathbb{N}_0^n \to \mathbb{N}_0$ is in $\mathcal{R}$.
- If partial functions $g : \mathbb{N}_0^n \to \mathbb{N}_0$ and $h : \mathbb{N}_0^{n+2} \to \mathbb{N}_0$ lie in $\mathcal{R}$, the primitive recursion $f = \mathrm{pr}(g,h) : \mathbb{N}_0^{n+1} \to \mathbb{N}_0$ is contained in $\mathcal{R}$.
- If a partial function $f : \mathbb{N}_0^{n+1} \to \mathbb{N}_0$ is in $\mathcal{R}$, the partial function $\mu f : \mathbb{N}_0^n \to \mathbb{N}_0$ obtained by minimalization belongs to $\mathcal{R}$.

Thus the class $\mathcal{R}$ consists of all partial recursive functions obtained from the base functions by finitely many applications of composition, primitive recursion, and minimalization. In particular, each total partial recursive function is called a *recursive function*, and the subclass of all total partial recursive functions is denoted by $\mathcal{T}$. It is clear that the class $\mathcal{P}$ of primitive recursive functions is a subclass of the class $\mathcal{T}$ of recursive functions.

Theorem 3.2. *Each partial recursive function is URM computable.*

Proof. It is sufficient to show that for each URM computable function $f : \mathbb{N}_0^{k+1} \to \mathbb{N}_0$ the corresponding minimalization μf is URM computable. For this, let P_f be an URM program for the function f. This program can be modified to provide an URM program P_f' with the following property:

$$|P_f'|(0, \boldsymbol{x}, y, 0, 0, \ldots) = \begin{cases} (0, \boldsymbol{x}, y, f(\boldsymbol{x}, y), 0, 0, \ldots) & \text{if } (\boldsymbol{x}, y) \in \mathrm{dom}(f), \\ \uparrow & \text{otherwise.} \end{cases} \tag{3.2}$$

Consider the URM program

$$P_{\mu f} = P_f';\ (Ak+1; (Sk+2)k+2; P_f')k+2;\ (S1)1; \ldots; (Sk)k; R_{k+1,1}. \tag{3.3}$$

The first block P_f' provides the computation in (3.2) for $y = 0$. The second block $(Ak+1; (Sk+2)k+2; P_f')k+2;$ iteratively calculates (3.2) for increasing values of y. This iteration stops when the function value becomes 0; in this case, the subsequent blocks reset the registers $R_1, \ldots, R_k$ to 0 and store the argument y in the first register. Otherwise, the program runs forever. It follows that the program $P_{\mu f}$ computes the minimalization of f, i.e., $\|P_{\mu f}\|_{k,1} = \mu f$. □

Note that $P_{\mu f}$ is generally not a LOOP program even if P_f' is a LOOP program.

3.2 GOTO Programs

GOTO programs offer another way to formalize the notion of computability. They are closely related to programs in BASIC or FORTRAN.

First, define the *syntax* of GOTO programs. For this, let $V = \{x_n \mid n \in \mathbb{N}\}$ be a set of variables. The *instructions* of a GOTO program are the following:

- incrementation:

$$(l, x_\sigma \leftarrow x_\sigma + 1, m), \quad l, m \in \mathbb{N}_0,\ x_\sigma \in V, \tag{3.4}$$

- decrementation:

$$(l, x_\sigma \leftarrow x_\sigma - 1, m), \quad l, m \in \mathbb{N}_0,\ x_\sigma \in V, \tag{3.5}$$

- conditionals:

$$(l, \texttt{if}\ x_\sigma = 0, k, m), \quad k, l, m \in \mathbb{N}_0,\ x_\sigma \in V. \tag{3.6}$$

The first component of an instruction, denoted by l, is called a *label*, the third component of an instruction $(l, x_\sigma+, m)$ or $(l, x_\sigma-, m)$, denoted by m, is termed *next label*, and the last two components of an instruction $(l, \texttt{if}\ x_\sigma = 0, k, m)$, denoted by k and m, are called *bifurcation labels*.

A *GOTO program* is given by a finite sequence of GOTO instructions

$$P = s_0; s_1; \ldots; s_q, \tag{3.7}$$

such that there is a unique instruction s_i which has the label $\lambda(s_i) = 0$, $0 \leq i \leq q$, and different instructions have distinct labels, i.e., for $0 \leq i < j \leq q$, $\lambda(s_i) \neq \lambda(s_j)$.

In the following, let $\mathcal{P}_{\mathrm{GOTO}}$ denote the class of all GOTO programs. Moreover, for each GOTO program P, let $V(P)$ depict the set of variables occurring in P and $L(P) = \{\lambda(s_i) \mid 0 \leq i \leq q\}$ describe the set of labels in P.

A GOTO program $P = s_0; s_1; \ldots; s_q$ is called *standard* if the ith instruction s_i carries the label $\lambda(s_i) = i$, $0 \leq i \leq q$.

Example 3.3. A standard GOTO program P_+ for the addition of two natural numbers is the following:

$$\begin{array}{llll} 0 & \texttt{if}\ x_2 = 0 & 3 & 1 \\ 1 & x_1 \leftarrow x_1 + 1 & 2 & \\ 2 & x_2 \leftarrow x_2 - 1 & 0 & \end{array}$$

The set of occurring variables is $V(P_+) = \{x_1, x_2\}$ and the set of labels is $L(P_+) = \{0, 1, 2\}$. ♢

Second, define the *semantics* of GOTO program. For this, the idea is to run a GOTO program on an URM. To this end, the variable x_σ in V is assigned the register R_σ of the URM for all $\sigma \geq 0$. In particular, the register R_0 serves as an instruction counter containing the label of the next instruction to be carried out. At the beginning, the instruction counter is set to 0 such that the execution starts with the instruction having label 0. The instruction $(l, x_\sigma \leftarrow x_\sigma + 1, m)$ increments the content of the register R_σ, the instruction $(l, x_\sigma \leftarrow x_\sigma - 1, m)$ decrements the content of the register R_σ provided that it contains a number greater than zero, and the conditional statement $(l, \texttt{if}\ x_\sigma = 0, k, m)$ provides a jump to the statement with label k if the content of register R_σ is 0; otherwise, a jump is performed to the statement with label m. The execution of a GOTO program *terminates* if the label of the instruction counter does not correspond to an instruction.

More specifically, let P be a GOTO program and k be a number. Define the partial function

$$\|P\|_{k,1} = \beta_1 \circ R_P \circ \alpha_k, \tag{3.8}$$

where α_k and β_1 are total functions given by

$$\alpha_k : \mathbb{N}_0^k \to \mathbb{N}_0^{n+1} : (x_1, x_2, \ldots, x_k) \mapsto (0, x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0) \tag{3.9}$$

and

$$\beta_1 : \mathbb{N}_0^{n+1} \to \mathbb{N}_0 : (x_0, x_1, \ldots, x_n) \mapsto x_1. \tag{3.10}$$

The function α_k loads the registers with the arguments and the function β_1 reads out the result. Note that the number n needs to be chosen large enough to provide sufficient workspace for the computation; that is,

$$n \geq \max\{k, \max\{\sigma \mid x_\sigma \in V(P)\}\}. \tag{3.11}$$

Finally, the function $R_P : \mathbb{N}_0^{n+1} \to \mathbb{N}_0^{n+1}$ providing the semantics of the program P will be formally described. For this, let $P = s_0; s_1; \ldots; s_q$ be a GOTO program. Each element $\boldsymbol{z} = (z_0, z_1, \ldots, z_n)$ in $\mathbb{N}_0^{n+1}$ is called a *configuration*. We say that the configuration $\boldsymbol{z}' = (z_0', z_1', \ldots, z_n')$ is *reached in one step* from the configuration $\boldsymbol{z} = (z_0, z_1, \ldots, z_n)$, written $\boldsymbol{z} \vdash_P \boldsymbol{z}'$, if z_0 is a label in P, say $z_0 = \lambda(s_i)$ for some $0 \leq i \leq q$, and

- if $s_i = (z_0, x_\sigma \leftarrow x_\sigma + 1, m)$,

$$\boldsymbol{z}' = (m, z_1, \ldots, z_{\sigma-1}, z_\sigma + 1, z_{\sigma+1}, \ldots, z_n), \tag{3.12}$$

- if $s_i = (z_0, x_\sigma \leftarrow x_\sigma - 1, m)$,

$$\boldsymbol{z}' = (m, z_1, \ldots, z_{\sigma-1}, z_\sigma \dot{-} 1, z_{\sigma+1}, \ldots, z_n), \tag{3.13}$$

- if $s_i = (z_0, \texttt{if}\ x_\sigma = 0, k, m)$,

$$\boldsymbol{z}' = \begin{cases} (k, z_1, \ldots, z_n) & \text{if } z_\sigma = 0, \\ (m, z_1, \ldots, z_n) & \text{otherwise.} \end{cases} \tag{3.14}$$

The configuration $\boldsymbol{z}'$ is called the *successor configuration* of $\boldsymbol{z}$. Moreover, if z_0 is not a label in P, there is no successor configuration of $\boldsymbol{z}$ and the successor configuration is *undefined*. The process to move from one configuration to the next one can be described by the *one-step function* $E_P : \mathbb{N}_0^{n+1} \to \mathbb{N}_0^{n+1}$ defined as

$$E_P : \boldsymbol{z} \mapsto \begin{cases} \boldsymbol{z}' & \text{if } \boldsymbol{z} \vdash_P \boldsymbol{z}', \\ \boldsymbol{z} & \text{otherwise.} \end{cases} \tag{3.15}$$

The function E_P is given by cases and has the following property.

Proposition 3.4. *For each GOTO program P and each number $n \geq \{\sigma \mid x_\sigma \in V(P)\}$, the one-step function $E_P : \mathbb{N}_0^{n+1} \to \mathbb{N}_0^{n+1}$ is primitive recursive in the sense that for each $0 \leq j \leq n$, $\pi_j^{(n+1)} \circ E_P$ is primitive recursive.*

The execution of the GOTO program P can be represented by a sequence $\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_2, \ldots$ of configurations such that $\boldsymbol{z}_i \vdash_P \boldsymbol{z}_{i+1}$ for all $i \geq 0$. During this process, a configuration $\boldsymbol{z}_t$ may eventually be reached whose label z_0 does not belong to $L(P)$. In this case, the program P terminates. However, such an event may eventually not happen. In this way, the *runtime function* of P is the partial function $Z_P : \mathbb{N}_0^{n+1} \to \mathbb{N}_0$ given by

$$Z_P : \boldsymbol{z} \mapsto \begin{cases} \min\{t \in \mathbb{N}_0 \mid (\pi_0^{(n+1)} \circ E_P^t)(\boldsymbol{z}) \notin L(P)\} & \text{if } \{\ldots\} \neq \emptyset, \\ \uparrow & \text{otherwise.} \end{cases} \tag{3.16}$$

The runtime function may not be primitive recursive as it corresponds to an unbounded search process.

Proposition 3.5. *For each GOTO program P, the runtime function Z_P is partial recursive.*

Proof. By Proposition 3.4, the one-step function E_P is primitive recursive. It follows that the iteration of E_P is also primitive recursive:

$$E_P' : \mathbb{N}_0^{n+2} \to \mathbb{N}_0^{n+1} : (\boldsymbol{z}, t) \mapsto E_P^t(\boldsymbol{z}). \tag{3.17}$$

But the set $L(P)$ is finite and so the characteristic function $\chi_{L(P)}$ is primitive recursive. Therefore, the subsequent function is also primitive recursive:

$$E_P'' : \mathbb{N}_0^{n+2} \to \mathbb{N}_0^{n} : (\boldsymbol{z}, t) \mapsto (\chi_{L(P)} \circ \pi_0^{(n+1)} \circ E_P')(\boldsymbol{z}, t). \tag{3.18}$$

This function has the following property:

$$E_P''(\boldsymbol{z}, t) = \begin{cases} 1 \text{ if } E_P^t(\boldsymbol{z}) = (z_0', z_1', \ldots, z_n') \text{ and } z_0' \in L(P), \\ 0 \text{ otherwise.} \end{cases} \tag{3.19}$$

But by definition, $Z_P = \mu E_P''$ and so the result follows. □

The *residual step function* of the GOTO program P is given by the partial function $R_P : \mathbb{N}_0^{n+1} \to \mathbb{N}_0^{n+1}$ that maps an initial configuration to the final configuration of the computation, if any:

$$R_P(\boldsymbol{z}) = \begin{cases} E_P'(\boldsymbol{z}, Z_P(\boldsymbol{z})) & \text{if } \boldsymbol{z} \in \mathrm{dom}(Z_P), \\ \uparrow & \text{otherwise.} \end{cases} \tag{3.20}$$

Proposition 3.6. *For each GOTO program P, the function R_P is partial recursive.*

Proof. For each configuration $\boldsymbol{z} \in \mathbb{N}_0^{n+1}$, $R_P(\boldsymbol{z}) = E_P'(\boldsymbol{z}, (\mu E_P'')(\boldsymbol{z}))$. Thus R_P is a composition of partial recursive functions and hence itself partial recursive. □

3.3 GOTO Computable Functions

A partial function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ is called *GOTO computable* if there is a GOTO program P that computes f in the sense that

$$f = \|P\|_{k,1}. \tag{3.21}$$

Let $\mathcal{F}_{\mathrm{GOTO}}$ denote the class of all (partial) GOTO computable functions and let $\mathcal{T}_{\mathrm{GOTO}}$ depict the class of all total GOTO computable functions.

Theorem 3.7. *Each GOTO computable function is partial recursive, and each total GOTO computable function is recursive.*

Proof. If f is GOTO computable, there is a GOTO program such that $f = \|P\|_{k,1} = \beta_1 \circ R_P \circ \alpha_k$. But the functions β_1 and α_k are primitive recursive and the function R_P is partial recursive. Thus the composition is partial recursive. In particular, if f is total, then R_P is also total and so f is recursive. □

Example 3.8. The GOTO program P_+ for the addition of two natural numbers in Example 3.3 gives rise to the following configurations if $x_2 > 0$:

$$\begin{array}{llllll}
(0, x_1, x_2) & 0 & \texttt{if}\ x_2 = 0 & 3 & 1 \\
(1, x_1, x_2) & 1 & x_1 \leftarrow x_1 + 1 & 2 & \\
(2, x_1 + 1, x_2) & 2 & x_2 \leftarrow x_2 - 1 & 0 & \\
(0, x_1 + 1, x_2 - 1) & 3 & \texttt{if}\ x_2 = 0 & 3 & 1 \\
\ldots & & & &
\end{array}$$

Clearly, this program provides the addition function $\|P_+\|_{2,1}(x_1, x_2) = x_1 + x_2$. ♢

Finally, it will be shown that URM programs can be translated into GOTO programs.

Theorem 3.9. *Each URM computable function is GOTO computable.*

Proof. Claim that each URM program P can be compiled into a GOTO program $\phi(P)$ such that both compute the same function, i.e., $\|P\|_{k,1} = \|\phi(P)\|_{k,1}$ for all $k \in \mathbb{N}$. Indeed, let P be an URM program. We may assume that it does not make use of the register R_0. Write P as a string $P = \tau_0\tau_1 \ldots \tau_q$, where each substring τ_i is of the form "$A\sigma$", "$S\sigma$", "(" or ")σ" with $\sigma \in Z \setminus \{0\}$. Note that each opening parenthesis "(" corresponds to a unique closing parenthesis ")σ".

Replace each string τ_i by a GOTO instruction s_i as follows:

- If $\tau_i =$ "$A\sigma$", put
$$s_i = (i, x_\sigma \leftarrow x_\sigma + 1, i + 1),$$
- if $\tau_i =$ "$S\sigma$", set
$$s_i = (i, x_\sigma \leftarrow x_\sigma - 1, i + 1),$$
- if $\tau_i =$ "(" and $\tau_j =$ ")σ" is the corresponding closing parenthesis, define
$$s_i = (i, \texttt{if}\ x_\sigma = 0, j + 1, i + 1),$$

- if $\tau_i = ")\sigma"$ and $\tau_j = "("$ is the associated opening parenthesis, put

$$s_i = (i, \texttt{if } x_\sigma = 0, i+1, j+1).$$

In this way, a GOTO program $\phi(P) = s_0; s_1; \ldots; s_q$ is established that has the required property $\|P\|_{k,1} = \|\phi(P)\|_{k,1}$, as claimed. □

Example 3.10. The URM program $P = (A1; S2)2$ provides the addition of two natural numbers. Its translation into a GOTO program first requires to identify the substrings:

$$\tau_0 = "(", \; \tau_1 = "A1", \; \tau_2 = "S2", \; \tau_3 = ")2".$$

These substrings give rise to the following GOTO program:

0	$\texttt{if } x_2 = 0$	4	1
1	$x_1 \leftarrow x_1 + 1$	2	
2	$x_2 \leftarrow x_2 - 1$	3	
3	$\texttt{if } x_2 = 0$	4	1

♢

3.4 GOTO-2 Programs

GOTO-2 programs are GOTO programs with two variables x_1 and x_2. Claim that each URM program can be simulated by an appropriate GOTO-2 program. For this, the set of states Ω of an URM is encoded by using the sequence of primes $(p_0, p_1, p_2, \ldots)$. To this end, define the function $G : \Omega \to \mathbb{N}_0$ that assigns to each state $\omega = (\omega_0, \omega_1, \omega_2, \ldots) \in \Omega$ the natural number

$$G(\omega) = p_0^{\omega_0} p_1^{\omega_1} p_2^{\omega_2} \ldots . \tag{3.22}$$

Clearly, this function is primitive recursive. The inverse functions $G_i : \mathbb{N}_0 \to \mathbb{N}_0$, $i \in \mathbb{N}_0$, are given by

$$G_i(x) = (x)_i, \tag{3.23}$$

where $(x)_i$ is the exponent of p_i in the prime factorization of x if $x > 0$. Define $G_i(0) = 0$ for all $i \in \mathbb{N}_0$. The functions G_i are primitive recursive by Proposition 2.35.

Claim that for each URM program P, there is a GOTO-2 program $\overline{P}$ with the same semantics; that is, for all states $\omega, \omega' \in \Omega$,

$$|P|(\omega) = \omega' \quad \Longleftrightarrow \quad |\overline{P}|(0, G(\omega)) = (0, G(\omega')). \tag{3.24}$$

To see this, first consider the GOTO-2 programs $M(k)$, $D(k)$, and $T(k)$, $k \in \mathbb{N}$, which have the following properties:

$$|M(k)|(0, x) = (0, k \cdot x), \tag{3.25}$$

$$|D(k)|(0, k \cdot x) = (0, x), \tag{3.26}$$

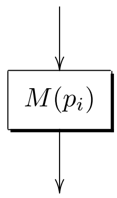
$$|T(k)|(0, x) = \begin{cases} (1, x) \text{ if } k \text{ divides } x, \\ (0, x) \text{ otherwise.} \end{cases} \tag{3.27}$$

For instance, the GOTO-2 program $M(k)$ can be implemented by two consecutive loops: Initially, put $x_1 = 0$ and $x_2 = x$. In the first loop, x_2 is decremented, while in each decremention step, x_1 is incremented k times. After this loop, $x_1 = k \cdot x$ and $x_2 = 0$. In the second loop, x_2 is incremented and x_1 is decremented such that upon termination, $x_1 = 0$ and $x_2 = k \cdot x$. This can be implemented by the following GOTO-2 program in standard form:

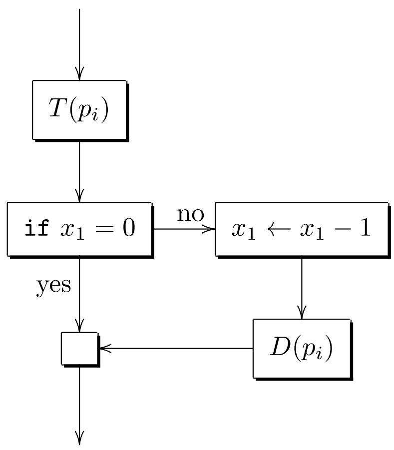
0	$\texttt{if}\ x_2 = 0$	$k+2$	1
1	$x_2 \leftarrow x_2 - 1$	2	
2	$x_1 \leftarrow x_1 + 1$	3	
	$\ldots$		
$k+1$	$x_1 \leftarrow x_1 + 1$	0	
$k+2$	$\texttt{if}\ x_1 = 0$	$k+5$	$k+3$
$k+3$	$x_2 \leftarrow x_2 + 1$	$k+4$	
$k+4$	$x_1 \leftarrow x_1 - 1$	$k+2$	

The other two kinds of programs can be analogously realized as GOTO-2 programs in standard form.

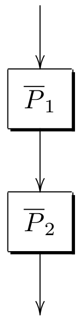
The assignment $P \mapsto \overline{P}$ will be established by making use of the inductive definition of URM programs. For this, flow diagrams will be employed to simplify the notation. First, the program $\overline{Ai}$ can be realized by the flow chart



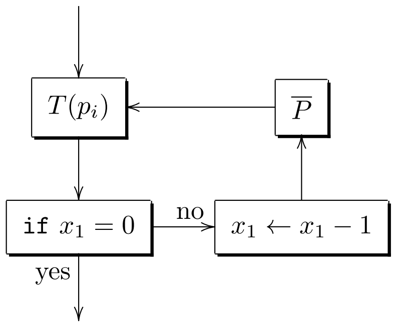
Second, the program $\overline{Si}$ is given by the flow diagram



Third, the program $\overline{P_1; P_2}$ can be decipted by the flow chart



Finally, the program $\overline{(P)i}$ can be represented by the flow diagram



All these GOTO-2 programs can be realized as standard GOTO programs.

Example 3.11. The URM program $P_+ = (A1; S2)2$ gets compiled into the following (standard) GOTO-2 program in pseudo code:

$0 : T(p_2)$
$1 :$ `if` $x_1 = 0$ `goto` 9
$2 : x_1 \leftarrow x_1 - 1$
$3 : M(p_1)$
$4 : T(p_2)$
$5 :$ `if` $x_1 = 0$ `goto` 8
$6 : x_1 \leftarrow x_1 - 1$
$7 : D(p_2)$
$8 :$ `goto` 0
$9 :$

$\diamondsuit$

By using the inductive definition of URM programs, the assignment $P \mapsto \overline{P}$ is well-defined. The computation of a function $f : \mathbb{N}_0^k \to \mathbb{N}_0^m$ by a GOTO-2 program requires to load the registers with the initial values and to identify the result. For this, define the primitive recursive functions

$$\hat{\alpha}_k : \mathbb{N}_0^k \to \mathbb{N}_0^2 : (x_1, \ldots, x_k) \mapsto (0, G(0, x_1, \ldots, x_k, 0, \ldots)) \tag{3.28}$$

and

$$\hat{\beta}_m : \mathbb{N}_0^2 \to \mathbb{N}_0^m : (a, b) \mapsto (G_1(b), \ldots, G_m(b)). \tag{3.29}$$

Proposition 3.12. *Each URM program P is (semantically) equivalent to the associated GOTO-2 program $\overline{P}$ in the sense that for all $k, m \in \mathbb{N}_0$:*

$$\|P\|_{k,m} = \hat{\beta}_m \circ R_{\overline{P}} \circ \hat{\alpha}_k. \tag{3.30}$$

3.5 Church's Thesis

Our attempts made so far to formalize the notion of computability are equivalent in the following sense.

