Curves, Cryptosystems, and Quantum Computing

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TUHH

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Preface

Today, data security is a huge asset of our society. Cryptography is a field that provides methods and techniques for the security and authenticity of data. It's a fantastic detective drama series about lovely Alice, crooked Bob, evil Eve and the like. Modern cryptography is heavily based on mathematics and computer science.

The slides at hand are a development of class notes of a four-hour lecture held for first-year Master students of Computer Science, Electrical Engineering, and Technomathematics at the Hamburg University of Technology in hot and dry summer 2018.

The ultimate goal of the course was to present the beautiful mathematics of elliptic curves and their use in cryptography. However, alarmed by recent announcements of leading computer companies that quantum computers with about 70 qubits are at the gates, I decided to include a short introduction to the intriguing quantum mechanics and the shrugging quantum algorithms of Grover and Shor which could render the current cryptographic protocols nearly useless in the future.

Preface (Cont'd)

I must apologize for giving this course fully based on slides and not by chalk and blackboard. This is an unexcusable mistake for a course which should be attributed to pure mathematics. However, the slides contain all the necessary steps to understand the arguments and I would have never come this far when writing on the board. Needless to say it would have cost me another made-to-measure suit.

*Starred material can be safely skipping on a first reading without loss of continuity. Note that literature is mentioned at several places in the document and an index will be separately available.

I would like to thank my collaborator Robert Leppert for useful comments and to help out when required. Finally, I would like to thank my students for their attention, their stimulating questions, and their dedicated work.

Hamburg, July 2019 *Karl-Heinz Zimmermann*

Preface

The second edition the body of the text has only been changed slightly. The reader might find the added improvements and clarifications quite useful.

Hamburg, July 2021 *Karl-Heinz Zimmermann*

Dedication

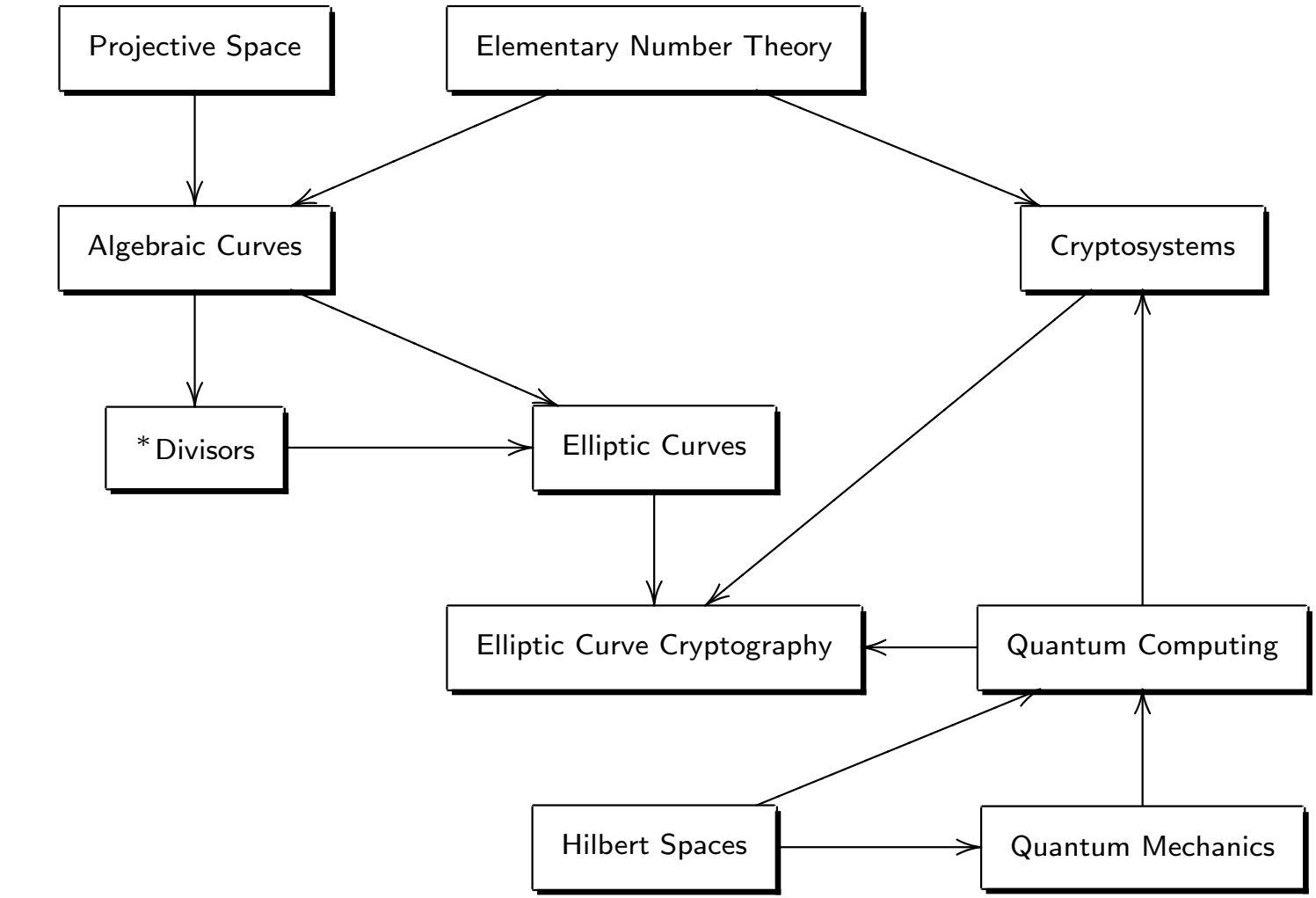
To my family
for sempiternal
support.

Contents

- Basic cryptography
- Theory of algebraic and elliptic curves
- Elliptic curve cryptography
- Quantum computing with emphasis on cryptography

Contents - Chapters

Contents – Road Map



Oral Exam – Topics

- Algebraic and elliptic curves in projective space
- Elliptic curves over finite fields
- Elliptic curve cryptography
- Quantum computing.

*No starred material, no `Singular`.

Formalities

- Pure Master course
- *Schedule*:
  - Tuesday, 10:00-11:30 am
  - Friday, 10:00-11:30 am
- *Classroom:* online
- *StudIP:* documents, appointments
- *Exam:* oral, online (20-25 min)

Literature

- F.L. Bauer, *Decrypted Secrets*, Springer, Berlin, 2000.
- Gert-Martin Greuel, Gerhard Pfister, *A Singular Introduction to Commutative Algebra*, Springer, Berlin, 2008.
- David J. Griffiths, Darrell F. Schroeter, *Introduction to Quantum Mechanics*, Cambridge Univ. Press, Cambridge, UK, 2018.
- Neal Koblitz, *A Course in Number Theory and Cryptography*, Springer, Berlin, 1994.
- Anthony Knapp, *Elliptic Curves*, Princeton Univ. Press, New York, 1992.
- Joseph Silverman, *The Arithmetic of Elliptic Curves*, Springer, Berlin, 2009.
- Lawrence Washington, *Elliptic Curves - Number Theory and Cryptography*, Chapman & Hall, Boca Raton, 2008.
- Annette Werner, *Elliptische Kurven*, Springer, 2013.

Notation

$\mathbb{Z}$	ring of integers
$\mathbb{Z}_n$	ring of integers modulo n
$\mathbb{Z}_n^*$	unit group of integers modulo n
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
i	imaginary unit
$\mathbb{K}$	field
$\bar{\mathbb{K}}$	algebraic closure of field $\mathbb{K}$
$\mathbb{F}_q$	finite field of order q
$\mathbb{K}^*$	unit group of field
$\mathbb{A}^n$	affine n-space
$\mathbb{P}^n$	projective n-space
$f^{(k)}$	degree k-part of homogenous polynomial f
f^h	homogenization of polynomial f
f^a	dehomogenization of homogeneous polynomial f

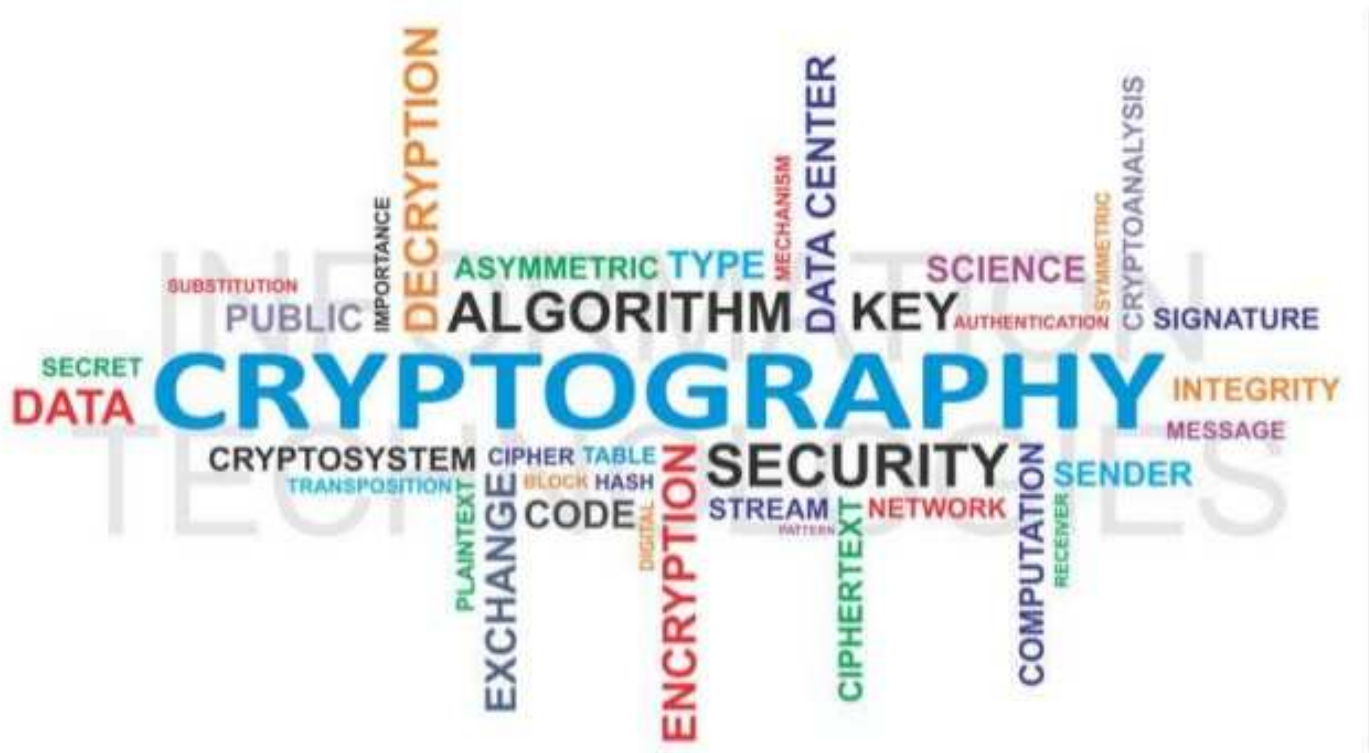
Notation (cont'd)

ℓ	linear homogeneous polynomial
$\mathcal{L}(\ell)$	projective line
$\mathcal{L}(\alpha, \beta, \gamma)$	projective line
$\mathcal{C}$	algebraic curve
$\mathcal{E}$	elliptic curve
$i(P, \mathcal{L}, \mathcal{C})$	intersection multiplicity
(a, b)	gcd of a and b
$a \equiv b \mod n$	congruence modulo n
$a \equiv b$ *mod* n	congruence modulo n, symmetric case
$\phi(n)$	Euler's totient function
φ	golden ratio
$[a_0, \ldots, a_n]$	Euler bracket
$\left(\frac{a}{n}\right)$	Jacobi symbol

Part I

Cryptosystems

Cryptosystems



Cryptosystems

> Nearly every inventor of a cipher system has been convinced of the unsolvability of his brainchild.

David Kahn, 1967

Cryptosystems

- Historical Account
- RSA
- Discrete logarithm
- *Attacking discrete logarithm
- *Pseudoprimality
- Factorization

Early History of Cryptosystems

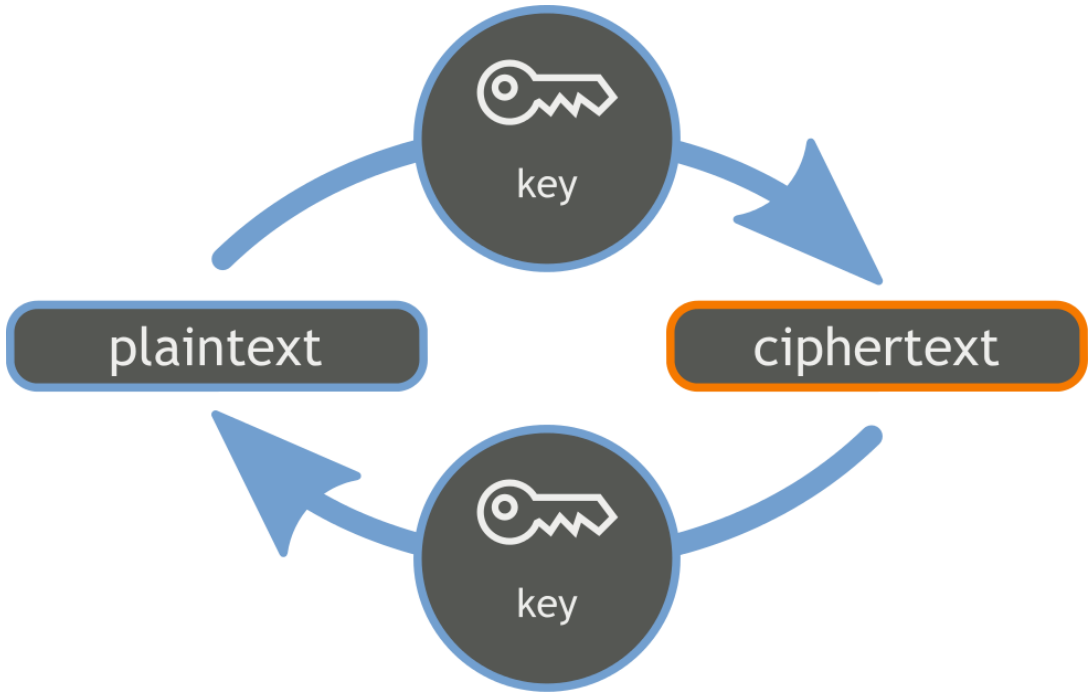
> A substitution with a CAESAR encryption step was introduced in 1915 in the Russian army after it turned out to be impossible to expect the staffs to use anything more complicated.

F.L. Bauer, 2000

Historical Account

- Symmetric cryptosystems
- Simple and polygraphic substitutions
- Transpositions
- Rotor crypto machines – Enigma
- Data encryption standard (DES)
- Asymmetric cryptosystems
- One-way and trapdoor functions
- Blockchain
- Cryptology

Symmetric Cryptosystems



Symmetric Cryptosystems

- The key agreed upon by two partners determines both the encryption and decryption.
- Cryptanalytic security depends on the secrecy of the key.
- Authentication is guaranteed as long as the secrecy of the key is guaranteed.

Symmetric Cryptosystems

Disadvantages:

- Sender of a message cannot prove to his partner that she sent the message (lack of judical protection).
- The key has to be communicated or negotiated on a cryptoanalytically secure channel.
- For a large number of partners wanting secure communication, the number of two-way channels and keys becomes quite large.
- A network with n partners requires $\binom{n}{2} = \frac{n(n-1)}{2}$ self-reciprocal keys or $n(n-1)$ symmetric keys.

Simple Substitions

- Monocyclic permutation:

```
abcdefghijklmnopqrstuvwxyz
bcdefghijklmnopqrstuvwxyza
```

- The 3rd power was used by Julius Caesar:

```
abcdefghijklmnopqrstuvwxyz
defghijklmnopqrstuvwxyzabc
```

  Key: number 3

- Non-selfreciprocal and non-cyclic permutation:

```
abcdefghijklmnopqrstuvwxyz
securityabdfghjklmnopqvwxz
```

  Key: security

Cipher Disk

Leon Battista Alberti (1466)

Polygraphic Substitions

Playfair cipher (1854)

- From a password, a permuted alphabet $\mathbb{Z}_{25}$ (say omitting 'J') is inscribed into a 5×5 square (thought as a closed torus):

S	E	C	U	R
A	B	D	F	G
H	I	K	L	M
N	O	P	Q	T
V	W	X	Y	Z

- If the letters of a bigram stand in the same line or column, each is replaced by the subsequent letter,

$$\mathtt{SH} \mapsto \mathtt{AN}, \quad \mathtt{CR} \mapsto \mathtt{US}.$$

Polygraphic Substitions

Kurzsignalheft of Kriegsmarine (since 1941):

```
AAAA Beabsichtige gemeldete Feindstreitkräfte anzugreifen
AAEE Beabsichtige Durchführung Unternehmung wie vorgesehen
AAFF Beabsichtige Durchführung Unternehmung mit vollem Einsatz
AAGG Beabsichtige gemeldete Feindstreitkräfte unter Vermeidung vollen Einsatzes
```

...

Four-letter code with position data and sender information encrypted by Enigma.

Transpositions

The plaintext is written in rows of chosen length k, the columns are reordered according to a permutation π of length k, and the ciphertext is read out column-wise.

- Plain text

  `the transposition method gives a nice mess`

- Matrices

  ```
  thetrans  snarteht
  position  noitisop
  methodgi  igdohtem
  vesanice  ecinasev
  messxxxx  xxxxssem
  ```

- Key $\pi = (8, 7, 6, 5, 4, 3, 2, 1)$ is called *Losung*.
- Cipher text

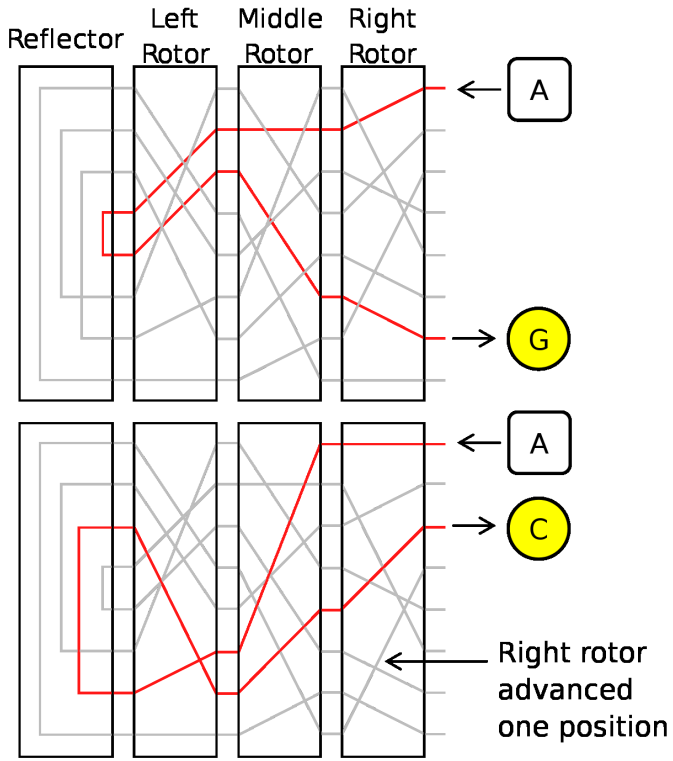
  `sniexnogcxaidixrtonxtihasestsshoeeetpmvm`

Rotor Crypto Machines

Enigma machine (Wehrmacht, World War II)

Rotor Crypto Machines

Enigma encryption



Rotor Crypto Machines

- Encryption in Engima C is an involution:

$$E = PR_1R_2R_3UR_3^{-1}R_2^{-1}R_1^{-1}P^{-1}$$

  with plugboard transformation P, rotors R_1, R_2, R_3, and reflector U.

- If the rotor R_i is moves j positions, the transformation is $\rho^j R_i \rho^{-j}$ with standard transformation $\rho = (a, b, c, \ldots, z)$.

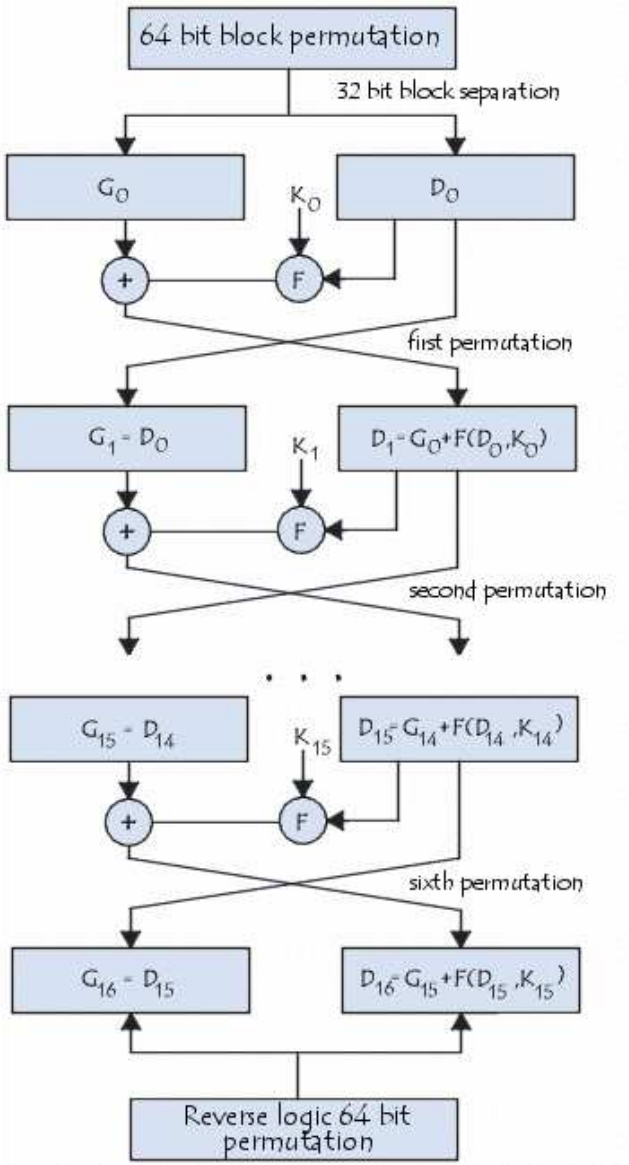
- The transformation becomes

$$\begin{aligned} E &= P(\rho^i R_1 \rho^{-i})(\rho^j R_2 \rho^{-j})(\rho^k R_3 \rho^{-k})U \\ &\quad (\rho^k R_3^{-1} \rho^{-k})(\rho^j R_2^{-1} \rho^{-j})(\rho^i R_1^{-1} \rho^{-i})P^{-1}. \end{aligned}$$

- Encryption and decryption work in the same way.

- Breaking the Enigma by the Polish and the British mainly due to problems with key negotiation and key administration.

Data Encyption Standard (DES)



DES Encyption

- The 8-byte plaintext block is subjected to a (key-independent) initial transposition π and then split into two 4-byte blocks L_0 and R_0.
- Next are 16 rounds, $1 \leq i \leq 16$,

$$L_i = R_{i-1} \quad \text{and} \quad R_i = L_{i-1} \oplus f(R_{i-1}, K_i)$$

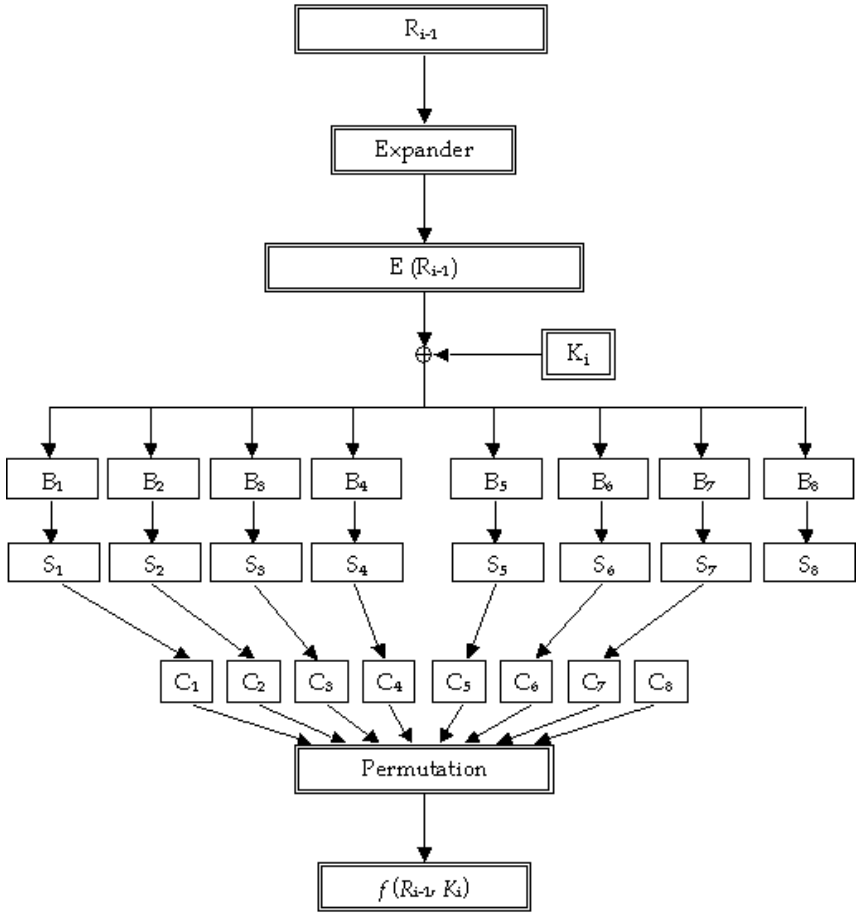
  where $\oplus$ is addition modulo 2 and K_i is a 48-bit key generated from the given key K.
- Final transposition π^{-1} ends encryption step.

DES Encyption

The function f is the central part of DES:

- The 32-bit block R_{i-1} is expanded into 48-bit block $E(R_{i-1})$ by duplication of certain bit positions and added modulo 2 to K_i.
- The 48-bit block is split into eight 6-bit groups as input of the eight substitution modules $S_1, \ldots, S_8$, called *S-boxes*.

Encryption Function f



S-Boxes

S_1	14	4	13	1	2	15	11	8	3	10	6	12	5	9	0	7
	0	15	7	4	14	2	13	1	10	6	12	11	9	5	3	8
	4	1	14	8	13	6	2	11	15	12	9	7	3	10	5	0
	15	12	8	2	4	9	1	7	5	11	3	14	10	0	6	13

S_2	15	1	8	14	6	11	3	4	9	7	2	13	12	0	5	10
	3	13	4	7	15	2	8	14	12	0	1	10	6	9	11	5
	0	14	7	11	10	4	13	1	5	8	12	6	9	3	2	15
	13	8	10	1	3	15	4	2	11	6	7	12	0	5	14	9

S_3	10	0	9	14	6	3	15	5	1	13	12	7	11	4	2	8
	13	7	0	9	3	4	6	10	2	8	5	14	12	11	15	1
	13	6	4	9	8	15	3	0	11	1	2	12	5	10	14	7
	1	10	13	0	6	9	8	7	4	15	14	3	11	5	2	12

S_4	7	13	14	3	0	6	9	10	1	2	8	5	11	12	4	15
	13	8	11	5	6	15	0	3	4	7	2	12	1	10	14	9
	10	6	9	0	12	11	7	13	15	1	3	14	5	2	8	4
	3	15	0	6	10	1	13	8	9	4	5	11	12	7	2	14

S-Boxes

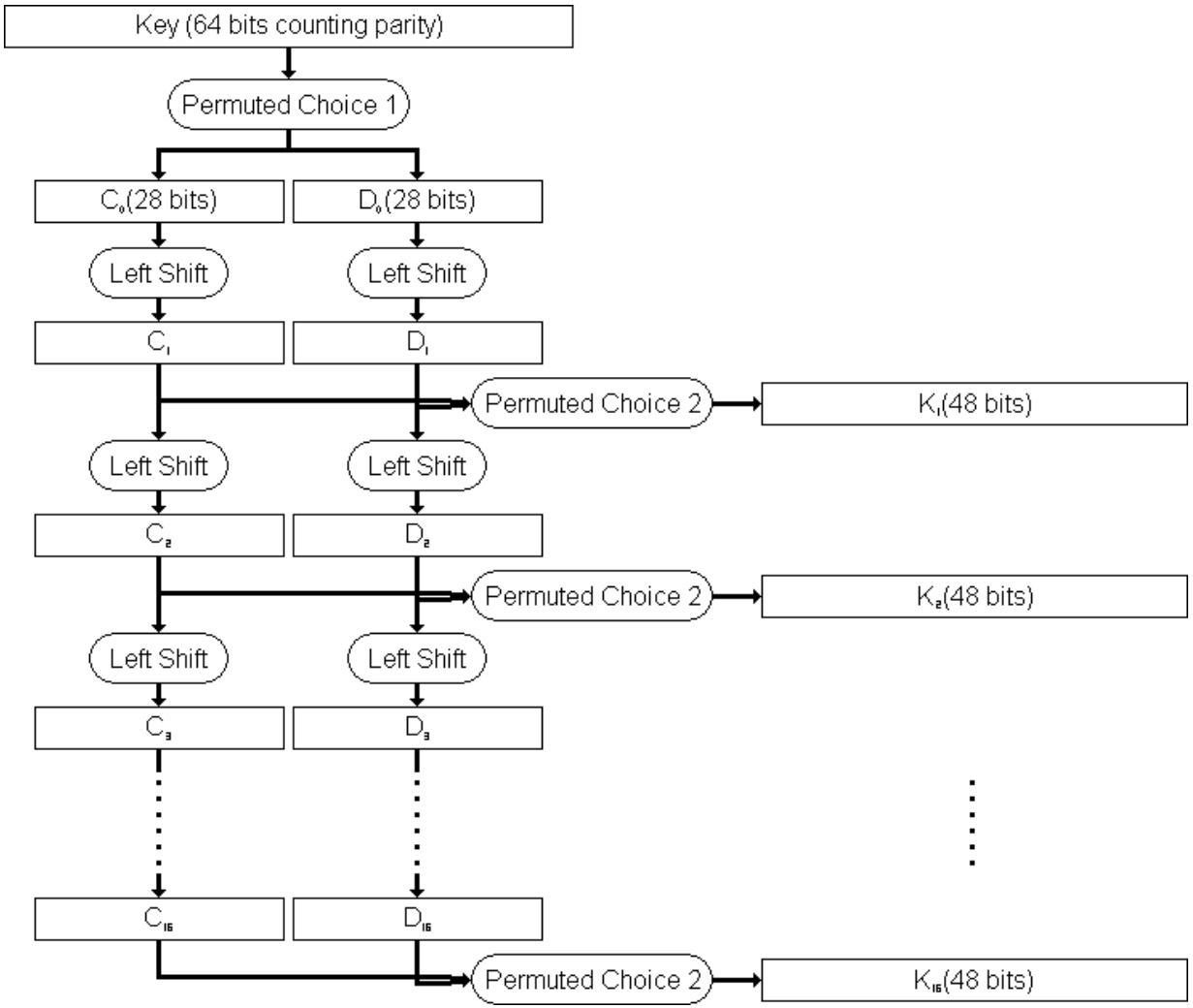
- Bit 1 and 6 of the 6-bit group, interpreted as binary numbers, determine the row.
- Bits 2 to 5 of the 6-bit group determine the column.
- In S-box S_1, the input 110010 gives row 2 (10) and column 9 (1001); the output is 1010 (10).

DES Subkey Generation

Generation of the subkeys:

- The parity bits of the input key K are removed giving a 56-bit word.
- The 56-bit word is transposed w.r.t. fixed prescription and split in two 28-bit blocks.
- These blocks are cyclically left-shifted in each round.
- From these blocks a 48-bit subkey K_i is generated.

DES Subkey Generation



DES Specification

- The rounds of encryption can be described by processing

$$h_i : (L, R) \mapsto (R, L \oplus f(R, K_i))$$

and swapping

$$g : (R, L) \mapsto (L, R).$$

- Both mappings are involutions

$$g(g(R, L)) = g(L, R) = (R, L)$$

and

$$\begin{aligned} h_i(h_i(R, L)) &= h_i(R, L \oplus f(R, K_i)) \\ &= (R, L \oplus f(R, K_i) \oplus f(R, K_i)) \\ &= (R, L). \end{aligned}$$

DES Encryption and Decryption

- Encryption

$$E = \pi^{-1} \circ h_{16} \circ g \circ h_{15} \circ \dots h_2 \circ g \circ h_1 \circ \pi.$$

- Decryption by reversing the order of the subkeys,

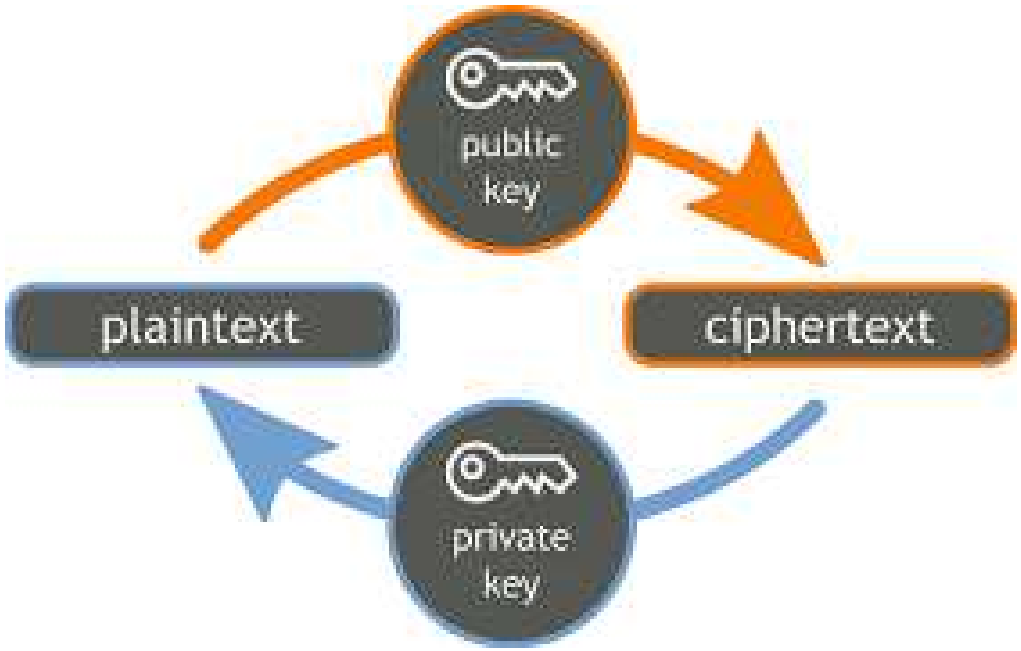
$$D = \pi^{-1} \circ h_1 \circ g \circ h_2 \circ \dots h_{15} \circ g \circ h_{16} \circ \pi.$$

- Since all mappings are self-reciprocal, the composition of E and D yields the identity.

DES Security

- *Avalanche effect:* After a few rounds, each bit of the intermediate result depends on each bit of the plaintext and the key.
- Main points of attack:
  - Design criteria of S-boxes not disclosed (trapdoor?).
  - Key length too small: $2^{56} \approx 72 \cdot 10^{15}$ different keys.
  - In the ECB mode (streaming cipher), the key is kept fixed for quite a while.
- First successful brute force attack in 1998.
- Today, Advanced Encyption Standard (AES) is used (three key lengths: 128, 192, and 256 bits).

Asymmetric Cryptosystems



Asymmetric Cryptosystems

- Each partner has an open (public) key and a private key.
- No negotiation of keys is necessary.
- For a large number of partners, the total number of keys is much less than in symmetric cryptography.
- Public keys of all participants can be stored in an open directory.
- The concept of an open encryption key system was first published by Diffie and Hellman (1976).
- First implementation of open encryption key system by RSA (1998).

Asymmetric Key Structure

- Let K_i and P_i be the public and private key of i-th partner, resp.
- K_i and P_i determine encryption E_i and decryption D_i methods.
- Both E_i and D_i have efficient implementations.
- $(K_i)_i$ is a public directory and P_i is only known to i-th partner.
- Deriving P_i from K_i is practically impossible (intractable).

Asymmetric Encryption

- Suppose the *asymmetric encryption property serving secrecy* holds:

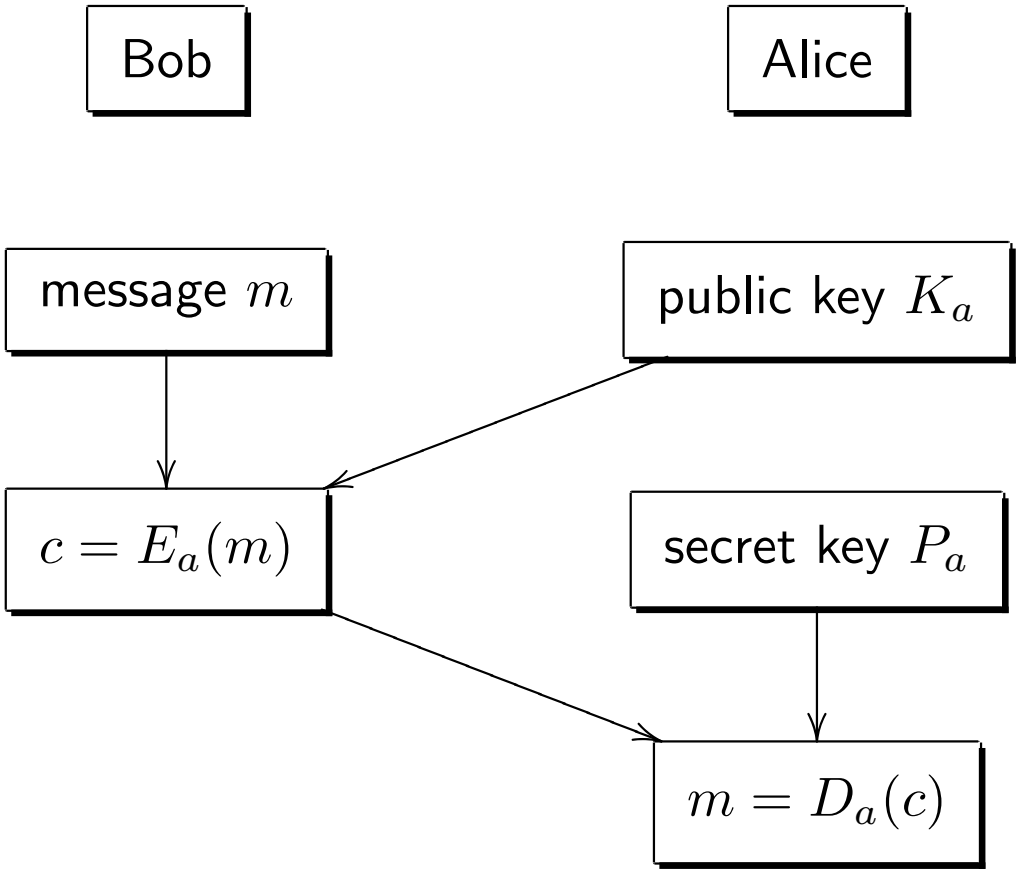
$$D_i(E_i(m)) = m$$

for each message m.

- Bob wants to send message m to Alice:
  - Bob takes the public key K_a of Alice and sends $c = E_a(m)$ to Alice.
  - Alice can recover message m by computing

$$D_a(c) = D_a(E_a(m)) = m.$$

Asymmetric Encryption



Asymmetric Signature Method

- Suppose the *asymmetric signature method serving authentication* holds:

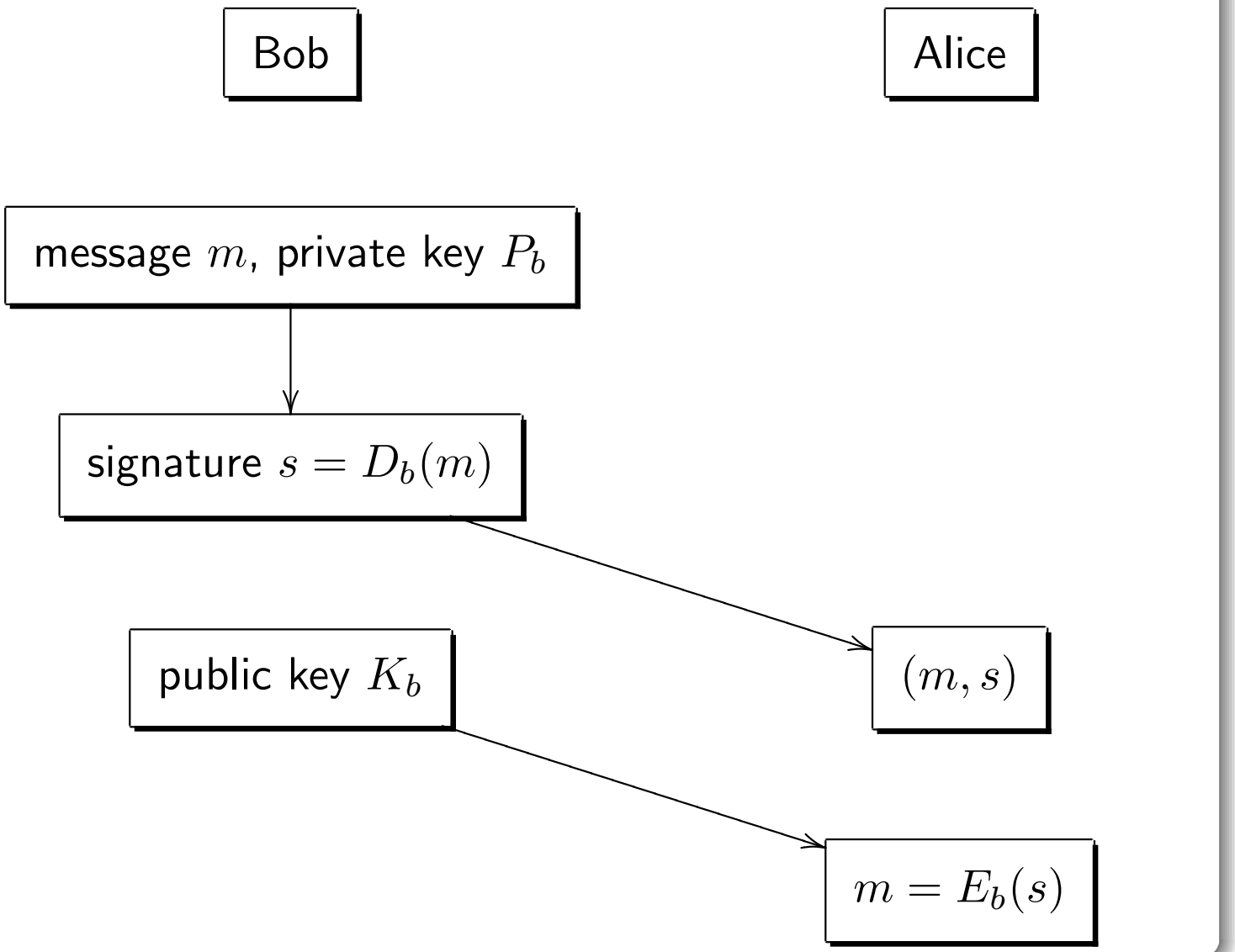
$$E_i(D_i(c)) = c$$

for each cipher c.

- Bob wants to send message m signed by his signature "bOb" to Alice.
  - Bob takes his private key P_b, computes the signature of the message $s = D_b(m)$ and sends (m, s) to Alice.
  - Alice takes Bob's public key K_b to verify the signature by computing

$$E_b(s) = E_b(D_b(m)) = m.$$

Asymmetric Signature Method



Asymmetric Signature Method

- Suppose the *asymmetric signature method serving authentication* holds:

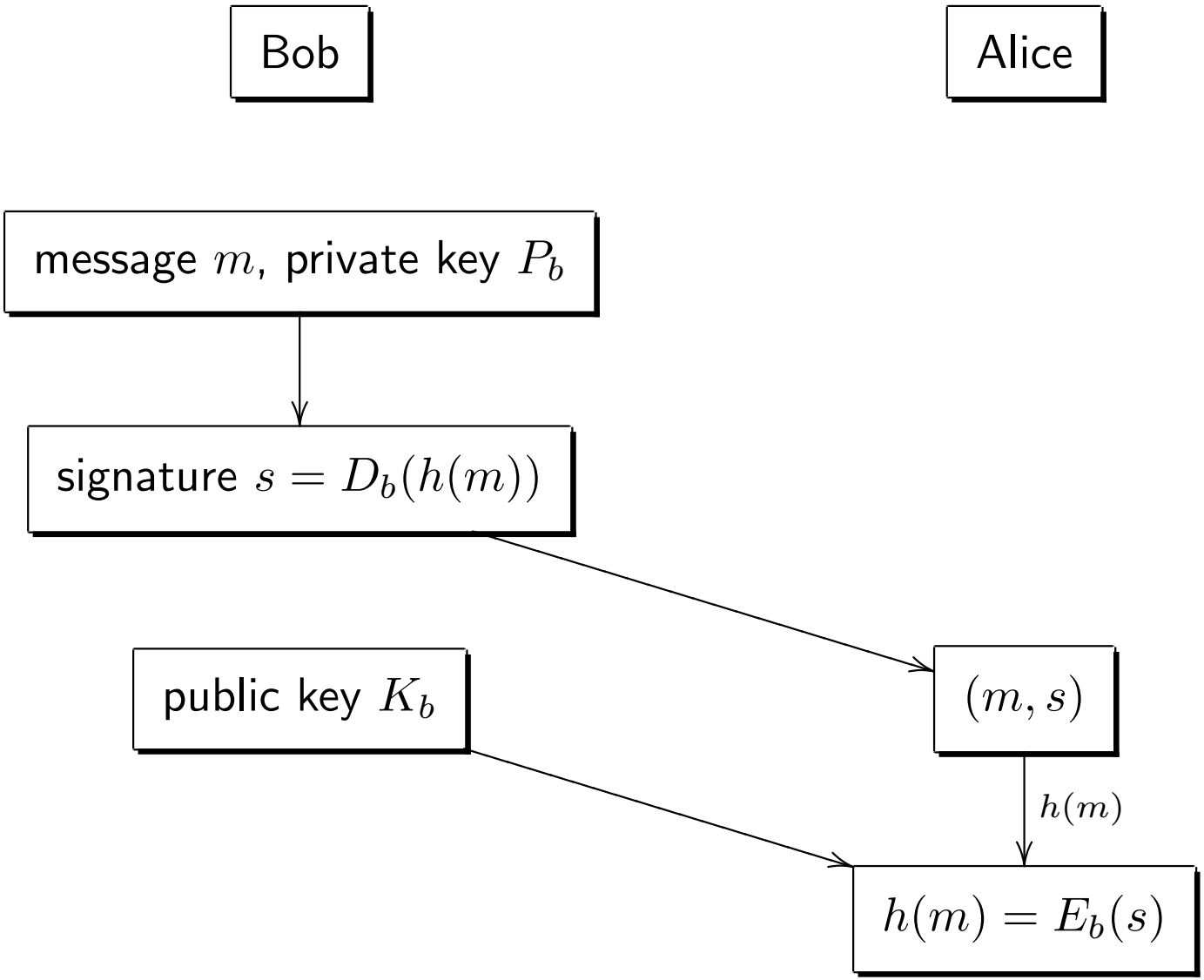
$$E_i(D_i(c)) = c$$

for each cipher c.

- A hash function h is used to map message m of arbitrary size to fixed-length message $h(m)$ before enciphering.
- Bob wants to send message m signed by his signature "bOb" to Alice.
  - Bob takes his private key P_b, computes the signature of the *fingerprinted* message $s = D_b(h(m))$ and sends (m, s) to Alice.
  - Alice takes Bob's public key K_b to verify the signature by computing $h(m)$ and comparing it with

$$E_b(s) = E_b(D_b(h(m))) = h(m).$$

Asymmetric Signature Method – Common Hash Function h



Hash Functions

A *hash function* h has the following properties:

- Applicable to messages of each size.
- Produces fixed-length output.
- For any message m, $h(m)$ is easy to compute.
- Given c, it is computationally infeasible to find m with $h(m) = c$ (one-way property).
- Given x, it is computationally infeasible to find y with $h(x) = h(y)$ (weak collision).

Today, hash functions in use are SHA-1 and MD5.

One-Way Functions

An injective function $f : X \to Y$ is a *one-way function* if the following holds:

- There is an efficient method to calculate $f(x)$ for given $x \in X$.
- There is no efficient method to compute the inverse $x \in X$ from given $y \in \operatorname{ran}(f)$.

One-Way Functions – Example

For a letter L, *some* name starting with L is looked up in the telephone dictionary of a large city, and a 7-digit telephone number listed under this name is the cryptotext.

Encryption of "kindergarten":

k	Koch	8202310	g	Greith	2730661
i	Ivanisevic	8119896	a	Aranyi	2603760
n	Nadler	6926286	r	Rexroth	5328563
d	Dicklberger	5702035	t	Tecins	6703008
e	Esau	8348578	e	Eisenhauer	7913174
r	Remy	7256575	n	Neunzig	3002123

Encryption is a sequence of twelve 7-digit codegroups:

8202310	8119896	6926286	5702035	8348578	7256575
2730661	2603760	5328565	6703008	7913174	3002123

Trapdoor is the (legally established) inverse dictionary.

One-Way Functions

- One-way functions cannot be used in a reasonable way for encryption of messages followed up by decryption.
- One-way functions can be used for authentication.
  - A password is encrypted by a one-way function and stored in this form.
  - Any time access is required the password presented is encrypted and compared with the stored cryptotext (UNIX^{TM} operating system).

Trapdoor Functions

An injective function $f : X \to Y$ is a *trapdoor function* if the following holds:

- There is an efficient method to compute $f(x)$ for given $x \in X$.
- There is no efficient method to calculate the inverse $x \in X$ from given $y \in \operatorname{ran}(f)$, unless an additional secret information (trapdoor) is available.

One-Way Function – Example

A one-way function without trapdoor: multiplication of primes.

Let $X = \{(x_1, x_2) \mid x_1, x_2 \text{ prime}, K \leq x_1 \leq x_2\}$ for sufficiently large K. The injective function

$$f : X \to \mathbb{N} : (x_1, x_2) \mapsto x_1 \cdot x_2$$

is one-way (multiplication of large numbers takes only seconds). No trapdoors (factorization) are known.

One-Way Function – Example

A one-way function without trapdoor: exponentiation in $\mathbb{Z}_p$.

Let p be a prime. For a fixed element $a \in \mathbb{Z}_p^*$ define the a-exponential function

$$f_a : \mathbb{Z}_p^* \to \mathbb{Z}_p^* : n \mapsto a^n \mod p$$

for sufficiently large p and a.
The function f_a is one-way (fast modular exponentiation). No trapdoors (discrete logarithm) are known.

Trapdoor One-Way Function – Example

Let e, n be positive integers.

- The exponentiation (RSA encryption)

$$f(m) = m^e \mod n$$

  is one-way (fast modular exponentiation).

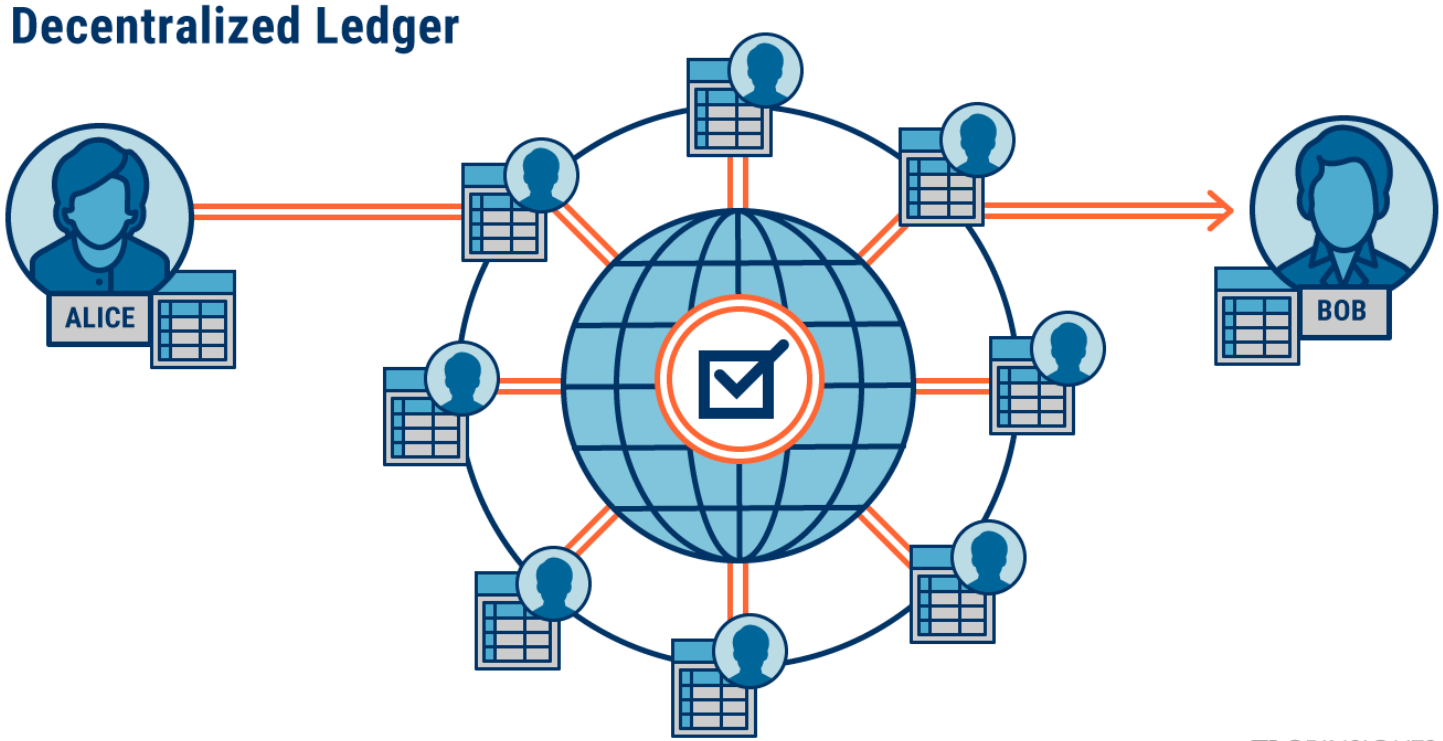
- Given $y = f(m) = m^e \mod n$. Trapdoor is the inverse of e mod $\phi(n)$, i.e.,

$$ed \equiv 1 \mod \phi(n).$$

  Then

$$y^d = m^{ed} \equiv m \mod n.$$

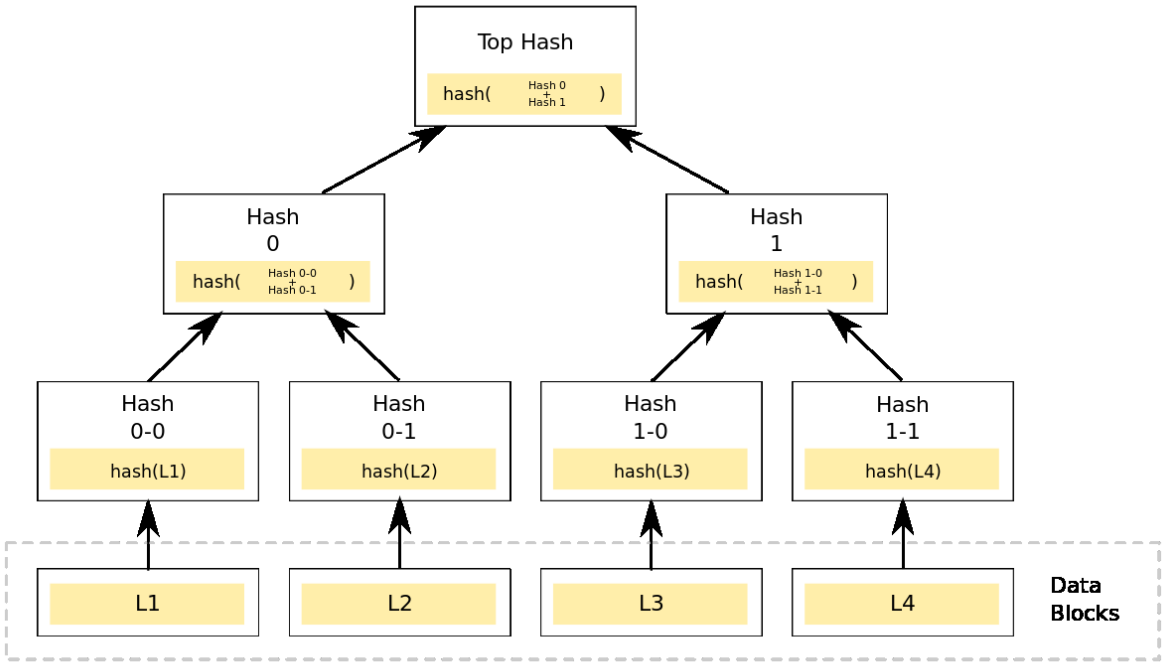
Blockchain



Blockchain – Technology

- Continually extendable list of data records (blocks) linked using cryptography.
- Each block contains cryptographic hash of previous block, time stamp, and transaction data (represented by Merkle tree).
- Blockchain implements distributed ledger technology (consensus of replicated, shared and synchronized digital data geographically spread across multiple sites without central administration).
- First application: Bitcoin (Satoshi Nakamoto, 2008)

Merkle Tree – Binary Hash Tree (1979)



- Hashes 0-0 and 0-1 are hash values of data blocks L1 and L2, resp.
- Hash 0 is the hash of the concatenation of the hashes 0-0 and 0-1, and so on.

Blockchain – Properties

- Consensus mechanism: New blocks are added by decentral consensus (replaces trustworthy third party).
- Concatenation principle: Addition of new blocks via linked list.
- Decentral storage: Participants can store their own copy.
- Security against manipulation: Data in a block cannot be altered retroactively (requires consensus of network majority).
- Transparency: Blocks are visible to the participants, but content can be encrypted.
- Authentication: Blocks can use digital signatures.

Cryptology – Breaking a Cryptosystem

Two types of information required:

- System structure (often leaks out over a period of time).
- Enciphering key (by frequency analysis of intercepted encrypted messages).

> `Deciphering is in my opinion one of the most fascinating of arts, and I fear I have wasted upon it more time than it deserves.`
>
> Charles Babbage, 1864

Cryptology Axioms

- One should not underrate the adversary.
- Only the cryptanalyst, if anybody, can judge the security of a cryptosystem.
- A cryptosystem should be secure even if everything about the system except the key is public knowledge (Augustine Kerckhoff, 19th century). Equivalently, the enemy knows the system being used (Claude Shannon, 1949).
- Superficial complications can be illusory (e.g., Umkehrwalze in the Enigma).
- Cryptographic faults should be taken into account in judging the encryption security (e.g., probable words and phrases like "by order of the Führer", repetition of encrypted message in plain).

Cryptology – Efficiency Boundary

- Technological progress shifts the border line between "intractable" and "efficient".
- Roughly, every two years the computer speed doubles and every 15 months the computer costs halve.
- Cryptologists counteract this by suitably increasing some of the encryption parameters.
- See the Wikipedia page "Integer factorization records".

RSA

In cryptography, no rule is absolute.

Étienne Bazeries, 1901

RSA

- RSA key generation
- Confidential message transmission
- RSA correctness
- RSA parameters
- Attacks against plain RSA

RSA

- Inventors Ron Rivest, Adi Shamir, Leonard Adleman (1978)
- One of the first public-key cryptosystems
- Security relies on mathematical problems
- Usage in passing shared keys for symmetric key cryptography and digital signature

Key Generation

Each user U generates a public/private key pair:

- Select two large primes at random, p and q
- Compute the Eulerian totient value $\phi(pq) = (p-1)(q-1)$
- Compute the system modulus $n = p \cdot q$
- Select at random the encryption key e,

$$1 < e < \phi(n) \quad \text{and} \quad (e, \phi(n)) = 1$$

- Calculate the decryption key d,

$$0 \leq d \leq n \quad \text{and} \quad ed \equiv 1 \mod \phi(n).$$

Public encryption key $U_P = (e, n)$ and secret decryption key $U_R = \{d\}$.

Exercise (RSA)

- Choose randomly two primes p and q with arity ≥ 100.
- Compute $\phi(pq)$.
- Choose randomly a prime e with arity ≥ 20.
- Compute the inverse d of e in $\mathbb{Z}_{\phi(pq)}$.

Exercise (RSA) – Solution

```
> with(numtheory):
> r1 := rand(10^100..10^101):
> a := r1():
> p := nextprime(a):
> b := r1():
> q := nextprime(b):
> pq_phi := (p-1)*(q-1):
> r2 := rand(10^20..10^21):
> c := r2():
> e := nextprime(c):
> igcdex(pq_phi,e,'r','s'):
> d := modp(s,pq_phi):
```

Exercise (RSA) – Toy Example

```
> with(numtheory):
> r1 := rand(100..1000):
> p := nextprime(r1());
> q := nextprime(r1());
```

$$p = 163, \quad q = 439$$

```
> pq_phi := (p-1)*(q-1);
```

$$70956$$

```
> r2 := rand(10..100):
> e := nextprime(r2());
```

$$e = 13$$

```
> igcdex(pq_phi,e,'r','s'):
> d := modp(s,pq_phi);
```

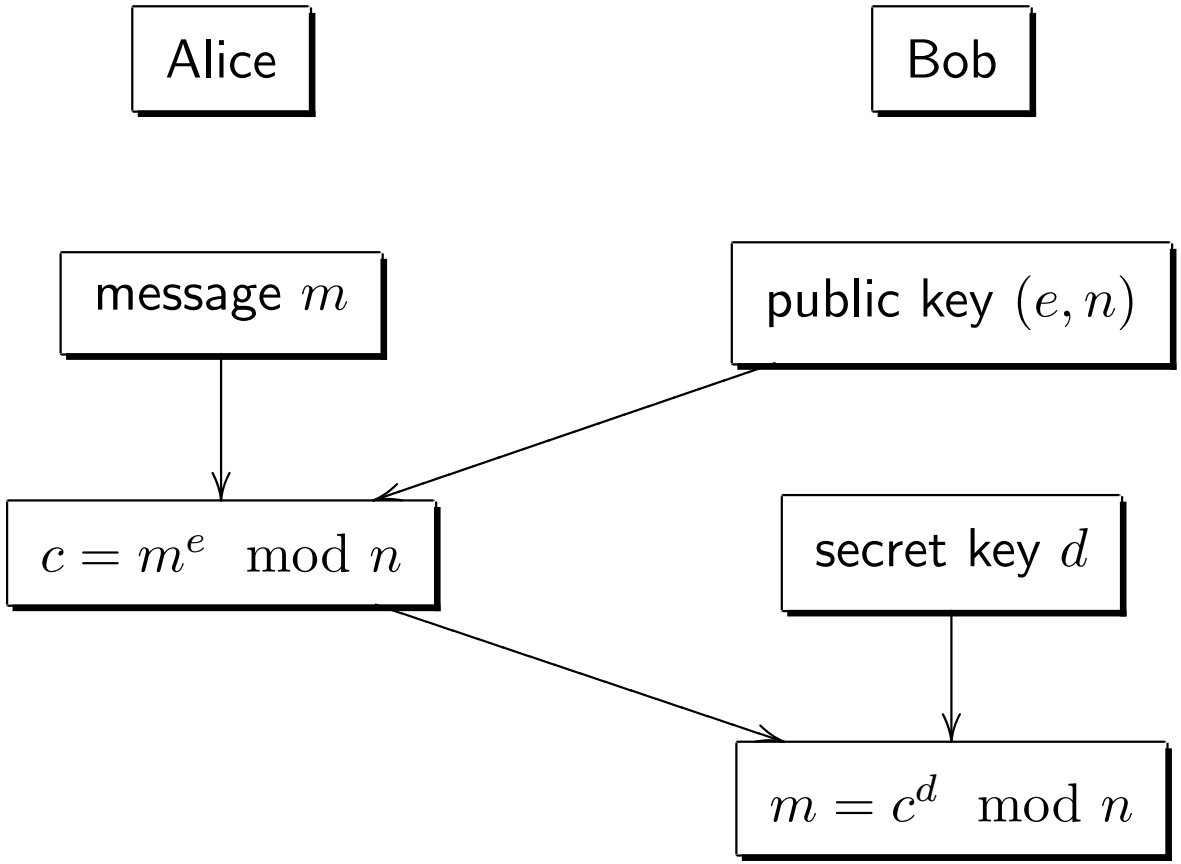
$$d = 32749$$

RSA – Confidential Message Transmission

Alice has message m she wants to send to Bob.

- Alice encrypts the message m:
  - Alice obtains the public key of Bob: $B_P = (e, n)$
  - Alice computes the ciphertext $c = m^e \mod n$, where $0 \leq m < n$
  - Alice sends the ciphertext c to Bob.
- Bob decypts the ciphertext c:
  - Bob takes his private key $B_R = \{d\}$.
  - Bob computes $m = c^d \mod n$.

RSA – Confidential Message Transmission



RSA – Correctness

For each $m \in \mathbb{Z}_n$,

$$m^{ed} \equiv m \mod n. \tag{1}$$

Proof.

Since $ed \equiv 1 \mod \phi(n)$, $ed = 1 + k\phi(n)$ for some integer k.

- Let $(m, n) = 1$. Then by Euler's theorem,

$$m^{ed} = m^{1+k\phi(n)} = m\left(m^{\phi(n)}\right)^k = m \cdot 1^k = m.$$

- Let $(m, n) \neq 1$. Then $p|m$ or $q|m$. Take the ring isomorphism

$$\psi : \mathbb{Z}_n \to \mathbb{Z}_p \times \mathbb{Z}_q : m \mapsto (m \mod p, m \mod q).$$

Suppose $p|m$, $q \nmid m$. Then $\psi(m^{ed}) = (0, m^{ed} \mod q) = (0, m \mod q) = (m \mod p, m \mod q) = \psi(m)$ as in the first case and so $m^{ed} \equiv m \mod n$. □

RSA – Parameters

Let n be the product of two primes. Then knowing $\phi(n)$ is sufficient to recover both primes.

Proof.

Let $n = pq$ with primes p and q. Then

$$n-\phi(n)+1 = n-(p-1)(q-1)+1 = n-pq+p+q-1+1 = p+q.$$

Suppose $p > q$. Then

$$p - q = \sqrt{(p-q)^2} = \sqrt{(p+q)^2 - 4n}.$$

Thus $p+q$ and $p-q$ are known and so

$$\begin{aligned} p &= \frac{1}{2}[(p+q)+(p-q)], \\ q &= \frac{1}{2}[(p+q)-(p-q)]. \end{aligned}$$

RSA – Parameters

- The numbers $p-1$ and $q-1$ should have only very small common factors, besides the necessary 2.

  Indeed, any common factors of $p-1$ and $q-1$ are present in the factorization of $n-1$, since

$$n-1 = pq-1 = (p-1)(q-1)+(p-1)+(q-1).$$

- Use of small value for e with small number of binary digits, such as the Fermat prime

$$2^{2^4}+1 = 65537,$$

  speeds up the encryption process.

  Indeed, exponentiation can be computed by the Horner scheme.

Security of RSA

The security of RSA cryptosystem is based on two mathematical problems:

- Factorization of large numbers.
- Computation of discrete logarithm (RSA problem).

No efficient algorithms exist for solving them.

Practical RSA implementations prepare message m by using structured randomized padding before encryption.

Attacks against plain RSA

- Small exponents (e.g., $e = 3$) and small values of m (e.g., $m < n^{1/e}$) give $m^e < n$. Decryption of c may be possible by taking its e-th root.
- Send the same encryption of m to $\geq e$ users with same encryption key e but different p, q and therefore n. The Chinese Remainder theorem may be used for decryption.
- Encrypt likely plaintexts under the public key and test if they are equal to the ciphertext (chosen plaintext attack).

Avoid plain RSA attacks by using a padding scheme such as the standard PKCS#1.

Discrete Log

> Even in cryptology, silence is golden.
>
> Laurence D. Smith

Discrete Log

- Definition of discrete logarithm
- Diffie-Hellman key exchange
- Massey-Omura cryptosystem
- ElGamal cryptosystem

Discrete Log

- Given finite group G, $g \in G$, and $y \in G$ which is a power of g. The *discrete logarithm* of y to base g is any integer x such that
$$g^x = y.$$
Write $x = \log_g y$ for the minimum integer $x \geq 0$.
- *Special case:* Given finite cyclic group G, a generator g of G and $y \in G$. The *discrete logarithm* of y to base g is any integer x such that
$$g^x = y.$$
Write $x = \log_g y$ for the minimum integer $x \geq 0$.

  Example: The multiplicative group of each finite field $\mathbb{F}_q$ is cyclic: $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$.

Example

The cyclic group $G = \mathbb{F}_{19}^* = \mathbb{Z}_{19}^*$ is generated by $b = 2$,

$$\begin{aligned} 2^1 &= 2, & 2^2 &= 4, \\ 2^3 &= 8, & 2^4 &= 16, \\ 2^5 &= 13, & 2^6 &= 7, \\ 2^7 &= 14, & 2^8 &= 9, \\ 2^9 &= 18, & 2^{10} &= 17, \\ 2^{11} &= 15, & 2^{12} &= 11, \\ 2^{13} &= 3, & 2^{14} &= 6, \\ 2^{15} &= 12, & 2^{16} &= 5, \\ 2^{17} &= 10, & 2^{18} &= 1. \end{aligned}$$

Thus $\log_2 7 = 6$.

Example

Take the cyclic group $G = \mathbb{F}_8^*$ generated by the root α of $X^3 + X + 1 \in \mathbb{F}_2[X]$. Since $\alpha^3 + \alpha + 1 = 0$, i.e., $\alpha^3 = \alpha + 1$, the elements of G are

$$\begin{aligned}
\alpha^0 &= 1 \\
\alpha^1 &= \alpha \\
\alpha^2 &= \alpha^2 \\
\alpha^3 &= \alpha + 1 \\
\alpha^4 &= \alpha^2 + \alpha \\
\alpha^5 &= \alpha^2 + \alpha + 1 \\
\alpha^6 &= \alpha^2 + 1 \\
\alpha^7 &= 1.
\end{aligned}$$

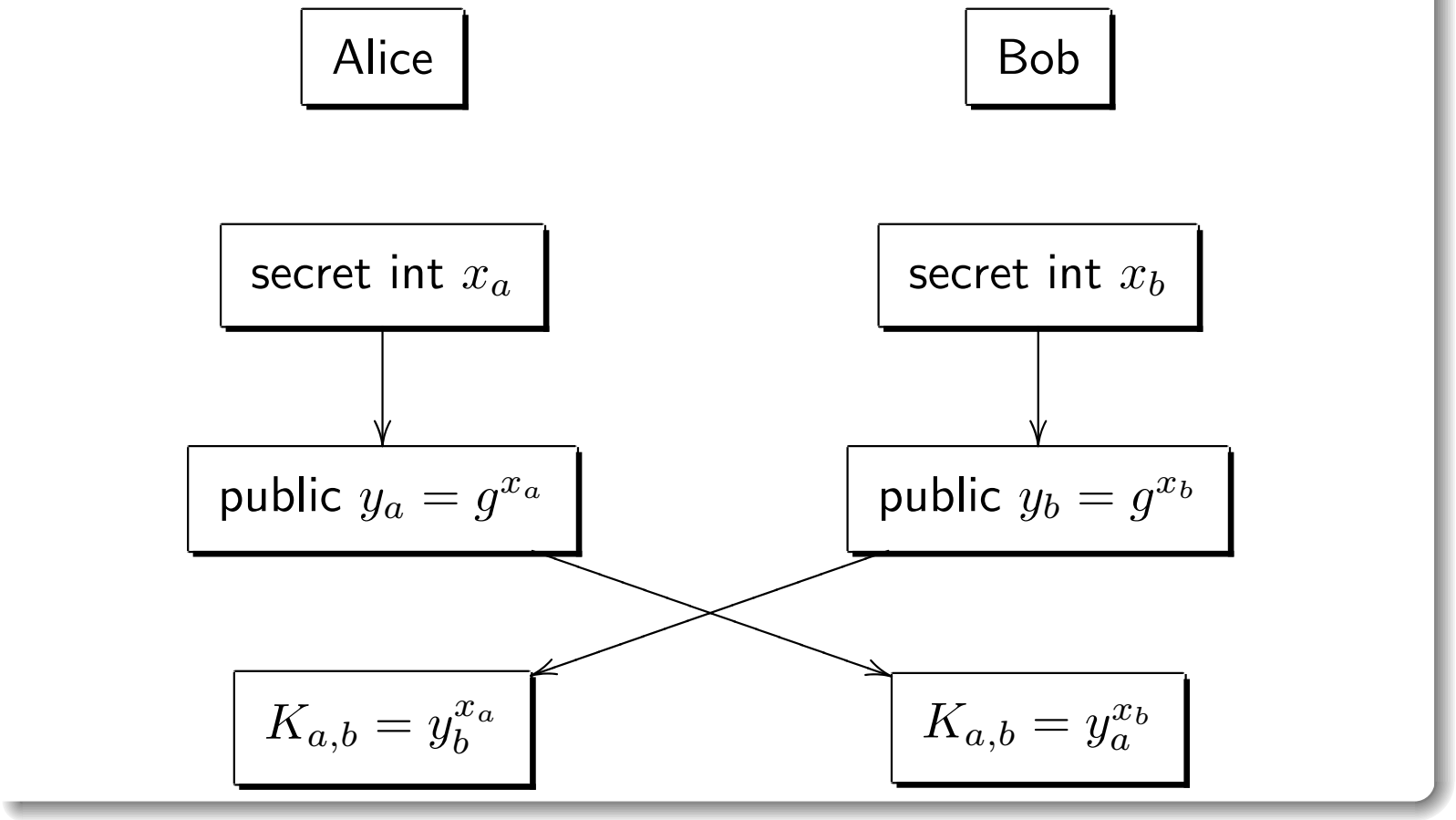
Thus $\log_\alpha(\alpha^2 + \alpha) = 4$.