Theorem 3.13. *The class of partial recursive functions equals the class of URM computable functions and the class of GOTO computable functions. In particular, the class of recursive functions is equal to the class of total URM computable functions and to the class of total GOTO computable functions.*

Proof. By Theorem 3.9, each URM computable function is GOTO computable and by Theorem 3.7, each GOTO computable function is partial recursive. On the other hand, by Theorem 3.2, each partial recursive function is URM computable. A similar statement holds for total functions. □

This result has led scientists to believe that the concept of computability is accurately characterized by the class of partial recursive functions. The *Church thesis* proposed by Alonso Church (1903-1995) in the 1930s states that the class of computable partial functions (in the intuitive sense) coincides with the class of partial recursive functions, equivalently, with the class of URM computable functions and the class of GOTO computable functions. Church's thesis characterizes the nature of computation and cannot be formally proved. Nevertheless, today, it has universal acceptance. Church's thesis is often practically used in the sense that if a (partial) function is intuitively computable, it is assumed to be partial recursive. In this way, the thesis may lead to more intuitive and less rigorous proofs (see Theorem 5.10).

4

A Recursive Function

The primitive recursive functions are total and computable. The Ackermann function, named after the German mathematician Wilhelm Ackermann (1986-1962), was the earliest-discovered example of a total computable function that is not primitive recursive.

4.1 Small Ackermann Functions

The small Ackermann functions form an interesting class of primitive recursive functions. They can be used to define the "big" Ackermann function and provide upper bounds on the runtime of LOOP programs. The latter property allows to show that the hierarchy of LOOP-computable functions is strict.

Define a sequence (B_n) of monadic total functions inductively as follows:

$$B_0 : x \mapsto \begin{cases} 1 & \text{if } x = 0, \\ 2 & \text{if } x = 1, \\ x+2 & \text{otherwise}, \end{cases} \tag{4.1}$$

and

$$B_{n+1} : x \mapsto B_n^x(1), \quad x, n \in \mathbb{N}_0, \tag{4.2}$$

where B_n^x denotes the x-fold power of the function B_n. For each $n \geq 0$, the function B_n is called the *n-th small Ackermann function.*

Proposition 4.1. *The small Ackermann functions have the following properties for all $x, n, p \in \mathbb{N}_0$:*

$$B_1(x) = \begin{cases} 1 & \textit{if } x = 0, \\ 2x & \textit{otherwise}, \end{cases} \tag{4.3}$$

$$B_2(x) = 2^x, \tag{4.4}$$

$$B_3(x) = \begin{cases} 1 & \textit{if } x = 0, \\ 2^{B_3(x-1)} & \textit{otherwise}, \end{cases} \tag{4.5}$$

$$x < B_n(x), \tag{4.6}$$

$$
\begin{aligned}
B_n(x) &< B_n(x+1), &(4.7)\\
B_0^p(x) &\leq B_1^p(x), &(4.8)\\
B_n(x) &\leq B_{n+1}(x), &(4.9)\\
B_n^p(x) &< B_n^p(x+1), &(4.10)\\
B_n^p(x) &< B_n^{p+1}(x), &(4.11)\\
B_n^p(x) &\leq B_{n+1}^p(x), &(4.12)\\
2^{p+1}x &\leq B_1^{p+1}(x), &(4.13)\\
2B_n^p(x) &\leq B_n^{p+1}(x), \quad n \geq 1. &(4.14)
\end{aligned}
$$

Proposition 4.2. *The small Ackermann functions B_n, $n \in \mathbb{N}_0$, are primitive recursive.*

Proof. The function B_0 is defined by cases:

$$
B_0(x) = \begin{cases} (\nu \circ c_0^{(1)})(x) & \text{if } x = 0, \\ (\nu \circ \nu \circ c_0^{(1)})(x) & \text{if } |x \dot{-} 1| = 0, \\ (\nu \circ \nu \circ \pi_1^{(1)})(x) & \text{otherwise.} \end{cases} \tag{4.15}
$$

By Eq. (4.3), the function B_1 follows the primitive recursive scheme

$$
\begin{aligned}
B_1(0) &= 1, &(4.16)\\
B_1(x+1) &= B_1(x) \cdot \operatorname{sgn}(x) + 2, \quad x \in \mathbb{N}_0. &(4.17)
\end{aligned}
$$

Finally, if $n \geq 2$, B_n is given by the primitive recursive scheme

$$
\begin{aligned}
B_n(0) &= 1, &(4.18)\\
B_n(x+1) &= B_{n-1}(B_n(x)), \quad x \in \mathbb{N}_0. &(4.19)
\end{aligned}
$$

By induction, the small Ackermann functions are primitive recursive. □

The $n+1$th small Ackermann function grows faster than any power of the nth Ackermann function in the following sense.

Proposition 4.3. *For all number n and p, there is a number x_0 such that for all numbers $x \geq x_0$,*

$$
B_n^p(x) < B_{n+1}(x). \tag{4.20}
$$

Proof. Let $n = 0$. For each $x \geq 2$, $B_0^p(x) = x + 2p$. On the other hand, for each $x \geq 1$, $B_1(x) = 2x$. Put $x_0 = 2p + 1$. Then for each $x \geq x_0$,

$$
B_0^p(x) = x + 2p \leq 2x = B_1(x).
$$

Let $n > 0$. First, let $p = 0$. Then for each $x \geq 0$,

$$
B_n^0(x) = x < B_n(x) \leq B_{n+1}(x)
$$

by (4.6) and (4.9). Second, let $p > 0$ and assume that $B_n^p(x) < B_{n+1}(x)$ holds for all $x \geq x_0'$. Put $x_0 = x_0' + 5$. Then

$$
\begin{aligned}
B_n^{p+1}(x) &< B_n^{p+1}(2 \cdot (x \dot{-} 2)), \quad x \geq 5, \text{ by (4.10)}, \\
&= B_n^{p+1}(B_1(x \dot{-} 2)) \\
&\leq B_n^{p+1}(B_n(x \dot{-} 2)), \quad \text{by (4.12)}, \\
&= B_n^{p+2}(x \dot{-} 2) \\
&= B_n^2(B_n^p(x \dot{-} 2)) \\
&< B_n^2(B_{n+1}(x \dot{-} 2)), \quad \text{by induction, } x \geq x_0' + 2, \\
&= B_n^2(B_n^{x \dot{-} 2}(1)) \\
&= B_n^x(1) \\
&= B_{n+1}(x), \quad x \geq 2.
\end{aligned}
$$

□

Proposition 4.4. *For each $n \geq 1$, the n-th small Ackermann function B_n is LOOP-n computable.*

Proof. First, define the LOOP-1 program

$$P_1 = \bar{C}(1;2); \bar{C}(1;3); Z1; A1; [Z1]3; [A1; A1]2. \tag{4.21}$$

For each input x, the program evaluates as follows:

0	1	2	3	4	... registers
0	x	0	0	0	... initially
0	x	x	0	0	...
0	x	x	x	0	...
0	0	x	x	0	...
0	1	x	x	0	...
0	1	0	0	0	... $x = 0$finally
0	$2x$	0	0	0	... $x \neq 0$finally

By (4.3), the program satisfies $\|P_1\|_{1,1} = B_1$.

Suppose there is a normal LOOP-n program P_n that computes B_n; that is, $\|P_n\|_{1,1} = B_n$, for $n \geq 1$. Put $m = n(P_n) + 1$ and consider the LOOP-$n+1$ program

$$P_{n+1} = [Am]1; A1; [P_n]m. \tag{4.22}$$

Note that the register R_m is unused in the program P_n. The program P_{n+1} computes $B_n^x(1)$ as follows:

0	1	2	3	4	...	m	... registers
0	x	0	0	0	...	0	... initially
0	x	0	0	0	...	x	...
0	1	0	0	0	...	x	...
0	$B_n^x(1)$	0	0	0	...	0	... finally

But $B_{n+1}(x) = B_n^x(1)$ and so $\|P_{n+1}\|_{1,1} = B_{n+1}$. □

4.2 Runtime of LOOP Programs

The LOOP programs will be extended in a way that they do not only perform their task but simultaneously compute their runtime. This will finally allow us to show that Ackermann's function is not primitive recursive.

For this, assume that the LOOP programs do not use the register R_0 in order to perform their task. This is not an essential restriction since it can always be the achieved by transformation of variables. The register R_0 will then solely be used to calculate the runtime of the LOOP program. To this end, the objective is to assign to each LOOP program P another LOOP program $\gamma(P)$ that performs the computation of P and simultaneously computes the runtime of P. The runtime of a program is essentially determined by the number of elementary operations (incrementation, zero setting, and copying):

$$\gamma(A\sigma) = A\sigma; A0, \quad \sigma \neq 0, \tag{4.23}$$
$$\gamma(Z\sigma) = Z\sigma; A0, \quad \sigma \neq 0, \tag{4.24}$$
$$\gamma(\bar{C}(\sigma;\tau)) = \bar{C}(\sigma;\tau); A0, \quad \sigma \neq \tau,\ \sigma \neq 0 \neq \tau, \tag{4.25}$$
$$\gamma(P;Q) = \gamma(P); \gamma(Q), \tag{4.26}$$
$$\gamma([P]\sigma) = [\gamma(P)]\sigma, \quad \sigma \neq 0. \tag{4.27}$$

This definition immediately leads to the following

Proposition 4.5. *Let $n \geq 0$.*

- *If P is a LOOP-n program, $\gamma(P)$ is also LOOP-n program.*
- *Let P be a LOOP-n program and let $|\gamma(P)| \circ \alpha_k(\boldsymbol{x}) = (\omega_n)$, where $\boldsymbol{x} \in \mathbb{N}_0^k$. Then $\omega_1 = \|P\|_{k,1}(\boldsymbol{x})$ is the result of the computation of P and ω_0 is the number of elementary operations made during the computation of P.*

The *runtime program* of a LOOP program P is the LOOP program P' given by

$$P' = \gamma(P); \bar{C}(0;1). \tag{4.28}$$

The corresponding function $\|P'\|_{k,1}$ with $k \geq n(P)$ is called the *runtime function* of P.

Proposition 4.6. *If P is a LOOP-n program, P' is also a LOOP-n program and has the property*

$$\|P'\|_{k,1} = \gamma(P), \quad k \geq n(P). \tag{4.29}$$

Example 4.7. The program $S1$ is given by the LOOP-1 program

$$P = \bar{C}(1;3); [\bar{C}(2;1); A2]3. \tag{4.30}$$

The corresponding runtime program is

$$P' = \bar{C}(1;3); A0; [\bar{C}(2;1); A0; A2; A0]3; \bar{C}(0;1). \tag{4.31}$$

$\diamondsuit$

Next, consider a function $\lambda : \mathcal{P}_{\text{LOOP}} \to \mathbb{N}_0$ that assigns to each LOOP program a measure of complexity. Here elementary operations have unit complexity, while loops contribute with a higher complexity:

$$\begin{aligned}
\lambda(A\sigma) &= 1, \quad \sigma \in \mathbb{N}_0, & (4.32)\\
\lambda(Z\sigma) &= 1, \quad \sigma \in \mathbb{N}_0, & (4.33)\\
\lambda(\bar{C}(\sigma;\tau)) &= 1, \quad \sigma \neq \tau,\, \sigma,\tau \in \mathbb{N}_0, & (4.34)\\
\lambda(P;Q) &= \lambda(P) + \lambda(Q), \quad P,Q \in \mathcal{P}_{\text{LOOP}}, & (4.35)\\
\lambda([P]\sigma) &= \lambda(P) + 2, \quad P \in \mathcal{P}_{\text{LOOP}},\, \sigma \in \mathbb{N}_0. & (4.36)
\end{aligned}$$

Note that for each LOOP program P, the *complexity measure* $\lambda(P)$ is the number of LOOP-0 subprograms of P plus twice the number of iterations of P.

Example 4.8. The LOOP program $P = \bar{C}(1;3); [\bar{C}(2;1); A2]3$ has the complexity

$$\begin{aligned}
\lambda(P) &= \lambda(\bar{C}(1;3)) + \lambda([\bar{C}(2;1); A2]3)\\
&= 1 + \lambda(\bar{C}(2;1); A2) + 2\\
&= 3 + \lambda(\bar{C}(2;1)) + \lambda(A2)\\
&= 5.
\end{aligned}$$

♢

A k-ary total function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ is said to be *bounded* by a monadic total function $g : \mathbb{N}_0 \to \mathbb{N}_0$ if for each $\boldsymbol{x} \in \mathbb{N}_0^k$,

$$f(\boldsymbol{x}) \leq g(\max(\boldsymbol{x})), \tag{4.37}$$

where $\max(\boldsymbol{x}) = \max\{x_1, \ldots, x_k\}$ if $\boldsymbol{x} = (x_1, \ldots, x_k) \in \mathbb{N}_0^k$.

Proposition 4.9. *For each LOOP-n program P and each input $\boldsymbol{x} \in \mathbb{N}_0^k$ with $k \geq n(P)$,*

$$\max(\|P\|_{k,k}(\boldsymbol{x})) \leq B_n^{\lambda(P)}(\max(\boldsymbol{x})). \tag{4.38}$$

Proof. First, let P be a LOOP-0 program; that is, $P = P_1; P_2; \ldots; P_m$, where each block P_i is an elementary operation, $1 \leq i \leq m$. Then

$$\max(\|P\|_{k,k}(\boldsymbol{x})) \leq m + \max(\boldsymbol{x}) = \lambda(P) + \max(\boldsymbol{x}) \leq B_0^{\lambda(P)}(\max(\boldsymbol{x})). \tag{4.39}$$

Suppose the assertion holds for LOOP-n programs. Let P be a LOOP program of the form $P = Q; R$, where Q and R are LOOP-$n+1$ programs. Then for $k \geq \max\{n(Q), n(R)\}$,

$$\begin{aligned}
\max(\|Q;R\|_{k,k}(\boldsymbol{x})) &= \max(\|R\|_{k,k}(\|Q\|_{k,k}(\boldsymbol{x})))\\
&\leq B_{n+1}^{\lambda(R)}(\|Q\|_{k,k}(\boldsymbol{x})), \text{ by induction,}\\
&\leq B_{n+1}^{\lambda(R)}(B_{n+1}^{\lambda(Q)}(\max(\boldsymbol{x}))), \text{ by induction,}\\
&= B_{n+1}^{\lambda(R)+\lambda(Q)}(\max(\boldsymbol{x}))\\
&= B_{n+1}^{\lambda(P)}(\max(\boldsymbol{x})).
\end{aligned} \tag{4.40}$$

Finally, let P be a LOOP program of the form $P = [Q]\sigma$, where Q is a LOOP-n program. Then for $k \geq \max\{n(Q), \sigma\}$,

$$\begin{aligned}
\max(\|[Q]\sigma\|_{k,k}(\boldsymbol{x})) &= \max(\|Q; S\sigma\|_{k,k}^{x_\sigma}(\boldsymbol{x})) \\
&\leq \max(\|Q\|_{k,k}^{x_\sigma}(\boldsymbol{x})) \\
&\leq B_n^{x_\sigma \cdot \lambda(Q)}(\max(\boldsymbol{x})), \text{ by induction,} \\
&\leq B_{n+1}^{\lambda(Q)+2}(\max(\boldsymbol{x})), \text{see (4.42)} \\
&= B_{n+1}^{\lambda(P)}(\max(\boldsymbol{x})).
\end{aligned} \tag{4.41}$$

The last inequality follows from the assertion

$$B_n^{x \cdot a}(y) \leq B_{n+1}^{a+2}(y), \quad x \leq y, \tag{4.42}$$

which can be proved by using the properties of the small Ackermann functions in Proposition 4.1 as follows:

$$\begin{aligned}
B_n^{x \cdot a}(y) &\leq B_n^{y \cdot a}(y) \\
&\leq B_n^{y \cdot a}(B_{n+1}(y)) \\
&= B_n^{y \cdot a}(B_n^y(1)) \\
&= B_n^{y(a+1)}(1) \\
&= B_{n+1}(y(a+1)) \\
&\leq B_{n+1}(y \cdot 2^{a+1}) \\
&= B_{n+1}(B_1^{a+1}(y)) \\
&\leq B_{n+1}(B_{n+1}^{a+1}(y)) \\
&= B_{n+1}^{a+2}(y)).
\end{aligned}$$

□

Corollary 4.10. *The runtime function of a LOOP-n program is bounded by the n-th small Ackermann function.*

Proof. Let P be a LOOP-n program. By Proposition 4.6, the runtime program $\gamma(P)$ is also LOOP-n. Thus by Proposition 4.9, there is an integer m such that for all inputs $\boldsymbol{x} \in \mathbb{N}_0^k$ with $k \geq n(P)$,

$$\max(\|\gamma(P)\|_{k,k}(\boldsymbol{x})) \leq B_n^m(\max(\boldsymbol{x})). \tag{4.43}$$

But by definition, the runtime program P' of P satisfies $\|P'\|_{k,1}(\boldsymbol{x}) \leq \max(\|\gamma(P)\|_{k,k}(\boldsymbol{x}))$ for all $\boldsymbol{x} \in \mathbb{N}_0$ and so the result follows. □

For sake of completeness, the converse assertion also holds which is attributed to Meyer and Ritchie (1967).

Theorem 4.11. *(Meyer-Ritchie) Let $n \geq 2$. A LOOP computable function f is LOOP-n computable if and only if there is a LOOP-n program P such that $\|P\| = f$ and the associated runtime program P' is bounded by a LOOP-n computable function.*

Finally, the runtime functions provide a way to prove that the LOOP hierarchy is proper.

Proposition 4.12. *For each $n \in \mathbb{N}_0$, the class of LOOP-n functions is a proper subclass of the class of LOOP-$n+1$ functions.*

Proof. By Proposition 4.4, the n-th small Ackermann function B_n is LOOP-n computable. Assume that B_{n+1} is LOOP-n computable. Then by Proposition 4.9, there is an integer $m \geq 0$ such that $B_{n+1}(x) \leq B_n^m(x)$ for all $x \in \mathbb{N}_0$. This contradicts Proposition 4.3, which shows that B_{n+1} grows faster than any power of B_n. ☐

4.3 Ackermann's Function

As already remarked earlier, there are total computable functions that are not primitive recursive. The most prominent example is *Ackermann's function* $A : \mathbb{N}_0^2 \to \mathbb{N}_0$ defined by the equations

$$A(0, y) = y + 1, \tag{4.44}$$

$$A(x + 1, 0) = A(x, 1), \tag{4.45}$$

$$A(x + 1, y + 1) = A(x, A(x + 1, y)). \tag{4.46}$$

Proposition 4.13. *Ackermann's function is a total function on $\mathbb{N}_0^2$.*

Proof. The term $A(0, y)$ is certainly defined for all y. Suppose that $A(x, y)$ is defined for all y. Then $A(x+1, 0) = A(x, 1)$ is defined. Assume that $A(x+1, y)$ is defined. Then by induction, $A(x+1, y+1) = A(x, A(x+1, y))$ is defined. It follows that $A(x+1, y)$ is defined for all y. In turn, it follows that $A(x, y)$ is defined for all x and y. ☐

Proposition 4.14. *The monadic functions $A_x : \mathbb{N}_0 \to \mathbb{N}_0 : y \mapsto A(x, y)$, $x \in \mathbb{N}_0$, have the form*

- $A_0(y) = y + 1 = \nu(y + 3) - 3$,
- $A_1(y) = y + 2 = f_+(2, y + 3) - 3$,
- $A_2(y) = 2y + 3 = f_.(2, y + 3) - 3$,
- $A_3(y) = 2^{y+3} - 3 = f_{\exp}(2, y + 3) - 3$,
- $A_4(y) = 2^{2^{\cdot^{\cdot^{\cdot^2}}}} - 3 = f_{\mathrm{itexp}}(2, y + 3) - 3$, *where there are $y + 3$ instances of the symbol 2 on the right-hand side.*

Proof.

- By definition, $A_0(y) = A(0, y) = y + 1$.
- First, $A_1(0) = A(0, 1) = 2$. Suppose $A_1(y) = y + 2$. Then $A_1(y + 1) = A(1, y + 1) = A(0, A(1, y)) = A(1, y) + 1 = (y + 1) + 2$.
- Plainly, $A_2(0) = A(1, 1) = 3$. Suppose $A_2(y) = 2y+3$. Then $A_2(y+1) = A(2, y+1) = A(1, A(2, y)) = A(2, y) + 2 = (2y + 3) + 2 = 2(y + 1) + 3$.
- Clearly, $A_3(0) = A(2, 1) = 5$. Suppose $A_3(y) = 2^{y+3} - 3$. Then $A_3(y + 1) = A(3, y + 1) = A(2, A(3, y)) = 2 \cdot (2^{y+3} - 3) + 3 = 2^{(y+1)+3} - 3$.

- First, $A_4(0) = A(3,1) = 2^{2^2} - 3$. Suppose that

  $$A_4(y) = 2^{2^{\cdot^{\cdot^{\cdot^2}}}} - 3,$$

  where there are $y+3$ instances of the symbol 2 on the right-hand side. Then

  $$A_4(y+1) = A(3, A(4,y)) = 2^{A(4,y)+3} - 3 = 2^{2^{\cdot^{\cdot^{\cdot^2}}}} - 3,$$

  where there are $(y+1)+3$ instances of the symbol 2 on the far right of these equations.

$\square$

The small Ackermann functions can be combined into a dyadic function $A : \mathbb{N}_0^2 \to \mathbb{N}_0$ given by

$$A(x,y) = B_x(y), \quad x, y \in \mathbb{N}_0. \tag{4.47}$$

This function is a variant of the original Ackermann function.

Proposition 4.15. *The function A in (4.47) is given by the equations*

$$A(0,y) = B_0(y), \tag{4.48}$$
$$A(x+1,0) = 1, \tag{4.49}$$
$$A(x+1,y+1) = A(x, A(x+1,y)). \tag{4.50}$$

Proof. We have $A(0,y) = B_0(y)$, $A(x{+}1,0) = B_{x+1}(0) = B_x^0(1) = 1$, and $A(x{+}1,y{+}1) = B_{x+1}(y{+}1) = B_x^{y+1}(1) = B_x(B_x^y(1)) = B_x(B_{x+1}(y)) = B_x(A(x+1,y) = A(x, A(x+1,y))$. $\square$

The Ackermann function grows very quickly. This can be seen by expanding a simple expression using the defining rules.

Example 4.16.

$$\begin{aligned}
A(2,3) &= A(1, A(2,2)) \\
&= A(1, A(1, A(2,1))) \\
&= A(1, A(1, A(1, A(2,0)))) \\
&= A(1, A(1, A(1,1))) \\
&= A(1, A(1, A(0, A(1,0)))) \\
&= A(1, A(1, A(0,1))) \\
&= A(1, A(1,2)) \\
&= A(1, A(0, A(1,1))) \\
&= A(1, A(0, A(0, A(1,0)))) \\
&= A(1, A(0, A(0,1))) \\
&= A(1, A(0,2)) \\
&= A(1, A(0,2)) \\
&= A(1,4)
\end{aligned}$$

$$
\begin{aligned}
&= A(0, A(1,3)) \\
&= A(0, A(0, A(1,2))) \\
&= A(0, A(0, A(0, A(1,1)))) \\
&= A(0, A(0, A(0, A(0, A(1,0))))) \\
&= A(0, A(0, A(0, A(0,1)))) \\
&= A(0, A(0, A(0,2))) \\
&= A(0, A(0,4)) \\
&= A(0,6) \\
&= 8.
\end{aligned}
$$

$\diamondsuit$

Proposition 4.17. *Ackermann's function is not primitive recursive.*

Proof. Assume that the function A is primitive recursive. Then A is LOOP-n computable for some $n \geq 0$. Thus by Proposition 4.9, there is an integer $p \geq 0$ such that for all $x, y \in \mathbb{N}_0$,

$$A(x,y) \leq B_n^p(\max\{x,y\}). \tag{4.51}$$

But by Proposition 4.3, there is a number $y_0 \geq 0$ such that for all numbers $y \geq y_0$,

$$B_n^p(y) < B_{n+1}(y). \tag{4.52}$$

It can be assumed that $y_0 \geq n+1$. Taking $x = n+1$ and $y \geq y_0$ leads to a contradiction:

$$A(n+1,y) \leq B_n^p(\max\{n+1,y\}) = B_n^p(y) < B_{n+1}(y) = A(n+1,y). \tag{4.53}$$

$\square$

The lattice of function classes (under inclusion) considered in this chapter is the following:

$$
\begin{array}{c}
\mathcal{T} \\
| \\
\mathcal{P} = \mathcal{F}_{\mathrm{LOOP}} \\
| \\
\cdots \\
| \\
\mathcal{F}_{\mathrm{LOOP}(1)} \\
| \\
\mathcal{F}_{\mathrm{LOOP}(0)}
\end{array}
$$

All inclusions are strict.

5

Acceptable Programming Systems

Acceptable programming systems form the basis for the development of the theory of computation. This chapter provides several basic theoretical results of computability, such as the existence of universal functions, the parametrization theorem known as smn theorem, and Kleene's normal form theorem. These results make use of the Gödel numbering of partial recursive functions.

5.1 Gödel Numbering of GOTO Programs

In mathematical logic, Gödel numbering refers to a function that assigns to each well-formed formula of some formal language a unique natural number called its Gödel number. This concept was introduced by the logician Kurt Gödel (1906-1978) for the proof of incompleteness of elementary arithmetic (1931). Here Gödel numbering is used to provide an encoding of GOTO programs.

For this, let $\mathbb{N}_0^*$ denote the union of all cartesian products $\mathbb{N}_0^k$, $k \geq 0$. In particular, $\mathbb{N}_0^0 = \{\epsilon\}$, where ϵ is the empty string. The Cantor pairing function $J_2 : \mathbb{N}_0^2 \to \mathbb{N}_0$ is used to define an encoding $J : \mathbb{N}_0^* \to \mathbb{N}_0$ as follows:

$$J(\epsilon) = 0, \tag{5.1}$$

$$J(x) = J_2(0, x) + 1, \quad x \in \mathbb{N}_0, \tag{5.2}$$

$$J(\boldsymbol{x}, y) = J_2(J(\boldsymbol{x}), y) + 1, \quad \boldsymbol{x} \in \mathbb{N}_0^*, y \in \mathbb{N}_0. \tag{5.3}$$

Note that the second equation is a special case of the third one, since for each $y \in \mathbb{N}_0$,

$$J(\epsilon, y) = J_2(J(\epsilon), y) + 1 = J_2(0, y) + 1 = J(y). \tag{5.4}$$

Example 5.1. We have

$$J(1,3) = J_2(J(1), 3) + 1 = J_2(J_2(0,1) + 1, 3) + 1 = J_2(2,3) + 1 = 17 + 1 = 18.$$

$\diamondsuit$

Proposition 5.2. *The encoding function J is a primitive recursive bijection.*

Proof. First, claim that J is primitive recursive. Indeed, the function J is primitive recursive for strings of length ≤ 1, since J_2 is primitive recursive. Assume that J is primitive recursive for strings of length $\leq k$, where $k \geq 1$. For strings of length $k+1$, the function J can be written as a composition of primitive recursive functions:

$$J = \nu \circ J_2(J(\pi_1^{(k+1)}, \ldots, \pi_k^{(k+1)}), \pi_{k+1}^{(k+1)}). \tag{5.5}$$

By induction hypothesis, J is primitive recursive for strings of length $\leq k+1$ and thus the claim follows.

Second, let A be the set of all numbers $n \in \mathbb{N}_0$ such that there is a unique string $\boldsymbol{x} \in \mathbb{N}_0^*$ with $J(\boldsymbol{x}) = n$. Claim that $A = \mathbb{N}_0$. Indeed, 0 lies in A since $J(\epsilon) = 0$ and $J(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \epsilon$ Let $n > 0$ and assume that the assertion holds for all numbers $m < n$. Define

$$u = K_2(n-1) \quad \text{and} \quad v = L_2(n-1). \tag{5.6}$$

Then $J_2(u, v) = J_2(K_2(n-1), L_2(n-1)) = n-1$. By construction, $K_2(z) \leq z$ and $L_2(z) \leq z$ for all $z \in \mathbb{N}_0$. Thus $u = K_2(n-1) < n$ and hence $u \in A$. By induction, there is exactly one string $\boldsymbol{x} \in \mathbb{N}_0^k$ such that $J(\boldsymbol{x}) = u$. Then $J(\boldsymbol{x}, v) = J_2(J(\boldsymbol{x}), v)) + 1 = J_2(u, v) + 1 = n$.

Assume that $J(\boldsymbol{x}, v) = n = J(\boldsymbol{y}, w)$ for some $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}_0^*$ and $v, w \in \mathbb{N}_0$. Then by definition, $J_2(J(\boldsymbol{x}), v) = J_2(J(\boldsymbol{y}), w)$. But the Cantor pairing function is bijective and thus $J(\boldsymbol{x}) = J(\boldsymbol{y})$ and $v = w$. Since $J(\boldsymbol{x}) < n$ it follows by induction that $\boldsymbol{x} = \boldsymbol{y}$. Thus $n \in A$ and so by the induction axiom, $A = \mathbb{N}_0$. It follows that J is bijective. ☐

The encoding function J gives rise to two functions $K, L : \mathbb{N}_0 \to \mathbb{N}_0$ defined as

$$K(n) = K_2(n \dot{-} 1) \quad \text{and} \quad L(n) = L_2(n \dot{-} 1), \quad n \in \mathbb{N}_0. \tag{5.7}$$

Note that the following marginal conditions hold:

$$K(1) = K(0) = L(0) = L(1) = 0. \tag{5.8}$$

Proposition 5.3. *The functions K and J are primitive recursive, and for each number $n \geq 1$, there are unique $\boldsymbol{x} \in \mathbb{N}_0^*$ and $y \in \mathbb{N}_0$ with $J(\boldsymbol{x}, y) = n$ such that*

$$J(\boldsymbol{x}) = K(n) \quad \textit{and} \quad y = L(n). \tag{5.9}$$

Proof. By Proposition 2.27, the functions K_2 and L_2 are primitive recursive and so K and L are primitive recursive, too.

Let $n \geq 1$. By Proposition 5.2, there are unique $\boldsymbol{x} \in \mathbb{N}_0^*$ and $y \in \mathbb{N}_0$ such that $J(\boldsymbol{x}, y) = n$. Thus $J_2(J(\boldsymbol{x}), y) = n-1$. But $J_2(K_2(n-1), L_2(n-1)) = n-1$ and the Cantor pairing function J_2 is bijective. Thus $K_2(n-1) = J(\boldsymbol{x})$ and $L_2(n-1) = y$, and so $K(n) = J(\boldsymbol{x})$ and $L(n) = y$. ☐

The length of a string can be determined by its encoding.

Proposition 5.4. *If $\boldsymbol{x}$ is a string in $\mathbb{N}_0^*$ of length k, then k is the smallest number such that*

$$K^k(J(\boldsymbol{x})) = 0.$$

Proof. In view of the empty string, $K^0(J(\epsilon)) = J(\epsilon) = 0$. Let $\boldsymbol{x} = x_1 \ldots x_k$ be a non-empty string. Then $J(\boldsymbol{x}) = n \geq 1$. By Proposition 5.3, $J(x_1 \ldots x_{k-1}) = K(n)$. By induction, $J(x_1 \ldots x_i) = K^{k-i}(n)$ for each $0 \leq i \leq k$. In particular, for each $0 \leq i < k$, $K^{k-i}(n)$ is non-zero, since $x_1 \ldots x_i$ is not the empty string. Moreover, $K^k(n) = J(\epsilon) = 0$. The result follows. ☐

Note that the converse of the above assertion is also valid. As an example, take a number $x \in \mathbb{N}_0$, a string of length one. Then by (5.4), $J(\epsilon, x) = J(x) = J_2(0, x) + 1 = n \geq 1$ and so by (5.9), $K(n) = J(\epsilon) = 0$ and $L(n) = x$.

In view of Proposition 5.4, define the mapping $f : \mathbb{N}_0^* \times \mathbb{N}_0 \to \mathbb{N}_0$ as

$$f : (\boldsymbol{x}, k) \mapsto K^k(J(\boldsymbol{x})). \tag{5.10}$$

Minimalization of this function yield the *length function* $\lg : \mathbb{N}_0^* \to \mathbb{N}_0$ given by

$$\lg(\boldsymbol{x}) = \mu f(\boldsymbol{x}) = \begin{cases} k \text{ if } k \text{ smallest with } f(\boldsymbol{x}, k) = 0 \text{ and } f(\boldsymbol{x}, i) > 0 \text{ for } 0 \leq i \leq k, \\ \uparrow \text{ otherwise.} \end{cases} \tag{5.11}$$

Proposition 5.5. *The length function* lg *is primitive recursive.*

Proof. The length function is partial recursive, since it is obtained by minimalization of a primitive recursive function. But each string $\boldsymbol{x}$ has finite length and so the repeated application of K to $J(\boldsymbol{x})$ yields 0 after a finite number of steps. Hence, the length function is total. Moreover, the length of a string $\boldsymbol{x} = x_1 \ldots x_n$ is not larger than an upper bound of its decimal equivalent given by 10^n and so the minimalization is actually bounded. Hence, the length function is primitive recursive. □

Proposition 5.6. *Let $n \geq 1$. The inverse value $J^{-1}(n)$ is given by*

$$(K^{k-1}(n), L \circ K^{k-2}(n), \ldots, L \circ K(n), L(n)), \tag{5.12}$$

where k is the smallest number such that $K^k(n) = 0$.

Proof. Let $x \in \mathbb{N}_0$. Then $J(x) = n \geq 1$ and as shown above, $K(n) = 0$ and $L(n) = x$. Hence, $J^{-1}(n) = L(n) = x$.

Let $\boldsymbol{x} \in \mathbb{N}_0^k$, $k \geq 1$, and $y \in \mathbb{N}_0$. Then $J(\boldsymbol{x}, y) = n \geq 1$. By Proposition 5.3, $K(n) = J(\boldsymbol{x})$ and $L(n) = y$. Since $K(n) < n$, induction shows that $\boldsymbol{x} = J^{-1}(K(n))$ is given by

$$(K^k(n), L \circ K^{k-1}(n), \ldots, L \circ K(n)).$$

Hence, $(\boldsymbol{x}, y)$ has the form

$$(K^k(n), L \circ K^{k-1}(n), \ldots, L \circ K(n), L(n))$$

as required. □

The primitive recursive bijection J allows to encode the *standard GOTO programs*, SGOTO programs for short. These are GOTO programs $P = s_0; s_1; \ldots; s_q$ that have a canonical labelling in the sense that $\lambda(s_l) = l$, $0 \leq l \leq q$. It is clear that for each GOTO program there is a (semantically) equivalent SGOTO program. SGOTO programs are used in the following since they will permit a slightly simpler Gödel numbering than GOTO program. In the following, let $\mathcal{P}_{\text{SGOTO}}$ denote the class of SGOTO programs.

Take an SGOTO program $P = s_0; s_1; \ldots; s_q$. For each $0 \leq l \leq q$, put

$$I(s_l) = \begin{cases} 3 \cdot J(i-1, k) & \text{if } s_l = (l, x_i \leftarrow x_i + 1, k), \\ 3 \cdot J(i-1, k) + 1 & \text{if } s_l = (l, x_i \leftarrow x_i - 1, k), \\ 3 \cdot J(i-1, k, m) + 2 & \text{if } s_l = (l, \texttt{if } x_i = 0, k, m). \end{cases} \tag{5.13}$$

The number $I(s_l)$ is called the *Gödel number* of the instruction s_l, $0 \leq l \leq q$. The function I is primitive recursive, since it is defined by cases and the functions involved are also primitive recursive.

Note that the function I allows to identify the lth instruction of an SGOTO program given its Gödel number e. Indeed, the residue of e modulo 3 provides the type of instruction and the quotient of e modulo 3 gives the encoding of the parameters of the instruction. More concretely, write

$$e = 3n + t, \tag{5.14}$$

where $n = \div(e, 3)$ and $t = \mathrm{mod}(e, 3)$. Then the instruction can be decoded by using Proposition 5.6 as follows:

$$s_l = \begin{cases} (l, x_{K(n)+1} \leftarrow x_{K(n)+1} + 1, L(n)) & \text{if } t = 0, \\ (l, x_{K(n)+1} \leftarrow x_{K(n)+1} - 1, L(n)) & \text{if } t = 1, \\ (l, \texttt{if } x_{K^2(n)+1} = 0, L(K(n)), L(n)) & \text{if } t = 2. \end{cases} \tag{5.15}$$

The *Gödel number* of an SGOTO program $P = s_0; s_1; \ldots; s_q$ is defined as

$$\Gamma(P) = J(I(s_0), I(s_1), \ldots, I(s_q)). \tag{5.16}$$

Proposition 5.7. *The function $\Gamma : \mathcal{P}_{\mathrm{SGOTO}} \to \mathbb{N}_0$ is bijective and primitive recursive.*

Proof. The mapping Γ is bijective since J is bijective and the instructions encoded by I are uniquely determined as shown above. Moreover, the function Γ is a composition of primitive recursive functions and thus is primitive recursive. $\square$

The SGOTO program P with Gödel number e is denoted by P_e. This Gödel numbering provides a list of all SGOTO programs

$$P_0, P_1, P_2, \ldots. \tag{5.17}$$

Conversely, each number e can be assigned the SGOTO program P such that $\Gamma(P) = e$. For this, the length of the string encoded by e is first determined by the minimalization given in Proposition 5.4. Suppose the string has length $n + 1$, where $n \geq 0$. Then the task is to find $\boldsymbol{x} \in \mathbb{N}_0^n$ and $y \in \mathbb{N}_0$ such that $J(\boldsymbol{x}, y) = n$. But by (5.9), $K(n) = J(\boldsymbol{x})$ and $L(n) = y$ and so the preimage of n under J can be repeatedly determined. Finally, when the string is given, the preimage (instruction) of each number is established as described in (5.15).

For each number e and each number $n \geq 0$, denote the n-ary partial recursive function computing the SGOTO program with Gödel number e by

$$\phi_e^{(n)} = \|P_e\|_{n,1}. \tag{5.18}$$

If f is an n-ary partial recursive function, each number $e \in \mathbb{N}_0$ with the property $f = \phi_e^{(n)}$ is called an *index* of f. The index of a partial recursive function f provides the Gödel number of an SGOTO program computing it. The list of all SGOTO program in (5.17) yiels a list of all n-ary partial recursive functions:

$$\phi_0^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}, \ldots. \tag{5.19}$$

Note that the list contain repetitions, since each n-ary partial recursive function has infinitely many indices.

5.2 Parametrization

The parametrization theorem, also called smn theorem, is a cornerstone of computability theory. It was first proved by Kleene (1943) and refers to computable functions in which some arguments are considered as parameters. The smn theorem does not only tell that the resulting function is computable but also shows how to compute an index for it. A special case is considered first. The basic form applies to dyadic computable functions.

Proposition 5.8. *For each dyadic partial recursive function f, there is a monadic primitive recursive function g such that*

$$f(x,\cdot) = \phi^{(1)}_{g(x)}, \quad x \in \mathbb{N}_0. \tag{5.20}$$

Proof. Let $P_e = s_0; s_1; \ldots; s_q$ be an SGOTO program computing the function f. For each number $x \in \mathbb{N}_0$, consider the following SGOTO program Q_x:

$$\begin{array}{llll}
0 & \texttt{if}\ x_1 = 0 & 3 & 1 \\
1 & x_2 \leftarrow x_2 + 1 & 2 & \\
2 & x_1 \leftarrow x_1 - 1 & 0 & \\
3 & x_1 \leftarrow x_1 + 1 & 4 & \\
4 & x_1 \leftarrow x_1 + 1 & 5 & \\
 & \vdots & & \\
2+x & x_1 \leftarrow x_1 + 1 & 3+x & \\
 & s'_0 & & \\
 & s'_1 & & \\
 & \vdots & & \\
 & s'_q & &
\end{array}$$

where $P'_e = s_0; s_1; \ldots; s_q$ is the SGOTO program that is derived from P by replacing each label j with $j+3+x$, $0 \leq j \leq q$. The milestones of the computation of Q_x are given by the following configurations:

$$\begin{array}{llll}
0 & y & 0\ 0\ 0 \ldots & \text{(initially)} \\
0 & 0 & y\ 0\ 0 \ldots & \text{(step 3)} \\
0 & x & y\ 0\ 0 \ldots & \text{(step } 3+x\text{)} \\
0 & f(x,y) & \ldots\ldots & \text{(finally)}
\end{array}$$

It follows that $\|Q_x\|_{1,1}(y) = f(x,y)$ for all $x, y \in \mathbb{N}_0$.

Take the function $g : \mathbb{N}_0 \to \mathbb{N}_0$ defined by $g(x) = \Gamma(Q_x)$, i.e., $g(x)$ is the Gödel number of the program Q_x. This function is primitive recursive by Proposition 5.7. But by definition, $\phi^{(1)}_{g(x)} = \|Q_x\|_{1,1}$ and thus the result follows. $\square$

This assertion is a special case of the socalled smn theorem. The unimaginative name originates from Kleene's notation $s_{m,n}$ for the primitive recursive function playing the key role.

Theorem 5.9. *(**smn Theorem**) For each pair of numbers $m, n \geq 1$, there is an $m+1$-ary primitive recursive function $s_{m,n}$ such that*

$$\phi^{(m+n)}_e(\boldsymbol{x},\cdot) = \phi^{(n)}_{s_{m,n}(e,\boldsymbol{x})}, \quad \boldsymbol{x} \in \mathbb{N}_0^m,\ e \in \mathbb{N}_0. \tag{5.21}$$

Proof. The idea is quite similar to that in the proof of the previous proposition. Take an SGOTO program $P_e = s_0; s_1; \ldots; s_q$ calculating $\phi_e^{(m+n)}$. For each input $\boldsymbol{x} \in \mathbb{N}_0^m$, extend the program P_e to an SGOTO program $Q_{e,\boldsymbol{x}}$ providing the following intermediate configurations:

$$\begin{array}{llll}
0 & \boldsymbol{y} & 0\,0\,0\ldots & \text{(initially)} \\
0 & \boldsymbol{0} & \boldsymbol{y}\,0\,0\ldots & \text{(reload } \boldsymbol{y}\text{)} \\
0 & \boldsymbol{x} & \boldsymbol{y}\,0\,0\ldots & \text{(generate parameter } \boldsymbol{x}\text{)} \\
0 & \phi_e^{(m+n)}(\boldsymbol{x},\boldsymbol{y}) & \ldots\ldots & \text{(finally)}
\end{array}$$

It follows that $\|Q_{e,\boldsymbol{x}}\|_{n,1}(\boldsymbol{y}) = \phi_e^{(m+n)}(\boldsymbol{x},\boldsymbol{y})$ for all $\boldsymbol{x} \in \mathbb{N}_0^m$ and $\boldsymbol{y} \in \mathbb{N}_0^n$.

Consider the function $s_{m,n} : \mathbb{N}_0^{m+1} \to \mathbb{N}_0$ defined by $s_{m,n}(e,\boldsymbol{x}) = \Gamma(Q_{e,\boldsymbol{x}})$; that is, $s_{m,n}(e,\boldsymbol{x})$ is the Gödel number of the program $Q_{e,\boldsymbol{x}}$. This function is primitive recursive by Proposition 5.7. But by definition, $\phi_{s_{m,n}(e,\boldsymbol{x})}^{(n)} = \|Q_{e,\boldsymbol{x}}\|_{n,1}$ and thus the result follows. ☐

5.3 Universal Functions

Another basic result of computability theory is the existence of a computable function called universal function that is capable of computing any other computable function. Let $n \geq 1$ be a number. A *universal function* for n-ary partial recursive functions is an $n+1$-ary function $\psi_{\text{univ}}^{(n)} : \mathbb{N}_0^{n+1} \to \mathbb{N}_0$ such that

$$\psi_{\text{univ}}^{(n)}(e,\cdot) = \phi_e^{(n)}, \quad e \in \mathbb{N}_0. \tag{5.22}$$

Theorem 5.10. *For each number $n \geq 1$, the universal function $\psi_{\text{univ}}^{(n)}$ exists and is partial recursive.*

Proof. The existence will be proved in several steps. First, define the *one-step function* $E : \mathbb{N}_0^2 \to \mathbb{N}_0^2$ that describes the process to move from one configuration of the URM to the next one during the computation of an SGOTO program. For this, consider the following diagram:

$$\begin{tikzcd}
e,\xi \arrow[r, "E"'] \arrow[d, "G^{-1}"'] & e,\xi' \\
P_e,\omega \arrow[r, "s_l"'] & P_e,\omega' \arrow[u, "G"']
\end{tikzcd}$$

The one-step function E takes a pair (e,ξ) and recovers from the second component ξ the corresponding configuration of the URM given by $\omega = G^{-1}(\xi)$. Then the next instruction given by the SGOTO program $P = P_e$ with Gödel number e is executed providing a new configuration ω' of the URM which is then encoded as $\xi' = G(\omega')$. In this way, the function E is defined as

$$E(e,\xi) = (e,\xi'). \tag{5.23}$$

This function can be described in a higher programming language as illustrated by algorithm 5.1. More specifically, the algorithm first represents the given number ξ as a product of prime powers

$$\xi = p_0^l p_1^{\omega_1} p_2^{\omega_2} p_3^{\omega_3} \cdots = 2^l 3^{\omega_1} 5^{\omega_2} 7^{\omega_3} \cdots \tag{5.24}$$

Algorithm 5.1 One-step function.

Require: (e, ξ)
Ensure: (e, ξ')
  if $\xi = 0$ **then**
    $\xi' \leftarrow 0$
  else
    $l \leftarrow G_0(\xi)$ {label, $\xi = 2^l \ldots$}
    $q \leftarrow lg(e) \dot{-} 1$
    if $l > q$ **then**
      $\xi' \leftarrow \xi$
    else
      $(\sigma_0, \sigma_1, \ldots, \sigma_q) \leftarrow J^{-1}(e)$
      $\xi_0 \leftarrow \xi \div 2^l$ {$\xi = 2^l \xi_0$}
      $n \leftarrow \sigma_l \div 3$ {$\sigma_l = 3n + t$, $0 \leq t \leq 2$ }
      if $t = 0$ **then**
        $i \leftarrow K(n) + 1$ {$s_l = (l, x_i \leftarrow x_i + 1, k)$}
        $k \leftarrow L(n)$
        $\xi' \leftarrow \xi_0 \cdot 2^k \cdot p_i$
      end if
      if $t = 1$ **then**
        $i \leftarrow K(n) + 1$ {$s_l = (l, x_i \leftarrow x_i - 1, k)$}
        $k \leftarrow L(n)$
        if $G_i(\xi) = 0$ **then**
          $\xi' \leftarrow \xi_0 \cdot 2^k$
        else
          $\xi' \leftarrow (\xi_0 \cdot 2^k) \div p_i$
        end if
        if $t = 2$ **then**
          $i \leftarrow K^2(n) + 1$ {$s_l = (l, \texttt{if } x_i = 0, k, m)$}
          $k \leftarrow L(K(n))$
          $m \leftarrow L(n)$
          if $G_i(\xi) = 0$ **then**
            $\xi' \leftarrow \xi_0 \cdot 2^k$
          else
            $\xi' \leftarrow \xi_0 \cdot 2^m$
          end if
        end if
      end if
    end if
  end if
  return (e, ξ')

and extracts the associated configuration of the URM given by the state

$$\omega = (0, \omega_1, \omega_2, \omega_3, \ldots) \tag{5.25}$$

and the label $l = G_0(\xi)$ of the instruction to be executed. Second, it provides the preimage of the program's Gödel number e using the length function and Proposition 5.6:

$$J^{-1}(e) = (\sigma_0, \sigma_1, \ldots, \sigma_q). \tag{5.26}$$

Third, the algorithm decodes the number σ_l into the associated instruction s_l using (5.14) and (5.15), executes it and then provides the encoding of the next configuration:

- If $s_l = (l, x_i \leftarrow x_i + 1, k)$, the next state of the URM is

$$\omega' = (0, \omega_1, \ldots, \omega_{i-1}, \omega_i + 1, , \omega_{i+1}, \ldots) \tag{5.27}$$

  and the next label is k. So the next configuration is encoded as

$$\xi' = \xi \cdot 2^{k-l} \cdot p_i. \tag{5.28}$$

- If $s_l = (l, x_i \leftarrow x_i - 1, k)$, the next state of the URM is

$$\omega' = (0, \omega_1, \ldots, \omega_{i-1}, \omega_i \dot{-} 1, , \omega_{i+1}, \ldots) \tag{5.29}$$

  and the next label is k. Thus the next configuration is given as

$$\xi' = \begin{cases} \xi \cdot 2^{k-l} & \text{if } \omega_i = 0, \\ \xi \cdot 2^{k-l} \cdot p_i^{-1} & \text{otherwise.} \end{cases} \tag{5.30}$$

- If $s_l = (l, \texttt{if } x_i = 0, k, m)$, the state of the URM remains unchanged and the next label is either k or l depending on whether the value x_i is zero or not. The next configuration is

$$\xi' = \begin{cases} \xi \cdot 2^{k-l} & \text{if } \omega_i = 0, \\ \xi \cdot 2^{m-l} & \text{otherwise.} \end{cases} \tag{5.31}$$

The algorithm shows that the function E is primitive recursive.