Diffie-Hellman Key Exchange (1976)

- Alice and Bob agree on large finite group G and element $g \in G$ of order n (public).
- Alice takes secret integer x_a with $1 \leq x_a \leq n-1$ and computes public $y_a = g^{x_a} \in G$.
- Bob takes secret integer x_b with $1 \leq x_b \leq n-1$ and computes public $y_b = g^{x_b} \in G$.
- Alice and Bob establish common secret key:
  - Alice computes $y_b^{x_a}$.
  - Bob computes $y_a^{x_b}$.
  - Common secret key $K_{a,b} = g^{x_a x_b}$.
- Correctness:

$$y_b^{x_a} = (g^{x_b})^{x_a} = K_{a,b} = (g^{x_a})^{x_b} = y_a^{x_b}.$$

Diffie-Hellman Key Exchange



Diffie-Hellman Key Exchange

- *Diffie-Hellman assumption:* It is computationally infeasible to compute $K_{a,b} = g^{x_a x_b}$ when the evesdropper only knows the transmitted messages

$$y_a = g^{x_a} \quad \text{and} \quad y_b = g^{x_b}.$$

- Discrete logarithms:

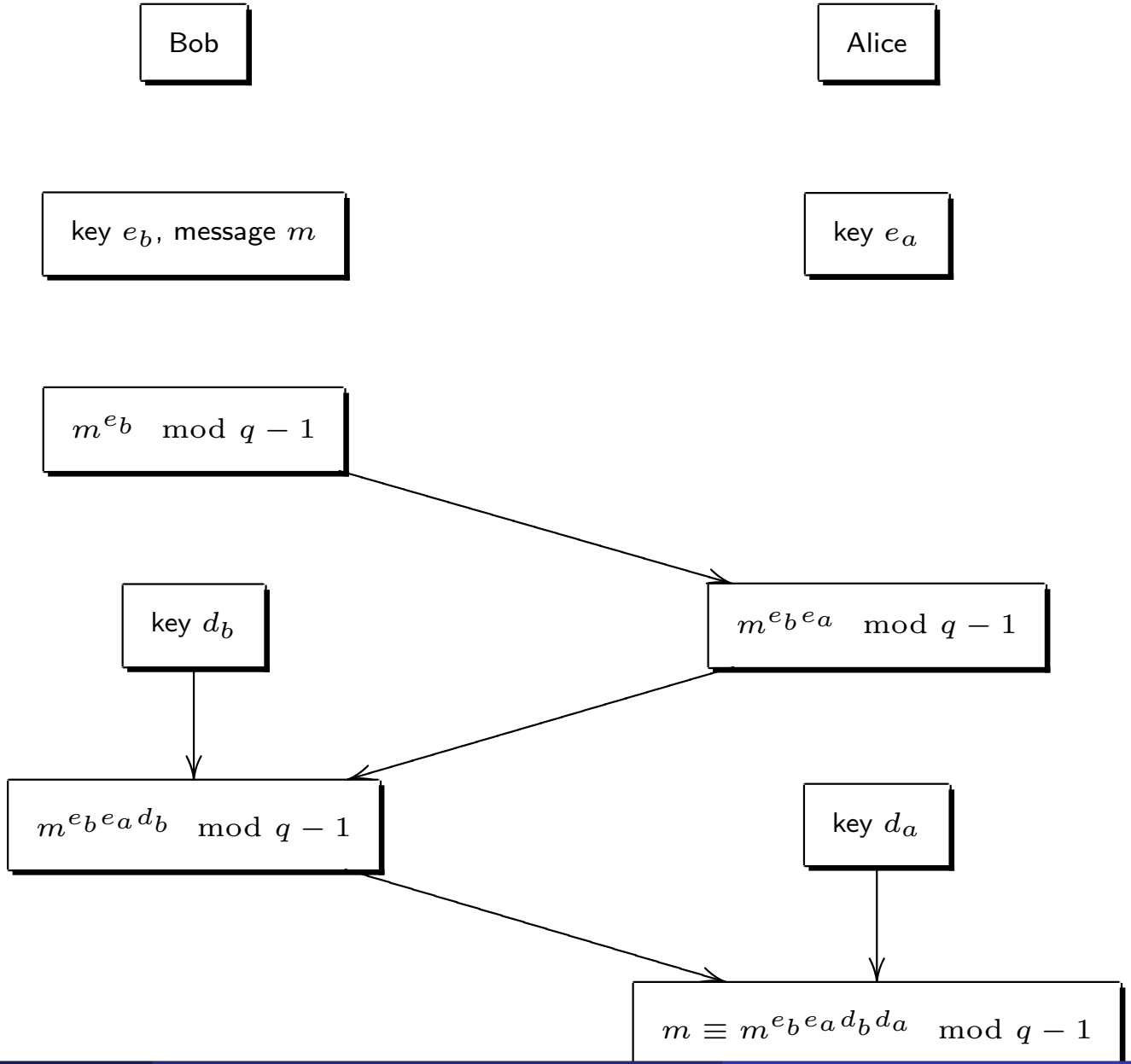
$$x_a = \log_g y_a \quad \text{and} \quad x_b = \log_g y_b.$$

If discrete logarithms are computable, the Diffie-Hellman assumption will fail.

Massey-Omura Cryptosystem (1993)

- Public data: large prime power q.
- Each user takes secret random integer e, $1 \leq e \leq q-1$, with $(e, q-1) = 1$ and computes its inverse $d = e^{-1} \mod q-1$; i.e., $ed \equiv 1 \mod q-1$.
- Bob has message m, $0 \leq m < q$, and communicates with Alice:
  - Bob sends $m^{e_b} \mod q-1$ to Alice.
  - Alice sends $m^{e_b e_a} \mod q-1$ back.
  - Bob sends $m^{e_b e_a d_b} = m^{e_a} \mod q-1$ back.
  - Alice computes $m^{e_a d_a} \equiv m \mod q-1$.
- Correctness follows from (1).
- Security is based on computing discrete logarithms.
- Method is rarely used in practice.

Massey-Omura Cryptosystem



ElGamal Cryptosystem (1985)

- Public data: large prime power q and generator $g \in \mathbb{F}_q^*$.
- Each user takes random private key x_u with $1 \leq x_u \leq q-1$ and computes public key $y_u = g^{x_u}$.
- Bob wants to send message m, $0 \leq m < q$, to Alice:
  - Bob chooses random number r with $1 \leq r \leq q-1$ and sends $(g^r, m \cdot y_a^r)$ to Alice.
  - Alice computes $s = (g^r)^{x_a} = g^{rx_a}$.
  - Alice recovers message:

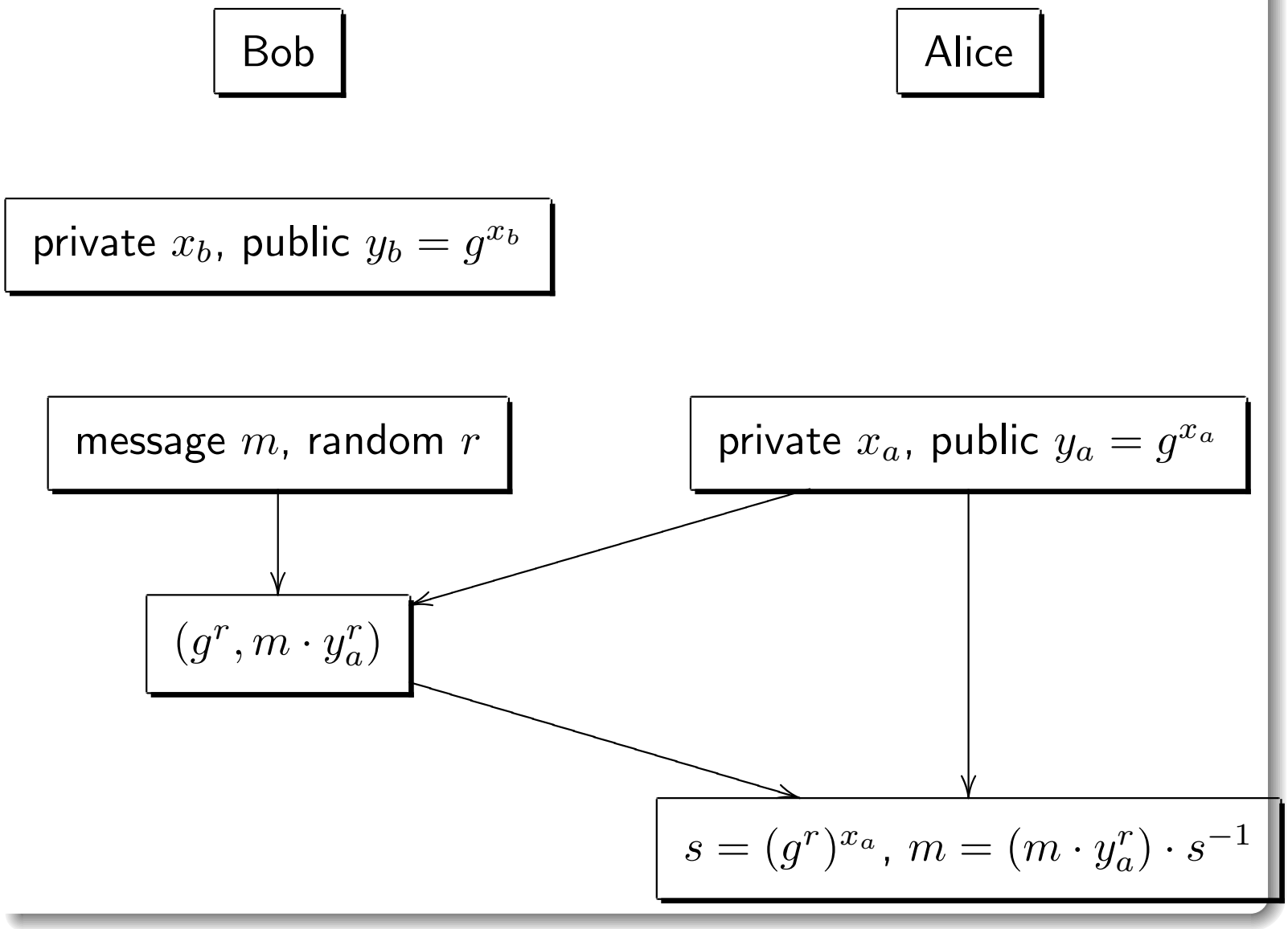
$$(m \cdot y_a^r) \cdot s^{-1} = m \cdot (g^{x_a r}) \cdot (g^{rx_a})^{-1} = m.$$

Direct computation of inverses:

$$(g^a)^{-1} = g^{q-1-a}, \quad 1 \leq a \leq q-1.$$

- Security is based on computing discrete logarithms.

ElGamal Cryptosystem



Attacking Discrete Logorithm

> Deciphering is an affair of time, ingenuity, and patience.

Charles Babbage, 1864

Attacking Discrete Logorithm

- Complete enumeration
- Baby-step giant-step algorithm
- Pohlig-Hellman algorithm
- Pollard's rho method

Complete Enumeration

The simplest attacking method is complete enumeration.

Require: Given cyclic group G of order n, generator g and group element y.
Ensure: Integer x with $y = g^x$.
  for $x \leftarrow 1$ to n **do**
    if $y = g^x$ **then**
      return x
    end if
  end for

This attack is extremely time-consuming. The other attacks here are much quicker.

Baby-Step Giant-Step Algorithm (1978)

Meet-in-the-middle algorithm for computing the discrete logarithm.

- Given cyclic group G of order n, generator g and group element y. Find an integer x with $y = g^x$.
- The BSGS algorithm is based on rewriting x as $x = qm + r$ with $m = \lceil \sqrt{n} \rceil$ and $0 \leq r \leq m - 1$.
- *Baby steps:* compute list of pairs $(y \cdot g^{-r}, r)$, $0 \leq r \leq m - 1$, in table; for fast retrieval use hash table or binary search in sorted array.
- *Giant steps:* compute g^{qm} for $1 \leq q \leq m - 1$.
- Time complexity $O(\sqrt{n})$.

Baby-Step Giant-Step Algorithm

Require: Cyclic group G of order n with generator g, group element y.
Ensure: Integer x with $g^x = y$.

$m \leftarrow \lceil \sqrt{n} \rceil$
for $r \leftarrow 0$ to $m-1$ **do**
  store pairs $(y \cdot g^{-r}, r)$ in table
end for
if $y \cdot g^{-r} = 1$ for some r **then**
  return $x \leftarrow r$
end if
for $q \leftarrow 1$ to $m-1$ **do**
  if g^{qm} equals the first element $y \cdot g^{-r}$ in list **then**
    return $x \leftarrow qm + r$
  end if
end for

Baby-Step Giant-Step Algorithm – Example

```
> n:=19: g:=2: y:=15: m:=4:
> for r from 0 to m-1 do print(modp(y/g^r,n)) end do;
15
17
18
9
> for q from 1 to m do print(modp(g^(q*m),n)) end do;
16
9
11
5
> q:=2: r:= 3: x:=q*m+r; modp(g^x,n);
11
15
```

Pohlig-Hellman Algorithm (1978)

Computation of discrete logarithms in finite abelian group whose order is a smooth integer.

- A *smooth integer* is an integer that factors completely into small prime numbers.
- Special algorithm for cyclic groups with prime-power order.
- General algorithm for cyclic groups using the Chinese Remainder theorem.

Pohlig-Hellman Algorithm

Given cyclic group G of order $n = p^e$ with generator g, group element y with $y = g^x$ and x, $0 \leq x \leq n-1$, unknown.

- Put $h = g^{n/p} = g^{p^{e-1}}$ and $y_0 = y^{n/p}$.
- Then $y_0 = (g^x)^{n/p} = h^x$ and h has order p, i.e., $h^p = 1$.
- If the discrete log problem is solvable in $\langle h \rangle$, there is x_0, $0 \leq x_0 \leq p-1$, with
$$y_0 = h^{x_0}.$$

Pohlig-Hellman Algorithm

- Consider the p-adic expansion of x, $0 \le x \le p^e - 1$,

$$x = x_0 + x_1 p + \ldots + x_{e-1} p^{e-1}$$

with $0 \le x_0, \ldots, x_{e-1} \le p - 1$.

- Use $y = g^{x_0 + x_1 p + \ldots + x_{e-1} p^{e-1}}$ to compute

$$\begin{aligned} y_j &= \left(y \cdot g^{-(x_0 + \ldots + x_{j-1} p^{j-1})} \right)^{n/p^{j+1}} \\ &= \left(g^{x_j p^j + \ldots + x_{e-1} p^{e-1}} \right)^{n/p^{j+1}} \\ &= (g^{x_j})^{n/p} = h^{x_j}, \end{aligned}$$

where $h = g^{n/p}$ has order p.

- If the discrete log problem is solvable in $\langle h \rangle$, there is x_j, $0 \le x_j \le p - 1$, with $y_j = h^{x_j}$.

Pohlig-Hellman Algorithm (PH I)

Require: Cyclic group G of order $n = p^e$ with generator g, group element y.

Ensure: Unique integer x, $0 \leq x \leq n-1$, with $g^x = y$.

$y_0 \leftarrow y^{n/p}$

$h \leftarrow g^{n/p}$

Use BSGS algorithm to compute x_0, $0 \leq x_0 \leq p-1$, with $h^{x_0} = y_0$

$x \leftarrow x_0$

for $j \leftarrow 1$ to $e-1$ **do**

$y_j \leftarrow \left(y \cdot g^{-(x_0 + \ldots + x_{j-1} p^{j-1})}\right)^{n/p^{j+1}}$

$h \leftarrow g^{n/p}$

Use BSGS algorithm to compute x_j, $0 \leq x_j \leq p-1$, with $h^{x_j} = y_j$

$x \leftarrow x + p^j x_j$

end for

return x

Pohlig-Hellman Algorithm (PH II)

Require: Cyclic group G of order n with generator g, group element y, and factorization $n = \prod_{i=1}^{r} p_i^{e_i}$.
Ensure: Unique integer x, $0 \leq x \leq n-1$, with $g^x = y$.
  for $i \leftarrow 1$ to r **do**
    $g_i \leftarrow g^{n/p_i^{e_i}}$ $\{g_i$ has order $p_i^{e_i}\}$
    $y_i \leftarrow y^{n/p_i^{e_i}}$ $\{y_i \in \langle g_i \rangle\}$
    Use algorithm PH I in $H_i = \langle h_i \rangle$ to compute x_i with $h_i^{x_i} = y_i$
  end for
  Solve the simultaneous congruences $x \equiv x_i \mod p_i^{e_i}$ for $1 \leq i \leq r$ {The CRT gives a unique solution x with $0 \leq x \leq n-1$.}
  return x

Pollard's rho Method (1975)

Computation of discrete logarithms in finite abelian group.

- Special algorithm for cyclic groups with prime-power order.
- General algorithm for cyclic groups similar to Pohlig-Hellman.
- Time complexity $O(\sqrt{n})$ but needs less space than BSGS.

Pollard's rho Method

Given cyclic group G of order $n = p^e$ with generator g, group element y with $y = g^x$, $0 \leq x \leq n-1$, unknown.

- Compute elements

$$h_i = g^{a_i} \cdot y^{b_i}, \quad 1 \leq i \leq s,$$

where a_i, b_i are random integers.

- Take surjective function $f : G \to \{1, \ldots, s\}$ partitioning G into disjoint subsets

$$G_i = f^{-1}(\{i\}), \quad 1 \leq i \leq s.$$

- Use starting point

$$z_0 = g^{u_0} \cdot y^{v_0}$$

to define the sequence

$$z_{l+1} = z_l \cdot h_{f(z_l)}, \quad l \geq 0.$$

Pollard's rho Method

- Each element z_l has the form

$$z_l = g^{u_l} \cdot y^{v_l}, \quad l \geq 0.$$

- Since G is finite, there are indices $l < m$ with $z_l = z_m$. Then

$$g^{u_l - u_m} = y^{v_m - v_l} = g^{(v_m - v_l)x}.$$

Thus

$$u_l - u_m \equiv (v_m - v_l)x \mod n.$$

- If $(v_m - v_l, n) = 1$, then $v_m - v_l$ is invertible in $\mathbb{Z}_n$ and so

$$x = (u_l - u_m) \cdot (v_m - v_l)^{-1}.$$

Pollard's rho Method

- If $d = (v_m - v_l, n) > 1$, then $(v_m - v_l)/d$ is invertible in $\mathbb{Z}_n$ and so for some integer v', $v'(v_m - v_l) \equiv d \mod n$.
- Since $u_l - u_m \equiv (v_m - v_l)x \mod n$ and $v_m - v_l$ is multiple of d, $u_l - u_m$ is a multiple of d, i.e., $u_l - u_m = du'$ for some integer u'.
- Thus $du'v' \equiv dx \mod n$ and hence

$$x \equiv u'v' + k\frac{n}{d} \mod n$$

  for some $k = 0, \frac{n}{d}, \ldots, (d-1)\frac{n}{d}$.
- If d is too large, take new starting point.
- The (smallest) equality $z_l = z_m$ can be expected after $O(\sqrt{n})$ steps.

Pseudoprimes

> Today, the Department of Computer Science at the University of Washington announced that $2^{58,111,625,031} + 8$ is even. This is the largest non-prime yet reported.

Bathroom graffiti, University of Washington

> The largest known prime number is $2^{77,232,916} - 1$, a Mersenne number with $23,249,425$ digits.

January, 2018

Pseudoprimes

- Trial division
- Pseudoprimes
- Strong pseudoprimes
- Rabin-Miller test

Trial Division

The simplest primality test is trial division.

Require: Odd integer $n > 0$.
Ensure: Return 1 if n is prime, otherwise 0.

```
for m = 3 to ⌊√n⌋ by 2 do
    if m divides n then
        return 0
    end if
end for
return 1
```

Time complexity $O(\sqrt{n})$.

Pseudoprimes

Given odd composite number n and integer b. Then n is a *pseudoprime to base* b if $(n, b) = 1$ and

$$b^{n-1} \equiv 1 \mod n. \tag{2}$$

If n is not prime, (2) is not very likely.

Fermat's Little Theorem

If n is prime, then for each integer b with $(n, b) = 1$,

$$b^{n-1} \equiv 1 \mod n. \tag{3}$$

Example

- $n = 91$ is a pseudoprime to base 3, since $3^{90} \equiv 1 \mod 91$.
- $n = 91$ is not a pseudoprime to base 2, since $2^{90} \equiv 64 \mod 91$.

Pseudoprimes

Let n be an odd composite integer.

- If n is a pseudoprime to base b, then $b \in \mathbb{Z}_n^*$, since $(n, b) = 1$, and the order of b in $\mathbb{Z}_n^*$ divides $n-1$, since $b^{n-1} \equiv 1 \mod n$.
- If n is a pseudoprime to base b_1, then n is also a pseudoprime to base b_1^{-1}.
- If n is a pseudoprime to bases b_1 and b_2, then n is also a pseudoprime to base $b_1 b_2$.
- If n fails the test (2) for some base $b \in \mathbb{Z}_n^*$, then n fails for at least one-half of the possible bases in $\mathbb{Z}_n^*$.

Proof (Part 4).

- Let $\{b_1, \ldots, b_s\}$ be the set of bases for which n is a pseudoprime.
- Let b be a base for which n is not a pseudoprime.
- Then n cannot be a pseudoprime for any of the bases

$$bb_1, \ldots, bb_s.$$

  If n is a pseudoprime for base bb_i, then (by part 2) n will be a pseudoprime for base $b \equiv (bb_i)b_i^{-1} \mod n$.
- So there are at least as many bases in $\mathbb{Z}_n^*$ for which n fails to be a pseudoprime as there are bases for which (2) holds.

□

Strong Pseudoprimes

Given odd composite integer n with

$$n - 1 = 2^s t, \quad t \text{ odd}. \tag{4}$$

Let $b \in \mathbb{Z}_n^*$. Then n is a *strong pseudoprime to base* b if either

$$b^t \equiv 1 \mod n \tag{5}$$

or there exists r, $0 \leq r < s$, with

$$b^{2^r t} \equiv -1 \mod n. \tag{6}$$

Strong Pseudoprimes – Example

- $n = 2047$ is a strong pseudoprime to base 2, since $2046 = 2 \cdot 1023$ and $2^{1023} \equiv 1 \mod 2047$.
- $n = 121$ is a strong pseudoprime to base 3, since $120 = 2^3 \cdot 15$ and $3^{15} \equiv 1 \mod 121$.
- Let $n = 91$, $90 = 2 \cdot 45$. Here n is a strong pseudoprime for the bases

$$9,\ 16,\ 22,\ 53,\ 74,\ 79,\ 81.$$

Miller-Rabin Primality Test

Decide whether a large odd integer $n > 0$ is prime or composite.

- Consider the sequence modulo n,

$$(b^t, b^{2t}, \ldots, b^{2^{s-1}t}, b^{2^s t}).$$

- If n is prime, then by Euler's theorem,

$$b^{2^s t} = b^{n-1} \equiv 1 \mod n$$

and the sequence ends with 1.

- If n is prime, $x^2 \equiv 1 \mod n$ iff $x \equiv \pm 1 \mod n$ and so the penultimate sequence position is ± 1.
- Test if the sequence begins with 1 or has ± 1 at least at the penultimate position. If so, n is prime or a strong pseudoprime to base b.

Miller-Rabin Primality Test

- If n is an odd composite, then n is a strong pseudoprime to base b for at most 25% of all $0 < b < n$; i.e., the probability that n passes the test (strong pseudoprime but not prime) is $< \frac{1}{4}$.
- Repeating the test for k independently chosen numbers b with $0 < b < n$, the probability that n passes the test is $< \frac{1}{4^k}$.
- For many odd composite numbers, the fraction of bases passing the test is much smaller than $\frac{1}{4}$.

Factoring

> The work of the professional cryptologist is thankless; he is not allowed to celebrate his success in public [...] Such restrictions usually persist even after active duty.

F.L. Bauer, 2000

Factoring

- Fermat factoring
- Factor bases
- Quadratic sieve
- Continued fractions

Fermat Factoring

Let $n > 0$ be an odd integer.

There is a 1-to-1 correspondence between factorizations of $n = pq$, where $p \geq q > 0$, and representations of n in the form $u^2 - v^2$, where u, v are non-negative integers.

The correspondence is given by

$$u = \frac{p+q}{2}, \quad v = \frac{p-q}{2}, \quad p = u + v, \quad q = u - v. \tag{7}$$

Note that if p and q are close together, then $v = (p-q)/2$ is small and so u is only slightly larger than $\sqrt{n}$.

Example

$n = 200819$ with $\sqrt{n} \approx 448$ factors into $n = 491 \cdot 409$ and has the representation $n = 450^2 - 41^2$.

Fermat Factoring – Algorithm

Fermat's factoring works well if n is a product of two primes close to each other.

Require: Odd integer $n > 0$ with $n = pq$ and p, q primes close together.
Ensure: Values p, q
$u \leftarrow \lfloor \sqrt{n} \rfloor + 1$
repeat
$u \leftarrow u + 1$
until $u^2 - n$ is a perfect square
$v \leftarrow \sqrt{u^2 - n}$
$p \leftarrow u + v$
$q \leftarrow u - v$
return p, q

Fermat Factoring – Example

Let $n = 200819$.

- Take $u = \lfloor \sqrt{200819} \rfloor + 1 = 449$. Then $449^2 - 200819 = 782 = 2 \cdot 17 \cdot 23$, not perfect square.
- Take $u = 449 + 1 = 450$. Then $450^2 - 200819 = 1681 = 41^2$, perfect square. Thus

$$u = 450, \quad v = 41, \quad p = 491, \quad q = 409.$$

Factor Bases (1974)

- A *factor basis* is a set $B = \{p_1, \ldots, p_h\}$ of distinct primes, except $p_1 = -1$.
- Let $n \geq 2$ be an odd integer (modulus).

  A square of an integer b is a B-*number* for n if the least absolute residue b^2 *mod* n can be written as a product of numbers from B.
- The *least absolute residue* of a number a modulo n is an integer from the interval $-n/2$ to $n/2$ (not from 0 to $n-1$) to which a is congruent, written a *mod* n.

  For $n = 5$, the least absolute residues are $-2, -1, 0, 1, 2$.

Example

Let $n = 4633$ and $B = \{-1, 2, 3\}$. The squares of 67, 68, and 69 are B-numbers for n, since $67^2 \equiv -144$ *mod* 4633, $68^2 \equiv -9$ *mod* 4633, and $69^2 \equiv 128$ *mod* 4633, with $-144 = (-1)2^4 3^2$, $-9 = (-1)3^2$, and $128 = 2^7$.

Factor Bases

Given n and factor basis $B = \{p_1, \ldots, p_h\}$.

- Write each B-number $a = b^2 \ mod \ n$ as

$$a = \prod_{j=1}^{h} p_j^{\alpha_j}, \quad \alpha_j \geq 0. \tag{8}$$

- For each B-number $a = b^2 \ mod \ n$ define $\epsilon_b = (\epsilon_{b1}, \ldots, \epsilon_{bh}) \in \mathbb{F}_2^h$ as

$$\epsilon_{bj} = \begin{cases} 0 & \text{if } \alpha_j \text{ is even in } a, \\ 1 & \text{otherwise.} \end{cases} \tag{9}$$

Factor Bases – Example

Let $n = 4633$ and $B = \{-1, 2, 3\}$.

- For 67, $-144 = (-1) \cdot 2^4 \cdot 3^2$ and so $\epsilon_{67} = (1, 0, 0)$.
- For 68, $-9 = (-1) \cdot 3^2$ and so $\epsilon_{68} = (1, 0, 0)$.
- For 69, $128 = 2^7$ and so $\epsilon_{69} = (0, 1, 0)$.

Factor Bases

Given n and factor basis $B = \{p_1, \ldots, p_h\}$.

- If b_1^2 *mod* n and b_2^2 *mod* n are B-numbers, then $(b_1 b_2)^2$ *mod* n is a B-number.
- The set of all B-numbers forms an additive subgroup of $\mathbb{F}_2^h$, since multiplying B-numbers means adding the corresponding binary vectors modulo 2 (in the exponents).

Example

For 67, $-144 = (-1) \cdot 2^4 \cdot 3^2$ and so $\epsilon_{67} = (1, 0, 0)$, and for 68, $-9 = (-1) \cdot 3^2$ and so $\epsilon_{68} = (1, 0, 0)$.

Then for $67 \cdot 68$, $(-144) \cdot (-9) = 2^4 \cdot 3^4$ and

$$\epsilon_{67 \cdot 68} = \epsilon_{67} + \epsilon_{68} = (1, 0, 0) + (1, 0, 0) = (0, 0, 0).$$

Factor Bases

Given n and factor basis $B = \{p_1, \ldots, p_h\}$.

Suppose the squares of the numbers $b_1, \ldots, b_s$ are B-numbers for n, where

$$a_i = b_i^2 \ mod \ n = \prod_{j=1}^{h} p_j^{\alpha_{ij}}. \tag{10}$$

Table:

		p_1	$\ldots$	p_j	$\ldots$	p_h
b_1	$b_1^2 \ mod \ n$	α_{11}	$\ldots$	α_{1j}	$\ldots$	α_{1h}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	
b_i	$b_i^2 \ mod \ n$	α_{i1}	$\ldots$	α_{ij}	$\ldots$	α_{ih}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	
b_s	$b_s^2 \ mod \ n$	α_{s1}	$\ldots$	α_{sj}	$\ldots$	α_{sh}

Factor Bases

Take B-numbers for n,

$$a_{i_1} = b_{i_1}^2 \ mod\ n, \ldots, a_{i_k} = b_{i_k}^2 \ mod\ n, \tag{11}$$

whose corresponding binary vectors add up to the zero vector in the space $\mathbb{F}_2^h$.

More concretely, write $a_{i_l} = \prod_{j=1}^{h} p_j^{\alpha_{i_l j}}$, $1 \leq l \leq k$. Then

$$\prod_{l=1}^{k} a_{i_l} = \prod_{j=1}^{h} p_j^{\sum_{l=1}^{k} \alpha_{i_l j}}, \tag{12}$$

where each summand $\sum_l \alpha_{i_l j}$ is even.

Factor Bases

Write

$$b = b_{i_1} \dots b_{i_k} \ mod\ n \tag{13}$$

and

$$c = \prod_{j=1}^{h} p_j^{\sum_{l=1}^{k} \alpha_{i_l j}/2} \ mod\ n. \tag{14}$$

Then by (12),

$$b^2 \equiv c^2 \ mod\ n. \tag{15}$$

Factor Bases

Let $b^2 \equiv c^2 \mod n$.

- If $b \not\equiv \pm c$ *mod* n, there will be a factor of n by computing $(b+c, n)$ or $(b-c, n)$.

  Indeed, n divides $b^2 - c^2 = (b+c)(b-c)$.

  But by hypothesis, n is not divisible by $b+c$ or $b-c$.

  Thus $(b+c, n)$ or $(b-c, n)$ must be a proper factor of n.

- If $b \equiv \pm c$ *mod* n, there will be no further information.

Factor Bases – Example

Let $n = 4633$.

- Take $B = \{-1, 2, 3\}$.
- The squares of 67, 68 and 69 are B-numbers for n; these numbers are close to $\sqrt{4633} \approx 68$.
- The squares of 67, 68 and 69 are B-numbers for n, since

$$67^2 \equiv -144 \ mod\ 4633,$$
$$68^2 \equiv -9 \ mod\ 4633,$$
$$69^2 \equiv 128 \ mod\ 4633.$$

- Table:

	-1	2	3	ϵ
67	1	4	2	$(1,0,0)$
68	1	0	2	$(1,0,0)$
69	0	7	0	$(0,1,0)$

Factor Bases – Example (cont'd)

- Table:

	-1	2	3	ϵ
67	1	4	2	$(1,0,0)$
68	1	0	2	$(1,0,0)$
69	0	7	0	$(0,1,0)$

- The first two binary vectors are both $(1,0,0)$ and add up to the zero vector.
- Compute $b = 67 \cdot 68 \ mod\ 4633 = -77$.
- Compute $c = 2^{4/2}3^{4/2} = 36$, where $(-144) \cdot (-9) = 2^4 3^4$.
- Then

$$(-77)^2 \equiv 36^2 \ mod\ 4633.$$

- We have $-77 \not\equiv \pm 36 \ mod\ 4633$ and so

$$(-77 + 36, 4633) = 41$$

gives a factor of 4633.

Factor Bases – Example

Let $n = 1829$.

- Take as b_i the integers of the form $\lfloor\sqrt{1829 \cdot k}\rfloor$ and $\lfloor\sqrt{1829 \cdot k}\rfloor + 1$ for $k = 1, 2, 3, 4$ such that b_i^2 *mod* n is a product of primes ≤ 19.
- Write b_i^2 *mod* $n = \prod_j p_j^{\alpha_{ij}}$ and tabulate the α_{ij}:

	-1	2	3	5	7	11	13	19
42	1			1			1	
43		2		1				
61			2		1			
74	1					1		
85	1				1		1	
86		4		1				

Factor Bases – Example (cont'd)

Let $n = 1829$.

- The 2nd and 6th row sum up to the even row $-6 - 2 - --$.
- Thus $b = b_2 \cdot b_6$ *mod* n and $c = 2^{6/2}5^{2/2}$ *mod* n and hence

$$(43 \cdot 86)^2 \equiv 40^2 \text{ mod } 1829.$$

- Since $43 \cdot 86 \equiv 40$ *mod* 1829, resume with another even row sum.

Factor Bases – Example (cont'd)

Let $n = 1829$.

- The 1st, 2nd, 3rd, and 5th row sum up to $22222 - 2-$.
- Thus

$$(42 \cdot 43 \cdot 61 \cdot 85)^2 \equiv (2 \cdot 3 \cdot 5 \cdot 7 \cdot 13)^2 \ \textit{mod}\ 1829,$$

  i.e., $1459^2 \equiv 901^2 \ \textit{mod}\ 1829$.
- Since $1459 \not\equiv \pm 901 \ \textit{mod}\ 1829$,

$$(1459 + 901, 1829) = 59$$

  is a factor of 1829.

Factor Bases – Algorithm

Require: Odd composite 50-decimal-digit integer $n > 0$
Ensure: Divisor of n
  repeat
    Choose a random integer y with 5 decimal digits
    Let B consist of -1 and all primes $\leq y$
    Choose a large number of random b_i such that b_i^2 *mod* n can be expressed as a product of primes in B (B-numbers)
    Compute the corresponding set of binary vectors ϵ_{b_i}
    Find a subset of the b_i whose associated sum of binary vectors is zero
    $b \leftarrow \prod_i b_i \ mod \ n$
    $c \leftarrow \prod_j p_j^{\gamma_j} \ mod \ n$
    if $b \not\equiv \pm c \ mod \ n$ **then**
      return $(b + c, n)$
    end if
  until time elapsed

Factoring an r-bit integer n requires $O(e^{C\sqrt{r \log r}})$ bit operations, where C is a constant; e.g., $\exp(\sqrt{50 \cdot \log(50)}) \approx 1185586$.

Quadratic Sieve Method (1981)

- Fastest for integers below 100 decimal digits.
- General running time $O(e^{\sqrt{\log n \log\log n}})$.
- Second fastest method known (after number field sieve).
- Number field sieve (1994) has general running time $e^{O((\log n)^{1/3}(\log\log n)^{2/3})}$, superior for larger integers.

Quadratic Sieve Algorithm

Given odd composite number n.

- Choose bounds P and A of order $L(n) = e^{\sqrt{\log n \log\log n}}$ with $P < A < P^2$. If $n \approx 10^6$, then $L(n) \approx 400$.
- Factor base B consists of 2 and all odd primes $p < P$ for which n is a quadratic residue modulo p.
- Define $|A| \times |B|$ matrix with rows labelled by $t = \lfloor\sqrt{n}\rfloor + 1, \ldots, \lfloor\sqrt{n}\rfloor + A$ and columns labelled by $p \in B$.
- The (t, p) matrix entry is the maximal $i \geq 0$ such that p^i divides $t^2 - n$.
- Delete all rows numbered t for which $t^2 - n$ is not a B-number.
- Use factor basis B to factor n; otherwise, restart.

Example

Factorize $n = 1042387$.

- Take bounds $P = 50$ and $A = 500$; i.e., $P < A < P^2$.
- Factor basis $B = \{2, 3, 11, 17, 19, 23, 43, 47\}$ with n quadratic residue; e.g., $n \mod 11 = 5$ and $5 \equiv 4^2 \mod 11$.
- $\lfloor \sqrt{n} \rfloor = 1020$.
- Take 500×8 array with rows numbered by $t = 1021, \ldots, 1520$ and columns numbered by $p \in B$.
- Delete all rows numbered by t for which $t^2 - n$ is not a B-number, i.e., 11 rows will remain.
- Apply the factor-basis algorithm to the resulting 11×8 table.

Example (cont'd)

t	$t^2 - n$	2	3	11	17	19	23	43	47
1021	54	1	3						
1027	12342	1	1	2	1				
1030	18513		2	2	1				
1061	83334	1	1		1	1		1	
1112	194157		5		1				1
1129	232254	1	3	1	1		1		1
1148	275517		2	3			1		
1157	338238	1	2			1	1	1	
1217	438702	1	1	1	2		1		
1390	889713		2	2		1		1	
1520	1268013		1		1		2		1

Example: $54^2 \mod n = 2916 = 2^2 3^6$.

Example (cont'd)

- The first three rows sum up to an even number in each column giving twice the row

$$1\ 3\ 2\ 1\ -\ -\ -\ -.$$

- The corresponding congruence is

$$(1021 \cdot 1027 \cdot 1030)^2 \equiv (2 \cdot 3^3 \cdot 11^2 \cdot 17)^2 \ \textit{mod}\ 1042387.$$

- Both numbers are $\equiv 111078$ *mod* 1042387 providing the trivial factorization.

Example (cont'd)

- The fifth and the last row sum up to an even number with corresponding congruence

$$(1112 \cdot 1520)^2 \equiv (3^2 \cdot 17 \cdot 23 \cdot 47)^2 \ \textit{mod}\ 1042387.$$

Thus $647853^2 \equiv 496179^2 \ \textit{mod}\ 1042387$.

- But $647853 \not\equiv \pm 496179 \ \textit{mod}\ 1042387$ and so a nontrivial factor is

$$(647853 - 496179, 1042387) = 1487.$$

Continued Fraction Method (1983)

Given an odd composite integer n.

- The factor-base method works best if there are many integers b, $1 < b < n$, such that $b^2 \ mod \ n$ is a product of small primes; this is most likely if $|b^2 \ mod \ n|$ is small.
- The continued fraction method gives integers b with $|b^2 \ mod \ n| < 2\sqrt{n}$.

Continued Fractions

Let x be an irrational number. Construct its continued fraction expansion as follows:

- Write
$$x = x_0 = a_0 + 1/x_1,$$
where $a_0 = \lfloor x_0 \rfloor$ and $x_1 > 1$.
- For $n \geq 1$, write
$$x_n = a_n + 1/x_{n+1},$$
where $a_n = \lfloor x_n \rfloor$ and $x_{n+1} > 1$.
- The process never stops, since each x_n is irrational.

Continued Fractions

Combining these equations gives

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots \cfrac{1}{a_n + x_{n+1}}}} \tag{16}$$

or shorter

$$x = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_n+}\frac{1}{x_{n+1}} \tag{17}$$

or shorter

$$x = [a_0, a_1, a_2, \ldots, a_n, x_{n+1}], \tag{18}$$

where $a_1, \ldots, a_n \in \mathbb{N}$ and $a_0 \in \mathbb{Z}$.

Continued Fractions – Example

```
> X := sqrt(3):
> n := 6:
> seq(x[i],i=1..n): seq(a[i],i=1..n):
> a[0] := floor(X): x[0] := evalf(X-a[0]):
> print(0,x[0],1/x[0],a[0]);
> for i from 1 to n do
    a[i] := floor(1/x[i-1]});
    x[i] := evalf(1/x[i-1]}-a[i]);
    print(i,x[i],1/x[i],a[i])
> end:
0, 0.732050808, 1.366025403, 1
1, 0.366025403, 1.366025394, 1
2, 0.732050808, 2,732050881, 2
3, 0.366025403, 1.366025267, 1
4, 0.732050808, 2.732051829, 2
5, 0.366025403, 1.365998859, 1
6, 0.732050808, 2.732065033, 2
```

Continued Fractions – Example (cont'd)

- Calculation:

```
0, 0.732050808, 1.366025403, 1
1, 0.366025403, 1.366025394, 1
2, 0.732050808, 2,732050881, 2
3, 0.366025403, 1.366025267, 1
4, 0.732050808, 2.732051829, 2
5, 0.366025403, 1.365998859, 1
6, 0.732050808, 2.732065033, 2
```

- We have

$$\sqrt{3} = 1 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\cdots.$$

  Conjecture that the a_i's alternate between 1 and 2.

- Let x be the infinite continued fraction with alternating 1's and 2's. Then

$$x = 1 + \frac{1}{1 + \frac{1}{1+x}}.$$

  Its expansion gives $x^2 - 3 = 0$ and so $x = \sqrt{3}$ (> 0 in our setting).

Continued Fractions

For any irrational number x, the *n-th convergent* of x in (16) is the rational number

$$\frac{b_n}{c_n} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_{n-1}+}\frac{1}{a_n}. \tag{19}$$

The convergents $\frac{b_n}{c_n}$ form an infinite sequence of rational numbers.

Continued Fractions – Example

Convergents of $x = \sqrt{3}$:

```
> cf := cfrac(sqrt(3)):
> for n from 1 to 6 do
>   nthconver(cf, n)
> end;
```

Output

$$2, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}.$$

Continued Fractions

By Euler's rule,

$$\begin{aligned} x &= a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_{n-1}+}\frac{1}{x_{n+1}} \\ &= \frac{[a_0, a_1, \ldots, a_n, x_{n+1}]}{[a_1, a_2, \ldots, a_n, x_{n+1}]}. \end{aligned} \tag{20}$$

Continued Fractions

Recurrence relation

$$\begin{aligned}[a_0,\ldots,a_n,x_{n+1}] &= x_{n+1}[a_0,\ldots,a_n]+[a_0,\ldots,a_{n-1}]\\ &= x_{n+1}b_n+b_{n-1},\end{aligned} \tag{21}$$

and

$$\begin{aligned}[a_1,\ldots,a_n,x_{n+1}] &= x_{n+1}[a_1,\ldots,a_n]+[a_1,\ldots,a_{n-1}]\\ &= x_{n+1}c_n+c_{n-1}.\end{aligned} \tag{22}$$

Hence, by (20),

$$x=\frac{x_{n+1}b_n+b_{n-1}}{x_{n+1}c_n+c_{n-1}}. \tag{23}$$

Continued Fractions

Given a purely periodic continued fraction

$$x = a_0 + \frac{1}{a_1+} \cdots \frac{1}{a_n+} \frac{1}{x}. \tag{24}$$

Then

$$x = \frac{b_n x + b_{n-1}}{c_n x + c_{n-1}}. \tag{25}$$

Define the continued fraction y by reversing the period,

$$y = a_n + \frac{1}{a_{n-1}+} \cdots \frac{1}{a_0+} \frac{1}{y}. \tag{26}$$

Then

$$y = \frac{b_n y + c_n}{b_{n-1} y + c_{n-1}}. \tag{27}$$

The numbers x and $-1/y$ are conjugate to each other (Galois, 1828).

Continued Fractions – Example

- Consider the purely periodic continued fraction

$$x = 4 + \frac{1}{1+}\frac{1}{3+}\frac{1}{x}.$$

Then $4x^2 - 18x - 5 = 0$ with solutions $\frac{9}{4} \pm \frac{1}{4}\sqrt{101}$.

- Take the continued fraction by reversing the period,

$$y = 3 + \frac{1}{1+}\frac{1}{4+}\frac{1}{y}.$$

Then $5y^2 - 18y - 4 = 0$ with solutions $\frac{9}{5} \pm \frac{1}{5}\sqrt{101}$.

- Both equations are related by the setting $x = -1/y$:

$$\begin{aligned} -1/(\frac{9}{4} + \frac{1}{4}\sqrt{101}) &= \frac{9}{5} - \frac{1}{5}\sqrt{101}, \\ -1/(\frac{9}{4} - \frac{1}{4}\sqrt{101}) &= \frac{9}{5} + \frac{1}{5}\sqrt{101}. \end{aligned}$$

Continued Fractions (Example)

Some purely periodic continued fractions:

$$\begin{aligned}
\sqrt{\varphi} &= [1;\overline{1},\ldots], \\
\sqrt{2} &= [1;\overline{2},\ldots], \\
\sqrt{3} &= [1;\overline{1,2},\ldots], \\
\sqrt{5} &= [2;\overline{4},\ldots], \\
\sqrt{6} &= [2;\overline{2,4},\ldots], \\
\sqrt{7} &= [2;\overline{1,1,1,4},\ldots].
\end{aligned}$$

Continued Fractions

For each real number $x > 1$ with convergents (b_n/c_n),

$$\left| x - \frac{b_n}{c_n} \right| < \frac{1}{c_n c_{n+1}}. \tag{28}$$

Proof.

We have

$$\begin{aligned} x - \frac{b_n}{c_n} &= \frac{x_{n+1} b_n + b_{n-1}}{x_{n+1} c_n + c_{n-1}} - \frac{b_n}{c_n} \\ &= \frac{b_{n-1} c_n - c_{n-1} b_n}{c_n (x_{n+1} c_n + c_{n-1})} \\ &= \frac{\pm 1}{c_n (x_{n+1} c_n + c_{n-1})}. \end{aligned}$$

But $x_{n+1} > a_{n+1}$ and $c_{n+1} = a_{n+1} c_n + c_{n-1}$ and so the result follows. □

Continued Fractions

For each real number $x > 1$ with convergents (b_n/c_n),

$$\left|b_n^2 - x^2 c_n^2\right| < 2x. \tag{29}$$

Corollary

Let $n > 1$ be an integer which is not a perfect square and let (b_m/c_m) be the convergents of $\sqrt{n}$. Then for each $m \geq 1$,

$$\left|b_m^2 \ mod \ n\right| < 2\sqrt{n}. \tag{30}$$

Proof.

Take $x = \sqrt{n}$. Then $b_m^2 \equiv b_m^2 - nc_m^2 \ mod \ n$ and so by (29), the latter is less than $2\sqrt{n}$ in absolute value. □

Continued Fractions – Maple

```
> n := 9073:
> k := 6:
> seq(x[i],i=0..k):
> seq(a[i],i=0..k):
> seq(b[i],i=-1..k):
> seq(c[i],i=-1..k):
> b[-1] := 1: c[-1] := 0: a[0] := floor(sqrt(n)):
> b[0] := a[0]: c[0] := 1: x[0] := evalf(sqrt(n)-a[0]):
> print(0,x[0],a[0],b[0],c[0],mods(b[0]^2,n));
> for i from 0 to k do
    a[i] := floor(1/x[i-1]);
    x[i] := evalf(1/x[i-1]-a[i]);
    b[i] := a[i]*b[i-1]+b[i-2];
    c[i] := a[i]*c[i-1]+c[i-2];
    print(i,x[i],a[i],b[i],c[i],mods(b[i]^2,n))
  end:
```

Continued Fractions – Maple Output

Let $n = 9073$:

$$\sqrt{9073} = 95 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{26+}\,\frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{1+}\cdots$$

i	0	1	2	3	4	5	6
a_i	95	3	1	26	2	6	1
b_i	95	286	381	1119	2619	7760	1306
c_i	1	3	4	107	218	1415	1633
$b_i^2 \bmod n$	-48	139	-7	87	-27	99	-88

Continued Fractions – Example (cont'd)

- Choose the factor basis $B = \{-1, 2, 3, 7\}$.
- For $i = 0, 2, 4$,

$$b_i^2 \ mod\ 9073 = -48,\ -7,\ -27$$

are B-numbers with vectors

$$(1, 4, 1, 0),\ (1, 0, 0, 1),\ (1, 0, 3, 0).$$

- The sum of the first and third is $(2, 4, 4, 0)$ and so an even vector.
- Then $b = b_0 b_4 = 95 \cdot 2619 \equiv 3834\ mod\ 9073$ and $c = 2^{4/2} \cdot 3^{4/2} = 36$.
- Thus $3834^2 \equiv 36^2\ mod\ 9073$.
- But $3834 \not\equiv \pm 36\ mod\ 9073$ and so a nontrivial factor is $(3834 + 36, 9073) = 43$. Indeed, $9073 = 43 \cdot 211$.

Part II

Projective Space

Projective Space

- Projective n-space
- Projective line
- Projective plane
- Homogeneous polynomials
- Lines in projective plane
- *Projective transformations

Projective n-Space

> The idea of projective space relates to perspective, more precisely to the way an eye or a camera projects a 3D scene to a 2D image.

Wikipedia, 2018

Affine n-Space

Let $\mathbb{K}$ be a field. Put $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$.

- *Affine n-space* over $\mathbb{K}$,

$$\mathbb{A}^n(\mathbb{K}) = \mathbb{K}^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{K}\}. \tag{31}$$

- *Projective n-space* $\mathbb{P}^n(\mathbb{K})$ over $\mathbb{K}$ is the set of all lines in $\mathbb{K}^{n+1}$ passing through the origin 0; i.e., the set of all one-dimensional subspaces of $\mathbb{K}^{n+1}$.

Projective n-Space

- Define an equivalence relation $\sim$ on $\mathbb{K}^{n+1} \setminus \{0\}$,

$$\begin{aligned}(p_0, \ldots, p_n) \sim (q_0, \ldots, q_n) \\ :\Longleftrightarrow \quad (p_0, \ldots, p_n) = \kappa(q_0, \ldots, q_n)\end{aligned} \tag{32}$$

  for some $\kappa \in \mathbb{K}^*$, where

$$\kappa(q_0, \ldots, q_n) = (\kappa q_0, \ldots, \kappa q_n). \tag{33}$$

- Two points P and Q in $\mathbb{K}^{n+1}$ are equivalent iff they lie on the same line.
- *Projective n-space* $\mathbb{P}^n(\mathbb{K})$ is the quotient space of the $\mathbb{K}$-vector space $\mathbb{K}^{n+1}$ given by the set of equivalence classes of the nonzero points in $\mathbb{K}^{n+1}$.

Homogeneous Coordinates

- A point P in $\mathbb{P}^n(\mathbb{K})$ is determined by the equivalence class of an $n+1$-tuple $(p_0, \ldots, p_n) \in \mathbb{K}^{n+1}$, written

$$P = (p_0 : \ldots : p_n).$$

  Each $(n+1)$-tuple belonging to this equivalence class is referred to as *homogeneous coordinates* of P.

- Each point in $\mathbb{P}^n(\mathbb{K})$ has many homogeneous coordinates; e.g.,

$$(0 : i : \sqrt{2}) = (0 : -\sqrt{2} : 2i)$$

  is the same point in $\mathbb{P}^2(\mathbb{C})$, since for $\kappa = i\sqrt{2}$,

$$i\sqrt{2} \cdot (0, i, \sqrt{2}) = (0, -\sqrt{2}, 2i),$$

  where $i \in \mathbb{C}$ is the imaginary unit with $i^2 = -1$.

Decomposition of Projective n-Space

For each $n \geq 2$, the projective n-space decomposes as

$$\mathbb{P}^n(\mathbb{K}) = \mathbb{A}^n(\mathbb{K}) \cup \mathbb{P}^{n-1}(\mathbb{K}). \tag{34}$$

Proof.

Let $P = (p_0 : p_1 : \cdots : p_n)$ be a point in $\mathbb{P}^n(\mathbb{K})$.

- If $p_n \neq 0$, then $P = (p'_0 : \cdots : p'_{n-1} : 1)$ with $p'_i = p_i/p_n$, $0 \leq i \leq n$. We have

$$\begin{aligned} (p_0 : \cdots : p_{n-1} : 1) = (q_0 : \cdots : q_{n-1} : 1) \\ \iff \quad (p_0, \ldots, p_{n-1}) = (q_0, \ldots, q_{n-1}). \end{aligned} \tag{35}$$

  Thus the points P with $p_n \neq 0$ form the affine n-space.

- If $p_n = 0$, then $P = (p_0 : \cdots : p_{n-1} : 0)$, where the p_i are defined only up to a common scalar. Thus the points P with $p_n = 0$ form the projective $n-1$-space, called *points at infinity*. □

Affine Subspaces of Projective n-Space

For each $0 \leq i \leq n$, define the subset of $\mathbb{P}^n(\mathbb{K})$,

$$\begin{aligned} U_i &= \{(p_0 : \ldots : p_i : \ldots : p_n) \in \mathbb{P}^n(\mathbb{K}) \mid p_i \neq 0\} \\ &= \{(p_0 : \ldots : p_{i-1} : 1 : p_{i+1} : \ldots : p_n) \in \mathbb{P}^n(\mathbb{K}) \mid p_i = 1\}. \end{aligned} \tag{36}$$

- U_i is an open subset of $\mathbb{P}^n(\mathbb{K})$ ($\rightarrow$ Algebraic Geometry).
- The mapping $\phi : \mathbb{K}^n \rightarrow U_i$ given by

$$\begin{aligned} &\phi(p_0, \ldots, p_{i-1}, p_{i+1}, \ldots : p_n) \\ &= (p_0, \ldots : p_{i-1} : 1 : p_{i+1} : \ldots : p_n) \end{aligned} \tag{37}$$

  is a bijection.

Affine Covering of Projective n-Space

The projective n-space $\mathbb{P}^n(\mathbb{K})$ has the following properties:

- For each $0 \leq i \leq n$, there is a bijection between U_i and $\mathbb{K}^n$.
- The projective n-space $\mathbb{P}^n(\mathbb{K})$ has an open covering of $n+1$ affine n-spaces

$$\mathbb{P}^n(\mathbb{K}) = \bigcup_{i=0}^{n} U_i. \tag{38}$$

- For each $0 \leq i \leq n$, the complement $\mathbb{P}^n(\mathbb{K}) \setminus U_i$ is a closed subset of $\mathbb{P}^n(\mathbb{K})$ and can be identified with $\mathbb{P}^{n-1}(\mathbb{K})$.

Order of Finite Projective n-Space

Let $\mathbb{F}_q$ be the Galois field with q elements. Then

$$|\mathbb{P}^n(\mathbb{F}_q)| = \frac{q^{n+1}-1}{q-1}. \tag{39}$$

Proof.

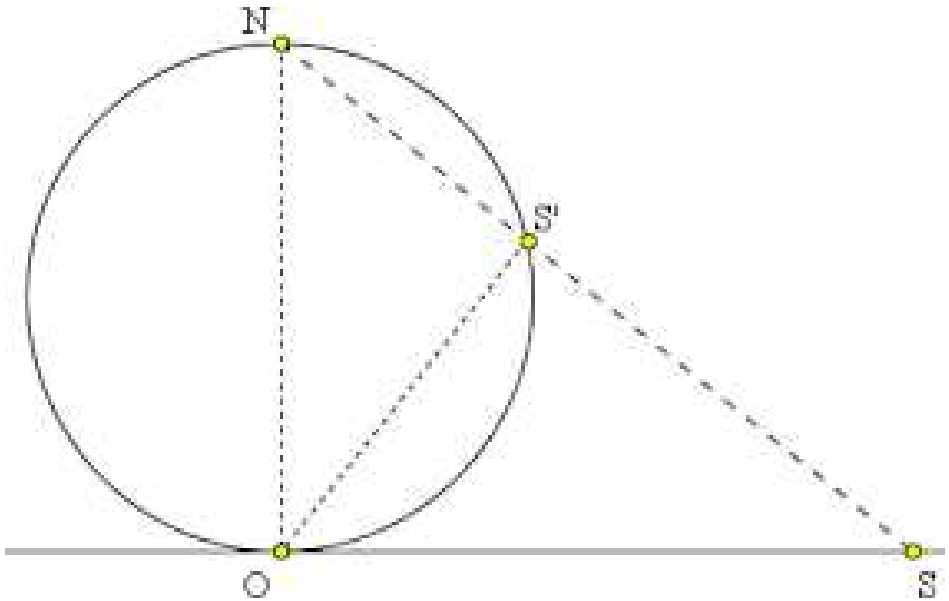
For the projective line,

$$|\mathbb{P}^1(\mathbb{F}_q)| = q+1 = \frac{q^2-1}{q-1}.$$

For the projective $n+1$-space,

$$\begin{aligned} |\mathbb{P}^{n+1}(\mathbb{F}_q)| &= |\mathbb{A}^{n+1}(\mathbb{F}_q)| + |\mathbb{P}^n(\mathbb{F}_q)| \\ &= q^{n+1} + \frac{q^{n+1}-1}{q-1} \\ &= \frac{q^{n+2}-1}{q-1}. \end{aligned}$$

Projective Line



Stereographic projection of real projective line.

Decomposition of Projective Line

Projective line over $\mathbb{K}$ decomposes as

$$\mathbb{P}^1(\mathbb{K}) = \mathbb{A}^1(\mathbb{K}) \cup \{(1:0)\}, \tag{40}$$

where

$$\mathbb{A}^1(\mathbb{K}) = \{(p:1) \mid p \in \mathbb{K}\} \tag{41}$$

is the *affine line*.

Proof.

Let $P = (p_0 : p_1)$ in $\mathbb{P}^1(\mathbb{K})$.

- If $p_1 \neq 0$, then $P = (p:1)$ with $p = p_0/p_1$. We have

$$(p:1) = (q:1) \quad \Longleftrightarrow \quad p = q. \tag{42}$$

  Thus the points P with $p_1 \neq 0$ form the affine line.

- If $p_1 = 0$, then $p_0 \neq 0$ and so $P = (1:0)$, called *point at infinity*.

□

Projective Line over $\mathbb{F}_4$

- The Galois field $\mathbb{F}_4 = \mathbb{Z}_2[X]/\langle X^2 + X + 1\rangle$ has the elements

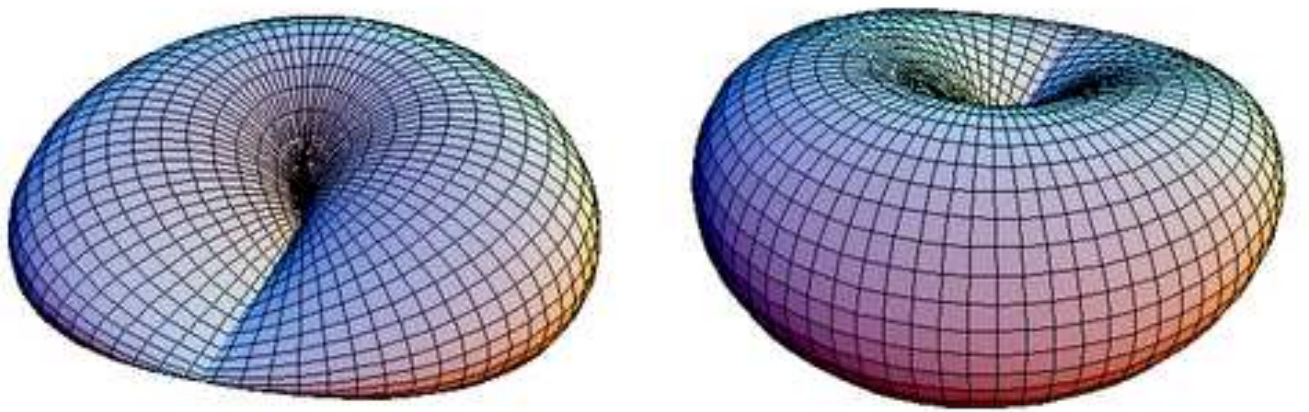
$$0,\ 1,\ \alpha,\ \alpha^2,$$

  where α is a zero of $X^2 + X + 1$ over $\mathbb{Z}_2$; i.e., $\alpha^2 + \alpha + 1 = 0$ or $\alpha^2 = \alpha + 1$.

- The projective line over $\mathbb{F}_4$ has the elements

$$(0:1),\ (1:1),\ (\alpha:1),\ (\alpha^2:1),\ \text{and}\ (1:0).$$

Projective Plane



Real projective plane

Decomposition of Projective Plane

Projective plane $\mathbb{P}^2(\mathbb{K})$ decomposes as

$$\mathbb{P}^2(\mathbb{K}) = \mathbb{A}^2(\mathbb{K}) \cup \mathbb{L}(\mathbb{K}) \cup \{(1:0:0)\}, \tag{43}$$

where $\mathbb{A}^2(\mathbb{K})$ is the *affine plane*,

$$\mathbb{A}^2(\mathbb{K}) = \{(a:b:1) \mid a, b \in \mathbb{K}\}, \tag{44}$$

and $\mathbb{L}(\mathbb{K})$ is a line,

$$\mathbb{L}(\mathbb{K}) = \{(a:1:0) \mid a \in \mathbb{K}\}. \tag{45}$$

The points $(a:b:c)$ with $c = 0$ are the *points at infinity*.

Projective Plane over $\mathbb{F}_4$

- The Galois field $\mathbb{F}_4 = \mathbb{Z}_2[X]/\langle X^2 + X + 1\rangle$ has the elements

$$0,\ 1,\ \alpha,\ \alpha^2,$$

where α is a zero of $X^2 + X + 1$ over $\mathbb{Z}_2$, i.e., $\alpha^2 + \alpha + 1 = 0$.

- The projective plane over $\mathbb{F}_4$ decomposes into

$$\mathbb{P}^2(\mathbb{F}_4) = \mathbb{A}^2(\mathbb{F}_4) \cup \mathbb{L}(\mathbb{F}_4) \cup \{(1:0:0)\}$$

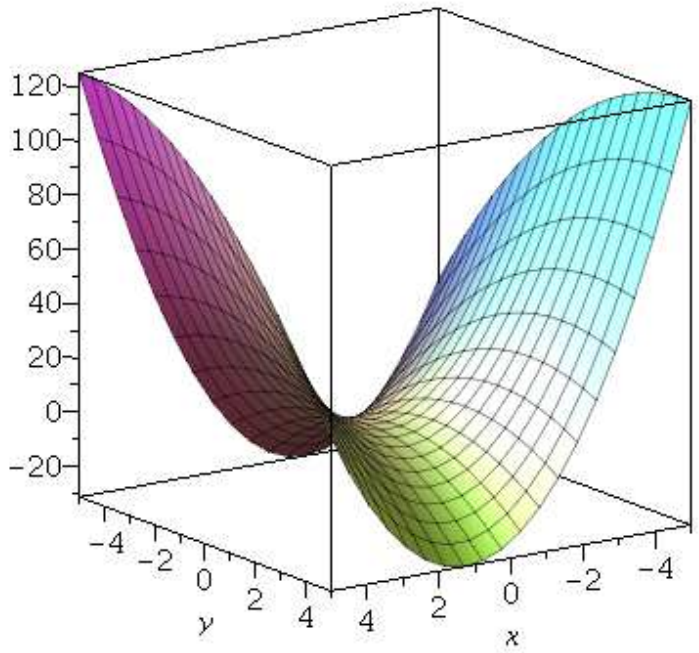
with

$$\begin{aligned} \mathbb{A}^2(\mathbb{F}_4) &= \{(a:b:1) \mid a, b \in \mathbb{F}_4\}, \\ \mathbb{L}(\mathbb{F}_4) &= \{(a:1:0) \mid a \in \mathbb{F}_4\}. \end{aligned}$$

- We have

$$|\mathbb{P}^2(\mathbb{F}_4)| = |\mathbb{A}^2(\mathbb{F}_4)| + |\mathbb{L}(\mathbb{F}_4)| + 1 = 16 + 4 + 1 = 21.$$

Homogeneous Polynomials




```
> with(plots):
> plot3d(4*x^2-2*x*y-y^2,x=-5..5,y=-5..5,grid=[20,20]);
```

Zeros of Homogeneous Polynomial

If $f \in \mathbb{K}[X_0, \ldots, X_n]$ is a homogeneous polynomial of total degree d that vanishes on some homogeneous coordinates of a point $P \in \mathbb{P}^n(\mathbb{K})$, then f vanishes on all homogeneous coordinates of P.

Proof.

Let $P = (p_0 : \ldots : p_n) \in \mathbb{P}^n(\mathbb{K})$. Then by homogeneity,

$$f(\kappa p_0, \ldots, \kappa p_n) = \kappa^d f(p_0, \ldots, p_n) \tag{46}$$

for each $\kappa \in \mathbb{K}^*$. Thus

$$f(p_0, \ldots, p_n) = 0 \quad \Longleftrightarrow \quad f(\kappa p_0, \ldots, \kappa p_n) = 0, \tag{47}$$

since $\kappa^d \in \mathbb{K}^*$. □

Zeros of Homogeneous Polynomial

Let

$$f = XY - Y^2 + XZ \in \mathbb{K}[X, Y, Z]$$

and $P = (1 : 1 : 0) \in \mathbb{P}^2(\mathbb{K})$.
Then $f(P) = 0$ and for each $Q = \kappa P = (\kappa : \kappa : 0)$ with $\kappa \neq 0$,

$$f(Q) = \kappa^2 f(P) = 0.$$

For a homogeneous polynomial f, an equation of the form

$$f = a$$

with $a \in \mathbb{K}$ makes only sense for $a = 0$.

Homogenization

Given a polynomial $f \in \mathbb{K}[X_1, \ldots, X_n]$ of degree d and its expansion as a sum of homogeneous components,

$$f = \sum_{k=0}^{d} f^{(k)}, \tag{48}$$

where $f^{(k)}$ is the degree-k part. Then

$$f^h(X_0, \ldots, X_n) = \sum_{k=0}^{d} X_0^{d-k} f^{(k)}(X_1, \ldots, X_n) \tag{49}$$

is a homogeneous polynomial in $\mathbb{K}[X_0, X_1, \ldots, X_n]$ of total degree d, called *homogenization* of f.

Homogenization – Example

The polynomial

$$f = X^2Y + X^2 + XY + Y + 2 \in \mathbb{K}[X, Y]$$

of degree 3 has the homogeneous components

$$f^{(0)} = 2,\ f^{(1)} = Y,\ f^{(2)} = X^2 + XY,\ f^{(3)} = X^2Y.$$

Homogenization yields the homogeneous polynomial

$$f^h = X^2Y + X^2Z + XYZ + YZ^2 + 2Z^3 \in \mathbb{K}[X, Y, Z].$$

Dehomogenization

Let $f \in \mathbb{K}[X_0, \ldots, X_n]$ be a homogeneous polynomial of total degree d. Then

$$f^a(X_1, \ldots, X_n) = f(1, X_1, \ldots, X_n) \tag{50}$$

is a polynomial in $\mathbb{K}[X_1, \ldots, X_n]$ obtained by the substitution $X_0 = 1$, called *dehomogenization* of f.

Dehomogenization – Example

The homogeneous polynomial

$$f = X^2Y + X^2Z + XYZ + YZ^2 + 2Z^3 \in \mathbb{K}[X, Y, Z]$$

is dehomogenized by setting $Z = 1$,

$$f^a = X^2Y + X^2 + XY + Y + 2 \in \mathbb{K}[X, Y].$$

Homogenization and Dehomogenization

- If f is a polynomial in $\mathbb{K}[X_1 \ldots, X_n]$ of degree d, then the homogenization of f satisfies

$$f^h = X_0^d \cdot f\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right), \tag{51}$$

  where the right-hand side is evaluated as rational function.

- If f is a polynomial in $\mathbb{K}[X_1, \ldots, X_n]$, then

$$(f^h)^a = f. \tag{52}$$

- If f is a homogeneous polynomial in $\mathbb{K}[X_0, \ldots, X_n]$ and $X_0^{e_0}$ is the highest power of X_0 dividing f, then

$$f = X_0^{e_0} \cdot (f^a)^h. \tag{53}$$

Homogenization and Dehomogenization – Example

Let

$$f = X^2Y + X^2 + Y + 2 \in \mathbb{K}[X, Y].$$

Then

$$\begin{aligned} f^h &= Z^3 f\left(\frac{X}{Z}, \frac{Y}{Z}\right) \\ &= Z^3\left[\left(\frac{X}{Z}\right)^2\left(\frac{Y}{Z}\right) + \left(\frac{X}{Z}\right)^2 + \left(\frac{Y}{Z}\right) + 2\right] \\ &= X^2Y + X^2Z + YZ^2 + 2Z^3 \in \mathbb{K}[X, Y, Z]. \end{aligned}$$

Moreover,

$$\begin{aligned} (f^h)^a &= (X^2Y + X^2Z + YZ^2 + 2Z^3)^a \\ &= X^2Y + X^2 + Y + 2 = f. \end{aligned}$$

Homogenization and Dehomogenization – Example

Let

$$f = X_0^3 + X_0X_1X_2 \in \mathbb{K}[X_0, X_1, X_2].$$

The highest power $X_0^{e_0}$ of X_0 dividing f is X_0^1, so $e_0 = 1$. Then

$$\begin{aligned} X_0^{e_0} \cdot (f^a)^h &= X_0 \cdot (1 + X_1X_2)^h \\ &= X_0 \cdot (X_0^2 + X_1X_2) \\ &= X_0^3 + X_0X_1X_2 = f. \end{aligned}$$

Homogeneous Polynomials in Singular

```
> ring r = 0, (x,y,z), dp;
> poly f = x3y3 + x7y + 2xy;
> poly fh = homog(f,z);  // homogenization of f
> fh;
x7y+x3y3z2 + x7y+ 2xyz6;
> homog(f);                  // test for homogeneity
0
> homog(fh);                 // test for homogeneity
1
```

Lines in Projective Plane

Projective Lines

- Given homogeneous polynomial $\ell \in \mathbb{K}[X, Y, Z]$ of total degree 1. Then

$$\ell = \alpha X + \beta Y + \gamma Z, \tag{54}$$

where $\alpha, \beta, \gamma \in \mathbb{K}$ with $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.

- The corresponding zero locus

$$\mathcal{L}(\ell) = \{(a : b : c) \in \mathbb{P}^2(\mathbb{K}) \mid \ell(a, b, c) = 0\} \tag{55}$$

over $\mathbb{K}$ is called *projective line* over $\mathbb{K}$, also written $\mathcal{L}_{\mathbb{K}}(\alpha, \beta, \gamma)$ or simpler $\mathcal{L}(\alpha, \beta, \gamma)$.

Nonsingularity of Lines

A projective line $\mathcal{L}(\alpha, \beta, \gamma)$ is always non-singular.

Proof.

For each point P on the line,

$$\frac{\partial \ell}{\partial X}(P) = \alpha, \quad \frac{\partial \ell}{\partial Y}(P) = \beta, \quad \frac{\partial \ell}{\partial Z}(P) = \gamma. \tag{56}$$

These partial derivatives do not vanish altogether, since $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. □

Lines and Dehomogenization

Given the projective line $\mathcal{L}_{\mathbb{K}}(\ell)$ by the linear polynomial

$$\ell = \alpha X + \beta Y + \gamma Z \in \mathbb{K}[X, Y, Z].$$

Dehomogenization yields the dehomogenized polynomial

$$\ell^a = \alpha X + \beta Y + \gamma \in \mathbb{K}[X, Y]$$

with zero locus

$$\mathcal{L}(\ell^a) = \{(a, b) \in \mathbb{K}^2 \mid \ell^a(a, b) = 0\}.$$

Affine and Projective Lines – Summary

- Affine line

$$\ell_1 : Y = mX + b, \quad m \neq 0,$$

projective line

$$\ell_1^h : Y = mX + bZ,$$

zero locus

$$\mathcal{L}(\ell_1^h) = \{(x : mx + b : 1) \mid x \in \mathbb{K}\} \cup \{(\frac{1}{m} : 1 : 0)\}.$$

- Affine line

$$\ell_2 : X = a,$$

projective line

$$\ell_2^h : X = aZ,$$

zero locus

$$\mathcal{L}(\ell_2^h) = \{(a : y : 1) \mid y \in \mathbb{K}\} \cup \{(0 : 1 : 0)\}.$$

Line Through Two Points

Two distinct points in $\mathbb{P}^2(\mathbb{K})$ lie on a unique projective line.

Proof.

Given two distinct points $P_1 = (a_1 : b_1 : c_1), P_2 = (a_2 : b_2 : c_2)$ in space $\mathbb{P}^2(\mathbb{K})$. A line $\mathcal{L}(\alpha, \beta, \gamma)$ over $\mathbb{K}$ with $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ between P_1, P_2 satisfies

$$\alpha a_1 + \beta b_1 + \gamma c_1 = \alpha a_2 + \beta b_2 + \gamma c_2.$$

Equivalently, there is a solution (α, β, γ) of the homogeneous linear system of equations with coefficient matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

Since the points are distinct, one row is not the scalar multiple of the other and thus the rows of A are linearly independent. Thus A has rank 2 and hence its kernel is one-dimensional giving a unique line $\mathcal{L}(\alpha, \beta, \gamma)$ between the points up to a scalar multiple. □

Line Through Two Points – Example

Given the points $P_1 = (0:1:0)$ and $P_2 = (1:1:0)$ in $\mathbb{P}^2(\mathbb{K})$. The matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

has rank 2 and its kernel is generated by the vector $(0,0,1)$. Thus the homogeneous polynomial $\ell = Z$ gives the unique projective line $\mathcal{L}(0,0,1)$ between these points.

Intersection of Two Lines

Two distinct projective lines in $\mathbb{P}^2(\mathbb{K})$ meet in exactly one point.

Proof.

Let $\mathcal{L}(\alpha_1, \beta_1, \gamma_1)$ and $\mathcal{L}(\alpha_2, \beta_2, \gamma_2)$ be distinct projective lines in space $\mathbb{P}^2(\mathbb{K})$. Then the matrix

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{pmatrix}$$

has rank 2 and the kernel of the matrix is one-dimensional yielding a unique vector (a, b, c) up to a scalar multiple. This is the unique point $P = (a : b : c)$ lying on both projective lines. □

Intersection of Two Lines – Example

Given the projective lines $\mathcal{L}(0,1,0)$ and $\mathcal{L}(1,1,0)$ in $\mathbb{P}^2(\mathbb{K})$. The corresponding polynomial are

$$\ell = Y \quad \text{and} \quad \ell = X + Y.$$

The matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

has rank 2 and its kernel is generated by the vector $(0,0,1)$. Thus the unique point lying on both projective lines is $P = (0:0:1)$.

Intersection of Two Lines – Example

Consider the parallel affine lines over $\mathbb{K}$ given by

$$l_0 = Y - X \quad \text{and} \quad l_1 = Y - X - 1.$$

- The affine loci are

$$\begin{aligned} \mathcal{L}(l_0) &= \{(a, a) \mid a \in \mathbb{K}\}, \\ \mathcal{L}(l_1) &= \{(a, a+1) \mid a \in \mathbb{K}\}. \end{aligned}$$

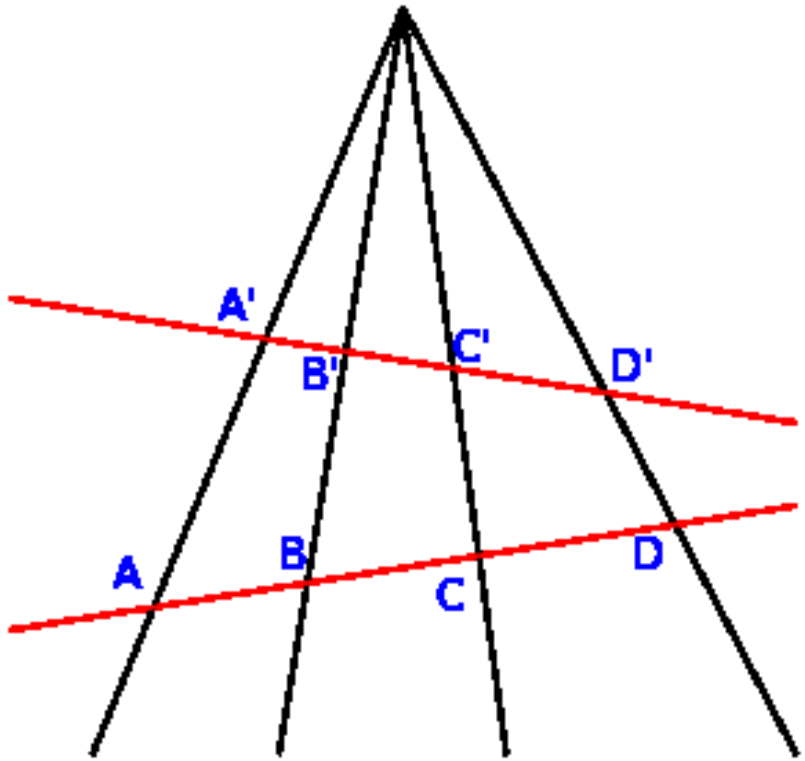
  These lines are parallel in $\mathbb{A}^2(\mathbb{K})$: $\mathcal{L}(l_0) \cap \mathcal{L}(l_1) = \emptyset$.

- By homogenization,

$$l_0^h = Y - X \quad \text{and} \quad l_1^h = Y - X - Z.$$

  These lines meet at $P = (1 : 1 : 0)$: $\mathcal{L}(l_0^h) \cap \mathcal{L}(l_1^h) = \{P\}$.

*Projective Transformations



Affine Transformations

An *affine transformation* T of the affine space $\mathbb{A}^2(\mathbb{K})$ is given by an invertible linear mapping followed by a translation, i.e.,

$$T : \mathbb{A}^2(\mathbb{K}) \to \mathbb{A}^2(\mathbb{K}) : (x, y) \mapsto (x', y'), \tag{57}$$

where

$$x' = ax + by + c, \quad y' = dx + ey + f, \quad ad - bc \neq 0. \tag{58}$$

That is,

$$\begin{aligned} (x, y) &\mapsto \begin{pmatrix} a & b \\ d & e \end{pmatrix} (x, y) = (ax + by, dx + ey) \\ &\mapsto (ax + by + c, dx + ey + f). \end{aligned}$$

The affine transformations of $\mathbb{A}^2(\mathbb{K})$ form a group under the composition of maps.

Projective Transformations

An invertible 3×3 matrix $A = (a_{ij})$ over $\mathbb{K}$ acts on the projective plane $\mathbb{P}^2(\mathbb{K})$ via

$$A \cdot (x : y : z) = (x' : y' : z'), \tag{59}$$

where

$$(x', y', z') = (x, y, z) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{60}$$

This mapping is well-defined, since

$$\begin{aligned} A \cdot (\lambda x : \lambda y : \lambda z) &= \lambda(x, y, z)A = \lambda(x', y', z') \\ &= (\lambda x' : \lambda y' : \lambda z'), \quad \lambda \in \mathbb{K}^*. \end{aligned} \tag{61}$$

Projective Transformations – Example

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

We have

$$\mathbb{P}^2(\mathbb{K}) = \mathbb{A}^2(\mathbb{K}) \cup \mathbb{L}(\mathbb{K}) \cup \{(1:0:0)\}.$$

Then

$$\begin{aligned} A \cdot (a:b:1) &= (a+b+1:b+1:1), \\ A \cdot (a:1:0) &= (a+1:1:0), \\ A \cdot (1:0:0) &= (1:0:0). \end{aligned}$$

Projective Transformations

- The diagonal matrix $\mathrm{diag}(\lambda, \lambda, \lambda)$, $\lambda \in \mathbb{K}^*$, fixes each element $(x : y : z) \in \mathbb{P}^2(\mathbb{K})$.
- The group of all diagonal matrices $\mathrm{diag}(\lambda, \lambda, \lambda)$, $\lambda \in \mathbb{K}^*$, is isomorph to $\mathbb{K}^*$.
- The *general linear group* $\mathrm{GL}_3(\mathbb{K})$ is the group of all invertible 3×3 matrices over $\mathbb{K}$.
- The *projective linear group* $\mathrm{PGL}_3(\mathbb{K})$ is the quotient group

$$\mathrm{PGL}_3(\mathbb{K}) = \mathrm{GL}_3(\mathbb{K})/\mathbb{K}^*.$$

  The elements of $\mathrm{PGL}_3(\mathbb{K})$ are called *projective transformations*.
- Two projective transformations A, B are equal iff $B = \lambda A$ for some $\lambda \in \mathbb{K}^*$.

Projective Transformations

Let $A = (a_{ij}) \in \mathrm{PGL}_3(\mathbb{K})$ be a projective transformation. The following are equivalent:

1. The restriction of A to $\mathbb{A}^2(\mathbb{K}) = \{(x : y : 1) \mid x, y \in \mathbb{K}\}$ is an affine transformation.
2. $a_{13} = a_{23} = 0$.
3. A fixes the projective line given by $\ell = Z$.

Projective Transformations – Proof

- $1 \Rightarrow 2$: We have $(x : y : 1)A = (x' : y' : z')$ with $z' = a_{13}x + a_{23}y + a_{33}$. If A induces an affine transformation, then $z' \neq 0$ for all $x, y \in \mathbb{K}$. Thus $a_{13} = a_{23} = 0$ and as $\det A \neq 0$ (by rescaling) $a_{33} = 1$.
- $2 \Rightarrow 1$: If $a_{13} = a_{23} = 0$ and $a_{33} = 1$, then $(x : y : 1)A = (x' : y' : 1)$, where $x' = a_{11}x + a_{21}y + a_{31}$ and $y' = a_{12}x + a_{22}y + a_{32}$. This is an affine transformation.

Projective Transformations – Proof (cont'd)

- $2 \Rightarrow 3$: If $a_{13} = a_{23} = 0$ and $a_{33} = 1$, then $(x : y : 0)A = (x' : y' : 0)$ and so the line $\mathcal{L}(0, 0, 1)$ is preserved.
- $3 \Rightarrow 2$: If $(x : y : 0)A = (x' : y' : 0)$ for all $x, y \in \mathbb{K}$, then $a_{13} = a_{23} = 0$.

□

Projective Transformations

The group $\mathrm{PGL}_3(\mathbb{K})$ acts on the set of projective cubic curves over $\mathbb{K}$ in the sense that

$$f^A = A \cdot f = f((X, Y, Z)A^{-1}), \tag{62}$$

where $A \in \mathrm{PGL}_3(\mathbb{K})$ and $f \in \mathbb{K}[X, Y, Z]$ is a homogeneous polynomial of degree 3.

Point $(x : y : z)$ on $\mathcal{C}(f)$ will get mapped by A to point $(x' : y' : z') = (x : y : z)A$ on $\mathcal{C}(f^A)$.
Indeed,

$$f^A(x' : y' : z') = f^A((x, y, z)A) = f((x, y, z)AA^{-1}) = f(x, y, z).$$

Group Action

We have

$$I \cdot f = f \quad \text{and} \quad (AB) \cdot f = A \cdot (B \cdot f) \tag{63}$$

where $A, B \in \mathrm{PGL}_3(\mathbb{K})$ and I is the unit 3×3 matrix over $\mathbb{K}$.

Proof.

We have

$$(I \cdot f)(X, Y, Z) = f((X, Y, Z)I) = f(X, Y, Z)$$

and

$$\begin{aligned} (AB \cdot f)(X, Y, Z) &= f((X, Y, Z)(AB)^{-1}) \\ &= f((X, Y, Z)B^{-1}A^{-1}) \\ &= A \cdot (f((X, Y, Z)B^{-1})) \\ &= (A \cdot (B \cdot f))(X, Y, Z). \end{aligned}$$

Projective Transformations – Example

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$f = YZ - X^2. \tag{64}$$

Then

$$\begin{aligned} f^A &= A \cdot f = f((X,Y,Z)A^{-1}) \\ &= f(X-Y,Y,Z) \\ &= YZ - (X-Y)^2. \end{aligned}$$

Point $(0:1:0)$ on $\mathcal{C}(f)$ gets mapped to $(1:1:0) = (0:1:0)A$ on $\mathcal{C}(f^A)$.

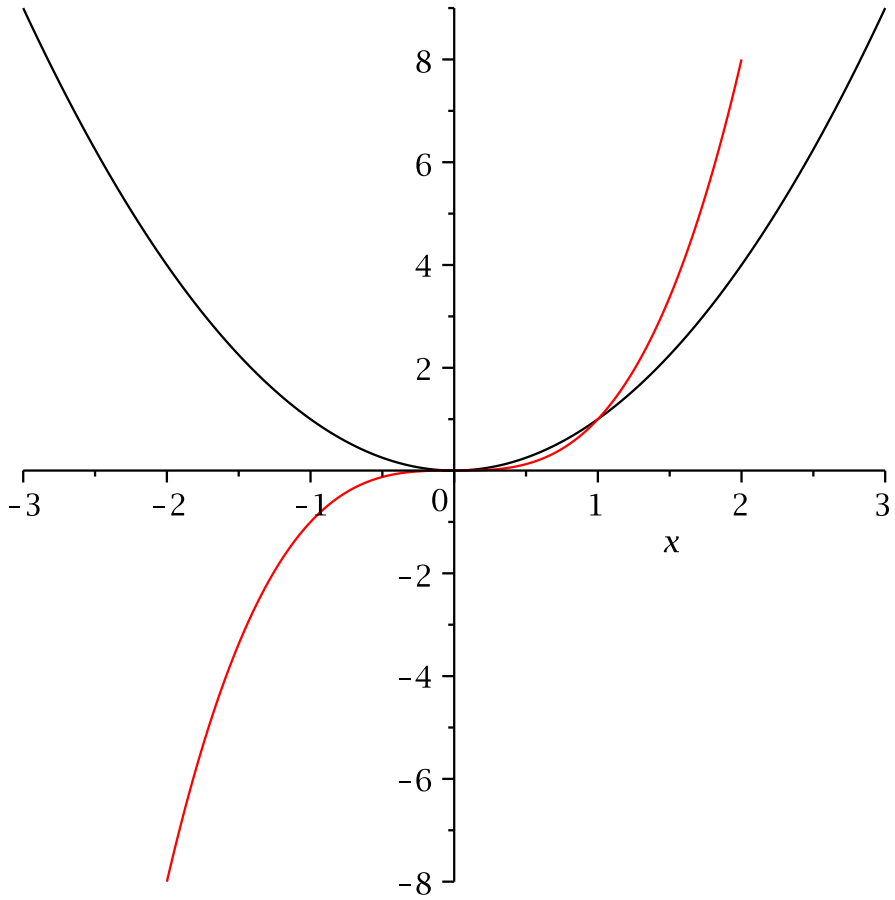
Part III

Algebraic Curves

Algebraic Curves

- Plane curves
- Singularies
- Rational mappings and birationality
- Tangents
- Intersection multiplicities
- Weierstrass equation

Plane Curves




```
> with(plots):
> plots[multiple](plot,[x^2,x=-3..3,color=black],
                      [y^3,y=-2..2,color=red]);
```

Plane Projective Curves

- A *plane projective curve of degree* $d \geq 1$ *over* $\mathbb{K}$ is the zero locus of a homogeneous polynomial $f \in \mathbb{K}[X, Y, Z]$ of degree d,

$$\mathcal{C}_{\mathbb{K}}(f) = \{(a : b : c) \in \mathbb{P}^2(\mathbb{K}) \mid f(a, b, c) = 0\}. \tag{65}$$

  If $\mathbb{L}$ is a field extension of $\mathbb{K}$, then $\mathcal{C}_{\mathbb{K}}(f)$ is a subset of $\mathcal{C}_{\mathbb{L}}(f)$

- Examples:
  - Unit circle $f = X^2 + Y^2 - Z^2$,
  - Parabola $g = X^2 - YZ$,
  - Elliptic curve $h = Y^2Z - X^3 - XZ^2$.

Plane Affine Curves

- Dehomogenization of a homogeneous polynomial $f \in \mathbb{K}[X, Y, Z]$ obtained by setting $Z = 1$ gives a polynomial f^a in $\mathbb{K}[X, Y]$ and the corresponding zero locus is a *plane affine curve*,

$$\mathcal{C}_{\mathbb{K}}(f^a) = \{(a, b) \in \mathbb{A}^2(\mathbb{K}) \mid f^a(a, b) = 0\}. \tag{66}$$

- Examples:
  - Unit circle $f^a = X^2 + Y^2 - 1$,
  - Parabola $g^a = X^2 - Y$,
  - Elliptic curve $h^a = Y^2 - X^3 - X$.

Projective Lines and Conics

- A projective curve of degree 1 is a *projective line* defined as zero locus of the linear homogeneous polynomial

$$f = aX + bY + cZ, \tag{67}$$

where $a, b, c \in \mathbb{K}$ with $(a, b, c) \neq (0, 0, 0)$.

- A projective curve of degree 2 is a *conic* defined as zero locus of the quadratic homogeneous polynomial

$$f = a_1X^2 + a_2XY + a_3XZ + a_4Y^2 + a_5YZ + a_6Z^2 \tag{68}$$

where $a_1, \ldots, a_6 \in \mathbb{K}$.

- Examples:
  - Unit circle $f = X^2 + Y^2 - Z^2$,
  - Parabola $g = X^2 - YZ$.

Unit Parabola – Example

Affine unit parabola is the zero locus of the polynomial $f = X^2 - Y \in \mathbb{R}[X, Y]$,

$$\mathcal{C}_{\mathbb{R}}(f) = \{(a, a^2) \mid a \in \mathbb{R}\}.$$

Projective unit parabola is the zero locus of the homogeneous polynomial $f^h = X^2 - YZ \in \mathbb{R}[X, Y, Z]$,

$$\mathcal{C}_{\mathbb{R}}(f^h) = \{(a : a^2 : 1) \mid a \in \mathbb{R}\} \cup \{(0 : 1 : 0)\}.$$

Cubics

- A projective curve of degree 3 is a *cubic* defined as zero locus of the cubic homogeneous polynomial

$$\begin{aligned} f \quad = \quad & a_1X^3 + a_2X^2Y + a_3X^2Z + a_4XY^2 \\ & + a_5XZ^2 + a_6XYZ + a_7Y^3 + a_8Y^2Z \\ & + a_9YZ^2 + a_{10}Z^3, \end{aligned} \tag{69}$$

  where $a_1, \ldots, a_{10} \in \mathbb{K}$.

- Example: elliptic curve in Weierstrass form

$$f = Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3.$$

Unit Cubic – Example

Affine unit cubic is the zero locus of the polynomial $f = X^3 - Y \in \mathbb{R}[X, Y]$,

$$\mathcal{C}_{\mathbb{R}}(f) = \{(a, a^3) \mid a \in \mathbb{R}\}.$$

Projective unit cubic is the zero locus of the homogeneous polynomial $f^h = X^3 - YZ^2 \in \mathbb{R}[X, Y, Z]$,

$$\mathcal{C}_{\mathbb{R}}(f^h) = \{(a : a^3 : 1) \mid a \in \mathbb{R}\} \cup \{(0 : 1 : 0)\}.$$

Homogenization

Let f be a nonzero polynomial in $\mathbb{K}[X, Y]$ of degree d. Homogenization of f is the homogeneous polynomial f^h of degree d in $\mathbb{K}[X, Y, Z]$ such that

$$f^h(a, b, 1) = f(a, b) \text{ for all } (a, b) \in \mathbb{A}^2(\mathbb{K}), \tag{70}$$

i.e.,

$$\mathcal{C}_{\mathbb{K}}(f^h) \cap \mathbb{A}^2(\mathbb{K}) = \mathcal{C}_{\mathbb{K}}(f). \tag{71}$$

Dehomogenization

Let f be a nonzero homogeneous polynomial in $\mathbb{K}[X, Y, Z]$ of degree d.
Dehomogenization of f is the polynomial f^a of degree at most d in $\mathbb{K}[X, Y]$ such that

$$f^a(a, b) = f(a, b, 1) \text{ for all } (a, b) \in \mathbb{A}^2(\mathbb{K}), \tag{72}$$

i.e.,

$$\mathcal{C}_{\mathbb{K}}(f^a) = \mathcal{C}_{\mathbb{K}}(f) \cap \mathbb{A}^2(\mathbb{K}). \tag{73}$$

Homogenization and Dehomogenization – Example

- Polynomial $f = X^3 - Y$ corresponds to the homogeneous polynomial $f^h = X^3 - YZ^2$ with

$$\mathcal{C}_{\mathbb{R}}(f^h) \cap \mathbb{A}^2(\mathbb{R}) = \mathcal{C}_{\mathbb{R}}(f) = \{(a, a^3) \mid a \in \mathbb{R}\}.$$

- Homogeneous polynomial $f = X^3 - YZ^2$ corresponds to the dehomogenized polynomial $f^a = X^3 - Y$ with

$$\mathcal{C}_{\mathbb{R}}(f) \cap \mathbb{A}^2(\mathbb{R}) = \mathcal{C}_{\mathbb{R}}(f^a) = \{(a, a^3) \mid a \in \mathbb{R}\}.$$

*Factorization of Homogeneous Polynomials

Let $f \in \mathbb{K}[X,Y,Z]$ be a homogeneous polynomial of degree $d \geq 1$, and let $f = gh$, where g and h are non-constant polynomials in $\mathbb{K}[X,Y,Z]$. Then g and h are homogeneous polynomials, too.

Proof.

Consider the rational function field $\mathbb{L} = \mathbb{K}(X,Y,Z)$ and take a new variable T. We have

$$\begin{aligned} f(TX,TY,TZ) &= f(X,Y,Z) \cdot T^d \in \mathbb{L}[T], \\ g(TX,TY,TZ) &= \sum_i g_i(X,Y,Z) \cdot T^i \in \mathbb{L}[T], \\ h(TX,TY,TZ) &= \sum_j h_j(X,Y,Z) \cdot T^j \in \mathbb{L}[T], \end{aligned}$$

where the g_i and h_j are the homogeneous components of g and h, respectively.

*Proof (cont'd).

Since $f = gh$, comparing coefficients gives

$$T^d = \left(\sum_i \frac{g_i(X,Y,Z)}{f(X,Y,Z)} T^i \right) \left(\sum_j \frac{h_j(X,Y,Z)}{f(X,Y,Z)} T^j \right) \in \mathbb{L}[T].$$

It follows that the first factor corresponds to T^k for some $k \geq 0$ and the second factor corresponds to T^l for some $l \geq 0$ with $d = k + l$. Thus all the homogeneous components g_i except for g_k are zero and all the homogeneous components h_j except for h_l are zero. Hence, the polynomials $g = g_k$ and $h = h_l$ are homogeneous. □

*Factorization of Homogeneous Polynomials – Example

Let $g = XY + Z^2$ and $h = Y + Z$. Then
$f = gh = XY^2 + XYZ + YZ^2 + Z^3$ and

$$\begin{aligned} f(TX, TY, TZ) &= T^3 f(X, Y, Z), \\ g(TX, TY, TZ) &= T^2 g(X, Y, Z), \\ h(TX, TY, TZ) &= T h(X, Y, Z). \end{aligned}$$

Thus

$$T^3 = \frac{g(X, Y, Z)}{f(X, Y, Z)} T^2 \cdot \frac{h(X, Y, Z)}{f(X, Y, Z)} T.$$

*Zero Loci of Homogeneous Polynomials

Let $f \in \mathbb{K}[X, Y, Z]$ be a homogeneous polynomial of degree $d \geq 1$. Given the decomposition of f as a product of powers of irreducible factors in $\bar{\mathbb{K}}[X, Y, Z]$,

$$f = \prod_{i=1}^{s} f_i^{e_i}, \quad e_i \geq 1. \tag{74}$$

Then all polynomials f_i are homogeneous (as shown) and we have

$$\mathcal{C}_{\bar{\mathbb{K}}}(f) = \bigcup_{i=1}^{s} \mathcal{C}_{\bar{\mathbb{K}}}(f_i). \tag{75}$$

The zero loci $\mathcal{C}_{\bar{\mathbb{K}}}(f_i)$ are the irreducible components of the curve $\mathcal{C}_{\bar{\mathbb{K}}}(f)$ and e_i is the multiplicity of the component $\mathcal{C}_{\bar{\mathbb{K}}}(f_i)$, $1 \leq i \leq s$.

*Projective Transformations

Projective transformations preserve the degree of curves.

Proof.

Given plane projective curve $\mathcal{C}(f)$ and projective transformation A of $\mathbb{P}^2(\mathbb{K})$. Consider the plane projective curve $\mathcal{C}(f^A)$ given by

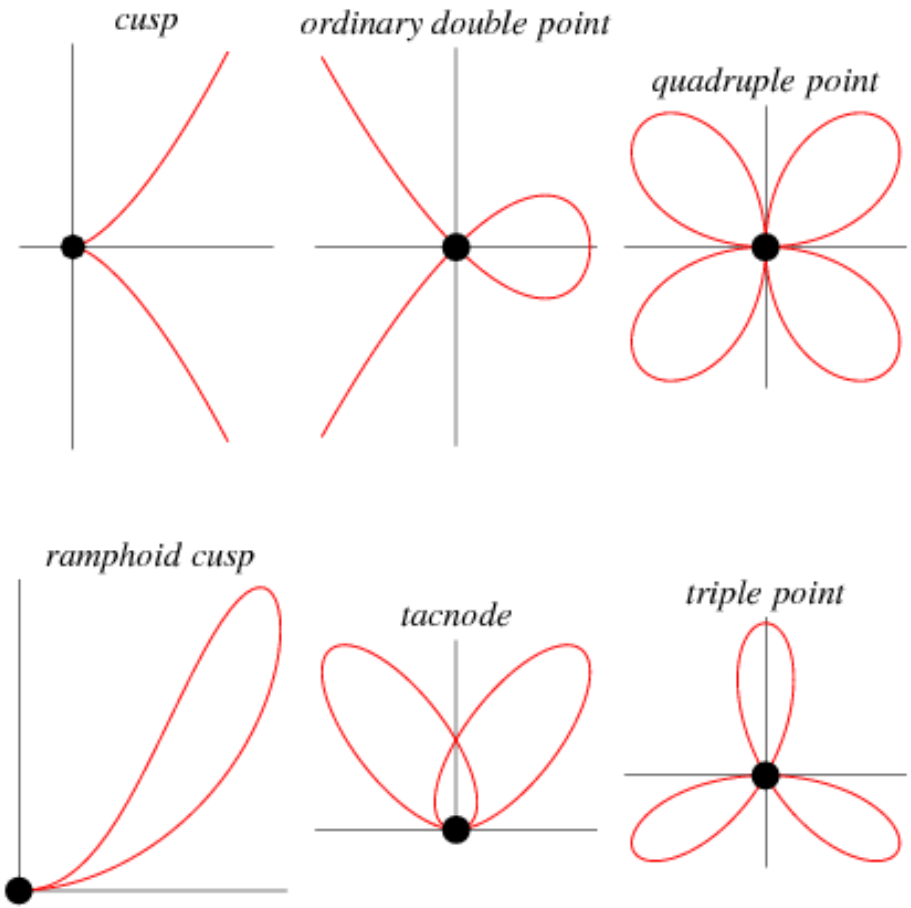
$$f^A(X,Y,Z) = f((X,Y,Z)A^{-1}).$$

Write $A^{-1} = (b_{ij})$. A monomial $X^iY^jZ^{d-i-j}$ of degree d involved in f is mapped to

$$(b_{11}X+b_{21}Y+b_{31}Z)^i(b_{12}X+b_{22}Y+b_{32}Z)^j(b_{13}X+b_{23}Y+b_{33}Z)^{d-i-j},$$

which is a homogeneous polynomial of degree d. Since f^A is a linear combination of such polynomials, it is itself homogeneous of degree d. □

Singularities



Definition of Singularity

Let f be a homogeneous polynomial in $\mathbb{K}[X, Y, Z]$ of degree d.

- The projective curve $\mathcal{C} = \mathcal{C}(f)$ over $\mathbb{K}$ is *singular* at the point $P = (a : b : c) \in \mathcal{C}$ if the first partial derivatives of f at P vanish,

$$\frac{\partial f}{\partial X}(P) = \frac{\partial f}{\partial Y}(P) = \frac{\partial f}{\partial Z}(P) = 0. \tag{76}$$

  Then P is a *singular point* of $\mathcal{C}$. Every other point in $\mathcal{C}$ is *non-singular*.

Well-definedness: f is homogeneous and so the partial derivatives are also homogeneous.

Definition of Singularity

Let f be a homogeneous polynomial in $\mathbb{K}[X, Y, Z]$ of degree d.

- A projective curve $\mathcal{C} = \mathcal{C}(f)$ is *non-singular* or *smooth* if its has no singular point.

The vanishing of all three derivatives of f at point $P = (a : b : c)$ is independent of the homogeneous coordinates of P.

Example

Homogeneous polynomial

$$\ell = \alpha X + \beta Y + \gamma Z \in \mathbb{K}[X, Y, Z]$$

of degree 1 gives rise to the projective line

$$\mathcal{L}(\alpha, \beta, \gamma) = \mathcal{L}(\ell) = \{(a : b : c) \in \mathbb{P}^2(\mathbb{K}) \mid \ell(a, b, c) = 0\}. \tag{77}$$

Line $\mathcal{L}(\ell)$ is non-singular, since for each point P on the line

$$\frac{\partial \ell}{\partial X}(P) = \alpha, \quad \frac{\partial \ell}{\partial Y}(P) = \beta, \quad \frac{\partial \ell}{\partial Z}(P) = \gamma \tag{78}$$

and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.

Example

Consider the projective conic given by

$$f = X^2 + Y^2 \in \mathbb{K}[X, Y, Z].$$

Then

$$\frac{\partial f}{\partial X} = 2X, \ \frac{\partial f}{\partial Y} = 2Y, \ \frac{\partial f}{\partial Z} = 0.$$

Point $P = (0 : 0 : 1)$ on the conic $\mathcal{C}(f)$ is singular.
If $\mathrm{char}(\mathbb{K}) = 2$, then every point is singular and so $\mathcal{C}(f)$ is singular.

Example

Consider the projective cubic given by

$$f = X^3 + Y^3 + Z^3 \in \mathbb{K}[X, Y, Z].$$

Then

$$\frac{\partial f}{\partial X} = 3X, \ \frac{\partial f}{\partial Y} = 3Y, \ \frac{\partial f}{\partial Z} = 3Z.$$

If $\mathrm{char}(\mathbb{K}) \neq 3$, the cubic $\mathcal{C}(f)$ is smooth.

If $\mathrm{char}(\mathbb{K}) = 3$, then

$$f = (X + Y + Z)^3$$

and so

$$\frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial Z} = 3(X + Y + Z)^2 = 0,$$

i.e., all points are singular and so $\mathcal{C}(f)$ is singular.

Example

Consider the twisted cubic given by

$$f = X^2Y - Z^3 \in \mathbb{K}[X, Y, Z].$$

Then

$$\frac{\partial f}{\partial X} = 2XY, \ \frac{\partial f}{\partial Y} = X^2, \ \frac{\partial f}{\partial Z} = -3Z^2.$$

It has one singularity at $P = (0 : 1 : 0)$, since

$$\frac{\partial f}{\partial X}(P) = \frac{\partial f}{\partial Y}(P) = \frac{\partial f}{\partial Z}(P) = 0.$$

All other points are non-singular. This can be checked by considering the points $(a : b : 1)$, $(a : 1 : 0)$ with $a \neq 0$, and $(1 : 0 : 0)$.

Intersection of Plane Curves

Let $\mathcal{C}_1 = \mathcal{C}(f_1)$ and $\mathcal{C}_2 = \mathcal{C}(f_2)$ be plane projective curves over $\mathbb{K}$, and let P be a point of the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$. Then P is a singular point of $\mathcal{C}(f_1 f_2)$.

Proof.

By the product rule,

$$\frac{\partial f_1 f_2}{\partial X}(P) = \frac{\partial f_1}{\partial X}(P) f_2(P) + f_1(P) \frac{\partial f_2}{\partial X}(P) = 0,$$

since $f_1(P) = 0 = f_2(P)$. The same holds for the other variables Y and Z. □

Intersection of Plane Curves – Example

Consider the curves $\mathcal{C}(f_1)$ and $\mathcal{C}(f_2)$ given by

$$f_1 = X^2 + Y^2 - 2Z^2 \quad \text{and} \quad f_2 = XY - Z^2.$$

They have a common point $P(1:1:1)$. In view of the product

$$f = f_1 f_2 = X^3Y + XY^3 - 2XYZ^2 - X^2Z^2 - Y^2Z^2 + 2Z^4,$$

the partial derivatives are

$$\begin{aligned} \frac{\partial f}{\partial X} &= 3X^2Y + Y^3 - 2YZ^2 - 2XZ^2, \\ \frac{\partial f}{\partial Y} &= X^3 + 3XY^2 - 2XZ^2 - 2YZ^2, \\ \frac{\partial f}{\partial Z} &= -4XYZ - 2X^2Z - 2Y^2Z + 8Z^3. \end{aligned}$$

The point P vanishes at all three derivatives.

Affine vs. Projective Singularities

Let f be a homogeneous polynomial in $\mathbb{K}[X, Y, Z]$ of degree d and let f^a denote the corresponding dehomogenized polynomial in $\mathbb{K}[X, Y]$.
Plane projective curve $\mathcal{C}(f)$ is singular at $P = (a : b : 1)$ iff plane affine curve $\mathcal{C}(f^a)$ is singular at $Q = (a, b)$.

Proof.

Point $P = (a : b : 1)$ lies in $\mathcal{C}(f)$ iff point $Q = (a, b)$ belongs to $\mathcal{C}(f^a)$. Write

$$f = \sum_{\substack{i,j \geq 0 \\ i+j \leq d}} a_{ij} X^i Y^j Z^{d-i-j}.$$

Then

$$f^a = \sum_{\substack{i,j \geq 0 \\ i+j \leq d}} a_{ij} X^i Y^j.$$

Proof (cont'd).

We have

$$\frac{\partial f}{\partial X} = \sum_{\substack{i>0, j\geq 0 \\ i+j\leq d}} i a_{ij} X^{i-1} Y^j Z^{d-i-j}$$

and

$$\frac{\partial f^a}{\partial X} = \sum_{\substack{i>0, j\geq 0 \\ i+j\leq d}} i a_{ij} X^{i-1} Y^j.$$

Thus

$$\frac{\partial f}{\partial X}(P) = \frac{\partial f^a}{\partial X}(Q).$$

Similarly,

$$\frac{\partial f}{\partial Y}(P) = \frac{\partial f^a}{\partial Y}(Q).$$

Proof (cont'd).

Moreover,

$$\frac{\partial f}{\partial Z} = \sum_{\substack{i,j \geq 0 \\ i+j<d}} (d-i-j) a_{ij} X^i Y^j Z^{d-i-j-1}.$$

Thus

$$\begin{aligned} \frac{\partial f}{\partial Z}(P) &= \sum_{\substack{i,j \geq 0 \\ i+j \leq d}} (d-i-j) a_{ij} a^i b^j \\ &= d \cdot f^a(Q) - a \cdot \frac{\partial f^a}{\partial X}(Q) - b \cdot \frac{\partial f^a}{\partial Y}(Q). \end{aligned}$$

In the next to the last sum, we can allow $i + j \leq d$ since the term with $d = i + j$ vanishes in the derivative. From these equations the result follows. □

Example

- Projective conic given by

$$f = X^2 + Y^2 \in \mathbb{K}[X, Y, Z]$$

is singular at $P = (0 : 0 : 1)$, since the partial derivatives are

$$\frac{\partial f}{\partial X} = 2X, \quad \frac{\partial f}{\partial Y} = 2Y, \quad \frac{\partial f}{\partial Z} = 0.$$

If $\mathrm{char}(\mathbb{K}) = 2$, each point is singular.

- Affine conic given by

$$f^a = X^2 + Y^2 \in \mathbb{K}[X, Y]$$

is singular at $Q = (0, 0)$, since the partial derivatives are

$$\frac{\partial f^a}{\partial X} = 2X, \quad \frac{\partial f^a}{\partial Y} = 2Y.$$

If $\mathrm{char}(\mathbb{K}) = 2$, each point is singular.

Example

- Projective conic given by

$$f = XZ + YZ \in \mathbb{K}[X, Y, Z]$$

is non-singular at $P = (a : -a : 1)$, since the partial derivatives are

$$\frac{\partial f}{\partial X} = Z, \quad \frac{\partial f}{\partial Y} = Z, \quad \frac{\partial f}{\partial Z} = X + Y$$

with

$$\frac{\partial f}{\partial X}(P) = 1, \quad \frac{\partial f}{\partial Y}(P) = 1, \quad \frac{\partial f}{\partial Z}(P) = 0.$$

- Affine line given by $f^a = X + Y \in \mathbb{K}[X, Y]$ is non-singular at $Q = (a, -a)$, since the partial derivatives are

$$\frac{\partial f^a}{\partial X} = 1, \quad \frac{\partial f^a}{\partial Y} = 1.$$

*Projective Transformations

Projective transformations preserve singularities.

Proof.

Given plane projective curve $\mathcal{C}(f)$ and projective transformation A of $\mathbb{P}^2(\mathbb{K})$. Consider the plane projective curve $\mathcal{C}(f^A)$ given by

$$f^A(X,Y,Z) = f((X,Y,Z)A^{-1}).$$

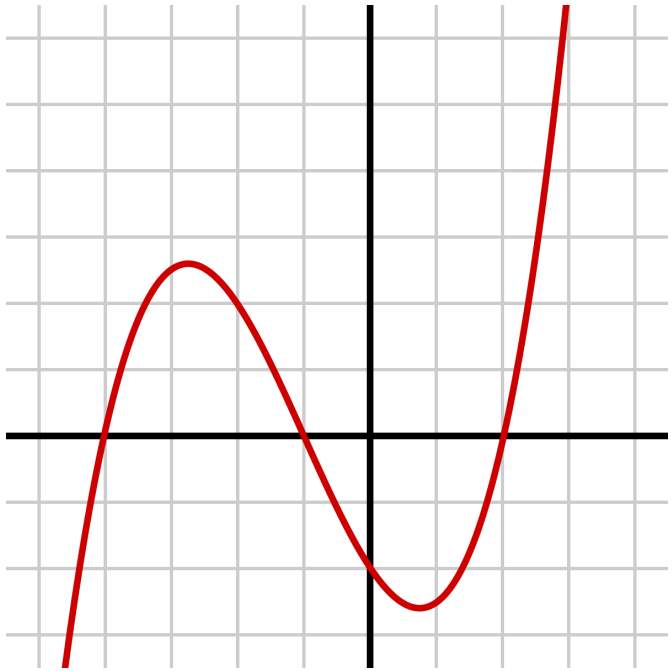
By the chain rule,

$$\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right) = \left(\frac{\partial f^A}{\partial X}, \frac{\partial f^A}{\partial Y}, \frac{\partial f^A}{\partial Z}\right) \cdot A. \tag{79}$$

Since A is invertible, $\mathcal{C}(f)$ is non-singular iff $\mathcal{C}(f^A)$ is non-singular. □

Rational Mappings

A curve is *rational* if it is birationally equivalent to a line.



Rational Mappings

Given two plane projective curves $\mathcal{C}_1$ and $\mathcal{C}_2$ over $\mathbb{K}$ and homogeneous polynomials $g_0, g_1, g_2, h_0, h_1, h_2 \in \mathbb{K}[X, Y, Z]$.

A mapping $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ is *rational* over $\mathbb{K}$ if for almost all points $(x : y : z) \in \mathcal{C}_1(\bar{\mathbb{K}})$, the expression

$$\phi(x : y : z) = \left(\frac{g_0(x, y, z)}{h_0(x, y, z)} : \frac{g_1(x, y, z)}{h_1(x, y, z)} : \frac{g_2(x, y, z)}{h_2(x, y, z)} \right) \tag{80}$$

is defined and lies in $\mathcal{C}_2(\bar{\mathbb{K}})$.

Example

Take the line $f_1 = Z$ and the conic $f_2 = Y^2 - XZ$ in $\mathbb{K}[X, Y, Z]$.

- Zero loci are

$$\mathcal{C}(f_1) = \{(x : 1 : 0) \mid x \in \bar{\mathbb{K}}\} \cup \{(1 : 0 : 0)\}$$

and

$$\mathcal{C}(f_2) = \{(x^2 : x : 1) \mid x \in \bar{\mathbb{K}}\} \cup \{(1 : 0 : 0)\}.$$

- Polynomials $g_0 = X^2$, $g_1 = XY$, and $g_2 = Y^2$ provide a rational mapping

$$\phi : \mathcal{C}(f_1) \to \mathcal{C}(f_2) : (x : y : z) \mapsto (x^2 : xy : y^2)$$

with $\phi(x : 1 : 0) = (x^2 : x : 1)$ and $\phi(1 : 0 : 0) = (1 : 0 : 0)$.

Birational Mappings

Given two plane projective curves $\mathcal{C}_1$ and $\mathcal{C}_2$ over $\mathbb{K}$.

- A rational mapping $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ defined over $\mathbb{K}$ is *birational* if there is a rational mapping $\psi : \mathcal{C}_2 \to \mathcal{C}_1$ defined over $\mathbb{K}$ such that for almost all points of $\mathcal{C}_1(\bar{\mathbb{K}})$ and $\mathcal{C}_2(\bar{\mathbb{K}})$, the mappings $\phi \circ \psi$ and $\psi \circ \phi$ are defined and are equal to the identity; i.e., a birational mapping is bijective almost everywhere.
- Two plane projective curves $\mathcal{C}_1$ and $\mathcal{C}_2$ over $\mathbb{K}$ are *birationally equivalent over* $\mathbb{K}$ if there is a birational mapping $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ defined over $\mathbb{K}$.

Example

Zero loci of the line $f_1 = Z$ and the conic $f_2 = Y^2 - XZ$ in $\mathbb{K}[X, Y, Z]$ are birationally equivalent over $\mathbb{K}$ by the rational mappings

$$\phi(x : y : z) = (x^2 : xy : y^2)$$

and

$$\psi(x : y : z) = (y : z : 0),$$

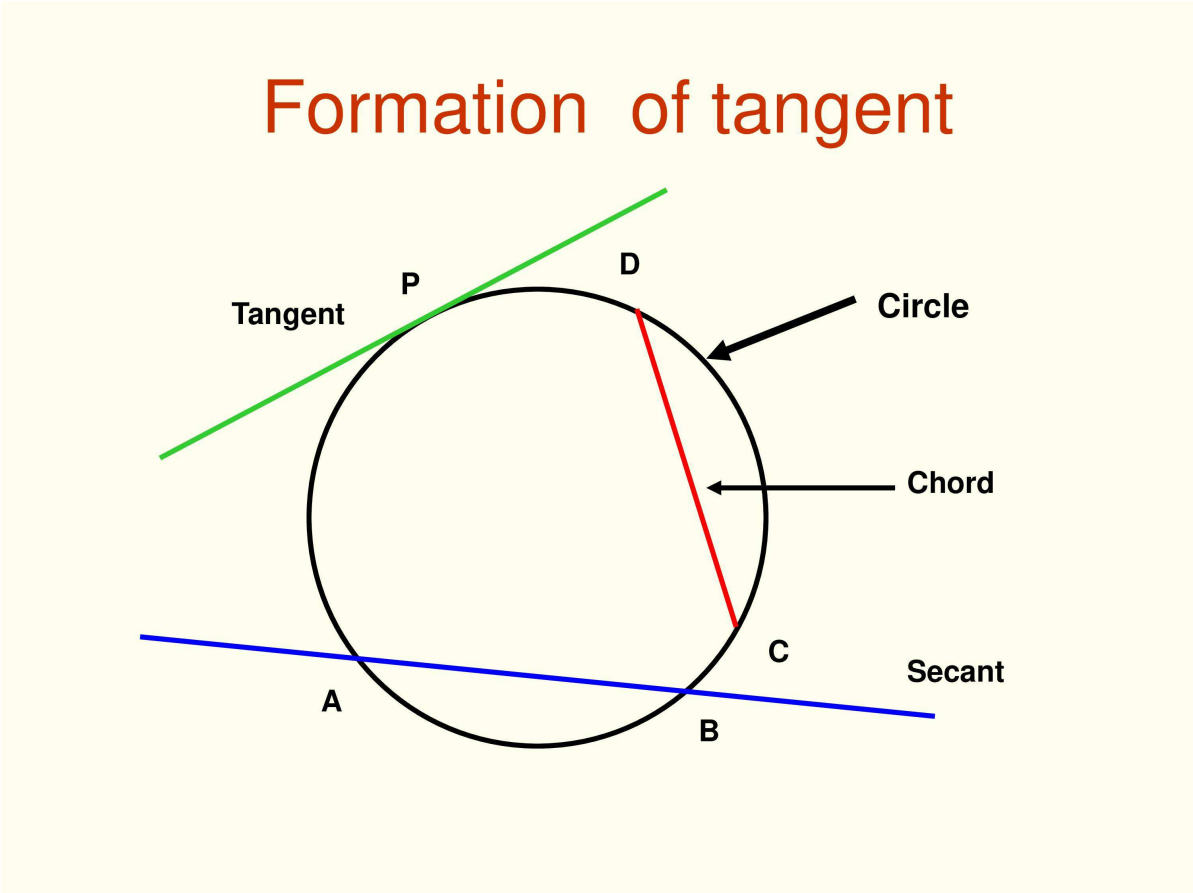
since

$$\phi(x : 1 : 0) = (x^2 : x : 1), \quad \phi(1 : 0 : 0) = (1 : 0 : 0)$$

and

$$\psi(x^2 : x : 1) = (x : 1 : 0), \quad \psi(1 : 0 : 0) \text{ undefined.}$$

Tangents



Tangents

Let $\mathcal{C} = \mathcal{C}(f)$ be a plane projective curve in $\mathbb{P}^2(\mathbb{K})$ and let P be a non-singular point on $\mathcal{C}$ defined over $\mathbb{K}$.
Projective line

$$\mathcal{L} = \mathcal{L}\left(\frac{\partial f}{\partial X}(P), \frac{\partial f}{\partial Y}(P), \frac{\partial f}{\partial Z}(P)\right) \tag{81}$$

is the *tangent line* to $\mathcal{C}$ at P (see (77)).

The tangent line $\mathcal{L}$ is well-defined, since P is non-singular and so the partial derivatives do not vanish altogether at P.

Example

Cubic $\mathcal{C}(f)$ given by

$$f = Y^2Z - X^3 - XZ^2 \in \mathbb{R}[X, Y, Z]$$

is non-singular with partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial X} &= -3X^2 - Z^2, \\ \frac{\partial f}{\partial Y} &= 2YZ, \\ \frac{\partial f}{\partial Z} &= Y^2 - 2XZ. \end{aligned}$$

Tangent line to $\mathcal{C}$ at $P = (0 : 1 : 0)$ is

$$\mathcal{L}([-3X^2 - Z^2](P), [2YZ](P), [Y^2 - 2XZ](P)) = \mathcal{L}(0, 0, 1),$$

given by the homogeneous polynomial $\ell = Z$.

*Example

- Projective unit parabola $\mathcal{C} = \mathcal{C}(f)$ with $f = YZ - X^2 \in \mathbb{R}[X, Y, Z]$ is non-singular with partial derivatives

$$\frac{\partial f}{\partial X} = -2X, \quad \frac{\partial f}{\partial Y} = Z, \quad \frac{\partial f}{\partial Z} = Y.$$

  Point $P = (0 : 1 : 0) \in \mathcal{C}(f)$ has tangent line $\mathcal{L}(0, 0, 1)$.

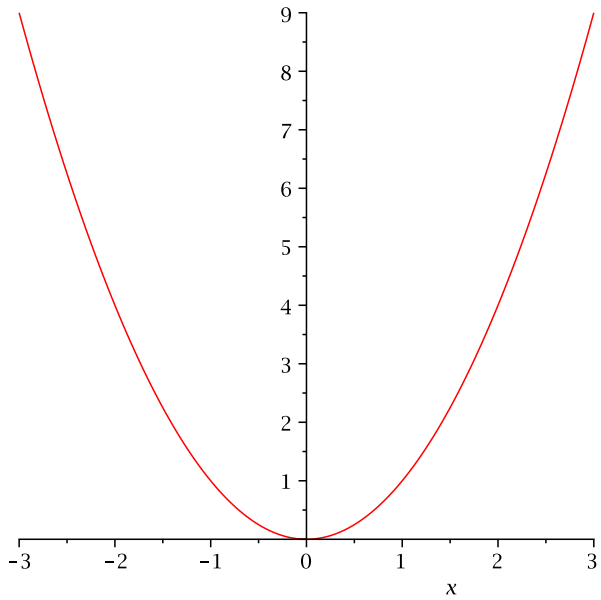
- In view of the projective transformation A in (64), the conic $\mathcal{C}^A = \mathcal{C}(f^A)$ with $f^A(X, Y, Z) = YZ - (X - Y)^2$ is non-singular with partial derivatives

$$\frac{\partial f^A}{\partial X} = -2(X - Y), \quad \frac{\partial f^A}{\partial Y} = Z + 2(X - Y), \quad \frac{\partial f^A}{\partial Z} = Y.$$

  Point $P' = PA = (1 : 1 : 0) \in \mathcal{C}(f^A)$ has tangent line $\mathcal{L}(0, 0, 1)$.

Intersection Multiplicities

Intersection of X-axis with parabola $Y = X^2$ has degree 2.



Intersection Multiplicities

Given plane projective curve $\mathcal{C} = \mathcal{C}(f)$ in $\mathbb{P}^2(\mathbb{K})$ and projective line $\mathcal{L}$ in $\mathbb{P}^2(\mathbb{K})$.

- Fix a point $P = (a : b : c) \in \mathcal{L}$ and an auxiliary point $P' = (a' : b' : c') \in \mathcal{L}$.
- The *intersection multiplicity* of $\mathcal{L}$ and $\mathcal{C}$ at P, denoted $i(P, \mathcal{L}, \mathcal{C})$, is the order of zero of the polynomial

$$\phi(t) = f(a + ta', b + tb', c + tc') \in \mathbb{K}[t]. \tag{82}$$

Order of Zero

- The *order of zero* of a polynomial

$$\phi(t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_m t^m \in \mathbb{K}[t]$$

is the smallest index $j \geq 0$ such that

$$\alpha_0 = 0,\ \alpha_1 = 0,\ \ldots,\ \alpha_{j-1} = 0,\ \text{and}\ \alpha_j \neq 0.$$

- Equivalently, the order of zero of $\phi(t)$ is the smallest index $j \geq 0$ such that the derivatives of $\phi(t)$ satisfy

$$\phi(0) = 0,\ \phi'(0) = 0,\ \ldots,\ \phi^{(j-1)}(0) = 0,\ \text{and}\ \phi^{(j)}(0) \neq 0.$$

- Example: The polynomials $2t + t^2$ and $t^3 - t^5$ in $\mathbb{K}[t]$ have order of zero 1 and 3, respectively.

Basic Facts

- The definition of intersection multiplicity is independent of the choice of auxiliary point P'.
- The constant term of polynomial $\phi(t)$ is $f(a, b, c)$ and thus the point $P = (a : b : c)$ lies on the curve $\mathcal{C}$ iff $\phi(0) = 0$, i.e., $i(P, \mathcal{L}, \mathcal{C}) \geq 1$ (see Explanation).
- Thus for each point P on the line $\mathcal{L}$, which does not belong to the curve $\mathcal{C}$, we have $\phi(0) \neq 0$, i.e., $i(P, \mathcal{L}, \mathcal{C}) = 0$.
- Define $i(P, \mathcal{L}, \mathcal{C}) = 0$ if the point P does not belong to the line $\mathcal{L}$.
- If $\mathcal{C}(f)$ is a plane projective curve of degree $d \geq 1$, then

$$i(P, \mathcal{L}, \mathcal{C}) \leq d, \tag{83}$$

since the arguments of f as polynomials in t are linear and so $\phi(t)$ has degree $\leq d$ (see Explanation).

Basic Facts – Explanation

Take the homogeneous polynomial of degree d,

$$f = \sum_{\alpha} a_\alpha X^\alpha \in \mathbb{K}[X_0, X_1, X_2],$$

where $a_\alpha \in \mathbb{K}$, $\alpha_0 + \alpha_1 + \alpha_2 = d$, and $X^\alpha = X_0^{\alpha_0} X_1^{\alpha_1} X_0^{\alpha_2}$. Then

$$\begin{aligned} \phi(t) &= f(a + ta', b + tb', c + tc') \\ &= \sum_{\alpha} a_\alpha (a + ta')^{\alpha_0} (b + tb')^{\alpha_1} (c + tc')^{\alpha_2} \\ &= f(a, b, c)t^0 + u_1 t^1 + \ldots + u_d t^d, \end{aligned}$$

where $u_1, \ldots, u_d \in \mathbb{K}$.

Example

- Consider the conic $\mathcal{C} = \mathcal{C}(f)$ given by $f = X^2 - YZ$ in $\mathbb{R}[X, Y, Z]$.
- Take the projective line $\mathcal{L} = \mathcal{L}(1, -1, 0)$, i.e., $\ell = X - Y$.
- The point $P = (0 : 0 : 1)$ lies in the intersection $\mathcal{C} \cap \mathcal{L}$.
- Fix the auxiliary point $P' = (1 : 1 : 1) \in \mathcal{L}$.
- Then

$$\begin{aligned} \phi(t) &= f(0 + 1 \cdot t, 0 + 1 \cdot t, 1 + 1 \cdot t) \\ &= f(t, t, 1 + t) = t^2 - t(1 + t) = -t \end{aligned}$$

and so $i(P, \mathcal{L}, \mathcal{C}) = 1$.

Example

- Consider the cubic $\mathcal{C} = \mathcal{C}(f)$ given by $f = Y^2Z - X^3 - XZ^2$ in $\mathbb{R}[X, Y, Z]$.
- Take the projective line $\mathcal{L} = \mathcal{L}(1, 0, 1)$, i.e., $\ell = X + Z$.
- The point $P = (0 : 1 : 0)$ lies in the intersection $\mathcal{C} \cap \mathcal{L}$.
- Fix the auxiliary point $P' = (1 : 0 : -1) \in \mathcal{L}$.
- Then

$$\begin{aligned} \phi(t) &= f(0 + 1 \cdot t, 1 + 0 \cdot t, 0 + (-1) \cdot t) \\ &= f(t, 1, -t) = (-t) - t^3 - t(-t)^2 = -t - 2t^3 \end{aligned}$$

and so $i(P, \mathcal{L}, \mathcal{C}) = 1$.

Intersection Multiplicities – Tangents

Let $\mathcal{L} = \mathcal{L}(\alpha, \beta, \gamma)$ be a tangent line to plane projective curve $\mathcal{C} = \mathcal{C}(f)$ at non-singular point $P = (a : b : c) \in \mathcal{C}$ defined over $\mathbb{K}$. Then

$$i(P, \mathcal{L}, \mathcal{E}) \geq 2. \tag{84}$$

Proof.

Since P lies on the curve $\mathcal{C}$, $\phi(0) = f(a,b,c) = 0$.
Moreover, $\mathcal{L} = \mathcal{L}(\alpha,\beta,\gamma)$, i.e., $\ell = \alpha X + \beta Y + \gamma Z$, is the tangent line to $\mathcal{C}$ at P and so $\mathcal{L}$ and f have the same slope at point P.
Thus

$$\alpha = \frac{\partial f}{\partial X}(a,b,c), \quad \beta = \frac{\partial f}{\partial Y}(a,b,c), \quad \gamma = \frac{\partial f}{\partial Z}(a,b,c).$$

By taking another point $P' = (a' : b' : c') \in \mathcal{L}$, the chain rule gives

$$\begin{aligned} \phi'(0) &= \frac{\partial f}{\partial X}(a,b,c) \cdot a' + \frac{\partial f}{\partial Y}(a,b,c) \cdot b' + \frac{\partial f}{\partial Z}(a,b,c) \cdot c' \\ &= \alpha a' + \beta b' + \gamma c' \end{aligned}$$

which equals 0 since $P' \in \mathcal{L}$. Hence, $i(P,\mathcal{L},\mathcal{E}) \geq 2$. □

Example

- Consider the conic $\mathcal{C} = \mathcal{C}(f)$ given by $f = X^2 - YZ$.
- Take the point $P = (0 : 0 : 1) \in \mathcal{C}$.
- The corresponding tangent line at P is $\mathcal{L} = \mathcal{L}(0, -1, 0)$, i.e., $\ell = -Y$, since

$$\frac{\partial f}{\partial X}(P) = 0, \quad \frac{\partial f}{\partial Y}(P) = -1, \quad \frac{\partial f}{\partial Z}(P) = 0.$$

- Take the auxiliary point $P' = (1 : 0 : 1) \in \mathcal{L}$.
- Then

$$\phi(t) = f(0 + 1 \cdot t, 0 + 0 \cdot t, 1 + 1 \cdot t) = f(t, 0, 1 + t) = t^2$$

and so $i(P, \mathcal{L}, \mathcal{C}) = 2$.

Example

- The cubic $\mathcal{C}(f)$ given by $f = Y^2Z - X^3 - XZ^2$ is non-singular at the point $P = (0:1:0)$.
- The corresponding tangent line is $\mathcal{L} = \mathcal{L}(0,0,1)$, i.e., $\ell = Z$, since
$$\frac{\partial f}{\partial X}(P) = 0,\ \frac{\partial f}{\partial Y}(P) = 0,\ \frac{\partial f}{\partial Z}(P) = 1.$$
- Take the auxiliary point $P' = (1:1:0) \in \mathcal{L}$.
- Then
$$\phi(t) = f(0 + 1 \cdot t, 1 + 1 \cdot t, 0 + 0 \cdot t) = f(t, 1 + t, 0) = -t^3$$
and so $i(P, \mathcal{L}, \mathcal{C}) = 3$; i.e., P is a *flex* of $\mathcal{C}$.

Example – Circle

Consider the complex conic $\mathcal{C} = \mathcal{C}(f)$ given by

$$f = X^2 + Y^2 - Z^2 \in \mathbb{C}[X, Y, Z].$$

- The conic consists of the affine points

$$\left(\frac{a}{\sqrt{a^2 + b^2}} : \frac{b}{\sqrt{a^2 + b^2}} : 1 \right), \quad (a, b) \neq (0, 0),$$

  and the point at infinity $(i : 1 : 0)$.

- The conic is smooth, since the partial derivatives do not vanish altogether,

$$\frac{\partial f}{\partial X} = 2X, \quad \frac{\partial f}{\partial Y} = 2Y, \quad \frac{\partial f}{\partial Z} = -2Z.$$

Example (cont'd)

- The affine point $P = \left(\frac{a}{\sqrt{a^2+b^2}} : \frac{b}{\sqrt{a^2+b^2}} : 1\right) \in \mathcal{C}$ has tangent line

$$\begin{aligned} & \mathcal{L}(2X(P), 2Y(P), -2Z(P)) \\ = \quad & \mathcal{L}\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, -1\right), \end{aligned}$$

i.e.,

$$\ell = \frac{a}{\sqrt{a^2+b^2}} X + \frac{b}{\sqrt{a^2+b^2}} Y - Z.$$

Taking the auxiliary point $P' = (-b : a : 0) \in \mathcal{L}$ gives

$$\begin{aligned} \phi(t) &= f\left(\frac{a}{\sqrt{a^2+b^2}} + (-b)t, \frac{b}{\sqrt{a^2+b^2}} + at, 1\right) \\ &= (a^2+b^2)t^2 \end{aligned}$$

and so $i(P, \mathcal{L}, \mathcal{C}) = 2$.

Example (cont'd)

- The point at infinity $Q = (i : 1 : 0) \in \mathcal{C}$ has tangent line

$$\mathcal{L}(2X(Q), 2Y(Q), -2Z(Q)) = \mathcal{L}(i, 1, 0),$$

i.e.,

$$\ell = iX + Y.$$

Taking the auxiliary point $Q' = (0 : 0 : 1) \in \mathcal{L}$ gives

$$\phi(t) = f(i+0\cdot t, 1+0\cdot t, 0+1\cdot t) = f(i, 1, t) = i^2+1^2-t^2 = -t^2$$

and so $i(Q, \mathcal{L}, \mathcal{C}) = 2$.

*Projective Transformations

Projective transformations preserve multiplicities.

Proof.

Given plane projective curve $\mathcal{C}(f)$ and projective transformation A of $\mathbb{P}^2(\mathbb{K})$. Consider the plane projective curve $\mathcal{C}(f^A)$ given by

$$f^A(X,Y,Z) = f((X,Y,Z)A^{-1}).$$

The intersection multiplicities $i(P,\mathcal{L},\mathcal{C}(f))$ and $i(PA,\mathcal{L}^A,\mathcal{C}(f^A))$ coincide, where $\mathcal{L}=\mathcal{L}(\ell)$ and $\mathcal{L}^A=\mathcal{L}(\ell^A)$ with $\ell^A(X,Y,Z)=\ell((X,Y,Z)A^{-1})$. □

Weierstrass Form

Karl Weierstrass (1815-1897)

General Cubics

A general cubic over $\mathbb{K}$ has the form

$$\begin{aligned} f \quad = \quad & a_{yyy}Y^3 + a_{xyy}XY^2 + a_{xxy}X^2Y + a_{yyz}Y^2Z \\ & + a_{xyz}XYZ + a_{yzz}YZ^2 + a_{xxx}X^3 + a_{xxz}X^2Z \\ & + a_{xzz}XZ^2 + a_{zzz}Z^3. \end{aligned} \tag{85}$$

Geometric Condition on Cubics I

The cubic (85) is to pass through the point at infinity $O = (0 : 1 : 0)$. Then

$$a_{yyy} = 0. \tag{86}$$

Proof.

We have $f(O) = a_{yyy}$ and so $a_{yyy} = 0$. □

Geometric Condition on Cubics II

Let $a_{yyy} = 0$.
The cubic (85) is to have $O = (0 : 1 : 0)$ as non-singular point.
Then

$$a_{xyy} \neq 0 \text{ or } a_{yyz} \neq 0. \tag{87}$$

Proof.

We have

$$\frac{\partial f}{\partial X}(O) = a_{xyy}, \quad \frac{\partial f}{\partial Y}(O) = 3a_{yyy} = 0, \quad \frac{\partial f}{\partial Z}(O) = a_{yyz}.$$

Thus $a_{xyy} \neq 0$ or $a_{yyz} \neq 0$. □

Geometric Condition on Cubics III

Let $a_{yyy} = 0$, and $a_{xyy} \neq 0$ or $a_{yyz} \neq 0$.
The cubic (85) is to have $\mathcal{L} = \mathcal{L}(0, 0, 1)$, i.e., $\ell = Z$, as tangent line at $O = (0 : 1 : 0)$. Then

$$a_{xyy} = 0 \text{ and so } a_{yyz} \neq 0. \tag{88}$$

Proof.

We have

$$\frac{\partial f}{\partial X}(O) = a_{xyy}, \quad \frac{\partial f}{\partial Y}(O) = 3a_{yyy} = 0, \quad \frac{\partial f}{\partial Z}(O) = a_{yyz}.$$

Thus the tangent line is $\mathcal{L} = \mathcal{L}(a_{xyy}, 0, a_{yyz})$, i.e.,

$$\ell = a_{xyy}X + a_{yyz}Z.$$

By hypothesis, $\mathcal{L} = \mathcal{L}(0, 0, 1)$, i.e., $\ell = Z$, and so $a_{xyy} = 0$ and $a_{yyz} \neq 0$. □

Geometric Condition on Cubics IV

Let $a_{yyy} = 0$, $a_{xyy} = 0$ and $a_{yyz} \neq 0$.
The tangent line $\mathcal{L} = \mathcal{L}(0, 0, 1)$, i.e., $\ell = Z$, to the cubic (85) at $O = (0 : 1 : 0)$ is to be a flex; i.e., $i(O, \mathcal{L}, \mathcal{C}) = 3$. Then

$$a_{xxy} = 0 \text{ and } a_{xxx} \neq 0. \tag{89}$$

Proof.

Taking the auxiliary point $P' = (1 : 1 : 0) \in \mathcal{L}$ gives

$$\begin{aligned}
\phi(t) &= f(0 + 1 \cdot t, 1 + 1 \cdot t, 0 + 0 \cdot t) \\
&= f(t, 1 + t, 0) \\
&= a_{yyy}(1 + t)^3 + a_{xyy}t(1 + t)^2 + a_{xxy}t^2(1 + t) + a_{xxx}t^3 \\
&= a_{xxy}t^2(1 + t) + a_{xxx}t^3 \\
&= a_{xxy}t^2 + (a_{xxy} + a_{xxx})t^3.
\end{aligned}$$

By hypothesis, O is a flex and so $a_{xxy} = 0$ and $a_{xxx} \neq 0$. □

Weierstrass Form

When the conditions I-IV are met, i.e.,

$$a_{yyy} = 0,\ a_{xyy} = 0,\ a_{xxy} = 0,\ a_{yyz} \neq 0,\ a_{xxx} \neq 0, \tag{90}$$

the cubic (85) becomes

$$\begin{aligned} f \quad = \quad & a_{yyz}Y^2Z + a_{xyz}XYZ + a_{yzz}YZ^2 \\ & + a_{xxx}X^3 + a_{xxz}X^2Z + a_{xzz}XZ^2 + a_{zzz}Z^3 \end{aligned} \tag{91}$$

with $a_{yyz} \neq 0$ and $a_{xxx} \neq 0$.
By a projective transformation, the cubic becomes

$$\begin{aligned} f \quad = \quad & Y^2Z + a_1XYZ + a_3YZ^2 \\ & - X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3. \end{aligned} \tag{92}$$

This cubic is in *Weierstrass form*.

Weierstrass Form

There is a projective transformation that transforms the cubic (91) to the cubic (92).

Proof.

Let $t \in \mathbb{K}^*$. Put $A = \operatorname{diag}(t^{-1}, t^{-1}, 1)$. Then

$$\begin{aligned} f^A(X,Y,Z) &= f((X,Y,Z)A^{-1}) \\ &= f(tX,tY,Z) \\ &= a_{yyz}(t^2Y^2Z) + \ldots + a_{xxx}(t^3X^3) + \ldots. \end{aligned}$$

Putting $t = -a_{yyz}/a_{xxx}$ shows that the coefficients of Y^2Z and X^3 have the same value and different sign, i.e.,

$$a_{yyz}\left(\frac{-a_{yyz}}{a_{xxx}}\right)^2 = \frac{a_{yyz}^3}{a_{xxx}^2} \text{ and } a_{xxx}\left(\frac{-a_{yyz}}{a_{xxx}}\right)^3 = \frac{-a_{yyz}^3}{a_{xxx}^2}.$$

This gives the required transformation. □

Part IV

Introduction to Elliptic Curves

Elliptic Curves

- Weierstrass form
- Discriminant
- *j-Invariant
- Singular points
- Intersection multiplicities
- Group law: abstract and in coordinates
- Use of `Singular`

Weierstrass Form

Karl Weierstrass (1815-1897)

Weierstrass Polynomials and Equations

Let $\mathbb{K}$ be a field.

- The polynomial

$$\begin{aligned} f &= Y^2Z + a_1XYZ + a_3YZ^2 \\ &\quad -X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3 \end{aligned} \tag{93}$$

  in $\mathbb{K}[X, Y, Z]$ is called *Weierstrass polynomial* over $\mathbb{K}$.

- The equation

$$\begin{aligned} Y^2Z + a_1XYZ + a_3YZ^2 = \\ X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \end{aligned} \tag{94}$$

  is called *Weierstrass equation* over $\mathbb{K}$.

Basic Facts

Consider the decomposition of the projective plane

$$\mathbb{P}^2(\mathbb{K}) = \mathbb{A}^2(\mathbb{K}) \cup \mathbb{L}(\mathbb{K}) \cup \{(1:0:0)\}. \tag{95}$$

- For each point $P = (a:1:0) \in \mathcal{E}(f) \cap \mathbb{L}(\mathbb{K})$ on the line, $f(a,1,0) = -a^3 = 0$ and so $a = 0$.
- The point $O = (0:1:0)$ lies in $\mathcal{E}(f)$ called *base point*.
- The point $(1:0:0)$ does not lie in $\mathcal{E}(f)$, since $f(1,0,0) = -1$.
- The base point O is the only point at infinity in $\mathcal{E}(f)$; i.e., the only point on the curve not in the affine plane $\mathbb{A}^2(\mathbb{K})$.
- The base point O is non-singular:

$$\frac{\partial f}{\partial Z}(O) = \tag{96}$$
$$[Y^2 + a_1XY + 2a_3YZ - a_2X^2 - 2a_4X - 3a_6Z^2](O) = 1.$$

Elliptic Curves

- An *elliptic curve* $\mathcal{E}(f)$ over $\mathbb{K}$ is a non-singular cubic over $\mathbb{K}$ given in Weierstrass form f.
- A curve in Weierstrass form can be tested for non-singularity by testing only the affine points $(a : b : 1) \in \mathbb{A}^2(\mathbb{K})$ as the base point O is always non-singular.
- Simplification of Weierstrass form depends on the characteristic of $\mathbb{K}$.

Characteristic of a Field

Characteristic of a field $\mathbb{K}$ is the smallest integer $n \geq 1$ such that

$$n \cdot 1 = 1 + \ldots + 1 = 0, \quad n \text{ times.}$$

If there is no such integer n, then $\mathbb{K}$ has characteristic 0.
Characteristic of a field $\mathbb{K}$ is denoted by $\mathrm{char}(\mathbb{K})$.

Characteristic of a field is 0 or a prime, since if $n = ab$ in $\mathbb{Z}$, then

$$0 = n \cdot 1 = (a \cdot 1)(b \cdot 1) \quad \text{in } \mathbb{K}.$$

But a field has no zero-divisors and so n is prime.

$\mathbb{Z}_p$ is the smallest field of characteristic p and $\mathbb{Q}$ is the smallest field of characteristic 0. Both are called prime fields.

$\mathrm{char}(\mathbb{Z}_p) = \mathrm{char}(\mathrm{GF}(p^n)) = p,$
$\mathrm{char}(\mathbb{Q}) = \mathrm{char}(\mathbb{R}) = \mathrm{char}(\mathbb{C}) = 0.$

Simplification of Weierstrass Form

If $\mathrm{char}(\mathbb{K}) \neq 2$, the elliptic curve $\mathcal{E}(f)$ is birationally equivalent to the elliptic curve $\mathcal{E}(h_1)$, where

$$h_1 = Y^2Z - X^3 - \frac{1}{4}b_2X^2Z - \frac{1}{2}b_4XZ^2 - \frac{1}{4}b_6Z^3 \tag{97}$$

and

$$b_2 = a_1^2 + 4a_2, \tag{98}$$
$$b_4 = 2a_4 + a_1a_3, \tag{99}$$
$$b_6 = a_3^2 + 4a_6. \tag{100}$$

Proof.

Birational equivalence

$$\phi_1 : \mathbb{P}^2 \to \mathbb{P}^2 : (a : b : c) \mapsto \left(a : b + \frac{a_1}{2}a + \frac{a_3}{2}c : c\right)$$

with inverse

$$\psi_1 : \mathbb{P}^2 \to \mathbb{P}^2 : (a : b : c) \mapsto \left(a : b - \frac{a_1}{2}a - \frac{a_3}{2}c : c\right).$$

Note that the denominators are nonzero in $\mathbb{K}$.
Projective transformation:

$$h_1(X, Y, Z) = f\left(X, Y - \frac{a_1}{2}X - \frac{a_3}{2}Z, Z\right)$$

i.e., $h_1(X, Y, Z) = f^A(X, Y, Z) = f((X, Y, Z)A^{-1})$ with appropriate matrix A. □

Simplification of Weierstrass Form in Maple

Implementation of rational mapping:

```
> f := Y^2*Z + a_1*X*Y*Z + a_3*Y*Z^2 - X^3
       - a_2*X^2*Z - a_4*X*Z^2 - a_6*Z^3;
> subs( { Y = Y - a_1/2 * X - a_3/2 * Z }, f);
> simplify(%);
> h_1 := collect(%, [X,Y,Z]);
```

Simplification of Weierstrass Form

If $\mathrm{char}(\mathbb{K}) \neq 2, 3$, the elliptic curve $\mathcal{E}(f)$ is birationally equivalent to the elliptic curve $\mathcal{E}(h_2)$, where

$$h_2 = Y^2Z - X^3 + 27c_4XZ^2 + 54c_6Z^3 \tag{101}$$

and

$$\begin{aligned} c_4 &= b_2^2 - 24b_4, & (102)\\ c_6 &= -b_2^3 + 36b_2b_4 - 216b_6. & (103) \end{aligned}$$

Proof.

Birational equivalence

$$\phi_2 : \mathbb{P}^2 \to \mathbb{P}^2 : (a : b : c) \mapsto (36a + 3b_2c : 216b : c)$$

with inverse

$$\psi_2 : \mathbb{P}^2 \to \mathbb{P}^2 : (a : b : c) \mapsto \left(\frac{1}{36}a - \frac{b_2}{12}c : \frac{1}{216}b : c\right).$$

Note that the denominators are nonzero in $\mathbb{K}$, since $36 = 2^2 3^2$, $12 = 2^2 3$, and $216 = 2^3 3^3$.
Projective transformation:

$$h_2(X, Y, Z) = 2^6 3^6 h_1\left(\frac{1}{36}X - \frac{b_2}{12}Z, \frac{1}{216}Y, Z\right),$$

i.e., $h_2(X, Y, Z) = h_1^A(X, Y, Z) = h_1((X, Y, Z)A^{-1})$ with appropriate matrix A. □

Simplification of Weierstrass Form

If $\mathrm{char}(\mathbb{K}) = 2$ and $a_1 \neq 0$, the elliptic curve $\mathcal{E}(f)$ is birationally equivalent to the elliptic curve $\mathcal{E}(h_3)$, where

$$h_3 = Y^2Z + XYZ - X^3 - a_2'X^2Z - a_6'Z^3 \tag{104}$$

and

$$a_2' = \frac{a_3 + a_1a_2}{a_1^3}, \tag{105}$$

$$a_6' = \frac{a_1^6a_6 + a_1^5a_3a_4 + a_1^4a_2a_3^2 + a_1^4a_4^2 + a_1^3a_3^3 + a_3^4}{a_1^{12}}. \tag{106}$$

Proof.

Birational equivalence

$$\phi_3 : \mathbb{P}^2 \to \mathbb{P}^2 : (a : b : c) \mapsto \left(\frac{1}{a_1^2} a - \frac{a_3}{a_1^3} c : \frac{1}{a_1^3} b - \frac{a_1^2 a_4 + a_3^2}{a_1^6} c : c \right)$$

with inverse

$$\psi_3 : \mathbb{P}^2 \to \mathbb{P}^2 : (a : b : c) \mapsto \left(a_1^2 a + \frac{a_3}{a_1} c : a_1^3 b + \frac{a_1^2 a_4 + a_3^2}{a_1^3} c : c \right).$$

Note that the denominators are by assumption nonzero in $\mathbb{K}$.
Projective transformation:

$$a_1^6 h_3(X, Y, Z) = f\left(a_1^2 X + \frac{a_3}{a_1} Z, a_1^3 Y + \frac{a_1^2 a_4 + a_3^2}{a_1^3} Z, Z \right),$$

i.e., $h_3(X, Y, Z) = f^A(X, Y, Z) = f((X, Y, Z)A^{-1})$ with appropriate matrix A.

□

Affine Weierstrass Equations

- $\mathrm{char}(\mathbb{K}) \neq 2$:

$$Y^2 = X^3 + \frac{1}{4}b_2X^2 + \frac{1}{2}b_4X + \frac{1}{4}b_6. \tag{107}$$

- $\mathrm{char}(\mathbb{K}) \neq 2, 3$:

$$Y^2 = X^3 - 27c_4X - 54c_6. \tag{108}$$

- $\mathrm{char}(\mathbb{K}) = 2$, $a_1 \neq 0$:

$$Y^2 + XY = X^3 + a_2'X^2 + a_6'. \tag{109}$$

Computation of Points

Consider the elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{Q}$ given by the affine Weierstrass polynomial

$$f^a = Y^2 + a_1XY + a_3Y - X^3 - a_2X^2 - a_4X - a_6.$$

For each $x \in \mathbb{Q}$, find the points $P = (x : y : 1) \in \mathcal{E}(f)$.

Substitute $X = x$,

$$Y^2 + (a_1x + a_3)Y = x^3 + a_2x^2 + a_4x + a_6$$

Put $c = a_1x + a_3$ and $d = x^3 + a_2x^2 + a_4x + a_6$. By completing the square,

$$\left(Y + \frac{c}{2}\right)^2 = d + \frac{c^2}{4}$$

Put $e = d + \frac{c^2}{4}$. Then

$$y_{1,2} = -\frac{c}{2} \pm \sqrt{e} \in \mathbb{Q}(\sqrt{e}) = \{a + b\sqrt{e} \mid a, b \in \mathbb{Q}\}.$$

Example

Given the elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{Q}$ by

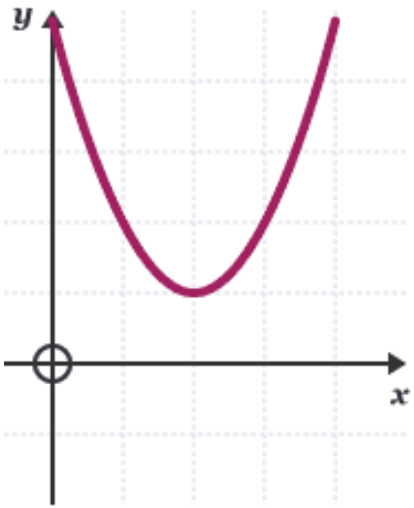
$$Y^2 + Y = X^3 - X.$$

```
> solveE := proc(x)
    local y; solve( y^2+y = x^3-x, y): print(x, %);
  end:
> for x from 1 to 7 do solveE(x) end;
```

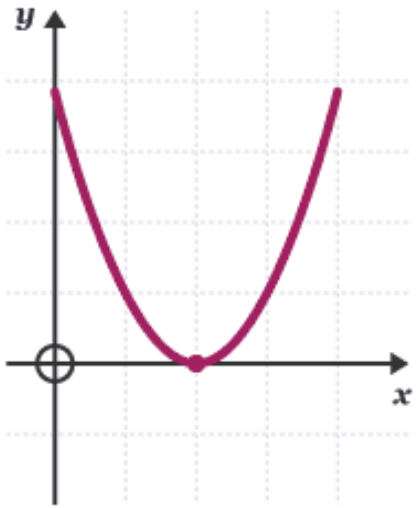
Output:

input x	output y
1	$0, -1$
2	$2, -3$
3	$-\frac{1}{2} \pm \frac{1}{2}\sqrt{97},$
4	$-\frac{1}{2} \pm \frac{1}{2}\sqrt{241},$
5	$-\frac{1}{2} \pm \frac{1}{2}\sqrt{481},$
6	$14, -15$
7	$-\frac{1}{2} \pm \frac{1}{2}\sqrt{1345}$

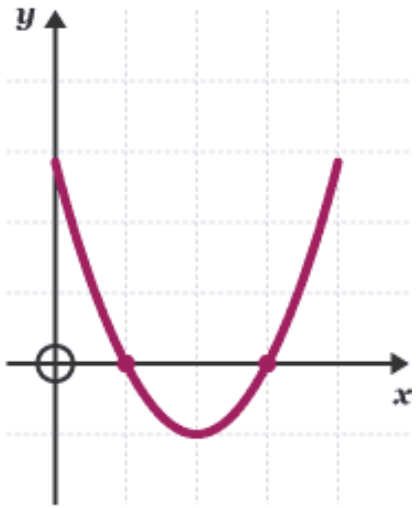
Discriminant



$b^2 - 4ac < 0$
there are no real roots



$b^2 - 4ac = 0$
the roots are real and equal



$b^2 - 4ac > 0$
the roots are real and unequal

Cubic Polynomials

Given a monic cubic polynomial in $\mathbb{K}[X]$,

$$\begin{aligned} f(X) &= X^3 - aX^2 + bX - c \\ &= (X - r_1)(X - r_2)(X - r_3). \end{aligned} \tag{110}$$

By comparing coefficients, the coefficients of f are elementary symmetric polynomials in the roots $r_1, r_2, r_3 \in \overline{\mathbb{K}}$,

$$a = r_1 + r_2 + r_3, \tag{111}$$

$$b = r_1 r_2 + r_1 r_3 + r_2 r_3, \tag{112}$$

$$c = r_1 r_2 r_3. \tag{113}$$

The *discriminant* of the polynomial f is defined as

$$D(f) = (r_1 - r_2)^2 (r_1 - r_3)^2 (r_2 - r_3)^2. \tag{114}$$

Cubic Polynomials

We have

$$\det \begin{pmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{pmatrix} = (r_3 - r_2)(r_3 - r_1)(r_2 - r_1) \tag{115}$$

and

$$\begin{pmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{pmatrix} \begin{pmatrix} 1 & r_1 & r_1^2 \\ 1 & r_2 & r_2^2 \\ 1 & r_3 & r_3^2 \end{pmatrix} = \begin{pmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix}, \tag{116}$$

where each s^i is a power sum in r_1, r_2, r_3,

$$s_i = r_1^i + r_2^i + r_3^i, \quad 1 \leq i \leq 4. \tag{117}$$

Cubic Polynomials

We have

$$D(f) = \det \begin{pmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix}, \tag{118}$$

where

$$\begin{aligned} s_1 &= a, && (119)\\ s_2 &= a^2 - 2b, && (120)\\ s_3 &= a^3 - 3ab + 3c, && (121)\\ s_4 &= a^4 - 4a^2b + 2b^2 + 4ac. && (122) \end{aligned}$$

Example

The cubic polynomial $f = X^3 + pX + q$ has discriminant

$$D(f) = -4p^3 - 27q^2. \tag{123}$$

Proof.