Second, the execution of the SGOTO program P_e can be described by the *iterated one-step function* $E' : \mathbb{N}_0^3 \to \mathbb{N}_0$ defined as

$$E'(e, \xi, t) = (\pi_2^{(2)} \circ E^t)(e, \xi). \tag{5.32}$$

This function is primitive recursive, since it is given by applying a projection to an iteration of a primitive recursive function.

Third, the computation of the SGOTO program P_e terminates if it reaches a non-existing label. That is, if $\lg(e)$ denotes the length of the SGOTO program P_e, termination is reached if and only if

$$G_0(\xi) > \lg(e) \dot{-} 1. \tag{5.33}$$

The *runtime function* $Z : \mathbb{N}_0^2 \to \mathbb{N}_0$ assigns to the program P_e and the state $\omega = G^{-1}(\xi)$ the number of steps required to reach termination,

$$Z(e,\xi) = \mu_t(\operatorname{csg}(G_0(E'(e,\xi,t)) \dot{-} (\lg(e) \dot{-} 1))), \tag{5.34}$$

where the minimalization $\mu = \mu_t$ is subject to the variable t counting the number of steps. This function is only partial recursive since the minimalization is unbounded, i.e., the program P_e may not terminate.

Fourth, the *residual step function* is defined by the partial function $R : \mathbb{N}_0^2 \to \mathbb{N}_0$ which assigns to each SGOTO program P_e and each initial state $\omega = G^{-1}(\xi)$ the result of computation given by the content of the first register:

$$R(e,\xi) = G_1(E'(e,\xi,Z(e,\xi))). \tag{5.35}$$

This function is partial recursive, since the function Z is partial recursive.

Summing up, the desired partial recursive function is given as

$$\psi_{\text{univ}}^{(n)}(e,\boldsymbol{x}) = \phi_e^{(n)}(\boldsymbol{x}) = \|P_e\|_{n,1}(\boldsymbol{x}) = R(e, G(0,x_1,\ldots,x_n,0,0,\ldots)), \tag{5.36}$$

where e is a Gödel number and $\boldsymbol{x} = (x_1,\ldots,x_n) \in \mathbb{N}_0^n$ an input. $\square$

The existence of universal functions implies the existence of universal URM programs, and vice versa.

An *acceptable programming system* is considered to be an enumeration of n-ary partial recursive functions $\psi_0, \psi_1, \psi_2, \ldots$ for which both the smn theorem and the theorem for universal function hold. For instance, the enumeration of URM (or GOTO) computable functions forms an acceptable programming system.

5.4 Kleene's Normal Form

Kleene (1943) introduced the T predicate that tells whether an SGOTO program will halt when run with a particular input and if so, a corresponding function provides the result of computation. Similar to the smn theorem, the original notation used by Kleene has become standard terminology.

First note that the definition of the universal function allows to define the *Kleene set* $S_n \subseteq \mathbb{N}_0^{n+3}$, $n \geq 1$, given as

$$(e,\boldsymbol{x},z,t) \in S_n \quad :\Longleftrightarrow \quad \operatorname{csg}[G_0(E'(e,\xi_{\boldsymbol{x}},t)) \dot{-} (\lg(e) \dot{-} 1)] = 0 \quad \wedge \quad G_1(E'(e,\xi_{\boldsymbol{x}},t)) = z, \tag{5.37}$$

where $\xi_{\boldsymbol{x}} = (0,x_1,\ldots,x_n,0,0,\ldots)$ is the initial state comprising the input $\boldsymbol{x} = (x_1,\ldots,x_n) \in \mathbb{N}_0^n$. Clearly, $(e,\boldsymbol{x},z,t) \in S_n$ if and only if the program ϕ_e with input $\boldsymbol{x}$ terminates after t steps with the result z. The set S_n is primitive recursive since the number of steps is explicitly given.

Let $A \subseteq \mathbb{N}_0^{n+1}$ be a relation. First, the *unbounded minimalization* of A is the function $\mu A : \mathbb{N}_0^n \to \mathbb{N}_0$ defined as

$$\mu A(\boldsymbol{x}) = \mu(\operatorname{csg} \circ \chi_A)(\boldsymbol{x}) = \begin{cases} y \text{ if } (\boldsymbol{x},y) \in A \text{ and } (\boldsymbol{x},i) \notin A \text{ for all } 0 \leq i < y, \\ \uparrow \text{ otherwise.} \end{cases} \tag{5.38}$$

We write

$$\mu A(\boldsymbol{x}) = \mu y[(\boldsymbol{x},y) \in A], \quad \boldsymbol{x} \in \mathbb{N}_0^n. \tag{5.39}$$

Second, the *unbounded existential quantification* of A is the relation $E_A \subseteq \mathbb{N}_0^n$ given by

$$E_A = \{\boldsymbol{x} \mid \exists y \in \mathbb{N}_0 : (\boldsymbol{x}, y) \in A\}. \tag{5.40}$$

Put

$$\exists y[(\boldsymbol{x}, y) \in A] \quad :\Longleftrightarrow \quad \boldsymbol{x} \in E_A, \quad \boldsymbol{x} \in \mathbb{N}_0^n. \tag{5.41}$$

Third, the *unbounded universal quantification* of A is the relation $U_A \subseteq \mathbb{N}_0^n$ defined by

$$U_A = \{\boldsymbol{x} \mid \forall y \in \mathbb{N}_0 : (\boldsymbol{x}, y) \in A\}. \tag{5.42}$$

Set

$$\forall y[(\boldsymbol{x}, y) \in A] \quad :\Longleftrightarrow \quad \boldsymbol{x} \in E_A, \quad \boldsymbol{x} \in \mathbb{N}_0^n. \tag{5.43}$$

Theorem 5.11. *(Kleene) For each number $n \geq 1$, there is a primitive recursive set $T_n \subseteq \mathbb{N}_0^{n+2}$ called* Kleene predicate *such that for all $\boldsymbol{x} \in \mathbb{N}_0^n$:*

$$\boldsymbol{x} \in \operatorname{dom} \phi_e^{(n)} \Longleftrightarrow (e, \boldsymbol{x}) \in \exists y[(e, \boldsymbol{x}, y) \in T_n]. \tag{5.44}$$

If $\boldsymbol{x} \in \operatorname{dom} \phi_e^{(n)}$, there is a primitive recursive function $U : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\phi_e^{(n)}(\boldsymbol{x}) = U(\mu y[(e, \boldsymbol{x}, y) \in T_n]). \tag{5.45}$$

Proof. Define the relation $T_n \subseteq \mathbb{N}_0^{n+2}$ as follows:

$$(e, \boldsymbol{x}, y) \in T_n \quad :\Longleftrightarrow \quad (e, \boldsymbol{x}, K_2(y), L_2(y)) \in S_n. \tag{5.46}$$

That is, the component y encodes both, the result of computation $z = K_2(y)$ and the number of steps $t = L_2(y)$. Since the relation S_n and the functions K_2 and L_2 are primitive recursive, it follows that the relation T_n is primitive recursive, too.

Let $\boldsymbol{x} \in \mathbb{N}_0^n$ lie in the domain of $\phi_e^{(n)}$ and let $\phi_e^{(n)}(\boldsymbol{x}) = z$. Then there is a number $t \geq 0$ such that $(e, \boldsymbol{x}, z, t) \in S_n$. Putting $y = J_2(z, t)$ yields $(e, \boldsymbol{x}, y) \in T_n$; that is, $(e, \boldsymbol{x}) \in \exists y[(e, \boldsymbol{x}, y) \in T_n]$.

Conversely, let $(e, \boldsymbol{x}) \in \exists y[(e, \boldsymbol{x}, y) \in T_n]$ and let $y \geq 0$ be a number such that $(e, \boldsymbol{x}, y) \in T_n$. Then by definition, $(e, \boldsymbol{x}, K_2(y), L_2(y)) \in S_n$ and hence $\boldsymbol{x}$ belongs to the domain of $\phi_e^{(n)}$.

Finally, let $\boldsymbol{x} \in \operatorname{dom} \phi_e^{(n)}$. Then there is a number $y \geq 0$ such that $(e, \boldsymbol{x}, y) \in T_n$. Define the function $U : \mathbb{N}_0 \to \mathbb{N}_0$ by putting $U(y) = K_2(y)$. The function U is primitive recursive and by definition of S_n yields the result of computation. $\square$

Kleene's normal form implies that any partial recursive function can be defined by using a single instance of the μ (minimalization) operator applied to a primitive recursive function. In the context of programming, this means that any program can be written with a single `while` loop.

6

Turing Machine

The Turing machine was described by the British mathematician Alan Turing (1912-1954) in 1937 as a thought experiment representing a computing machine. Despite its simplicity, a Turing machine is a universal device of computation.

6.1 The Machinery

A Turing machine consists of an infinite tape and an associated read/write head connected to a control mechanism. The tape is divided into denumerably many cells, each of which containing a symbol from a tape alphabet. This alphabet contains the special symbol "b" signifying that a cell is blank or empty. The cells are scanned, one at a time, by the read/write head which is able to move in both directions. At any given time instant, the machine will be in one of a finite number of states. The behaviour of the read/write head and the change of the machine's state are governed by the present state of the machine and by the symbol in the cell under scan.

The machine operates on words over an input alphabet. The symbols forming a word are written, in order, in consecutive cells of the tape from left to right. When the machine enters a state, the read/write head scans the symbol in the controlled cell, and writes in this cell a symbol from the tape alphabet; it then moves one cell to the left, or one cell to the right, or not at all; after that, the machine enters a new state.

A *Turing machine* is a quintuple $M = (\Sigma, Q, T, q_0, q_F)$ consisting of

- a finite alphabet Σ, called *tape alphabet*, containing a distinguished *blank* symbol b; the subset $\Sigma_I = \Sigma \setminus \{b\}$ is called *input alphabet*.
- a finite set Q of *states*,
- a partial function $T : Q \times \Sigma \to Q \times \Sigma \times \{L, R, \Lambda\}$, the *state transition function*,
- a *start state* $q_0 \in Q$, and
- a *halt state* $q_F \in Q$ such that $T(q_F, \sigma)$ is undefined for all σ in Σ.

The symbols L, R, and Λ are interpreted as *left move*, *right move*, and *no move*, respectively. The tape cells can be numbered by the set of integers $\mathbb{Z}$, and the tape contents can be considered as a mapping $\tau : \mathbb{Z} \to \Sigma$ which assigns blank to almost every cell; that is, only a finite portion of the tape contains symbols from the input alphabet.

A *configuration* of the Turing machine M consists of the contents of the tape which may be given by the finite portion of the tape containing symbols from the input alphabet, the cell controlled by the read/write head, and the state of the machine. Thus a configuration can be pictured as follows:

$$\begin{array}{c} - \, a_{i_1} \; a_{i_2} \; \ldots \; a_{i_j} \; \ldots \; a_{i_l} \, - \\ \uparrow \\ q \end{array}$$

where $q \in Q$ is the current state, the cell controlled by the read/write head is marked by the arrow, and all cells to the left of a_{i_1} and to the right of a_{i_l} contain the blank symbol. This configuration will also be written as a triple $(a_{i_1} \ldots a_{i_{j-1}}, q, a_{i_j} \ldots a_{i_l})$.

The equation $T(q, a) = (q', a', D)$ means that if the machine is in state $q \in Q$ and reads the symbol $a \in \Sigma$ from the cell controlled by the read/write head, it writes the symbol $a' \in \Sigma$ into this cell, moves to the left if $D = L$, or moves to the right if $D = R$, or moves not at all if $D = \Lambda$, and enters the state $q' \in Q$. Given a configuration (uc, q, av) where $a, c \in \Sigma$, $u, v \in \Sigma^*$, and $q \in Q$, $q \neq q_F$. The configuration *reached in one step* from it is given as follows:

$$(u', q', v') = \begin{cases} (uca', q', v) \text{ if } T(q, a) = (q', a', R), \\ (u, q', ca'v) \text{ if } T(q, a) = (q', a', L), \\ (uc, q', a'v) \text{ if } T(q, a) = (q', a', \Lambda). \end{cases} \tag{6.1}$$

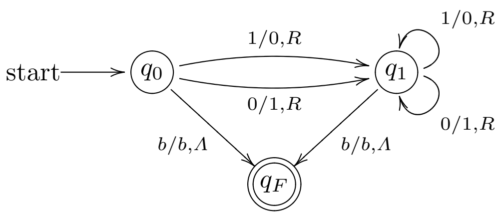
In the following, the notation $(u, q, v) \vdash (u', q', v')$ will signify that (u', q', v') is reached from (u, q, v) in one step.

A *computation* of the Turing machine M is started by writing, in order, the symbols of the input word $\boldsymbol{x} = x_1 x_2 \ldots x_n \in \Sigma_I^*$ in consecutive cells of the tape from left to right, while all other cells contain the blank symbol. Moreover, the machine is initially in the state q_0 and the read/right head controls the leftmost cell of the input; that is, the *initial configuration* can be illustrated as follows:

$$\begin{array}{l} - \, x_1 \; x_2 \; \ldots \; x_n \, - \\ \;\;\;\uparrow \\ \;\;\;q_0 \end{array}$$

Then the machine eventually performs a sequence of transitions as given by the state transition function. If the machine reaches the state q_F, it stops and the output of the computation is given by the collection of symbols on the tape.

Example 6.1. Consider the Turing machine $M = (\Sigma, Q, T, q_0, q_F)$, where $\Sigma = \{0, 1, b\}$, $Q = \{q_0, q_1, q_F\}$, and T is given by the following state diagram:



In such a diagram, the encircled nodes represent the states and an arrow joining a state q to a state q' and bearing the label $a/a', D$ indicates the transition $T(q, a) = (q', a', D)$. For instance, the computation of the machine on the input 0011 is the following:

$$(b, q_0, 0011) \vdash (1, q_1, 011) \vdash (11, q_1, 11) \vdash (110, q_1, 0) \vdash (1100, q_1, b) \vdash (1100, q_F, b).$$

In general, the machine calculates the bitwise complement of the given binary word. $\diamondsuit$

6.2 Post-Turing Machine

A Turing machine specified by state diagrams is rather hard to follow. Therefore, a program formulation of the Turing machine known as Post-Turing machine invented by Martin Davis (born 1928) will be considered.

A *Post-Turing machine* uses a binary alphabet $\Sigma = \{1, b\}$, an infinite tape of binary storage locations, and a primitive programming language with instructions for bidirectional movement among the storage locations, alteration of cell content one at a time, and conditional jumps. The instructions are as follows:

- `write` a, where $a \in \Sigma$,
- `move left`,
- `move right`, and
- `if read` a `then goto` A, where $a \in \Sigma$ and A is a label.

A *Post-Turing program* consists of a finite sequence of labelled instructions which are sequentially executed starting with the first instruction. Each instruction may have a label that must be unique in the program. Since there is no specific instruction for termination, the machine will stop when it has arrived at a state at which the program contains no instruction telling the machine what to do next.

First, several Post-Turing programs will be introduced used as macros later on. The first macro is `move left to next blank`:

$$\begin{aligned} A: &\ \texttt{move left} \\ &\ \texttt{if read } 1 \texttt{ then goto } A \end{aligned} \tag{6.2}$$

This program moves the read/write head to next blank on the left. If this program is started in the configuration

$$\begin{array}{c} \text{—}\; b\; 1\; 1\; 1\; \ldots\; 1 \;\text{—} \\ \phantom{\text{—}\; b\; 1\; 1\; 1\; \ldots\;} \uparrow \phantom{\;\text{—}} \end{array}$$

it will end in the configuration

$$\begin{array}{c} \text{—}\; b\; 1\; 1\; 1\; \ldots\; 1 \;\text{—} \\ \phantom{\text{—}\;} \uparrow \phantom{\; 1\; 1\; 1\; \ldots\; 1 \;\text{—}} \end{array}$$

The macro `move right to next blank` is similarly defined:

$$\begin{aligned} A: &\ \texttt{move right} \\ &\ \texttt{if read } 1 \texttt{ then goto } A \end{aligned} \tag{6.3}$$

This program moves the read/write head to next blank on the right. If this program begins in the configuration

$$\begin{array}{c} \text{—}\; 1\; 1\; 1\; \ldots\; 1\; b \;\text{—} \\ \phantom{\text{—}\;} \uparrow \phantom{\; 1\; 1\; \ldots\; 1\; b \;\text{—}} \end{array}$$

it will stop in the configuration

$$\begin{array}{c} \text{—}\ 1\ 1\ 1\ \ldots\ 1\ b\ \text{—} \\ \qquad\qquad\quad\uparrow \end{array}$$

The macro `write` $b1$ is defined as

```
write b
move right
write 1
move right
```
(6.4)

If this macro begins in the configuration

$$\begin{array}{c} \text{—}\ a_{i_0}\ a_{i_1}\ a_{i_2}\ a_{i_3}\ \text{—} \\ \quad\uparrow \end{array}$$

it will terminate in the configuration

$$\begin{array}{c} \text{—}\ a_{i_0}\ b\ 1\ a_{i_3}\ \text{—} \\ \qquad\quad\uparrow \end{array}$$

The macro `move block right` is given as

```
write b
move right to next blank
write 1
move right
```
(6.5)

This program shifts a block of 1's by one cell to the right such that it merges with the subsequent block of 1's to right. If this macro starts in the configuration

$$\begin{array}{c} \text{—}\ b\ 1\ 1\ 1\ \ldots\ 1\ b\ 1\ 1\ 1\ \ldots\ 1\ b\ \text{—} \\ \uparrow\qquad\qquad\qquad\qquad\qquad\qquad \end{array}$$

it will halt in the configuration

$$\begin{array}{c} \text{—}\ b\ b\ 1\ 1\ \ldots\ 1\ 1\ 1\ 1\ 1\ \ldots\ 1\ b\ \text{—} \\ \qquad\qquad\qquad\uparrow\qquad\qquad \end{array}$$

The macro `move block left` is analogously defined. The unconditional jump `goto` A stands for the Post-Turing program

```
if read b then goto A
if read 1 then goto A
```
(6.6)

Another useful macro is `erase` given as

```
A : if read b then goto B
    write b
    move left
    goto A
B : move left
```
(6.7)

This program deletes a block of 1's from right to left. If it starts in the configuration

$$— a\ b\ 1\ 1\ 1\ \ldots\ \underset{\uparrow}{1}\ b\ —$$

where a is an arbitrary symbol, it will stop in the configuration

$$— \underset{\uparrow}{a}\ b\ b\ b\ b\ \ldots\ b\ b\ —$$

Finally, the repetition of a statement such as

```
move left
move left
move left
```
(6.8)

will be abbreviated by denoting the statement and the number of repetitions in parenthesis such as

`move left` (3) (6.9)

For instance, the routine `move block right` (2) shifts two consecutive blocks by one cell to the right. It turns the configuration

$$— b\ \underset{\uparrow}{1}\ 1\ \ldots\ 1\ b\ 1\ 1\ \ldots\ 1\ b\ 1\ 1\ \ldots\ 1\ b\ —$$

into the configuration

$$— b\ b\ 1\ \ldots\ 1\ 1\ b\ 1\ \ldots\ 1\ 1\ \underset{\uparrow}{1}\ 1\ \ldots\ 1\ b\ —$$

6.3 Turing Computable Functions

The Turing machine has the same computational capabilities as the unrestricted register machine. To prove this, it will be shown that the URM computable functions are computable by Post-Turing programs. For this, let $f : \mathbb{N}_0^k \to \mathbb{N}_0$ be an URM computable function. It may be assumed that there is a GOTO program P that computes the function f and uses the variables $x_1, \ldots, x_n$, where $n \geq k$. The goal is to provide a Post-Turing program P' that simulates the computation of P. The program P' consists of three subroutines:

```
start: initiate
       simulate
       clean_up
```
(6.10)

The routine `initiate` presets the program for the computation of the function f. An input $\boldsymbol{x} = (x_1, \ldots, x_k) \in \mathbb{N}_0^k$ of the function f is encoded on the tape in unary format; that is, a natural number x is represented by a block of $x + 1$ ones and a sequence of k numbers is described by k blocks such that consecutive blocks are separated by one blank. Thus the initial tape looks as follows:

$$— \overbrace{11\ldots1}^{x_1+1}\, b\, \overbrace{11\ldots1}^{x_2+1}\, b \ldots b\, \overbrace{11\ldots1}^{x_k+1} — \tag{6.11}$$
$$\uparrow$$

The idea is that the blocks will correspond to the registers $R_1, R_2, \ldots, R_k$ of the URM. The remaining registers used during the computation will be initialized by zeros. This will be accomplished by the routine `initiate`:

$$\begin{array}{ll} \textit{init:} & \texttt{move right to next blank}\ (k) \\ & \texttt{write}\ b1\ (n-k) \\ & \texttt{move left to next blank}\ (n) \\ & \texttt{move right} \end{array} \tag{6.12}$$

If the procedure `initiate` is started in the configuration (6.11), it will end in the configuration

$$— b\, \overbrace{11\ldots1}^{x_1+1}\, b\, \overbrace{11\ldots1}^{x_2+1}\, b \ldots b\, \overbrace{11\ldots1}^{x_k+1}\, b\, 1\, b\, 1\, b \ldots b\, 1\, b\, 1\, b — \tag{6.13}$$
$$\uparrow$$

The tape can now be interpreted in a way that it contains the numbers $x_1, \ldots, x_k$ followed by $n-k$ zeros.

The routine `simulate` is the heart of the simulation program. The idea is to simulate each GOTO instruction by a corresponding Post-Turing program such that the Post-Turing programs concatenated in order, constitute the procedure `simulate`. For this, the simulation should keep track of the registers $R_1, R_2, \ldots, R_n$ used to carry out the program P. To this end, the tape will always contain n blocks of ones which will in turn correspond to these registers.

The GOTO instruction $(l, x_i \leftarrow x_i + 1, l')$ is replaced by the Post-Turing program

$$\begin{array}{ll} Al: & \texttt{move right to next blank}\ (i) \\ Bl: & \texttt{write}\ 1 \\ & \texttt{move right} \\ & \texttt{if read}\ b\ \texttt{then goto}\ Cl \\ & \texttt{write}\ b \\ & \texttt{move right to next blank} \\ & \texttt{goto}\ Bl \\ Cl: & \texttt{move left to next blank}\ (n) \\ & \texttt{move right} \\ & \texttt{goto}\ Al' \end{array} \tag{6.14}$$

This program enlarges the ith block by an additional 1 and shifts all blocks to the right by one cell to the right.