Here $a = 0$, $b = p$, $c = -q$. Thus

$$\begin{aligned} D(f) &= \det \begin{pmatrix} 3 & 0 & -2p \\ 0 & -2p & -3q \\ -2p & -3q & 2p^2 \end{pmatrix} \\ &= -4p^3 - 27q^2. \end{aligned}$$

Basic Facts

- For a general cubic f, $D(f) = 0$ iff f has at least two equal roots by (114).
- For a cubic $f \in \mathbb{R}[X]$, $D(f) = 0$ iff f has repeated roots.
  - If all roots of $f \in \mathbb{R}[X]$ are real, then $D(f) \geq 0$.
  - Otherwise, $f \in \mathbb{R}[X]$ has one real root r_1 and two complex conjugate roots $r_2, r_3 = \bar{r}_2$, since complex conjugation

$$z = a + ib \mapsto \overline{z} = a - ib \tag{124}$$

is an automorphism of $\mathbb{C}$. Indeed, if $z \in \mathbb{C}$ is a zero of a polynomial $f \in \mathbb{R}[X]$, then

$$f(\overline{z}) = \overline{f(z)} = \overline{0} = 0. \tag{125}$$

Then $(r_1 - r_2)(r_1 - \bar{r}_2)$ is real and $(r_2 - \bar{r}_2)$ is imaginary. Since $D(f)$ is the square, $D(f) \leq 0$.

Discriminants vs. Singularities

Let $\mathrm{char}(\mathbb{K}) \neq 2$. The affine curve over $\mathbb{K}$ given by

$$f = Y^2 - (X^3 - aX^2 + bX - c), \tag{126}$$

is non-singular iff the polynomial

$$g = X^3 - aX^2 + bX - c \tag{127}$$

has distinct roots in $\overline{\mathbb{K}}$.

Proof.

The affine curve given by f is singular iff there is a $\overline{\mathbb{K}}$-rational point $(x_0 : y_0 : 1)$ on the curve such that

$$\begin{aligned} f(x_0, y_0) &= y_0^2 - (x_0^3 - ax_0^2 + bx_0 - c) = 0, \\ \frac{\partial f}{\partial X}(x_0, y_0) &= 3x_0^2 - 2ax_0 + b = 0, \\ \frac{\partial f}{\partial Y}(x_0, y_0) &= 2y_0 = 0. \end{aligned}$$

Equivalently, $y_0 = 0$ and $g(x_0) = g'(x_0) = 0$.

- Candidates for singular points over $\overline{\mathbb{K}}$ are only $(x_0 : 0 : 1)$, where x_0 is a root of g.
- The point $(x_0 : 0 : 1)$ is singular iff x_0 is multiple root of g, i.e., the first derivative of g at x_0 also vanishes.

Note that if $\mathrm{char}(\mathbb{K}) = 2$, then $2y_0 = 0$ with y_0 arbitrary. □

Discriminant of Cubic

Given the Weierstrass polynomial over $\mathbb{K}$,

$$f = Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3.$$

The *discriminant* of the curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ is defined as

$$\Delta(\mathcal{E}) = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, \tag{128}$$

where

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, && (129)\\ b_4 &= 2a_4 + a_1a_3, && (130)\\ b_6 &= a_3^2 + 4a_6, && (131)\\ b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2. && (132) \end{aligned}$$

Discriminant – Criterion

Given the Weierstrass polynomial over $\mathbb{K}$,

$$f = Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3.$$

The curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ is non-singular iff the discriminant $\Delta(\mathcal{E})$ is nonzero.

Proof.

The point at infinity $O = (0 : 1 : 0)$ is not singular. Thus the curve $\mathcal{E}(f)$ is non-singular iff the affine curve $\mathcal{E}(f^a)$ is non-singular,[a] where $(Z = 1)$

$$f^a = Y^2 + a_1XY + a_3Y - X^3 - a_2X^2 - a_4X - a_6.$$

The zero locus $\mathcal{E}(f^a)$ contains a singular point $(x_0, y_0) \in \overline{\mathbb{K}}$ iff the following holds,

$$\begin{aligned} f^a(x_0, y_0) &= y_0^2 + a_1x_0y_0 + a_3y_0 - x_0^3 - a_2x_0^2 - a_4x_0 - a_6 = 0, \\ \frac{\partial f^a}{\partial X}(x_0, y_0) &= a_1y_0 - 3x_0^2 - 2a_2x_0 - a_4 = 0, \\ \frac{\partial f^a}{\partial Y}(x_0, y_0) &= 2y_0 + a_1x_0 + a_3 = 0. \end{aligned}$$

Consider several cases depending on the characteristic of $\mathbb{K}$.

[a] See Singularities in Algebraic Curves

Proof (cont'd).

Let $\mathrm{char}(\mathbb{K}) \neq 2, 3$. The elliptic curve $\mathcal{E}(f)$ corresponds to $\mathcal{E}(h_2)$, where

$$h_2 = Y^2Z - X^3 + 27c_4XZ^2 + 54c_6Z^3.$$

The discriminant of the curve $\mathcal{E}(h_2)$ is

$$\Delta = 2^6 3^9 (c_4^3 - c_6^2). \tag{133}$$

Indeed, the coefficients

$$a_1 = a_2 = a_3 = 0,\ a_4 = -27c_4,\ a_6 = -54c_6$$

give

$$b_2 = 0,\ b_4 = -54c_4,\ b_6 = -216c_6,\ b_8 = -27^2c_4^2$$

and thus

$$\Delta = 2^6 3^9 (c_4^3 - c_6^2). \tag{134}$$

Proof (cont'd).

The curve $\mathcal{E}(h_2)$ over $\overline{\mathbb{K}}$ has a singular point iff the polynomial

$$h_2^a = X^3 - 27c_4X - 54c_6$$

has a repeated root in $\overline{\mathbb{K}}$ (see (127)). By (123), the discriminant of h_2^a is

$$D(h_2^a) = -4(-27c_4)^3 - 27(-54c_6)^2 = 2^2 3^9 (c_4^3 - c_6^2).$$

Thus $D(h_2^a) = 0$ iff $c_4^3 - c_6^2 = 0$ iff $\Delta = 0$. □

Discriminant

Let $\mathrm{char}(\mathbb{K}) \neq 2, 3$. The affine cubic $\mathcal{E}$ over $\mathbb{K}$ given by the Weierstrass equation

$$Y^2 = X^3 + pX + q \tag{135}$$

has the discriminant

$$\Delta = -2^4(4p^3 + 27q^2). \tag{136}$$

Proof.

By (108), $p = -27c_4$ and $q = -54c_6$. Thus by (134),

$$\begin{aligned} \Delta &= 2^6 3^9 (c_4^3 - c_6^2) \\ &= 2^6 3^9 ((-3^{-3}p)^3 - (2^{-1}3^{-3}q)^2) \\ &= -2^4(4p^3 + 27q^2). \end{aligned}$$

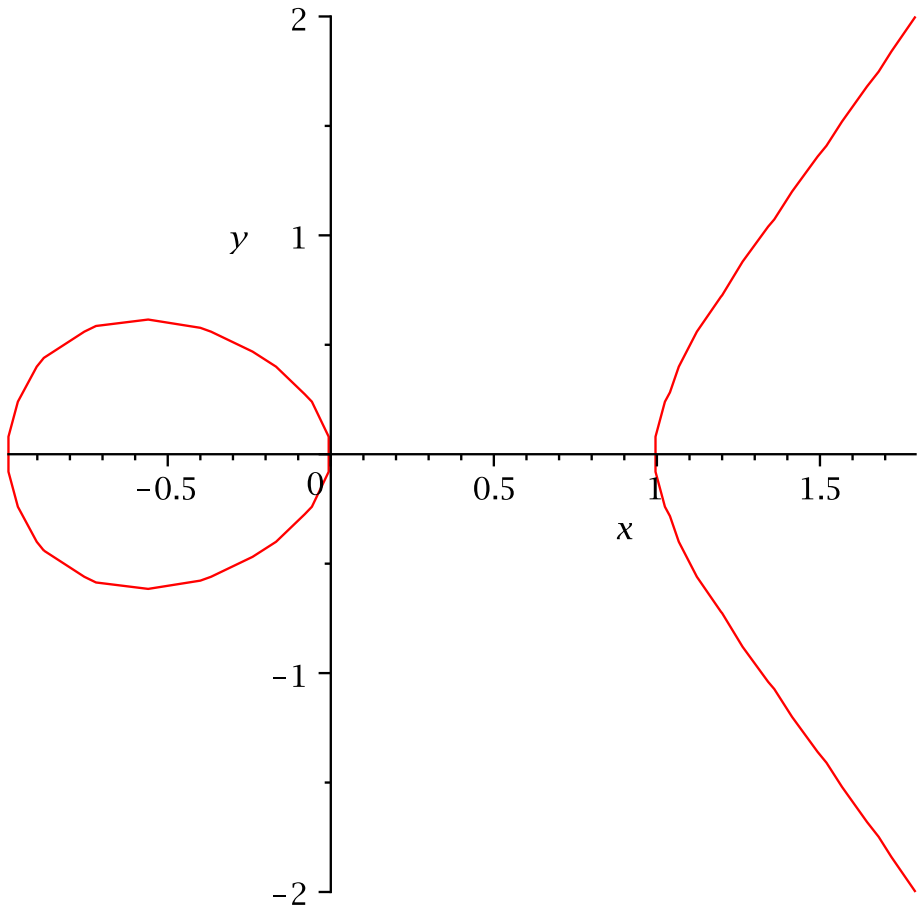
Discriminants in Maple

Discriminant of cubic given by $Y^2 = X^3 + pX + q$ in Maple:

```
> discrEllipt := proc(f)
    local p, q, D;
    p := coeff( f(x),x,1);
    q := coeff( f(x),x,0);
    D := -2^4*(4*p^3+27*q^2);
    RETURN( D )
  end:
> f := x -> x^3+x: discrEllipt(f);
-64
> f := x -> x^3-2*x+1: discrEllipt(f);
80
```

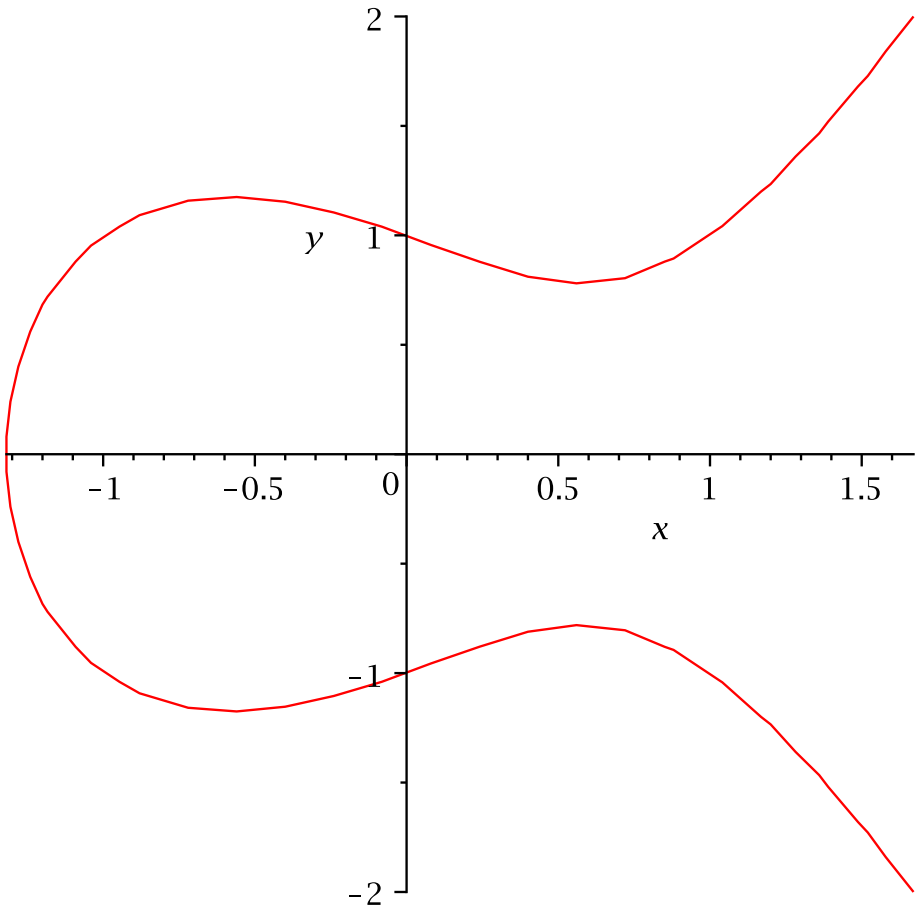
Example

Elliptic curve $Y^2 = X^3 - X$ with discriminant $\Delta = 64$.



Example

Elliptic curve $Y^2 = X^3 - X - 1$ with discriminant $\Delta = -368$.



Negative Discriminants

Some elliptic curves over $\mathbb{Q}$ with small negative discriminant:

Elliptic curve	Δ	c_4	c_6
$Y^2+Y=X^3-X^2$	-11	16	-152
$Y^2+XY=X^3-2X^2+X$	-15	1	-161
$Y^2+Y=X^3+X^2+X$	-19	-32	8
$Y^2+XY+Y=X^3$	-26	-23	-181
$Y^2+Y=X^3$	-27	0	-216
$Y^2+XY-Y=X^3$	-28	25	-253
$Y^2+Y=X^3+X^2-X$	-35	64	-568
$Y^2+XY=X^3+X^2+X$	-39	-2	235
$Y^2+Y=X^3+X^2$	-43	16	-280
$Y^2=X^3-X^2+X$	-48	-32	-224

Positive Discriminants

Some elliptic curves over $\mathbb{Q}$ with small positive discriminant:

Elliptic curve	Δ	c_4	c_6
$Y^2 + 7XY + 2Y = X^3 + 4X^2 + X$	15	3841	-238049
$Y^2 + 3XY = X^3 + X$	17	33	-81
$Y^2 + Y = X^3 - X$	37	48	-216
$Y^2 + 2XY - 3Y = X^3 - 1$	37	160	-2008
$Y^2 + XY = X^3 - X^2 + X$	57	73	539
$Y^2 + 9XY - 9Y = X^3 + 9X^2 - 5X - 3$	62	15873	-1999809
$Y^2 = X^3 - X$	64	48	0
$Y^2 + XY = X^3 - X$	65	49	-73
$Y^2 + XY = X^3 - X^2 - X$	73	49	243
$Y^2 + 3XY - Y = X^3 - X^2$	79	97	-881

Example

Compute the integral solutions of the equation

$$\binom{n}{2} = \binom{m}{3},$$

i.e.,

$$\frac{n(n-1)}{2} = \frac{m(m-1)(m-2)}{6}.$$

Clearing denominators and multiplying with $3^3 = 27$ gives

$$81n^2 - 81n = 27m^3 - 81m^2 + 54m.$$

Setting $n = (Y-5)/9$ and $m = (X-3)/3$ gives the Weierstrass equation

$$Y^2 + Y = X^3 - 9X + 20,$$

which defines an elliptic curve over $\mathbb{Q}$ with discriminant $\Delta = -130491$.

Siegel's Theorem (1929)

A smooth algebraic curve $\mathcal{C}$ of genus $g > 0$ over a number field $\mathbb{K}$ has only finitely many points on $\mathcal{C}$ with coordinates in the ring of integers $\mathcal{O}$ of $\mathbb{K}$. Example: $\mathbb{K} = \mathbb{Q}$ and $\mathcal{O} = \mathbb{Z}$.

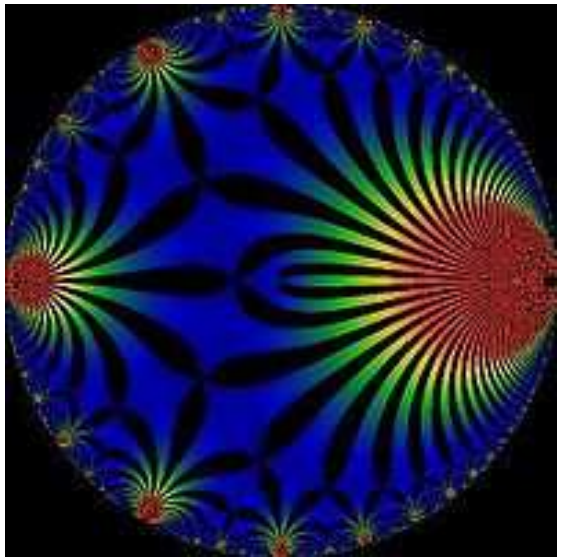
Example (cont'd)

The integral points of the elliptic curve are

$$(-3, 4), (-2, 5), (0, 4), (1, 3), (3, 4), (6, 13), (10, 30),$$
$$(12, 40), (27, 139), (63, 499), (105, 1075).$$

Only one of each pair $\pm P$ is listed. For more information see `www.Imfdb.org/EllipticCurve/Q/1611/a/1`.
Backtransformation $Y = 9n - 5$ and $X = 3m - 3$ shows that the initial equation has only finitely many solutions.

The j-Invariant



Definition of j-Invariant

For an elliptic curve $\mathcal{E}$ with discriminant Δ, the quantity

$$j = \frac{c_4^3}{\Delta} \tag{137}$$

is the *j-invariant* of $\mathcal{E}$; well-defined since $\Delta \neq 0$.

j-Invariant

Let $\mathrm{char}(\mathbb{K}) \neq 2, 3$. Consider the elliptic curve $\mathcal{E}(f)$ over $\mathbb{K}$ given by

$$f = Y^2Z - X^3 - pXZ^2 - qZ^3.$$

Then

$$a_1 = a_2 = a_3 = 0,\ a_4 = p,\ a_6 = q$$

and so

$$b_2 = 0,\ b_4 = 2p,\ b_6 = 4q.$$

Thus

$$\Delta = -8b_4^3 - 27b_6^2 = -2^4(4p^3 + 27q^2)$$

and so

$$j = \frac{c_4^3}{\Delta} = 2^6 3^3 \frac{4p^3}{4p^3 + 27q^2}, \tag{138}$$

where $c_4 = b_2^2 - 24b_4 = -48p = -2^4 3p$.

Examples

- Each elliptic curve $\mathcal{E}(f)$ over $\mathbb{K}$ given by

$$f = Y^2Z - X^3 - pXZ^2 \tag{139}$$

with $q = 0$ has j-invariant $j = 2^6 3^3 = 1728$.

- Each elliptic curve $\mathcal{E}(f)$ over $\mathbb{K}$ given by

$$f = Y^2Z - X^3 - qZ^3 \tag{140}$$

with $p = 0$ has j-invariant $j = 0$.

j-Invariants in Maple

The j-invariant of cubic given by $Y^2 = X^3 + pX + q$ in Maple:

```
> discrEllipt := proc(f)
    local p, q, c4, D;
    p := coeff( f(x),x,1);
    q := coeff( f(x),x,0);
    c4 := -48*p;
    D := -2^4*(4*p^3+27*q^2);
    if (D = 0) then RETURN(D)
      else RETURN(D, c4^3/D)
    end if
  end:
> f := x -> x^3+x: discrEllipt(f);
-64, 1728
> f := x -> x^3-2*x+1: discrEllipt(f);
80, 55296/5
```

Structure of j-Invariants

Let $\mathrm{char}(\mathbb{K}) \neq 2, 3$. For each $j_0 \in \mathbb{K}$, there is an elliptic curve over $\mathbb{K}$ with j-invariant j_0.

Proof.

- The cases $j_0 = 0$ and $j_0 = 1728$ have appeared in (139) and (140).
- For the other values, consider the Weierstrass form

$$f = Y^2Z - X^3 + \frac{27}{4}\frac{j_0}{j_0 - 1728}XZ^2 + \frac{27}{4}\frac{j_0}{j_0 - 1728}Z^3,$$

where

$$p = q = -\frac{27}{4}\frac{j_0}{j_0 - 1728}.$$

Then by (138),

$$j = 1728\frac{4p^3}{4p^3 + 27q^2} \stackrel{!}{=} j_0.$$

Admissible Change of Variables

An *admissible change of variables* in a Weierstrass form

$$f = Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3$$

over $\mathbb{K}$ is the projective transformation

$$f^A(X, Y, Z) = f((X, Y, Z)A^{-1}) \tag{141}$$

where

$$A = \begin{pmatrix} \frac{1}{u^2} & -\frac{s}{u^3} & 0 \\ 0 & \frac{1}{u^3} & 0 \\ -\frac{r}{u^2} & \frac{rs-t}{u^3} & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} u^2 & su^2 & 0 \\ 0 & u^3 & 0 \\ r & t & 1 \end{pmatrix} \tag{142}$$

with $u, r, s, t \in \mathbb{K}$ and $u \neq 0$.

Admissible Change of Variables in Maple

```
> f := Y^2-X^3-pX-q:
> subs( {X=u^2*x+r, Y=u^3*y+s*u^3*x+t}, f):
> g := collect(simplify(%), [x,y]);
```

Resulting polynomial:

$$\begin{aligned} g \quad = \quad & u^6y^2 + 2u^5sxy + 2u^3ty \\ & -u^6x^3 + (s^2u^4 - 3u^4r)x^2 + (-pu^2 + 2su^2t - 3u^2r^2)x \\ & +(t^2 - pr - q - r^3). \end{aligned}$$

Under normalization (multiplication with u^{-6}), the coefficients of y^2 and x^3 become 1 and -1, resp.

Basic Facts I

- Under an admissible change of variables and the normalization (multiplication with u^{-6}), a cubic $\mathcal{E}(f)$ in Weierstrass form is mapped to cubic $\mathcal{E}(g)$ in Weierstrass form.
- An admissible change of variables fixes $O = (0 : 1 : 0)$, since

$$A \cdot (0 : 1 : 0) = (0 : 1 : 0)A = (0 : \frac{1}{u^3} : 0)$$

  and $u^3 \neq 0$.
- Two elliptic curves are called *isomorphic* if they are related by an admissible change of variables.

Basic Facts II

- The assignment $f \mapsto f^A$ maps the coefficients of f,

$$a_1, a_2, \ldots, b_2, \ldots, c_4, c_6,$$

to new coefficients of f^A,

$$a'_1, a'_2, \ldots, b'_2, \ldots, c'_4, c'_6.$$

Each primed coefficient is u^{-i} times an expression in $r, s, t, a_1, \ldots, c_6$ not involving u. The coefficient is then said to have *weight* i.

In particular, $c'_4 = u^{-4}c_4$ and $c'_6 = u^{-6}c_6$ with r, s, t not involved.

Structure of j-Invariants

Let $\mathrm{char}(\mathbb{K}) \neq 2, 3$. If two elliptic curves are related by an admissible change of variables, they have the same j-invariant.

Proof.

We have $c'_4 = u^{-4}c_4$ and $c'_6 = u^{-6}c_6$ with r, s, t not involved. By (133),

$$\Delta(f) = 2^6 3^9 (c_4^3 - c_6^2)$$

and so

$$\Delta(f^A) = 2^6 3^9 (c'^3_4 - c'^2_6) = 2^6 3^9 u^{-12} (c_4^3 - c_6^2) = u^{-12}\Delta(f).$$

Thus

$$j = \frac{c_4^3}{\Delta(f)} = \frac{u^{-12}c_4^3}{u^{-12}\Delta(f)} = \frac{c'^3_4}{\Delta(f^A)}.$$

Admissible Change of Variables in Maple

```
> f := Y^2-X^3-pX-q:
> subs( {X=u^2*x, Y=u^3*y}, f):
> g := collect(simplify(%), [x,y]);
```

Resulting polynomial:

$$g = u^6y^2 - u^6x^3 - pu^2x - q.$$

Under normalization (multiplication with u^{-6}), we obtain

$$f^A = y^2 - x^3 - pu^{-4}x - qu^{-6}. \tag{143}$$

Structure of j-Invariants

Let $\mathrm{char}(\mathbb{K}) \neq 2, 3$. If two elliptic curves over $\overline{\mathbb{K}}$ have the same j-invariant, they are related by an admissible change of variables.

Proof.

Take two elliptic curves defined by $f = Y^2 - X^3 - pX - q$ and $g = Y^2 - X^3 - p'X - q'$ which have equal j-invariants

$$\frac{4p^3}{4p^3 + 27q^2} = \frac{4p'^3}{4p'^3 + 27q'^2}. \tag{144}$$

Find $u \neq 0$ such that $p' = pu^{-4}$ and $q' = qu^{-6}$. Then (144) holds and by (143), the curves are related by an admissible change of variables.

Proof (cont'd).

Find $u \neq 0$ such that

$$p' = pu^{-4} \quad \text{and} \quad q' = qu^{-6}. \tag{145}$$

Three cases:

- Let $p' = 0$. Then $q' \neq 0$ by non-singularity, and so $p = 0$. Put $u = (q/q')^{1/6}$. Then (145) holds.
- Let $q' = 0$. Then $p' \neq 0$ by non-singularity, and so $q = 0$. Put $u = (p/p')^{1/4}$. Then (145) holds.
- Let $p'q' \neq 0$. Put $u = (p/p')^{1/4}$. Then $p' = pu^{-4}$ and so $q^2 = (p/p')^3 q'^2 = u^{12} q'^2$. Thus $q = u^6 q'$ or $q = -u^6 q'$.
  - If $q = u^6 q'$, then (145) holds.
  - If $q = -u^6 q'$, replace u by iu, where $i = \sqrt{-1}$, so that $q = u^6 q'$. Then (145) holds.

□

Example

- The elliptic curves over $\mathbb{R}$ given by the cubics

$$Y^2 = X^3 - 2X + 1 \quad \text{and} \quad Y^2 = X^3 - 2X - 1$$

have the same j-invariant $j = \frac{2^{11}3^3}{59}$.

- By (143), the admissible change of variables

$$X \mapsto u^2 X \quad \text{and} \quad Y \mapsto u^3 Y$$

leads from

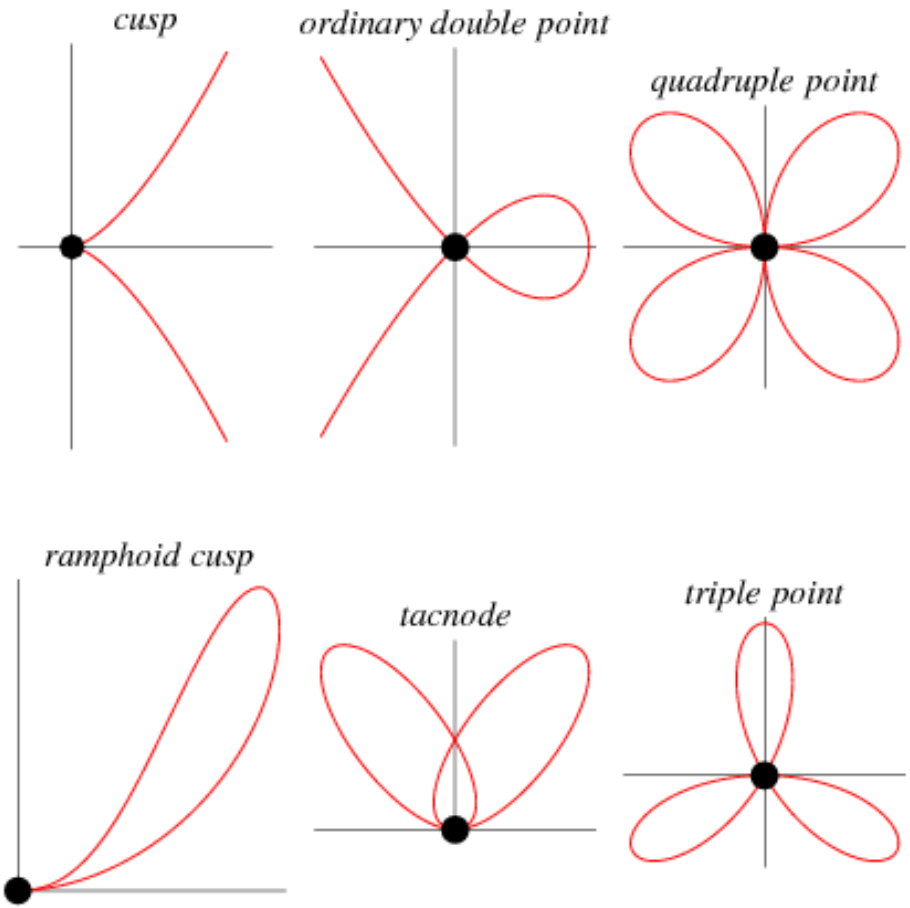
$$Y^2 - X^3 - pX - q \quad \text{to} \quad Y^2 - X^3 - pu^{-4}X - qu^{-6}.$$

Choosing $u = i$ (primitive 4th root of unity) shows that

$$Y^2 = X^3 - 2X + 1 \quad \text{and} \quad Y^2 = X^3 - 2X - 1$$

are related by admissible change of variables.

Singular Points



Singular Points - Projective Transformation I

Let $\mathcal{E}(f)$ be a singular Weierstrass curve over $\mathbb{K}$ with

$$f = Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3.$$

- The base point $O = (0 : 1 : 0)$ is non-singular.
- Given singular point $P = (a : b : 1)$ on $\mathcal{E}(f)$.
- Translate P to $(0 : 0 : 1)$ by the projective transformation A with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix},$$

i.e., $A \cdot P = PA = (0 : 0 : 1)$ and

$$f^A(X, Y, Z) = f((X, Y, Z)A^{-1}) = f(X + aZ, Y + bZ, Z).$$

Projective transformations preserve singularity by (79).

Singular Points - Projective Transformation II

Let

$$f^A(X,Y,Z) = f(X + aZ, Y + bZ, Z).$$

Then

- $f^A(0,0,1) = [a_6Z^3 + \ldots](0,0,1) = 0$ gives $a_6 = 0$.
- $\frac{\partial f^A}{\partial X}(0,0,1) = [a_4Z^2 + \ldots](0,0,1) = 0$ gives $a_4 = 0$.
- $\frac{\partial f^A}{\partial Y}(0,0,1) = [a_3Z^2 + \ldots](0,0,1) = 0$ gives $a_3 = 0$.

Thus the (affine) curve has the form

$$(f^A)^a = Y^2 + a_1XY - X^3 - a_2X^2.$$

Cusps and Nodes

Consider the (affine) curve given by

$$Y^2 + a_1XY = X^3 + a_2X^2. \tag{146}$$

- Factorization of $Y^2 + a_1XY - a_2X^2$ over $\overline{\mathbb{K}}$ gives

$$(Y - \alpha X)(Y - \beta X) = X^3, \quad \alpha, \beta \in \overline{\mathbb{K}}. \tag{147}$$

- Singular point $(0 : 0 : 1)$ is a *cusp* if $\alpha = \beta$, and a *node* (*double point*) if $\alpha \neq \beta$.

Unique Singularity

Each singular curve $\mathcal{E}(f)$ over $\mathbb{K}$ with Weierstrass polynomial f has a unique singular point $P = (a : b : 1) \in \mathbb{A}^2(\mathbb{K})$.

Point P is a cusp if $c_4 = 0$ and a node if $c_4 \neq 0$.

Proof.

Let $\mathrm{char}(\mathbb{K}) \neq 2$. By (107), each (affine) curve over $\mathbb{K}$ is given by

$$Y^2 = X^3 - aX^2 + bX - c, \quad a, b, c \in \mathbb{K}.$$

This curve is singular iff $g = X^3 - aX^2 + bX - c$ has repeated roots (see (127)). In the latter case, g and g' have gcd h over $\mathbb{K}$ with degree ≥ 1.

- The singular points are $(x_0 : 0 : 1)$ where x_0 ranges over the roots of h; see proof of (126,127).
- If h has degree 1, its unique root is $x_0 \in \mathbb{K}$ and $(x_0 : 0 : 1)$ is the unique singular point.
- If h has degree 2, its roots x_0, x_0' are equal since otherwise x_0, x_0' would each have multiplicity ≥ 2 (with total multiplicity ≥ 4) of a cubic (degree 3).

It follows that $(x_0 : 0 : 1)$ is the *unique* singular point.

Proof (cont'd).

Let $\mathrm{char}(\mathbb{K}) \neq 2$.

- Under the projective transformation A that moves $(a : b : 1)$ to $(0 : 0 : 1)$, we obtain (with $a_3 = a_4 = a_6 = 0$)

$$c_4 = (a_1^2 + 4a_2)^2.$$

- By comparing coefficients between (146) and (147),

$$a_1 = -(\alpha + \beta) \quad \text{and} \quad a_2 = -\alpha\beta \tag{148}$$

and

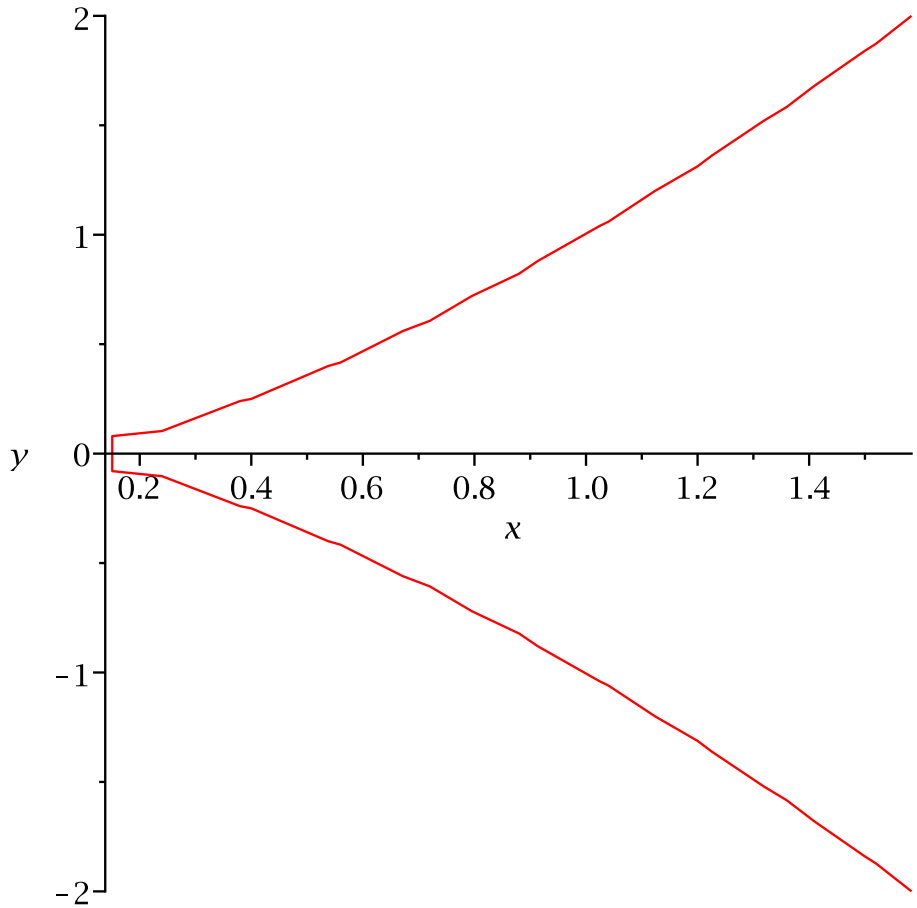
$$\beta = \gamma \quad \text{and} \quad \alpha = -a_1 - \gamma, \tag{149}$$

where $\gamma = \mathrm{RootOf}(Z^2 + a_1 Z - a_2)$.

- Thus $c_4 = 0$ iff $0 = a_1^2 + 4a_2 = (\alpha - \beta)^2$ iff $\alpha = \beta$ (cusp case). □

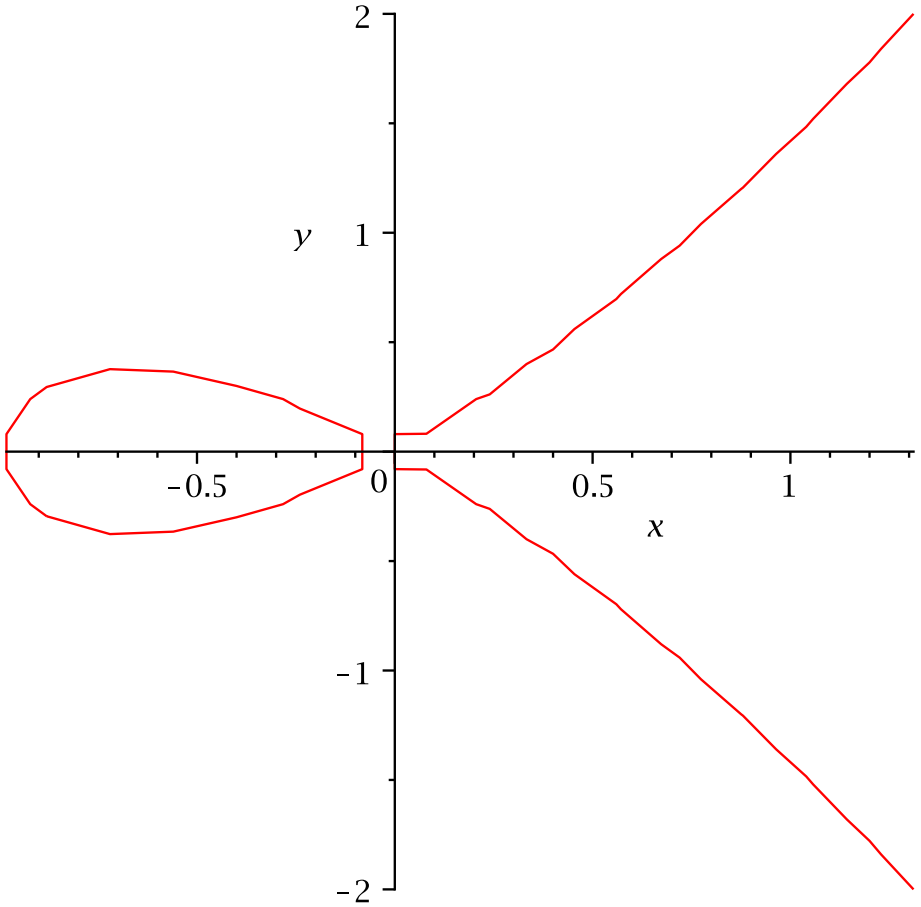
Example

Plane projective curve $Y^2 = X^3$ ($a_1 = a_2 = 0$ and so $c_4 = 0$) with discriminant $\Delta = 0$ and a cusp ($\alpha = \beta = 0$).

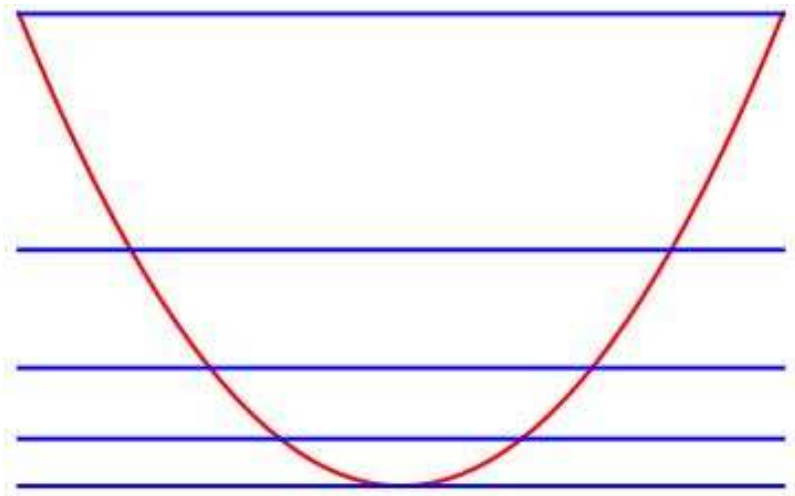


Example

Plane projective curve $Y^2 = X^3 + X^2$ ($a_1 = 0, a_2 = 1$ and so $c_4 = 16$) with discriminant $\Delta = 0$ and a node $(\alpha = 1, \beta = -1)$.



Intersection Multiplicities



Structure of Intersections

Let $\mathcal{E} = \mathcal{E}(f)$ be an elliptic curve over $\mathbb{K}$ and $\mathcal{L} = \mathcal{L}(\alpha, \beta, \gamma)$ be a projective line. Then

$$\sum_{P \in \mathbb{P}^2(\mathbb{K})} i(P, \mathcal{L}, \mathcal{E}) \in \{0, 1, 3\}; \tag{150}$$

i.e., a projective line and an elliptic curve intersect in 0, 1, or 3 points with multiplicities counted.

Proof.

Let $\mathcal{E} = \mathcal{E}(f)$ be an elliptic curve given by the Weierstrass polynomial f.

- The points in $\mathcal{E}$ are (affine) points of the form $(a : b : 1)$ and the base point $O = (0 : 1 : 0)$.
- $i(P, \mathcal{L}, \mathcal{E}) = 0$ for all points P in the projective plane not in $\mathcal{L} \cap \mathcal{E}$.
- For each point $P \in \mathcal{L} \cap \mathcal{E}$ the intersection multiplicity satisfies $i(P, \mathcal{L}, \mathcal{E}) \leq 3$, since the polynomial

$$\phi(t) = f(a + ta', b + tb', c + tc') \in \mathbb{K}[t]$$

has degree at most 3, see (83).

Three cases occur for $\mathcal{L} = \mathcal{L}(\alpha, \beta, \gamma)$, i.e., $\ell = \alpha X + \beta Y + \gamma Z$.

Proof (cont'd).

1 Let $\alpha = \beta = 0$. Then $\mathcal{L} = \mathcal{L}(0, 0, 1)$, i.e., $\ell = Z$, and the only point in $\mathcal{L} \cap \mathcal{E}$ is $O = (0 : 1 : 0)$. Take the auxiliary point $P' = (1 : 0 : 0) \in \mathcal{L}$. Then

$$\phi(t) = f(0 + 1 \cdot t, 1 + 0 \cdot t, 0 + 0 \cdot t) = f(t, 1, 0) = -t^3$$

and so

$$i(O, \mathcal{L}, \mathcal{E}) = 3,$$

i.e., O is a flex point.

Proof (cont'd).

2 Let $\alpha \neq 0$, $\beta = 0$: Then $\mathcal{L} = (\alpha, 0, \gamma)$ for some $\gamma \in \mathbb{K}$, i.e., $\ell = \alpha X + \gamma Z$.

First, the only point at infinity that lies in $\mathcal{L} \cap \mathcal{E}$ is $O = (0 : 1 : 0)$. Take the auxiliary point $P' = (-\gamma : 0 : \alpha) \in \mathcal{L}$. Then

$$\begin{aligned} \phi(t) &= f(0 - \gamma \cdot t, 1 + 0 \cdot t, 0 + \alpha \cdot t) \\ &= f(-\gamma t, 1, \alpha t) \\ &= \alpha t + \text{terms of higher order in } t \end{aligned}$$

and so $i(O, \mathcal{L}, \mathcal{E}) = 1$.

Proof (cont'd).

2 Second, an affine point $P = (x : y : 1)$ lies in $\mathcal{L} \cap \mathcal{E}$ iff $x = -\frac{\gamma}{\alpha}$ and y is a root of the polynomial

$$h(Y) = f(-\frac{\gamma}{\alpha}, Y, 1) \in \mathbb{K}[Y].$$

Since h has degree 2 and coefficient 1 in Y^2, there are $r_1, r_2 \in \overline{\mathbb{K}}$ with

$$h(Y) = (Y - r_1)(Y - r_2).$$

If none of r_1, r_2 lies in $\mathbb{K}$, then

$$\sum_P i(P, \mathcal{L}, \mathcal{E}) = i(O, \mathcal{L}, \mathcal{E}) = 1.$$

Otherwise, both r_1, r_2 lie in $\mathbb{K}$ and so with $y_{1,2} = r_1, r_2$,

$$\sum_P i(P, \mathcal{L}, \mathcal{E}) = 3.$$

Roots of Quadratic Polynomials

Let $f = X^2 - aX + b \in \mathbb{K}[X]$.

- If $r_1, r_2 \in \overline{\mathbb{K}}$ are the roots of f, then

$$f = (X - r_1)(X - r_2) = X^2 - (r_1 + r_2)X + r_1 r_2.$$

- By comparing coefficients,

$$a = r_1 + r_2 \in \mathbb{K}.$$

- Thus either both roots lie in $\mathbb{K}$ or none of them.

Proof (cont'd).

3 Let $\beta \neq 0$ with $\ell = \alpha X + \beta Y + \gamma Z$.
Then $O = (0 : 1 : 0) \notin \mathcal{L}$.
An affine point $P = (x : y : 1)$ lies in $\in \mathcal{L} \cap \mathcal{E}$ iff

$$y = -\frac{\gamma}{\beta} - \frac{\alpha}{\beta} x$$

and x is a root of the polynomial

$$h(X) = f(X, -\frac{\gamma}{\beta} - \frac{\alpha}{\beta} X, 1) \in \mathbb{K}[X].$$

Since h has degree 3 and coefficient -1 in X^3, there are $r_1, r_2, r_3 \in \overline{\mathbb{K}}$ with

$$h(X) = -(X - r_1)(X - r_2)(X - r_3) \in \mathbb{K}[X].$$

Proof (cont'd).

3 If none of $r_1, r_2, r_3 \in \mathbb{K}$, then

$$\sum_P i(P, \mathcal{L}, \mathcal{E}) = 0.$$

If exactly one of $r_1, r_2, r_3 \in \mathbb{K}$, then for some $x = r_i$,

$$\sum_P i(P, \mathcal{L}, \mathcal{E}) = 1.$$

If two of $r_1, r_2, r_3 \in \mathbb{K}$, then all three lie in $\mathbb{K}$ and so with $x_{1,2,3} = r_1, r_2, r_3$,

$$\sum_P i(P, \mathcal{L}, \mathcal{E}) = 3.$$

□

Roots of Cubic Polynomials

Let $f = X^3 - aX^2 + bX - c \in \mathbb{K}[X]$.

- If $r_1, r_2, r_3 \in \overline{\mathbb{K}}$ are the roots of f, then

$$\begin{aligned} f &= (X - r_1)(X - r_2)(X - r_3) \\ &= X^3 - (r_1 + r_2 + r_3)X^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)X \\ &\quad - r_1 r_2 r_3. \end{aligned}$$

- By comparing coefficients,

$$a = r_1 + r_2 + r_3 \in \mathbb{K}.$$

- Thus it cannot happen that two of the roots lie in $\mathbb{K}$ and the third one does not.

Example

Consider the elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ given by

$$f = Y^2 - X^3 - X$$

and the line $\mathcal{L} = \mathcal{L}(\alpha, \beta, \gamma)$, i.e., $\ell = \alpha X + \beta Y + \gamma Z$.

1 Let $\alpha = \beta = 0$: Then $\mathcal{L} = \mathcal{L}(0, 0, 1)$, i.e., $\ell = Z$.

- The only point in the intersection $\mathcal{L} \cap \mathcal{E}$ is the base point $O = (0 : 1 : 0)$.
- The intersection multiplicity of O with $\mathcal{L}$ and $\mathcal{C}$ is 3 (flex point).
- Thus

$$\sum_P i(P, \mathcal{L}, \mathcal{E}) = i(O, \mathcal{L}, \mathcal{E}) = 3.$$

Example (cont'd)

Consider the elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ with $\mathbb{Q} \subseteq \mathbb{K}$ given by

$$f = Y^2 - X^3 - X$$

and the line $\mathcal{L} = \mathcal{L}(\alpha, \beta, \gamma)$, i.e., $\ell = \alpha X + \beta Y + \gamma Z$.

2 Let $\alpha \neq 0$ and $\beta = 0$. Take $\mathcal{L} = \mathcal{L}(1, 0, 1)$, i.e., $\ell = X + Z$.

- The point O lies in $\mathcal{L} \cap \mathcal{E}$ with $i(O, \mathcal{L}, \mathcal{E}) = 1$.
- The point $P = (x : y : 1)$ lies in $\mathcal{L} \cap \mathcal{E}$ iff $x = -1$ and y is a root of

$$h(Y) = Y^2 - x^3 - x = Y^2 + 2 = (Y - \sqrt{-2})(Y + \sqrt{-2}).$$

  Thus the roots of $h(Y)$ are $\pm i\sqrt{2}$.
- Hence,

$$\sum_P i(P, \mathcal{L}, \mathcal{E}) = \begin{cases} 3 & \text{if } i\sqrt{2} \in \mathbb{K}, \\ 1 & \text{otherwise.} \end{cases}$$

Example (cont'd)

Consider the elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ with $\mathbb{Q} \subseteq \mathbb{K}$ given by

$$f = Y^2 - X^3 - X$$

and the line $\mathcal{L} = \mathcal{L}(\alpha, \beta, \gamma)$, i.e., $\ell = \alpha X + \beta Y + \gamma Z$.

3 Let $\beta \neq 0$. Take $\mathcal{L} = \mathcal{L}(0, 1, 0)$, i.e., $\ell = Y$.

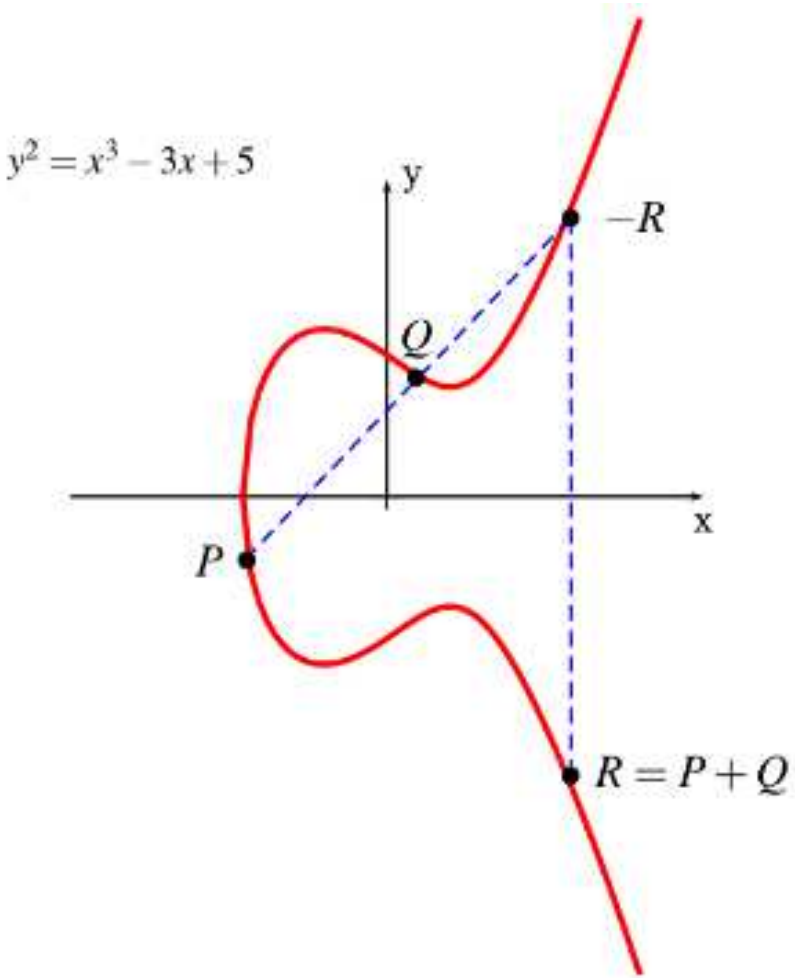
- The point O does not lie in $\mathcal{L} \cap \mathcal{E}$, so $i(O, \mathcal{L}, \mathcal{E}) = 0$.
- The point $P = (x : y : 1)$ lies in $\mathcal{L} \cap \mathcal{E}$ iff $y = 0$ and x is a root of

$$h(X) = -X^3 - X = -X(X^2 + 1) = -X(X - i)(X + i).$$

  Thus the roots of $h(X)$ are 0 and $\pm i$.
- Hence,

$$\sum_P i(P, \mathcal{L}, \mathcal{E}) = \begin{cases} 3 & \text{if } i \in \mathbb{K}, \\ 1 & \text{if } i \notin \mathbb{K}. \end{cases}$$

The Group Law



Secant-Tangent Composition Law

Let $\mathcal{E} = \mathcal{E}(f)$ be an elliptic curve over $\mathbb{K}$.

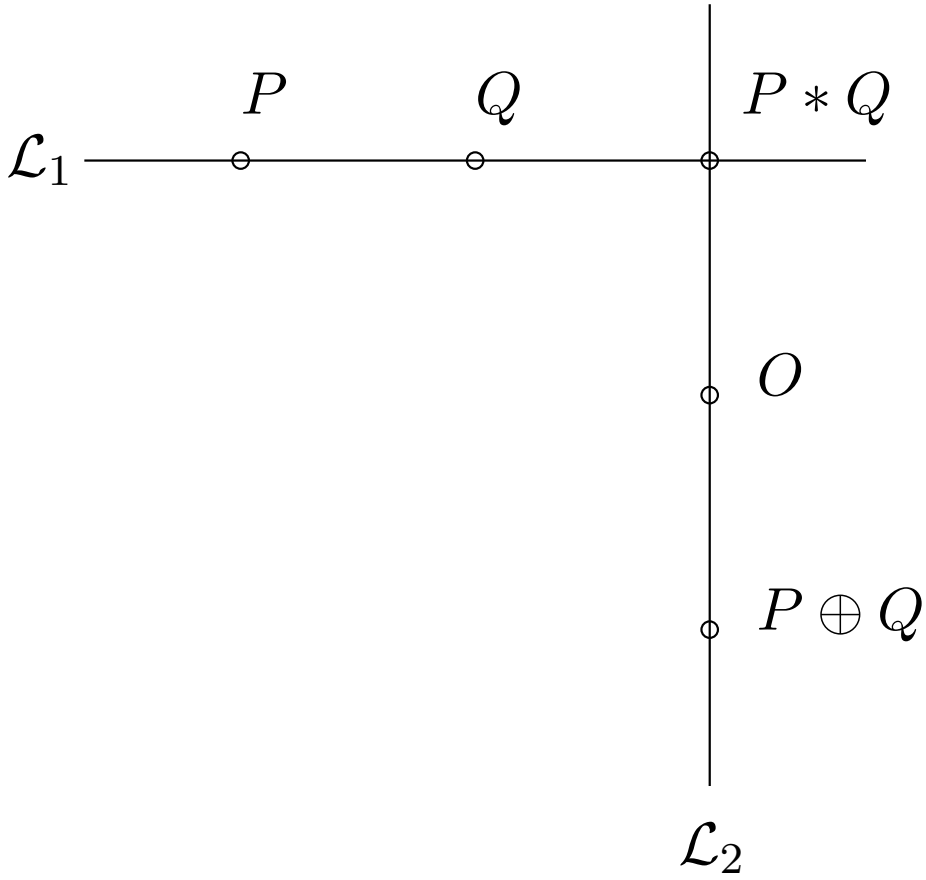
- *Secant case:* Let P and Q be distinct points in $\mathcal{E}$ and let $\mathcal{L}$ be the projective line connecting these points. Then by (150) there is a third point $R \in \mathcal{L} \cap \mathcal{E}$ if multiplicities are counted. The third point may coincide with P or Q.
- *Tangent case:* Let $\mathcal{L}$ be the tangent line to $\mathcal{E}$ at P. Then by (84), $i(P, \mathcal{L}, \mathcal{E}) \geq 2$ and so by (150) there is a third point $R \in \mathcal{L} \cap \mathcal{E}$ if multiplicities are counted. The third point will be $R = P$ if P is flex point of $\mathcal{E}$.

Secant Case

Let $\mathcal{E} = \mathcal{E}(f)$ be an elliptic curve over $\mathbb{K}$.
For two distinct $\mathbb{K}$-rational points P, Q on $\mathcal{E}$ define the $\mathbb{K}$-rational point $P \oplus Q$:

- Take the unique projective line $\mathcal{L}_1$ through the points P and Q. The line $\mathcal{L}_1$ intersects the curve $\mathcal{E}$ in another point $P * Q$.
- Take the unique projective line $\mathcal{L}_2$ through the point $P * Q$ and the base point O. If $P * Q = O$, take the tangent line at $\mathcal{E}$ in O. The line $\mathcal{L}_2$ intersects the curve $\mathcal{E}$ in another point $P \oplus Q$.

Secant Case

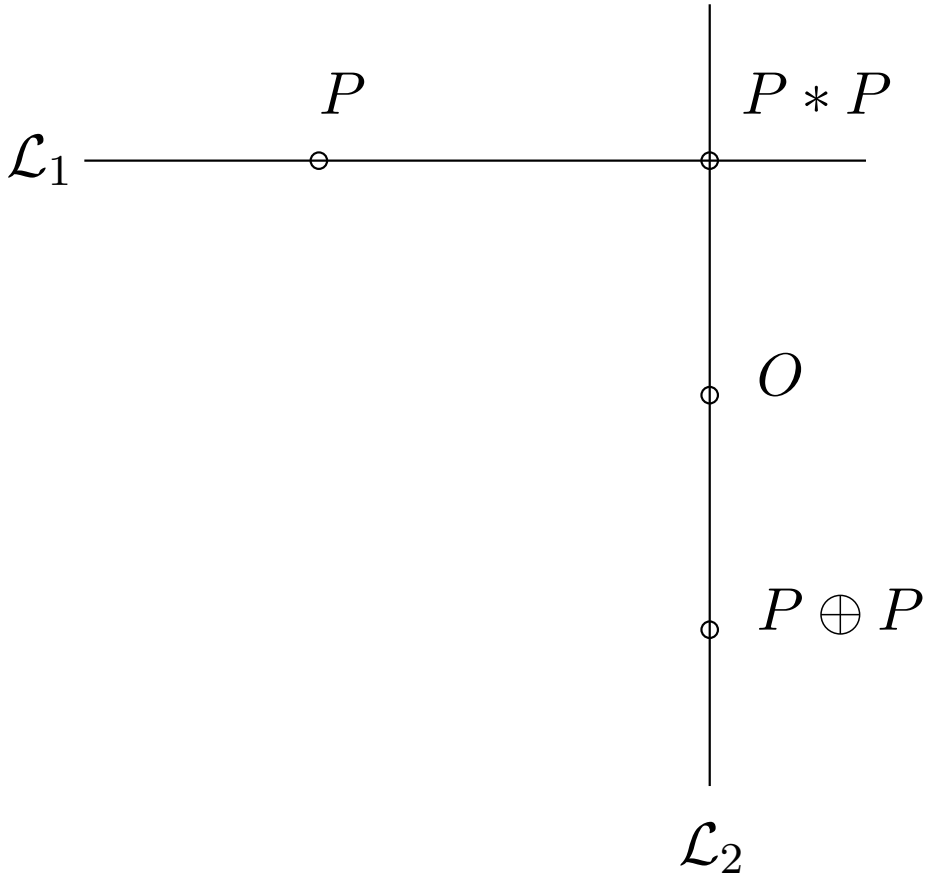


Tangent Case

Let $\mathcal{E} = \mathcal{E}(f)$ be an elliptic curve over $\mathbb{K}$.
For a $\mathbb{K}$-rational point P on $\mathcal{E}$ define the $\mathbb{K}$-rational point $P \oplus P$:

- Let $\mathcal{L}_1$ be the tangent line at $\mathcal{E}$ in the point P. The line $\mathcal{L}_1$ intersects the curve $\mathcal{E}$ in another point $P * P$.
- Take the unique projective line $\mathcal{L}_2$ through the point $P * P$ and the base point O. If $P * P = O$, take the tangent line at $\mathcal{E}$ in O. The line $\mathcal{L}_2$ intersects the curve $\mathcal{E}$ in another point $P \oplus P$.

Tangent Case



Unit Element

Let $\mathcal{E} = \mathcal{E}(f)$ be an elliptic curve over $\mathbb{K}$. Each $\mathbb{K}$-rational point P on $\mathcal{E}$ satisfies

$$P \oplus O = P. \tag{151}$$

Proof.

- Let $P = O$. The tangent line $\mathcal{L}_1$ at $\mathcal{E}$ in O is $\mathcal{L}_1 = \mathcal{L}(0,0,1)$, i.e., $\ell = Z$, since the Weierstrass polynomial f shows that

$$\frac{\partial f}{\partial X}(0,1,0) = 0, \quad \frac{\partial f}{\partial Y}(0,1,0) = 0, \quad \frac{\partial f}{\partial Y}(0,1,0) = 1.$$

  By the discussion of intersection multiplicities (case 1), $i(O, \mathcal{L}_1, \mathcal{E}) = 3$. Thus the third intersection point is O and hence $O * O = O$.

  Moreover, the line $\mathcal{L}_2$ is the tangent line at $\mathcal{E}$ in O and thus $\mathcal{L}_2 = \mathcal{L}(0,0,1)$, i.e., $\ell = Z$. Similarly, the third intersection point is O and hence $O \oplus O = O$.

Proof (cont'd).

- Let $P \neq O$. Take the line $\mathcal{L}_1$ through P and O which intersects the curve in a third point $P * O$.

  Moreover, the line $\mathcal{L}_2$ intersects O and $P * O$. But there is a unique projective line between any two points and so $\mathcal{L}_1 = \mathcal{L}_2$. Thus the third intersection point of $\mathcal{L}_2$ with $\mathcal{E}$ is P and hence $P \oplus O = P$.

□

Collinearity

Let $\mathcal{E} = \mathcal{E}(f)$ be an elliptic curve over $\mathbb{K}$. Let P, Q, and R be distinct $\mathbb{K}$-rational points of $\mathcal{E}$ which lie on a projective line $\mathcal{L}$ over $\mathbb{K}$. Then

$$(P \oplus Q) \oplus R = O. \tag{152}$$

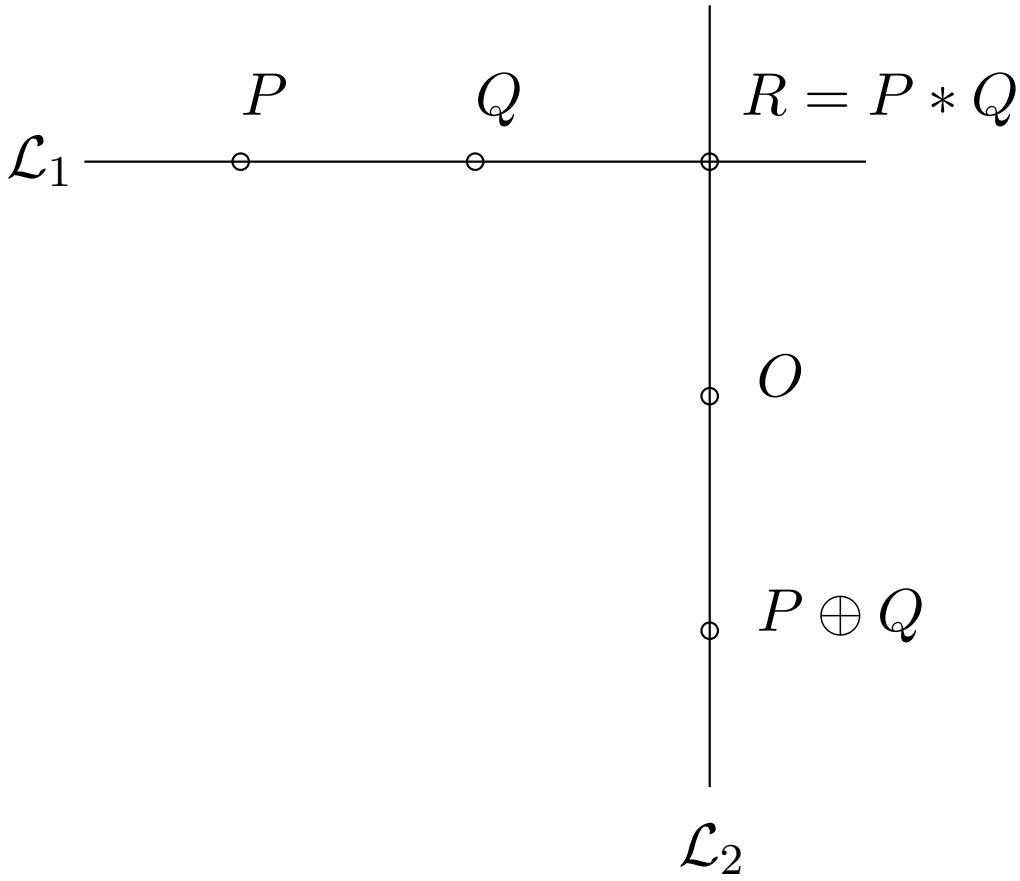
Proof.

- Add the points P and Q:

  Take the line $\mathcal{L}_1$ through the points P and Q. By uniqueness, $\mathcal{L}_1 = \mathcal{L}$ and by the intersection multiplicities, $R = P * Q$ is the third point on the line.

  Moreover, take the line $\mathcal{L}_2$ through R and O. The third point on the line $\mathcal{L}_2$ is $P \oplus Q$.

Proof (cont'd).



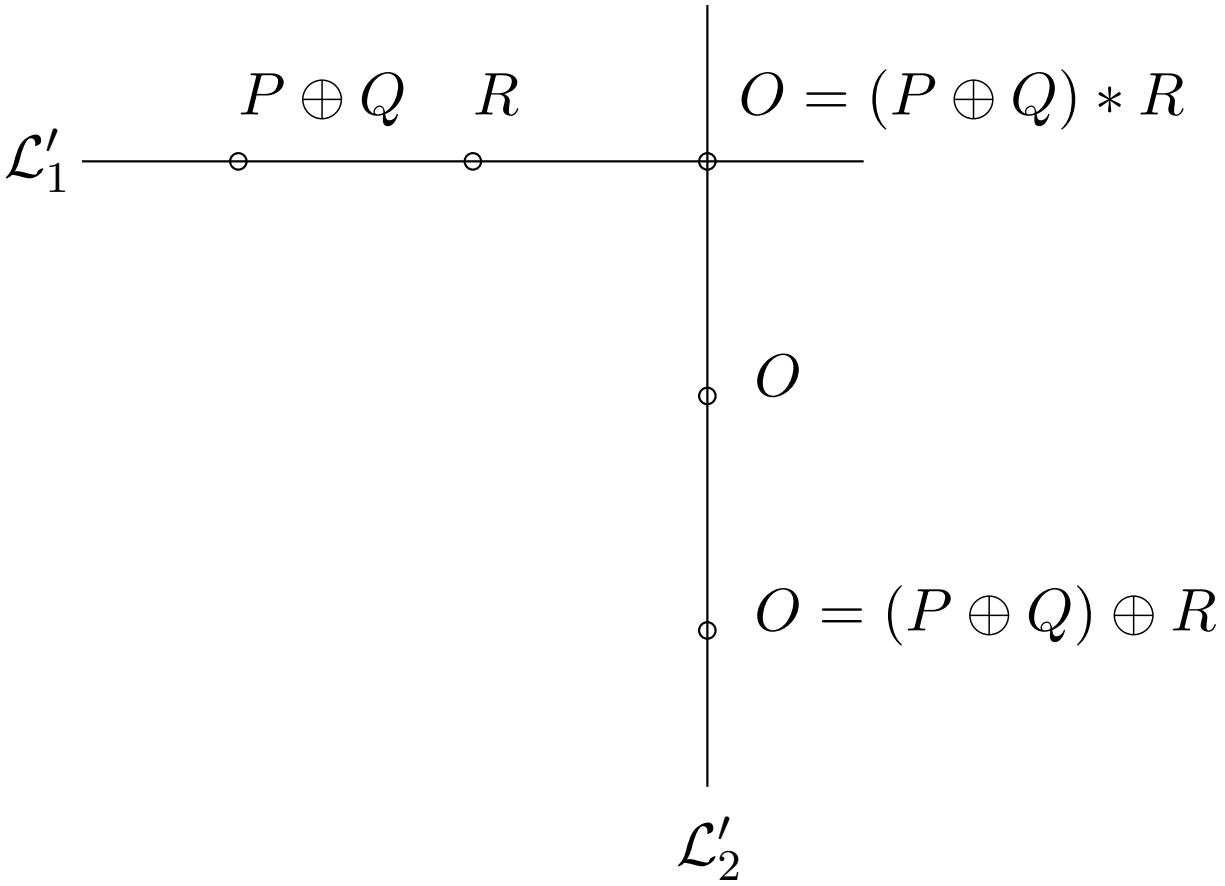
Proof (cont'd).

- Add the points $P \oplus Q$ and R:

  Take the line $\mathcal{L}_1'$ through the points $P \oplus Q$ and R. By uniqueness, $\mathcal{L}_1' = \mathcal{L}_2$ and the third intersection point is O.

  Moreover, the tangent line $\mathcal{L}_2'$ at $\mathcal{E}$ in flex point O has intersection multiplicity 3 and so the third intersection point is O giving $(P \oplus Q) \oplus R = O$.

Proof (cont'd).



Group Structure

An elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ forms together with the operation

$$\oplus : \mathcal{E} \times \mathcal{E} \to \mathcal{E} : (P, Q) \mapsto P \oplus Q \tag{153}$$

an abelian group with unit element O.

Proof.

- The operation is well-defined thanks to the secant-tangent composition law.
- The operation is commutative, since the construction of the point $P \oplus Q$ is independent of the order of the given points P and Q.
- The base point O is the unit element by (151).

Proof (cont'd).

- Each $\mathbb{K}$-rational point P on $\mathcal{E}$ has a $\mathbb{K}$-rational inverse point P' on $\mathcal{E}$, i.e.,

$$P \oplus P' = O. \tag{154}$$

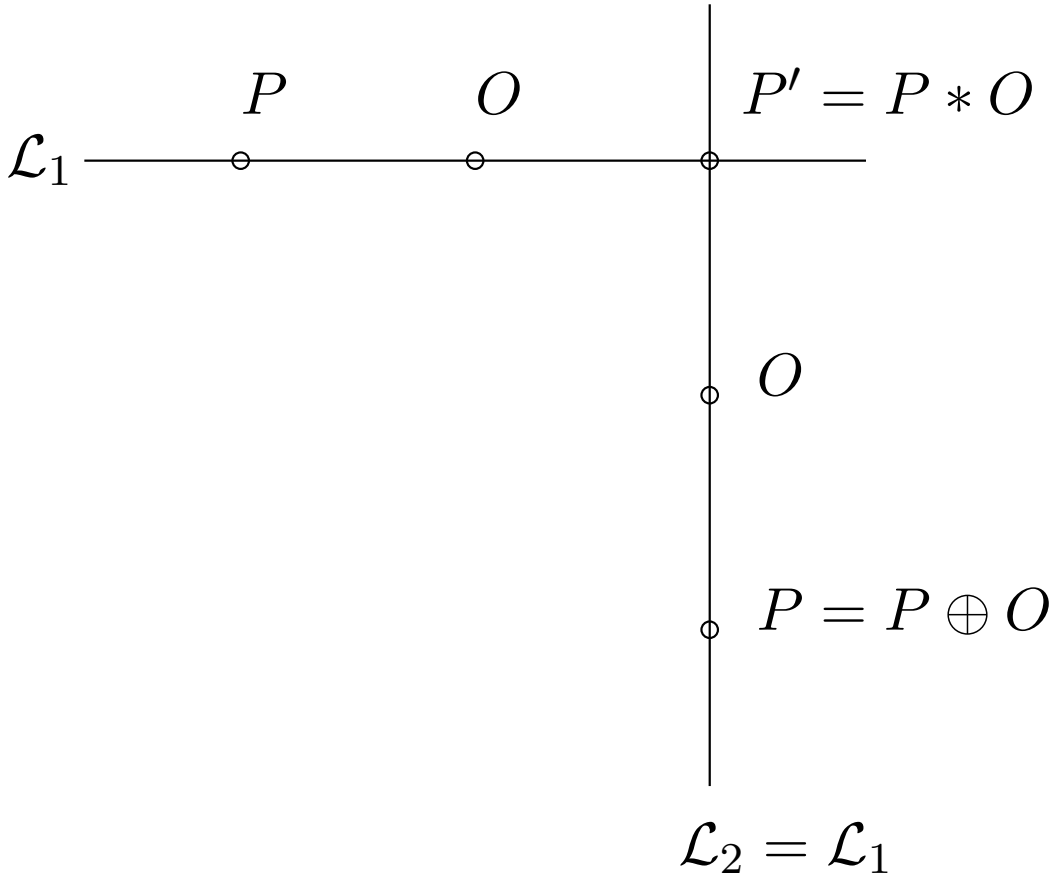
To see this, assume that $P \neq O$.

Take the line $\mathcal{L}_1$ through P and O. By the intersection multiplicities (case 2), O has multiplicity $i(O, \mathcal{L}, \mathcal{E}) = 1$. Thus there is a third intersection point $P' = P * O$ on the line. Then by collinearity (152) and (151),

$$O = (P \oplus O) \oplus P' = P \oplus P'.$$

- The operation is associative. This can be proved by using explicit coordinates or the famous Nine point lemma.

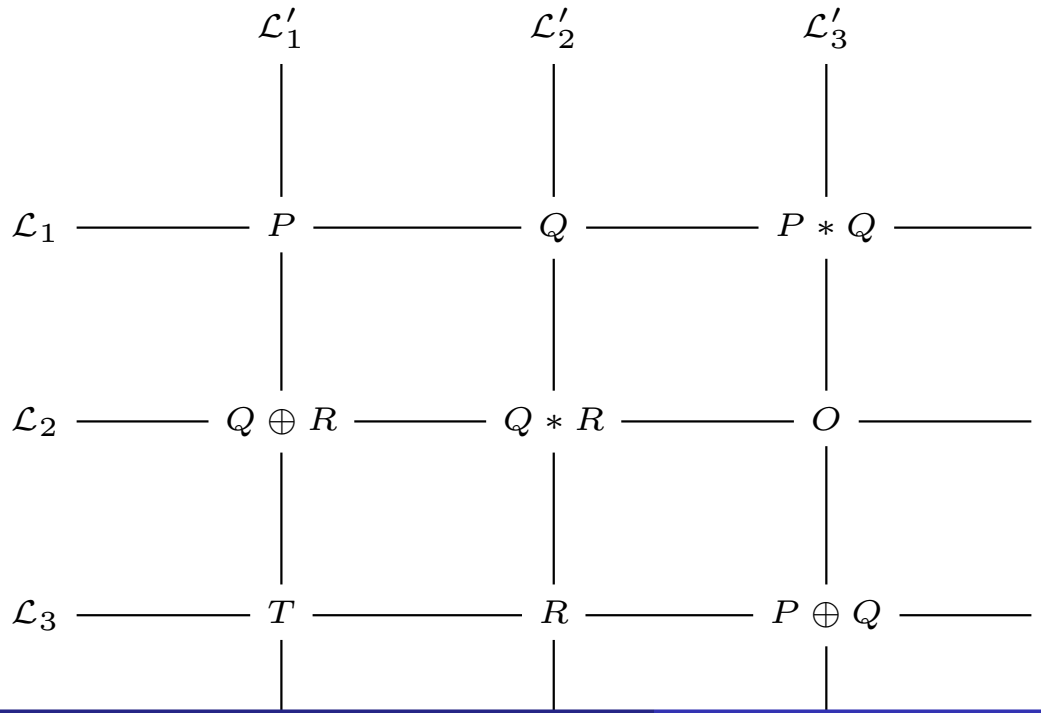
Proof (cont'd).



Proof (cont'd).

- *Nine point lemma:* Suppose the eight $\mathbb{K}$-rational points O, P, Q, R, $P * Q$, $Q * R$, $P \oplus Q$, and $Q \oplus R$ in $\mathbb{P}^2(\mathbb{K})$ are pairwise distinct and are different from the $\mathbb{K}$-rational points $P * (Q \oplus R)$ and $(P \oplus Q) * R$. Then there is a unique $\mathbb{K}$-rational point T (below) which equals

$$P * (Q \oplus R) = (P \oplus Q) * R.$$



Notation

- Define $P + Q$ as $P \oplus Q$.
- Denote the inverse of P as $-P$.
- For each integer l, the *l-fold* of a point $P \in \mathcal{E}$ is

$$\begin{aligned} 0P &= O, \\ lP &= (l-1)P + P \text{ for } l \geq 1, \\ lP &= (-l)(-P) \text{ for } l \leq -1. \end{aligned}$$

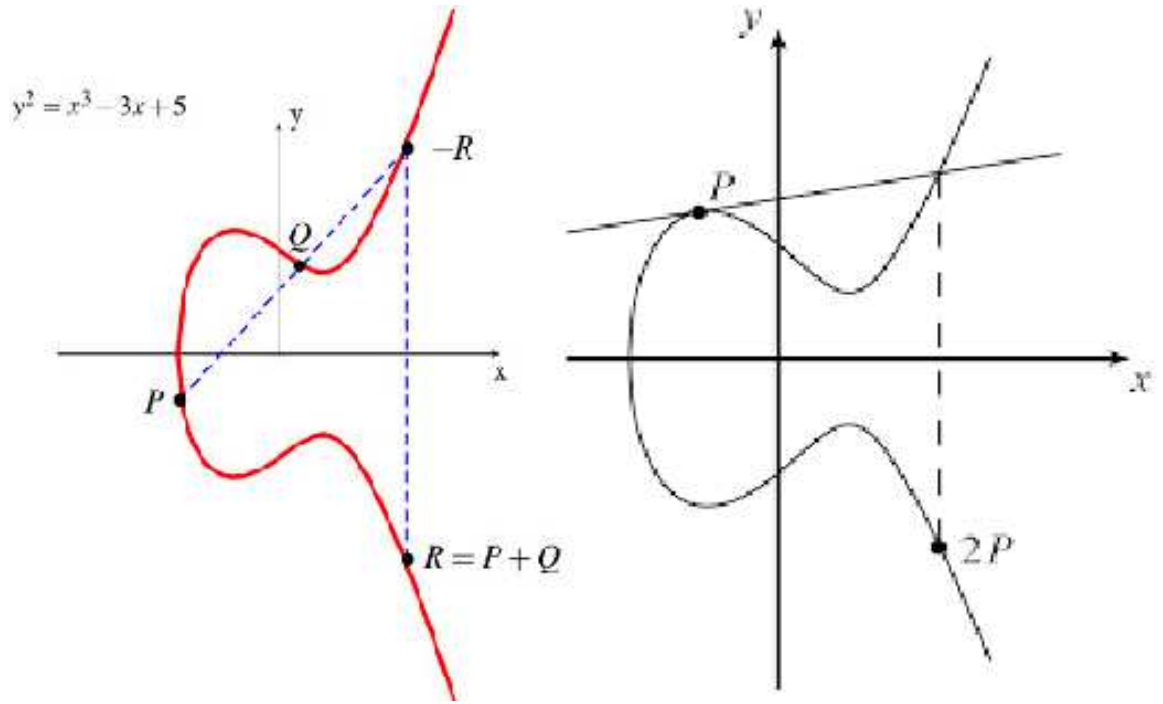
Thus for each integer $l \geq 1$,

$$lP = P + \ldots + P, \quad l\text{-times},$$

and

$$(-l)P = (-P) + \ldots + (-P) \quad l\text{-times}.$$

Group Law in Coordinates



Inverse Point

Let $P = (x : y : 1)$ be a $\mathbb{K}$-rational point on an elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ with Weierstrass polynomial f. The inverse of P is the $\mathbb{K}$-rational point

$$-P = (x : -y - a_1 x - a_3 : 1). \tag{155}$$

Proof.

The line through P and O is $\mathcal{L}(1, 0, -x)$, i.e., $\ell = X - xZ$. The third intersection point on the line is

$$P * O = (x : -y - a_1 x - a_3 : 1),$$

which is the inverse of P, see (154). □

Inverse Point – Special Case

Let $P = (x : y : 1)$ be a $\mathbb{K}$-rational point on an elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ with affine Weierstrass polynomial

$$f = Y^2 - X^3 - pX - q \in \mathbb{K}[X].$$

The inverse of P is the $\mathbb{K}$-rational point

$$-P = (x : -y : 1), \tag{156}$$

since $a_1 = a_3 = 0$; i.e., $-P$ is obtained from P by reflection at the x-axis!

Line Through Two Points

Let $P_1 = (x_1 : y_1 : 1)$ and $P_2 = (x_2 : y_2 : 1)$ be $\mathbb{K}$-rational points on an elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ with $P_2 \neq -P_1$; otherwise, $P_1 + P_2 = O$.

The line through P_1 and P_2 is given by

$$Y = mX + bZ = \begin{cases} \text{line through } P_1, P_2 & \text{if } P_1 \neq P_2 \text{ (secant case)}, \\ \text{tangent line at } P_1 & \text{if } P_1 = P_2 \text{ (tangent case)}, \end{cases}$$

where

$$m = \begin{cases} \frac{y_2 - y_1}{x_2 - x_1} & \text{in secant case}, \\ \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3} & \text{in tangent case}, \end{cases} \tag{157}$$

and

$$b = \begin{cases} \frac{y_1x_2 - y_2x_1}{x_2 - x_1} & \text{in secant case}, \\ \frac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3} & \text{in tangent case}. \end{cases} \tag{158}$$

Proof.

Secant case: The system of equations

$$y_1 = mx_1 + b \quad \text{and} \quad y_2 = mx_2 + b$$

yields

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and

$$b = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}.$$

Proof (cont'd).

Tangent case: Take the affine Weierstrass equation

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$

By implicit differentiation,[a]

$$2YY' + a_1XY' + a_1Y + a_3Y' = 3X^2 + 2a_2X + a_4.$$

Putting $Y' = m$ and $(X, Y) = (x_1, y_1)$ gives

$$2y_1m + a_1x_1m + a_1y_1 + a_3m = 3x_1^2 + 2a_2x_1 + a_4.$$

Thus

$$m = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3}.$$

[a] For a real-valued implicit function $f(X, Y) = 0$, the derivative is given by the chain rule $\frac{\partial f}{\partial X} + \frac{\partial f}{\partial Y}\frac{dY}{dX} = 0$.

Proof (cont'd).

Tangent case: We have

$$m = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3}.$$

Since $b = y_1 - mx_1$, we obtain

$$b = \frac{2y_1^2 + a_1x_1y_1 + a_3y_1 - 3x_1^3 - 2a_2x_1^2 - a_4x_1 + a_1x_1y_1}{2y_1 + a_1x_1 + a_3}.$$

Substituting $y_1^2 = -a_1x_1y_1 - a_3y_1 + x_1^3 + a_2x_1^2 + a_4x_1 + a_6$ gives

$$b = \frac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3}.$$

□

Example – Secant Case

Consider the elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{Q}$ given by

$$f = Y^2 - X^3 + 36X,$$

where $a_1 = a_2 = a_3 = a_6 = 0$ and $a_4 = -36$.

- $P_1 = (-3, 9)$ and $P_2 = (-2, 8)$ lie on $\mathcal{E}$.
- The line through P_1 and P_2 is

$$Y = -X + 6,$$

since

$$m = \frac{y_2 - y_1}{x_2 - x_1} = -1 \quad \text{and} \quad b = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1} = 6.$$

Example – Tangent Case

Consider the elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{Q}$ given by

$$f = Y^2 - X^3 + 36X,$$

where $a_1 = a_2 = a_3 = a_6 = 0$ and $a_4 = -36$.

- $P_1 = (-3, 9)$ lies on $\mathcal{E}$.
- The tangent line of $\mathcal{E}$ at P_1 is

$$Y = -\frac{1}{2}X + \frac{15}{2},$$

since

$$m = \frac{3x_1^2 - 36}{2y_1} = -\frac{1}{2} \quad \text{and} \quad b = \frac{-x_1^3 - 36x_1}{2y_1} = \frac{15}{2}.$$

Point Addition

Let $P_1 = (x_1 : y_1 : 1)$ and $P_2 = (x_2 : y_2 : 1)$ be distinct $\mathbb{K}$-rational points on elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$ with $P_2 \neq -P_1$; otherwise, $P_1 + P_2 = O$.
Then

$$P_3 = P_1 + P_2$$

is a $\mathbb{K}$-rational point $P_3 = (x_3 : y_3 : 1)$ with

$$x_3 = m^2 + a_1 m - a_2 - x_1 - x_2, \tag{159}$$

$$y_3 = -(m + a_1)x_3 - b - a_3. \tag{160}$$

Proof.

Take the affine Weierstrass equation

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$

- Let $F(X, Y)$ denote the difference between the right-hand side and the left-hand side of the Weierstrass equation.

  Along the line $Y = mX + b$, this expression becomes $F(X, mX + b)$ which amounts to a monic cubic polynomial in X over $\mathbb{K}$.

- The points P_1, P_2, and $P_1 * P_2$ lie on the line $Y = mX + b$. By (155), x_3 equals the x-coordinate of $P_1 * P_2$. Since x_1, x_2, and x_3 are roots of $F(X, mX + b)$, we have

$$F(X, mX + b) = (X - x_1)(X - x_2)(X - x_3).$$

Proof (cont'd).

- Substituting $Y = mX + b$ into the Weierstrass equation gives

$$\begin{aligned} & X^3 + a_2X^2 + a_4X + a_6 - (mX+b)^2 - a_1X(mX+b) - a_3(mX+b) \\ = \quad & X^3 - (x_1+x_2+x_3)X^2 + \text{ terms of lower order in } X. \end{aligned}$$

Comparing the coefficients of X^2 gives

$$a_2 - m^2 - a_1m = -(x_1+x_2+x_3)$$

and so

$$x_3 = m^2 + a_1m - a_2 - x_1 - x_2.$$

For the point $P_1 * P_2 = (x_3 : y_3' : 1)$, $y_3' = mx_3 + b$ and so by (155),

$$y_3 = -y_3' - a_1x_3 - a_3 = -(m+a_1)x_3 - b - a_3.$$

□

Example – Point Addition

Consider the elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{Q}$ given by

$$f = Y^2 - X^3 + 36X,$$

where $a_1 = a_2 = a_3 = a_6 = 0$ and $a_4 = -36$.

- $P_1 = (-3, 9)$ and $P_2 = (-2, 8)$ lie on $\mathcal{E}$.
- The line through P_1 and P_2 is

$$Y = -X + 6.$$

- Point addition gives $P_3 = P_1 + P_2 = (6, 0)$, since

$$x_3 = m^2 - x_1 - x_2 = 6 \quad \text{and} \quad y_3 = -mx_3 - b = 0.$$

Point Addition – Maple

Given elliptic curve $\mathcal{E}$ over $\mathbb{Q}$ by Weierstrass form

$$Y^2 = X^3 + pX + q$$

with $\mathbb{Q}$-rational points $P_1 = (x_1 : y_1 : 1)$ and $P_2 = (x_2 : y_2 : 1)$ on $\mathcal{E}$.

```
> pointAdditionEllipt := proc( x1,y1, x2,y2, f)
      p := coeff( f(x),x,1);
      q := coeff( f(x),x,0);
      m := (y2-y1)/(x2-x1);
      b := (y1*x2-y2*x1)/(x2-x1);
      x3 := m^2-x1-x2;
      y3 := -m*x3-b;
      print(m, b, x3, y3);
  end:
> f := x -> x^3-36*x;
> pointAdditionEllipt( -3,9, -2,8, f);
-1, 6, 6, 0
```

Point Doubling

Let $P = (x_1 : y_1 : 1)$ be a $\mathbb{K}$-rational point on elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{K}$.
Then

$$2P = P \oplus P$$

is a $\mathbb{K}$-rational point $2P = (x_3 : y_3 : 1)$ with

$$x_3 = \frac{x_1^4 - b_4 x_1^2 - 2b_6 x_1 - b_8}{4x_1^3 + b_2 x_1^2 + 2b_4 x_1 + b_6}, \tag{161}$$

$$y_3 = -(m + a_1)x_3 - b - a_3. \tag{162}$$

Proof.

We are in the tangent case ($P_1 = P_2$), where (see (159-160))

$$\begin{aligned} x_3 &= m^2 + a_1 m - a_2 - 2x_1, \\ y_3 &= -(m + a_1)x_3 - b - a_3. \end{aligned}$$

Substituting m in the tangent case (157),

$$m = \frac{3x_1^2 + 2a_2 x_1 + a_4 - a_1 y_1}{2y_1 + a_1 x_1 + a_3},$$

yields

$$x_3 = \frac{x_1^4 - b_4 x_1^2 - 2b_6 x_1 - b_8}{4x_1^3 + b_2 x_1^2 + 2b_4 x_1 + b_6},$$

where b_2, b_4, b_6, b_8 are as in (129-132).
The coordinate y_3 depends only on x_3 and so is given as above. □

Example – Point Doubling

Consider the elliptic curve $\mathcal{E} = \mathcal{E}(f)$ over $\mathbb{Q}$ given by

$$f = Y^2 - X^3 + 36X,$$

where $a_1 = a_2 = a_3 = a_6 = 0$ and $a_4 = -36$.

- $P_1 = (-3, 9)$ lies on $\mathcal{E}$.
- The tangent line of $\mathcal{E}$ at P_1 is

$$Y = -\frac{1}{2}X + \frac{15}{2}.$$

- Point doubling gives $2P_1 = \left(\frac{25}{4}, -\frac{35}{8}\right)$, since $a_4 = -36$, $b_2 = 0$, $b_4 = 2a_4$, $b_6 = 0$, $b_8 = -a_4^2$ and so

$$x_3 = \frac{x_1^4 - b_4 x_1^2 - b_8}{4x_1^3 + 2b_4 x_1} = \frac{25}{4} \quad \text{and} \quad y_3 = -mx_3 - b = -\frac{35}{8}.$$

Point Doubling – Maple

Given elliptic curve $\mathcal{E}$ over $\mathbb{Q}$ by Weierstrass form $Y^2 = X^3 + pX + q$ and a $\mathbb{Q}$-rational point $P = (x_1 : y_1 : 1) \in \mathcal{E}$.

```
> pointDoublingEllipt := proc( x1, y1, f )
    local, p, q, b4, b6, b8, m, b, x3, y3;
      p := coeff( f(x),x,1);
      q := coeff( f(x),x,0);
      b4 := 2*p; b6 := 4*q; b8 := -p^2;
      m := (3*x1^2+p)/(2*y1);
      b := (-x1^3+p*x1+2*q)/(2*y1);
      x3 := m^2-2*x1;
      y3 := -m*x3-b;
      print(m, b, x3, y3);
  end:
> f := x -> x^3 - 36*x;
> pointDoublingEllipt( -3, 9, f);
-1/2, 15/2, 25/4, -35/8
```

Fast Point Doubling

Require: Point P on elliptic curve $\mathcal{E}$, positive integer k
Ensure: Point kP on elliptic curve $\mathcal{E}$
  $a \leftarrow k$, $B \leftarrow O$, $C \leftarrow P$
  while $a > 0$ **do**
    if a is even **then**
      $a \leftarrow a/2$, $B \leftarrow B$, $C \leftarrow 2C$
    else
      $a \leftarrow a - 1$, $B \leftarrow B + C$, $C \leftarrow C$
    end if
  end while
  return B

Time complexity $O(\log_2 k)$.

Fast Point Doubling

Trace for the computation of the point $B = 19P$:

a	B	C
19	O	P
18	P	P
9	P	$2P$
8	$P + 2P$	$2P$
4	$P + 2P$	$4P$
2	$P + 2P$	$8P$
1	$P + 2P$	$16P$
0	$P + 2P + 16P$	$16P$

The computation makes use of the Horner scheme.

Summary - Special Case

Let $\mathrm{char}(\mathbb{K}) \neq 2, 3$. Let $\mathcal{E} = \mathcal{E}(f)$ be an elliptic curve over $\mathbb{K}$ given by

$$f = Y^2Z - X^3 - pXZ^2 - qZ^3.$$

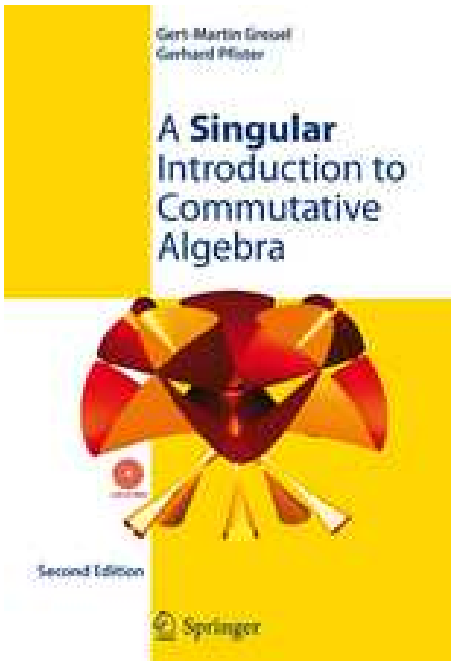
- Let $P = (x : y : 1)$ be a $\mathbb{K}$-rational point of $\mathcal{E}$. Then $-P$ is the $\mathbb{K}$-rational point $-P = (x : -y : 1)$ of $\mathcal{E}$.
- Let $P_1 = (x_1 : y_1 : 1)$ and $P_2 = (x_2 : y_2 : 1)$ be $\mathbb{K}$-rational points of $\mathcal{E}$ with $P_1 \neq -P_2$. Then $P_3 = P_1 + P_2$ is a $\mathbb{K}$-rational point $P_3 = (x_3 : y_3 : 1)$ of $\mathcal{E}$, where

$$x_3 = m^2 - x_1 - x_2, \quad y_3 = -mx_3 - b \tag{163}$$

with

$$m = \begin{cases} \frac{y_2 - y_1}{x_2 - x_1}, \\ \frac{3x_1^2 + p}{2y_1}, \end{cases} \qquad b = \begin{cases} \frac{y_1x_2 - y_2x_1}{x_2 - x_1} & \text{if } P_1 \neq P_2, \\ \frac{-3x_1^3 + px_1 + 2q}{2y_1} & \text{if } P_1 = P_2. \end{cases} \tag{164}$$

Elliptic Curves in Singular



Generation of Random Elliptic Curve

Generation of random elliptic curve over ring $\mathbb{Z}_N$.

```
> LIB "crypto.lib";
> ring r = 0,z,dp;
> ellipticRandomCurve(11); // N=11
[1]:
   5                              // a
[2]:
   8                              // b
[3]:
   1                              // discriminant
```

Returned is a list of two random numbers a, b and $\Delta = 4a^3 + 27b^2 \mod N$.
The curve is given by the equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3.$$

Test for Point Membership

Test whether a point lies on a curve:

```
> isOnCurve(11, 5, 8, list(0,1,0));
1                                   // on curve
> isOnCurve(11, 5, 8, list(5,5,1));
0                                   // not on curve
```

Generation of Random Point

Generate a random point on an elliptic curve:

```
> ellipticRandomPoint(11, 5, 8);
[1]:
   6
[2]:
   1
[3]:
   1
```

Number of Points

Provide the number of points on a curve:

```
> countPoints(11, 5, 8);
15
```

List of Points

List of all points of an elliptic curve:

```
> list L = ellipticAllPoints(11, 5, 8);
> size(L);
15
> L[1];
[1]:
   0
[2]:
   1
[3]:
   0
```

Point Addition

Addition of two points on an elliptic curve:

```
> list P,Q;
> P[1]=1;
> P[2]=5;
> P[3]=1;
> Q[1]=1;
> Q[2]=6;
> Q[3]=1;
> ellipticAdd(11, 5, 8, P, Q); // P+Q
[1]:
   0
[2]:
   1
[3]:
   0
```

Point Multiple

Multiple of a point on an elliptic curve:

```
> list P;
> P[1]=1;
> P[2]=5;
> P[3]=1;
> ellipticMult(11, 5, 8, P, 6); // 6*P
[1]:
   6
[2]:
   1
[3]:
   1
```

Addition Table

The elliptic curve $\mathcal{C}$ over $\mathbb{Z}_7$ given by $f = Y^2Z - X^3 - 3XZ^2 - 5Z^3$ has the points

$$P_1 = O = [0:1:0], \quad P_2 = [1:4:1], \quad P_3 = [1:3:1],$$
$$P_4 = [4:2:1], \quad P_5 = [4:5:1],$$
$$P_6 = [6:1:1], \quad P_7 = [6:6:1].$$

The addition table is as follows:

$+$	O	P_2	P_3	P_4	P_5	P_6	P_7
O	O	P_2	P_3	P_4	P_5	P_6	P_7
P_2		P_6	O	P_5	P_7	P_4	P_3
P_3			P_7	P_6	P_4	P_2	P_5
P_4				P_3	O	P_7	P_2
P_5					P_2	P_3	P_6
P_6						P_5	O
P_7							P_4

Division Polynomials

```
> LIB "crypto.lib";
> ring r = 0, (x,y), dp;
> ellipticRandomCurve(11);
[1]:
   4
[2]:
   9
[3]:
   1
> generateG(4,9,2);
2x
> generateG(4,9,3);
3y4+24y2+108y-16
```

Part V

Theory of Elliptic Curves

Elliptic Curves

- Endomorphisms
- Frobenius endmorphism
- Torsion subgroup
- Division polynomials
- Weil pairing
- *Divisors
- Use of Singular

Endomorphisms

Ferdinand Georg Frobenius (1849-1917).

Endomorphisms

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$.
A mapping $\phi : \mathcal{E}(\bar{\mathbb{K}}) \to \mathcal{E}(\bar{\mathbb{K}})$ is an *endomorphism* of $\mathcal{E}$ if

- ϕ is a homomorphism, i.e., for all $P_1, P_2 \in \mathcal{E}(\bar{\mathbb{K}})$

$$\phi(P_1 + P_2) = \phi(P_1) + \phi(P_2), \tag{165}$$

- there are rational functions (quotients of polynomials) $r_1(X, Y), r_2(X, Y)$ with coefficients in $\bar{\mathbb{K}}$ such that for all $(x, y) \in \mathcal{E}(\bar{\mathbb{K}})$ (in affine coordinates)

$$\phi(x, y) = (r_1(x, y), r_2(x, y)). \tag{166}$$

For each homomorphism ϕ,

$$\phi(O) = O. \tag{167}$$

The *trivial endomorphism* maps each point to the base point O and is also denoted by 0.

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2, 3$ given by

$$Y^2 = X^3 + pX + q.$$

An endomorphism of $\mathcal{E}$ is given by the doubling operation

$$\phi : \mathcal{E}(\bar{\mathbb{K}}) \to \mathcal{E}(\bar{\mathbb{K}}) : P \mapsto 2P. \tag{168}$$

- ϕ is a homomorphism,

$$\begin{aligned} \phi(P_1 + P_2) &= 2(P_1 + P_2) = 2P_1 + 2P_2 \\ &= \phi(P_1) + \phi(P_2). \end{aligned} \tag{169}$$

Tangent Case – Revisited

Let $P = (x : y : 1)$ be a $\mathbb{K}$-rational point of $\mathcal{E}$.

- The inverse point is the $\mathbb{K}$-rational point $-P = (x : -y : 1)$ of $\mathcal{E}$.
- The doubling point is the $\mathbb{K}$-rational point $2P = (x_3 : y_3 : 1)$ of $\mathcal{E}$, where by (163) and (164),

$$x_3 = m^2 - 2x, \quad y_3 = -mx_3 - b \tag{170}$$

with

$$m = \frac{3x^2 + p}{2y}, \quad b = \frac{-3x^3 + px_1 + 2q}{2y}. \tag{171}$$

Example (cont'd)

- There are rational functions by (170) and (171),

$$r_1(X,Y) = \left(\frac{3X^2+p}{2Y}\right)^2 - 2X, \tag{172}$$

$$r_2(X,Y) = \left(\frac{3X^2+p}{2Y}\right)\left(2X - \left(\frac{3X^2+p}{2Y}\right)^2\right) - \frac{-3X^3+pX+2q}{2Y} \tag{173}$$

such that

$$\phi(x,y) = (r_1(x,y), r_2(x,y)). \tag{174}$$

Endomorphisms

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{K}$ given by

$$Y^2 = X^3 + pX + q.$$

Each endomorphism ϕ of $\mathcal{E}$ can be written as

$$\phi(X, Y) = (r_1(X), r_2(X)Y) \tag{175}$$

with rational functions $r_1(X), r_2(X)$.

Proof.

Using $Y^2 = X^3 + pX + q$ replace each power Y^{2m} by $(X^3 + pX + q)^m$ and each power Y^{2m+1} by $(X^3 + pX + q)^m Y$. Thus each rational function $r(X, Y)$ over $\bar{\mathbb{K}}$ has the form

$$r(X, Y) = \frac{p_1(X) + p_2(X)Y}{p_3(X) + p_4(X)Y}.$$

Proof (cont'd).

Multiplying numerator and denominator by $p_3(X) - p_4(X)Y$ and using $Y^2 = X^3 + pX + q$ gives

$$r(X,Y) = \frac{q_1(X) + q_2(X)Y}{q_3(X)}$$

for some rational functions $q_1(X), q_2(X), q_3(X)$.
Take the endomorphism ϕ of $\mathcal{E}$ given by

$$\phi(X,Y) = (r_1(X,Y), r_2(X,Y)).$$

We have

$$\phi(X,-Y) = \phi(-(X,Y)),$$

since $-(X,Y) = (X,-Y)$ is the inverse point, and

$$\phi(-(X,Y)) = -\phi(X,Y),$$

since ϕ is a homomorphism.

Proof (cont'd).

It follows that

$$\phi(X,-Y) = (r_1(X,-Y), r_2(X,-Y)) = -\phi(X,Y)$$

and so

$$r_1(X,-Y) = r_1(X,Y) \quad \text{and} \quad r_2(X,-Y) = -r_2(X,Y).$$

Write

$$r_1(X,Y) = \frac{q_1(X) + q_2(X)Y}{q_3(X)} \text{ and } r_2(X,Y) = \frac{q_1'(X) + q_2'(X)Y}{q_3'(X)}.$$

Since $r_1(X,-Y) = r_1(X,Y)$, $q_2(X) = 0$, and since $r_2(X,-Y) = -r_2(X,Y)$, $q_1'(X) = 0$. Hence, $\phi(X,Y)$ has the required form. □

Degree

Given an endomorphism ϕ of an elliptic curve $\mathcal{E}$ over $\mathbb{K}$ by

$$\phi(X, Y) = (r_1(X), r_2(X)Y) \tag{176}$$

with

$$r_1(X) = \frac{p(X)}{q(X)}, \tag{177}$$

where $p(X)$ and $q(X)$ are polynomials without common factor.

If ϕ is nontrivial, the *degree* of ϕ is

$$\deg(\phi) = \max\{\deg p(X), \deg q(X)\}. \tag{178}$$

The degree of the trivial endomorphism $\phi = 0$ is $\deg(0) = 0$.

Separability

Given a nontrivial endomorphism ϕ of an elliptic curve $\mathcal{E}$ over $\mathbb{K}$ by

$$\phi(X, Y) = (r_1(X), r_2(X)Y) \tag{179}$$

with

$$r_1(X) = \frac{p(X)}{q(X)}, \tag{180}$$

where $p(X)$ and $q(X)$ are polynomials without common factor.

The endomorphism ϕ is *separable* if the derivative $r_1'(X)$ is not identically zero.

- If $\mathrm{char}(\mathbb{K}) = 0$, each nonconstant polynomial has nonzero derivative.
- If $\mathrm{char}(\mathbb{K}) = p > 0$, the polynomials with zero derivative are of the form $g(X^p)$, since $\frac{d}{dX} g(X^p) = pX^{p-1} g'(X^p)$.

Example (cont'd)

Reconsider the elliptic curve $\mathcal{E}$ over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2, 3$ given by $Y^2 = X^3 + pX + q$ and endomorphism $\phi(P) = 2P$.
Then

$$r_1(X, Y) = \left(\frac{3X^2 + p}{2Y}\right)^2 - 2X. \tag{181}$$

Since $Y^2 = X^3 + pX + q$,

$$r_1(X) = \frac{X^4 - 2pX^2 - 8qX + p^2}{4(X^3 + pX + q)}. \tag{182}$$

Thus the endomorphism ϕ has degree $\deg(\phi) = 4$.

The derivative

$$r_1'(X) = \frac{X^6 + 5pX^4 + 20qX^3 - 5p^2X^2 - 4pqX - p^3 - 8q^2}{4(X^3 + pX + q)^2} \tag{183}$$

is a rational function in X (not identically 0) and so ϕ is separable.

Endomorphisms

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$ with endomorphism $\phi \neq 0$. The *kernel* of ϕ is

$$\ker(\phi) = \{P \in \mathcal{E}(\bar{\mathbb{K}}) \mid \phi(P) = O\}. \tag{184}$$

- If ϕ is separable,

$$\deg \phi = \sharp \ker(\phi). \tag{185}$$

- If ϕ is not separable,

$$\deg \phi > \sharp \ker(\phi). \tag{186}$$

(See Washington, section 2.9)

Endomorphisms

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$. Each nonzero endomorphism $\phi : \mathcal{E}(\bar{\mathbb{K}}) \to \mathcal{E}(\bar{\mathbb{K}})$ is surjective.

Proof.

Let $P \in \mathcal{E}(\bar{\mathbb{K}})$. Since $\phi(O) = O$, we may assume that $P \neq O$. For the affine part, take $P = (a, b)$ and write

$$\phi(X, Y) = (r(X), s(X)Y).$$

Write $r(X) = g(X)/h(X)$. If $g(X) - ah(X)$ is not a constant polynomial, it has a root $x_0 \in \bar{\mathbb{K}}$; i.e., $g(x_0) = ah(x_0)$. As $g(X), h(X)$ have no common root, $h(x_0) \neq 0$. Take $y_0 \in \bar{\mathbb{K}}$ as a square root of $x_0^3 + px_0 + q$. Then $\phi(x_0, y_0) = (a, c)$ with $c^2 = a^3 + pa + q = b^2$ and so $c = \pm b$. If $c = b$, we are done; otherwise, $\phi(x_0, -y_0) = (a, -c) = (a, b)$. Thus ϕ is surjective. The case where $g - ah$ is constant is similar. □

Endomorphisms

Let ϕ_1, ϕ_2, ϕ_3 be nonzero endomorphisms of elliptic curve $\mathcal{E}$ over $\mathbb{K}$ with $\phi_1 + \phi_2 = \phi_3$. Write

$$\phi_i(X, Y) = (r_i(X), s_i(X)Y), \quad 1 \leq i \leq 3. \tag{187}$$

If there are constants c_1, c_2 with

$$\frac{r_1'(X)}{s_1(X)} = c_1 \quad \text{and} \quad \frac{r_2'(X)}{s_2(X)} = c_2, \tag{188}$$

then

$$\frac{r_3'(X)}{s_3(X)} = c_1 + c_2. \tag{189}$$

Proof uses differentials (Washington, section 2.9).

Endomorphisms

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$ and let $n \neq 0$ be an integer. If multiplication by n on $\mathcal{E}$ is given by

$$[n](X, Y) = (r_n(X), s_n(X)Y), \tag{190}$$

then

$$\frac{r_n'(X)}{s_n(X)} = n. \tag{191}$$

Thus multiplication by n is separable iff $\mathrm{char}(\mathbb{K}) \not| \, n$.

Proof.

The assertion holds for $n = 1$, since $r_1(X) = X$ and $s_1(X) = 1$. If it holds for $n \geq 1$, then also for $n + 1$, which by induction hypothesis and (189) is the sum of $c_1 = 1$ and $c_2 = n$. □

Frobenius Endomorphism

Ferdinand Georg Frobenius (1849-1917).

Frobenius Endomorphism

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F} = \mathbb{F}_q$. The mapping

$$\phi_q : \mathcal{E}(\bar{\mathbb{F}}) \to \mathcal{E}(\bar{\mathbb{F}}) : (a : b : c) \mapsto (a^q : b^q : c^q) \tag{192}$$

is an endomorphism of $\mathcal{E}$ of degree q, but ϕ_q is not separable. ϕ_q is called *Frobenius endomorphism* of $\mathcal{E}$.

Frobenius Map

Let $\mathbb{F}$ be a field of characteristic $p > 0$. The mapping

$$\phi : \mathbb{F} \to \mathbb{F} : a \mapsto a^p \tag{193}$$

is a ring endomorphism, called *Frobenius map*. We have

$$\phi(1) = 1, \quad \phi(ab) = (ab)^p = a^p b^p = \phi(a)\phi(b) \tag{194}$$

and (Freshman's dream)

$$\begin{aligned} \phi(a+b) &= (a+b)^p = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i} = a^p + b^p \\ &= \phi(a) + \phi(b), \end{aligned} \tag{195}$$

since in the binomial coefficient $\binom{p}{i} = \frac{p!}{i!(p-i)!}$ with $1 \le i \le p-1$, the numerator is divisible by p but the denominator is not.

Frobenius Map – Example

The field $\mathbb{F}_8 = \mathbb{Z}_2[X]/\langle X^3 + X + 1\rangle$ has the elements

$$\begin{array}{llll} 0, & \alpha, & \alpha^3 = \alpha + 1, & \alpha^5 = \alpha^2 + \alpha + 1, \\ 1, & \alpha^2, & \alpha^4 = \alpha^2 + \alpha, & \alpha^6 = \alpha^2 + 1, \end{array}$$

where $\alpha^7 = 1$ and α is a zero of $X^3 + X + 1$ in $\mathbb{F}_8$, i.e.,

$$\alpha^3 = \alpha + 1.$$

The Frobenius map $\mathbb{F}_8 \to \mathbb{F}_8 : a \mapsto a^2$ gives

$$\begin{array}{ll} \phi(0) = 0, & \phi(1) = 1, \\ \phi(\alpha) = \alpha^2, & \phi(\alpha^2) = \alpha^4, \\ \phi(\alpha^3) = \alpha^6, & \phi(\alpha^4) = \alpha, \\ \phi(\alpha^5) = \alpha^3, & \phi(\alpha^6) = \alpha^5. \end{array}$$

It fixes the prime field $\mathbb{Z}_2$: $0^2 = 0$ and $1^2 = 1$.

Frobenius Map – Example

- The Frobenius map $\phi : \mathbb{Z}_p \to \mathbb{Z}_p$ is the identity mapping, since by Little Fermat,

$$a^p = a \text{ for each } a \in \mathbb{Z}_p. \tag{196}$$

- The Frobenius map $\phi : \mathbb{Z}_p[X] \to \mathbb{Z}_p[X]$ fulfills by Little Fermat and Freshman's Dream

$$\left(\sum_i a_i X^i\right)^p = \sum_i a_i X^{pi}, \quad a_i \in \mathbb{Z}_p. \tag{197}$$

Frobenius Map

Let $\mathbb{F}$ be a field of characteristic $p > 0$. The Frobenius map

$$\phi : \mathbb{F} \to \mathbb{F} : a \mapsto a^p \tag{198}$$

is injective.

Proof.

If $\phi(a) = a^p = 0$ for some $a \in \mathbb{F}$, then a is nilpotent. But a field has no zero divisors and so $a = 0$. □

A field $\mathbb{F}$ of characteristic $p > 0$ is *perfect* if the Frobenius map of $\mathbb{F}$ is surjective and so an automorphism.

Frobenius Map

Each finite field $\mathbb{F}$ is perfect.

Proof.

The Frobenius map $\phi : \mathbb{F} \to \mathbb{F}$ is injective. But each injective mapping between two finite sets of the same cardinality is also surjective and so bijective. □

Frobenius Map

The rational function field

$$\mathbb{F} = \mathbb{Z}_p(X) = \left\{ \frac{f(X)}{g(X)} \mid f(X), g(X) \in \mathbb{Z}_p[X],\ g(X) \neq 0 \right\}$$

is not perfect.

Proof.

The unknown X does not lie in the image of the Frobenius map. Indeed, suppose $X = \left(\frac{f(X)}{g(X)}\right)^p$ for some $f(X), g(X) \in \mathbb{Z}_p[X]$, $g(X) \neq 0$. Then $g(X)^p X = f(X)^p$.
Write $f(X) = \sum_i a_i X^i$ and $g(X) = \sum_i b_i X^i$. Then by (197),

$$\sum_i b_i X^{ip+1} = \sum_i a_i X^{ip}.$$

By comparing coefficients, $a_i = b_i = 0$ contradicting $g(X) \neq 0$. □

Frobenius Map

Consider the prime field $\mathbb{F}_p = \mathbb{Z}_p$ of characteristic $p > 0$.

By Little Fermat, each element $a \in \mathbb{F}_p$ satisfies $a^p = a$. More generally, the elements of $\mathbb{F}_p$ are exactly the roots of the polynomial

$$X^p - X \in \mathbb{F}_p[X]. \tag{199}$$

Thus if $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_p$ such as the algebraic closure of $\mathbb{F}_p$, then $\mathbb{F}_p$ is the *fixed field* of the Frobenius map $\mathbb{F} \to \mathbb{F} : a \mapsto a^p$.

Frobenius Map – Example

Consider the prime field $\mathbb{F}_3 = \{0, 1, 2\}$ $(p = 3)$.

The polynomial $g = X^2 + X + 2 \in \mathbb{F}_3[X]$ is irreducible.

The extension field $\mathbb{F} = \mathbb{F}_9 = \mathbb{F}_3[X]/\langle g \rangle$ is given by

$$\mathbb{F}_9 = \{a\alpha + b \mid a, b \in \mathbb{F}_3\}$$

where $\alpha^2 + \alpha + 2 = 0$; i.e., $\alpha^2 = 2\alpha + 1$ $(2 = -1,\ 3 = 0$ in $\mathbb{F}_3)$.
Addition

$$(a\alpha + b) + (c\alpha + d) = (a + c)\alpha + (b + d).$$

Multiplication

$$\begin{aligned}(a\alpha + b)(c\alpha + d) &= ac\alpha^2 + ad\alpha + bc\alpha + bd \\ &= (2ac + ad + bc)\alpha + (ac + bd).\end{aligned}$$

The automorphism $\phi : \mathbb{F}_9 \to \mathbb{F}_9 : a \mapsto a^3$ fixes the elements of $\mathbb{F}_3 \subseteq \mathbb{F}_9$.

Frobenius Map

Let $\mathbb{F}_q$ denote the finite field with $q = p^r$ elements.

The *r-th iterate* of the Frobenius automorphism has similar properties. For this, note that the elements of $\mathbb{F}_q$ are exactly the roots of the polynomial

$$X^q - X \in \mathbb{F}_q[X]. \tag{200}$$

Thus if $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_q$, the r-th iterate of the Frobenius map,

$$\phi : \mathbb{F} \to \mathbb{F} : a \mapsto a^q, \tag{201}$$

has the *fixed field* $\mathbb{F}_q$.
The rth iterate Frobenius map $\phi : \mathbb{F}_q[X] \to \mathbb{F}_q[X]$ fulfills

$$\left(\sum_i a_i X^i\right)^q = \sum_i a_i X^{qi}, \quad a_i \in \mathbb{F}_q. \tag{202}$$

Frobenius Map – Example

Consider the field $\mathbb{F}_4 = \{0, 1, \beta, \beta^2\}$, where $\beta^2 = \beta + 1$ $(q = 4)$.

The polynomial $g = X^2 + X + \beta \in \mathbb{F}_4[X]$ is irreducible.

The extension field $\mathbb{F} = \mathbb{F}_{16} = \mathbb{F}_4[X]/\langle g \rangle$ is given by

$$\mathbb{F}_{16} = \{a\alpha + b \mid a, b \in \mathbb{F}_4\}$$

where $\alpha^2 + \alpha + \beta = 0$; i.e., $\alpha^2 = \alpha + \beta$.
Addition

$$(a\alpha + b) + (c\alpha + d) = (a + c)\alpha + (b + d).$$

Multiplication

$$\begin{aligned}(a\alpha + b)(c\alpha + d) &= ac\alpha^2 + ad\alpha + bc\alpha + bd \\ &= (ac + ad + bc)\alpha + (ac\beta + bd).\end{aligned}$$

The automorphism $\phi : \mathbb{F}_{16} \to \mathbb{F}_{16} : a \mapsto a^4$ fixes the elements of $\mathbb{F}_4 \subseteq \mathbb{F}_{16}$.

Proof (Frobenius map).

- The mapping ϕ_q is well-defined. Indeed, since the Frobenius map of $\mathbb{F}$ is injective, $a^q = b^q = c^q = 0$ implies $a = b = c = 0$.
- The mapping ϕ_q preserves the equivalence classes defining the projective space $\mathbb{P}^2(\bar{\mathbb{F}})$.

  Indeed, if (a, b, c) and (e, d, f) represent the same point in the projective plane $\mathbb{P}^2(\bar{\mathbb{F}})$, then

$$(e, d, f) = (ta, tb, tc) = t(a, b, c)$$

  for some nonzero $t \in \bar{\mathbb{F}}$. Then

$$\begin{aligned} (e^q, d^q, f^q) &= ((ta)^q, (tb)^q, (tc)^q) = (t^q a^q, t^q b^q, t^q c^q) \\ &= t^q (a^q, b^q, c^q) \end{aligned}$$

  and (a^q, b^q, c^q) represent the same point as well.

Proof (cont'd).

- Let the elliptic curve $\mathcal{E}(\mathbb{F})$ be given by the general Weierstrass polynomial $f \in \mathbb{F}[X, Y, Z]$, and let $P = (a : b : c) \in \mathcal{E}(\bar{\mathbb{F}})$. Then $f(a, b, c) = 0$ and thus by the Frobenius map of $\mathbb{F}$ (see (202)),

$$0 = 0^q = f(a, b, c)^q = f(a^q, b^q, c^q).$$

  Hence, $\phi_q(P) = (a^q : b^q : c^q)$ is also a point of $\mathcal{E}(\bar{\mathbb{F}})$.

- Claim that the mapping commutes with the group operation. Indeed, the base point $O = (0 : 1 : 0)$ satisfies

$$\phi_q(0 : 1 : 0) = (0 : 1 : 0).$$

  Since O is the identity element, each point P of $\mathcal{E}(\bar{\mathbb{F}})$ satisfies

$$\phi_q(P + O) = \phi_q(P) = \phi_q(P) + O = \phi_q(P) + \phi_q(O).$$

Proof (cont'd).

- Let P_1 and P_2 be two points of $\mathcal{E}(\bar{\mathbb{F}})$ different from the base point O. Then they lie in the affine plane and can be written as $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$.

  First consider the point addition $P_1 + P_2 = P_3$ with $P_1 \neq -P_2$. Then $P_3 = (x_3, y_3)$, where by (159) and (160),

$$\begin{aligned} x_3 &= m^2 + a_1 m - a_2 - x_1 - x_2, \\ y_3 &= -(m + a_1)x_3 - b - a_3 \end{aligned}$$

  where $m, b \in \bar{\mathbb{F}}$. By (157) and (158),

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  and

$$b = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}.$$

Proof (cont'd).

Applying the Frobenius map of $\mathbb{F}$ gives

$$\begin{aligned} x_3^q &= (m^q)^2 + a_1 m^q - a_2 - x_1^q - x_2^q, \\ y_3^q &= -(m^q + a_1)x_3^q - b^q - a_3, \end{aligned}$$

since $a_1, a_2, a_3 \in \mathbb{F}$.
On the other hand,

$$\phi_q(P_1) + \phi_q(P_2) = (x_1^q, y_1^q) + (x_2^q, y_2^q) = (x_4, y_4),$$

where by (159) and (160),

$$\begin{aligned} x_4 &= \hat{m}^2 + a_1\hat{m} - a_2 - x_1^q - x_2^q, \\ y_4 &= -(\hat{m} + a_1)x_4^q - \hat{b} - a_3 \end{aligned}$$

for some $\hat{m}, \hat{b} \in \bar{\mathbb{F}}$.

Proof (cont'd).

By (157), (158) and Freshman's dream,

$$\hat{m} = \frac{y_2^q - y_1^q}{x_2^q - x_1^q} = m^q$$

and

$$\hat{b} = \frac{y_1^q x_2^q - y_2^q x_1^q}{x_2^q - x_1^q} = b^q.$$

Hence,

$$\begin{aligned} \phi_q(P_1) + \phi_q(P_2) &= (x_4, y_4) \\ &= (x_3^q, y_3^q) \\ &= \phi_q(P_3) \\ &= \phi_q(P_1 + P_2). \end{aligned}$$

The other cases are similar.

Proof (cont'd).

- The discriminant $\Delta \neq 0$ of $\mathcal{E}$ is an element of $\mathbb{F}_q$, since it only depends on the coefficients of Weierstrass polynomial $f \in \mathbb{F}_q[X, Y, Z]$, which are elements of $\mathbb{F}_q$.

  Thus $\Delta^q = \Delta$ and hence the image of $\mathcal{E}$ is nonsingular.

- The Frobenius endomorphism

$$\phi_q(X, Y) = (r_1(X), r_2(Y)) = (X^q, Y^q) \tag{203}$$

  has degree $\deg r_1(X) = q$.

- Since $q = 0$ in $\mathbb{F}_q$, the derivative of $r_1(X) = X^q$ satisfies

$$r_1'(X) = qX^{q-1} = 0 \tag{204}$$

  and so ϕ_q is not separable.

□

Endomorphisms

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$ with $q = p^r$, $p \neq 2$, given by $Y^2 = X^3 + AX + B$, and let m, n be integers, not both 0. The endomorphism

$$[m]\phi_q + [n], \tag{205}$$

written $m\phi_q + n$, is separable iff $p \not| n$.

Proof.

By (232), we can write

$$[m](X, Y) = (r_m(X), s_m(X)Y).$$

Then as q is odd,

$$\begin{aligned}
&(r_{m\phi_q}(X), s_{m\phi_q}(X)Y) \\
&\quad = \quad \phi_q[m](X, Y) = (r_m^q(X), s_m^q(X)Y^q) \\
&\quad = \quad \left(r_m^q(X), s_m^q(X) \cdot Y(X^3 + AX + B)^{(q-1)/2}\right).
\end{aligned}$$

Proof (cont'd).

We have

$$(r_{m\phi_q}(X), s_{m\phi_q}(X)Y) = \left(r_m^q(X), s_m^q(X)(X^3 + AX + B)^{(q-1)/2}Y\right).$$

Thus

$$c_{m\phi_q} = r'_{m\phi_q}/s_{m\phi_q} = qr_m^{q-1}r'_m/s_{m\phi_q} = 0.$$

Moreover, by (191),

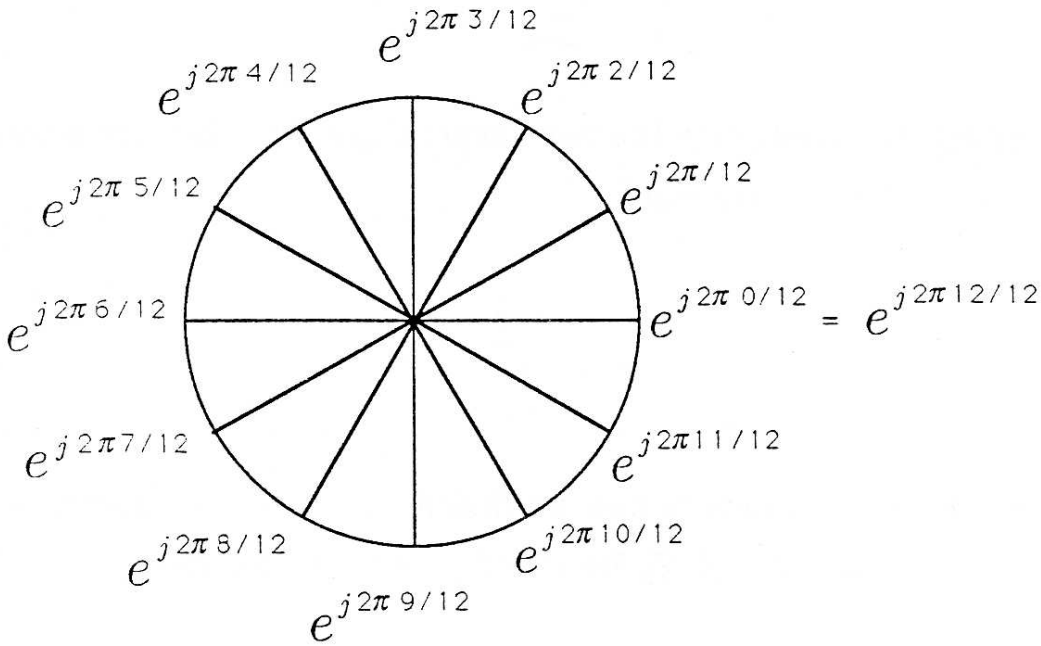
$$c_n = r'_n/s_n = n$$

and so by (189),

$$r'_{m\phi_q+n}/s_{m\phi_q+n} = c_{m\phi_q+n} = c_{m\phi_q} + c_n = 0 + n = n.$$

Hence, $r'_{m\phi_q+n} \neq 0$ iff $p \nmid n$. □

Torsion Subgroup



Torsion Subgroup

Let $\mathcal{E}$ be an elliptic curve over field $\mathbb{K}$ and let $n \geq 0$ be an integer. The mapping

$$[n] : \mathcal{E}(\bar{\mathbb{K}}) \to \mathcal{E}(\bar{\mathbb{K}}) : P \mapsto [n]P = nP \tag{206}$$

is a group homomorphism with kernel

$$\mathcal{E}[n] = \{P \in \mathcal{E}(\bar{\mathbb{K}}) \mid nP = O\} \tag{207}$$

called *n-torsion subgroup* of $\mathcal{E}$.
The *torsion subgroup* of $\mathcal{E}$ is given by the points of finite order

$$\mathcal{E}_{\text{tor}} = \bigcup_{n=0}^{\infty} \mathcal{E}[n]. \tag{208}$$

The mapping $[n]$ is an endomorphism of $\mathcal{E}$ (see division polynomials).

Torsion Subgroup

Let $\mathcal{E}$ be an elliptic curve over field $\mathbb{K}$ and let $n \geq 2$ be an integer. If $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) \not| \, n$, then

$$\mathcal{E}[n] \simeq \mathbb{Z}_n \oplus \mathbb{Z}_n. \tag{209}$$

If $\mathrm{char}(\mathbb{K}) = p > 0$ and $p \mid n$, write $n = p^r m$ with $p \not| \, m$, then

$$\mathcal{E}[n] \simeq \mathbb{Z}_m \oplus \mathbb{Z}_m \quad \text{or} \quad \mathcal{E}[n] \simeq \mathbb{Z}_n \oplus \mathbb{Z}_m. \tag{210}$$

Proof.

Let $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) \not| \, n$.

- By (232), we have

$$[n](X, Y) = (r(X), s(X)Y) \tag{211}$$

for some rational functions $r(X), s(X)$. Then using division polynomials,

$$r(X) = \frac{X^{n^2} + \text{lower degree terms}}{n^2 X^{n^2-1} + \text{lower degree terms}}. \tag{212}$$

By hypothesis on $\mathbb{K}$, $r'(X) \neq 0$ and so the endomorphism $[n]$ is separable.

- By (212), the endomorphism $[n]$ has degree n^2 and so by (185), $\mathcal{E}[n] = \ker([n])$ has order n^2.

Proof (cont'd)

- By the structure theorem for finite abelian groups, $\mathcal{E}[n]$ is isomorphic to
$$\mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k},$$
where $n_1, \ldots, n_k$ are integers with $n_i \mid n_{i+1}$ for all i.
- Let ℓ be a prime dividing n_1. Then $\ell \mid n_i$ for all i and so $\mathcal{E}[\ell] \subseteq \mathcal{E}[n]$ has order ℓ^k, since $\mathcal{E}[\ell]$ is a subgroup of each $\mathbb{Z}_{n_i}$ and $\mathcal{E}[n]$ is a direct sum of those subgroups. But $\mathcal{E}[\ell]$ has order ℓ^2 and so $k = 2$.
- We have $\mathcal{E}[n] = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$. Since $\mathbb{Z}_{n_2}$ is a subgroup of $\mathcal{E}[n]$, we have $n_2 \mid n$ (Lagrange's theorem).
- We have $n^2 = \sharp\mathcal{E}[n] = n_1 n_2$ and so $n_1 = n_2 = n$. Therefore, $\mathcal{E}[n] \simeq \mathbb{Z}_n \oplus \mathbb{Z}_n$.

The case $\mathrm{char}(\mathbb{K}) = p > 0$ and $p \mid n$ is quite similar. □

Torsion Subgroup

The direct product group $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$ is defined by the componentwise operation

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2). \tag{213}$$

In particular, if $(m, n) = 1$, there exists an isomorphism

$$\mathbb{Z}_{mn} \to \mathbb{Z}_m \oplus \mathbb{Z}_n \tag{214}$$

given by

$$a \mapsto (a \mod m, a \mod n). \tag{215}$$

Surjectivity is proved by the Chinese remainder theorem.

Examples

$\mathbb{Z}_6$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ are isomorphic by (214), but $\mathbb{Z}_4$ (cyclic) and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (noncyclic) are not.

Torsion Subgroup

Suppose $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) \not| \, n$.

- By (209), $\mathcal{E}[n] \simeq \mathbb{Z}_n \oplus \mathbb{Z}_n$ and so $\mathcal{E}[n]$ has a *basis* $\{\beta_1, \beta_2\}$; i.e., each element of $\mathcal{E}[n]$ can be uniquely written in the form

$$k_1\beta_1 + k_2\beta_2 \tag{216}$$

  with integers k_1, k_2 (uniquely determined mod n).

- A homomorphism $\phi : \mathcal{E}(\bar{\mathbb{K}}) \to \mathcal{E}(\bar{\mathbb{K}})$ maps $\mathcal{E}[n]$ into $\mathcal{E}[n]$. Indeed, if $P \in \mathcal{E}[n]$, then

$$n\phi(P) = \phi(nP) = \phi(O) = O$$

  and so $\phi(P) \in \mathcal{E}[n]$.

Torsion Subgroup (cont'd)

- There are $a, b, c, d \in \mathbb{Z}$ such that

$$\phi(\beta_1) = a\beta_1 + c\beta_2 \quad \text{and} \quad \phi(\beta_2) = b\beta_1 + d\beta_2. \tag{217}$$

- Each homomorphism $\phi : \mathcal{E}(\bar{\mathbb{K}}) \to \mathcal{E}(\bar{\mathbb{K}})$ is represented by an integral 2×2 transformation matrix

$$\phi^{(n)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{218}$$

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{R}$ given by $f = Y^2Z - X^3 + 2Z^3$, i.e., $f^a = Y^2 - X^3 + 2$, and let $n = 2$. Then

$$\mathcal{E}[2] = \{O, (\sqrt[3]{2} : 0 : 1), (\xi\sqrt[3]{2} : 0 : 1), (\xi^2\sqrt[3]{2} : 0 : 1)\},$$

where $\xi = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2} \in \bar{\mathbb{R}} = \mathbb{C}$ is a primitive third root of unity.

- A basis for $\mathcal{E}[2]$ is

$$\{\beta_1 = (\sqrt[3]{2} : 0 : 1), \beta_2 = (\xi\sqrt[3]{2} : 0 : 1)\}.$$

- We have

$$(\sqrt[3]{2} : 0 : 1) + (\xi\sqrt[3]{2} : 0 : 1) = (\xi^2\sqrt[3]{2} : 0 : 1).$$

- We have $\mathcal{E}[2] \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Division Polynomials

Singular computation:

```
> LIB "crypto.lib";
> ring r = 0, (y,x), dp;
> generateG(4,9,2);
2y
> generateG(4,9,3);
3x4+24x2+108x-16
```

Division Polynomials

Define the division polynomials $\psi_m \in \mathbb{Z}[X, Y, A, B]$, $m \geq 0$, as the rational functions

$$\begin{aligned}
\psi_0 &= 0, && (219)\\
\psi_1 &= 1, && (220)\\
\psi_2 &= 2Y, && (221)\\
\psi_3 &= 3X^4 + 6AX^2 + 12BX - A^2, && (222)\\
\psi_4 &= 4Y(X^6 + 5AX^4 + 20BX^3 - 5A^2X^2 \\
&\quad -4ABX - 8B^2 - A^3) && (223)\\
\psi_{2m+1} &= \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3, && (224)\\
&\quad m \geq 2,\\
\psi_{2m} &= \frac{1}{2Y}\psi_m(\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2), && (225)\\
&\quad m \geq 3.
\end{aligned}$$

Division Polynomials

- ψ_n is a polynomial in $\mathbb{Z}[X, Y^2, A, B]$ if n is odd.
- ψ_n is a polynomial in $2Y\mathbb{Z}[X, Y^2, A, B]$ if n is even.

Proof.

The assertion holds for $n \leq 4$. Assume that $2m > 4$, so $m > 2$. Then $2m > m + 2$. Thus all polynomials appearing in the definition of ψ_{2m} fulfill the induction hypothesis.

- If m is even, then $\psi_m, \psi_{m+2}, \psi_{m-2} \in 2Y\mathbb{Z}[X, Y^2, A, B]$ by induction hypothesis and so $\psi_{2m} \in 2Y\mathbb{Z}[X, Y^2, A, B]$.
- If m is odd, then $\psi_{m+1}, \psi_{m-1} \in 2Y\mathbb{Z}[X, Y^2, A, B]$ by induction hypothesis and so $\psi_{2m} \in 2Y\mathbb{Z}[X, Y^2, A, B]$.

Thus the assertion holds for $n = 2m$. Similar for $n = 2m + 1$. □

Division Polynomials

Define the rational functions

$$\phi_n = X\psi_n^2 - \psi_{n-1}\psi_{n+1}, \tag{226}$$

$$\omega_n = \frac{1}{4Y}\left(\psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2\right). \tag{227}$$

We have

- $\phi_n \in \mathbb{Z}[X, Y^2, A, B]$ for all n.
- $\omega_n \in Y\mathbb{Z}[X, Y^2, A, B]$ if n is odd, and $\omega_n \in \mathbb{Z}[X, Y^2, A, B]$ if n is even.

Proof.

- Case ϕ_n:
  - If n is odd, then $\psi_{n-1}, \psi_{n+1} \in 2Y\mathbb{Z}[X, Y^2, A, B]$ and so $\psi_{n-1} \cdot \psi_{n+1} \in \mathbb{Z}[X, Y^2, A, B]$. Moreover, $\psi_n \in \mathbb{Z}[X, Y^2, A, B]$ and so $\phi_n \in \mathbb{Z}[X, Y^2, A, B]$.
  - If n is even, then $\psi_{n-1}, \psi_{n+1} \in \mathbb{Z}[X, Y^2, A, B]$ and so $\psi_{n-1} \cdot \psi_{n+1} \in \mathbb{Z}[X, Y^2, A, B]$. Moreover, $\psi_n \in 2Y\mathbb{Z}[X, Y^2, A, B]$ and so $\phi_n \in \mathbb{Z}[X, Y^2, A, B]$.
- Case ω_n:
  - If n is odd, then $\psi_{n+2}, \psi_{n-2} \in \mathbb{Z}[X, Y^2, A, B]$ and $\psi_{n+1}, \psi_{n-1} \in 2Y\mathbb{Z}[X, Y^2, A, B]$. Thus $\psi_{n+2}\psi_{n-1}^2, \psi_{n-2}\psi_{n+1}^2 \in 4Y^2\mathbb{Z}[X, Y^2, A, B]$ and so $\omega_n \in Y\mathbb{Z}[X, Y^2, A, B]$.
  - If n is even, then $\psi_{n+2}, \psi_{n-2} \in 2Y\mathbb{Z}[X, Y^2, A, B]$ and $\psi_{n+1}, \psi_{n-1} \in \mathbb{Z}[X, Y^2, A, B]$. Thus $\psi_{n+2}\psi_{n-1}^2, \psi_{n-2}\psi_{n+1}^2 \in 2Y\mathbb{Z}[X, Y^2, A, B]$ and so $\omega_n \in \frac{1}{2}\mathbb{Z}[X, Y^2, A, B]$.

Proof (cont'd).

- In $\omega_n \in \frac{1}{2}\mathbb{Z}[X, Y^2, A, B]$, the factor $\frac{1}{2}$ can be eliminated by observing by induction that

$$\psi_n \equiv (X^2 + A)^{(n^2-1)/4} \mod 2, \quad n \text{ odd},$$

and

$$\frac{1}{2Y}\psi_n \equiv \frac{n}{2}(X^2 + A)^{(n^2-4)/4} \mod 2, \quad n \text{ even}.$$

□

Division Polynomials

Consider the elliptic curve $\mathcal{E}$ given by the affine Weierstrass polynomial

$$Y^2 = X^3 + AX + B, \tag{228}$$

where A, B are taken as variables and by nonsingularity the discriminant (136) is

$$4A^3 + 27B^2 \neq 0. \tag{229}$$

The polynomials in $\mathbb{Z}[X, Y^2, A, B]$ can be regarded as polynomials in $\mathbb{Z}[X, A, B]$ or in $\mathbb{Z}[A, B][X]$ when Y^2 is replaced by $X^3 + AX + B$.

Division Polynomials

ϕ_n and ψ_n^2 are polynomials in X with

$$\phi_n = X^{n^2} + \text{lower degree terms in } X, \tag{230}$$
$$\psi_n^2 = n^2 X^{n^2-1} + \text{lower degree terms in } X. \tag{231}$$

Proof.

We have

- $\phi_n \in \mathbb{Z}[X, Y^2, A, B]$ for all n.
- $\psi_n \in \mathbb{Z}[X, Y^2, A, B]$ for odd n and $\psi_n \in 2Y\mathbb{Z}[X, Y^2, A, B]$ for even n.

Replacing Y^2 by $X^3 + AX + B$ shows that ϕ_n and ψ_n^2 are polynomials in $\mathbb{Z}[A, B][X]$.

Proof (cont'd).

The remaining assertions follow from the claim

$$\psi_n = \begin{cases} Y(nX^{(n^2-4)/2} + \ldots) & \text{if } n \text{ is even,} \\ nX^{(n^2-1)/2} + \ldots & \text{if } n \text{ is odd.} \end{cases}$$

To see this, consider several cases. For instance, let $n = 2m + 1$ with m even. Then by induction, the leading term of $\psi_{m+2}\psi_m^3$ is

$$(m+2)m^3Y^4X^{((m+2)^2-4)/2+(3m^2-12)/2}.$$

Replacing Y^2 by $X^3 + AX + B$ gives

$$(m+2)m^3X^{((2m+1)^2-1)/2}.$$

Similarly, the leading term of $\psi_{m-1}\psi_{m+1}^3$ is

$$(m-1)(m+1)^3X^{((2m+1)^2-1)/2}.$$

By definition, $\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3$ and so the leading term is $nX^{(n^2-1)/2}$. □

Point Multiplication

Let $P = (x, y)$ be an affine point on the elliptic curve $\mathcal{E}$ given by $Y^2 = X^3 + AX + B$ over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$, and let $n \geq 1$ be an integer. Then

$$[n]P = \left(\frac{\phi_n(x)}{\psi_n^2(x)}, \frac{\omega_n(x,y)}{\psi_n(x,y)^3} \right). \tag{232}$$

(See Washington, section 9.5)

Corollary

Let $\mathcal{E}$ be an elliptic curve as above. The endomorphism $[n]$ of $\mathcal{E}$ given by multiplication by n has degree n^2.

Proof (Sketch).

By (232), the degree of the endomorphism $[n]$ is

$$\max\{\phi_n(X), \psi_n^2(X)\}$$

if the quotient $\phi_n(X)/\psi_n^2(X)$ is reduced. It can be shown that $\phi_n(X)$ and $\psi_n^2(X)$ have no common roots. Thus by (230) and (231) the degree is

$$\max\{\deg(X^{n^2}), \deg(n^2 X^{n^2-1})\} = n^2.$$

□

Weil Pairing

André Weil (1906-1998).

Roots of Unity

Let $n \geq 1$ be an integer.

- An *nth root of unity* is a number $z \in \mathbb{C}$ such that

$$z^n = 1. \tag{233}$$

- The nth roots of unity are

$$\exp\left(\frac{2k\pi i}{n}\right) = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad 0 \leq k \leq n-1. \tag{234}$$

The fourth roots of unity are $\pm 1, \pm i$ with (Euler identity)

$$e^{\pi i} = -1. \tag{235}$$

- An nth root of unity z is *primitive* if

$$z^n = 1 \quad \text{and} \quad z^k \neq 1, \quad 1 \leq k \leq n-1. \tag{236}$$

Roots of Unity

Let $n \geq 1$ be an integer.

- The set of nth roots of unity

$$U_n = \{z \in \mathbb{C} \mid z^n = 1\} \tag{237}$$

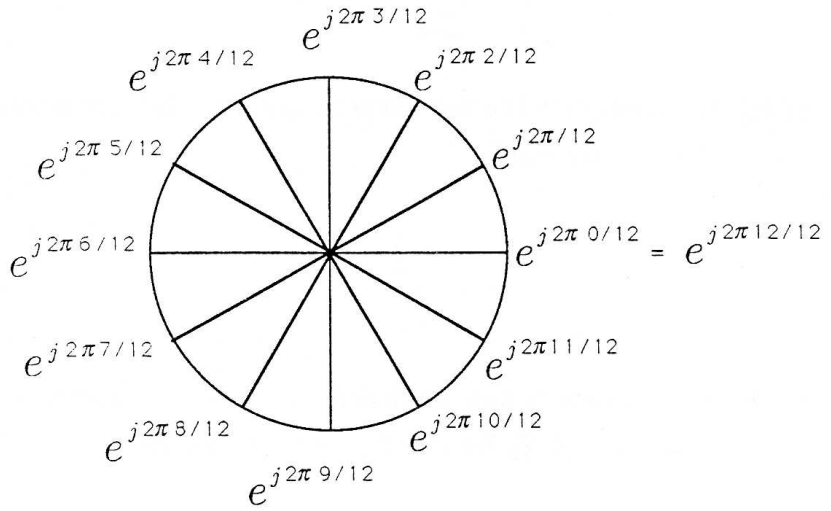
forms a cyclic group of order n, i.e., for some $\xi \in U_n$,

$$U_n = \langle \xi \rangle = \{1, \xi, \ldots, \xi^{n-1}\}, \quad \xi^n = 1. \tag{238}$$

- Each primitive nth root of unity ξ is a generator of U_n.
- If ξ is a primitive nth root of unity, then ξ^k with $(k, n) = 1$ is also a primitive nth root of unity.
- The number of primitive nth roots of unity is $\phi(n)$, where ϕ is Euler's totient function.

Roots of Unity – Example

Consider the 12th roots of unity.



Primitive 12th roots of unity: $\xi = e^{2\pi i/12}, \xi^5, \xi^7, \xi^{11}$, with $\phi(12) = 4$.

Roots of Unity

Let $n \geq 1$ be an integer.

- Definition of roots of unity is meaningful over any field.
- Each nonzero element in a finite field $\mathbb{F}_q$ is a root of unity, since each nonzero element of $\mathbb{F}_q$ satisfies by (200),

$$X^{q-1} - 1 = 0. \tag{239}$$

- For a field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = p > 0$, we have by Freshman's Dream

$$X^{np} - 1 = (X^n - 1)^p. \tag{240}$$

Roots of Unity – Example

Consider the field

$$\mathbb{F}_8 = \mathbb{Z}_2[X]/\langle X^3 + X + 1\rangle.$$

Let α be a zero of $X^3 + X + 1$ in $\mathbb{F}_8$, i.e., $\alpha^3 + \alpha + 1 = 0$, and so

$$\alpha^3 = \alpha + 1.$$

Then α is a primitive 7th root of unity, since

$$\begin{array}{llll} 0, & \alpha, & \alpha^3 = \alpha + 1, & \alpha^5 = \alpha^2 + \alpha + 1, \\ 1, & \alpha^2, & \alpha^4 = \alpha^2 + \alpha, & \alpha^6 = \alpha^2 + 1, \end{array}$$

and $\alpha^7 = 1$.

Weil Pairing

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$ and $n \geq 1$ with $\mathrm{char}(\mathbb{K}) \not| \, n$.

- The set of *n-th roots of unity* in $\bar{\mathbb{K}}$,

$$U_n = \{x \in \bar{\mathbb{K}} \mid x^n = 1\}, \tag{241}$$

  forms a cyclic group of order n, since $\mathrm{char}(\mathbb{K}) \not| \, n$ (see 240).

- A generator ξ of U_n is a *primitive n-th root of unity*,

$$U_n = \langle \xi \rangle = \{1, \xi, \xi^2, \ldots, \xi^{n-1}\}, \quad \xi^n = 1. \tag{242}$$

Weil Pairing

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$ and $n \geq 1$ with $\mathrm{char}(\mathbb{K}) \not| \, n$. There is a mapping

$$e_n : \mathcal{E}[n] \times \mathcal{E}[n] \to U_n, \tag{243}$$

called *Weil pairing*, such that

- e_n is bilinear; i.e., $e_n(S + S', T) = e_n(S, T)e_n(S', T)$ and $e_n(S, T + T') = e_n(S, T)e_n(S, T')$ for all $S, S', T, T' \in \mathcal{E}[n]$.
- e_n is nondegenerate; i.e., if $e_n(T, S) = 1$ for all $S \in \mathcal{E}[n]$, then $T = O$.
- $e_n(T, T) = 1$ for all $T \in \mathcal{E}[n]$.
- $e_n(T, S) = e_n(S, T)^{-1}$ for all $S, T \in \mathcal{E}[n]$.
- $e_n(\sigma S, \sigma T) = \sigma(e_n(S, T))$ for all automorphisms σ of $\bar{\mathbb{K}}$ which are constant on $\mathbb{K}$.
- $e_n(\phi(S), \phi(T)) = e_n(S, T)^{\deg(\phi)}$ for all (separable or not) endomorphisms ϕ of $\mathcal{E}$.

Weil Pairing

If $\{T_1, T_2\}$ is a basis of $\mathcal{E}[n]$, then $e_n(T_1, T_2)$ is a primitive n-th root of unity.

Proof.

- Let $e_n(T_1, T_2) = \xi$ with $\xi^d = 1$. Then by bilinearity, $e_n(T_1, [d]T_2) = e_n(T_1, T_2)^d = \xi^d = 1$. Similarly, $e_n(T_2, [d]T_2) = e_n(T_2, T_2)^d = 1^d = 1$.
- Let $S \in \mathcal{E}[n]$. Then by hypothesis,

$$S = aT_1 + bT_2$$

for some integers a, b. Thus by bilinearity,

$$e_n(S, [d]T_2) = e_n(T_1, [d]T_2)^a e_n(T_2, [d]T_2)^b = 1^a 1^b = 1.$$

This holds for all $S \in \mathcal{E}[n]$. Since e_n is nondegenerate, $[d]T_2 = O$. But T_2 has order n by (209) and so $[d]T_2 = O$ iff $n \mid d$. Hence, ξ is a primitive n-th root of unity. □

Weil Pairing

If $\mathcal{E}[n] \subseteq \mathcal{E}(\mathbb{K})$, then $U_n \subset \mathbb{K}$.
$\mathcal{E}[n] \subseteq \mathcal{E}(\mathbb{K})$ means that all points in $\mathcal{E}[n]$ have coordinates in $\mathbb{K}$.

Proof.

By definition, the points in $\mathcal{E}[n]$ have coordinates in $\bar{\mathbb{K}}$. By hypothesis, these points have coordinates in $\mathbb{K}$.
Let $\sigma : \bar{\mathbb{K}} \to \bar{\mathbb{K}}$ be an automorphism, which is the identity on $\mathbb{K}$.
Let $\{T_1, T_2\}$ be a basis of $\mathcal{E}[n]$. By hypothesis, T_1, T_2 have coordinates in $\mathbb{K}$ and so $\sigma(T_1) = T_1$ and $\sigma(T_2) = T_2$. Thus

$$\xi = e_n(T_1, T_2) = e_n(\sigma(T_1), \sigma(T_2)) = \sigma(e_n(T_1, T_2)) = \sigma(\xi).$$

Since ξ is fixed by all automorphisms σ, it follows that $\xi \in \mathbb{K}$.
Since by the previous result ξ is a primitive n-th root of unity, it generates U_n and so $U_n \subset \mathbb{K}$. □

Weil Pairing

Let ϕ be an endomorphism of an elliptic curve $\mathcal{E}$ over $\mathbb{K}$, and let n be a positive integer with $\mathrm{char}(\mathbb{K}) \not| \, n$.

- By (209), $\mathcal{E}[n] \simeq \mathbb{Z}_n \oplus \mathbb{Z}_n$.
- There is a transformation matrix

$$\phi^{(n)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_n^{2\times 2} \tag{244}$$

describing the action of ϕ on a basis $\{T_1, T_2\}$ of $\mathcal{E}[n]$; i.e.,

$$\phi(T_1) = aT_1 + cT_2 \quad \text{and} \quad \phi(T_2) = bT_1 + dT_2. \tag{245}$$

- We have

$$\det\left(\phi^{(n)}\right) \equiv \deg(\phi) \mod n. \tag{246}$$

Proof.