The GOTO instruction $(l, x_i \leftarrow x_i \dot{-} 1, l')$ is substituted by the program

$$
\begin{array}{ll}
Al: & \texttt{move right to next blank}\ (i) \\
 & \texttt{move left}\ (2) \\
 & \texttt{if read}\ 1\ \texttt{then goto}\ Bl \\
 & \texttt{goto}\ Cl \\
Bl: & \texttt{move left to next blank}\ (i) \\
 & \texttt{move right} \\
 & \texttt{move block right}\ \ (i-1) \\
 & \texttt{write}\ b \\
 & \texttt{move left to next blank}\ (i-1) \\
 & \texttt{move right} \\
 & \texttt{goto}\ Al' \\
Cl: & \texttt{move left to next blank}\ (i-1) \\
 & \texttt{move right} \\
 & \texttt{goto}\ Al'
\end{array}
\tag{6.15}
$$

This subroutine shortens the ith block by a single 1 if it contains at least two 1's which corresponds to a non-zero number; otherwise, the block is left invariant. Then all blocks to the left are shifted by one cell to the right.

The GOTO instruction $(l, \texttt{if}\ x_i = 0, l', l'')$ is superseded by the program

$$
\begin{array}{ll}
Al: & \texttt{move right to next blank}\ (i) \\
 & \texttt{move left}\ (2) \\
 & \texttt{if read}\ 1\ \texttt{then goto}\ Bl \\
 & \texttt{goto}\ Cl \\
Bl: & \texttt{move left to next blank}\ (i) \\
 & \texttt{move right} \\
 & \texttt{goto}\ Al'' \\
Cl: & \texttt{move left to next blank}\ (i-1) \\
 & \texttt{move right} \\
 & \texttt{goto}\ Al'
\end{array}
\tag{6.16}
$$

This program checks if the ith block contains one or more 1's and jumps accordingly to the label Al' or Al''.

Note that when one of these subroutines ends, the read/write head always points to the first cell of the first block.

Suppose the routine `simulate` starts with the configuration (6.11) and terminates; this will exactly be the case when the input $\boldsymbol{x} = (x_1, \ldots, x_k)$ belongs to the domain of the function f. In this case, the tape will contain in the first block the unary encoding of the result $f(\boldsymbol{x})$:

$$
\begin{array}{l}
- \, b \, \overbrace{11\ldots1}^{f(\boldsymbol{x})+1} \, b \, \overbrace{11\ldots1}^{y_2+1} \, b \ldots b \, \overbrace{11\ldots1}^{y_n+1} \, b \, - \\
\quad\ \ \uparrow
\end{array}
\tag{6.17}
$$

Finally, the subroutine `clean_up` will rectify the tape such that upon termination the tape will only contain $f(\boldsymbol{x})$ ones. This will be achieved by the code

$$
\begin{array}{ll}
\textit{clean:} & \texttt{move right to next blank } (n) \\
& \texttt{move left} \\
& \texttt{erase } (n-1) \\
& \texttt{write } b \\
& \texttt{move left to next blank} \\
& \texttt{move right}
\end{array}
\tag{6.18}
$$

This subroutine produces the tape:

$$
\begin{array}{c}
- \, b \, \overbrace{11 \ldots 1}^{f(\boldsymbol{x})} \, b \, - \\
\uparrow
\end{array}
\tag{6.19}
$$

This kind of simulation captures the notion of Post-Turing computability. A function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ is *Post-Turing computable* if there is a Post-Turing program P such that for all elements $\boldsymbol{x} = (x_1, \ldots, x_k)$ in the domain of f, the program P started with the initial tape (6.11) will terminate with the final tape (6.19); otherwise, the program P will not stop. Summing up, the following result has been established.

Theorem 6.2. *Each GOTO computable function is Post-Turing computable.*

6.4 Gödel Numbering of Post-Turing Programs

This section provides a Gödel numbering of Post-Turing programs similar to that of SGOTO programs. This numbering will be used to show that Post-Turing computable functions are partial recursive.

To this end, let $\Sigma_{s+1} = \{b = a_0, a_1, \ldots, a_s\}$ be the tape alphabet containing the blank symbol b, and let $P = \sigma_0; \sigma_1; \ldots; \sigma_q$ be a Post-Turing program given by a sequence of instructions σ_j, $0 \leq j \leq q$. A *configuration* of the Post-Turing program P consists of the contents of the tape, the position of the read/write head, and the instruction σ_j to be performed. Such a configuration can be pictured as follows:

$$
\begin{array}{c}
- \, a_{i_1} \; a_{i_2} \; \ldots \; a_{i_v} \; \ldots \; a_{i_t} \, - \\
\uparrow \\
\sigma_j
\end{array}
\tag{6.20}
$$

where all cells to the left of a_{i_1} and to the right of a_{i_t} contain the blank symbol.

A *Gödel numbering* of such a configuration can be considered as a triple (u, v, j) consisting of

- the Gödel numbering $u = J(i_1, i_2, \ldots, i_t)$ of the contents of the tape,
- the position v of the read/write head, with $1 \leq v \leq t = \mu k(K^k(u) = 0)$, and
- the number j of the next instruction σ_j.

First, define a *one-step function* $E : \mathbb{N}_0^3 \to \mathbb{N}_0^3$ that describes the process of moving from one configuration to the next one during the computation of program P. For this, let $\boldsymbol{z} = (u, v, j)$ be a configuration of the program given as in (6.20). The configuration $\boldsymbol{z}' = (u', v', j')$ is *reached in one step* from $\boldsymbol{z}$, written $\boldsymbol{z} \vdash_P \boldsymbol{z}'$, if one of the following holds:

- If σ_j is `write` a_i,

$$(u', v', j') = (J(i_1, \ldots, i_{v-1}, i, i_{v+1}, \ldots, i_t), v, j+1). \tag{6.21}$$

- If σ_j is `move left`,

$$(u', v', j') = \begin{cases} (u, v-1, j+1) & \text{if } v > 1, \\ (J(0, i_1, \ldots, i_t), v, j+1) & \text{otherwise.} \end{cases} \tag{6.22}$$

- If σ_j is `move right`,

$$(u', v', j') = \begin{cases} (u, v+1, j+1) & \text{if } v < \mu k(K^k(u) = 0), \\ (J(i_1, \ldots, i_t, 0), v+1, j+1) & \text{otherwise.} \end{cases} \tag{6.23}$$

- If σ_j is `if read` a_i `then goto` A, where the label A is given as an instruction number,

$$(u', v', j') = \begin{cases} (u, v, A) & \text{if } i_v [= L(K^{t-v}(u)] = i, \\ (u, v, j+1) & \text{otherwise.} \end{cases} \tag{6.24}$$

The function $E : \boldsymbol{z} \mapsto \boldsymbol{z}'$ is defined by cases and primitive recursive operations. It follows that E is primitive recursive.

Second, the execution of the GOTO program P is given by a sequence $\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_2, \ldots$ of configurations such that $\boldsymbol{z}_i \vdash_P \boldsymbol{z}_{i+1}$ for each $i \geq 0$. During this process, a configuration $\boldsymbol{z}_t$ may eventually be reached such that the label of the involved instruction does not belong to the set of instruction numbers $L(P) = \{0, 1, \ldots, q\}$. In this case, the program P terminates. Such an event may eventually not happen. In this way, the *runtime function* of P is a partial function $Z_P : \mathbb{N}_0^3 \to \mathbb{N}_0$ given by

$$Z_P : \boldsymbol{z} \mapsto \begin{cases} \min\{t \in \mathbb{N}_0 \mid (\pi_2^{(3)} \circ E_P^t)(\boldsymbol{z}) \notin L(P)\} & \text{if } \{\ldots\} \neq \emptyset, \\ \uparrow & \text{otherwise.} \end{cases} \tag{6.25}$$

The runtime function may not be primitive recursive as it corresponds to an unbounded search process. Claim that the runtime function Z_P is partial recursive. Indeed, the one-step function E_P is primitive recursive and thus its iteration is primitive recursive:

$$E_P' : \mathbb{N}_0^4 \to \mathbb{N}_0^3 : (\boldsymbol{z}, t) \mapsto E_P^t(\boldsymbol{z}). \tag{6.26}$$

But the set $L(P)$ is finite and so the characteristic function $\chi_{L(P)}$ is primitive recursive. Therefore, the following function is also primitive recursive:

$$E_P'' : \mathbb{N}_0^4 \to \mathbb{N}_0 : (\boldsymbol{z}, t) \mapsto (\chi_{L(P)} \circ \pi_2^{(3)} \circ E_P')(\boldsymbol{z}, t). \tag{6.27}$$

This function has the property that

$$E_P''(\boldsymbol{z}, t) = \begin{cases} 1 \text{ if } E_P^t(\boldsymbol{z}) = (u, v, j) \text{ and } j \in L(P), \\ 0 \text{ otherwise.} \end{cases} \tag{6.28}$$

By definition, $Z_P = \mu E_P''$ and thus the claim follows.

The *residual step function* of the GOTO program P is given by the partial function $R_P : \mathbb{N}_0^3 \to \mathbb{N}_0^3$ that maps an initial configuration to the final configuration of the computation, if any:

$$R_P(\boldsymbol{z}) = \begin{cases} E'_P(\boldsymbol{z}, Z_P(\boldsymbol{z})) & \text{if } \boldsymbol{z} \in \text{dom}(Z_P), \\ \uparrow & \text{otherwise.} \end{cases} \tag{6.29}$$

This function is also partial recursive, since for each $\boldsymbol{z} \in \mathbb{N}_0^3$, $R_P(\boldsymbol{z}) = E'_P(\boldsymbol{z}, (\mu E''_P)(\boldsymbol{z}))$.

Define the total functions

$$\alpha_k : \mathbb{N}_0^k \to \mathbb{N}_0^3 : (x_1, \ldots, x_k) \mapsto (J(x_1, \ldots, x_k), 1, 0) \tag{6.30}$$

and

$$\omega_1 : \mathbb{N}_0^3 \to \mathbb{N}_0 : (u, v, j) \mapsto J^{-1}(u). \tag{6.31}$$

Both functions are primitive recursive; α_k and ω_1 provide the initial configuration and the result of the computation, respectively. For each arity $k \in \mathbb{N}_0$, the Post-Turing program P provides the partial recursive function

$$\|P\|_{k,1} = \omega_1 \circ R_P \circ \alpha_k. \tag{6.32}$$

It follows that each Post-Turing computable function is partial recursive. Hence, the Theorems 3.13 and 6.2 yield the following.

Theorem 6.3. *The class of Post-Turing computable functions equals the class of partial recursive functions.*

7
Undecidability

In computability theory, undecidable problems refer to decision problems which are yes-or-no questions on an input set. An undecidable problem does not allow to construct a general algorithm that always leads to a correct yes-or-no answer. Prominent examples are the halting problem, the word problems in formal language theory and group theory, and Hilbert's tenth problem.

7.1 Undecidable Sets

A set of natural numbers is decidable, computable or recursive if there is an algorithm which terminates after a finite amount of time and correctly decides whether or not a given number belongs to the set. More formally, a set A in $\mathbb{N}_0^k$ is called *decidable* if its characteristic function χ_A is recursive. An algorithm for the computation of χ_A is called a *decision procedure* for A. A set A which is not decidable is called *undecidable*.

Example 7.1.

- Every finite set A of natural numbers is computable, since

$$\chi_A(x) = \text{sgn} \circ \sum_{a \in A} \chi_=(a, x), \quad x \in \mathbb{N}_0. \tag{7.1}$$

  In particular, the empty set is computable.
- The entire set of natural numbers is computable, since $\mathbb{N}_0 = \overline{\emptyset}$ (see Proposition 7.2).
- The set of prime numbers is computable (see Proposition 2.35).

$\diamondsuit$

Proposition 7.2.

- *If A is a decidable set, the complement of A is decidable.*
- *If A and B are decidable sets, the sets $A \cup B$, $A \cap B$, and $A \setminus B$ are decidable.*

Proof. We have

$$\begin{aligned}
\chi_{\overline{A}} &= \mathrm{csg} \circ \chi_A, && (7.2)\\
\chi_{A\cup B} &= \mathrm{sgn} \circ (\chi_A + \chi_B), && (7.3)\\
\chi_{A\cap B} &= \chi_A \cdot \chi_B, && (7.4)\\
\chi_{A\setminus B} &= \chi_A \cdot \chi_{\overline{B}}. && (7.5)
\end{aligned}$$

□

There are two general methods to prove that a set is undecidable. The first is *diagonalization* similar to Georg Cantor's (1845-1918) famous diagonalization proof showing the denumerability of the rational numbers, and the second is *reduction* of an already known undecidable problem to the problem under consideration.

Here is a first undecidable set. Note the proof of undecidability requires diagonalization due to the lack of another undecidable problem at hand.

Proposition 7.3. *The set* $K = \{x \in \mathbb{N}_0 \mid x \in \mathrm{dom}\,\phi_x\}$ *is undecidable.*

Proof. Assume the set K would be decidable; i.e., the function χ_K would be recursive. Then the function $f : \mathbb{N}_0 \to \mathbb{N}_0$ given by

$$f(x) = \begin{cases} 0 \text{ if } \chi_K(x) = 0, \\ \uparrow \text{ if } \chi_K(x) = 1, \end{cases} \tag{7.6}$$

is partial recursive. To see this, take the function $g : \mathbb{N}_0^2 \to \mathbb{N}_0$ defined by $g(x, y) = \chi_K(x)$. The function g is recursive and thus $f = \mu g$ is partial recursive. It follows that the function f has an index e, i.e., $f = \phi_e$. Then $e \in \mathrm{dom}\,\phi_e$ is equivalent to $f(e) = 0$, which in turn is equivalent to $e \notin K$, which means that $e \notin \mathrm{dom}\,\phi_e$ contradicting the hypothesis. □

Note that in opposition to the function f used in the proof, the function $h : \mathbb{N}_0 \to \mathbb{N}_0$ defined by

$$h(x) = \begin{cases} 0 \text{ if } \chi_K(x) = 1, \\ \uparrow \text{ if } \chi_K(x) = 0, \end{cases} \tag{7.7}$$

is partial recursive. To see this, observe that for each $x \in \mathbb{N}_0$,

$$h(x) = 0 \cdot \phi_x(x) = 0 \cdot \psi_{\mathrm{univ}}^{(1)}(x, x). \tag{7.8}$$

Moreover, the function $h' : \mathbb{N}_0 \to \mathbb{N}_0$ given by

$$h'(x) = \begin{cases} x \text{ if } \chi_K(x) = 1, \\ \uparrow \text{ if } \chi_K(x) = 0, \end{cases} \tag{7.9}$$

is partial recursive. Indeed, for each $x \in \mathbb{N}_0$,

$$h'(x) = x \cdot \mathrm{sgn}(\phi_x(x) + 1) = x \cdot \mathrm{sgn}(\phi_{\mathrm{univ}}^{(1)}(x, x) + 1). \tag{7.10}$$

It is interesting to note that the domain and range of the function h' are undecidable sets, since

$$\operatorname{dom} h' = \operatorname{ran} h' = K. \tag{7.11}$$

The *halting problem* is one of the famous undecidabilty results. It states that given a program and an input to the program, decide whether the program finishes or continues to run forever when run with that input. Alan Turing proved in 1936 that a general algorithm to solve the halting problem for all possible program-input pairs cannot exist. By Church's thesis, the halting problem is undecidable not only for Turing machines but for any formalism capturing the notion of computability.

Proposition 7.4. *The set* $H = \{(x, y) \in \mathbb{N}_0^2 \mid y \in \operatorname{dom} \phi_x\}$ *is undecidable.*

Proof. Suppose the characteristic function of H would be recursive. Then the function $g : \mathbb{N}_0 \to \mathbb{N}_0$ given by $x \mapsto \chi_H(x, x)$ is also recursive and equals the characteristic function of K, i.e., for all $x \in \mathbb{N}_0$, $g(x) = \chi_K(x)$. But the function χ_K is not recursive contradicting the hypothesis. □

We have used reduction in order to prove that the halting problem is undecidable. More generally, a subset A of $\mathbb{N}_0^k$ is said to be *reducible* to a subset B of $\mathbb{N}_0^l$ if there is a recursive function $f : \mathbb{N}_0^k \to \mathbb{N}_0^l$, called *reduction function*, such that

$$\boldsymbol{x} \in A \quad \Longleftrightarrow \quad f(\boldsymbol{x}) \in B, \quad \boldsymbol{x} \in \mathbb{N}_0^k. \tag{7.12}$$

This assertion is equivalent to

$$\chi_A(\boldsymbol{x}) = \chi_B(f(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathbb{N}_0^k. \tag{7.13}$$

This means that if B is decidable, A is also decidable; or by contraposition, if A is undecidable, B is also undecidable. For instance, in the proof of the halting problem, the set K has been reduced to the set H by the reduction function $f : x \mapsto (x, x)$.

The next undecidability result makes use of the smn theorem.

Proposition 7.5. *The set* $C = \{x \in \mathbb{N}_0 \mid \phi_x = c_0^{(1)}\}$ *is undecidable.*

Proof. Take the function $f : \mathbb{N}_0^2 \to \mathbb{N}_0$ given by

$$f(x, y) = \begin{cases} 0 \text{ if } \chi_K(x) = 1, \\ \uparrow \text{ if } \chi_K(x) = 0. \end{cases} \tag{7.14}$$

This function is partial recursive, since it can be written as $f = h \circ \pi_1^{(2)}$, where h is the function given in (7.7). By the smn theorem, there is a monadic recursive function g such that $f(x, y) = \phi_{g(x)}(y)$ for all $x, y \in \mathbb{N}_0$. Consider two cases:

- If $x \in K$, then $f(x, y) = 0$ for all $y \in \mathbb{N}_0$ and so $\phi_{g(x)}(y) = c_0^{(1)}(y)$ for all $y \in \mathbb{N}_0$. Hence, $g(x) \in C$.
- If $x \notin K$, then $f(x, y)$ is undefined for all $y \in \mathbb{N}_0$ and thus $\phi_{g(x)}(y)$ is undefined for all $y \in \mathbb{N}_0$. Hence, $g(x) \notin C$.

It follows that the recursive function g provides a reduction of the set K to the set C. But K is undecidable and so C is also undecidable. □

Proposition 7.6. *The set* $E = \{(x, y) \in \mathbb{N}_0^2 \mid \phi_x = \phi_y\}$ *is undecidable.*

Proof. Let c be an index for the function $c_0^{(1)}$; i.e., $\phi_c = c_0^{(1)}$. Define the function $f : \mathbb{N}_0 \to \mathbb{N}_0^2$ given by $f(x) = (x, c)$. This function is clearly primitive recursive. Moreover, for each $x \in \mathbb{N}_0$, $x \in C$ is equivalent to $\phi_x = c_0^{(1)}$ which in turn is equivalent to $f(x) \in E$. Thus the recursive function f provides a reduction of the set C to the set E. Since C is undecidable, it follows that E is undecidable. □

Proposition 7.7. *For each number $a \in \mathbb{N}_0$, the sets $I_a = \{x \in \mathbb{N}_0 \mid a \in \operatorname{dom} \phi_x\}$ and $O_a = \{x \in \mathbb{N}_0 \mid a \in \operatorname{ran} \phi_x\}$ are undecidable.*

Proof. Take the function $f : \mathbb{N}_0^2 \to \mathbb{N}_0$ defined by

$$f(x, y) = \begin{cases} y \text{ if } \chi_K(x) = 1, \\ \uparrow \text{ if } \chi_K(x) = 0. \end{cases} \tag{7.15}$$

This function is partial recursive, since it can be written as

$$f(x, y) = y \cdot \operatorname{sgn}(\phi_x(x) + 1) = y \cdot \operatorname{sgn}(\psi_{\text{univ}}^{(1)}(x, x) + 1). \tag{7.16}$$

By the smn theorem, there is a monadic recursive function g such that

$$f(x, y) = \phi_{g(x)}(y), \quad x, y \in \mathbb{N}_0. \tag{7.17}$$

Consider two cases:

- If $x \in K$, then $f(x, y) = y$ for all $y \in \mathbb{N}_0$ and thus $\operatorname{dom} \phi_{g(x)} = \mathbb{N}_0 = \operatorname{ran} \phi_{g(x)}$.
- If $x \notin K$, then $f(x, y)$ is undefined for all $y \in \mathbb{N}_0$ and so $\operatorname{dom} \phi_{g(x)} = \emptyset = \operatorname{ran} \phi_{g(x)}$.

It follows that for each $a \in \mathbb{N}_0$, $x \in K$ is equivalent to both, $g(x) \in I_a$ and $g(x) \in O_a$. Thus the recursive function g provides a simultaneous reduction of K to both, I_a and O_a. Since the set K is undecidable, the result follows. □

Proposition 7.8. *The set $T = \{x \in \mathbb{N}_0 \mid \phi_x$ is total$\}$ is undecidable.*

Proof. Assume that T would be decidable. Pick the function $f : \mathbb{N}_0 \to \mathbb{N}_0$ defined as

$$f(x) = \begin{cases} \phi_x(x) + 1 & \text{if } \chi_T(x) = 1, \\ 0 & \text{if } \chi_T(x) = 0. \end{cases} \tag{7.18}$$

This function is recursive, since $\phi_x(x)$ is only evaluated if ϕ_x is total. Thus there is an index $e \in T$ such that $f = \phi_e$. But then $\phi_e(e) = f(e) = \phi_e(e) + 1$ contradicting the hypothesis. □

The undecidability results established so far have a number of practical implications, which will be briefly summarized using the formalism of GOTO programs:

- The problem whether an GOTO program halts with a given input is undecidable.
- The problem whether an GOTO program computes a specific function (here $c_0^{(1)}$) is undecidable.
- The problem whether two GOTO programs and are semantically equivalent, i.e., exhibit the same input-output behaviour, is undecidable.
- The problem whether an GOTO program halts for a specific input is undecidable.
- The problem whether an GOTO program always halts is undecidable.

These undecidability results refer to the input-output behaviour or the semantics of GOTO programs. The following result of Henry Gordon Rice (born 1920) is a milestone in computability theory. It states that for any non-trivial property of partial recursive functions, there is no general and effective method to decide whether an algorithm computes a partial recursive function with that property. Here a property of partial recursive functions is called *trivial* if it holds for all partial recursive functions or for none of them.

Theorem 7.9. *(Rice, 1953) If $\mathcal{A}$ is a proper subclass of monadic partial recursive functions, the corresponding index set*

$$\mathrm{prog}(\mathcal{A}) = \{x \in \mathbb{N}_0 \mid \phi_x \in \mathcal{A}\} \tag{7.19}$$

is undecidable.

By Example 7.1, if $\mathcal{A}$ is a class of monadic partial recursive functions, $\mathrm{prog}(\mathcal{A})$ is decidable if and only if $\mathcal{A}$ is either empty or consists of all partial recursive functions.

Proof. By Proposition 7.2, if a set is decidable, its complement is also decidable. Therefore, it may be assumed that the nowhere defined function $f_\uparrow$ does not belong to $\mathcal{A}$. Take any function $f \in \mathcal{A}$ and define the function $h : \mathbb{N}_0^2 \to \mathbb{N}_0$ as follows:

$$h(x,y) = \begin{cases} f(y) & \text{if } \chi_K(x) = 1, \\ \uparrow & \text{if } \chi_K(x) = 0. \end{cases} \tag{7.20}$$

This function is partial recursive, since

$$h(x,y) = f(y) \cdot \mathrm{sgn}(\phi_x(x) + 1) = f(y) \cdot \mathrm{sgn}(\psi_{\mathrm{univ}}^{(1)}(x,x) + 1). \tag{7.21}$$

Therefore by the smn theorem, there is a monadic recursive function g such that

$$h(x,y) = \phi_{g(x)}(y), \quad x, y \in \mathbb{N}_0 \tag{7.22}$$

Consider two cases:

- If $x \in K$, then $h(x,y) = f(y)$ for all $y \in \mathbb{N}_0$ and thus $\phi_{g(x)} = f$. Hence, $g(x) \in \mathrm{prog}(\mathcal{A})$.
- If $x \notin K$, then $h(x,y)$ is undefined for all $y \in \mathbb{N}_0$ and so $\phi_{g(x)} = f_\uparrow$. Hence, by hypothesis, $g(x) \notin \mathrm{prog}(\mathcal{A})$.