We have that $\xi = e_n(T_1, T_2)$ is primitive n-th root of unity. Then by (245) and the properties of the Weil pairing,

$$\begin{aligned}
\xi^{\deg(\phi)} &= e_n(\phi(T_1), \phi(T_2)) \\
&= e_n(aT_1 + cT_2, bT_1 + dT_2) \\
&= e_n(aT_1, bT_1)e_n(aT_1, dT_2)e_n(cT_2, bT_1)e_n(cT_2, dT_2) \\
&= e_n(T_1, T_1)^{ab}e_n(T_1, T_2)^{ad}e_n(T_2, T_1)^{cb}e_n(T_2, T_2)^{cd} \\
&= 1^{ab}e_n(T_1, T_2)^{ad}e_n(T_2, T_1)^{cb}1^{cd} \\
&= e_n(T_1, T_2)^{ad}e_n(T_2, T_1)^{cb} \\
&= \xi^{ad-bc}.
\end{aligned}$$

By (244), we have $\det\left(\phi^{(n)}\right) = ad - bc$. Since ξ is a primitive root of unity, $\deg(\phi) \equiv \det\left(\phi^{(n)}\right) \mod n$. □

Endomorphisms

Let ϕ, ψ be endomorphisms of an elliptic curve $\mathcal{E}$ over $\mathbb{K}$ and let r, s be integers. The mapping $r\phi + s\psi$ given by

$$\begin{aligned} (r\phi + s\psi)(P) &= r\phi(P) + s\psi(P) \\ &= [r](\phi(P)) + [s](\psi(P)) \end{aligned} \tag{247}$$

is an endomorphism of $\mathcal{E}$.

Proof.

This mapping is a group homomorphism, since $\phi, \psi, [r], [s]$ are group homomorphisms and the composition of group homomorphisms is also a group homomorphism.

Each step of the computation (applying ϕ, ψ to P, applying $[r], [s]$, and addition) can be described by rational functions (for ϕ, ψ this holds by definition, for $[r], [s]$ this holds by (232), and for addition this holds by the coordinate representation) and so the composition is given by rational functions. □

Endomorphisms

Let ϕ, ψ be endomorphisms of an elliptic curve $\mathcal{E}$ over $\mathbb{K}$ and let r, s be integers. Then

$$\begin{aligned} &\deg(r\phi + s\psi) \\ &= \quad r^2 \deg\phi + s^2 \deg\psi + rs[\deg(\phi+\psi) - \deg\phi - \deg\psi]. \end{aligned} \tag{248}$$

Proof.

Let $n \geq 2$ be not divisible by $\mathrm{char}(\mathbb{K})$. Represent ϕ and ψ by transformation matrices $\phi^{(n)}$ and $\psi^{(n)}$ w.r.t. some basis of $\mathcal{E}[n]$. Transformation matrix $r\phi^{(n)} + s\psi^{(n)}$ describes the action of $r\phi + s\psi$ on $\mathcal{E}[n]$ as in (244,245).
By straightforward calculation,

$$\begin{aligned}\det\left(r\phi^{(n)} + s\psi^{(n)}\right) &= r^2 \det\phi^{(n)} + s^2 \det\psi^{(n)} + \\ &\quad rs\left[\det\left(\phi^{(n)} + \psi^{(n)}\right) - \det\phi^{(n)} - \det\psi^{(n)}\right].\end{aligned}$$

Thus by (246),

$$\begin{aligned}\deg(r\phi + s\psi) &\equiv r^2 \deg\phi + s^2 \deg\psi + \\ &\quad rs[\deg(\phi + \psi) - \deg\phi - \deg\psi] \mod n.\end{aligned}$$

Since this holds for infinitely many n, it must be an equality (see (249)). □

Residue Class Computation

Let a, b be integers. If

$$a \equiv b \mod n \tag{249}$$

holds for infinitely many moduli $n \geq 2$, then $a = b$.

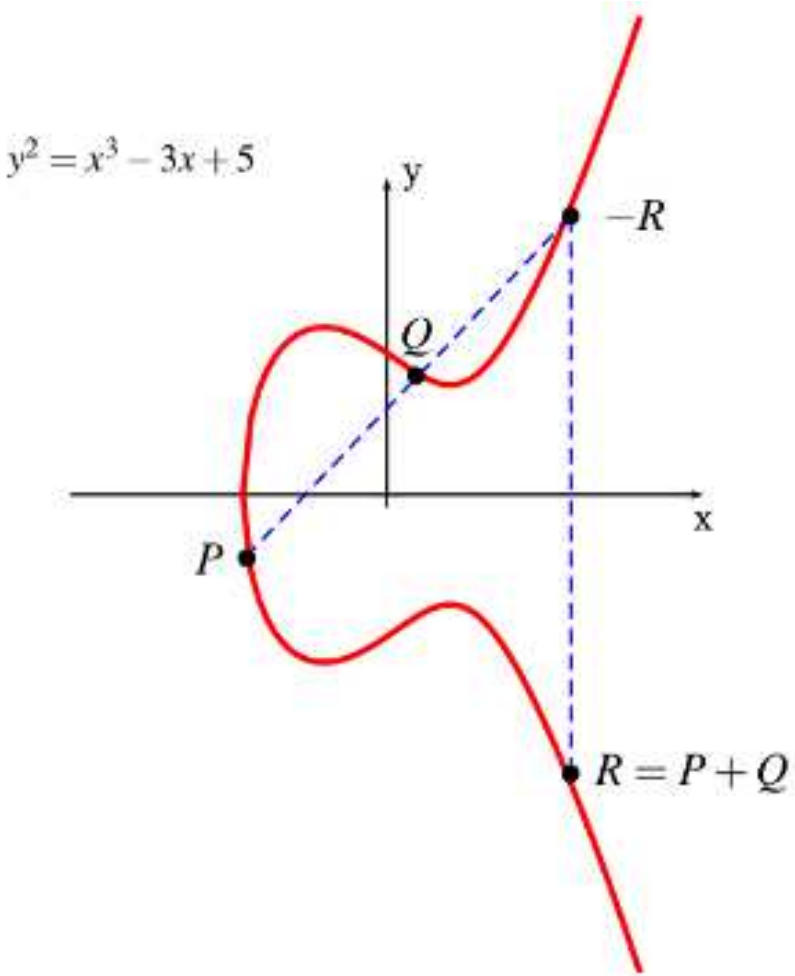
Proof.

By hypothesis, n divides $a - b$ for infinitely many n.
Consider the unique factorization of $a - b$ into prime powers,

$$a - b = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}.$$

Then the number of divisors of $a - b$ is $n_1 n_2 \cdots n_r$ and so finite.
Hence, $a - b = 0$, i.e., $a = b$. □

Divisors



Divisors

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$.

- For each point $P \in \mathcal{E}(\bar{\mathbb{K}})$ define the formal symbol $[P]$.
- A *divisor* D on $\mathcal{E}$ is a finite linear combination of such symbols with integer coefficients; i.e.,

$$D = \sum_{\substack{i \\ \text{finite}}} a_i [P_i], \quad a_i \in \mathbb{Z}. \tag{250}$$

- The set of all divisors on $\mathcal{E}$ forms a free abelian group generated by the symbols $[P]$, written $\mathrm{Div}(\mathcal{E})$.

Definition of Weil pairing makes use of divisors.

Divisors

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$.

- The *degree* of a divisor D is

$$\deg(D) = \deg\left(\sum_i a_i[P_i]\right) = \sum_i a_i \in \mathbb{Z}. \tag{251}$$

- The *sum* of a divisor D is

$$\mathrm{sum}(D) = \mathrm{sum}\left(\sum_i a_i[P_i]\right) = \sum_i a_i P_i \in \mathcal{E}(\bar{\mathbb{K}}). \tag{252}$$

Divisors – Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_{11}$ given by

$$Y^2 = X^3 + 3X + 25.$$

This curve has nine points and discriminant $\Delta = 1$.
Points on the curve are $P = (4 : 2 : 1)$ and $Q = (1 : 3 : 1)$.

- Divisor
$$D = [P] + 2[Q]$$
has degree $1 + 2 = 3$.
- Sum
$$P + 2Q = (0 : 7 : 1).$$

Divisors

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$.

- The mapping $\deg : \mathrm{Div}(\mathcal{E}) \to \mathbb{Z}$ is a surjective homomorphism with kernel

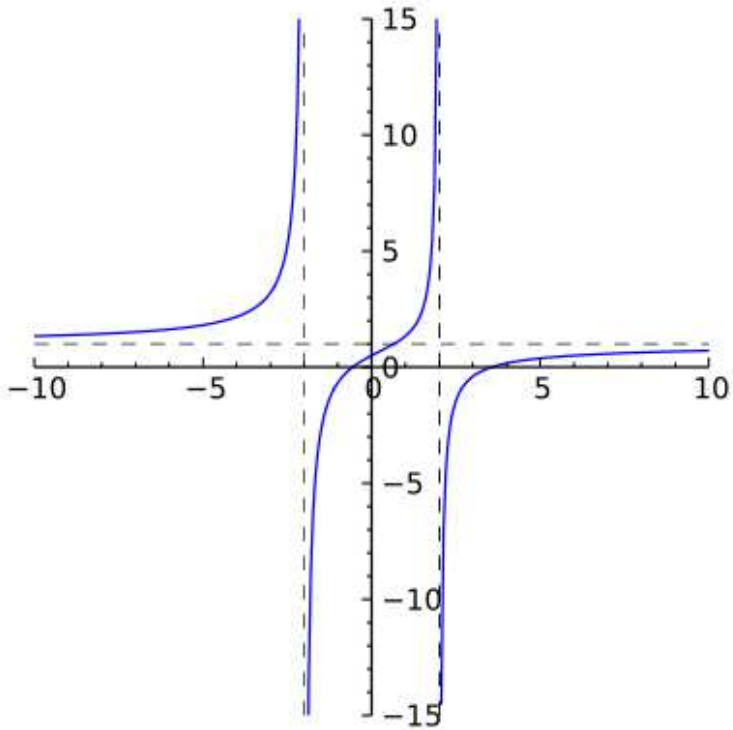
$$\mathrm{Div}^0(\mathcal{E}) = \{D \in \mathrm{Div}(\mathcal{E}) \mid \deg(D) = 0\}. \tag{253}$$

  $\mathrm{Div}^0(\mathcal{E})$ is a subgroup of $\mathrm{Div}(\mathcal{E})$.

- The mapping $\mathrm{sum} : \mathrm{Div}(\mathcal{E}) \to \mathcal{E}(\bar{\mathbb{K}})$ is a surjective homomorphism.

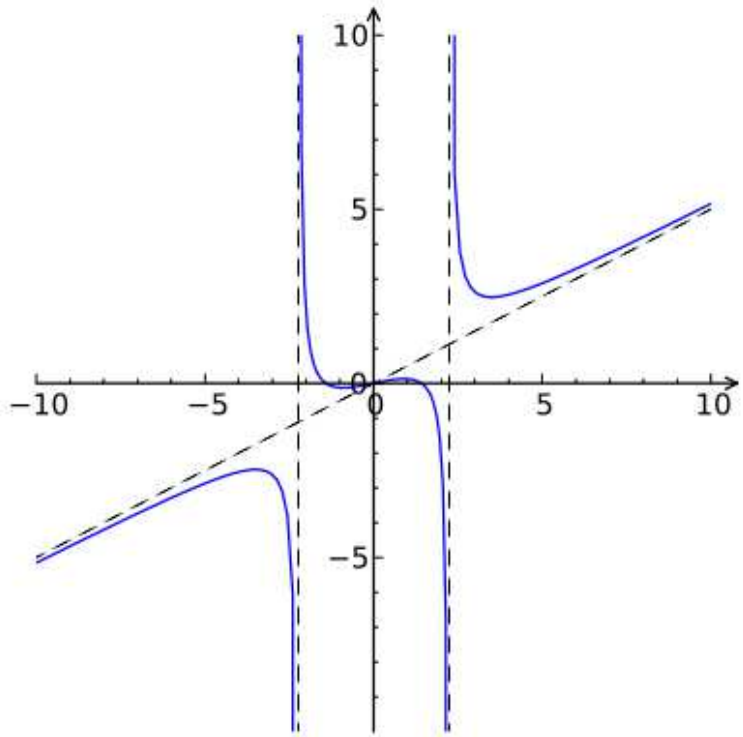
Rational Functions

The rational function $f(X) = \frac{X^2-3X-2}{X^2-4}$ of degree 2 has zeros $\frac{3}{2} \pm \frac{1}{2}\sqrt{17}$ and poles ± 2. It is asymptotic to 1 as x approaches infinity.



Rational Functions

The rational function $f(X) = \frac{X^3-2X}{2(X^2-5)}$ of degree 3 has zeros $0, \pm\sqrt{2}$ and poles $\pm\sqrt{5}$. It is asymptotic to $\frac{x}{2}$ as x approaches infinity.



Divisors

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$.

- A *function* on $\mathcal{E}$ is a rational function

$$f(X,Y) = \frac{r(X,Y)}{s(X,Y)} \in \bar{\mathbb{K}}(X,Y) \tag{254}$$

  which is defined for at least one point $P \in \mathcal{E}(\bar{\mathbb{K}})$.
- The rational function $f(X,Y)$ has values in $\bar{\mathbb{K}} \cup \{\infty\}$.
- If $\mathcal{E}$ is defined by $Y^2 = X^3 + AX + B$, the function $1/(Y^2 - X^3 - AX - B)$ is not allowed.

Divisors

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$ and $P \in \mathcal{E}(\bar{\mathbb{K}})$.

- A function f has a *zero* at P if $f(P) = 0$.
- A function f has a *pole* at P if $f(P) = \infty$.
- There exists a function u_P on $\mathcal{E}$, called *uniformizer* at P, with $u_P(P) = 0$ and such that each function f on $\mathcal{E}$ with $f(P) = 0$ has the form

$$f = u_P^r g, \tag{255}$$

where $r \in \mathbb{Z}$ and $g(P) \neq 0, \infty$.

- The *order* of the function f at P is

$$\mathrm{ord}_P(f) = r. \tag{256}$$

This number is independent of the choice of the uniformizer at P.

Divisors – Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{K}$ given by $Y^2 = X^3 - X$.

- The function $u_P = Y$ is a uniformizer at $P = (0 : 0 : 1)$.
- We have

$$X = Y^2 \frac{1}{X^2 - 1}$$

and $X^2 - 1$ is nonzero and finite at P. Thus

$$\mathrm{ord}_P(X) = 2.$$

Similarly,

$$\frac{X}{Y} = Y \frac{1}{X^2 - 1}$$

and so

$$\mathrm{ord}_P(X/Y) = 1.$$

Divisors

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{K}$ given by $Y^2 = X^3 + AX + B$.

- For each affine point $P = (a : b : 1) \in \mathcal{E}(\bar{\mathbb{K}})$, a uniformizer at P is
$$u_P = \begin{cases} X - a & \text{if } b \neq 0, \\ Y & \text{otherwise.} \end{cases}$$
- For the base point $O = (0 : 1 : 0)$, a uniformizer at O is
$$u_O = \frac{X}{Y}.$$

Divisors – Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{K}$ given by $Y^2 = X^3 + 72$.

- For the point $P = (-2 : 8 : 1) \in \mathcal{E}$, a uniformizer is $u_P = X + 2$.
- The function $f(X, Y) = X + Y - 6$ vanishes at P (but at what order?).

  For this, rewrite the equation of the curve as

$$(Y + 8)(Y - 8) = (X + 2)^3 - 6(X + 2)^2 + 12(X + 2).$$

  Then

$$\begin{aligned} f(X, Y) &= (X + 2) + (Y - 8) \\ &= u_P \cdot \left(1 + \frac{(X + 2)^2 - 6(X + 2) + 12}{Y + 8}\right). \end{aligned}$$

  The function in parenthesis is finite and does not vanish at P, so $\mathrm{ord}_P(f) = 1$.

Divisors

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$ and $f \neq 0$ be a function on $\mathcal{E}$.

- The *divisor* of f is defined as

$$\operatorname{div}(f) = \sum_{P \in \mathcal{E}(\bar{\mathbb{K}})} \operatorname{ord}_P(f)[P] \in \operatorname{Div}(\mathcal{E}), \tag{257}$$

  called the *principal divisor* of f. Since f has only finitely many zeros and poles, $\operatorname{div}(f)$ is well-defined (i.e., sum is finite).

- f has as many zeros as poles when counted with multiplicities, i.e.,

$$\deg(\operatorname{div}(f)) = 0, \quad \text{i.e., } \operatorname{div}(f) \in \operatorname{Div}^0(\mathcal{E}). \tag{258}$$

- If f has no zeros or poles, then f is a constant and so $\operatorname{div}(f) = 0$.

(See Washington, section 11.1)

Divisors – Example

Consider an elliptic curve $\mathcal{E}$ over $\mathbb{K}$.

- Suppose P_1, P_2, P_3 are three collinear points on $\mathcal{E}$.
- There exists a homogeneous linear polynomial

$$\ell = \alpha X + \beta Y + \gamma Z$$

  with zeros P_1, P_2, P_3.
- If $\beta \neq 0$, then ℓ has the triple pole at O with

$$\operatorname{div}(\ell) = [P_1] + [P_2] + [P_3] - 3[O].$$

Divisors – Example (cont'd)

- The line through $P_3 = (a_3 : b_3 : 1)$ and $-P_3 = (a_3 : -b_3 : 1)$ is $X - a_3$ with divisor

$$\operatorname{div}(X - a_3) = [P_3] + [-P_3] - 2[O].$$

Thus

$$\begin{aligned} \operatorname{div}\left(\frac{\alpha X + \beta Y + \gamma Z}{X - a_3}\right) & \\ = \quad & \operatorname{div}(\alpha X + \beta Y + \gamma Z) - \operatorname{div}(X - a_3) \\ = \quad & [P_1] + [P_2] + [P_3] - ([P_3] + [-P_3] - 2[O]) \\ = \quad & [P_1] + [P_2] - [-P_3] - [O]. \end{aligned}$$

- Since $P_1 + P_2 = -P_3$ on $\mathcal{E}$, we obtain

$$[P_1] + [P_2] = [P_1 + P_2] + [O] + \operatorname{div}\left(\frac{\alpha X + \beta Y + \gamma Z}{X - a_3}\right).$$

Divisors

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$ and D be a divisor on $\mathcal{E}$ with $\deg(D) = 0$.
There exists a function f on $\mathcal{E}$ such that

$$\operatorname{div}(f) = D \quad \Longleftrightarrow \quad \operatorname{sum}(D) = O. \tag{259}$$

The proof provides an algorithm that finds for each divisor D with $\deg(D) = 0$ and $\operatorname{sum}(D) = O$ a function f on $\mathcal{E}$ such that $\operatorname{div}(f) = D$.

Proof.

- The above example shows that

$$[P_1] + [P_2] = [P_1 + P_2] + [O] + \operatorname{div}(g)$$

for some function g on $\mathcal{E}$. Moreover,

$$\operatorname{sum}(\operatorname{div}(g)) = P_1 + P_2 - (P_1 + P_2) - O = O.$$

- In particular, if $P_1 + P_2 = O$, then

$$[P_1] + [P_2] = 2[O] + \operatorname{div}(g)$$

for some function g on $\mathcal{E}$. Moreover,

$$\operatorname{sum}(\operatorname{div}(g)) = P_1 + P_2 - 2O = O.$$

Proof (cont'd).

- Therefore, the sum of all terms in D with positive coefficients equals a single symbol $[P]$ plus a multiple of $[O]$ plus a divisor of a function. A similar result holds for the sum of the terms with negative coefficients.
- Thus there are points P and Q on $\mathcal{E}$, a function h on $\mathcal{E}$, and an integer n such that

$$D = [P] - [Q] + n[O] + \operatorname{div}(h).$$

- Since h is the product of functions g with $\operatorname{sum}(\operatorname{div}(g)) = O$, it follows that

$$\operatorname{sum}(\operatorname{div}(h)) = O.$$

- By (258), $\deg(\operatorname{div}(h)) = 0$ and so by hypothesis

$$0 = \deg(D) = 1 - 1 + n + 0 = n.$$

Proof (cont'd).

- Thus
$$D = [P] - [Q] + \operatorname{div}(h)$$
and
$$\operatorname{sum}(D) = P - Q + \operatorname{sum}(\operatorname{div}(h)) = P - Q.$$

Proof:

- Let $\operatorname{sum}(D) = O$. Then $P - Q = O$ and so $D = \operatorname{div}(h)$.
- Let $D = \operatorname{div}(f)$ for some function f on $\mathcal{E}$. Then
$$[P] - [Q] = \operatorname{div}\left(\frac{f}{h}\right).$$
The next result will show that $P = Q$ and so $\operatorname{sum}(D) = O$. $\square$

Divisors

Let $P, Q \in \mathcal{E}(\bar{\mathbb{K}})$. If there exists a function h on $\mathcal{E}$ with

$$\operatorname{div}(h) = [P] - [Q],$$

then $P = Q$.

(See Washington, Lemma 11.3)

Divisors – Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{Z}_{11}$ given by $Y^2 = X^3 + 4X$. Take the divisor

$$D = [(0,0)] + [(2,4)] + [(4,5)] + [(6,3)] - 4[O].$$

We have

$$\deg(D) = 0 \quad \text{and} \quad \operatorname{sum}(D) = O.$$

Find a function f on $\mathcal{E}$ with $\operatorname{div}(f) = D$.

Divisors – Example (cont'd)

1 The line through $(0,0)$ and $(2,4)$ is $Y = 2X$ which is tangent at $(2,4)$, so

$$\operatorname{div}(Y - 2X) = [(0,0)] + 2[(2,4)] - 3[O].$$

The vertical line through $(2,4)$ is $X = 2$ and so

$$\operatorname{div}(X - 2) = [(2,4)] + [(2,-4)] - 2[O].$$

Thus

$$D = [(2,-4)] + \operatorname{div}\left(\frac{Y - 2X}{X - 2}\right) + [(4,5)] + [(6,3)] - 3[O].$$

Divisors – Example (cont'd)

2 The line through $(4,5)$ and $(6,3)$ is $Y = -X - 2$ with third point $(2,-4)$, so

$$\operatorname{div}(Y + X + 2) = [(4,5)] + [(6,3)] + [(2,-4)] - 3[O].$$

The vertical line through $(2,-4)$ is $X = 2$ and so

$$\operatorname{div}(X - 2) = [(2,-4)] + [(2,4)] - 2[O].$$

Thus

$$[(4,5)] + [(6,3)] = [(2,4)] + [O] + \operatorname{div}\left(\frac{Y + X + 2}{X - 2}\right).$$

Divisors – Example (cont'd)

Thus the divisor is

$$\begin{aligned} D &= [(2,-4)] + \operatorname{div}\left(\frac{Y-2X}{X-2}\right) + [(2,4)] + \operatorname{div}\left(\frac{Y+X+2}{X-2}\right) - 2[O] \\ &= \operatorname{div}(X-2) + \operatorname{div}\left(\frac{Y-2X}{X-2}\right) + \operatorname{div}\left(\frac{Y+X+2}{X-2}\right) \\ &= \operatorname{div}\left(\frac{(Y-2X)(Y+X+2)}{X-2}\right). \end{aligned}$$

Simplification using $Y^2 = X^3 + 4X$ gives

$$\begin{aligned} (Y-2X)(Y+X+2) &= Y^2 - XY - 2X^2 + 2Y - 4X \\ &= X^3 - XY - 2X^2 + 2Y \\ &= (X-2)(X^2 - Y) \end{aligned}$$

and so

$$D = \operatorname{div}(X^2 - Y).$$

Divisors

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{K}$.

- The mapping $\mathrm{sum} : \mathrm{Div}^0(\mathcal{E}) \to \mathcal{E}(\bar{\mathbb{K}})$ is a surjective homomorphism.

  Indeed, for any point $P \in \mathcal{E}(\bar{\mathbb{K}})$, $\deg([P] - [O]) = 1 - 1 = 0$ and $\mathrm{sum}([P] - [O]) = P - O = P$.

- We have $D \in \ker(\mathrm{sum})$ iff $\deg(D) = 0$ and $\mathrm{sum}(D) = O$ iff $D = \mathrm{div}(f)$ for some function f on $\mathcal{E}$ by (259), i.e., $\ker(\mathrm{sum})$ is the given by the group of principle divisors $P_0(\mathcal{E})$ of $\mathcal{E}$.

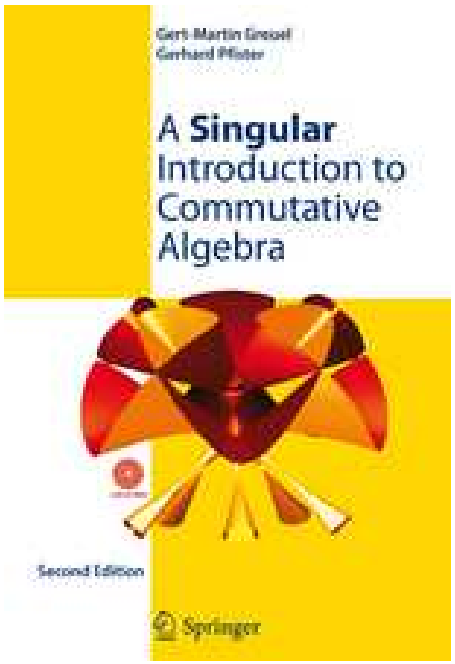
- By the First Isomorphism theorem, the induced mapping

$$\mathrm{Div}^0(\mathcal{E})/P_0(\mathcal{E}) \to \mathcal{E}(\bar{\mathbb{K}}) : D + P_0(\mathcal{E}) \mapsto \mathrm{sum}(D)$$

  is an isomorphism of groups.

- The group law on $\mathcal{E}(\bar{\mathbb{K}})$ corresponds to the group law on $\mathrm{Div}^0(\mathcal{E})$ modulo the principal divisors.

Elliptic Curves in Singular



Generation of Random Elliptic Curve

Generation of random elliptic curve over ring $\mathbb{Z}_N$.

```
> LIB "crypto.lib";
> ring r = 0,z,dp;
> ellipticRandomCurve(11); // N=11
[1]:
   5                              // a
[2]:
   8                              // b
[3]:
   1                              // discriminant
```

Returned is a list of two random numbers a, b and $\Delta = 4a^3 + 27b^2 \mod N$.
The curve is given by the equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3.$$

Test for Point Membership

Test whether a point lies on a curve:

```
> isOnCurve(11, 5, 8, list(0,1,0));
1                                   // on curve
> isOnCurve(11, 5, 8, list(5,5,1));
0                                   // not on curve
```

Generation of Random Point

Generate a random point on an elliptic curve:

```
> ellipticRandomPoint(11, 5, 8);
[1]:
   6
[2]:
   1
[3]:
   1
```

Number of Points

Provide the number of points on a curve:

```
> countPoints(11, 5, 8);
15
```

List of Points

List of all points on an elliptic curve:

```
> list L = ellipticAllPoints(11, 5, 8);
> size(L);
15
> L[1];
[1]:
   0
[2]:
   1
[3]:
   0
```

Point Addition

Addition of two points on an elliptic curve:

```
> list P,Q;
> P[1]=1;
> P[2]=5;
> P[3]=1;
> Q[1]=1;
> Q[2]=6;
> Q[3]=1;
> ellipticAdd(11, 5, 8, P, Q); // P+Q
[1]:
   0
[2]:
   1
[3]:
   0
```

Point Multiple

Multiple of a point on an elliptic curve:

```
> list P;
> P[1]=1;
> P[2]=5;
> P[3]=1;
> ellipticMult(11, 5, 8, P, 6); // 6*P
[1]:
   6
[2]:
   1
[3]:
   1
```

Point Addition

The elliptic curve $\mathcal{E}$ over $\mathbb{Z}_7$ given by

$$f = Y^2Z - X^3 - 3XZ^2 - 5Z^3$$

has the points

$$P_1 = O = [0:1:0], \quad P_2 = [1:4:1], \quad P_3 = [1:3:1],$$
$$P_4 = [4:2:1], \quad P_5 = [4:5:1],$$
$$P_6 = [6:1:1], \quad P_7 = [6:6:1].$$

Addition table:

$+$	O	P_2	P_3	P_4	P_5	P_6	P_7
O	O	P_2	P_3	P_4	P_5	P_6	P_7
P_2		P_6	O	P_5	P_7	P_4	P_3
P_3			P_7	P_6	P_4	P_2	P_5
P_4				P_3	O	P_7	P_2
P_5					P_2	P_3	P_6
P_6						P_5	O
P_7							P_4

Division Polynomials

```
> LIB "crypto.lib";
> ring r = 0, (y,x), dp;
> ellipticRandomCurve(11);
[1]:
   4
[2]:
   9
[3]:
   1
> generateG(4,9,2);
2y
> generateG(4,9,3);
3x4+24x2+108x-16
```

Part VI

Elliptic Curves over Finite Fields

Elliptic Curves over Finite Fields

- Group structure
- Frobenius endomorphism
- Bounds on group order
- Group order
- Zeta function
- Supersingular curves
- Schoof's algorithm

Group Structure

Niels Henrik Abel (1802-1829)

Group Structure

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$. Then

$$\mathcal{E}(\mathbb{F}_q) \simeq \mathbb{Z}_n \quad \text{or} \quad \mathcal{E}(\mathbb{F}_q) \simeq \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \tag{260}$$

for some integer $n \geq 1$ or some integers $n_1, n_2 \geq 1$ with $n_1 \mid n_2$.

Proof.

- Each finite abelian group is isomorphic to a direct sum of cyclic groups
$$\mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k},$$
where $n_1, \ldots, n_k$ are integers with $n_i \mid n_{i+1}$ for all i.
- Since $\mathbb{Z}_{n_i}$ has n_1 elements of order dividing n_1, $\mathcal{E}(\mathbb{F}_q)$ has n_1^k elements of order dividing n_1.
- By the result on $\mathcal{E}[n_1]$ in (209,210) there are at most n_1^2 such points and so $k \leq 2$.
- Thus $\mathcal{E}(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ with $n_1 \mid n_2$ or by (214) isomorphic to $\mathbb{Z}_n$.

□

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_5$ given by $Y^2 = X^3 + X + 1$.

The affine $\mathbb{F}_5$-rational points are

x	$x^3 + x + 1$	y	Points
0	1	± 1	$(0:1:1), (0:4:1)$
1	3		
2	1	± 1	$(2:1:1), (2:4:1)$
3	1	± 1	$(3:1:1), (3:4:1)$
4	4	± 2	$(4:2:1), (4:3:1)$

Note that the squares modulo 5 are 1 and 4, since $1^2 = 1$, $2^2 = 4$, $3^2 = 4$, and $4^2 = 1$, but 3 is not a square modulo 5.

Together with the base point O, $\mathcal{E}(\mathbb{F}_5)$ has order 9.

The group $\mathcal{E}(\mathbb{F}_5)$ is cyclic with generator $P = (0:1:1)$, i.e., $\mathcal{E}(\mathbb{F}_5) \simeq \mathbb{Z}_9$.

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_7$ given by $Y^2 = X^3 + 2$.

The $\mathbb{F}_7$-rational points are

$$\begin{array}{lll} (0:1:0), & (0:3:1), & (0:4:1), \\ (3:1:1), & (3:6:1), & (5:1:1), \\ (5:6:1), & (6:1:1), & (6:6:1). \end{array}$$

All points P satisfy $3P = O$. Thus $\mathcal{E}(\mathbb{F}_7) \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Check via Singular:

```
> LIB "crypto.lib";
> countPoints(7,0,2);
> list L = ellipticAllPoints(7,0,2);
> ellipticMult(7,0,2,L[2],3);
```

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_2$ given by $Y^2 + XY = X^3 + 1$.

The $\mathbb{F}_2$-rational points are

$$(0:1:0),\ (1:0:1),\ (0:1:1),\ (1:1:1).$$

The group $\mathcal{E}(\mathbb{F}_2)$ is cyclic of order 4, i.e., $\mathcal{E}(\mathbb{F}_2) \simeq \mathbb{Z}_4$.

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_4$ given by $Y^2 + XY = X^3 + 1$.

Write $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ with $\alpha^2 + \alpha + 1 = 0$.

The $\mathbb{F}_4$-rational points are

$$\begin{array}{llll} (0:1:0), & (0:1:1), & (1:0:1), & (1:1:1), \\ (\alpha:0:1), & (\alpha:\alpha:1), & (\alpha^2:0:1), & (\alpha^2:\alpha^2:1). \end{array}$$

The group $\mathcal{E}(\mathbb{F}_2)$ is cyclic of order 8, i.e., $\mathcal{E}(\mathbb{F}_4) \simeq \mathbb{Z}_8$.

Frobenius Endomorphism

Evariste Galois (1811-1832)

Frobenius Endomorphism

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$.

Recall that the mapping $\phi_q : \mathcal{E}(\bar{\mathbb{F}}_q) \to \mathcal{E}(\bar{\mathbb{F}}_q)$ given by

$$(a : b : c) \mapsto (a^q : b^q : c^q) \tag{261}$$

is a nonseparable endomorphism of degree q, called *Frobenius endomorphismus*, as described in (192).

Frobenius Endomorphism

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$ and let $(x, y) \in \mathcal{E}(\bar{\mathbb{F}}_q)$. Then

$$\phi_q(x, y) \stackrel{\text{def}}{=} (x^q, y^q) \in \mathcal{E}(\bar{\mathbb{F}}_q) \tag{262}$$

and

$$(x, y) \in \mathcal{E}(\mathbb{F}_q) \quad \Longleftrightarrow \quad \phi_q(x, y) = (x, y). \tag{263}$$

Proof.

- Suppose the affine point (x, y) satisfies the Weierstrass equation over $\mathbb{F}_q$,

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Applying the Frobenius map $z \mapsto z^q$ gives

$$(y^q)^2 + a_1x^qy^q + a_3y^q = (x^q)^3 + a_2(x^q)^2 + a_4x^q + a_6,$$

since $a_1, a_2, a_3, a_5, a_6 \in \mathbb{F}_q$. Thus $(x^q, y^q) \in \mathcal{E}(\bar{\mathbb{F}}_q)$.

Proof (cont'd).

- We have

$$\begin{aligned}(x,y) \in \mathcal{E}(\mathbb{F}_q) &\iff (x,y) \in \mathcal{E}(\bar{\mathbb{F}}_q) \wedge x, y \in \mathbb{F}_q \\ &\iff (x,y) \in \mathcal{E}(\bar{\mathbb{F}}_q) \wedge x^q = x,\ y^q = y \\ &\iff \phi_q(x,y) \stackrel{\text{def}}{=} (x^q, y^q) = (x,y).\end{aligned}$$

□

Frobenius Endomorphism

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$, and let $n \geq 1$.

The n*th iterate* of the Frobenius endomorphism

$$\phi_q^n : \mathcal{E}(\bar{\mathbb{F}}_q) \to \mathcal{E}(\bar{\mathbb{F}}_q) \tag{264}$$

is given by

$$\phi_q^n(a : b : c) = (a^{q^n} : b^{q^n} : c^{q^n}). \tag{265}$$

It is a nonseparable endomorphism of degree q^n.

Proof.

- The Frobenius endomorphism $\phi_q(X, Y) = (X^q, Y^q)$ is an endomorphism.
- The iterate ϕ_{q^n} is also an endomorphism, by iteration of the proof of the Frobenius endomorphism.
- The iterate $\phi_q^n(X, Y) = (X^{q^n}, Y^{q^n})$ has degree q^n, as X^{q^n} has degree q^n, and is nonseparable, since the derivative of $r(X) = X^{q^n}$ is $r'(X) = q^n X^{q^n - 1} = 0$.

Frobenius Endomorphism

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$ and let $n \geq 1$. Then

$$\ker(\phi_q^n - [1]) = \mathcal{E}(\mathbb{F}_{q^n}) \tag{266}$$

and $\phi_q^n - [1]$ is a separable endomorphism, so

$$\sharp\mathcal{E}(\mathbb{F}_{q^n}) = \deg(\phi_q^n - [1]). \tag{267}$$

Note that for each $P \in \mathbb{P}^2(\bar{\mathbb{F}}_q)$,

$$(\phi_q^n - [1])(P) = \phi_q^n(P) - P, \tag{268}$$

where $[1]$ is the identity mapping.

Proof.

- Let $(x, y) \in \mathcal{E}(\bar{\mathbb{F}}_{q^n})$. Then by (263),

$$(x, y) \in \mathcal{E}(\mathbb{F}_{q^n}) \quad \Longleftrightarrow \quad \phi_q^n(x, y) = (x, y)$$

and by (268),

$$\phi_q^n(x, y) = (x, y) \quad \Longleftrightarrow \quad (x, y) \in \ker(\phi_q^n - [1]).$$

- By (205), the endomorphism $\phi_q^n + [-1] = \phi_q^n - [1]$ is separable, since $p \not| -1$. Thus by (266) and (185),

$$\sharp\mathcal{E}(\mathbb{F}_{q^n}) = \sharp \ker(\phi_q^n - [1]) = \deg(\phi_q^n - [1]).$$

Bounds on Group Order

Helmut Hasse (1898-1979)

Bound on Group Order

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$. Then

$$\sharp\mathcal{E}(\mathbb{F}_q) \leq 2q + 1. \tag{269}$$

Proof.

- The curve contains the base point O which is the only point on the curve not in affine space $\mathbb{A}^2(\mathbb{F}_q)$.
- For the remaining points it is sufficient to consider the affine Weierstrass equation $f^a(X, Y) = 0$ over $\mathbb{F}_q$.
- Since the equation $f^a(X, Y) = 0$ is quadratic in Y, for each $X = x \in \mathbb{F}_q$ there are at most two solutions $Y = \pm y \in \mathbb{F}_q$. So there are at most $2q$ affine points on the curve.

□

The Famous Hasse Bound

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$. Then

$$|q + 1 - \sharp\mathcal{E}(\mathbb{F}_q)| \leq 2\sqrt{q}. \tag{270}$$

Proof.

By (267), we have

$$t \stackrel{\text{def}}{=} q + 1 - \sharp\mathcal{E}(\mathbb{F}_q) = q + 1 - \deg(\phi_q - [1]). \tag{271}$$

Proof (cont'd).

- For integers r, s with $(s, q) = 1$, we have

$$\deg([r]\phi_q - [s]) = r^2 q + s^2 - rst. \tag{272}$$

Indeed, by (248),

$$\begin{aligned}\deg(r\phi_q + s[-1]) &= r^2 \deg \phi_q + s^2 \deg[-1] + \\ &\quad rs\{\deg(\phi_q + [-1]) - \deg \phi_q - \deg[-1]\}.\end{aligned}$$

Since $\deg \phi_q = q$ and $\deg[-1] = 1$ as $[-1](X, Y) = (X, \cdot)$, by (271),

$$\begin{aligned}&rs\{\deg(\phi_q + [-1]) - \deg \phi_q - \deg[-1]\} \\ &= rs \cdot (\deg(\phi_q + [-1]) - q - 1) \\ &= rs \cdot (-t).\end{aligned}$$

Proof (cont'd).

- We have $\deg([r]\phi_q - [s]) = r^2 q + s^2 - rst$.
- Since $\deg([r]\phi_q - [s]) \geq 0$, multiplication with $1/s^2$ gives

$$q\left(\frac{r}{s}\right)^2 - t\left(\frac{r}{s}\right) + 1 \geq 0 \tag{273}$$

  for all r, s with $(s, q) = 1$.
- The set of rational numbers r/s with $(s, q) = 1$ is dense in $\mathbb{R}$. Indeed, the rational numbers of the form $r/2^m$ or $r/3^m$ (choose according to q) are dense in $\mathbb{R}$.
- By a limiting process in (273),

$$qx^2 - tx + 1 \geq 0$$

  for all real numbers x. Thus the discriminant Δ of the polynomial $qX^2 - tX + 1$ is ≤ 0, i.e., $\Delta = t^2 - 4q \leq 0$. Hence, $|t| \leq 2\sqrt{q}$.

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_4$ given by $Y^2 = X^3 - Y$.

Write $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ with $\alpha^2 + \alpha + 1 = 0$.

The $\mathbb{F}_4$-rational points are

$$\begin{array}{lll} (0:1:0), & (0:0:1), & (0:1:1), \\ (1:\alpha:1), & (1:\alpha^2:1), & (\alpha:\alpha:1), \\ (\alpha:\alpha^2:1), & (\alpha^2:\alpha:1), & (\alpha^2:\alpha^2:1). \end{array}$$

Thus $\sharp\mathcal{E}(\mathbb{F}_4) = 9$ and so

$$|4 + 1 - 9| = 4 = 2\sqrt{4}.$$

Hence, the Hasse bound (270) is attained with equality!

Frobenius Trace

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$ and let

$$t = q + 1 - \sharp\mathcal{E}(\mathbb{F}_q). \tag{274}$$

Then

$$\phi_q^2 - [t]\phi_q + [q] = 0 \tag{275}$$

is an endomorphism of $\mathcal{E}$ and t is the unique integer such that this equality holds.

By (275), for each $P = (x : y : 1) \in \mathcal{E}(\bar{\mathbb{F}}_q)$,

$$\begin{aligned} &\left(\phi_q^2 - [t]\phi_q + [q]\right)(P) \\ &= \left(x^{q^2} : y^{q^2} : 1\right) - t(x^q : y^q : 1) + q(x : y : 1) = O. \end{aligned} \tag{276}$$

Proof.

If the endomorphism $\psi = \phi_q^2 - [t]\phi_q + [q]$ is not 0, its kernel is finite as $\deg \psi$ is finite and by (185,186) $\deg \psi \geq \sharp \ker(\psi)$. We show that the kernel is infinite and so $\psi = 0$ (trivial endomorphism).

- Let $m \geq 1$ with $(m, q) = 1$. The endomorphism ϕ_q induces a transformation matrix $\phi_q^{(m)}$ describing the action of ϕ_q on $\mathcal{E}[m]$ as in (218). Write

$$\phi_q^{(m)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

- Since $\phi_q - [1]$ is separable by (205), we obtain

$$\begin{aligned} \sharp \ker(\phi_q - [1]) &= \deg(\phi_q - [1]) \text{ by (185)}, \\ &\equiv \det\left(\phi_q^{(m)} - I\right) \text{ by (246)}, \\ &= ad - bc - (a + d) + 1 \mod m. \end{aligned}$$

Proof (cont'd)

- By (246), $ad - bc = \det\left(\phi_q^{(m)}\right) \equiv \deg(\phi_q) = q \mod m$.
- By (266, 274), $\sharp \ker(\phi_q - [1]) = \sharp \mathcal{E}(\mathbb{F}_q) = q + 1 - t$ and so by the last three identities,

$$\operatorname{trace}\left(\phi_q^{(m)}\right) = a + d \equiv t \mod m.$$

- By the Cayley-Hamilton theorem of linear algebra,

$$A^2 - \operatorname{trace}(A)A + \det(A)I = 0$$

for any 2×2 matrix A and so for $A = \phi_q^{(m)}$,

$$\left(\phi_q^{(m)}\right)^2 - t\phi_q^{(m)} + qI \equiv 0 \mod m.$$

Thus the endomorphism $\psi = \phi_q^2 - [t]\phi_q + [q]$ is identically zero on $\mathcal{E}[m]$. Since there are infinitely many choices for m, the kernel of ψ is infinite as claimed.

Proof (cont'd)

- Let t, t' be distinct integers with

$$\phi_q^2 - [t]\phi_q + [q] = 0 \quad \text{and} \quad \phi_q^2 - [t']\phi_q + [q] = 0.$$

Then

$$[t - t']\phi_q = (\phi_q^2 - [t']\phi_q + [q]) - (\phi_q^2 - [t]\phi_q + [q]) = 0.$$

Since $\phi_q : \mathcal{E}(\bar{\mathbb{F}}_q) \to \mathcal{E}(\bar{\mathbb{F}}_q)$ is surjective, the endomorphism $[t - t']$ annihilates $\mathcal{E}(\bar{\mathbb{F}}_q)$ and thus also $\mathcal{E}[m]$ for each $m \geq 1$. If $(m, q) = 1$, then by (209), $\mathcal{E}[m]$ has points P of order m. For such a point P, $[t - t'](P) = (t - t')P = O$ and so $t - t'$ is multiple of m, i.e., $t - t' \equiv 0 \mod m$. Since there are infinitely many such m, $t - t' = 0$ and so t is unique as described in (249). □

Frobenius Trace

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$ and let $t = q + 1 - \sharp\mathcal{E}(\mathbb{F}_q)$.
Let $m \geq 1$ with $(m, q) = 1$ and let $\phi_q^{(m)}$ be the transformation matrix describing the action of ϕ_q on $\mathcal{E}[m]$.
Then

$$\operatorname{trace}\left(\phi_q^{(m)}\right) \equiv t \mod m \tag{277}$$

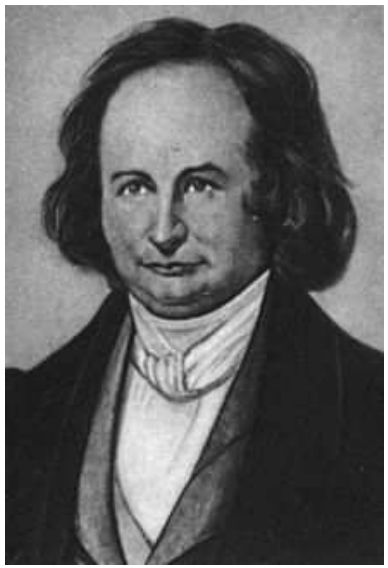
and

$$\det\left(\phi_q^{(m)}\right) \equiv q \mod m. \tag{278}$$

See former proof.

The number t is the *Frobenius trace* of ϕ_q and $X^2 - tX + q$ is the *characteristic polynomial of Frobenius*.

Group Order



Carl Jacobi (1804-1851)

Example (Maple)

Consider the algebraic curves $\mathcal{E}$ over $\mathbb{F}_{11}$ given by $Y^2 = X^3 + AX + B$ with $A, B \in \mathbb{F}_{11}^*$.
The number of $\mathbb{F}_{11}$-rational points of $\mathcal{E}$ depending on A, B were calculated by `countPoints`:

$A\backslash B$	1	2	3	4	5	6	7	8	9	10
1	14	16	7	9	11	13	15	17	8	10
2	16	9	13	6	10	14	18	11	15	8
3	7	13	8	14	9	15	10	16	11	17
4	9	17	14	11	8	16	13	10	7	15
5	11	10	9	8	7	17	16	15	14	13
6	13	14	15	16	6	18	8	9	10	11
7	15	18	10	13	16	8	11	14	6	9
8	6	11	16	10	15	9	14	8	13	18
9	8	15	11	7	14	10	17	13	9	16
10	10	8	6	15	13	11	9	18	16	14

The curves with the following parameters (A, B) are singular:
$(2, 3), (2, 8), (6, 1), (6, 10), (7, 4), (7, 7), (8, 2), (8, 9), (10, 5), (10, 6)$.

Group Order (Using Frobenius Trace)

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$. Write

$$t = q + 1 - \sharp\mathcal{E}(\mathbb{F}_q) \tag{279}$$

and

$$X^2 - tX + q = (X - \alpha)(X - \beta) \tag{280}$$

over $\mathbb{C}$ such that by comparing coefficients,

$$t = \alpha + \beta \quad \text{and} \quad q = \alpha\beta. \tag{281}$$

Then for each $n \geq 1$,

$$\sharp\mathcal{E}(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n). \tag{282}$$

Thus if $\sharp\mathcal{E}(\mathbb{F}_q)$ is known, then $\sharp\mathcal{E}(\mathbb{F}_{q^n})$ can be calculated.

Group Order (Using Frobenius Trace)

For each $n \geq 0$, put

$$s_n = \alpha^n + \beta^n. \tag{283}$$

Then

$$s_0 = 2,\ s_1 = t,\ \text{and } s_{n+1} = ts_n - qs_{n-1} \text{ for } n \geq 1. \tag{284}$$

Proof.

We have $s_0 = \alpha^0 + \beta^0 = 2$ and $s_1 = \alpha + \beta = t$ by (281).

Let $n \geq 2$. Since α is a zero of $X^2 - tX + q$, we have

$$\alpha^2 - t\alpha + q = 0.$$

Multiplying with α^{n-1} gives

$$\alpha^{n+1} - t\alpha^n + q\alpha^{n-1} = 0.$$

Proof (cont'd)

We have

$$\alpha^{n+1} - t\alpha^n + q\alpha^{n-1} = 0.$$

Similarly,

$$\beta^{n+1} - t\beta^n + q\beta^{n-1} = 0.$$

Adding both relations gives

$$(\alpha^{n+1} + \beta^{n+1}) - t(\alpha^n + \beta^n) + q(\alpha^{n-1} + \beta^{n-1}) = 0.$$

Thus by definition,

$$s_{n+1} - ts_n + qs_{n-1} = 0.$$

□

Since q, t are integers, the recurrence (284) shows that $\alpha^n + \beta^n$ is an integer for each $n \geq 0$.

Proof of Eq. (282).

Let

$$\begin{aligned} f(X) &= (X^n - \alpha^n)(X^n - \beta^n) \\ &= X^{2n} - (\alpha^n + \beta^n)X^n + q^n. \end{aligned}$$

The polynomial $g(X) = X^2 - tX + q = (X - \alpha)(X - \beta)$ divides $f(X)$. This can be proved using `Maple`:

```
> f(X) := X^(2*n)-(a^n+b^n)*X+q^n;
> g(X) := X^2-(a+b)*X+q;
> rem(f,g,X);
0
```

Since $f(X)$ and $g(X)$ have integer coefficients and $g(X)$ is monic, dividing $g(X)$ into $f(X)$ yields a quotient polynomial $h(X)$ with integer coefficients.

Proof (cont'd).

- Substituting the Frobenius endomorphism $X = \phi_q$ gives

$$\begin{aligned} \left(\phi_q^n\right)^2 - [\alpha^n + \beta^n]\phi_q^n + [q^n] &= f(\phi_q) \qquad (285) \\ &= h(\phi_q)(\phi_q^2 - [t]\phi_q + [q]) \\ &= 0 \end{aligned}$$

  as endomorphisms, where ϕ_q^n is the nth iterate of ϕ_q and the last equation follows from (275).

- By (275), there is exactly one integer t with

$$\left(\phi_q^n\right)^2 - [t]\phi_{q^n} + [q^n] = 0,$$

  which is determined by $t = q^n + 1 - \sharp\mathcal{E}(\mathbb{F}_{q^n})$. Thus by (285),

$$\alpha^n + \beta^n = q^n + 1 - \sharp\mathcal{E}(\mathbb{F}_{q^n}).$$

□

Example

The elliptic curve $\mathcal{E}$ over $\mathbb{F}_2$ given by $Y^2 + XY = X^3 + 1$ has $\sharp\mathcal{E}(\mathbb{F}_2) = 4$ points:

$$(0:1:0),\ (1:0:1),\ (0:1:1),\ (1:1:1).$$

Thus

$$t = q + 1 - \sharp\mathcal{E}(\mathbb{F}_2) = 2 + 1 - 4 = -1.$$

Consider the polynomial

$$X^2 - tX + q = X^2 + X + 2 = (X - \alpha)(X - \beta)$$

where

$$\alpha = \frac{-1 + \sqrt{-7}}{2} \quad \text{and} \quad \beta = \frac{-1 - \sqrt{-7}}{2}.$$

Example (cont'd)

In view of the extension field $\mathbb{F}_4$, we obtain by (282)

$$\begin{aligned} \sharp\mathcal{E}(\mathbb{F}_4) &= q^2 + 1 - (\alpha^2 + \beta^2) \\ &= 4 + 1 - \left(\frac{-1+\sqrt{-7}}{2}\right)^2 - \left(\frac{-1-\sqrt{-7}}{2}\right)^2. \end{aligned}$$

Alternatively, use the recurrence

$$s_0 = q = 2,\ s_1 = t = -1,\ s_2 = ts_1 - qs_0 = -3$$

to obtain

$$\sharp\mathcal{E}(\mathbb{F}_4) = q^2 + 1 - s_2 = 4 + 1 - (-3) = 8.$$

Example (cont'd)

Consider small extension fields of $\mathbb{F}_2$:

- Recurrence: $s_0 = q = 2$, $s_1 = t = -1$, and for $n \geq 2$,

$$s_n = ts_{n-1} - qs_{n-2} = -s_{n-1} - 2s_{n-2}.$$

- Define group order: $N_n = \sharp\mathcal{E}(\mathbb{F}_{2^n}) = 2^n + 1 - s_n$.
- `Maple` computation:

n	0	1	2	3	4	5	6	7	8	9	10
s_n	2	-1	-3	5	1	-11	9	13	-31	5	57

and

n	1	2	3	4	5	6	7	8	9	10
N_n	4	8	4	16	44	56	116	288	508	968

Example (cont'd)

Let $n = 101$. Then by using `Maple`,

$$\begin{aligned} s_{101} &= \left(\frac{-1+\sqrt{-7}}{2}\right)^{101} + \left(\frac{-1-\sqrt{-7}}{2}\right)^{101} \\ &= 2.969292316 \cdot 10^{15} \end{aligned}$$

and so

$$\begin{aligned} \sharp\mathcal{E}(\mathbb{F}_{2^{101}}) &= 2^{101} + 1 - 2.969292316 \cdot 10^{15} \\ &= 2.535301200 \cdot 10^{30}. \end{aligned}$$

Group Order (Using Hasse Bound)

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$. By the Hasse bound (270),

$$q + 1 - 2\sqrt{q} \leq \sharp\mathcal{E}(\mathbb{F}_q) \leq q + 1 + 2\sqrt{q}. \tag{286}$$

The *order* of a point $P \in \mathcal{E}(\mathbb{F}_q)$ is the smallest integer $k \geq 1$ such that $kP = O$.

- Point order divides group order: Point order of P is the order of the cyclic subgroup $\langle P \rangle$ of $\mathcal{E}(\mathbb{F}_q)$ generated by P and by Lagrange the order of subgroup $\langle P \rangle$ divides the group order.
- By (286), the group order $\sharp\mathcal{E}(\mathbb{F}_q)$ lies within an interval of length $4\sqrt{q}$.
- *Group order criterion*: If there is a point P whose order M lies in the interval

$$q + 1 - 2\sqrt{q}, \ldots, q + 1 + 2\sqrt{q} \tag{287}$$

and no other multiple of M lies in this interval, then $\sharp\mathcal{E}(\mathbb{F}_q) = M$.

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_{101}$ given by $Y^2 = X^3 + 7X + 1$.

- The point $P = (0 : 1 : 1)$ has order 116, so $\sharp\mathcal{E}(\mathbb{F}_{101})$ is a multiple of 116.
- By the Hasse bound (286),

$$101 + 1 - 2\sqrt{101} \leq \sharp\mathcal{E}(\mathbb{F}_{101}) \leq 101 + 1 + 2\sqrt{101},$$

  i.e., $82 \leq \sharp\mathcal{E}(\mathbb{F}_{101}) \leq 122$.
- The only multiple of 116 in this interval is 116 and so $\sharp\mathcal{E}(\mathbb{F}_{101}) = 116$.
- Since P has order 116, it follows that the group $\mathcal{E}(\mathbb{F}_{101})$ is cyclic with generator P.

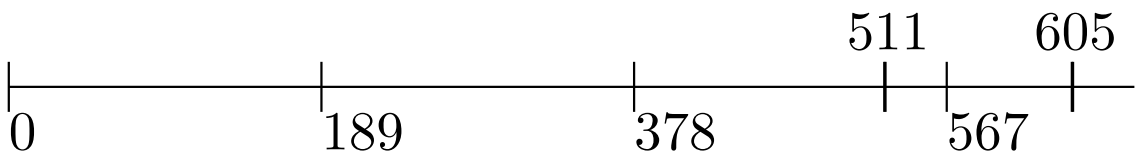
Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_{557}$ given by $Y^2 = X^3 - 10X + 21$.

- The point $P = (2 : 3 : 1)$ has order 189, so $\sharp\mathcal{E}(\mathbb{F}_{557})$ is a multiple of 189.
- By the Hasse bound (286),

$$557 + 1 - 2\sqrt{557} \leq \sharp\mathcal{E}(\mathbb{F}_{557}) \leq 557 + 1 + 2\sqrt{557},$$

  i.e., $511 \leq \sharp\mathcal{E}(\mathbb{F}_{557}) \leq 605$.
- The only multiple of 189 in this interval is $3 \cdot 189 = 567$ and so $\sharp\mathcal{E}(\mathbb{F}_{557}) = 567$.



Baby Step, Giant Step – Point Order

Consider elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ and let $O \neq P \in \mathcal{E}(\mathbb{F}_q)$.
Find the order of P.

- Let $\sharp\mathcal{E}(\mathbb{F}_q) = N$ (unknown). By Lagrange, $[N]P = O$.
- By the Hasse bound (270),

$$q + 1 - 2\sqrt{q} \leq N \leq q + 1 + 2\sqrt{q}.$$

- Try all values of N in this interval to see if $[N]P = O$; traversing the interval takes $4\sqrt{q}$ steps. Speed up by BSGS.
- The *order* of $P \neq O$ is the smallest integer $k \geq 1$ such that $[k]P = O$. Then by Lagrange $k \mid N$.

Baby Step, Giant Step – Algorithm

1. Compute $Q = [q+1]P$.
2. *Baby steps:* Take $m = \lceil \sqrt[4]{q} \rceil$, compute and store
$$[0]P, [1]P, [2]P, \ldots, [m]P.$$
3. *Giant steps:* Compute
$$Q + [k(2m)]P \quad \text{for } k = -m, -(m-1), \ldots, m-1, m$$
until there is a match $Q + [i(2m)]P = [\pm j]P$ with a point (or its negative) in the stored list.
4. Set $M = q + 1 + 2mi \mp j$. We have $[M]P = O$.
5. Factorize $M = p_1^{e_1} \cdots p_r^{e_r}$ into prime powers.
6. Compute $[M/p_i]P$ for $1 \leq i \leq r$. If $[M/p_i]P = O$ for some i, resume with the factorization of $M = M/p_i$. Otherwise, M is the order of P.

Baby Step, Giant Step – Correctness

There is a match in step 3.

Proof.

- By the Frobenius trace formula (274),

$$[q+1]P - [\sharp\mathcal{E}(\mathbb{F}_q)]P = [k]P$$

for some integer $k(=t)$ with $|k| \leq 2m^2 = 2\sqrt{q}$ by (270). Since by Lagrange, element order divides group order, $[\sharp\mathcal{E}(\mathbb{F}_q)]P = O$ and so

$$[q+1]P = [k]P.$$

Proof (cont'd)

- There are integers k_0, k_1 with $-m < k_0 \leq m$, $-m \leq k_1 \leq m$ such that

$$k = k_0 + 2mk_1.$$

Indeed, let $k_0 \equiv k \mod 2m$ with $-m < k_0 \leq m$ and $k_1 = (k - k_0)/(2m)$. Then

$$|k_1| \leq (2m^2 + m)/(2m) < m + 1.$$

- Let $k = k_0 + 2mk_1$ and $i = -k_1$. Then

$$\begin{aligned} Q + [i(2m)]P &= [q + 1 - 2mk_1]P \\ &= [q + 1 - k + k_0]P \\ &= [q + 1 - k]P + [k_0]P \\ &= O + [k_0]P = [\pm j]P, \end{aligned}$$

where $j = |k_0| \geq 0$. Thus there is a match. □

Baby Step, Giant Step – Correctness

Step 6 yields the order of P.

Let G be an additive group and let $g \in G$. Suppose $Mg = 0$ for some $M \geq 1$ and let $M = p_1^{e_1} \cdots p_r^{e_r}$ be the prime factorization. If $(M/p_i)g \neq 0$ for all i, then M is the order of g.

Proof.

Let $Mg = 0$. Suppose M is not the order of g. Then M is a multiple of the order of g. Thus there must be a prime divisor p_i of M such that $(M/p_i)g = 0$. □

Baby Step, Giant Step – Point Order

- Running time has the order of $m = \sqrt[4]{q}$ (giant steps).
- *Baby steps*: Store only the x-coordinate of $[j]P$ (along with j), since looking for a match with $[\pm j]P$ requires only the x-coordinate. When a match is found, the two possible y-coordinates can be computed.
- *Giant steps*: Precompute Q and $[2m]P$. Then $Q + [k+1]([2m]P)$ can be computed from $Q + [k]([2m]P)$ by adding $[2m]P$.
- The factorization of M may be difficult to achieve. If so, find all small prime factors p_i of M and check $[M/p_i]P \neq O$. If so, M will be a good candidate for the order of P.

Example

Reconsider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_{557}$ given by $Y^2 = X^3 - 10X + 21$ and let $P = (2 : 3 : 1) \in \mathcal{E}(\mathbb{F}_{557})$.

- $Q = [558]P = (418 : 33 : 1)$.
- Take $m = \lceil \sqrt[4]{557} \rceil = 5$ and store the points $[j]P$, $0 \leq j \leq 5$:

$$O, \quad (2 : 3 : 1), \quad (58 : 164 : 1), \quad (44 : 294 : 1),$$
$$(56 : 339 : 1), \quad (132 : 364 : 1).$$

- Match $Q + [i(2m)]P = [\pm j]P$ is given for $i = 1 = j$.
  So $[q + 1 + 2mi - j]P = [567]P = O$.
- Factorize $567 = 3^4 \cdot 7$ and compute $[567/3]P = [189]P = O$.
- Factorize $189 = 3^3 \cdot 7$ and compute
  $[189/3]P = (38 : 535 : 1) \neq O$,
  $[189/7]P = (136 : 360 : 1) \neq O$.
  So P has order 189.

Baby Step, Giant Step – Group Order

Determine the group order $\sharp\mathcal{E}(\mathbb{F}_q)$.

- Repeat the steps 1-6 with randomly chosen points P in $\mathcal{E}(\mathbb{F}_q)$ until the least common multiple (lcm) of the orders of the points divides only one integer N with

$$q + 1 - 2\sqrt{q} \leq N \leq q + 1 + 2\sqrt{q}.$$

  Then $N = \sharp\mathcal{E}(\mathbb{F}_q)$.

- Note that if P_1 has order M_1 and P_2 has order M_2, then $P_1 + P_2$ has order $\mathrm{lcm}(M_1, M_2)$.

Quadratic Character

Define the mapping $\chi : \mathbb{F}_q^* \to \{\pm 1\}$ by

$$\chi(x) = \begin{cases} 1 & \text{if } x \text{ is a square in } \mathbb{F}_q, \\ -1 & \text{otherwise.} \end{cases} \tag{288}$$

The multiplicative group $\mathbb{F}_q^*$ is cyclic of order $q-1$, i.e., for some generator ξ of $\mathbb{F}_q^*$,

$$\mathbb{F}_q^* = \langle \xi \rangle = \{\xi, \xi^2, \ldots, \xi^{q-1} = 1\}.$$

Thus for each integer $m \geq 0$,

$$\chi(\xi^m) = \begin{cases} 1 & \text{if } m \text{ is even,} \\ -1 & \text{otherwise.} \end{cases} \tag{289}$$

Quadratic Character

The mapping χ is a group homomorphism, called *quadratic character* of $\mathbb{F}_q^*$, i.e., for all $x, y \in \mathbb{F}_q^*$,

$$\chi(xy) = \chi(x)\chi(y). \tag{290}$$

The mapping χ can be extended to $\chi : \mathbb{F}_q \to \{\pm 1, 0\}$ by setting $\chi(0) = 0$.

Quadratic Character – Example

Consider the Galois field $\mathbb{F}_8 = \mathbb{F}_2[X]/\langle X^3 + X + 1\rangle$. The elements are

$$\begin{aligned} 0, \quad & \alpha^3 = \alpha + 1, \\ 1, \quad & \alpha^4 = \alpha^2 + \alpha, \\ \alpha, \quad & \alpha^5 = \alpha^2 + \alpha + 1, \\ \alpha^2, \quad & \alpha^6 = \alpha^2 + 1, \end{aligned}$$

where α is a zero of $X^3 + X + 1$, i.e., $\alpha^3 + \alpha + 1 = 0$.
The squares of $\mathbb{F}_8^*$ are

$$1^2 = 1,\ (\alpha)^2 = \alpha^2,\ (\alpha^2)^2 = \alpha^4,\ (\alpha^3)^2 = \alpha^6$$

and the non-squares are

$$\alpha,\ \alpha^3,\ \alpha^5.$$

Group Order (Using Characters)

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$ with characteristic $p \neq 2, 3$ given by $Y^2 = X^3 + AX + B \in \mathbb{F}_q[X]$. Put

$$h(X) = X^3 + AX + B. \tag{291}$$

Then

$$\sharp\mathcal{E}(\mathbb{F}_q) = 1 + q + \sum_{x \in \mathbb{F}_q} \chi(h(x)). \tag{292}$$

Proof.

Let $x \in \mathbb{F}_q$.

- If $h(x) = 0$, then $y = 0$ is the only solution.
- If $h(x) \neq 0$ is a square in $\mathbb{F}_q$, the equation $Y^2 = h(X)$ has two solutions (x, y) and $(x, -y)$.
- If $h(x) \neq 0$ is not a square in $\mathbb{F}_q$, the equation $Y^2 = h(X)$ has no solution.

Thus for each $x \in \mathbb{F}_q$, the equation $Y^2 = h(X)$ has $\chi(h(x)) + 1$ solutions y in $\mathbb{F}_q$.
The number of such solutions is

$$\sum_{x \in \mathbb{F}_q} (\chi(h(x)) + 1) = q + \sum_{x \in \mathbb{F}_q} \chi(h(x)).$$

Adding the base point O yields the result. □

Group Order – Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_5$ given by $Y^2 = X^3 + X + 1$. Then

$$\begin{aligned} \sharp\mathcal{E}(\mathbb{F}_5) &= 1 + 5 + \sum_{x=0}^{4} \chi(x^3 + x + 1) \\ &= 6 + \chi(1) + \chi(3) + \chi(1) + \chi(1) + \chi(4) \\ &= 6 + 1 - 1 + 1 + 1 + 1 = 9, \end{aligned}$$

since the squares modulo 5 are 1 and 4; i.e., $1^2 = 1$, $2^2 = 4$, $3^2 = 4$, and $4^2 = 1$ in $\mathbb{Z}_5$, and the nonsquares modulo 5 are 2 and 3.

Group Order – Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_{31}$ given by $Y^2 = X^3 - X$.

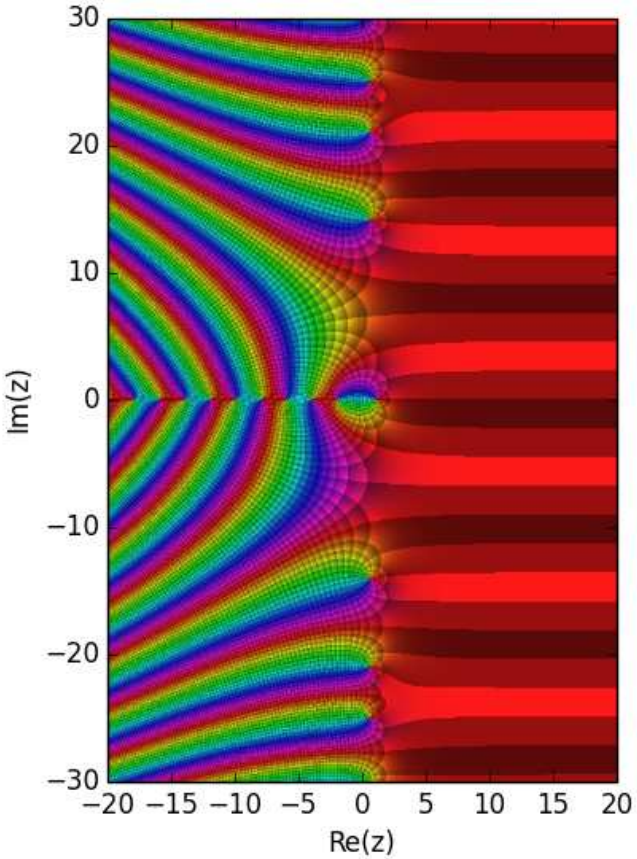
- -1 is not a square in $\mathbb{F}_{31}$, i.e., $\chi(-1) = -1$, since $31 \equiv 3 \mod 4$.
- For each $x \in \mathbb{F}_{31}$ with $x^3 - x \neq 0$,

$$\begin{aligned} \chi((-x)^3 - (-x)) &= \chi(-(x^3 - x)) = \chi(-1)\chi(x^3 - x) \\ &= -\chi(x^3 - x). \end{aligned}$$

- The equation $X^3 - X = 0$ has three solutions $x = 0, 1, -1$ in $\mathbb{F}_{31}$ and in these cases we have $\chi(x^3 - x) = 0$.
- Otherwise, the above equation shows that either $x^3 - x$ or $(-x)^3 - (-x)$ is a square in $\mathbb{F}_{31}$. Thus $\chi(x^3 - x) + \chi((-x)^3 - (-x)) = 0$.
- It follows that

$$\sharp\mathcal{E}(\mathbb{F}_{31}) = 1 + 31 + \sum_{x \in \mathbb{F}_{31}} \chi(x^3 - x) = 32.$$

Zeta-Function



Riemann zeta function $\zeta(z)$ plotted with domain coloring.

Zeta-Function

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$. Write

$$N_n = \sharp\mathcal{E}(\mathbb{F}_{q^n}), \quad n \geq 1. \tag{293}$$

The *Z-function* of $\mathcal{E}$ is the formal power series in $\mathbb{Q}[[T]]$ defined by

$$Z(T; \mathcal{E}/\mathbb{F}_q) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n}{n} T^n\right), \tag{294}$$

where $\exp(t) = \sum_n t^n/n!$ is the exponential function.

Zeta-Function

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$ with $\sharp\mathcal{E}(\mathbb{F}_q) = q + 1 - t$. Then

$$Z(T; \mathcal{E}/\mathbb{F}_q) = \frac{qT^2 - tT + 1}{(1 - T)(1 - qT)}. \tag{295}$$

Proof.

Factorize

$$X^2 - tX + q = (X - \alpha)(X - \beta)$$

over $\mathbb{C}$ as in (280). Then by comparing coefficients,

$$t = \alpha + \beta \quad \text{and} \quad q = \alpha\beta$$

and by (282),

$$N_n = \sharp\mathcal{E}(\mathbb{F}_{q^n}) = q^n + 1 - \alpha^n - \beta^n, \quad n \geq 1.$$

Proof (cont'd).

$$
\begin{aligned}
& Z(T;\mathcal{E}/\mathbb{F}_q) \\
&= \exp\left(\sum_{n=1}^{\infty}\frac{N_n}{n}T^n\right) \\
&= \exp\left(\sum_{n=1}^{\infty}(q^n+1-\alpha^n-\beta^n)\frac{T^n}{n}\right) \\
&= \exp\left(-\log(1-qT)-\log(1-T)+\log(1-\alpha T)+\log(1-\beta T)\right) \\
&= \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)} \\
&= \frac{qT^2-tT+1}{(1-T)(1-qT)},
\end{aligned}
$$

where $\log(x)=\sum_n(-1)^{n-1}(x-1)^n/n$ by Taylor series expansion and so $-\log(1-x)=\sum_n x^n/n$. □

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_2$ given by $Y^2 + Y = X^3$. The $\mathbb{F}_2$-rational points are $O, (0:0:1), (0:1:1)$. Thus $t = q + 1 - \sharp\mathcal{E}(\mathbb{F}_2) = 2 + 1 - 3 = 0$ and $N_1 = 3$. Therefore,

$$X^2 - tX + q = X^2 + 2 = (X + i\sqrt{2})(X - i\sqrt{2})$$

and so

$$N_n = 2^n + 1 - (i\sqrt{2})^n - (-i\sqrt{2})^n, \quad n \geq 1.$$

`Maple` computation of series (295),

```
> series( (2*x^2+1)/((1-x)*(1-2*x)), x=0,8);
```

$$1 + 3x + 9x^2 + 21x^3 + 45x^4 + 93x^5 + 189x^6 + 381x^7 + O(x^8).$$

Explicitly,

$$N_n = \begin{cases} 2^n + 1 & \text{if } n \text{ is odd,} \\ 2^n + 1 - 2(-2)^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Zeta-Function

The *zeta function* of $\mathcal{E}$ over $\mathbb{F}_q$ is

$$\zeta_{\mathcal{E}}(s) = Z(q^{-s}; \mathcal{E}/\mathbb{F}_q), \tag{296}$$

where s is a complex variable; i.e., for $T = q^{-s}$ in (295),

$$\zeta_{\mathcal{E}}(s) = \frac{q^{1-2s} - tq^{-s} + 1}{(1 - q^{-s})(1 - q^{1-s})}. \tag{297}$$

*The zeta function $\zeta_{\mathcal{E}}(s)$ has properties similar to the classical Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{298}$$

Zeta-Function

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$. Then

$$\zeta_{\mathcal{E}}(s) = \zeta_{\mathcal{E}}(1-s). \tag{299}$$

Proof.

By (297) we have

$$\begin{aligned} \zeta_{\mathcal{E}}(s) &= \frac{q^{1-2s} - tq^{-s} + 1}{(1-q^{-s})(1-q^{1-s})}, \quad \text{expand by } q^{2s-1}, \\ &= \frac{1 - tq^{s-1} + q^{-1+2s}}{(q^s-1)(q^{s-1}-1)} \\ &= \zeta_{\mathcal{E}}(1-s). \end{aligned}$$

□

Zeta-Function

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$. If $\zeta_{\mathcal{E}}(s) = 0$, then $\Re(s) = \frac{1}{2}$.

Proof.

We have $X^2 - tX + q = (X - \alpha)(X - \beta)$. By comparing coefficients, $q = \alpha\beta$ and $t = \alpha + \beta$. So the numerator of $Z(T; \mathcal{E}/\mathbb{F}_q)$ in (295) can be written as

$$qT^2 - tT + 1 = (1 - \alpha T)(1 - \beta T).$$

Thus by using $T = q^{-s}$,

$$\zeta_{\mathcal{E}}(s) = 0 \quad \Longleftrightarrow \quad \alpha = q^s \text{ or } \beta = q^s.$$

By the quadratic formula, the zeros of $X^2 - tX + q$ are

$$\alpha, \beta = \frac{1}{2}\left(t \pm \sqrt{t^2 - 4q}\right).$$

Proof (cont'd).

By the Hasse bound (270),

$$|t| \leq 2\sqrt{q}$$

and so $t^2 - 4q \leq 0$. Thus α and β are complex conjugates. Then

$$q = \alpha\beta = \alpha\overline{\alpha} = |\alpha|^2$$

and so

$$|\alpha| = |\beta| = \sqrt{q}.$$

If $q^s = \alpha$ or β, then

$$q^{\Re(s)} \stackrel{!}{=} |q^s| = \sqrt{q}.$$

Hence, $\Re(s) = \frac{1}{2}$. □

*Zeta Function

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$.

- Define the *degree* of $P \in \mathcal{E}(\bar{\mathbb{F}}_q)$, written $\deg(P)$, as the smallest integer $n \geq 1$ such that $P \in \mathcal{E}(\mathbb{F}_{q^n})$.
- The set

$$S_P = \{P, \phi_q(P), \phi_q^2(P), \ldots, \phi_q^{n-1}(P)\} \tag{300}$$

has $n = \deg(P)$ elements and $\phi_q^n(P) = P$. Each point in S_P has also degree n.

*Zeta Function

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$. Then

$$\zeta_{\mathcal{E}}(s) = \prod_{S_P} \left(1 - \frac{1}{q^{s \cdot \deg(P)}}\right)^{-1}, \tag{301}$$

where the product is over the points $P \in \mathcal{E}(\bar{\mathbb{F}}_q)$ with exactly one point from each set S_P.

*Proof.

If $\deg(P) = m$, then P and all other points in S_P have coordinates in $\mathbb{F}_{q^m}$.
Since $\mathbb{F}_{q^m}$ is a subfield of $\mathbb{F}_{q^n}$ iff $m \mid n$, it follows that S_P contributes m points to $N_n = \sharp\mathcal{E}(\mathbb{F}_{q^n})$ iff $m \mid n$; otherwise, it contributes no points to N_n. Thus

$$N_n = \sum_{m|n} \sum_{\substack{S_P \\ \deg(P)=m}} m.$$

*Proof (cont'd)

We have

$$\begin{aligned}
\log Z(T;\mathcal{E}/\mathbb{F}_q) &= \sum_{n=1}^{\infty} \frac{N_n}{n} T^n \\
&= \sum_{n=1}^{\infty} \frac{1}{n} T^n \sum_{m|n} \sum_{\substack{S_P \\ \deg(P)=m}} m \\
&= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mj} \sum_{\substack{S_P \\ \deg(P)=m}} mT^{mj}, \ (mj=n), \\
&= \sum_{j=1}^{\infty} \sum_{S_P} \frac{1}{j} T^{j\cdot \deg(P)} \\
&= -\sum_{S_P} \log\left(1 - T^{\deg(P)}\right).
\end{aligned}$$

Substitute $T = q^{-s}$ and exponentiate to obtain the result. □

Supersingular Curves

> Hasse (1936) discovered supersingular elliptic curves during his work on the Riemann hypothesis for elliptic curves by observing that in positive characteristic elliptic curves could have endomorphism rings of unusually large rank.

Wikipedia (2018)

Supersingular Curves

An elliptic curve $\mathcal{E}$ over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = p > 0$ is *supersingular* if it has no points of order p; i.e.,

$$\mathcal{E}[p] = \{O\}. \tag{302}$$

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$ with $q = p^r$, p prime and $r \geq 1$.

Let $t = q + 1 - \sharp\mathcal{E}(\mathbb{F}_q)$ be the Frobenius trace. Then

$$\mathcal{E} \text{ is supersingular} \iff t \equiv 0 \mod p, \tag{303}$$

i.e., $\sharp\mathcal{E}(\mathbb{F}_q) \equiv 1 \mod p$.

Proof.

- Write $X^2 - tX + q = (X - \alpha)(X - \beta)$. Then by (282),

$$\sharp\mathcal{E}(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n).$$

- Consider the sequence $(s_n)_{n\geq 0}$ given by $s_n = \alpha^n + \beta^n$. Then by (284), $s_0 = 2$, $s_1 = t$, and $s_{n+1} = ts_n - qs_{n-1}$ for all $n \geq 1$.
- Suppose $t \equiv 0 \mod p$. Then $s_1 = t \equiv 0 \mod p$ and so $s_{n+1} \equiv 0 \mod p$ for all $n \geq 1$ by the recurrence. Thus

$$\sharp\mathcal{E}(\mathbb{F}_{q^n}) = q^n + 1 - s_n \equiv 1 \mod p.$$

By Lagrange, element order divides group order and so $\mathcal{E}(\mathbb{F}_{q^n})$ has no point of order p. Since $\bar{\mathbb{F}}_q = \bigcup_{n\geq 1} \mathbb{F}_{q^n}$, $\mathcal{E}(\bar{\mathbb{F}}_q)$ has no point of order p. Hence, $\mathcal{E}$ is supersingular over $\bar{\mathbb{F}}_q$.

Proof (cont'd).

- Suppose $t \not\equiv 0 \mod p$. By the recurrence, $s_{n+1} \equiv t s_n \mod p$ for all $n \geq 1$. Since $s_1 = t$, we have $s_n \equiv t^n \mod p$ for all $n \geq 1$. Thus

$$\sharp\mathcal{E}(\mathbb{F}_{q^n}) = q^n + 1 - s_n \equiv 1 - t^n \mod p.$$

By Fermat's little theorem, $x^{p-1} \equiv 1 \mod p$ for all nonzero points $x \in \mathbb{F}_p$. Thus

$$\sharp\mathcal{E}(\mathbb{F}_{q^{p-1}}) \equiv 1 - t^{p-1} \equiv 0 \mod p.$$

So the order of $\mathcal{E}(\mathbb{F}_{q^{p-1}})$ is divisible p and thus $\mathcal{E}(\mathbb{F}_{q^{p-1}})$ has a point of order p. Hence, $\mathcal{E}$ is not supersingular.

□

Supersingular Curves

Let $\mathcal{E}$ is an elliptic curve over $\mathbb{F}_p$ with $p \geq 5$ prime. Then

$$\mathcal{E} \text{ is supersingular} \iff t = 0, \tag{304}$$

i.e., $\sharp\mathcal{E}(\mathbb{F}_p) = p + 1$.

Proof.

- Let $t = 0$. Then by (303), $\mathcal{E}(\mathbb{F}_p)$ is supersingular.
- Let $\mathcal{E}$ be supersingular with $t \neq 0$. Then by (303) , $t \equiv 0 \mod p$ and so $|t| \geq p$. By the Hasse bound (270), $|t| \leq 2\sqrt{p}$ and so $p \leq 2\sqrt{p}$, which is only valid for $p \leq 4$. □

Example

- The elliptic curve $\mathcal{E}$ over $\mathbb{F}_2$ given by $Y^2 + Y = X^3 + X$ has five $\mathbb{F}_2$-rational points,

$$O,\ (0:0:1),\ (0:1:1),\ (1:0:1),\ (1:1:1).$$

  Here $t = p + 1 - \sharp\mathcal{E}(\mathbb{F}_p) = 2 + 1 - 5 = -2$.

- The elliptic curve $\mathcal{E}$ over $\mathbb{F}_3$ given by $Y^2 = X^3 - X + 2$ has only the base point O as $\mathbb{F}_3$-rational point.

  Here $t = p + 1 - \sharp\mathcal{E}(\mathbb{F}_p) = 3 + 1 - 1 = 3$.

Both curves are supersingular, but $t = p + 1 - \sharp\mathcal{E}(\mathbb{F}_p) \neq 0$. So the above restriction $p \geq 5$ is necessary.

Supersingular Curves

Let q be an odd prime power with $q \equiv 2 \mod 3$ and let $b \in \mathbb{F}_q^*$. Then the elliptic curve $\mathcal{E}(\mathbb{F}_q)$ given by

$$Y^2 = X^3 + b \tag{305}$$

is supersingular.

Proof.

- Consider the homomorphism $\phi : \mathbb{F}_q^* \to \mathbb{F}_q^* : x \mapsto x^3$. Since $q-1$ is not a multiple of 3, $\mathbb{F}_q^*$ has no element of order 3 and so $\ker \phi$ is trivial; i.e., there is no $x \in \mathbb{F}_q^*$ with $x^3 = 1$. Thus ϕ is one-to-one and so onto, i.e., bijective. Hence, each $x \in \mathbb{F}_q^*$ has a *unique* cubic root (preimage).
- For each $y \in \mathbb{F}_q$, there is a *unique* $x \in \mathbb{F}_q$ given by the cubic root of $y^2 - b$ such that $(x, y) \in \mathcal{E}(\mathbb{F}_q)$. For each of the q values for Y, we obtain q values for X. Including the base point O, we have $\sharp\mathcal{E}(\mathbb{F}_q) = q + 1$. Therefore, by (303), $\mathcal{E}(\mathbb{F}_q)$ is supersingular. □

Fast Point Multiplication

Let $\mathcal{E}$ be a supersingular elliptic curve over $\mathbb{F}_q$ and let $P = (x : y : 1) \in \mathcal{E}(\mathbb{F}_{q^n})$ for some (large) $n \geq 1$.
Given $k \geq 1$. Compute $[k]P$.

- Let $t = 0$. Then by (275),

$$\phi_q^2 + [q] = 0 \tag{306}$$

and so by plugging in P,

$$[q](x, y) = -\phi_q^2(x, y) = \left(x^{q^2}, -y^{q^2}\right). \tag{307}$$

- Compute x^{q^2} and y^{q^2} by finite field arithmetic.
- Express x and y by a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Then x^{q^2} and y^{q^2} can be computed by shift operations.

Fast Point Multiplication

1. Expand k in base q,

$$k = k_0 + k_1 q + \ldots + k_r q^r, \quad 0 \leq k_i < q. \tag{308}$$

2. Compute

$$[k_i]P = (x_i, y_i), \quad 0 \leq i \leq r. \tag{309}$$

3. Compute

$$[q^i]([k_i]P) = -\phi_{q^i}^2(x_i, y_i) = \left(x_i^{q^{2i}}, (-1)^i y_i^{q^{2i}}\right), 0 \leq i \leq r \tag{310}$$

4. Compute the sum of $[q^i]([k_i]P)$ for $0 \leq i \leq r$ to obtain $[k]P$.

The main saving is in step 3 in which finite field calculations can be used.

Normal Bases

Let $n \geq 1$. The finite field $\mathbb{F}_{q^n}$ is a vector space of dimension n over $\mathbb{F}_q$.
Let $\{\beta_1, \ldots, \beta_n\}$ be an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^n}$. Each element $\alpha \in \mathbb{F}_{q^n}$ has a unique representation of the form

$$\alpha = a_1\beta_1 + \ldots + a_n\beta_n \tag{311}$$

with $a_1, \ldots, a_n \in \mathbb{F}_q$.

There exists $\beta \in \mathbb{F}_{q^n}$ such that

$$\{\beta, \beta^q, \ldots, \beta^{q^{n-1}}\} \tag{312}$$

is an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^n}$, called *normal basis*.

Normal Bases

Let $\{\beta, \beta^q, \ldots, \beta^{q^{n-1}}\}$ be a normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.
For each $\alpha \in \mathbb{F}_{q^n}$ write

$$\alpha = a_1\beta + a_2\beta^q + \ldots + a_n\beta^{q^{n-1}} \tag{313}$$

with $a_1, \ldots, a_n \in \mathbb{F}_q$. Then

$$\begin{aligned} \alpha^q &= a_1\beta^q + a_2\beta^{q^2} + \ldots + a_n\beta^{q^n} \\ &= a_n\beta^{q^n} + a_1\beta^q + \ldots + a_{n-1}\beta^{q^{n-1}} \\ &= a_n\beta + a_1\beta^q + \ldots + a_{n-1}\beta^{q^{n-1}}, \end{aligned} \tag{314}$$

since $a_i^q = a_i$ and $\beta^{q^n} = \beta$.
Thus if α has coordinates $(a_1, \ldots, a_n)$, then α^q has (cyclically shifted) coordinates $(a_n, a_1, \ldots, a_{n-1})$.

Example

Let $q = 2$ and $n = 3$. The field $\mathbb{F}_8$ over $\mathbb{F}_2$ generated by α with $\alpha^3 + \alpha + 1 = 0$ has the elements

$$\begin{array}{ll} 0, & \alpha^4 = \alpha^2 + \alpha, \\ \alpha, & \alpha^5 = \alpha^2 + \alpha + 1, \\ \alpha^2, & \alpha^6 = \alpha^2 + 1, \\ \alpha^3 = \alpha + 1, & \alpha^7 = 1. \end{array}$$

Then $\{\alpha^3, \alpha^6, \alpha^5\}$ is a normal basis of $\mathbb{F}_8$ over $\mathbb{F}_2$,

$$\begin{array}{lllllll} 0 & = & (0,0,0), & \alpha^3 & = & (1,0,0), \\ 1 & = & (1,1,1), & \alpha^4 & = & (1,1,0), \\ \alpha & = & (0,1,1), & \alpha^5 & = & (0,0,1), \\ \alpha^2 & = & (1,0,1), & \alpha^6 & = & (0,1,0). \end{array}$$

We have

$$\begin{aligned} \alpha &= 0 \cdot \alpha^3 + 1 \cdot \alpha^6 + 1 \cdot \alpha^5, \\ \alpha^2 &= 1 \cdot \alpha^3 + 0 \cdot \alpha^6 + 1 \cdot \alpha^5, \\ \alpha^4 &= 1 \cdot \alpha^3 + 1 \cdot \alpha^6 + 0 \cdot \alpha^5. \end{aligned}$$

Schoof's Algorithm

The algorithm was published by René Schoof in 1985 and it was a theoretical breakthrough, as it was the first deterministic polynomial time algorithm for counting points on elliptic curves. Before Schoof's algorithm, approaches to counting points on elliptic curves such as the naive and baby-step giant-step algorithms were, for the most part, tedious and had an exponential running time.

Wikipedia (2018)

Schoof's Algorithm

Given elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) = p > 0$, p odd, given by $Y^2 = X^3 + AX + B$.

- By the Frobenius trace (274) and Hasse bound (270),

$$\sharp\mathcal{E}(\mathbb{F}_q) = q + 1 - t \tag{315}$$

  with $|t| \leq 2\sqrt{q}$.

- Let $S = \{2, 3, \ldots, L\}$ be a set of primes with $p \notin S$ and

$$\prod_{\ell \in S} \ell > 4\sqrt{q}. \tag{316}$$

- If we compute $t \mod \ell$ for all $\ell \in S$, then by the Chinese Remainder theorem we know

$$t \mod \prod_{\ell \in S} \ell;$$

  i.e., we know t.

Schoof's Algorithm

Case $\ell = 2$: $t \equiv \sharp\mathcal{E}(\mathbb{F}_q) \mod 2$, i.e., $\sharp\mathcal{E}(\mathbb{F}_q)$ even or odd.

- Suppose $X^3 + AX + B$ has no root in $\mathbb{F}_q$. For each $x \in \mathbb{F}_q$ with $y_0 = x^3 + Ax + B \neq 0$, there are either two points $(x, \pm\sqrt{y_0})$ or no point on the curve. The number of such points is even.
- Suppose $X^3 + AX + B$ has a root x_0 in $\mathbb{F}_q$. Over $\bar{\mathbb{F}}_q$,

$$X^3 + AX + B = (X - x_0)(X - x_1)(X - x_2) \tag{317}$$

with $x_1, x_2 \in \bar{\mathbb{F}}_q$. Since $\mathcal{E}$ is nonsingular, x_0, x_1, x_2 are pairwise distinct. By comparing the coefficients of X^2, $x_0 + x_1 + x_2 = 0$. So x_1, x_2 are either both in $\mathbb{F}_q$ or none of them is in $\mathbb{F}_q$. Thus there are either three points $(x_0, 0), (x_1, 0), (x_2, 0)$ or there is only one point $(x_0, 0)$ on the curve. The number of such points is odd.

Schoof's Algorithm

Case $\ell = 2$ (cont'd):

- If the base point O is added, $\sharp\mathcal{E}(\mathbb{F}_q) \equiv 1 \mod 2$ iff $X^3 + AX + B$ has no root in $\mathbb{F}_q$.
- The roots of $X^q - X$ are exactly the elements of $\mathbb{F}_q$; i.e.,

$$X^q - X = \prod_{\alpha \in \mathbb{F}_q} (X - \alpha). \tag{318}$$

- $X^3 + AX + B$ has a root in $\mathbb{F}_q$ iff it has a root in common with $X^q - X$.
- $(X^q - X, X^3 + AX + B) = 1$ iff $X^3 + AX + B$ has no root in $\mathbb{F}_q$.
- Use the Euclidean algorithm to compute this gcd.

Schoof's Symbolic Algorithm

Case $\ell > 2$:

- ψ_ℓ is a polynomial in X (ℓ odd) and for $(x, y) \in \mathcal{E}(\bar{\mathbb{F}}_q)$,

$$(x, y) \in \mathcal{E}[\ell] \quad \Longleftrightarrow \quad \psi_\ell(x) = 0;$$

  i.e., the roots of ψ_ℓ are exactly the x-coordinates of the points in $\mathcal{E}[\ell]$.

- Since $\phi_q^2 - t\phi_q + q = 0$, for each point (x, y) of order ℓ,

$$\left(x^{q^2}, y^{q^2}\right) + q(x, y) = t(x^q, y^q).$$

- Let $q_\ell \equiv q \mod \ell$ with $|q_\ell| < \ell/2$. Then

$$\left(x^{q^2}, y^{q^2}\right) + q_\ell(x, y) = t(x^q, y^q).$$

  Since (x^q, y^q) is also a point of order ℓ, this relation determines $t \mod \ell$.

Schoof's Symbolic Algorithm

Case $\ell > 2$ (cont'd):

- Suppose for some $(x, y) \in \mathcal{E}[\ell]$,

$$\left(x^{q^2}, y^{q^2}\right) \neq \pm q_\ell(x, y).$$

- Then put

$$(x', y') = \left(x^{q^2}, y^{q^2}\right) + q_\ell(x, y) \neq O.$$

  Thus $t \not\equiv 0 \mod \ell$.

- The x-coordinates of the summands on the right-hand side are distinct and so x' is found by the formula using the line through the two points.

Schoof's Symbolic Algorithm

Case $\ell > 2$ (cont'd):

- We have

$$x' = \left(\frac{y^{q^2} - y_{q_\ell}}{x^{q^2} - x_{q_\ell}} \right)^2 - x^{q^2} - x_{q_\ell}$$

and by using $y^2 = x^3 + Ax + B$,

$$\begin{aligned} & \left(y^{q^2} - y_{q_\ell}\right)^2 \\ &= \quad y^2 \left(y^{q^2-1} - r_{2,q_\ell}(x)\right)^2 \\ &= \quad (x^3 + Ax + B) \left((x^3 + Ax + B)^{(q^2-1)/2} - r_{2,q_\ell}(x)\right)^2. \end{aligned}$$

- Write

$$j(x, y) = (x_j, y_j), \quad j \in \mathbb{Z},$$

where $x_j = r_{1,j}(x)$ and $y_j = r_{2,j}(x)y$. Compute x_j, y_j using division polynomials.

Schoof's Symbolic Algorithm

Case $\ell > 2$ (cont'd):

- Find $j \in \mathbb{Z}$ such that

$$(x', y') = j(x^q, y^q) = (x_j^q, y_j^q).$$

- For $(x, y) \in \mathcal{E}[\ell]$,

$$(x', y') = \pm(x_j^q, y_j^q) \quad \Longleftrightarrow \quad x' = x_j^q.$$

- The roots of ψ_ℓ are the x-coordinates of the points in $\mathcal{E}[\ell]$ and so

$$(x', y') = \pm(x_j^q, y_j^q) \quad \Longleftrightarrow \quad x' - x_j^q \equiv 0 \mod \psi_\ell.$$

- The polynomial ψ_ℓ has degree $(\ell^2 - 1)/2$ and its roots are simple. Thus there are $(\ell^2 - 1)/2$ distinct x-coordinates of the points in $\mathcal{E}[\ell]$.

Schoof's Symbolic Algorithm

Case $\ell > 2$ (cont'd):

- Suppose we have found j such that

$$(x', y') = \pm(x_j^q, y_j^q) = (x_j^q, \pm y_j^q).$$

- Determine the sign by looking at the y-coordinates.
- Both y'/y and y_j^q/y can be written as functions of x. If

$$(y' - y_j^q)/y \equiv 0 \mod \psi_\ell,$$

then $t \equiv j \mod \ell$; otherwise, $t \equiv -j \mod \ell$.

- The remaining (rare) case is where

$$\left(x^{q^2}, y^{q^2}\right) = \pm q(x, y)$$

for all $(x, y) \in \mathcal{E}[\ell]$.

Schoof's Symbolic Algorithm

Given elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ with $q = p^r$, p odd prime, given by $Y^2 = X^3 + AX + B$. Find Frobenius trace t.

1. Choose a set of primes $S = \{2, 3, \ldots, L\}$ with $p \notin S$ and $\prod_{\ell \in L} \ell > 4\sqrt{q}$.
2. If $\ell = 2$, then $t \equiv 0 \mod 2$ iff $(X^3 + AX + B, X^q - X) \neq 1$.
3. For each odd prime $\ell \in S$:
   a. Let $q_\ell \equiv q \mod \ell$ with $|q_\ell| < \ell/2$.
   b. Compute the x-coordinate X' of

$$(X', Y') = \left(X^{q^2}, Y^{q^2}\right) + [q_l](X, Y) \mod \psi_\ell.$$

Schoof's Algorithm (cont'd)

c For $j = 1, 2, \ldots, (\ell - 1)/2$:

i Compute the x-coordinate of $(X_j, Y_j) = [j](X, Y)$ using division polynomials.

ii If $X' - X_j^q \equiv 0 \mod \psi_\ell$, goto step iii. Otherwise, try the next value of j. If all values of j have been tried, goto step d.

iii Compute Y' and Y_j. If $(Y' - Y_j^q)/Y \equiv 0 \mod \psi_\ell$, then $t \equiv j \mod \ell$; otherwise, $t \equiv -j \mod \ell$.

Schoof's Algorithm (cont'd)

d If all values of j have been tried without success, let $w^2 \equiv q \mod \ell$. If w does not exist, then $t \equiv 0 \mod \ell$.

e If $(\text{numerator}(X^q - X_w), \psi_\ell) = 1$, then $t \equiv 0 \mod \ell$. Otherwise, compute

$$(\text{numerator}((Y^q - Y_w)/Y), \psi_\ell).$$

If this gcd is not 1, then $t \equiv 2w \mod \ell$; otherwise, $t \equiv -2w \mod \ell$.

Schoof's Algorithm (cont'd)

4 Use the values $t \mod \ell$ computed for each $\ell \in S$ to calculate $t \mod \prod_{\ell \in S} \ell$ by the Chinese Remainder theorem. Choose the value of t that fulfills the congruence with $|t| \leq 2\sqrt{q}$. The number of points in $\mathcal{E}(\mathbb{F}_q)$ is $q + 1 - t$. Done.

Schoof's algorithm has polynomial running time.

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_{19}$ given by

$$Y^2 = X^3 + 2X + 1.$$

Then $\sharp\mathcal{E}(\mathbb{F}_{19}) = 19 + 1 - t$ with $|t| \leq 2\sqrt{19} \approx 8.718$.

Let $\ell = 2$.

- Compute

$$X^{19} \equiv X^2 + 13X + 14 \mod X^3 + 2X + 1.$$

Then

$$(X^{19} - X, X^3 + 2X + 1) = (X^2 + 12X + 14, X^3 + 2X + 1) = 1.$$

Thus $X^3 + 2X + 1$ has no roots in $\mathbb{F}_{19}$ and so (by adding the base point O) $t \equiv 1 \mod 2$.

Example (cont'd)

Let $\ell = 5$.

- We have $19 \equiv 4 \equiv -1 \mod 5$ and so $q_\ell = -1$.
- Thus $[19](X, Y) = -(X, Y) = (X, -Y)$ for all points in $\mathcal{E}[5]$.
- Check for $j = 2$,

$$\begin{aligned} (X', Y') &\overset{\text{def}}{=} (X^{361}, Y^{361}) + (X, -Y), \quad 19^2 = 361, \\ &\overset{?}{=} [\pm 2](X^{19}, Y^{19}) \overset{\text{def}}{=} (X'', Y''). \end{aligned}$$

- The fifth division polynomial is

$$\begin{aligned} \psi_5 &= 5X^{12} + 10X^{10} + 17X^8 + 5X^7 + X^6 + 9X^5 \\ &\quad + 12X^4 + 2X^3 + 5X^2 + 8X + 8. \end{aligned}$$

Example (cont'd)

- The x-coordinate of $(X^{361}, Y^{361}) + (X, -Y)$ is

$$X' = \left(\frac{Y^{361} - Y}{X^{361} - X}\right)^2 - X^{361} - X$$

$$= (X^3 + 2X + 1)\left(\frac{(X^3 + 2X + 1)^{180} - 1}{X^{361} - X}\right)^2 - X^{361} - X.$$

- The x-coordinate of $[\pm 2](X^{19}, Y^{19})$ is

$$X'' = \left(\frac{3X^{38} + 2}{2Y^{19}}\right)^2 - 2X^{19}.$$

Example (cont'd)

- Replacing Y^2 by $X^2 + 2X + 1$ gives a polynomial relation in X,

$$\left(\frac{Y^{361} - Y}{X^{361} - X}\right)^2 - X^{361} - X \stackrel{?}{\equiv} \left(\frac{3X^{38} + 2}{2Y^{19}}\right)^2 - 2X^{19} \mod \psi_5,$$

  which indeed holds.

- Thus $t \equiv \pm 2 \mod 5$.
- To determine the sign, look at the y-coordinates.

Example (cont'd)

- The y-coordinate Y' of $(X^{361}, Y^{361}) + (X, -Y)$ is

$$Y\left(9X^{11} + 13X^{10} + 15X^9 + 15X^7 + 18X^6 + 17X^5 + 8X^4 + 12X^3 + 8X + 6\right) \mod \psi_5.$$

- The y-coordinate Y'' of $[2](X^{19}, Y^{19})$ is

$$Y\left(13X^{10} + 15X^9 + 16X^8 + 13X^7 + 8X^6 + 6X^5 + 17X^4 + 18X^3 + 8X + 18\right) \mod \psi_5.$$

- We have

$$(Y' + Y''^{19})/Y \equiv 0 \mod \psi_5.$$

Thus $t \equiv -2 \mod 5$.

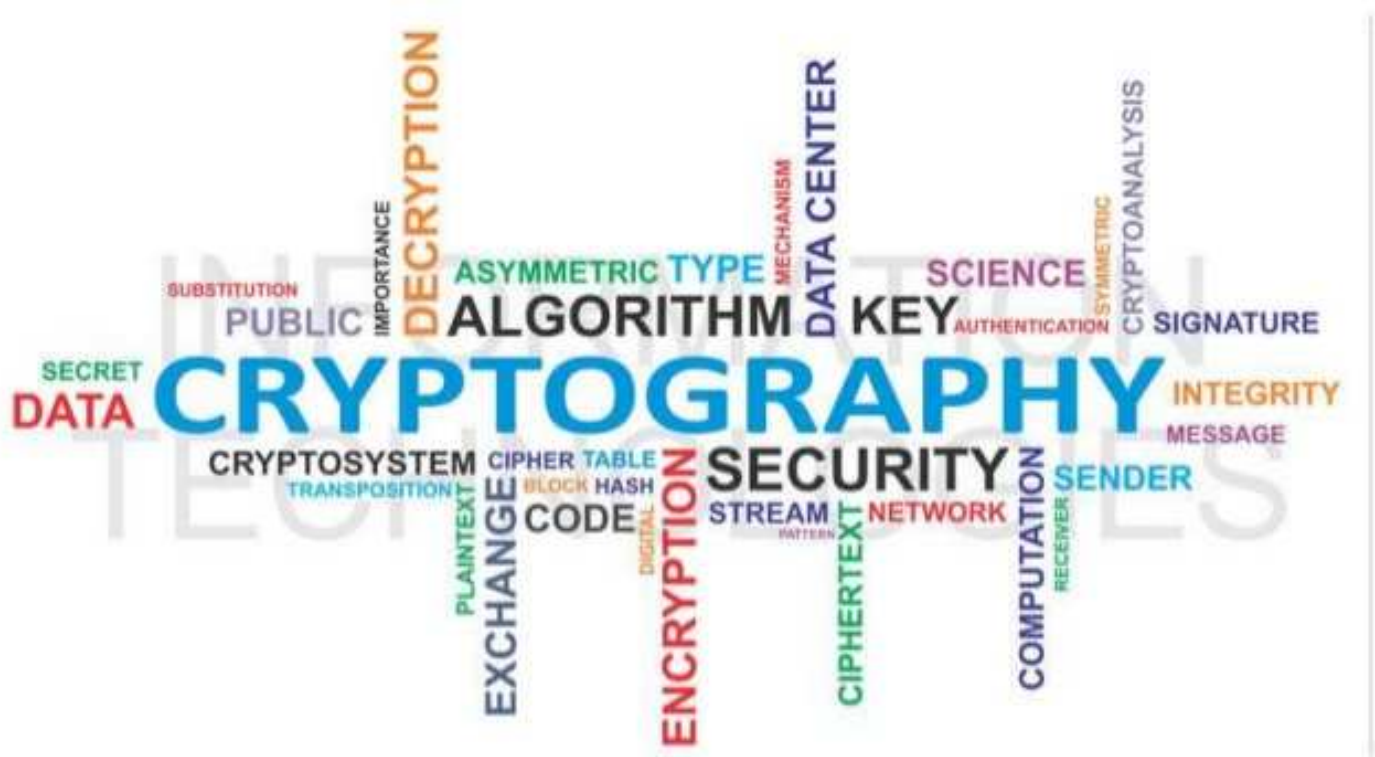
Example (cont'd)

- We have
$$t \equiv \begin{cases} 1 & \mod 2, \\ 2 & \mod 3, \\ 3 & \mod 5. \end{cases}$$
- We have $2 \cdot 3 \cdot 5 = 30 > 4\sqrt{19} \approx 17.435$ and so by the Chinese Remainder theorem,
$$t \equiv 23 \mod 30.$$
Since $|t| < 2\sqrt{19} < 9$, we have $t = -7$. Thus
$$\sharp\mathcal{E}(\mathbb{F}_{19}) = 19 + 1 - (-7) = 27.$$

Part VII

Elliptic Curve Cryptography

EC Cryptography



EC Cryptography

> The U.S. National Institute of Standards and Technology (NIST) has endorsed elliptic curve cryptography in its Suite B set of recommended algorithms, specifically elliptic curve Diffie-Hellman (ECDH) for key exchange and Elliptic Curve Digital Signature Algorithm (ECDSA) for digital signature. The U.S. National Security Agency (NSA) allows their use for protecting information classified up to top secret with 384-bit keys.

Wikipedia (2018).

EC Cryptography

- Inventors Neal Koblitz and Victor Miller (1985)
- ECC requires smaller keys compared to non-ECC cryptography
- Usage in key agreement and digital signatures

Contents

RSA Analog

COMPARABLE KEY SIZES FOR EQUIVALENT SECURITY

Symmetric scheme (key size in bits)	ECC-based scheme (size of n in bits)	RSA/DSA (modulus size in bits)
56	112	512
80	160	1024
112	224	2048
128	256	3072
192	384	7680
256	512	15360

RSA Analog

- Version of Menezes, Vanstone
- Version of Koyama, Maurer, Okamoto, Vanstone
- Recommended elliptic curves

Key Generation (Menezes, Vanstone, 1993)

Alice and Bob generate key:

- Choose (public) elliptic curve $\mathcal{E}$ over $\mathbb{F}_p$ with prime $p > 3$.
- Choose (public) random point $P \in \mathcal{E}(\mathbb{F}_p)$ such that P generates large subgroup of $\mathcal{E}(\mathbb{F}_p)$,

$$H = \langle P \rangle = \{P, 2P, 3P, \ldots, |H|P = O\}.$$

- Take (secret) random values $a, k \in \mathbb{Z}_{|H|}$.

Confidential Message Transmission

Alice has message m she wants to send to Bob. Suppose

$$m = (m_1, m_2) \in \mathbb{Z}_p^* \times \mathbb{Z}_p^*.$$

Alice encrypts the message m:

- $Q = [a]P$ with P and Q public.
- $Y = [k]Q$ with $Y = (y_1, y_2)$, $y_1 \neq 0 \neq y_2$.
- $C = [k]P$.
- $c_i \equiv y_i m_i \mod p$ for $i = 1, 2$.
- Alice sends the encrypted message $c = (C, c_1, c_2)$ to Bob.

Confidential Message Transmission

Bob decrypts the cipher $c = (C, c_1, c_2)$:

- $[a]C = Y$,
- $m = (c_1 y_1^{-1} \mod p, c_2 y_2^{-1} \mod p)$.

Correctness:

- $[a]C = [a][k]P = [k][a]P = [k]Q = Y$
- $y_i \neq 0$ is invertible in $\mathbb{Z}_p$ and so $m_i \equiv c_i y_i^{-1} \mod p$ for $i = 1, 2$.

Security is based on the mathematical problem of computing discrete logarithms.

Example

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{F}_{557}$ given by

$$Y^2 = X^3 - 10X + 21.$$

- Take the point $P = (2, 3)$ of order 189.
- Choose the private keys $k = 3$ and $a = 2$.
- Take plaintext $m = (12, 7)$.

Example (cont'd)

Encryption:

- $Q = [a]P = [2](2,3) = (58,164)$
- $Y = [k]Q = [3](58,164) = (162,24)$
- $C = [k]P = [3](2,3) = (44,294)$
- $c_1 \equiv y_1 m_1 = 12 \cdot 162 \equiv 273 \mod 557$
- $c_2 \equiv y_2 m_2 = 7 \cdot 24 \equiv 168 \mod 557$

Alice sends $((44,294), 273, 168)$ to Bob.

Example (cont'd)

Singular computation:

```
> list P, Q, Y, C;
> P[1] = 2;
> P[2] = 3;
> P[3] = 1;
> Q = ellipticMult( 557, -10, 21, P, 2);
    // Q = (58,164)
> Y = ellipticMult( 557, -10, 21, Q, 3);
    // Y = (162,24)
> C = ellipticMult( 557, -10, 21, P, 3);
    // C = (44,294)
```

Key Generation (Koyama et al., 1992)

Public key generation:

- Each user chooses two distinct large primes p, q with $p \equiv q \equiv 2 \mod 3$ and computes $n = pq$.
- Each user chooses integers e, d with

$$ed \equiv 1 \mod (p+1)(q+1).$$

- Each user makes n, e public and keeps d, p, q private.

Confidential Message Transmission

Alice wants to send message m to Bob.

- Alice downloads Bob's public key n, e.
- Alice represents m as pair $(m_1, m_2) \mod n$ and regards it as point P on elliptic curve $\mathcal{E}$ given by

$$Y^2 = X^3 + b \mod n.$$

  Note that $b = m_2^2 - m_1^3 \mod n$.
- Alice sends $C = [e]P$ to Bob.

Bob computes $P = [d]C$.

Properties of Elliptic Curve

- The formula for point addition does not require b. Eavesdropper Eve can compute it as $b = c_2^2 - c_1^3 \mod n$, since $C = (c_1, c_2) \in \mathcal{E}(\mathbb{Z}_n)$.
- By the Chinese Remainder theorem,

$$\mathcal{E}(\mathbb{Z}_n) = \mathcal{E}(\mathbb{F}_p) \oplus \mathcal{E}(\mathbb{F}_q). \tag{319}$$

- Example: The point $P = (11, 32)$ on the elliptic curve given by $Y^2 = X^3 + 8$ modulo 35 can be viewed as the pair of points $(1, 2)$ modulo 5 and $(4, 4)$ modulo 7.
- By (319), $\sharp\mathcal{E}(\mathbb{Z}_n) = \sharp\mathcal{E}(\mathbb{F}_p) \cdot \sharp\mathcal{E}(\mathbb{F}_q)$.
- Both curves $\mathcal{E}(\mathbb{F}_p)$ and $\mathcal{E}(\mathbb{F}_q)$ are supersingular by (305) and so by (304),

$$\sharp\mathcal{E}(\mathbb{F}_p) = p + 1 \quad \text{and} \quad \sharp\mathcal{E}(\mathbb{F}_q) = q + 1. \tag{320}$$

- Thus $[p+1]P = O \mod p$ and $[q+1]P = O \mod q$.

Correctness

Write

$$de = 1 + k(p+1)$$

for some integer k. Then

$$\begin{aligned} [d]C &= [de]P = [1 + k(p+1)]P \\ &= P + [k(p+1)]P = P + O = P \mod p. \end{aligned}$$

Similarly,

$$[d]C = P \mod q.$$

By (319) and the Chinese Remainder theorem, one obtains $[d]C = P \bmod n$. □

Basic Attacks

- If Eve could factor n as pq, she would know $(p+1)(q+1)$ and so could find d with $ed \equiv 1 \mod (p+1)(q+1)$. Then she could decrypt m from C and d similarly to RSA.
- If Eve could find out d, she would be able to factor n with some probability similarly to RSA; see Washington (pages 183-184).

NIST Curves

NIST recommended 5 prime curves and 10 binary curves for optimal security and implementation efficiency:

- Five prime fields $\mathbb{F}_p$ for primes p of size 192, 224, 256, 384, 521 bits. For each of these prime fields, one elliptic curve was recommended.
- Five binary fields $\mathbb{F}_{2^n}$ for n equal to 163, 233, 283, 409, and 571. For each of these fields, one elliptic curve and one Koblitz curve were recommended.

Discrete Logarithm

Elliptic curves over $\mathbb{F}_{2^n}$ are particularly advantageous, since the arithmetic processors for the underlying field are easy to construct and relatively simple to implement for large n.

F.L. Bauer (2000).

Discrete Log

- Analog of Diffie-Hellman
- Analog of Massey-Omura
- Analog of ElGamal
- Analog of Digital Signature Algorithm (DSA)
- EC integrated encryption scheme
- Trusted authority

Discrete Log

Given an elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ and points P, Q of $\mathcal{E}(\mathbb{F}_q)$, where Q is a multiple of P.

The *discrete logarithm* of Q to base P is any integer x such that

$$Q = [x]P.$$

Write $x = \log_P Q$ for the minimum $x \geq 0$.

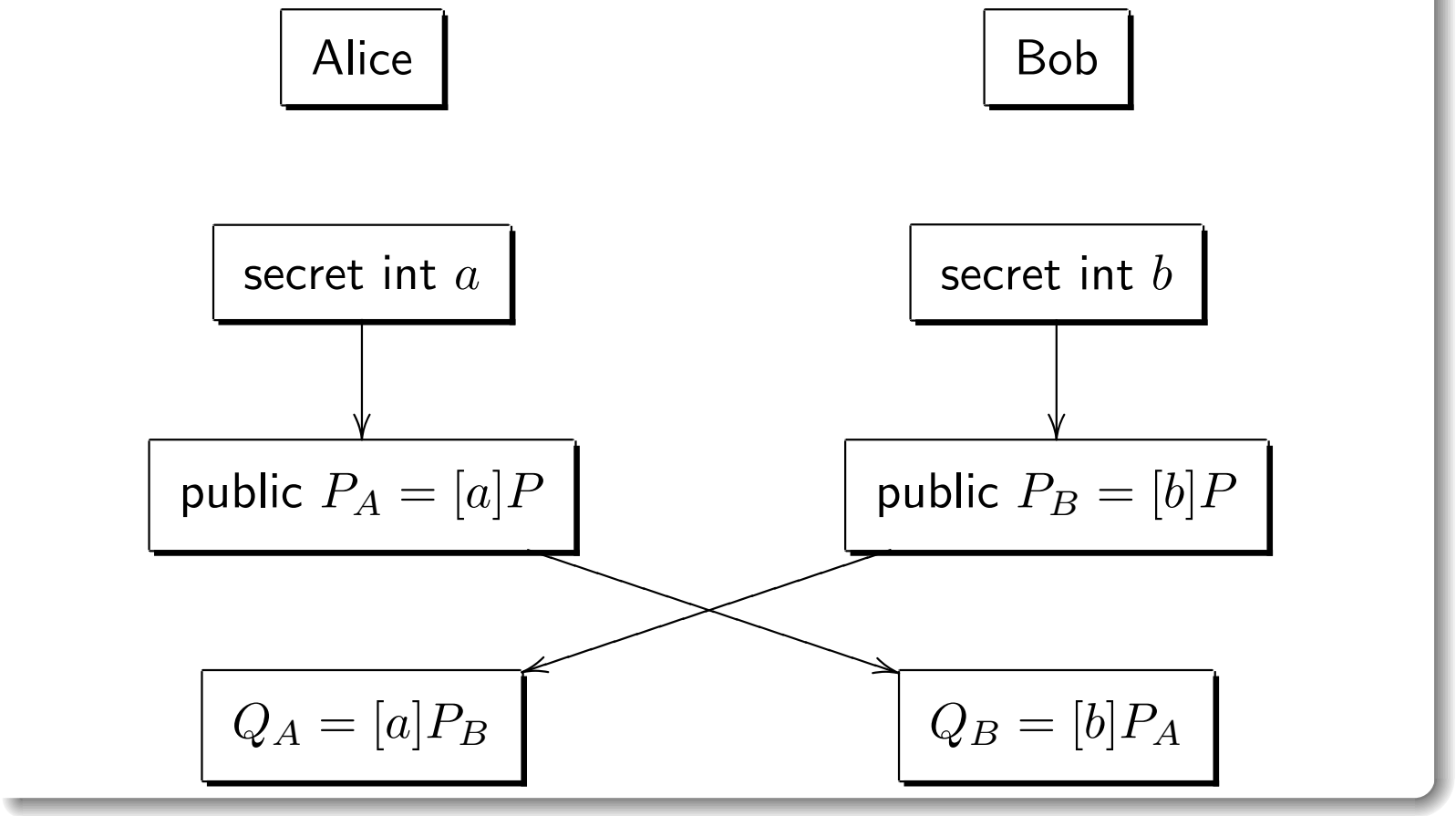
Analog of Diffie-Hellman – Protocol

- Alice and Bob agree on elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ and point $P \in \mathcal{E}(\mathbb{F}_q)$ (public).
- Alice chooses secret integer a, computes $P_A = [a]P$, and sends P_A to Bob.
- Bob chooses secret integer b, computes $P_B = [b]P$, and sends P_B to Alice.
- Alice and Bob establish common key:
  - Alice computes $Q_A = [a]P_B$.
  - Bob computes $Q_B = [b]P_A$.
  - Then

$$\begin{aligned} Q_A &= [a]P_B = [a][b]P \\ &= [b][a]P = [b]P_A = Q_B \end{aligned} \tag{321}$$

is the common key, $P_{A,B} = [ab]P$.

Diffie-Hellman Key Exchange



Diffie-Hellman Problem

Given P, P_A, and P_B in $\mathcal{E}(\mathbb{F}_q)$, compute $P_{A,B}$.

- Eavesdropper Eve knows curve $\mathcal{E}$, field $\mathbb{F}_q$, point P (public), and points P_A, P_B (transmitted data).
- If Eve could solve discrete logs in $\mathcal{E}(\mathbb{F}_q)$, she would be able to use P and P_A to find a, or P and P_B to find b.
- Then by (321) Eve could compute

$$[a]P_B = P_{A,B} \quad \text{or} \quad [b]P_A = P_{A,B}. \tag{322}$$

Decision Diffie-Hellman Problem

Given P, P_A, and P_B in $\mathcal{E}(\mathbb{F}_q)$, and given $Q \in \mathcal{E}(\mathbb{F}_q)$, determine whether or not $Q = P_{A,B}$.

Suppose Eve receives an anonymous tip telling her $P_{A,B}$. Can Eve show that this information is sound?

For some curves, Eve eventually can ...

Example

Consider the supersingular elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ with $q \equiv 2 \mod 3$ given by $Y^2 = X^3 + 1$ (see (305)).

- The field $\mathbb{F}_{q^2}$ contains a primitive 3rd root of unity ξ, since the cyclic group $\mathbb{F}_{q^2}^*$ has order $q^2 - 1$ and $q^2 \equiv 1 \mod 3$ implies $3 \mid q^2 - 1$.
- We have $\xi \notin \mathbb{F}_q$, since the cyclic group $\mathbb{F}_q^*$ has order $q - 1$ and $q - 1 \not\equiv 0 \mod 3$, i.e., $3 \nmid q - 1$.
- Given a third root of unity ξ, the mapping

$$\phi : \mathcal{E}(\bar{\mathbb{F}}_q) \to \mathcal{E}(\bar{\mathbb{F}}_q) : (x, y) \mapsto (\xi x, y) \tag{323}$$

with $\phi(O) = O$ is a group isomorphism; to prove this, use the addition law.

Modified Weil Pairing

Define the *modified Weil pairing*

$$\tilde{e}_n(P_1, P_2) = e_n(P_1, \phi(P_2)), \tag{324}$$

where e_n is the usual Weil pairing (243) and $P_1, P_2 \in \mathcal{E}[n]$.

- $\tilde{e}_n$ is well-defined, since if $P \in \mathcal{E}(\bar{\mathbb{F}}_q)$ has order n, then $\phi(P)$ has also order n as ϕ is an isomorphism.
- Since the Weil pairing is bilinear and ϕ is an isomorphism,

$$\begin{aligned} \tilde{e}_n([a]P_1, [b]P_2) &= e_n([a]P_1, \phi([b]P_2)) \\ &= e_n([a]P_1, [b]\phi(P_2)) \\ &= e_n(P_1, \phi(P_2))^{ab} \\ &= \tilde{e}_n(P_1, P_2)^{ab}. \end{aligned} \tag{325}$$

Modified Weil Pairing

If $P \in \mathcal{E}(\mathbb{F}_q)$ has order n and $3 \nmid n$, then $\tilde{e}_n(P, P)$ is a primitive n-th root of unity.

Proof (Sketch).

- The only relation of the form

$$[u]P = [v]\phi(P)$$

  with integers u, v is the trivial relation: $u, v \equiv 0 \mod n$.
- Thus $\{P, \phi(P)\}$ is a basis of $\mathcal{E}[n]$.
- Hence, $\tilde{e}_n(P, P) = e_n(P, \phi(P))$ is a primitive n-th root of unity as known from the Weil pairing.

□

Example (cont'd)

- Assume $3 \nmid n$. Then as shown $\tilde{e}_n(P,P) = \xi$ is a primitive nth root of unity.
- In view of the transmitted points P_A, P_B, we obtain by (325)

$$\tilde{e}_n(P_A, P_B) = \tilde{e}_n([a]P, [b]P) = \tilde{e}_n(P,P)^{ab} = \xi^{ab} \tag{326}$$

and in view of the common key $P_{A,B}$, we obtain by (325)

$$\tilde{e}_n(P, P_{A,B}) = \tilde{e}_n(P, [ab]P) = \tilde{e}_n(P,P)^{ab} = \xi^{ab}. \tag{327}$$

Thus

$$\tilde{e}_n(P_A, P_B) = \tilde{e}_n(P, P_{A,B}). \tag{328}$$

Example (cont'd)

- In view of the anonymous tip $Q = [k]P$ for some k, we obtain by (325)

$$\tilde{e}_n(P, Q) = \tilde{e}_n(P, [k]P) = \tilde{e}_n(P, P)^k = \xi^k. \tag{329}$$

- Eve computes both roots of unity,

$$\tilde{e}_n(P_A, P_B) = \xi^{ab} \quad \text{and} \quad \tilde{e}_n(P, Q) = \xi^k \tag{330}$$

If $\xi^{ab} = \xi^k$, i.e., $k \equiv ab \mod n$, then by (328) Eve may infer that $Q = P_{A,B}$.

Tripartite Diffie-Hellman

- Alice, Bob, and Claire agree publically on a supersingular elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ given by $Y^2 = X^3 + 1$ with $q \equiv 2 \mod 3$, and a point $P \in \mathcal{E}(\mathbb{F}_q)$ of order n (large prime).
- Alice, Bob, and Claire choose secret integers $a, b, c \mod n$, resp.
- Alice broadcasts $[a]P$, Bob broadcasts $[b]P$, and Claire broadcasts $[c]P$.
- Alice computes $\tilde{e}_n([b]P, [c]P)^a$, Bob computes $\tilde{e}_n([a]P, [c]P)^b$, and Claire computes $\tilde{e}_n([a]P, [b]P)^c$.
- Then by (325), all three users have computed the same root of unity,

$$\tilde{e}_n(P, P)^{abc}, \tag{331}$$

which can be used to produce a key.

Massey-Omura Analog – Protocol

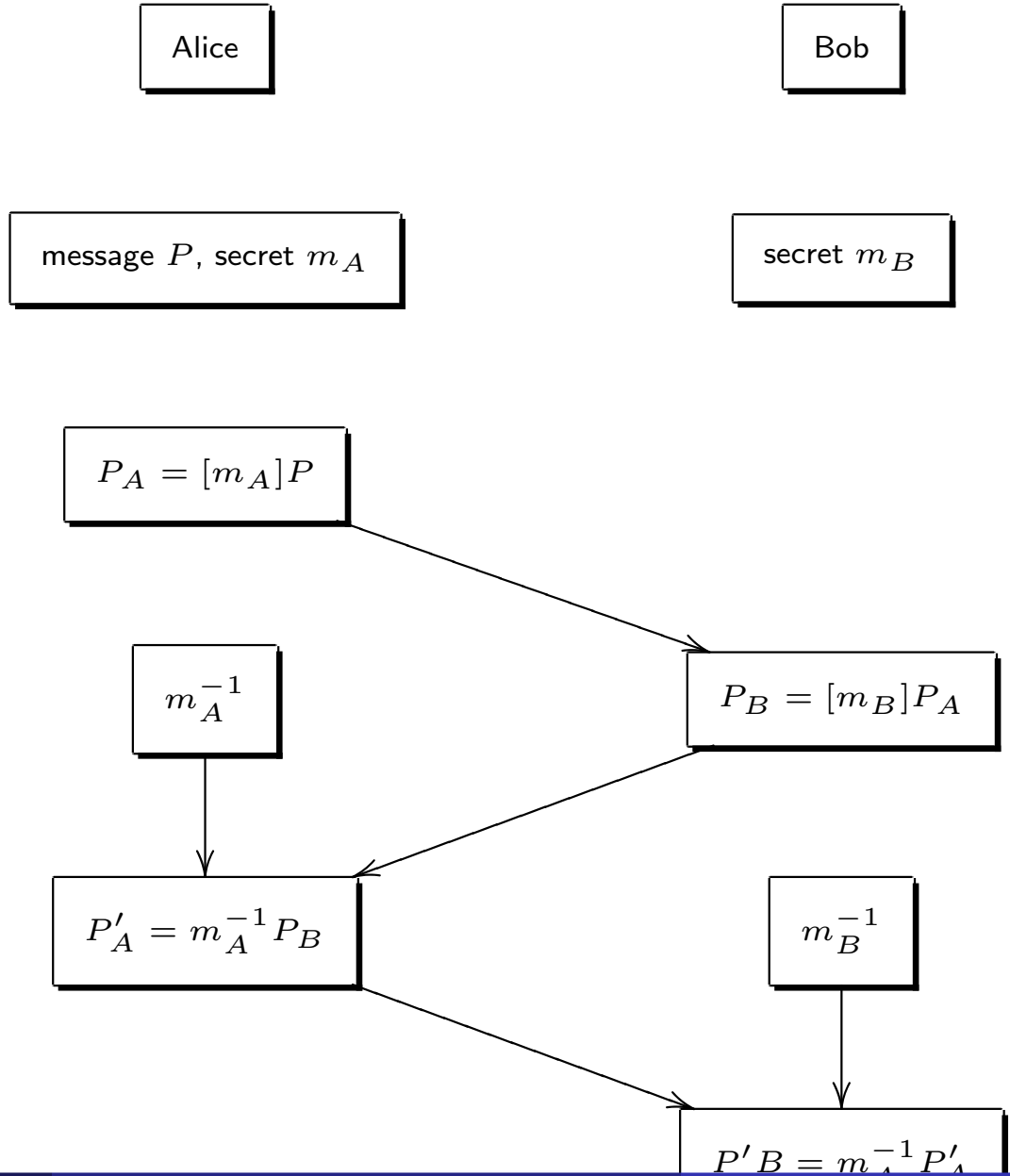
- Users agree on elliptic curve $\mathcal{E}(\mathbb{F}_q)$. Let $N = \sharp\mathcal{E}(\mathbb{F}_q)$.
- Each user takes secret random integer m_U with $(m_U, N) = 1$. Then by the extended Euclidean algorithm,

$$sm_U + rN = 1, \quad s, r \in \mathbb{Z}, \tag{332}$$

  and so $m_U^{-1} \equiv s \mod N$.
- Alice wants to send a message represented as a point $P \in \mathcal{E}(\mathbb{F}_q)$ to Bob.
  - Alice computes $P_A = [m_A]P$ and sends P_A to Bob.
  - Bob computes $P_B = [m_B]P_A$ and sends P_B back.
  - Alice computes $P'_A = [m_A^{-1}]P_B$ and sends P'_A back.
  - Bob computes $P'_B = [m_B^{-1}]P'_A$.

Massey-Omura public key system is rarely used in practice.

Massey-Omura Cryptosystem



Massey-Omura Analog – Correctness

We have

$$P'_B = [m_B^{-1} m_A^{-1} m_B m_A]P = P. \tag{333}$$

Proof.

- We have $m_A^{-1} m_A \equiv 1 \mod N$, i.e., $m_A^{-1} m_A = 1 + kN$ for some integer k.
- The group $\mathcal{E}(\mathbb{F}_q)$ has order N and so $[N]R = O$ for each point $R \in \mathcal{E}(\mathbb{F}_q)$.
- Thus $[m_A^{-1} m_A]R = [1 + kN]R = [1]R + [k][N]R = R$ for each point $R \in \mathcal{E}(\mathbb{F}_q)$.
- Therefore,

$$\begin{aligned} P'_B &= [(m_B^{-1} m_B)(m_A^{-1} m_A)]P \\ &= [m_B^{-1} m_B][m_A^{-1} m_A]P \\ &= [m_B^{-1} m_B]P = P. \end{aligned}$$

ElGamal Analog – Protocol

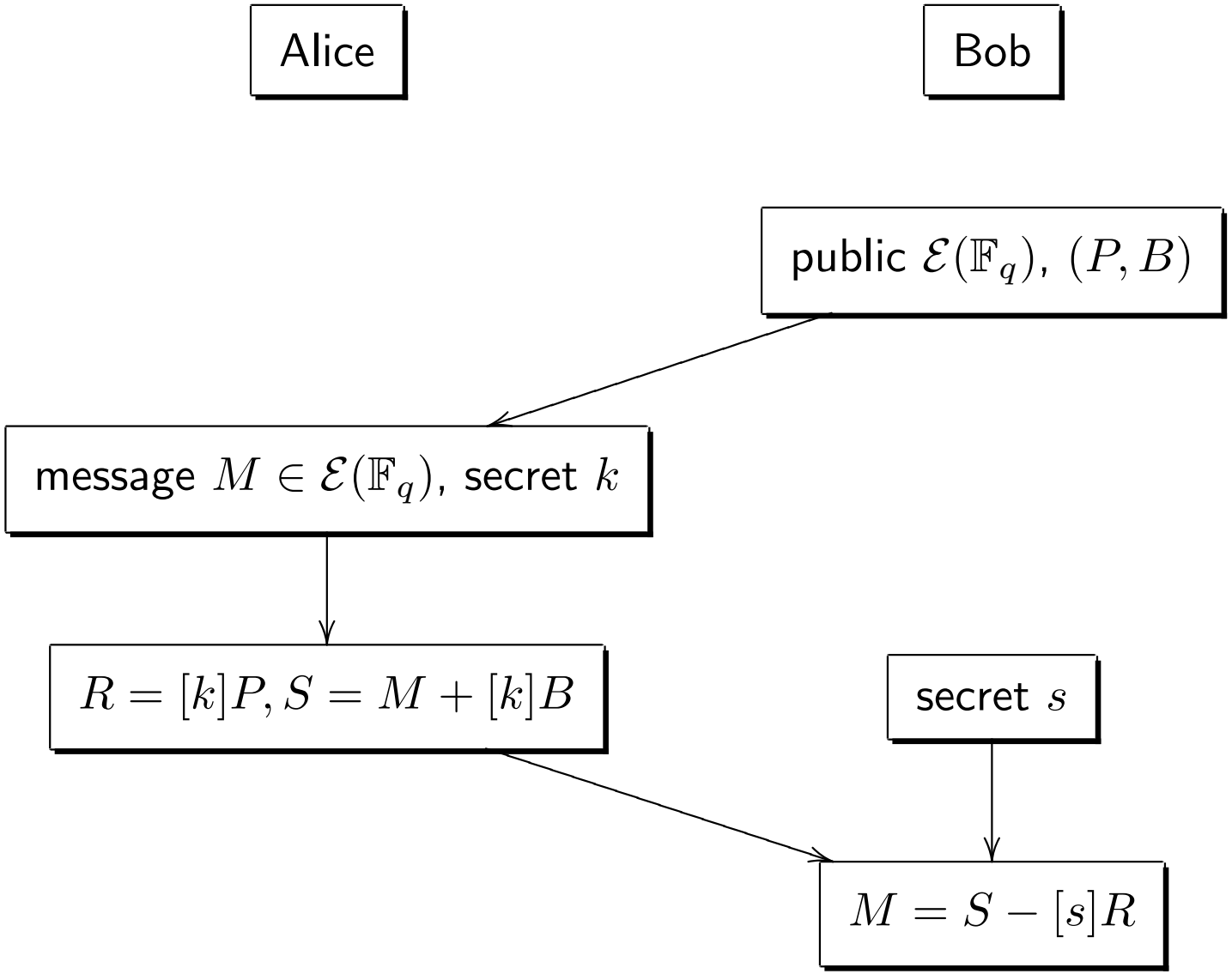
- Each user chooses an elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$, point $P \in \mathcal{E}(\mathbb{F}_q)$, integer s, and computes $B = [s]P$. All data are public up to private key s.
- Alice wants to send a message m to Bob.
  - Alice downloads Bob's public key given by the pair (P, B). Bob has private key s.
  - Alice expresses her message m by a point $M \in \mathcal{E}(\mathbb{F}_q)$.
  - Alice chooses a secret random integer k, computes $R = [k]P$, $S = M + [k]B$, and sends (R, S) to Bob.
  - Bob decrypts by computing

$$M = S - [s]R. \tag{334}$$

Correctness:

$$\begin{aligned} S - [s]R &= (M + [k]B) - [s][k]P \\ &= M + [k][s]P - [s][k]P = M. \end{aligned}$$

ElGamal Cryptosystem



ElGamal Analog – Eavesdropper

- Eavesdropper Eve knows Bob's public key pair (P, B) and the points R, S (transmitted data).
- If Eve is able to calculate discrete logs, she can use P and B to find Bob's private key s and then decrypt $S - [s]R = M$.
- Suppose Alice uses the same key $k = k'$ for two messages M and M'.
  - Eve can recognize this because then

$$R = [k]P = [k']P = R'.$$

  - Suppose M is a public announcement made public one day later, so Eve can find out M.
  - Then Eve finds also the other message M', since

$$M - S + S' = M - (M + [k]B) + (M' + [k'][B]) = M'.$$

ElGamal public key system is rarely used in practice.

Analog of Digital Signature Algorithm – Setup

Alice sets up her signature scheme:

- Alice chooses elliptic curve $\mathcal{E}(\mathbb{F}_q)$, where $\sharp\mathcal{E}(\mathbb{F}_q) = fr$ with r large prime and f small integer.
- Alice choose point $P \in \mathcal{E}(\mathbb{F}_q)$ of order r, secret integer a, and computes $Q = [a]P$.
- Alice makes public the quintuple

$$(\mathbb{F}_q, \mathcal{E}, r, P, Q). \tag{335}$$

No need to keep f secret as it can eventually be deduced from q and r using the Hasse bound (see section about group order).

Digital Signature Algorithm (DSA) is the basis of the Digital Signature Standard (DSS).

Analog of Digital Signature Algorithm – Signature

Alice signs the document m:

- Alice chooses random integer k with $1 \leq k < r$ and computes

$$R = [k]P = (x, y). \tag{336}$$

- Alice computes

$$s = k^{-1}(h(m) + ax) \mod r, \tag{337}$$

  where $h(m)$ is a hash value of m.

- Signed document is the triple

$$(m, R, s). \tag{338}$$

All users take the same cryptographic hash function h.

Analog of Digital Signature Algorithm – Verification

Bob verifies the signature:

- Bob receives (m, R, s) and computes

$$u = s^{-1}h(m) \mod r \quad \text{and} \quad v = s^{-1}x \mod r.$$

- Bob computes $V = [u]P + [v]Q$.
- Bob declares the signature valid if $V = R$.

Correctness:

$$\begin{aligned} V &= [u]P + [v]Q = [s^{-1}h(m)]P + [s^{-1}x][a]P \\ &= [s^{-1}(h(m) + xa)]P \\ &= [k]P = R \text{ by (337).} \end{aligned}$$

The verification process requires two point multiplications, while the ElGamal Digital Signature system needs three point multiplications.

EC Integrated Encryption Scheme (ECIES) – Setup

ECIES is a public key encryption system introduced by Bellare, Rogaway (2001).

- Each user chooses elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ and point $A \in \mathcal{E}(\mathbb{F}_q)$ of large prime order N.
- Each user chooses secret integer s and computes $B = [s]A$.
- Each user has public key

$$(q, \mathcal{E}, N, A, B). \tag{339}$$

- Public data: cryptographic hash functions h_1, h_2 and symmetric encryption function γ_k depending on a key k.

EC Integrated Encryption Scheme (ECIES) – Encryption

Alice encrypts message m as follows:

- Alice downloads Bob's public key $(q, \mathcal{E}, N, A, B)$.
- Alice chooses random integer k with $1 \leq k \leq N - 1$.
- Alice computes points $R = [k]A$ and $Z = [k]B$.
- Alice writes output of hash value $h_1(R, Z)$ as pair (k_1, k_2), where k_1 and k_2 have specific lengths.
- Alice computes ciphertext $c = \gamma_{k_1}(m)$ and hash value (finger print) $t = h_2(c, k_2)$.
- Alice sends (R, c, t) to Bob.

EC Integrated Encryption Scheme (ECIES) – Decryption

Bob decrypts (R, c, t) as follows:

- Bob computes $Z = [s]R$ using his secret key s.
- Bob computes $h_1(R, Z)$ and writes the output as (k_1, k_2).
- Bob computes $t' = h_2(c, k_2)$.
- If fingerprints t, t' are not equal, Bob rejects the ciphertext; otherwise, he continues.
- Bob computes $m = \delta_{k_1}(c)$, where δ_k is the decryption function corresponding to γ_k; i.e., $\delta_k \gamma_k = \mathrm{id}$ is the identity mapping.

Correctness:

$$[s]R = [s][k]A = [k]B = Z.$$

ECIES has the advantage over Massey-Omura, ElGamal and others that the message need not be represented as point on the curve.

Trusted Authority

Organization of trusted autority (TA):

- Each user has a public key based on its identity; e.g., email address.
- Authentication happens in the initial communication between user and TA.
- After this, the user is the only one who has the information to decrypt messages encrypted by its public key.

Trusted Authority – Setup

Consider the supersingular elliptic curve $\mathcal{E}$ over $\mathbb{F}_p$ with $p \equiv 2 \mod 3$ given by $Y^2 = X^3 + 1$ (see Diffie-Hellman).

- Let $\xi \in \mathbb{F}_{p^2}$ be a primitive 3rd root of unity; we have $\xi \notin \mathbb{F}_p$, since $\mathbb{F}_p^*$ has order $p-1$ and $p - 1 \not\equiv 0 \mod 3$.
- Reconsider the mapping

$$\phi : \mathcal{E}(\bar{\mathbb{F}}_{p^2}) \to \mathcal{E}(\bar{\mathbb{F}}_{p^2}) : (x, y) \mapsto (\xi x, y)$$

  with $\phi(O) = O$, which is a group isomorphism.
- Reconsider the modified Weil pairing

$$\tilde{e}_n(P_1, P_2) = e_n(P_1, \phi(P_2)),$$

  where e_n is the usual Weil pairing and $P_1, P_2 \in \mathcal{E}[n]$. $\tilde{e}_n$ is well-defined, since if $P \in \mathcal{E}(\bar{\mathbb{F}}_q)$ has order n, then $\phi(P)$ has also order n.

Trusted Authority – Setup

- If $P \in \mathcal{E}(\mathbb{F}_p)$ has order n and $3 \not| n$, then $\tilde{e}_n(P, P)$ is a primitive n-th root of unity.
- The curve $\mathcal{E}(\mathbb{F}_p)$ is supersingular by (305) and so by (304), $\sharp\mathcal{E}(\mathbb{F}_p) = p + 1$.
- Let $p = 6\ell - 1$ for some prime ℓ. Then for each $P \in \mathcal{E}(\mathbb{F}_p)$, $Q = [6]P$ has order ℓ or 1. Indeed,

$$[\ell]Q = [6\ell]P = [p+1]P = O.$$

- The following primes p, $3 \leq p \leq 1000$, qualify:

$$11,\ 17,\ 29,\ 41,\ 101,\ 113,\ 137,\ 257,\ 281,\ 317,\ 353,\\ 401,\ 617,\ 641,\ 653,\ 677,\ 761,\ 821,\ 941,\ 977,$$

since $11 = 6 \cdot 2 - 1$, $17 = 6 \cdot 3 - 1$, $29 = 6 \cdot 5 - 1$, etc.

Trusted Authority – Setup

TA sets up the system as follows:

- Chooses large prime $p = 6\ell - 1$ for some prime ℓ.
- Chooses point $P \in \mathcal{E}(\mathbb{F}_p)$ of order ℓ.
- Chooses hash functions h_1, h_2:
  - h_1 maps binary strings of arbitrary length to points of order ℓ on $\mathcal{E}(\mathbb{F}_p)$ (see Washington, Exercise 6.8).
  - h_2 maps ℓth roots of unity to binary strings of length n, where n is the message length.
- Chooses secret random value $s \in \mathbb{F}_\ell^*$ and computes $P_{\text{pub}} = [s]P$.
- Makes $p, h_1, h_2, n, P, P_{\text{pub}}$ public, keeps s secret.

Trusted Authority – Private Key Setup

TA acts as follows when user with identity ID wants a private key:

- Computes $Q_{\mathrm{ID}} = h_1(\mathrm{ID})$; this is a point of order ℓ on the curve.
- Computes $D_{\mathrm{ID}} = [s]Q_{\mathrm{ID}}$.
- After verifying that `ID` is the identification of the user with whom it is communicating, TA sends D_{ID} to the user.

D_{ID} is the user's private key.

Trusted Authority – Message Encryption

Alice wants to send message m to Bob.

- Looks up Bob's identity, such as $\mathtt{ID} = \mathtt{bob(at)tuhh.de}$ as binary string, and computes $Q_{\mathtt{ID}} = h_1(\mathtt{ID})$.
- Computes $g_{\mathtt{ID}} = \tilde{e}_\ell(Q_{\mathtt{ID}}, P_{\mathrm{pub}})$.
- Chooses random value r with $1 \leq r \leq \ell - 1$.
- Computes the ciphertext pair

$$c = ([r]P, m \oplus h_2(g_{\mathtt{ID}}^r)),$$

where $\oplus$ denotes the Boolean XOR operation, and sends it to Bob.

Trusted Authority – Message Decryption

Bob decrypts the ciphertext (u, v) given by

$$c = ([r]P, m \oplus h_2(g_{\mathtt{ID}}^r))$$

as follows:

- Uses his private key $D_{\mathtt{ID}}$ to compute $h_{\mathtt{ID}} = \tilde{e}_\ell(D_{\mathtt{ID}}, u)$.
- Computes $m' = v \oplus h_2(h_{\mathtt{ID}})$.

Correctness:

$$\begin{aligned} \tilde{e}_\ell(D_{\mathtt{ID}}, u) &= \tilde{e}_\ell([s]Q_{\mathtt{ID}}, [r]P) = \tilde{e}_\ell(Q_{\mathtt{ID}}, P)^{sr} \\ &= \tilde{e}_\ell(Q_{\mathtt{ID}}, P_{\mathrm{pub}})^r = g_{\mathtt{ID}}^r. \end{aligned}$$

Thus

$$\begin{aligned} m' &= v \oplus h_2(h_{\mathtt{ID}}) = v \oplus h_2(\tilde{e}_\ell(D_{\mathtt{ID}}, u)) \\ &= (m \oplus h_2(g_{\mathtt{ID}}^r)) \oplus h_2(g_{\mathtt{ID}}^r) = m. \end{aligned}$$

Attacking Discrete Log

> There is no provable lower bound for the amount of work of a cryptoanalyst analyzing a public-key cryptosystem.

Arto Salomaa (1990)

Attacking Discrete Log

- Complete enumeration
- Baby-step, giant-step
- Pollard's rho method
- Pohlig-Hellman method
- MOV attack
- Index calculus

Complete Enumeration

The simplest attacking method is complete enumeration.

Require: Given elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$, points P, Q on $\mathcal{E}(\mathbb{F}_q)$, where Q is a multiple of P, integer N is the order of $G = \langle P \rangle$.
Ensure: Integer k with $Q = [k]P$.

```
for k ← 1 to N do
    if Q = [k]P then
        return k
    end if
end for
```

Time complexity $O(N)$.

Baby-Step Giant-Step Algorithm

Given elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ and points P, Q on $\mathcal{E}(\mathbb{F}_q)$, where $Q = [k]P$ is a multiple of P for some integer k (unknown).
Let $G = \langle P \rangle$ with order N (known).

Procedure Outline (Shanks, 1971)

- Fix an integer $m \geq \sqrt{N}$ and compute the point $[m]P$.
- *Baby step:* Generate a list of points $[i]P$, $0 \leq i < m$.
- *Giant step:* Compute the points $Q - [jm]P$, $0 \leq j < m$, until a match with an element of the stored list is found.
- If $[i]P = Q - [jm]P$ is a match, then $Q = [k]P$ with $k \equiv i + jm \mod N$.

Time complexity $O(\sqrt{N})$.

Baby-Step Giant-Step Algorithm

Correctness:

- Since $m^2 > N$ (order), the answer k satisfies $0 \leq k < m^2$.
- Write

$$k = mk_1 + k_0 \tag{340}$$

with $k \equiv k_0 \mod m$, $0 \leq k_0 < m$, and $k_1 = (k - k_0)/m$ with $0 \leq k_1 < m$.

- If $i = k_0$ and $j = k_1$, there is a match:

$$\begin{aligned} Q - [k_1 m]P &= [k]P - [k_1 m]P \\ &= [k - k_1 m]P = [k_0]P. \end{aligned} \tag{341}$$

Baby-Step Giant-Step Algorithm

Require: Elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$, points $P, Q \in \mathcal{E}(\mathbb{F}_q)$, where Q is a multiple of P, integer N is the order of $G = \langle P \rangle$, list $L = []$.

Ensure: Integer k with $Q = [k]P$.

$m \leftarrow \lceil \sqrt{N} \rceil$
for $i = 0$ to $m - 1$ **do**
  Append $(i, [i]P)$ to list L
end for
for $j = 0$ to $m - 1$ **do**
  if $Q - [jm]P$ matches $[i]P$ in list L **then**
    $k \leftarrow i + jm \mod N$
    return k
  end if
end for

Baby-Step Giant-Step Algorithm

- *Baby step:* point addition

$$P + [i-1]P \mapsto [i]P. \tag{342}$$

- *Giant step:* point addition

$$-[mP] + (Q - [(j-1)m]P) \mapsto Q - [jm]P. \tag{343}$$

- The group order N is not needed, only an upper bound as given by the Hasse bound (270),

$$m^2 \geq q + 1 + 2\sqrt{q}. \tag{344}$$

Example

Consider the elliptic curve $\mathcal{E}(\mathbb{F}_{41})$ given by $Y^2 = X^3 + 2X + 1$ and let $P = (0, 1)$, $Q = (30, 40)$.

- By the Hasse bound, $\sharp\mathcal{E}(\mathbb{F}_{41}) \leq 41 + 1 + 2\sqrt{41} = 58.8$.
  For $m = 8$, $m^2 = 64 > 58.8$.
- Baby-step list $[i]P$, $1 \leq i \leq 7$,

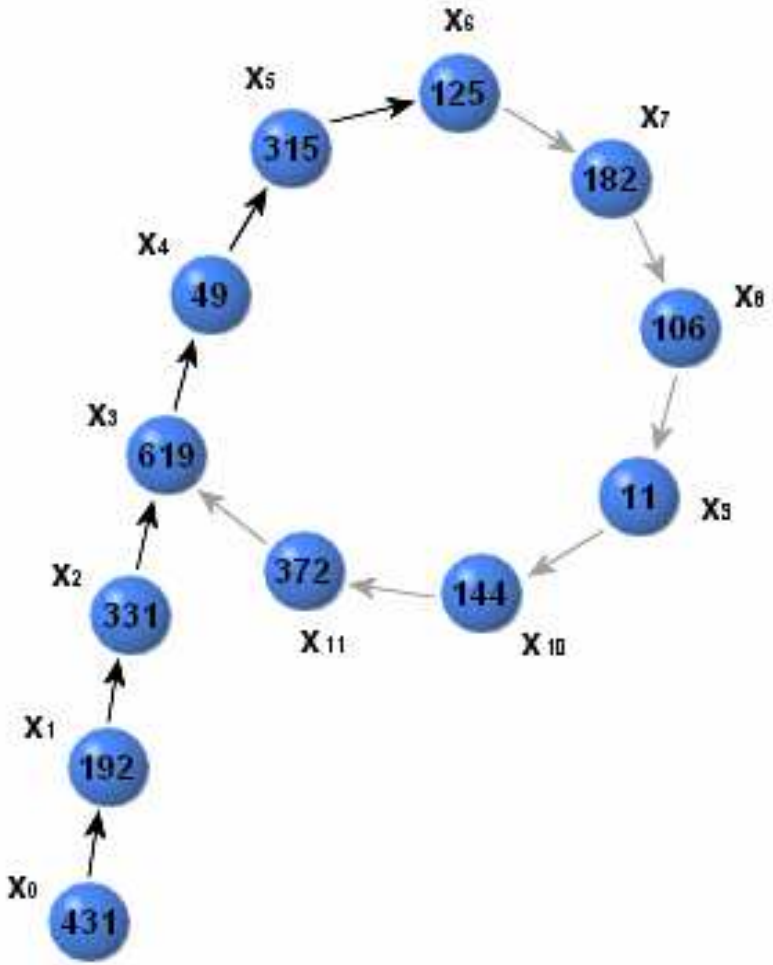
$$(0, 1),\ (1, 39),\ (8, 23),\ (38, 38),\ (23, 23),\ (20, 28),\ \underline{(26, 9)}.$$

- Giant-step list $Q - [8j]P$ for $j = 0, 1, 2$ gives

$$(30, 40),\ (9, 25),\ \underline{(26, 9)}.$$

  So there is a match for $i = 7$ and $j = 2$.
- Thus $Q = (30, 40) = [7 + 2 \cdot 8]P = [23]P$ and hence $k = 23$.

Pollard's rho Method



Pollard's rho Method

Given finite group G of order N and $P, Q \in G$, where Q is a multiple of P. Find an integer k with $Q = [k]P$.

Procedure Outline (Pollard, 1978)

- Choose function $f : G \to G$ that behaves randomly.
- Start with random $P_0 \in G$ and iterate $P_{i+1} = f(P_i)$.
- Since G is finite, there are $i_0 < j_0$ such that $P_{i_0} = P_{j_0}$. Then $P_{i_0+1} = f(P_{i_0}) = f(P_{j_0}) = P_{j_0+1}$. More generally,

$$P_{i_0+\ell} = P_{j_0+\ell}, \quad \ell \geq 0. \tag{345}$$

- The sequence (P_i) is periodic with period $j_0 - i_0$ (or a divisor of $j_0 - i_0$).
- Use Floyd's algorithm for cycle detection.
- Time complexity: The (smallest) equality $P_{i_0} = P_{j_0}$ can be expected after $O(\sqrt{N})$ steps.