Therefore, the recursive function g reduces the set K to the set $\mathrm{prog}(\mathcal{A})$. Since the set K is undecidable, the result follows. ∎

7.2 Semidecidable Sets

A set A of natural numbers is called computably enumerable, semidecidable or provable if there is an algorithm such that the set of input numbers for which the algorithm halts is exactly the set of numbers in A. More generally, a subset A of $\mathbb{N}_0^k$ is called *semidecidable* if the function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ defined by

$$f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in A, \\ \uparrow & \text{otherwise}, \end{cases} \tag{7.23}$$

is partial recursive.

Proposition 7.10. *A subset A of $\mathbb{N}_0^k$ is semidecidable if and only if the set A is the domain of a k-ary partial recursive function.*

Proof. Let A be semidecidable. Then the corresponding function f given in (7.23) has the property that $\operatorname{dom} f = A$. Conversely, let A be a subset of $\mathbb{N}_0^k$ for which there is a partial computable function $h : \mathbb{N}_0^k \to \mathbb{N}_0$ with the property that $\operatorname{dom} h = A$. Then the function $f = \nu \circ c_0^{(1)} \circ h$ is also partial recursive and coincides with function in (7.23). Hence the set A is semidecidable. □

Example 7.11. The prototype set K is semidecidable as it is the domain of the partial recursive function in (7.7). ♢

A program for the function f given in (7.23) provides a *partial decision procedure* for A:

Given $\boldsymbol{x} \in \mathbb{N}_0^k$. If $\boldsymbol{x} \in A$, the program started with input $\boldsymbol{x}$ will halt giving a positive answer. Otherwise, the program will not terminate in a finite number of steps.

Proposition 7.12. *Each decidable set is semidecidable.*

Proof. Let A be a decidable subset of $\mathbb{N}_0^k$. Then the function $g : \mathbb{N}_0^{k+1} \to \mathbb{N}_0$ defined by

$$g(\boldsymbol{x}, y) = (\mathrm{csg} \circ \chi_A)(\boldsymbol{x}) = \begin{cases} 0 \text{ if } \boldsymbol{x} \in A, \\ 1 \text{ otherwise,} \end{cases} \tag{7.24}$$

is recursive. Thus the function $f = \mu g$ is partial recursive. It has the property that $\mu g(\boldsymbol{x}) = y$ if $g(\boldsymbol{x}, y) = 0$ and $g(\boldsymbol{x}, i) \neq 0$ for all $0 \leq i < y$, and $\mu g(\boldsymbol{x})$ is undefined otherwise. It follows that $\mu g(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in A$ and $\mu g(\boldsymbol{x})$ is undefined otherwise. Hence, $A = \operatorname{dom} f$ as required. □

Proposition 7.13. *The halting problem is semidecidable.*

Proof. Consider the corresponding set H. The universal function $\psi_{\mathrm{univ}}^{(1)}$ has the property

$$\psi_{\mathrm{univ}}^{(1)}(x, y) = \begin{cases} \phi_x(y) & \text{if } y \in \operatorname{dom} \phi_x, \\ \uparrow & \text{otherwise.} \end{cases} \tag{7.25}$$

It follows that $H = \operatorname{dom} \psi_{\mathrm{univ}}^{(1)}$ as required. □

Proposition 7.14. *Let A be a subset of $\mathbb{N}_0^k$. If A is reducible to a semidecidable set, then A is semidecidable.*

Proof. Suppose A is reducible to a semidecidable subset B of $\mathbb{N}_0^l$. Then there is a recursive function $f : \mathbb{N}_0^k \to \mathbb{N}_0^l$ such that $\boldsymbol{x} \in A$ if and only if $f(\boldsymbol{x}) \in B$. Moreover, there is a partial recursive function $g : \mathbb{N}_0^l \to \mathbb{N}_0$ such that $B = \operatorname{dom} g$. Thus the composite function $g \circ f : \mathbb{N}_0^k \to \mathbb{N}_0$ is partial recursive. Furthermore, for each $\boldsymbol{x} \in \mathbb{N}_0^k$, $\boldsymbol{x} \in A$ is equivalent to $f(\boldsymbol{x}) \in B$ which in turn is equivalent that $g(f(\boldsymbol{x}))$ is defined. Hence, $A = \operatorname{dom} g \circ f$ as required. □

The next assertion states that each semidecidable set results from a decidable one by unbounded existential quantification. That is, a partial decision procedure can be formulated as an unbounded search to satisfy a decidable relation.

Proposition 7.15. *A set A is semidecidable if and only if there is a decidable set B such that $A = \exists y[(\boldsymbol{x}, y) \in B]$.*

Proof. Let B be a decidable subset of $\mathbb{N}_0^{k+1}$ and $A = \exists y[(\boldsymbol{x}, y) \in B]$. Consider the function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ given by

$$f(\boldsymbol{x}) = \mu(\mathrm{csg} \circ \chi_B)(\boldsymbol{x}) = \begin{cases} 0 \text{ if } (\boldsymbol{x}, y) \in B \text{ for some } y \in \mathbb{N}_0, \\ \uparrow \text{otherwise.} \end{cases} \tag{7.26}$$

This function is partial recursive and has the property that $\mathrm{dom}\, f = A$.

Conversely, let A be a semidecidable subset of $\mathbb{N}_0^k$. Then there is an index e such that $\mathrm{dom}\, \phi_e^{(k)} = A$. By Kleene's normal form theorem, an element $\boldsymbol{x} \in \mathbb{N}_0^k$ satisfies $\boldsymbol{x} \in A$ if and only if $\boldsymbol{x} \in \exists y[(e, \boldsymbol{x}, y) \in T_k]$, where the index e is kept fixed. Hence, $A = \exists y[(e, \boldsymbol{x}, y) \in T_k]$ as required. □

The next result shows that the class of semidecidable sets is closed under unbounded existential minimalization.

Proposition 7.16. *If B is semidecidable, then $A = \exists y[(\boldsymbol{x}, y) \in B]$ is semidecidable.*

Proof. Let B be a semidecidable subset of $\mathbb{N}_0^{k+1}$. By Proposition 7.15, there is a decidable subset C of $\mathbb{N}_0^{k+2}$ such that $B = \exists z[(\boldsymbol{x}, y, z) \in C]$. But the search for a pair (y, z) of numbers with $(\boldsymbol{x}, y, z) \in C$ can be replaced by the search for a number u such that $(\boldsymbol{x}, K_2(u), L_2(u)) \in C$. It follows that $A = \exists u[(\boldsymbol{x}, K_2(u), L_2(u)) \in C]$. Thus by Proposition 7.15, the set A is semidecidable. □

It follows that the class of semidecidable sets is closed under existential quantification. This is not true for the class of decidable sets. To see this, take the Kleene predicate T_1 which is primitive recursive by the Kleene normal form theorem. The prototype set K which is only semidecidable results from T_1 by existential quantification as follows:

$$K = \exists y[(x, x, y) \in T_1]. \tag{7.27}$$

Another useful connection between decidable and semidecidable sets is the following.

Proposition 7.17. *A set A is decidable if and only if A and $\overline{A}$ are semidecidable.*

Proof. If A is decidable, then by Proposition 7.2, the set $\overline{A}$ is also decidable. But each decidable set is semidecidable and so A and $\overline{A}$ are semidecidable.

Conversely, if A and $\overline{A}$ are semidecidable, a decision procedure for A can be established by using the partial decision procedures for A and $\overline{A}$ that are now simultaneously applied. One of these procedures will provide a positive answer in a finite number of steps giving an answer to the decision procedure for A. □

Example 7.18. The complement of the halting problem is given by the set

$$\overline{H} = \{(x, y) \in \mathbb{N}_0^2 \mid y \notin \mathrm{dom}\, \phi_x\}. \tag{7.28}$$

This set is not semidecidable, since the set H is semidecidable but not decidable. ♢

Recall that the *graph* of a partial function $f : \mathbb{N}_0^n \to \mathbb{N}_0$ is given as

$$\operatorname{graph}(f) = \{(\boldsymbol{x}, y) \in \mathbb{N}_0^{n+1} \mid f(\boldsymbol{x}) = y\}. \tag{7.29}$$

Proposition 7.19. *A function $f : \mathbb{N}_0^n \to \mathbb{N}_0$ is partial recursive if and only if the graph of f is semidecidable.*

Proof. Suppose the function f is partially recursive. Then there is an index e for f, i.e., $f = \phi_e^{(n)}$. The Kleene set S_n used to derive Kleene's normal form shows that $f(\boldsymbol{x}) = y$ is equivalent to $(\boldsymbol{x}, y) \in \exists t[(e, \boldsymbol{x}, y, t) \in S_n]$. Thus the set graph$(f)$ is obtained form the decidable set S_n with e being fixed by existential quantification and so is semidecidable.

Conversely, let the set graph(f) be semidecidable. Then there is a decidable set $A \subseteq \mathbb{N}_0^{n+2}$ such that graph(f) has the form $\exists z[(\boldsymbol{x}, y, z) \in A]$. To compute the function f, take an argument $\boldsymbol{x} \in \mathbb{N}_0^n$ and systematically search for a pair $(y, z) \in \mathbb{N}_0^2$ such that $(\boldsymbol{x}, y, z) \in A$, say by listing the elements of $\mathbb{N}_0^2$ as in (2.37). If such a pair exists, put $f(\boldsymbol{x}) = y$. Otherwise, $f(\boldsymbol{x})$ is undefined. ☐

7.3 Recursively Enumerable Sets

In this section, we restrict our attention to sets of natural numbers. To this end, the terminology will be changed a bit. A set A of natural numbers is called *recursive* if its characteristic function χ_A is recursive, and a set A of natural numbers is called *recursively enumerable* (*r.e.* for short) if its characteristic function χ_A is partial recursive.

The first result provides the relationship between decidable and recursive sets as well as semidecidable and recursively enumerable sets.

Proposition 7.20. *Let $J_k : \mathbb{N}_0^k \to \mathbb{N}_0$ be a primitive recursive bijection.*

- *A subset A of $\mathbb{N}_0^k$ is decidable if and only if $J_k(A)$ is recursive.*
- *A subset A of $\mathbb{N}_0^k$ is semidecidable if and only if $J_k(A)$ is recursively enumerable.*

Proof. Let $J_k(A)$ be recursive. Then for each number $x \in \mathbb{N}_0$,

$$\begin{aligned}
\chi_{J_k(A)}(x) &= \begin{cases} 1 \text{ if } x \in J_k(A), \\ 0 \text{ otherwise,} \end{cases} \\
&= \begin{cases} 1 \text{ if } \exists \boldsymbol{a} \in \mathbb{N}_0^k : \ J_k(\boldsymbol{a}) = x \wedge \chi_A(\boldsymbol{a}) = 1, \\ 0 \text{ otherwise,} \end{cases} \\
&= \begin{cases} 1 \text{ if } \chi_A \circ J_k^{-1}(x) = 1, \\ 0 \text{ otherwise.} \end{cases}
\end{aligned}$$

Thus $\chi_{J_k(A)} = \chi_A \circ J_k^{-1}$ and hence χ_A is recursive. Conversely, if A is recursive, then $\chi_A = \chi_{J_k(A)} \circ J_k$ and so $\chi_{J_k(A)}$ is recursive.

Second, if A is semidecidable given by the domain of a partial recursive function $f : \mathbb{N}_0^k \to \mathbb{N}_0$, the function $g = f \circ J_k^{-1}$ is partial recursive and has the domain $J_k(A)$. Conversely, if $J_k(A)$ is semidecidable defined by the domain of a partial recursive function $g : \mathbb{N}_0 \to \mathbb{N}_0$, the function $f = g \circ J_k$ is partial recursive with domain A. ☐

It follows that it is sufficient to consider subsets of $\mathbb{N}_0$ instead of subsets of $\mathbb{N}_0^k$ for any $k \geq 1$. Next, closure properties of recursive sets are studied.

Proposition 7.21. *If A and B are recursive sets, the sets $\overline{A}$, $A \cup B$, $A \cap B$, and $A \setminus B$ are recursive.*

Proof. Let the functions χ_A and χ_B be recursive. Then the functions $\chi_{\overline{A}} = \mathrm{csg} \circ \chi_A$, $\chi_{A \cup B} = \mathrm{sgn} \circ (\chi_A + \chi_B)$, $\chi_{A \cap B} = \chi_A \cdot \chi_B$, and $\chi_{A \setminus B} = \chi_A \cdot \chi_{\overline{B}}$ are also recursive. □

The Gödel numbering of GOTO programs yields an enumeration of all monadic partial recursive functions

$$\phi_0, \phi_1, \phi_2, \ldots \tag{7.30}$$

By taking the domains of these functions, i.e., $D_e = \mathrm{dom}\, \phi_e$, this list provides an enumeration of all recursively enumerable sets

$$D_0, D_1, D_2, \ldots \tag{7.31}$$

Let A be a recursively enumerable set. Then there is a Gödel number $e \in \mathbb{N}_0$ such that $D_e = A$. The number e is called an *index* for A.

Proposition 7.22. *For each set A of natural numbers, the following assertions are equivalent:*

- *A is recursively enumerable.*
- *$A = \emptyset$ or there is a monadic recursive function f where $A = \mathrm{ran}\, f$.*
- *There is a k-ary partial recursive function g where $A = \mathrm{ran}\, g$.*

Proof.

- First, let A be a non-empty recursively enumerable set. By the above discussion, the set A has an index e. Fix an element $a \in A$ and use Kleene's normal form theorem to define the monadic function
$$f : x \mapsto \begin{cases} K_2(x) & \text{if } (e, K_2(x), L_2(x)) \in T_1, \\ a & \text{otherwise.} \end{cases} \tag{7.32}$$
This function is primitive recursive and since $A = \mathrm{dom}\, \phi_e$ it follows that $A = \mathrm{ran}\, f$ as required.
- Second, let $A = \emptyset$ or $A = \mathrm{ran}\, f$ for some monadic recursive function f. Define the partial recursive function g such that g is the nowhere-defined function if $A = \emptyset$, and $g = f$ if $A \neq \emptyset$. Then $A = \mathrm{ran}\, g$ as required.
- Third, let g be a k-ary partial recursive function with $A = \mathrm{ran}\, g$. By Proposition 7.19, the set $B = \{(\boldsymbol{x}, y) \in \mathbb{N}_0^{k+1} \mid g(\boldsymbol{x}) = y\}$ is semidecidable and thus by Proposition 7.16, the set $A = \exists x_1 \ldots \exists x_k[(x_1, \ldots, x_k, y) \in B]$ is recursively enumerable. □

This result shows that a non-empty set of natural numbers A is recursively enumerable if and only if there is a monadic recursive function f that allows to enumerate the elements of A, i.e.,

$$A = \{f(0), f(1), f(2), \ldots\}. \tag{7.33}$$

Such a function f is called an *enumerator* for A. As can be seen from the proof, the enumerators can be chosen to be primitive recursive.

Proposition 7.23. *If A and B are recursively enumerable sets, the sets $A \cap B$ and $A \cup B$ are also recursively enumerable.*

Proof. First, let f and g be monadic partial recursive functions where $A = \mathrm{dom}\, f$ and $B = \mathrm{dom}\, g$. Then $f \cdot g$ is partial recursive with the property that $\mathrm{dom}\, f \cdot g = A \cap B$.

Second, let A and B be non-empty sets, and let f and g be monadic recursive functions where $A = \mathrm{ran}\, f$ and $B = \mathrm{ran}\, g$. Define the monadic function h as follows,

$$h : x \mapsto \begin{cases} f(\lfloor x/2 \rfloor) & \text{if } x \text{ is even,} \\ g(\lfloor x/2 \rfloor) & \text{otherwise.} \end{cases} \tag{7.34}$$

Thus $h(0) = f(0)$, $h(1) = g(0)$, $h(2) = f(1)$, $h(3) = g(1)$ and so on. The function h defined by cases is recursive and satisfies $\mathrm{ran}\, h = A \cup B$.

□

Proposition 7.24. *An infinite set A of natural numbers is recursive if and only if the set A has a strictly monotonous enumerator, i.e., $A = \mathrm{ran}\, f$ with $f(0) < f(1) < f(2) < \ldots$.*

Proof. Let A be an infinite and recursive set. Define the monadic function f by minimalization and primitive recursion as follows:

$$f(0) = \mu y[y \in A], \tag{7.35}$$

$$f(n+1) = \mu y[y \in A \wedge y > f(n)]. \tag{7.36}$$

This function is recursive, strictly monotonous and satisfies $\mathrm{ran}\, f = A$.

Conversely, let f be a strictly monotonous monadic recursive function where $A = \mathrm{ran}\, f$ is infinite. Then $f(n) = y$ implies $y \geq n$ and thus

$$y \in A \quad \Longleftrightarrow \quad \exists n[n \leq y \wedge f(n) = y]. \tag{7.37}$$

The relation on the right-hand side is decidable and so A is recursive. □

Corollary 7.25. *Each infinite recursively enumerable set contains an infinite recursive subset.*

7.4 Theorem of Rice-Shapiro

The theorem of Rice provides a class of sets that are undecidable. Now we present a similar result for recursively enumerable sets called the theorem of Rice-Shapiro, which was posed by Henry Gordan Rice and proved by Norman Shapiro (born 1932).

For this, a monadic function g is called an *extension* of an monadic function f, written $f \subseteq g$, if $\mathrm{dom}\, f \subseteq \mathrm{dom}\, g$ and $f(x) = g(x)$ for all $x \in \mathrm{dom}\, f$. The relation of extension is an order relation on the set of all monadic functions with smallest element given by the nowhere-defined function. The maximal elements are the monadic recursive functions.

A monadic function f is called *finite* if its domain is finite. Each finite function f is partial recursive, since

$$f(x) = \sum_{a \in \mathrm{dom}\, f} \mathrm{csg}(|x - a|) f(a), \quad x \in \mathbb{N}_0. \tag{7.38}$$

Theorem 7.26. *(Rice-Shapiro) Let $\mathcal{A}$ be a class of monadic partial recursive functions whose corresponding index set* $\text{prog}(\mathcal{A}) = \{x \in \mathbb{N}_0 \mid \phi_x \in \mathcal{A}\}$ *is recursively enumerable. Then a monadic partial recursive function f lies in $\mathcal{A}$ if and only if there is a finite function $g \in \mathcal{A}$ such that $g \subseteq f$.*

Proof. First, let $f \in \mathcal{A}$ and assume that no finite function g which is extended by f lies in $\mathcal{A}$. Take the recursively enumerable set $K = \{x \mid x \in \text{dom}\,\phi_x\}$, let e be an index for K, and let P_e be a GOTO program that computes ϕ_e. Define the function

$$g : (z,t) \mapsto \begin{cases} \uparrow & \text{if } P_e \text{ computes } \phi_e(z) \text{ in} \leq t \text{ steps,} \\ f(t) & \text{otherwise.} \end{cases} \tag{7.39}$$

The function g is partial recursive. Thus by the smn theorem, there is a monadic recursive function s such that

$$g(z,t) = \phi_{s(z)}(t), \quad t, z \in \mathbb{N}_0. \tag{7.40}$$

Hence, $\phi_{s(z)} \subseteq f$ for each $z \in \mathbb{N}_0$. Consider two cases:

- If $z \in K$, the program $P_e(z)$ halts after, say t_0 steps. Then

$$\phi_{s(z)}(t) = \begin{cases} \uparrow & \text{if } t_0 \leq t, \\ f(t) & \text{otherwise.} \end{cases} \tag{7.41}$$

  Thus $\phi_{s(z)}$ is finite and hence, by hypothesis, $\phi_{s(z)}$ does not belong to $\mathcal{A}$.
- If $z \notin K$, the program $P_e(z)$ does not halt and so $\phi_{s(z)} = f$ which implies that $\phi_{s(z)} \in \mathcal{A}$.

It follows that the function s reduces the non-recursively enumerable set $\overline{K}$ to the set $\text{prog}(\mathcal{A})$ contradicting the assumption that $\text{prog}(\mathcal{A})$ is recursively enumerable.

Conversely, let f be a monadic partial recursive function that does not belong to $\mathcal{A}$ and let g be a finite function in $\mathcal{A}$ with $g \subseteq f$. Define the function

$$h : (z,t) \mapsto \begin{cases} f(t) & \text{if } t \in \text{dom}\,g \text{ or } z \in K, \\ \uparrow & \text{otherwise.} \end{cases} \tag{7.42}$$

The function h is partial recursive. Thus by the smn theorem, there is a monadic recursive function s such that

$$h(z,t) = \phi_{s(z)}(t), \quad t, z \in \mathbb{N}_0. \tag{7.43}$$

Consider two cases:

- If $z \in K$, $\phi_{s(z)} = f$ and so $\phi_{s(z)} \notin \mathcal{A}$.
- If $z \notin K$, $\phi_{s(z)}(t) = f(t) = g(t)$ for all $t \in \text{dom}\,g$ and $\phi_{s(z)}$ is undefined elsewhere. Hence, $\phi_{s(z)} \in \mathcal{A}$.

It follows that the function s provides a reduction of the non-recursively enumerable set $\overline{K}$ to the set $\text{prog}(\mathcal{A})$ contradicting the hypothesis that $\text{prog}(\mathcal{A})$ is recursively enumerable. $\square$

Corollary 7.27. *Let* $\text{prog}(\mathcal{A}) = \{x \in \mathbb{N}_0 \mid \phi_x \in \mathcal{A}\}$ *be recursively enumerable. Then any extension of a function in $\mathcal{A}$ lies itself in $\mathcal{A}$.*

Proof. Let h be an extension of a function $f \in \mathcal{A}$. By the theorem of Rice-Shapiro, there is a finite function g which extends f. But then also h extends g and so it follows by the theorem of Rice-Shapiro that h lies in $\mathcal{A}$. □

Corollary 7.28. *Let* $\operatorname{prog}(\mathcal{A}) = \{x \in \mathbb{N}_0 \mid \phi_x \in \mathcal{A}\}$ *be recursively enumerable. If the nowhere-defined function is in* $\mathcal{A}$*, all monadic partial recursive functions lie in* $\mathcal{A}$*.*

Proof. Each monadic computable function extends the nowhere-defined function and so by the theorem of Rice-Shaprio lies in $\mathcal{A}$. □

Note that the theorem of Rice is a consequence of the theorem of Rice-Shapiro. To see this, let $\mathcal{A}$ be a set of monadic partial recursive functions. Suppose the set $\operatorname{prog}(\mathcal{A})$ is decidable. Then both, $\operatorname{prog}(\mathcal{A})$ and its complement are recursive enumerable. Thus we may assume that the set $\mathcal{A}$ contains the nowhere-defined function. By the theorem of Rice-Shapiro, it follows that $\mathcal{A}$ is not proper.

Example 7.29. The set $\mathcal{A} = \{\phi_x \mid \phi_x \text{ bijective}\}$ is not recursively enumerable. Indeed, suppose $\operatorname{prog}(\mathcal{A})$ would be recursive enumerable. Then by the theorem of Rice-Shapiro, the set $\operatorname{prog}(\mathcal{A})$ would contain a finite function. But finite functions are not bijective and so cannot belong to $\mathcal{A}$. A contradiction. ♢

7.5 Diophantine Sets

David Hilbert (1982-1943) presented a list of 23 mathematical problems at the International Mathematical Congress in Paris in 1900. The tenth problem can be stated as follows:

> Given a diophantine equation with a finite number of unknowns and with integral coefficients. Devise a procedure that determines in a finite number of steps whether the equation is solvable in integers.

In 1970, a result in mathematical logic known as Matiyasevich's theorem settled the problem negatively.

Let $\mathbb{Z}[X_1, X_2, \ldots, X_n]$ denote the commutative polynomial ring in the unknowns $X_1, X_2, \ldots, X_n$ with integer coefficients. Each polynomial p in $\mathbb{Z}[X_1, X_2, \ldots, X_n]$ gives rise to a *diophantine equation*

$$p(X_1, X_2, \ldots, X_n) = 0 \tag{7.44}$$

asking for *integer* solutions of this equation. By the *fundamental theorem of algebra*, every non-constant single-variable polynomial with complex coefficients has at least one complex root, or equivalently, the field of complex numbers is algebraically closed.

Example 7.30. Linear diophantine equations have the form $a_1X_1 + \ldots + a_nX_n = b$. If b is the greatest common divisor of $a_1, \ldots, a_n$ (or a multiple of it), the equation has an infinite number of solutions. This is Bezout's theorem and the solutions can be found by applying the extended Euclidean algorithm. On the other hand, if b is not a multiple of the greatest common divisor of $a_1, \ldots, a_n$, the diophantine equation has no solution. ♢

Let p be a polynomial in $\mathbb{Z}[X_1, \ldots, X_n]$. The *natural variety* of p is the zero set of p in $\mathbb{N}_0^n$,

$$V(p) = \{(x_1, \ldots, x_n) \in \mathbb{N}_0^n \mid p(x_1, \ldots, x_n) = 0\}. \tag{7.45}$$

Each polynomial function is defined by composition of addition and multiplication of integers and so leads the following result. Here we assume that addition and multiplication of integers are computable functions.

Proposition 7.31. *Each natural variety is a decidable set.*

A *diophantine set* results from a natural variety by existential quantification. More specifically, let p be a polynomial in $\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$. A diophantine set is an n-ary relation

$$\{(x_1, \ldots, x_n) \in \mathbb{N}_0^n \mid p(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \text{ for some } y_1, \ldots, y_m \in \mathbb{N}_0\}, \tag{7.46}$$

which will subsequently be denoted by

$$\exists y_1 \ldots \exists y_m [p(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0]. \tag{7.47}$$

Proposition 7.15 yields the following assertion.

Proposition 7.32. *Each diophantine set is semidecidable.*

Example 7.33.

- The set of positive integers is diophantine, since it is given by $\{x \mid \exists y[x = y + 1]\}$.
- The predicates $\leq$ and $<$ are diophantine, since $x \leq y$ if and only if $\exists z[y = x + z]$, and $x < y$ if and only if $\exists z[y = x + z + 1]$.
- The predicate $a \equiv b \bmod c$ is diophantine, since it can be written as $\exists x[(a - b)^2 = c^2 x^2]$. ♢

The converse of the above proposition shown by Yuri Matiyasevich (born 1947) in 1970 is also valid, but the proof is not constructive.

Theorem 7.34. *Each semidecidable set is diophantine.*

That is, for each semidecidable set A in $\mathbb{N}_0^n$ there is a polynomial p in $\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ such that

$$A = \exists y_1 \ldots \exists y_m [p(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0]. \tag{7.48}$$

The negative solution of Hilbert's tenth problem can be proved by using the four-square theorem due to Joseph-Louis Lagrange (1736-1813). For this, an identity due to Leonhard Euler (1707-1783) is needed which is proved by multiplying out and checking:

$$\begin{aligned}&(a^2 + b^2 + c^2 + d^2)(t^2 + u^2 + v^2 + w^2) = \\ &\quad (at + bu + cv + dw)^2 + (au - bt + cw - dv)^2 + (av - ct - bw + du)^2 + (aw - dt + bv - cu)^2.\end{aligned} \tag{7.49}$$

This equation implies that the set of numbers which are the sum of four squares is closed under multiplication.

Theorem 7.35. *(Lagrange, 1770) Each natural number can be written as a sum of four squares.*

For instance, $3 = 1^2 + 1^2 + 1^2 + 0^2$, $14 = 3^2 + 2^2 + 1^2 + 0^2$, and $39 = 5^2 + 3^2 + 2^2 + 1^2$.

Proof. By the above remark it is enough to show that all primes are the sum of four squares. Since $2 = 1^2 + 1^2 + 0^2 + 0^2$, the result is true for 2.

Let p be an odd prime. First, claim that there is some number m with $0 < m < p$ such that mp is a sum of four squares. Indeed, consider the $p + 1$ numbers a^2 and $-1 - b^2$ where $0 \leq a, b \leq (p - 1)/2$. Two of these numbers must have the same remainder when divided by p. However, a^2 and c^2 have the same remainder when divided by p if and only if p divides $a^2 - c^2$; that is, p divides $a + c$ or $a - c$. Thus the numbers a^2 must all have different remainders. Similarly, the numbers $-1 - b^2$ have different remainders. It follows that there must be a and b such that a^2 and $-1 - b^2$ have the same remainder when divided by p; equivalently, $a^2 + b^2 + 1^2 + 0^2$ is divisible by p. But a and b are at most $(p - 1)/2$ and so $a^2 + b^2 + 1^2 + 0^2$ has the form mp, where $0 < m < p$. This proves the claim.

Second, claim that if mp is the sum of four squares with $1 < m < p$, there is a number n with $1 \leq n < m$ such that np is also the sum of four squares. Indeed, let $mp = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and suppose first that m is even. Then either each x_i is even, or they are all odd, or exactly two of them are even. In the last case, it may be assumed that x_1 and x_2 are even. In all three cases each of $x_1 \pm x_2$ and $x_3 \pm x_4$ are even. So $(m/2)p$ can be written as $((x_1 + x_2)/2)^2 + ((x_1 - x_2)/2)^2 + ((x_3 + x_4)/2)^2 + ((x_3 - x_4)/2)^2$, as required.

Next let m be odd. Define numbers y_i by $x_i \equiv y_i \bmod m$ and $|y_i| < m/2$. Then $y_1^2 + y_2^2 + y_3^2 + y_4^2 \equiv x_1^2 + x_2^2 + x_3^2 - x_4^2 \bmod m$ and so $y_1^2 + y_2^2 + y_3^2 + y_4^2 = nm$ for some number $n \geq 0$. The case $n = 0$ is impossible, since this would make every y_i zero and so would make every x_i divisible by m. But then mp would be divisible by m^2, which is impossible since p is a prime and $1 < m < p$.

Clearly, $n < m$ since each y_i is less than $m/2$. Note that $m^2np = (x_1^2+x_2^2+x_3^2-x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)$. Use Euler's identity to write m^2np as a sum of four squares. Claim that each integer involved in this representation is divisibe by m. Indeed, one of the involved squares is $(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2$. But the sum $x_1y_1+x_2y_2+x_3y_3+x_4y_4$ is congruent mod m to $y_1^2+y_2^2+y_3^2+y_4^2$, since $x_i \equiv y_i \bmod m$. However, $y_1^2 + y_2^2 + y_3^2 + y_4^2 \equiv 0 \bmod m$, as needed. Similarly, the other three integers involved are divisible by m. Now m^2np is the sum of four squares each of which is divisible by m^2.p It follows that np itself is the sum of four squares, as required.

Finally, the process to write a multiple mp of p as a sum of four primes iterates leading to smaller multiples np of p. This process will end by reaching p. ☐

Let p be a polynomial in $\mathbb{Z}[X_1, \ldots, X_n]$. Define the integral polynomial q in the unknowns $T_1, \ldots, T_n$, $U_1, \ldots, U_n$, $V_1 \ldots, V_n$, $W_1, \ldots, W_n$ such that

$$\begin{aligned} &q(T_1, T_2, \ldots, T_n, U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n, W_1, W_2, \ldots, W_n) = \\ &\quad p(T_1^2 + U_1^2 + V_1^2 + W_1^2, T_2^2 + U_2^2 + V_2^2 + W_2^2, \ldots, T_n^2 + U_n^2 + V_n^2 + W_n^2). \end{aligned} \tag{7.50}$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{N}_0^n$ be a solution of the diophantine equation

$$p(X_1, \ldots, X_n) = 0 \tag{7.51}$$

and let $\boldsymbol{t} = (t_1, \ldots, t_n)$, $\boldsymbol{u} = (u_1, \ldots, u_n)$, $\boldsymbol{v} = (v_1, \ldots, v_n)$, $\boldsymbol{w} = (w_1, \ldots, w_n)$ be elements of $\mathbb{Z}^n$ such that by the theorem of Lagrange,

$$x_i = t_i^2 + u_i^2 + v_i^2 + w_i^2, \quad 1 \leq i \leq n. \tag{7.52}$$

Then $(\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \mathbb{Z}^{4n}$ is a solution of the diophantine equation

$$q(T_1, \ldots, T_n, U_1, \ldots, U_n, V_1, \ldots, V_n, W_1, \ldots, W_n) = 0. \tag{7.53}$$

Conversely, if $(\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \mathbb{Z}^{4n}$ is a solution of the diophantine equation (7.53), then $\boldsymbol{x} \in \mathbb{N}_0^n$ defined as in (7.52) provides an integer solution of the diophantine equation (7.51).

It follows that if there is an effective procedure to decide whether the diophantine equation (7.53) has *integer* solutions, there is one to decide if the diophantine equation (7.51) has *non-negative integer* solutions.

Theorem 7.36. *Hilbert's tenth problem is undecidable.*

Proof. The set $K = \{x \mid x \in \operatorname{dom} \phi_x\}$ is recursively enumerable and so by Matiyasevich's result there is a polynomial p in $\mathbb{Z}[X, Y_1, \ldots, Y_m]$ such that

$$K = \exists y_1 \ldots \exists y_m [p(x, y_1, \ldots, y_m) = 0]. \tag{7.54}$$

Suppose there is an effective procedure to decide whether or not a diophantine equation has non-negative integer solutions. Then the question whether a number $x \in \mathbb{N}_0$ lies in K or not can be decided by finding a non-negative integer solution of the equation $p(x, Y_1, \ldots, Y_m) = 0$. However, this contradicts the undecidability of K. $\square$

8

Word Problems

The undecidability of the halting problem has many consequences not only in computability theory but also in other branches of science. The word problems encountered in abstract algebra and formal language theory belong to the most prominent undecidability problems.

8.1 Semi-Thue Systems

The word problem for a set is the algorithmic problem of deciding whether two given representatives stand for the same element. In abstract algebra and formal language theory, sets have a presentation given by generators and relations which allows the word problem for a set to be decribed by utilizing its presentation.

A *string rewriting system*, historically called a *semi-Thue system*, is a pair (Σ, R) where Σ is an alphabet and R is a dyadic relation on non-empty strings over Σ, i.e., $R \subseteq \Sigma^+ \times \Sigma^+$. Each element $(u, v) \in R$ is called a *rewriting rule* and is written as $u \to v$. Semi-Thue systems were introduced by the Norwegian mathematician Axel Thue (1863-1922) in 1914.

The rewriting rules can be naturally extended to strings in Σ^* by allowing substrings to be rewritten accordingly. More specifically, the *one-step rewriting relation* $\to_R$ induced by R on Σ^* is a dyadic relation on Σ^* such that for any strings s and t in Σ^*,

$$s \to_R t \quad :\Longleftrightarrow \quad s = xuy,\ t = xvy,\ u \to v \text{ for some } x, y, u, v \in \Sigma^*. \tag{8.1}$$

That is, a string s is rewritten by a string t when there is a rewriting rule $u \to v$ such that s contains u as a substring and this substring is replaced by v giving the string t.

The pair $(\Sigma^*, \to_R)$ is called an *abstract rewriting system*. Such a system allows to form a finite or infinite sequence of strings which is produced by starting with an initial string $s_0 \in \Sigma^+$ and repeatedly rewriting it by using one-step rewriting. A *zero-or-more steps rewriting* or *derivation* like this is captured by the reflexive transitive closure of $\to_R$ denoted by $\stackrel{*}{\to}_R$. That is, for any strings $s, t \in \Sigma^+$, $s \stackrel{*}{\to}_R t$ if and only if $s = t$ or there is a finite sequence $s_0, s_1, \ldots, s_m$ of elements in Σ^+ such that $s_0 = s$, $s_i \to_R s_{i+1}$ for $0 \leq i \leq m-1$, and $s_m = t$.

Example 8.1. Take the semi-Thue system (Σ, R) with $\Sigma = \{a, b\}$ and $R = \{(ab, bb), (ab, a), (b, aba)\}$. The derivation $abb \to_R ab \to_R bb \to_R baba \to_R bbba$ shows that $abb \stackrel{*}{\to}_R bbba$. $\diamondsuit$

The *word problem* for semi-Thue systems can be stated as follows:

> Given a semi-Thue system (Σ, R) and two strings $s, t \in \Sigma^+$. Can the string s be transformed into the string t by applying the rules from R; that is, $s \xrightarrow{*}_R t$?

This problem is undecidable. To see this, the halting problem for SGOTO-2 programs will be reduced to this word problem. For this, let $P = s_0; s_1; \ldots; s_{n-1}$ be an SGOTO-2 program consisting of n instructions. Note that the label n which does not address an instruction is the only one used to signify termination.

A *configuration* of the two-register machine is given by a triple (j, x, y), where $0 \leq j \leq n-1$ is the actual instruction, x is the content of the first register, and y is the content of the second register. These numbers can be encoded in unary format

$$\overline{x} = \overbrace{LL\ldots L}^{x} \quad \text{and} \quad \overline{y} = \overbrace{LL\ldots L}^{y}. \tag{8.2}$$

In this way, each configuration of the two-register machine can be written as a string

$$a\overline{x}j\overline{y}b \tag{8.3}$$

over the alphabet $\Sigma = \{a, b, 0, 1, 2, \ldots, n, L\}$.

Define a semi-Thue system (Σ, R_P) that simulates the mode of operation of the SGOTO-2 program P. For this, each SGOTO-2 instruction is assigned an appropriate rewriting rule as follows:

GOTO-2 instructions	rewriting rules
$(j, x_1 \leftarrow x_1 + 1, k)$	(j, Lk)
$(j, x_2 \leftarrow x_2 + 1, k)$	(j, kL)
$(j, x_1 \leftarrow x_1 - 1, k)$	$(Lj, k),\ (aj, ak)$
$(j, x_2 \leftarrow x_2 - 1, k)$	$(jL, k),\ (jb, kb)$
$(j, \texttt{if}\ \ x_1 = 0, k, l)$	$(Lj, Ll),\ (aj, ak)$
$(j, \texttt{if}\ \ x_2 = 0, k, l)$	$(jL, lL),\ (jb, kb)$

(8.4)

Moreover, the semi-Thue system contains two clean-up rewriting rules

$$(Ln, n) \quad \text{and} \quad (anL, an). \tag{8.5}$$

Example 8.2. Consider the SGOTO-2 program P:

$$\begin{array}{l}
(0, \texttt{if}\ \ x_1 = 0, 3, 1) \\
(1, x_1 \leftarrow x_1 - 1, 2) \\
(2, x_1 \leftarrow x_2 - 1, 0) \\
(3, \texttt{if}\ \ x_2 = 0, 5, 4) \\
(4, x_2 \leftarrow x_2 + 1, 3)
\end{array}$$

This program computes the partial function

$$\|P\|_{2,1}(x, y) = \begin{cases} 0 \text{ if } x \geq y, \\ \uparrow \text{ otherwise.} \end{cases} \tag{8.6}$$

The corresponding semi-Thue system over the alphabet $\Sigma = \{a, b, 0, 1, 2, 3, 4, 5, L\}$ consists of the rewriting rules

$$(L0, L1), (a0, a3), (L1, 2), (a1, a2), (2L, 0), (2b, 0b), (3L, 4L), (3b, 5b), (4, 3L), (L5, 5), (a5L, a5). \quad (8.7)$$

Here is a sample computation, where the registers initially hold the values $x_1 = 3$ and $x_2 = 2$:

SGOTO-2 program	semi-Thue system
$(0, 3, 2)$	$aLLL0LLb$
$(1, 3, 2)$	$aLLL1LLb$
$(2, 2, 2)$	$aLL2LLb$
$(0, 2, 1)$	$aLL0Lb$
$(1, 2, 1)$	$aLL1Lb$
$(2, 1, 1)$	$aL2Lb$
$(0, 1, 0)$	$aL0b$
$(1, 1, 0)$	$aL1b$
$(2, 0, 0)$	$a2b$
$(0, 0, 0)$	$a0b$
$(3, 0, 0)$	$a3b$
$(5, 0, 0)$	$a5b$

♢

The contruction immediately yields the following result.

Proposition 8.3. *If (j, x, y) is a configuration of the SGOTO-2 program P with $j < n$, the semi-Thue system (Σ, R_P) provides the one-step rewriting rule*

$$a\overline{x}j\overline{y}b \;\rightarrow_{R_P}\; a\overline{u}k\overline{v}b, \quad (8.8)$$

where (k, u, v) is the successor configuration of (j, x, y). There is no other one-step rewriting rule in the semi-Thue system applicable to $a\overline{x}j\overline{y}b$.

The iterated application of this statement implies the following.

Proposition 8.4. *A configuration (j, x, y) of the SGOTO-2 program P with $j < n$ leads to the configuration (k, u, v) if and only if*

$$a\overline{x}j\overline{y}b \;\stackrel{*}{\rightarrow}_{R_P}\; a\overline{u}k\overline{v}b. \quad (8.9)$$

Moreover, if the SGOTO-2 program terminates, its final configuration is of the form (n, x, y). The corresponding word in the semi-Thue system is $a\overline{x}n\overline{y}b$ which can be further rewritten according to the clean-up rules (8.5) as follows:

$$a\overline{x}n\overline{y}b \;\stackrel{*}{\rightarrow}_{R_P}\; anb. \quad (8.10)$$

This establishes the following result.

Proposition 8.5. *An SGOTO-2 program P started in the configuration $(0, x, y)$ halts if and only if in the corresponding semi-Thue system,*

$$a\overline{x}0\overline{y}b \xrightarrow{*}_{R_P} anb. \tag{8.11}$$

This proposition yields an effective reduction of the halting problem for SGOTO-2 programs to the word problem for semi-Thue systems. But the halting problem for SGOTO-2 programs is undecidable and thus we have established the following.

Theorem 8.6. *The word problem for semi-Thue systems is undecidable.*

8.2 Thue Systems

The Thue systems form a subclass of semi-Thue systems. A *Thue system* is a semi-Thue system (Σ, R) whose relation R is symmetric, i.e., if $u \to v \in R$ then $v \to u \in R$. In a Thue system, the reflexive transitive closure $\xrightarrow{*}_R$ of the one-step rewriting relation $\to_R$ is also symmetric and thus an equivalence relation on Σ^+.

The *word problem* can be summarized as follows:

> Given a Thue system (Σ, R) and two strings $s, t \in \Sigma^+$. Can the string s be transformed into the string t by applying the rules from R; that is, $s \xrightarrow{*}_R t$?

This problem is also undecidable. To see this, Thue systems will be related to semi-Thue systems. For this, let (Σ, R) be a semi-Thue system. The *symmetric closure* of the rewriting relation R is the symmetric relation

$$R^{(s)} = R \cup R^{-1}, \tag{8.12}$$

where $R^{-1} = \{(v, u) \mid (u, v) \in R\}$ is the *inverse relation* of R. The set $R^{(s)}$ is the smallest symmetric relation containing R, and the pair $(\Sigma, R^{(s)})$ is a Thue system. The relation $\xrightarrow{*}_{R^{(s)}}$ is thus the reflexive transitive and symmetric closure of $\to_R$ and hence an equivalence relation on Σ^+. That is, for any strings $s, t \in \Sigma^+$, $s \xrightarrow{*}_{R^{(s)}} t$ if and only if $s = t$ or there is a finite sequence $s_0, s_1, \ldots, s_m$ of elements in Σ^+ such that $s = s_0$, $s_i \to_R s_{i+1}$ or $s_{i+1} \to_R s_i$ for $0 \leq i \leq m-1$, and $s_m = t$. Note that if $s \xrightarrow{*}_{R^{(s)}} t$ holds in a Thue system $(\Sigma, R^{(s)})$, neither $s \xrightarrow{*}_R t$ nor $t \xrightarrow{*}_R s$ need to be valid in the corresponding semi-Thue system (Σ, R).

Example 8.7. Consider the semi-Thue system (Σ, R) with $\Sigma = \{a, b\}$ and $R = \{(ab, b), (ba, a)\}$. The corresponding Thue system is $(\Sigma, R^{(s)})$, where $R^{(s)} = R \cup \{(b, ab), (a, ba)\}$.

In the semi-Thue system, rewriting is strictly antitone leading to smaller strings, i.e., if $u \xrightarrow{*}_R v$ and $u \neq v$, then $|u| > |v|$, e.g., $aabb \to_R abb \to_R bb$.

On the other hand, in the Thue system, $aabb \to_{R^{(s)}} abb \to_{R^{(s)}} abab$, but in the semi-Thue system neither $aabb \xrightarrow{*}_R abab$ nor $abab \xrightarrow{*}_R aabb$. ♢

Theorem 8.8. (Post's Lemma) *Let P be an SGOTO-2 program with n instructions, let (Σ, R_P) be the corresponding semi-Thue system, and let $(\Sigma, R_P^{(s)})$ be the associated Thue system. For each configuration (j, x, y) of the program P,*

$$a\overline{x}j\overline{y}b \xrightarrow{*}_{R_P} anb \quad \Longleftrightarrow \quad a\overline{x}j\overline{y}b \xrightarrow{*}_{R_P^{(s)}} anb. \tag{8.13}$$

Proof. The direction from left-to-right holds since $R \subseteq R^{(s)}$. Conversely, let (j, x, y) be a configuration of the program P. By hypothesis, there is a rewriting sequence in the Thue system such that

$$s_0 = a\overline{x}j\overline{y}b \to_{R_P^{(s)}} s_1 \to_{R_P^{(s)}} \ldots \to_{R_P^{(s)}} s_q = anb, \tag{8.14}$$

where it may be assumed that the length $q+1$ of the derivation is minimal. It is clear that each occurring string s_i corresponds to a configuration of the program P, $0 \leq i \leq q$.

Suppose the derivation (8.14) cannot be established by the semi-Thue system. That is, the sequence contains a rewriting step $s_p \leftarrow_{R_P} s_{p+1}$, $0 \leq p \leq q-1$. The index p can be chosen to be maximal with this property. Since there is no rewriting rule applicable to $s_q = anb$, we have $p+1 < q$. This leads to the following situation:

$$s_p \leftarrow_{R_P} s_{p+1} \to_{R_P} s_{p+2}. \tag{8.15}$$

However, the string s_{p+1} encodes a configuration of P and there is at most one rewriting rule applicable to it. Thus the words s_p and s_{p+2} must be identical and hence the derivation (8.14) can be shortened by deleting the string s_{p+1} contradicting the assumption. □

The above result due to the Jewish logician Emil Post (1897-1954) provides an effective reduction of the derivations in semi-Thue system to derivations in Thue systems. But the word problem for semi-Thue systems is undecidable and thus we obtain the following.

Theorem 8.9. *The word problem for Thue systems is undecidable.*

8.3 Semigroups

Word problems are also encountered in abstract algebra. A *semigroup* is an algebraic structure consisting of a non-empty set S together with an associative dyadic operation. For instance, the set of all non-empty strings Σ^+ over an alphabet Σ together with the concatenation of strings is a semigroup, called *free semigroup* over Σ.

Each Thue system gives rise to a semigroup in a natural way. To see this, let $(\Sigma, R^{(s)})$ be a Thue system. We already know that the rewriting relation $\xrightarrow{*}_{R^{(s)}}$ on Σ^+ is an equivalence relation. The *equivalence class* of a string $s \in \Sigma^+$ is the subset $[s]$ of all strings in Σ^+ that can be derived from s by a finite number of rewriting steps:

$$[s] = \{t \in \Sigma^+ \mid s \xrightarrow{*}_{R^{(s)}} t\}. \tag{8.16}$$

Proposition 8.10. *The set of equivalence classes $H = H(\Sigma, R^{(s)}) = \{[s] \mid s \in \Sigma^+\}$ forms a semigroup with the operation*

$$[s] \circ [t] = [st], \quad s, t \in \Sigma^+. \tag{8.17}$$

Proof. Claim that the operation is well-defined. Indeed, let $[s] = [s']$ and $[t] = [t']$, where $s, s', t, t' \in \Sigma^+$. Then $s \xrightarrow{*}_{R^{(s)}} s'$, $s' \xrightarrow{*}_{R^{(s)}} s$, $t \xrightarrow{*}_{R^{(s)}} t'$, and $t' \xrightarrow{*}_{R^{(s)}} t$. Thus $st \xrightarrow{*}_{R^{(s)}} s't \xrightarrow{*}_{R^{(s)}} s't'$ and $s't' \xrightarrow{*}_{R^{(s)}} st' \xrightarrow{*}_{R^{(s)}} st$. Hence, $[st] = [s't']$. □

The semigroup $H = H(\Sigma, R^{(s)})$ can be defined by generators and relations. To see this, let $\Sigma = \{a_1, \ldots, a_n\}$ and $R = \{(u_1, v_1), \ldots, (u_m, v_m)\}$. Then the semigroup H has a *presentation* in terms of generators and relations as follows:

$$H = \langle [a_1], \ldots, [a_n] \mid [u_1] = [v_1], \ldots, [u_m] = [v_m] \rangle.$$

The semigroup H is presented as the quotient of the free semigroup Σ^+ by the subsemigroup of Σ^+ generated by the relations $[u_i] = [v_i]$, $1 \leq i \leq m$.

Example 8.11. Consider the Thue system $(\Sigma, R^{(s)})$, where $\Sigma = \{a, b\}$ and $R = \{(ab, b), (ba, a)\}$. For instance, $a \to_{R^{(s)}} ba \to_{R^{(s)}} aba \to_{R^{(s)}} aa$ and $b \to_{R^{(s)}} ab \to_{R^{(s)}} bab \to_{R^{(s)}} bb$ and thus $[a] = [aa]$ and $[b] = [bb]$. The semigroup $H = H(\Sigma, R^{(s)})$ has the presentation

$$H = \langle [a], [b] \mid [b] = [ab], [a] = [ba] \rangle.$$

Any word of H is of the form $[a^{i_1} b^{j_1} \ldots a^{i_k} b^{j_k}]$, where $i_1, \ldots, i_k, j_1, \ldots, j_k \geq 0$ with at least one of these numbers > 0. But the relations $[b] = [ab]$ and $[a] = [ba]$ imply $[a] = [ba] = [aba] = [aa]$ and $[b] = [ab] = [bab] = [bb]$. Thus the semigroup H only consists of two elements $[a]$ and $[b]$ which are idempotent, i.e., $[a] \circ [a] = [a]$ and $[b] \circ [b] = [b]$, and satisfy $[a] \circ [b] = [b]$ and $[b] \circ [a] = [a]$. $\diamondsuit$

The *word problem* for the semigroup $H = H(\Sigma, R^{(s)})$ asks whether arbitrary strings $s, t \in \Sigma^+$ describe the same element $[s] = [t]$ or not. By definition,

$$[s] = [t] \quad \Longleftrightarrow \quad s \xrightarrow{*}_{R^{(s)}} t, \quad s, t \in \Sigma^+. \tag{8.18}$$

This equivalence provides an effective reduction of the word problem for Thue systems to the word problem for semigroups. This leads to the following result which was independently established by Emil Post (1987-1954) and Andrey Markov Jr. (1903-1979).

Theorem 8.12. *The word problem for semigroups is undecidable.*

8.4 Post's Correspondence Problem

The Post correspondence problem is an undecidable problem that was introduced by Emil Post in 1946. Due to its simplicity it is often used in proofs of undecidability.

A *Post correspondence system* (PCS) over an alphabet Σ is a finite set Π of pairs $(\alpha_i, \beta_i) \in \Sigma^+ \times \Sigma^+$, $1 \leq i \leq m$. For each finite sequence $\boldsymbol{i} = (i_1, \ldots, i_r) \in \{1, \ldots, m\}^+$ of indices, define the strings

$$\alpha(\boldsymbol{i}) = \alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_r} \tag{8.19}$$

and

$$\beta(\boldsymbol{i}) = \beta_{i_1} \circ \beta_{i_2} \circ \cdots \circ \beta_{i_r}. \tag{8.20}$$

A *solution* of the PCS Π is a sequence $\boldsymbol{i}$ of indices such that $\alpha(\boldsymbol{i}) = \beta(\boldsymbol{i})$.

Example 8.13. The PCS $\Pi = \{(\alpha_1, \beta_1) = (a, aaa), (\alpha_2, \beta_2) = (abaa, ab), (\alpha_3, \beta_3) = (aab, b)\}$ over the alphabet $\Sigma = \{a, b\}$ has the solution $\boldsymbol{i} = (2, 1, 1, 3)$, since

$$\begin{aligned}\alpha(\boldsymbol{i}) &= \alpha_2 \circ \alpha_1 \circ \alpha_1 \circ \alpha_3 = abaa \circ a \circ a \circ aab \\ &= abaaaaaab \\ &= ab \circ aaa \circ aaa \circ b = \beta_2 \circ \beta_1 \circ \beta_1 \circ \beta_3 = \beta(\boldsymbol{i}).\end{aligned}$$

$\diamondsuit$

The *word problem* for Post correspondence systems asks whether a Post correspondence system has a solution or not. This problem is undecidable. To see this, the word problem for semi-Thue systems will be reduced to this one. To this end, let (Σ, R) be a semi-Thue system, where $\Sigma = \{a_1, \ldots, a_n\}$ and $R = \{(u_i, v_i) \mid 1 \leq i \leq m\}$. Take a copy of the alphabet Σ given by $\Sigma' = \{a'_1, \ldots, a'_n\}$. In this way, each string $s = a_{i_1} a_{i_2} \ldots a_{i_r}$ over Σ can be assigned a copy $s' = a'_{i_1} a'_{i_2} \ldots a'_{i_r}$ over Σ'.

Put $q = m+n$ and let $s, t \in \Sigma^+$. Define the PCS $\Pi = \Pi(\Sigma, R, s, t)$ over the alphabet $\Sigma \cup \Sigma' \cup \{x, y, z\}$ by the following $2q + 4$ pairs:

$$\begin{aligned}(\alpha_i, \beta_i) &= (u_i, v'_i),\ 1 \leq i \leq m, \\ (\alpha_{m+i}, \beta_{m+i}) &= (a_i, a'_i),\ 1 \leq i \leq n, \\ (\alpha_{q+i}, \beta_{p+i}) &= (u'_i, v_i),\ 1 \leq i \leq m, \\ (\alpha_{q+m+i}, \beta_{p+m+i}) &= (a'_i, a_i),\ 1 \leq i \leq n, \\ (\alpha_{2q+1}, \beta_{2p+1}) &= (y, z), \\ (\alpha_{2q+2}, \beta_{2p+2}) &= (z, y), \\ (\alpha_{2p+3}, \beta_{2p+3}) &= (x, xsy), \\ (\alpha_{2p+4}, \beta_{2p+4}) &= (ztx, x).\end{aligned} \tag{8.21}$$

Example 8.14. Take the semi-Thue system (Σ, R), where $\Sigma = \{a, b\}$ and $R = \{(ab, bb), (ab, a), (b, aba)\}$. The corresponding PCS $\Pi = \Pi(\Sigma, R, s, t)$ over the alphabet $\{a, b, a', b', x, y, z\}$ consists of the following pairs:

$$\begin{aligned}&(ab, b'b'), \\ &(ab, a'), \\ &(b, a'b'a'), \\ &(a, a'), \\ &(b, b'), \\ &(a'b', bb), \\ &(a'b', a), \\ &(b', aba), \\ &(a', a), \\ &(b', b), \\ &(y, z), \\ &(z, y), \\ &(x, xsy), \\ &(ztx, x).\end{aligned}$$

$\diamondsuit$

First, observe that the derivations of a semi-Thue system can be mimicked by the corresponding PCS.

Proposition 8.15.

- *If* $s, t \in \Sigma^+$ *with* $s = t$ *or* $s \to_R t$*, there is a sequence* $\boldsymbol{i} \in \{1, \ldots, q\}^+$ *of indices such that* $\alpha(\boldsymbol{i}) = s$ *and* $\beta(\boldsymbol{i}) = t'$*, and there is a sequence* $\boldsymbol{i}' \in \{q+1, \ldots, 2q\}^+$ *of indices such that* $\alpha(\boldsymbol{i}') = s'$ *and* $\beta(\boldsymbol{i}') = t$*.*
- *If* $s, t \in \Sigma^+$ *such that there is a sequence* $\boldsymbol{i} \in \{1, \ldots, q\}^+$ *of indices such that* $\alpha(\boldsymbol{i}) = s$ *and* $\beta(\boldsymbol{i}) = t'$*, then* $s \xrightarrow{*}_R t$*.*
- *If* $s, t \in \Sigma^+$ *such that there is a sequence* $\boldsymbol{i}' \in \{q+1, \ldots, 2q\}^+$ *of indices such that* $\alpha(\boldsymbol{i}') = s'$ *and* $\beta(\boldsymbol{i}') = t$*, then* $s \xrightarrow{*}_R t$*.*

Example 8.16. (Cont'd) Let $s = aabb$ and $t = aab$. Then $s \to_R t$ by the rule $(ab, a) \in R$ and the sequences $\boldsymbol{i} = (4, 2, 5)$ and $\boldsymbol{i}' = (9, 7, 10)$ yield

$$\begin{aligned}
\alpha(\boldsymbol{i}) &= \alpha_4 \circ \alpha_2 \circ \alpha_5 = a \circ ab \circ b = aabb, \\
\beta(\boldsymbol{i}) &= \beta_4 \circ \beta_2 \circ \beta_5 = a' \circ a' \circ b' = a'a'b', \\
\alpha(\boldsymbol{i}') &= \alpha_9 \circ \alpha_7 \circ \alpha_{10} = a' \circ a'b' \circ b' = a'a'b'b', \\
\beta(\boldsymbol{i}') &= \beta_9 \circ \beta_7 \circ \beta_{10} = a \circ a \circ b = aab.
\end{aligned}$$

Let $s = bbaba = \alpha(5, 3, 2, 4)$ and $t' = b'a'b'a'a'a' = \beta(5, 3, 2, 4)$. Then $s = bbaba \to_R babaaba \to_R babaaa = t$ and so $s \xrightarrow{*}_R t$. $\diamondsuit$

Second, the solutions of the constructed PCS can always be canonically decomposed.

Proposition 8.17.

- *Let* $\boldsymbol{i} = (i_1, \ldots, i_r)$ *be a solution of the PCS* $\Pi = \Pi(\Sigma, R, s, t)$ *with* $w = \alpha(\boldsymbol{i}) = \beta(\boldsymbol{i})$*. If* $w = w_1 y w_2$ *for some* $w_1, w_2 \in \Sigma'^*$*, there is an index* j*,* $1 \leq j \leq r$*, such that*

$$\alpha(i_1, \ldots, i_{j-1}) = w_1,\ \alpha(i_j) = y,\ \textit{and}\ \alpha(i_{j+1}, \ldots, i_r) = w_2. \tag{8.22}$$

- *Let* $\boldsymbol{i} = (i_1, \ldots, i_r)$ *be a solution of the PCS* $\Pi = \Pi(\Sigma, R, s, t)$ *with* $w = \alpha(\boldsymbol{i}) = \beta(\boldsymbol{i})$*. If* $w = w_1 z w_2$ *for some* $w_1, w_2 \in \Sigma'^*$*, there is an index* j*,* $1 \leq j \leq r$*, such that either*

$$\alpha(i_1, \ldots, i_{j-1}) = w_1,\ \alpha(i_j) = z,\ \textit{and}\ \alpha(i_{j+1}, \ldots, i_r) = w_2, \tag{8.23}$$

or

$$\alpha(i_1, \ldots, i_{j-1}) = w_1,\ \alpha(i_j) = ztx,\ \alpha(i_{j+1}, \ldots, i_r) = u,\ \textit{and}\ w_2 = txu. \tag{8.24}$$

Example 8.18. (Cont'd) Let $s = b$ and $t = abaa$. A solution of the PCS Π is given by

$$\boldsymbol{i} = (13, 3, 11, 6, 9, 12, 3, 5, 4, 11, 9, 10, 7, 9, 14),$$

since

$$\begin{aligned}\alpha(\boldsymbol{i}) &= x \circ b \circ y \circ a'b' \circ a' \circ z \circ b \circ b \circ a \circ y \circ a' \circ b' \circ a'b' \circ a' \circ zabaax \\ &= xbya'b'a'zbbaya'b'a'b'a'zabaax \\ &= xby \circ a'b'a' \circ z \circ bb \circ a \circ y \circ a'b'a' \circ b' \circ a' \circ z \circ a \circ b \circ a \circ a \circ x \\ &= \beta(\boldsymbol{i}).\end{aligned}$$

Here we have

$$w_1 = xbya'b'a'zbba = \alpha(13, 3, 11, 6, 9, 12, 3, 5, 4)$$

and

$$w_2 = a'b'a'b'a'zabaax = \alpha(9, 10, 7, 9, 14),$$

or

$$w_1 = xbya'b'a' = \alpha(13, 3, 11, 6, 9)$$

and

$$w_2 = bbaya'b'a'b'a'zabaax = \alpha(3, 5, 4, 11, 9, 10, 7, 9, 14),$$

or

$$w_1 = xbya'b'a'zbbaya'b'a'b'a' = \alpha(13, 3, 11, 6, 9, 3, 5, 4, 11, 9, 10, 7, 9),$$

and

$$ztx = zabaax = \alpha(14) \text{ and } w_2 = abaax = tx.$$

$\diamondsuit$

Note that if the PCS $\Pi = \Pi(\Sigma, R, s, t)$ has a solution $\alpha(\boldsymbol{i}) = \beta(\boldsymbol{i})$ with $\boldsymbol{i} = (i_1, \ldots, i_r)$, the solution starts with $i_1 = 2q + 3$ and ends with $i_r = 2p + 4$, i.e., the corresponding string has the prefix xsy and the postfix ztx. The reason is that $(\alpha_{2p+3}, \beta_{2p+3}) = (x, xsy)$ is the only pair whose components have the same prefix and $(\alpha_{2p+4}, \beta_{2p+4}) = (ztx, x)$ is the only pair whose components have the same postfix.

Proposition 8.19. *Let $s, t \in \Sigma^+$. If there is a derivation $s \xrightarrow{*}_R t$ in the semi-Thue system (Σ, R), the PCS $\Pi = \Pi(\Sigma, R, s, t)$ has a solution.*

Proof. Let $s \xrightarrow{*}_R t$ with

$$s = s_1 \rightarrow_R s_2 \rightarrow_R \ldots \rightarrow_R s_{k-1} \rightarrow_R s_k = t, \tag{8.25}$$

where $s_i \rightarrow_R s_{i+1}$ or $s_i = s_{i+1}$, $0 \leq i \leq k - 1$. The inclusion of steps without effect allows to assume that k is odd. By Proposition 8.15, for each j, $1 \leq j < k$, there is a sequence $\boldsymbol{i}^{(j)}$ of indices such that $\alpha(\boldsymbol{i}^{(j)}) = s_j$ and $\beta(\boldsymbol{i}^{(j)}) = s'_{j+1}$ if j is odd and $\alpha(\boldsymbol{i}^{(j)}) = s'_j$ and $\beta(\boldsymbol{i}^{(j)}) = s_{j+1}$ if j is even.

Claim that a solution of the PCS is given by the sequence

$$\boldsymbol{i} = (2p+3, \boldsymbol{i}^{(1)}, 2p+1, \boldsymbol{i}^{(2)}, 2p+2, \boldsymbol{i}^{(3)}, 2p+1, \ldots, \boldsymbol{i}^{(k-2)}, 2p+1, \boldsymbol{i}^{(k-1)}, 2p+4). \tag{8.26}$$

Indeed, the sequence can be evaluated as follows:

$$\begin{array}{llllllllllllll}
\alpha(\boldsymbol{i}): & x & s_1 & y & s_2' & z & s_3 & y & \ldots & s_{k-2} & y & s_{k-1}' & zs_kx \\
 & . & . & . & . & . & . & . & & . & . & . & . \\
\boldsymbol{i}: & 2p+3 & \boldsymbol{i}^{(1)} & 2p+1 & \boldsymbol{i}^{(2)} & 2p+2 & \boldsymbol{i}^{(3)} & 2p+1 & \ldots & \boldsymbol{i}^{(k-2)} & 2p+1 & \boldsymbol{i}^{(k-1)} & 2p+4 \\
 & . & . & . & . & . & . & . & & . & . & . & . \\
\beta(\boldsymbol{i}): & xs_1y & s_2' & z & s_3 & y & s_4' & z & \ldots & s_{k-1}' & z & s_k & x
\end{array}$$

This proves the claim. □

Example 8.20. (Cont'd) Consider the derivation

$$s = aab \to_R abb \to abb \to_R aabab \to_R aaab \to_R aaa = t.$$

A solution of the PCS $\Pi = \Pi(\Sigma, R, aab, aaa)$ is given by the sequence

$$\boldsymbol{i} = (13, 4, 1, 11, 9, 8, 10, 12, 4, 2, 4, 5, 11, 9, 9, 7, 14),$$

where

$$\begin{array}{llllllllll}
\alpha(\boldsymbol{i}): & x & aab & y & a'b'b' & z & aabab & y & a'a'a'b' & zaaax \\
 & . & . & . & . & . & . & . & . & . \\
\boldsymbol{i}: & 13 & (4,1) & 11 & (9,8,10) & 12 & (4,2,4,5) & 11 & (9,9,7) & 14 \\
 & . & . & . & . & . & . & . & . & . \\
\beta(\boldsymbol{i}): & xaaby & a'b'b' & z & aabab & y & a'a'a'b' & z & aaa & x
\end{array}$$

♢

Proposition 8.21. *Let $s, t \in \Sigma^+$. If the PCS $\Pi = \Pi(\Sigma, R, s, t)$ has a solution, there is a derivation $s \xrightarrow{*}_R t$ in the semi-Thue system (Σ, R).*

Proof. Let $\boldsymbol{i} = (i_1, \ldots, i_r)$ be a minimal solution of the PCS Π. The observation prior to Proposition 8.19 shows that the string $w = \alpha(\boldsymbol{i}) = \beta(\boldsymbol{i})$ must have the form $w = xsy \ldots ztx$.

By Proposition 8.17, there is a sequence $\boldsymbol{i}^{(1)}$ of indices such that $\alpha(\boldsymbol{i}^{(1)}) = s = s_1$. Put $s_2' = \beta(\boldsymbol{i}^{(1)})$. It follows that $w = xs_1ys_2'z\ldots$ and by Proposition 8.15, $s_1 \xrightarrow{*}_R s_2$.

By Proposition 8.17, there is a sequence $\boldsymbol{i}^{(2)}$ of indices such that $\alpha(\boldsymbol{i}^{(2)}) = s_2'$. Set $s_3 = \beta(\boldsymbol{i}^{(2)})$. Thus $w = xs_1ys_2'zs_3\ldots$ and by Proposition 8.15, $s_2 \xrightarrow{*}_R s_3$.

By Proposition 8.17, there are two possibilities:

- Let $\boldsymbol{i} = (2q+3, \boldsymbol{i}^{(1)}, 2q+1, \boldsymbol{i}^{(2)}, 2q+2, \ldots)$, i.e., $w = xs_1ys_2'zs_3y\ldots$. Then the sequence can be continued as indicated above.
- Let $\boldsymbol{i} = (2q+3, \boldsymbol{i}^{(1)}, 2q+1, \boldsymbol{i}^{(2)}, 2q+4, \ldots)$. This is only possible if $s_3 = t$ and $w = xs_1ys_2'ztx$. It follows that $\boldsymbol{i} = (2q+3, \boldsymbol{i}^{(1)}, 2q+1, \boldsymbol{i}^{(2)}, 2q+4)$ is already a solution due to minimality:

$$\begin{array}{llllll}
\alpha(\boldsymbol{i}): & x & s_1 & y & s_2' & ztx \\
 & . & . & . & . & . \\
\boldsymbol{i}: & 2q+3 & \boldsymbol{i}^{(1)} & 2q+1 & \boldsymbol{i}^{(2)} & 2q+4 \\
 & . & . & . & . & . \\
\beta(\boldsymbol{i}): & xs_1y & s_2' & z & t & x
\end{array}$$

But $s = s_1 \xrightarrow{*}_R s_2$ and $s_2 \xrightarrow{*}_R s_3 = t$, and so $s \xrightarrow{*}_R t$.

By induction, there are strings $s = s_1, s_2, s_3, \ldots, s_k = t$ in Σ^+ with k odd such that

$$\alpha(\boldsymbol{i}) = x \circ s_1 \circ y \circ s_2' \circ z \circ s_3 \circ \ldots \circ s_{k-2} \circ y \circ s_{k-1}' \circ z \circ s_k \circ x = \beta(\boldsymbol{i})$$

and $s_i \overset{*}{\rightarrow}_R s_{i+1}$, $1 \leq i \leq k-1$. Hence, $s \overset{*}{\rightarrow}_R t$ as required. □

Example 8.22. (Cont'd) Let $s = b$ and $t = abaa$. A solution of the PCS Π is given by

$$\boldsymbol{i} = (13, 3, 11, 6, 9, 12, 3, 5, 4, 11, 9, 10, 7, 9, 14).$$

The string

$$\alpha(\boldsymbol{i}) = \beta(\boldsymbol{i}) = xbya'b'a'zbbaya'b'a'b'a'zabaax$$

provides a derivation in the semi-Thue system:

$$s = b \overset{*}{\rightarrow}_R aba \overset{*}{\rightarrow}_R bba \overset{*}{\rightarrow}_R ababa \overset{*}{\rightarrow}_R abaa = t.$$

♢

The last two propositions provide a reduction of the word problem for semi-Thue systems to the word problem for Post correspondence systems. But the word problem for semi-Thue systems is undecidable. This gives rise to the following result.

Theorem 8.23. *The word problem for Post correspondence systems is undecidable.*

Index

Why do we need a formalization of the notion of algorithm or effective computation? In order to show that a specific problem is algorithmically solvable, it is sufficient to provide an algorithm that solves it in a sufficiently precise manner. However, in order to prove that a problem is in principle not solvable by an algorithm, a rigorous formalism is necessary that allows mathematical proofs. The need for such a formalism became apparent in the works of David Hilbert (1900) on the foundations of mathematics and Kurt Gödel (1931) on the incompleteness of elementary arithmetic.

The book is a development of class notes for a lecture on computability held for Bachelor students of Computer Science at the Hamburg University of Technology in the summer term 2011. The aim of the course was to present the basic results of the field:

- mathematical models of computation (Turing machine, unlimited register machine, LOOP and GOTO programs),
- primitive recursive and partial recursive functions, Ackermann's function,
- Gödel numbering,
- acceptable programming systems (universal functions, smn theorem, Kleene's normal form),
- undecidable sets, theorems of Rice, and word problems (diophantine sets, semi-Thue systems, semigroups, Post's correspondence problem).

The Schickard machine is a non-programable calculating machine for integers invented by Wilhelm Schickard (1592-1646).

Front Page: The „Analytical Engine“ is a mechanical programmable machine designed by Charles Babbage (1792–1871).

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