Pollard's rho Method

Choice of suitable function f:

- Divide G into r disjoint subsets $S_1, \ldots, S_r$ of approximately equal size, say $r = 20$.
- Choose $2r$ random integers $a_i, b_i \mod N$ and put

$$M_i = [a_i]P + [b_i]Q. \tag{346}$$

- Define

$$f(R) = R + M_i, \quad \text{if } R \in S_i. \tag{347}$$

The iteration of f provides a random walk in G with steps given by the M_i's.

Pollard's rho Method

- Initially choose random integers $a_0, b_0 \mod N$ and put

$$P_0 = [a_0]P + [b_0]Q. \tag{348}$$

- Calculate the sequence $(P_j)_{j\geq 0}$ and keep track of the expression of the points P_j in terms of P and Q.
- If $P_j = [u_j]P + [v_j]Q$ and $f(P_j) = P_{j+1} = P_j + M_i$ with $P_j \in S_i$, then

$$P_{j+1} = P_j + M_i = [u_j + a_i]P + [v_j + b_i]Q \tag{349}$$

with

$$(u_{j+1}, v_{j+1}) = (u_j + a_i, v_j + b_i) = (u_j, v_j) + (a_i, b_i). \tag{350}$$

Pollard's rho Method

- If there is a match $P_{j_0} = P_{i_0}$, then

$$[u_{j_0}]P + [v_{j_0}]Q = [u_{i_0}]P + [v_{i_0}]Q \tag{351}$$

and so

$$[u_{i_0} - u_{j_0}]P = [v_{j_0} - v_{i_0}]Q. \tag{352}$$

- If $(v_{j_0} - v_{i_0}, N) = d$, then $(v_{j_0} - v_{i_0}, N/d) = 1$ and so

$$k \equiv (v_{j_0} - v_{i_0})^{-1}(u_{i_0} - u_{j_0}) \mod N/d. \tag{353}$$

- Since $1 \leq k \leq N$, this gives d choices for k: $k, 2k, \ldots, dk$. Often d is small and so all possibilities can be tried to get $Q = [k]P$.
- In cryptography, N is often prime and so $d = 1$ or $d = N$.

  If $d = N$, we need to start from the beginning; otherwise, $d = 1$ and we obtain directly k.

Pollard's rho Method

Require: Elliptic curve $\mathcal{E}(\mathbb{F}_q)$, points $P, Q \in \mathcal{E}(\mathbb{F}_q)$, where Q is a multiple of P, integer $N = \sharp\mathcal{E}(\mathbb{F}_q)$, point P_0, partition $S_1, \ldots, S_r$ of $G = \langle P \rangle$, and function $f : \mathcal{E}(\mathbb{F}_q) \to \mathcal{E}(\mathbb{F}_q)$.

Ensure: Integer k with $Q = [k]P$.

noMatch $\leftarrow$ TRUE

$i \leftarrow 0$

while noMatch **do**

$i \leftarrow i + 1$ {Use Floyd's algorithm}

$P_i \leftarrow f(P_{i-1})$

$P_{2i} \leftarrow f(f(P_{2(i-1)}))$

if $P_i = P_{2i}$ **then**

noMatch $\leftarrow$ FALSE

end if

end while

Compute k from P_i and P_{2i}.

Example

Consider the elliptic curve $\mathcal{E}(\mathbb{F}_{1093})$ given by $Y^2 = X^3 + X + 1$ and points $P = (0, 1)$ with order 1067 and $Q = (431, 959)$. Find an integer k with $Q = [k]P$.

- Let $r = 3$ and put

$$P_0 = [3]P + [5]Q, \quad M_0 = [4]P + [3]Q,$$
$$M_1 = [9]P + [17]Q, \quad M_2 = [19]P + [6]Q.$$

- Define $f : \mathcal{E}(\mathbb{F}_{1093}) \to \mathcal{E}(\mathbb{F}_{1093})$ by

$$f(x, y) = (x, y) + M_i, \quad \text{if } x \equiv i \mod 3,$$

where x is an integer with $0 \le x < 1093$. For instance,

$$f(P_0) = P_0 + M_2 = (727, 589)$$

since $P_0 = (326, 69)$ and $326 \equiv 2 \mod 3$.

Example (cont'd)

- Compute the sequence (P_j),

$$P_0 = (326, 69), \quad P_1 = (727, 589), \quad P_2 = (560, 365),$$
$$P_3 = (1070, 260), \quad P_4 = (473, 903), \quad \underline{P_5 = (1006, 951)},$$
$$\ldots$$
$$P_{57} = (895, 337), \quad \underline{P_{58} = (1006, 951)}, \quad P_{59} = (523, 938),$$
$$\ldots$$

  There is a match: $P_5 = P_{58}$.

- By keeping track of the coefficients of P and Q,

$$P_5 = [88]P + [46]Q \quad \text{and} \quad P_{58} = [685]P + [620]Q.$$

- Thus

$$O = P_{58} - P_5 = [597]P + [574]Q.$$

  Then $O = [597]P + [574][k]P = [597 + k \cdot 574]P$.

Example (cont'd)

- Since P has order 1067,

$$597 + k \cdot 574 \equiv 0 \mod 1067$$

or $597 \equiv k(-574) \mod 1067$. We have

$$(-574)^{-1} \cdot 597 \equiv 499 \mod 1067$$

and so $Q = [499]P$ with $k = 499$.

- The points $P_0, P_1, \ldots, P_{58}$ need to be stored until a match is found: $P_5 = P_{58}$. It can be done better ...
- By Floyd's cycle detection method, compute the pairs (P_i, P_{2i}) and store only the current pair. Then for $i = 53$ there is a match $P_{53} = P_{106}$ with

$$[620]P + [557]Q = P_{53} = P_{106} = [1217]P + [1131]Q.$$

Thus $[597]P + [574]Q = O$ (as above) and hence $k = 499$.

Floyd's Cycle Detection Algorithm (1969)

Given finite set S, function $f : S \to S$, and $x_0 \in S$.

- Define the sequence (x_i) as follows,

$$x_{i+1} = f(x_i), \quad i \geq 0. \tag{354}$$

- Since S is finite, there are smallest non-negative integers μ (initial segment) and λ (period) such that

$$x_\mu = x_{\lambda+\mu}. \tag{355}$$

Example

Let $S = \mathbb{Z}_{11}$, $f(x) = (x^2 + 1) \mod 11$, and $x_0 = 3$.
The sequence is

$$3, 10, 2, 5, 4, 6, 4, 6, 4, 6, \ldots$$

and so $\mu = 4$ and $\lambda = 2$, i.e., $x_4 = 4 = x_6$ with $6 = 4 + 2$.

Floyd's Cycle Detection Algorithm

- By (355), for each $i \geq \mu$ and $k \geq 0$,

$$x_i = x_{i+k\lambda}, \tag{356}$$

since looping k times the cycle of length λ ends in the same element of the cycle.

- By (356), we have

$$i = k\lambda \geq \mu \text{ for some } k \quad \Longleftrightarrow \quad x_i = x_{2i}. \tag{357}$$

- Existence of i in (357): Take the smallest $i \geq \mu$ which has the form $i = k\lambda$ inside the cycle. Then by (356), $x_i = x_{2i}$.

Example: $\mu = 5$ and $\lambda = 3$. Then $i = 2 \cdot 3 = 6 \geq 5$ and $2i = 12 = 6 + 2 \cdot 3$.

Floyd's Cycle Detection Algorithm

Floyd's algorithm:

1. Find in (x_i) the first repeated element x_μ, $\mu \leq i_0$.
2. Find in (x_i) the shortest repeated cycle $x_{\mu+\lambda}$, $\lambda \geq 1$.
3. Compute simultaneously (x_i) and (x_{2i}) to find the minimal $i_0 \geq \mu$ with $x_{i_0} = x_{2i_0}$; it is only necessary to store the current elements x_i and x_{2i}.

Example

Let $S = \mathbb{Z}_{11}$, $f(x) = (x^2 + 1) \mod 11$, and $x_0 = 3$.

i	0	1	2	3	**4**	5	6
x_i	3	10	2	5	4	6	4
x_{2i}	3	2	4	4	4	4	4

Pohlig-Hellman Method (1978)

Given finite group G and $P, Q \in G$, where P has order N and Q is a multiple of P. Find an integer k with $Q = [k]P$.

- Suppose N has known prime factorization

$$N = \prod_i p_i^{e_i}. \tag{358}$$

- Idea: Find $k \mod p_i^{e_i}$ for each i, use the Chinese Remainder theorem to combine these and obtain $k \mod N$.
- For prime p with $p \mid N$, write k in its base p expansion

$$k = k_0 + k_1 p + k_2 p^2 + \ldots \tag{359}$$

with $0 \leq k_i < p$.

- Idea: Evaluate $k \mod p^e$ by successively determining $k_0, \ldots, k_{e-1}$, where p^e is the exact power of p dividing N.

Pohlig-Hellman Method

For each prime p with p^e being the exact power of p dividing N:

1. Compute list L with elements
$$[jN/p]P, \quad 0 \le j \le p-1.$$
2. Determine k_0 with $[N/p]Q = [k_0 N/p]P$ using L.
3. If $e = 1$, stop; otherwise continue.
4. Let $Q = Q_0$. Suppose $k_0, \ldots, k_{r-1}$ and $Q_1, \ldots, Q_{r-1}$ have already been computed.
5. Let $Q_r = Q_{r-1} - [k_{r-1} p^{r-1}]P$.
6. Determine k_r with $[N/p^{r+1}]Q_r = [k_r N/p]P$ using L.
7. If $r = e - 1$, stop; otherwise, increment r and return to 4.

Then
$$k \equiv k_0 + k_1 p + \ldots + k_{e-1} p^{e-1} \mod p^e.$$

Pohlig-Hellman Method

Correctness:

- We have

$$\begin{aligned}[N/p]Q &= [(N/p)k]P = [(N/p)(k_0 + k_1 p + \ldots)]P \\ &= [k_0 N/p]P + [(k_1 + k_2 p + \ldots)N]P \\ &= [k_0 N/p]P,\end{aligned}$$

since $[N]P = O$. Thus step 2 finds k_0.

- Then

$$Q_1 = Q - [k_0]P = [k_1 p + k_2 p^2 + \ldots]P$$

and so

$$\begin{aligned}[N/p^2]Q_1 &= [(k_1 + k_2 p + \ldots)N/p]P \\ &= [k_1 N/p]P + [(k_2 + k_3 p + \ldots)N]P \\ &= [k_1 N/p]P,\end{aligned}$$

since $[N]P = O$. Thus step 6 finds k_1, and so on.

Pohlig-Hellman Method

Require: Elliptic curve $\mathcal{E}(\mathbb{F}_q)$, points $P, Q \in \mathcal{E}(\mathbb{F}_q)$, where P has order N and Q is a multiple of P, prime factorization $N = \prod_{i=1}^{m} p_i^{e_i}$.

Ensure: Integer k with $Q = [k]P$.

$Q_0 \leftarrow Q$

for $i = 1$ to m **do**

$L \leftarrow \{[jN/p_i]P \mid 0 \le j \le p_i - 1\}$.

Find k_0 with $[N/p_i]Q = [k_0 N/p_i]P$ using L.

for $r = 1$ to $e_r - 1$ **do**

$Q_r \leftarrow Q_{r-1} - [k_{r-1} p_i^{r-1}]P$

Find k_r with $[N/p_i^{r+1}]Q_r = [k_r N/p_i]P$ using L.

end for

$k^{(i)} \leftarrow k_0 + k_1 p_i + \ldots + k_{e_i - 1} p^{e_i - 1}$

end for

Use the Chinese Remainder theorem to obtain $k \mod N$ from $k^{(i)} \mod p_i^{e_i}$, $1 \le i \le m$.

Example

Consider the elliptic curve $\mathcal{E}(\mathbb{F}_{599})$ given by $Y^2 = X^3 + 1$. $P = (60, 19)$ has order $N = 600$ and $Q = (277, 239)$.

Find integer k with $Q = [k]P$.

- Prime factorization $600 = 2^3 \cdot 3 \cdot 5^2$.
- Compute k mod 8, mod 3, and mod 25, and combine the results by the Chinese Remainder theorem to obtain k mod 600.

Example (cont'd)

Computation k mod 8:

- Compute list L:

$$[0 \cdot N/2]P = O \quad \text{and} \quad [1 \cdot N/2]P = (598, 0).$$

- $[N/2]Q = O = [0 \cdot N/2]P$ and so $k_0 = 0$. Thus

$$Q_1 = Q - [0]P = Q.$$

- $[N/4]Q_1 = [150]Q_1 = (598, 0) = [1 \cdot N/2]P$ and so $k_1 = 1$. Thus

$$Q_2 = Q_1 - [1 \cdot 2]P = (35, 243).$$

- $[N/8]Q_2 = [75]Q_2 = O = [0 \cdot N/2]P$ and so $k_2 = 0$.
- Therefore,

$$k = 0 \cdot 1 + 1 \cdot 2 + 0 \cdot 4 \equiv 2 \mod 8.$$

Example (cont'd)

Computation k mod 3:

- Compute list L:

$$\begin{aligned} [0 \cdot N/3]P &= O, \\ [1 \cdot N/3]P &= (0, 1), \\ [2 \cdot N/3]P &= (0, 598). \end{aligned}$$

- $[N/3]Q = (0, 598) = [2 \cdot N/3]P$ and so $k_0 = 2$.
- Therefore,

$$k \equiv 2 \mod 3.$$

Example (cont'd)

Computation k mod 25:

- Compute list L:

$$\begin{aligned} [0 \cdot N/5]P &= O, \\ [1 \cdot N/5]P &= (84, 179), \\ [2 \cdot N/5]P &= (491, 134), \\ [3 \cdot N/5]P &= (491, 465), \\ [4 \cdot N/5]P &= (84, 420). \end{aligned}$$

- $[N/5]Q = (84, 179) = [1 \cdot N/5]P$ and so $k_0 = 1$. Thus

$$Q_1 = Q - [1]P = (130, 129).$$

- $[N/25]Q_1 = (491, 465) = [3 \cdot N/5]P$ and so $k_1 = 3$.
- Therefore,

$$k = 1 + 3 \cdot 5 \equiv 16 \mod 25.$$

Example (cont'd)

Simultaneous congruences

$$\begin{aligned} k &\equiv 2 \mod 8, \\ k &\equiv 2 \mod 3, \\ k &\equiv 16 \mod 25. \end{aligned}$$

By the Chinese Remainder theorem, $k = 266$ is uniquely determined modulo 600.

Pohlig-Hellman Method

- The method works well if all prime numbers dividing N are small.
- If p is a large prime factor, it will be hard to list all elements of L.
- For cryptographic reasons (discrete log), the group order should contain a large prime factor.

The MOV Attack

- Introduced by Menezes, Okamoto, Vanstone (1990).
- Uses Weil pairing to reduce the discrete log problem in $\mathcal{E}(\mathbb{F}_q)$ to one in $\mathbb{F}_{q^m}^*$.
- Discrete log problem in $\mathbb{F}_{q^m}^*$ can be attacked more efficiently (index calculus).
- For supersingular curves, the discrete log problem reduces to one in smaller group $(m = 2)$.

The MOV Attack

Given elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$ and $P, Q \in \mathcal{E}(\mathbb{F}_q)$, where N is the order of P and $(N, q) = 1$. Find an integer k with $Q = [k]P$.

There exists an integer k such that

$$Q = [k]P \quad \Longleftrightarrow \quad [N]Q = O \text{ and } e_N(P, Q) = 1. \tag{360}$$

Proof.

- Let $Q = [k]P$. Then $[N]Q = [N][k]P = [k][N]P = O$. Moreover,

$$e_N(P, Q) = e_N(P, [k]P) = e_N(P, P)^k = 1^k = 1.$$

- Let $[N]Q = O$ and $e_N(P, Q) = 1$. Then $Q \in \mathcal{E}[N]$. Since $(N, q) = 1$, by (209), $E[N] \simeq \mathbb{Z}_N \oplus \mathbb{Z}_N$. Choose R such that $\{P, R\}$ is a basis of $\mathcal{E}[N]$. Then there are integers a, b with

$$Q = [a]P + [b]R.$$

Since $\{P, R\}$ is a basis of $\mathcal{E}[N]$, $e_N(P, R) = \xi$ is a primitive N-th root of unity and so

$$1 = e_N(P, Q) = e_N(P, P)^a e_N(P, R)^b = 1^a \xi^b = \xi^b.$$

Thus $b \equiv 0 \mod N$ and so $[b]R = O$. Hence, $Q = [a]P$. □

The MOV Attack

- All points of $\mathcal{E}[N]$ have coordinates in the algebraic closure

$$\bar{\mathbb{F}}_q = \bigcup_{k \geq 1} \mathbb{F}_{q^k}.$$

  Since the set $\mathcal{E}[N]$ is finite, there exists $m \geq 1$ such that

$$E[N] \subset \mathcal{E}(\mathbb{F}_{q^m}).$$

- It follows that for the set of N-th roots of unity U_N,

$$U_N \subset \mathbb{F}_{q^m}^*.$$

- All calculations can be made in $\mathbb{F}_{q^m}$.

Procedure Outline

- Choose random point $T \in \mathcal{E}(\mathbb{F}_{q^m})$.
- Compute the order M of T.
- Let $d = (M, N)$ and $T_1 = [M/d]T$. Then T_1 has order d. Since d divides N, $[N]T_1 = O$ and so $T_1 \in \mathcal{E}[N]$.
- Compute $\xi_1 = e_N(P, T_1)$ and $\xi_2 = e_N(Q, T_1)$. Then ξ_1, ξ_2 are in $U_d \subset \mathbb{F}_{q^m}^*$, since $\xi_1^d = e_N(P, [d]T_1) = e_N(P, O) = 1$ and $\xi_2^d = e_N(Q, [d]T_1) = e_N(Q, O) = 1$.
- We have

$$\xi_2 = e_N(Q, T_1) = e_N([k]P, T_1) = e_N(P, T_1)^k = \xi_1^k$$

- Solve the discrete log problem $\xi_2 = \xi_1^k$ in $\mathbb{F}_{q^m}^*$. This gives $k \mod d$ as ξ_1, ξ_2 are d-th roots of unity.
- Repeat with random points T until the lcm of the various d's obtained is N. This gives $k \mod N$.

The MOV Attack

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$, let $t = q + 1 - \sharp\mathcal{E}(\mathbb{F}_q) = 0$, and let $N \geq 1$. If there exists a point $P \in \mathcal{E}(\mathbb{F}_q)$ of order N, then $\mathcal{E}[N] \subset \mathcal{E}(\mathbb{F}_{q^2})$.

Note: An elliptic curve over $\mathbb{F}_q$ with $q = p^r$ is supersingular if $t \equiv 0 \mod p$. This is equivalent to $t = 0$ when $q = p \geq 5$.

Proof.

- The Frobenius endomorphism ϕ_q fulfills $\phi_q^2 - [t]\phi_q + [q] = 0$. Since $t = 0$, we obtain

$$\phi_q^2 + [q] = 0.$$

- Let $P \in \mathcal{E}[N]$ have order N. We have $\sharp\mathcal{E}(\mathbb{F}_q) = q + 1$ and so $N \mid q + 1$ (as element order divides group order) and so $-q \equiv 1 \mod N$. Thus

$$\phi_q^2(P) = -[q]P = [1]P = P.$$

But by (263), $\phi_q^2(P) = P$ is equivalent to $P \in \mathcal{E}(\mathbb{F}_{q^2})$.

□

Index Calculus

Given prime p and generator g of cyclic group $\mathbb{F}_p^*$, i.e.,

$$\langle g \rangle = \{g, g^2, \ldots, g^{p-1} = 1\}. \tag{361}$$

- Each $h \in \mathbb{F}_p^*$ (order $p-1$) can be written as $h = g^k$ for some integer k which is uniquely determined mod $p-1$.
- Let $k = \log_g(h)$ denote the *discrete logarithm* of h w.r.t. g and p; i.e.,

$$g^{\log_g(h)} \equiv h \mod p. \tag{362}$$

- For nonzero group elements h_1, h_2,

$$g^{\log_g(h_1 h_2)} \equiv h_1 h_2 \equiv g^{\log_g(h_1)+\log_g(h_2)} \mod p \tag{363}$$

and so

$$\log_g(h_1 h_2) \equiv \log_g(h_1) + \log_g(h_2) \mod p-1. \tag{364}$$

Example

Let $p = 1217$. The cyclic group $\mathbb{F}_{1217}^*$ has generator $g = 3$. Solve

$$3^k \equiv 37 \mod 1217.$$

Outline:

- Choose factor basis $B = \{2, 3, 5, 7, 11, 13\}$.
- Find relations of the form

$$3^x = \pm\text{product of some primes in } B \mod 1217.$$

- Compute $3^j \cdot 37 \mod 1217$ for several random values j until an integer is obtained that factors into a product of primes in B.

Example (cont'd)

Find relations of the form $3^x \equiv \pm$product of primes in $B \mod p$:

$$\begin{aligned} 3^1 &\equiv 3 \mod 1217, \\ 3^{24} &\equiv -2^2 \cdot 7 \cdot 13 \mod 1217, \\ 3^{25} &\equiv 5^3 \mod 1217, \\ 3^{30} &\equiv -2 \cdot 5^2 \mod 1217, \\ 3^{54} &\equiv -5 \cdot 11 \mod 1217, \\ 3^{87} &\equiv 13 \mod 1217, \end{aligned}$$

and $3^{(p-1)/2} \equiv -1 \mod 1217$ with $(p-1)/2 = 608$.

Example (cont'd)

Convert congruences into discrete logs writing $L(h) = \log_g(h)$ with $g = 3$:

$$\begin{aligned}
1 &\equiv L(3) \mod 1216,\\
24 &\equiv 608 + 2L(2) + L(7) + L(13) \mod 1216,\\
25 &\equiv 3L(5) \mod 1216,\\
30 &\equiv 608 + L(2) + 2L(5) \mod 1216,\\
54 &\equiv 608 + L(5) + L(11) \mod 1216,\\
87 &\equiv L(13) \mod 1216,
\end{aligned}$$

and $L(-1) = (p-1)/2 = 608$.

Example (cont'd)

These congruences yield the discrete logs of the elements of the factor basis:

$$\begin{aligned} L(3) &\equiv 1 \mod 1216, \\ L(5) &\equiv 819 \mod 1216, \\ L(13) &\equiv 87 \mod 1216, \\ L(2) &\equiv 216 \mod 1216, \\ L(11) &\equiv 1059 \mod 1216, \\ L(7) &\equiv 113 \mod 1216. \end{aligned}$$

For instance, $25 \equiv 3L(5) \mod 1216$ and so

$$L(5) \equiv 3^{-1} \cdot 25 \equiv 811 \cdot 25 \equiv 819 \mod 1216.$$

Moreover, $30 \equiv 608 + L(2) + 2L(5) \mod 1216$ and so

$$L(2) \equiv 30 - 608 - 2 \cdot 819 \equiv 216 \mod 1216.$$

Example (cont'd)

Compute

$$3^j \cdot 37 \mod 1217$$

for several random values j until an integer is obtained that factors into a product of primes in B.

Here we find for $j = 16$,

$$3^{16} \cdot 37 \equiv 616 \equiv 2^3 \cdot 7 \cdot 11 \mod 1217.$$

Thus

$$L(37) \equiv 3L(2) + L(7) + L(11) - 16L(3) \equiv 588 \mod 1216.$$

Hence,

$$3^{588} \equiv 37 \mod 1217.$$

Index Calculus

- The choice of the factor basis B is important.
- Good method for producing relations of the form

$$g^x \equiv \text{product of primes in } B \text{ mod } p$$

  is the quadratic sieve with subexponential running time $O(\exp(\sqrt{2 \ln p \ln \ln p}))$.
- The presented methods for attacking discrete logs for elliptic curves over $\mathbb{F}_p$ have exponential running time $O(\sqrt{p})$ with $\sqrt{p} = \exp(\frac{1}{2} \ln p)$.
- The index calculus can be generalized to the algebraic number field sieve using algebraic number theory.
- There is an analog of the index calculus for groups given by hyperelliptic curves.

*Primality Testing

> The problems of factorization and primality testing are related, but very different in nature. The largest announced factorization up to the year 2007 was of an integer with 200 digits. However, it was at that time possible to prove primality of primes of several thousand digits.

Lawrence C. Washington (2008).

Primality Testing

Prove that a number n is composite without presenting a factor:

- Fermat's Little theorem says that if n is prime and a is any integer with $(a, n) = 1$, then $a^{n-1} \equiv 1 \mod n$.
- Thus if $a^{n-1} \not\equiv 1 \mod n$ for some integer a with $(a, n) = 1$, then n is composite.

Primality Testing

- When an integer n is composite, it can be proved by knowing a nontrivial factor or failing a pseudoprimality test.
- When n is prime, passing several pseudoprimality tests is an indication that n is probably prime.
- The Goldwasser-Kilian (GK) primality test is based on elliptic curves and was used to prove primality of numbers with more than 1000 decimal digits.
- A classical version of this test is the Pocklington primality test.

Pocklington Primality Test

Given integer $n > 1$. Suppose there exists a prime q with $q \mid n-1$ and $q > \sqrt{n} - 1$. If there exists an integer a with

1. $a^{n-1} \equiv 1 \mod n$ and
2. $\left(a^{(n-1)/q} - 1, n\right) = 1,$

then n is prime.

Proof.

If n is composite, there exists a prime p with $p \mid n$ and $p \leq \sqrt{n}$. Since $q > p-1$, $(q, p-1) = 1$ and so there exists an integer v with $vq \equiv 1 \mod p-1$. Then by condition 1,

$$a^{(n-1)/q} \equiv a^{vq(n-1)/q} = a^{v(n-1)} \equiv 1 \mod p.$$

Thus p divides $a^{(n-1)/q} - 1$ and n contradicting condition 2. □

Pocklington Primality Test

- The test is probabilistic in the sense that a randomly chosen a may or may not fulfill condition 2. If it fails to satisfy condition 1, then n is not prime.
- Once an a is found ($a = 2$ will usually work), the test exhibits that n is prime.
- Unlike the primality test of Miller-Rabin, Pocklington's test yields "n is prime" with certainty.
- There is a more general version which is based on the prime factorization of $n - 1$.

Pocklington Primality Test

Given integer $n > 1$ with $n - 1 = rs$ and $r \geq \sqrt{n}$. Suppose for each prime ℓ with $\ell \mid r$, there exists an integer a_ℓ with

1. $a_\ell^{n-1} \equiv 1 \mod n$ and
2. $\left(a_\ell^{(n-1)/\ell} - 1, n\right) = 1.$

Then n is prime.

Proof.

Let p be prime with $p \mid n$ and $r = r'\ell^e$ with $\ell \nmid r'$.

- Let $b \equiv a_\ell^{(n-1)/\ell^e} \mod p$. Then

$$b^{\ell^e} \equiv a_\ell^{n-1} \equiv 1 \mod p$$

and

$$b^{\ell^{e-1}} \equiv a_\ell^{(n-1)/\ell} \not\equiv 1 \mod p,$$

since $\left(a_\ell^{(n-1)/\ell} - 1, n\right) = 1$. So the order of b mod p is ℓ^e. Hence, $\ell^e \mid p-1$.

- For each prime power factor ℓ^e of r, $\ell^e \mid p-1$ and so $r \mid p-1$. Thus

$$p > r \geq \sqrt{n}.$$

If n is composite, it has a prime factor of at most $\sqrt{n}$. But this is not the case and so n is prime. □

Example

Let $n = 153533$. Then $n - 1 = 4 \cdot 131 \cdot 293$. Let $r = 4 \cdot 131$. The primes dividing r are $\ell = 2$ and $\ell = 131$.

- We have

$$2^{n-1} \equiv 1 \mod n \quad \text{and} \quad \left(2^{(n-1)/2} - 1, n\right) = 1$$

and so $a_2 = 2$.

- We have

$$2^{n-1} \equiv 1 \mod n \quad \text{and} \quad \left(2^{(n-1)/131} - 1, n\right) = 1$$

and so $a_{131} = 2$.

By the Pocklington test, 153533 is prime.

Goldwasser-Kilian Primality Test

Given integer $n > 1$ and elliptic curve $\mathcal{E}$ mod n. Suppose there exist distinct primes $\ell_1, \ldots, \ell_k$ and finite points $P_1, \ldots, P_k \in \mathcal{E}(\mathbb{Z}_n)$ such that

1. $[\ell_i]P_i = O$ for $1 \leq i \leq k$,
2. $\prod_{i=1}^{k} \ell_i > \left(n^{1/4} + 1\right)^2$.

Then n is prime.

Proof.

Let p be prime with $n = n'p^e$ with $p \not| n'$. Then by the Chinese Remainder theorem,

$$\mathcal{E}(\mathbb{Z}_n) = \mathcal{E}(\mathbb{Z}_{p^e}) \oplus \mathcal{E}(\mathbb{Z}_{n'}).$$

- The point P_i has finite order and so $P_i \mod p^e$ has finite order. By further reduction, $P_{i,p} = P_i \mod p$ has finite order. Since $[\ell_i]P_i = O$, $[\ell_i]P_{i,p} = O$ in $\mathcal{E}(\mathbb{F}_p)$. i.e., $P_{i,p}$ has order ℓ_i. Thus $\ell_i \mid \sharp\mathcal{E}(\mathbb{F}_p)$ for all i and hence $\prod_i \ell_i \mid \mathcal{E}(\mathbb{F}_p)$.
- By the Hasse bound (270),

$$\left(n^{1/4}+1\right)^2 < \prod_i \ell_i \leq \sharp\mathcal{E}(\mathbb{F}_p) \leq p+1+2\sqrt{p} = \left(p^{1/2}+1\right)^2.$$

Thus $p > \sqrt{n}$ and so all prime factors of n are $> \sqrt{n}$. Hence, n is prime. □

Example

Let $n = 907$ and $\mathcal{E}$ be the elliptic curve given by $Y^2 = X^3 + 10X - 2 \mod n$.

- Let $\ell = 71$. Then

$$\ell > \left(907^{1/4} + 1\right)^2 \approx 42.1.$$

  Let $P = (819, 784)$. Then $[71]P = O$ and so 907 is prime.

- How to find $\mathcal{E}$ and P?
  - First look at elliptic curves $\mathcal{E}$ mod 907 until one is found whose order is divisible by a prime ℓ with ℓ slightly larger than 42.1.
  - Take a known point Q on $\mathcal{E}$ mod 907. Here, say $Q = (1, 3)$. Using BSGS, Q has order $923 = 13 \cdot 71$. Then $P = [13]Q$ has order 71.

Factorization

> A key reason for the increasing interest in elliptic curves on the part of cryptographers is the recent ingenious use of elliptic curves by H.W. Lenstra to obtain a new factorization method that in many respects is better than the earlier known ones. The discovery of an improvement using an unexpected new device serves as a warning that one should never be too complacent about the supposed imperviousness of the factoring problem to dramatic breakthroughs.

Neal Koblitz (1994).

Pollard's $p-1$ Method

Require: Composite odd integer $n > 1$
Ensure: Factorization of n
  Choose integer bound B {Factor basis}
  $k \leftarrow \operatorname{lcm}(1, 2, \ldots, B)$
  Choose random integer a with $2 \leq a \leq n-1$
  Compute $a^k \mod n$ by repeated squaring
  Compute $d \leftarrow (a^k - 1, n)$ by Euclidean algorithm
  if $d > 1$ **then**
    Restart with n replaced by n/d {Factor of n is d}
  else
    Restart with new choice of B and a
  end if

Correctness

Let k be divisible by all positive integers $\leq B$.
Suppose p is prime with $p \mid n$ and $p-1$ is a product of small primes $\leq B$.
Then k is multiple of $p-1$. Thus by Fermat's Little theorem,

$$a^{p-1} \equiv 1 \mod p$$

and so

$$a^k \equiv 1 \mod p.$$

Hence, $p \mid (a^k - 1, n)$ provides a nontrivial factor of n. □

The Pollard $p-1$ method does not work if all prime divisors p of n have $p-1$ divisible by a large prime.

Example

Let $n = 540143$. Choose $B = 8$. Then

$$k = \operatorname{lcm}(1, 2, \ldots, 8) = 840.$$

Pick $a = 2$. Then

$$2^{840} \equiv 53047 \mod n$$

and

$$(53046, n) = 421.$$

Thus

$$540143 = 421 \cdot 1283$$

with 421 being a (prime) factor.

Lenstra's Method (H. Lenstra, 1987)

Require: Composite odd integer $n > 1$

Ensure: Factorization of n

Choose several random elliptic curves $\mathcal{E}_i$ mod n given by $Y^2 = X^3 + a_i X + b_i$ (say, 10 to 20)

Choose points $P_i \in \mathcal{E}_i$ mod n {See the GK primality test.}

Choose large integer B (say, 10^8)

Compute $[B!]P_i$ on $\mathcal{E}_i$ mod n for each i

if Computation of slope of $[B!]P_i$ for some i fails **then**

Factor of n found

else

Restart with increased B or newly choosen curves and points

end if

Quadratic sieve and number field sieve outperform Lenstra's method, but Lenstra's method is good for searching medium sized prime factors $\leq 10^{40}$.

Example

Factorize $n = 4453$. Take the elliptic curve $\mathcal{E}$ mod 4453 given by $Y^2 = X^3 + 10X - 2 \mod 4453$ and $P = (1, 3)$.

- Compute $[2]P$: The slope of the tangent line at P is

$$\frac{3X^2 + 10}{2Y}(P) = \frac{13}{6} \equiv 3713 \mod 4453,$$

since $(6, 4453) = 1$ and so 6 is invertible mod 4453 with $6^{-1} \equiv 3711 \mod 4453$. Thus $[2]P = (x, y)$ with

$$\begin{aligned} x &\equiv 3713^2 - 2 \equiv 4332 \mod 4453, \\ y &\equiv -3713(x - 1) \equiv 3230 \mod 4453. \end{aligned}$$

Example (cont'd)

- Compute $[3]P$ by adding $[2]P$ and P. The slope of the line through these points is

$$\frac{3230-3}{4332-1} = \frac{3227}{4331}.$$

But $(4331, 4453) = 61 \neq 1$ and so 4331 is not invertible mod 4453. However, 61 is a factor of 4453 with $4453 = 61 \cdot 73$.

Example (cont'd)

- By the Chinese Remainder theorem,

$$\mathcal{E}(\mathbb{Z}_{4453}) = \mathcal{E}(\mathbb{Z}_{61}) \oplus \mathcal{E}(\mathbb{Z}_{73}).$$

- Multiples of P mod 61:

$$P \equiv (1,3), [2]P \equiv (1,58), [3]P \equiv O, [4]P \equiv (1,3), \ldots.$$

  Multiples of P mod 73:

$$P \equiv (1,3), [2]P \equiv (25,18), [3]P \equiv (28,44), \ldots, [64]P \equiv O.$$

- The slope of $[3]P$ mod 4453 has 61 in the denominator and is thus infinite mod 61.
- Note that if $[3]P = O$ mod 73, the slope will be 0 mod 4453 in the denominator and the factorization will not be possible. But this case is unlikely.

EC Cryptography

> However, in August 2015, the U.S. National Security Agency (NSA) announced that it plans to replace Suite B with a new cipher suite due to concerns about quantum computing attacks on ECC.

Wikipedia (2018).

Part VIII

Quantum Computing

Quantum Computing

- Quantum computing is computing using quantum-mechanical phenomena.
- Large-scale quantum computers may be able to efficiently solve problems which are not practically feasible on classical computers.
- Post-quantum cryptography: cryptographic algorithms secure against QC attacks.
- Announcements (2017/2018): 50-qubit QC (IBM), 49-qubit QC (Intel), 72-qubit QC (Google).

Quantum Hype and Quantum Skepticism (2019)

- Quantum computing is one of the U.S. National Science Foundation's "Ten Big Ideas".
- Quantum computation offers an exponential advantage of classical computation in theory.
- Quantum computers that compromise PK cryptography are unlikely to be built within the next decade including infrastructure (protocols).
- Qubits kept in highly sensitive superpositions will inevitably be corrupted by interaction with outside world.
- Error-correcting methods are needed as near term quantum computers are likely to be error-prone.
- Thriving marketplace is needed to sustain a virtuous cycle of developing increasingly useful quantum computers.

Quantum Computing

- Quantum bits
- Multi-quantum bits
- Operations on multi-quantum bits
- Measurement

Literature

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- P. Kaye, R. Laflamme, M. Mosca, *An Introduction to Quantum Computing*, Oxford Univ. Press, Oxford, 2007.
- E. Rieffel, W. Polak: *Quantum Computing – A Gentle Introduction*, The MIT Press, Cambridge, MA, 2011.
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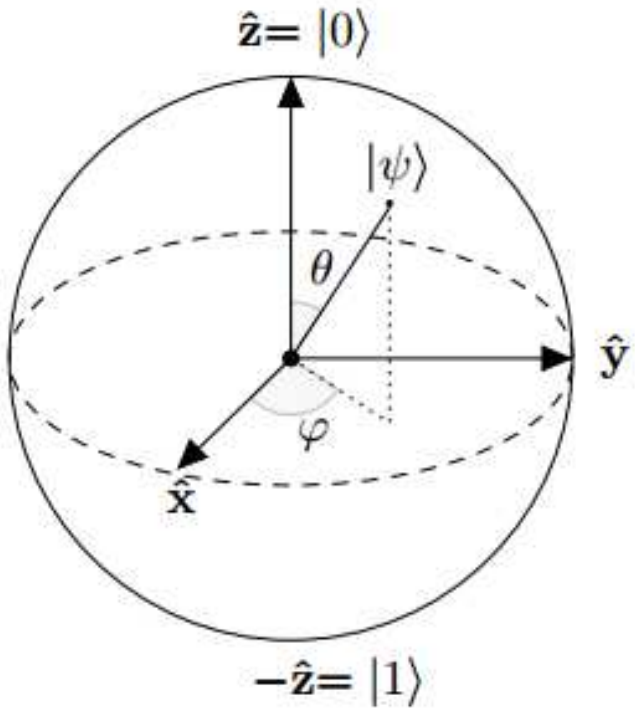
Quantum Bits



Block sphere, Felix Bloch (1905-1983)

Quantum Bits

- Qubits
- Hilbert space
- Representation of qubits
- Bloch sphere

Quantum Bits

- A *qubit* is a two-state quantum mechanical system.
- A qubit can be in superposition of both states at the same time.
- In the classical system, a bit is in one state or the other (0 or 1).

Quantum Bits

- A qubit can be measured in one of two *basis states* given by (ket notation),

$$|0\rangle \text{ and } |1\rangle.$$

- A *qubit* is a linear superposition of the basis states,

$$v = \lambda_0 |0\rangle + \lambda_1 |1\rangle \tag{365}$$

where $\lambda_0, \lambda_1 \in \mathbb{C}$ are the *probability amplitudes* with absolute magnitude (length)

$$\|v\| = \|(\lambda_0, \lambda_1)\| = \sqrt{|\lambda_0|^2 + |\lambda_1|^2} = 1, \tag{366}$$

where the *absolute value* of complex number $z = x + iy$ is

$$|z| = |x + iy| = \sqrt{x^2 + y^2}, \quad x, y \in \mathbb{R}. \tag{367}$$

- The *probability* of the state $|i\rangle$ in qubit v is $|\lambda_i|^2$ for $i = 0, 1$.

Quantum Bits – Examples

Qubits

$$\begin{aligned} v &= \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\,, \\ w &= \frac{1}{\sqrt{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|1\rangle\,, \end{aligned}$$

with lengths

$$\begin{aligned} \|v\| &= \sqrt{\left(\sqrt{\frac{1}{2}}\right)^2 + \left(\sqrt{\frac{1}{2}}\right)^2} = 1, \\ \|w\| &= \sqrt{\left(\sqrt{\frac{1}{3}}\right)^2 + \left(\sqrt{\frac{2}{3}}\right)^2} = 1. \end{aligned}$$

Pure Qubits

A *pure qubit* is a qubit v which is the multiple of a basis state; i.e, $v = \lambda_0 |0\rangle$ or $v = \lambda_1 |1\rangle$ and $\lambda_i \in \mathbb{C}$ with $|\lambda_i|^2 = 1$, $i = 0, 1$.

Example

$$\begin{aligned} v &= i|0\rangle, \\ w &= -i|1\rangle, \end{aligned}$$

with $\|v\| = \sqrt{|i|^2} = 1$ and $\|w\| = \sqrt{|-i|^2} = 1$, where $|i| = |-i| = 1$.

Quantum Bits

Let $\mathcal{H}$ denote a $\mathbb{C}$-Hilbert space with Hilbert basis $\{|0\rangle, |1\rangle\}$.

- A *qubit* of $\mathcal{H}$ is an element $v \in \mathcal{H}$ with $\|v\| = 1$. The set of all qubits forms the unit sphere

$$\mathcal{S}_{\mathcal{H}} = \{v \in \mathcal{H} \mid \|v\| = 1\}. \tag{368}$$

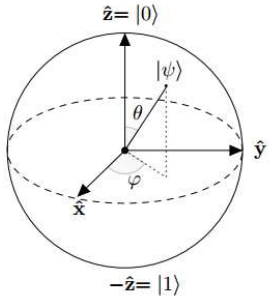
- The mapping $\mathcal{H} \to \mathbb{C}^2$ given by

$$|0\rangle \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{369}$$

  is a $\mathbb{C}$-vector space isomorphism; i.e.,

$$\lambda_0 |0\rangle + \lambda_1 |1\rangle \mapsto \lambda_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{370}$$

Bloch Sphere



Geometrical representation of qubit:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{371}$$

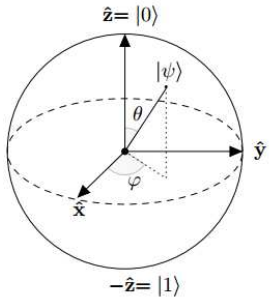
where $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$.

Note that in the general representation,

$$|\psi\rangle = e^{i\alpha}a\,|0\rangle + e^{i\beta}b\,|1\rangle = e^{i\alpha}\left(a\,|0\rangle + e^{i(\beta-\alpha)}b\,|1\rangle\right) \tag{372}$$

the overall phase is neglected.

Bloch Sphere



θ	φ	$\lvert\psi\rangle$
0	0	$\lvert 0\rangle$
$\pi/2$	0	$a\lvert 0\rangle + a\lvert 1\rangle$
π	0	$\lvert 1\rangle$
0	$\pi/2$	$\lvert 0\rangle$
$\pi/2$	$\pi/2$	$a\lvert 0\rangle + ai\lvert 1\rangle$
π	$\pi/2$	$i\lvert 1\rangle$

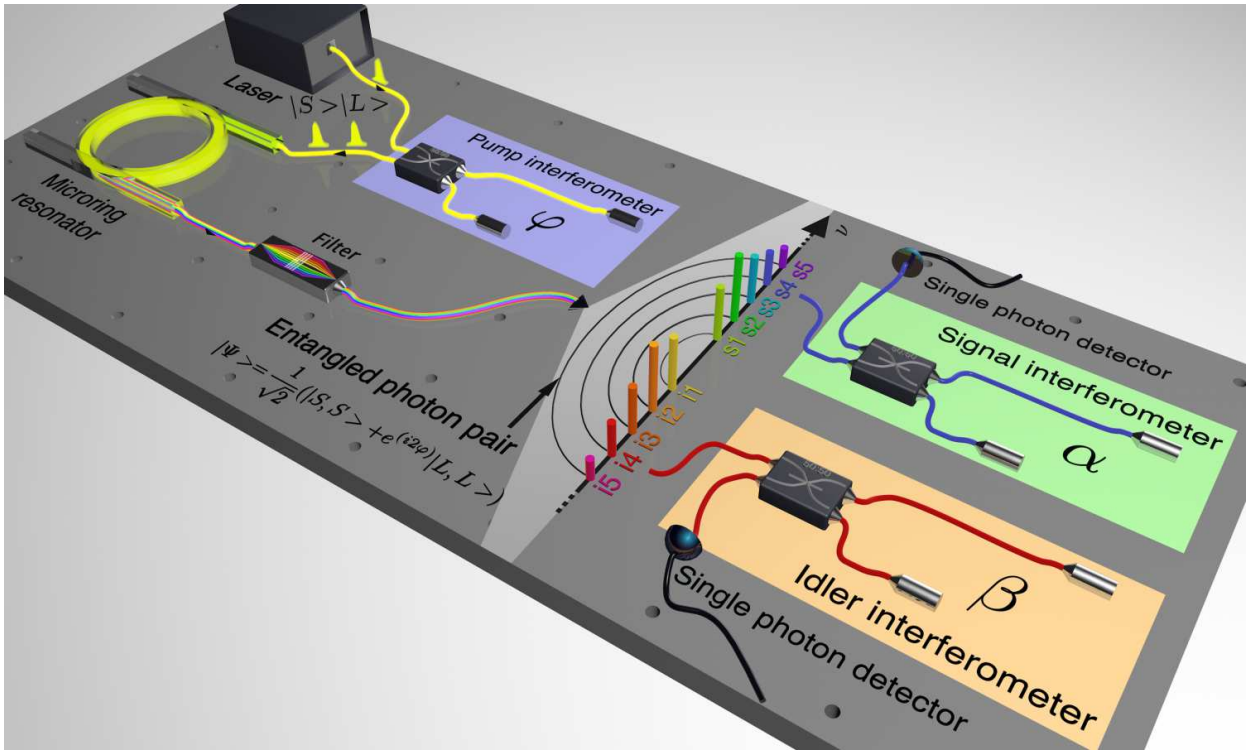
θ	φ	$\lvert\psi\rangle$
0	π	$\lvert 0\rangle$
$\pi/2$	π	$a\lvert 0\rangle - a\lvert 1\rangle$
π	π	$-\lvert 1\rangle$
0	$3\pi/2$	$\lvert 0\rangle$
$\pi/2$	$3\pi/2$	$a\lvert 0\rangle - ai\lvert 1\rangle$
π	$3\pi/2$	$-i\lvert 1\rangle$

where $a = \sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$.

Physical Realization of Qubit

Physical support	Name	$\|0\rangle$	$\|1\rangle$
Photon	polarization encoding	horizontal	vertical
Electron	electronic spin	up	down
Nucleus	nuclear spin (NMR)	up	down
Optical lattice	atomic spin	up	down
Quantum dot	dot spin	down	up

Multi-Quantum Bits



Multi-Quantum Bits

- Tensor product Hilbert space
- Representation of multi-qubits
- Pure, separable, and entangled multi-qubits

Multi-Quantum Bits

- A quantum computer performs calculations by manipulating qubits within a quantum register.
- A quantum register is a collection of qubits taken together.
- A quantum register is mathematically given by a tensor of a tensor product Hilbert space.
- In a classical system, a binary register is defined by the cartesian product of binary states.

Tensor Product Space

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with Hilbert basis $\{|0\rangle, |1\rangle\}$.

- Each element $v \in \mathcal{H}^{\otimes n}$ with $\|v\| = 1$ is an n-*qubit* and the set of all n-qubits is

$$S_{\mathcal{H}^{\otimes n}} = \{v \in \mathcal{H}^{\otimes n} \mid \|v\| = 1\}. \tag{373}$$

- For each integer x, $0 \leq x < 2^n$, with binary representation

$$x = x_{n-1} \dots x_1 x_0 = \sum_{j=0}^{n-1} 2^j x_j, \quad x_j \in \{0, 1\}, \tag{374}$$

defines the n-qubit $|x\rangle_n \in \mathcal{H}^{\otimes n}$ as

$$\begin{aligned} |x\rangle_n &= |x_{n-1} \dots x_1 x_0\rangle \\ &= |x_{n-1}\rangle \dots |x_1\rangle |x_0\rangle \\ &= |x_{n-1}\rangle \otimes \dots \otimes |x_1\rangle \otimes |x_0\rangle . \end{aligned} \tag{375}$$

Tensor Product Space – Notation

The tensor products

$$|0\rangle \otimes |0\rangle\,, \quad |0\rangle \otimes |1\rangle\,, \quad |1\rangle \otimes |0\rangle\,, \quad |1\rangle \otimes |1\rangle$$

can be written as

$$|0\rangle\,|0\rangle\,, \quad |0\rangle\,|1\rangle\,, \quad |1\rangle\,|0\rangle\,, \quad |1\rangle\,|1\rangle$$

or shorter as

$$|00\rangle\,, \quad |01\rangle\,, \quad |10\rangle \quad |11\rangle$$

or using decimals as

$$|0\rangle_2\,, \quad |1\rangle_2\,, \quad |2\rangle_2 \quad |3\rangle_2\,.$$

Note that $|3\rangle_2 = |11\rangle$ and $|3\rangle_3 = |011\rangle$.

Tensor Product Space

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with Hilbert basis $\{|0\rangle, |1\rangle\}$.

- The vector space $\mathcal{H}^{\otimes n}$ is a 2^n-dimensional Hilbert space over $\mathbb{C}$ with Hilbert basis

$$\{|x\rangle_n \mid 0 \leq x \leq 2^n - 1\}. \tag{376}$$

- Each n-qubit $v \in \mathcal{S}_{\mathcal{H}^{\otimes n}}$ can be uniquely written as

$$v = \sum_{j=0}^{2^n-1} \lambda_j |j\rangle_n \tag{377}$$

with $(\lambda_j) \in \mathbb{C}^{2^n}$ and

$$\|v\| = \sqrt{\sum_{i=0}^{2n-1} |\lambda_i|^2} = 1. \tag{378}$$

Bi-Qubits – Examples

$$\begin{aligned} u &= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle, \\ v &= \frac{1}{3}|00\rangle + \frac{1}{3}|01\rangle + \frac{\sqrt{5}}{3}|10\rangle + \frac{\sqrt{2}}{3}|11\rangle, \\ w &= \frac{1}{4}|00\rangle + \frac{\sqrt{5}}{4}|01\rangle + \frac{\sqrt{5}}{4}|10\rangle + \frac{\sqrt{5}}{4}|11\rangle. \end{aligned}$$

Note that

$$\|v\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} = 1.$$

Tensor Product Space

The mapping $\mathcal{H}^{\otimes n} \to \mathbb{C}^{2^n}$ given by

$$|x\rangle_n \mapsto e_{x+1} \tag{379}$$

is a $\mathbb{C}$-vector space isomorphism, where $e_1, \ldots, e_{2^n}$ is the standard basis of $\mathbb{C}^{2^n}$.

$$|6\rangle_3 = |110\rangle = |1\rangle \otimes |1\rangle \otimes |0\rangle \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Multi-Quantum Bits

- A *pure n-qubit* is an n-qubit $v \in \mathcal{S}_{\mathcal{H}^{\otimes n}}$ which is the multiple of a basis state (pure tensor); i.e.,

$$v = \lambda \left|x\right\rangle_n \tag{380}$$

  for some $x \in \{0, \ldots, 2^n - 1\}$ and $\lambda \in \mathbb{C}$ with $|\lambda|^2 = 1$.

- The assignment of an n-qubit to a pure n-qubit will be effected by measurement!

Tensor Permutation

Let σ, τ be bijections (permutations) of the set $\{0, \ldots, n-1\}$.

- The linear mapping $Q_\sigma : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ given by

$$\begin{aligned} & x_{n-1} \otimes \ldots \otimes x_1 \otimes x_0 \\ & \quad \mapsto x_{\sigma^{-1}(n-1)} \otimes \ldots \otimes x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(0)} \end{aligned} \tag{381}$$

  is called *tensor permutation* of $\mathcal{H}^{\otimes n}$.

- We have

$$Q_\sigma Q_\tau = Q_{\sigma\tau}. \tag{382}$$

Multi-Quantum Bits

Proof:

$$
\begin{aligned}
& Q_\sigma Q_\tau (x_{n-1} \otimes \ldots \otimes x_0) \\
= \quad & Q_\sigma (x_{\tau^{-1}(n-1)} \otimes \ldots \otimes x_{\tau^{-1}(0)}) \\
= \quad & Q_\sigma (d_{n-1} \otimes \ldots \otimes d_0), \quad d_i = x_{\tau^{-1}(i)}, \\
= \quad & (d_{\sigma^{-1}(n-1)} \otimes \ldots \otimes d_{\sigma^{-1}(0)}) \\
= \quad & (x_{\tau^{-1}(\sigma^{-1}(n-1))} \otimes \ldots \otimes x_{\tau^{-1}(\sigma^{-1}(0))}) \\
= \quad & (x_{(\sigma\tau)^{-1}(n-1)} \otimes \ldots \otimes x_{(\sigma\tau)^{-1}(0)}) \\
= \quad & Q_{\sigma\tau} (x_{n-1} \otimes \ldots \otimes x_0).
\end{aligned}
$$

□

Multi-Quantum Bits

- An n-qubit v is *separable* if there exists $1 \leq p < n$ such that

$$v = x \otimes y, \tag{383}$$

  where $x \in \mathcal{S}_{\mathcal{H}^{\otimes p}}$ and $y \in \mathcal{S}_{\mathcal{H}^{\otimes q}}$ with $q = n - p$.

- An n-qubit v is *indirectly separable* if there exists tensor permutation Q of $\mathcal{H}^{\otimes n}$ such that Qv is separable.

- An n-qubit v is *inseparable* or *entangled* if v is neither separable nor indirectly separable.

Multi-Quantum Bits – Examples

- The bi-qubit

$$v = ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle$$

is separable, since

$$v = (a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle).$$

- The 3-qubit

$$v = \frac{1}{\sqrt{2}}\,|010\rangle + \frac{1}{\sqrt{2}}\,|111\rangle$$

is indirectly separable, since if Q_σ is the tensor permutation given by $\sigma = (01)(2)$ (fix position 2, transpose positions 0,1), then

$$Q_\sigma v = \frac{1}{\sqrt{2}}\,|001\rangle + \frac{1}{\sqrt{2}}\,|111\rangle = \left(\frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle\right) \otimes |1\rangle\,.$$

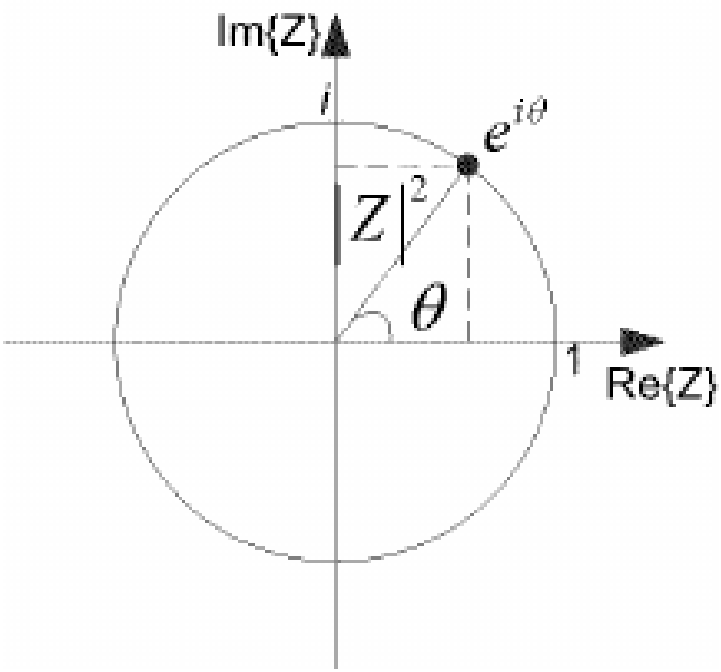
Multi-Quantum Bits – Example

The following bi-qubits known as *Bell qubits* are inseparable:

$$\begin{aligned}
\beta_{00} &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \\
\beta_{01} &= \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \\
\beta_{10} &= \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right), \\
\beta_{11} &= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right).
\end{aligned}$$

The vectors $\beta_{00}, \beta_{01}, \beta_{10}, \beta_{11}$ form a Hilbert basis of $\mathcal{H}^{\otimes 2}$.

Operations on Multi-Qubits



Operations on Multi-Qubits

- Gates for qubits, bi-qubits, and multi-qubits
- Gates for Boolean functions
- Quantum Fourier transform

Gates for Qubits

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$.

The $\mathbf{I}$-gate is the unitary operator $\mathbf{I} : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathbf{I}|0\rangle = |0\rangle \quad \text{and} \quad \mathbf{I}|1\rangle = |1\rangle. \tag{384}$$

Matrix representation of $\mathbf{I}$ w.r.t. the standard basis:

$$M_{\mathbf{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{385}$$

We have

$$M_{\mathbf{I}}^* = M_{\mathbf{I}}. \tag{386}$$

The gate realizes the identity operator.

Gates for Qubits

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$.

The $\mathbf{X}$-gate or *not-gate* is the unitary operator $\mathbf{X} : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathbf{X}|0\rangle = |1\rangle \quad \text{and} \quad \mathbf{X}|1\rangle = |0\rangle. \tag{387}$$

Matrix representation of $\mathbf{X}$ w.r.t. the standard basis:

$$M_{\mathbf{X}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{388}$$

We have

$$M_{\mathbf{X}}^* = M_{\mathbf{X}}. \tag{389}$$

The gate changes the pure states, measurement changes the bit value.

Gates for Qubits

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$.

The $\mathbf{Y}$-gate is the unitary operator $\mathbf{Y} : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathbf{Y}|0\rangle = i|1\rangle \quad \text{and} \quad \mathbf{Y}|1\rangle = -i|0\rangle. \tag{390}$$

Matrix representation of $\mathbf{Y}$ w.r.t. the standard basis:

$$M_{\mathbf{Y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{391}$$

We have

$$M_{\mathbf{Y}}^* = M_{\mathbf{Y}}. \tag{392}$$

The gate exchanges the pure states, but with different values.

Gates for Qubits

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$.

The $\mathbf{Z}$-gate is the unitary operator $\mathbf{Z} : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathbf{Z}|0\rangle = |0\rangle \quad \text{and} \quad \mathbf{Z}|1\rangle = -|1\rangle. \tag{393}$$

Matrix representation of $\mathbf{Z}$ w.r.t. the standard basis:

$$M_{\mathbf{Z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{394}$$

We have

$$M_{\mathbf{Z}}^* = M_{\mathbf{Z}}. \tag{395}$$

This matrix is called *Pauli matrix*.

Gates for Qubits

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$.

The $\mathbf{P}$-gate or *phase gate* is the unitary operator $\mathbf{P} : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathbf{P}|0\rangle = |0\rangle \quad \text{and} \quad \mathbf{P}|1\rangle = i|1\rangle. \tag{396}$$

Matrix representation of $\mathbf{P}$ w.r.t. the standard basis:

$$M_{\mathbf{P}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{397}$$

We have

$$M_{\mathbf{P}}^* = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{398}$$

The gate does not change the pure states.

Gates for Qubits

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$.

The $\mathbf{H}$-gate or *Hadamard gate* is the unitary operator $\mathbf{H} : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathbf{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \text{and} \quad \mathbf{H}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{399}$$

Matrix representation of $\mathbf{H}$ w.r.t. the standard basis:

$$M_{\mathbf{H}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{400}$$

We have

$$M_{\mathbf{H}}^* = M_{\mathbf{H}}. \tag{401}$$

The gate changes the pure states into states with equal probability.

Implementation of Hadamard Gate

A beamsplitter is an optical device that splits a beam of light into two separate beams:

$$\mathbf{B}\,|0\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{i}{\sqrt{2}}\,|1\rangle\,, \tag{402}$$

$$\mathbf{B}\,|1\rangle = \frac{i}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\,. \tag{403}$$

Matrix representation of $\mathbf{B}$ w.r.t. the standard basis:

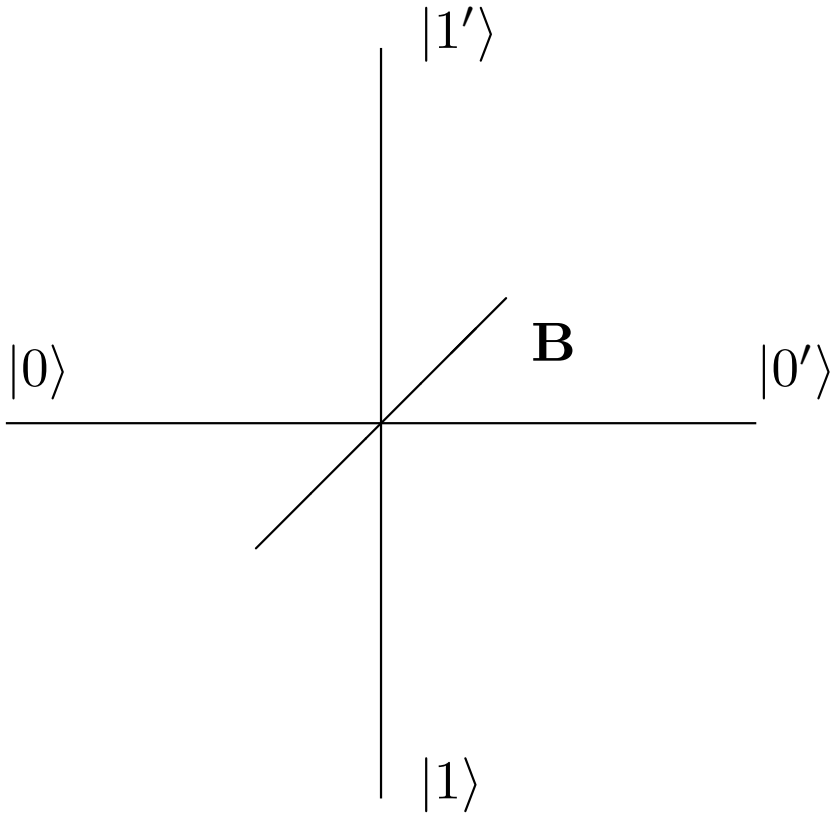
$$M_{\mathbf{B}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{404}$$

We have

$$M_{\mathbf{B}}^{*} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{405}$$

Implementation of Hadamard Gate

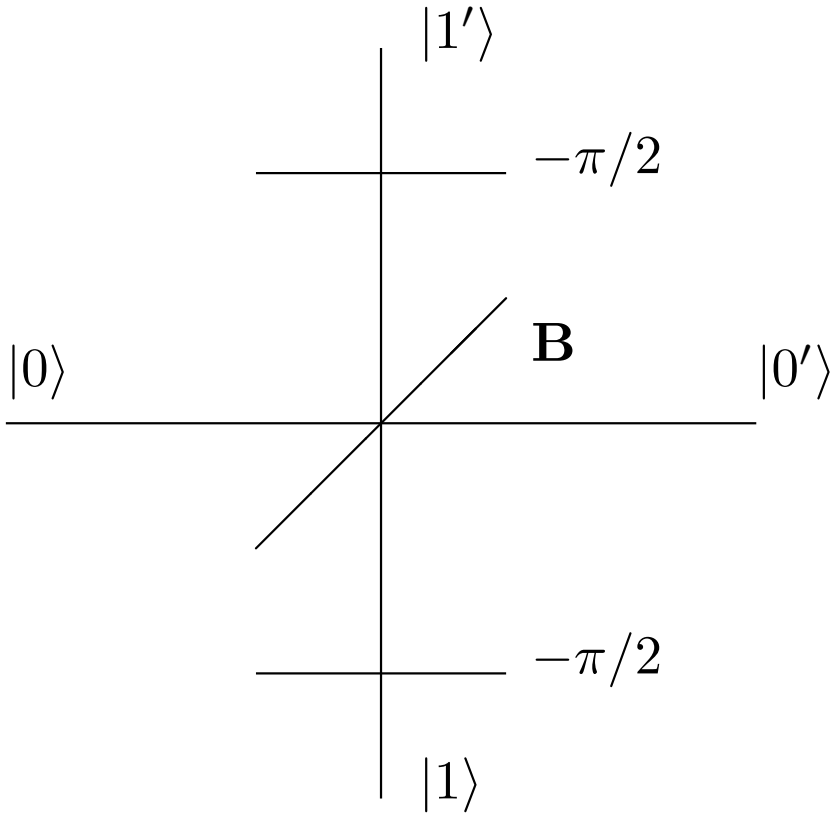
Beamsplitter splits a beam of light into separate beams:



$$\begin{aligned} \mathbf{B}\,|0\rangle &= \frac{1}{\sqrt{2}}\,|0\rangle + \frac{i}{\sqrt{2}}\,|1\rangle\,, \\ \mathbf{B}\,|1\rangle &= \frac{i}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\,. \end{aligned}$$

Implementation of Hadamard Gate

A Hadamard gate can be realized optically by a beamsplitter and two $-\pi/2$ phase shifters:



Implementation of Hadamard Gate

- A $-\pi/2$ *phase shifter* provides a multiplication with

$$e^{-i\pi/2} = -i. \tag{406}$$

- A Hadamard gate can be realized optically by a beamsplitter and two $-\pi/2$ phase shifters:

$$\begin{aligned} \mathbf{H}\,|0\rangle &= \frac{1}{\sqrt{2}}\,|0\rangle + e^{-i\pi/2}\frac{i}{\sqrt{2}}\,|1\rangle \\ &= \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\,, \end{aligned} \tag{407}$$

$$\begin{aligned} \mathbf{H}\,|1\rangle &= \frac{i}{\sqrt{2}}e^{-i\pi/2}\,|0\rangle + e^{-i\pi/2}\frac{1}{\sqrt{2}}e^{-i\pi/2}\,|1\rangle \\ &= \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle\,. \end{aligned} \tag{408}$$

Multi-Qubit Extension

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$ and $A, B : \mathcal{H} \to \mathcal{H}$ be linear operators given by

$$A|0\rangle = a_{11}|0\rangle + a_{21}|1\rangle, \tag{409}$$
$$A|1\rangle = a_{12}|0\rangle + a_{22}|1\rangle, \tag{410}$$
$$B|0\rangle = b_{11}|0\rangle + b_{21}|1\rangle, \tag{411}$$
$$B|1\rangle = b_{12}|0\rangle + b_{22}|1\rangle. \tag{412}$$

Matrix representations of A and B:

$$M_A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{413}$$
$$M_B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. \tag{414}$$

Multi-Qubit Extension (cont'd)

Matrix representation of *Kronecker product* $A \otimes B$:

$$M_{A\otimes B} = \begin{pmatrix} a_{11}M_B & a_{12}M_B \\ a_{21}M_B & a_{22}M_B \end{pmatrix} \tag{415}$$

$$= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}.$$

Tensor Permutations

Each tensor permutation is a unitary operation given by a permutation matrix.

Example

Let $\mathcal{H}$ be the $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$ and $P : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$ be the tensor permutation which permutes the tensor components; i.e.,

$$P|00\rangle = |00\rangle, \; P|01\rangle = |10\rangle, \; P|10\rangle = |01\rangle, \; P|11\rangle = |11\rangle.$$

Matrix representation of P w.r.t. the standard basis:

$$M_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Multi-Qubit Extension

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$ and let n, k be integers with $0 \leq k < n$.

The operator

$$\mathbf{H}_k^{(n)} = \underbrace{\mathbf{I} \otimes \ldots \otimes \mathbf{I}}_{n-k-1} \otimes \mathbf{H} \otimes \underbrace{\mathbf{I} \otimes \ldots \otimes \mathbf{I}}_{k} \tag{416}$$

is unitary on $\mathcal{H}^{\otimes n}$, where $\mathbf{H}$ is the Hadamard gate.

Gates for Bi-Qubits

The unitary operators

$$\mathbf{H}_0^{(2)} = \mathbf{I} \otimes \mathbf{H} \quad \text{and} \quad \mathbf{H}_1^{(2)} = \mathbf{H} \otimes \mathbf{I} \tag{417}$$

have the matrix representations w.r.t. the standard bases

$$M_{\mathbf{H}_0^{(2)}} = \begin{pmatrix} M_{\mathbf{H}} & 0 \\ 0 & M_{\mathbf{H}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{418}$$

$$M_{\mathbf{H}_1^{(2)}} = \begin{pmatrix} M_{\mathbf{I}} & M_{\mathbf{I}} \\ M_{\mathbf{I}} & -M_{\mathbf{I}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \tag{419}$$

Tri-Qubit Extension

We have

$$\begin{aligned}\mathbf{H}_1^{(3)}(|x_2\rangle \otimes |x_1\rangle \otimes |x_0\rangle) &= (\mathbf{I} \otimes \mathbf{H} \otimes \mathbf{I})(|x_2\rangle \otimes |x_1\rangle \otimes |x_0\rangle) \\ &= |x_2\rangle \otimes (\mathbf{H}\,|x_1\rangle) \otimes |x_0\rangle\,. \qquad (420)\end{aligned}$$

Example

$$\begin{aligned}&\mathbf{H}_1^{(3)}\left(\sqrt{\frac{1}{2}}\,|001\rangle + \sqrt{\frac{1}{2}}\,|111\rangle\right) \\ &= \sqrt{\frac{1}{2}}\,(|0\rangle \otimes (\mathbf{H}\,|0\rangle) \otimes |1\rangle) + \sqrt{\frac{1}{2}}\,(|1\rangle \otimes (\mathbf{H}\,|1\rangle) \otimes |1\rangle) \\ &= \sqrt{\frac{1}{2}}\left(|0\rangle \otimes \left(\sqrt{\frac{1}{2}}(|0\rangle + |1\rangle)\right) \otimes |1\rangle\right) + \sqrt{\frac{1}{2}}\left(|1\rangle \otimes \left(\sqrt{\frac{1}{2}}(|0\rangle - |1\rangle)\right) \otimes |1\rangle\right) \\ &= \frac{1}{2}\,(|001\rangle + |011\rangle + |101\rangle - |111\rangle)\,.\end{aligned}$$

Gates for Multi-Qubits

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$ and let $n \geq 1$ be an integer.

The linear operator $\mathbf{H}^{\otimes n} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ given by

$$\begin{aligned} \mathbf{H}^{\otimes n}(x_1 \otimes \ldots \otimes x_n) &= \mathbf{H}^{\otimes n} |x_1 \ldots x_n\rangle_n \\ &= \mathbf{H}|x_1\rangle \otimes \ldots \otimes \mathbf{H}|x_n\rangle \end{aligned} \tag{421}$$

is unitary.

Proof.

Since the linear operator $\mathbf{H}$ is unitary and the tensor product of unitary operators is unitary, the linear operator $\mathbf{H}^{\otimes n}$ is unitary. □

Gates for Multi-Qubits – Example

Case $n = 2$:

$$\begin{aligned} \mathbf{H}^{\otimes 2} |01\rangle_2 &= \mathbf{H}|0\rangle \otimes \mathbf{H}|1\rangle \\ &= \left(\sqrt{\frac{1}{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\sqrt{\frac{1}{2}}(|0\rangle - |1\rangle)\right) \\ &= \frac{1}{2}\left((|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)\right) \\ &= \frac{1}{2}\left(|00\rangle_2 - |01\rangle_2 + |10\rangle_2 - |11\rangle_2\right). \end{aligned} \tag{422}$$

Gates for Bi-Qubits

The unitary operator

$$\mathbf{H}^{\otimes 2} = \mathbf{H} \otimes \mathbf{H} \tag{423}$$

has the matrix representation w.r.t. the standard basis

$$\begin{aligned} M_{\mathbf{H}^{\otimes 2}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} M_{\mathbf{H}} & M_{\mathbf{H}} \\ M_{\mathbf{H}} & -M_{\mathbf{H}} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \end{aligned} \tag{424}$$

Gates for Multi-Qubits

If n Hadamard gates are prepared in parallel to n qubits each prepared in state $|0\rangle$, a superposition of all 2^n binary registers is obtained:

$$\begin{aligned} \mathbf{H}^{\otimes n} |0\rangle_n &= \mathbf{H}|0\rangle \otimes \ldots \otimes \mathbf{H}|0\rangle \\ &= \bigotimes_{i=1}^{n} \left(\sqrt{\frac{1}{2}} (|0\rangle + |1\rangle) \right) \\ &= \frac{1}{\sqrt{2^n}} \bigotimes_{i=1}^{n} (|0\rangle + |1\rangle) \\ &= 2^{-n/2} \sum_{j=0}^{2^n-1} |j\rangle_n . \end{aligned} \tag{425}$$

The Hadamard gate is responsible for the high degree of parallelism in quantum computing.

Gates for Multi-Qubits – Example

Case $n = 2$:

$$\begin{aligned} \mathbf{H}^{\otimes 2} |00\rangle_2 &= \mathbf{H}|0\rangle \otimes \mathbf{H}|0\rangle \\ &= \left(\sqrt{\frac{1}{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\sqrt{\frac{1}{2}}(|0\rangle + |1\rangle)\right) \\ &= \frac{1}{2}\left((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)\right) \\ &= \frac{1}{2}\left(|00\rangle_2 + |01\rangle_2 + |10\rangle_2 + |11\rangle_2\right). \end{aligned} \tag{426}$$

Gates for Multi-Qubits

Let $\mathbb{B} = \{0, 1\}$.

The *Hadamard function* $h : \mathbb{N}_0^2 \to \mathbb{B}$ is defined as

$$h(x, y) = \bigoplus_{j=0}^{m} x_j \odot y_j \tag{427}$$

where $x = \sum_{j=0}^{m} x_j 2^j$, $y = \sum_{j=0}^{m} y_j 2^j$, and $x_j, y_j \in \mathbb{B}$.
$\oplus$ and $\odot$ denote the addition and multiplication mod 2, resp.

We have

$$h(x, y) = 1 \tag{428}$$

iff in the binary representations of x and y the number of 1's at equal positions is odd.

Example

$h(0000, 0111) = 0$, $h(000\underline{1}, 011\underline{1}) = 1$, $h(00\underline{11}, 01\underline{11}) = 0$. $h(0\underline{111}, 0\underline{111}) = 1$.

Gates for Multi-Qubits

The unitary operator $\mathbf{H}^{\otimes n}$ satisfies

$$\mathbf{H}^{\otimes n} |x\rangle_n = 2^{-n/2} \sum_{y=0}^{2^n-1} (-1)^{h(x,y)} |y\rangle_n \tag{429}$$

for all $0 \leq x < 2^n$.

Matrix representation w.r.t. the standard basis:

$$M_{\mathbf{H}^{\otimes n}} = 2^{-n/2} \left((-1)^{h(i,j)} \right)_{i,j=0,\ldots,2^n-1}. \tag{430}$$

Proof.

Let $x = \sum_{j=0}^{n-1} x_j 2^j$, $x_j \in \mathbb{B}$. Then

$$\begin{aligned}
\mathbf{H}^{\otimes n} |x\rangle_n &= \bigotimes_{j=0}^{n-1} \mathbf{H} |x_{n-1-j}\rangle \\
&= 2^{-n/2} \bigotimes_{j=0}^{n-1} \left(|0\rangle + (-1)^{x_{n-1-j}} |1\rangle\right) \\
&= 2^{-n/2} \sum_{y=0}^{2^n-1} (-1)^{x_{n-1} \odot y_{n-1} \oplus \ldots \oplus x_0 \odot y_0} |y\rangle_n \\
&= 2^{-n/2} \sum_{y=0}^{2^n-1} (-1)^{h(x,y)} |y\rangle_n ,
\end{aligned}$$

where $y = \sum_{j=0}^{n-1} y_j 2^j$, $y_j \in \{0, 1\}$. □

Gates for Multi-Qubits – Example

Case $n = 3$:

$$
\begin{aligned}
&\mathbf{H}^{\otimes 3} \left|011\right\rangle_3 \\
&\quad = \quad \mathbf{H}\left|0\right\rangle \otimes \mathbf{H}\left|1\right\rangle \otimes \mathbf{H}\left|1\right\rangle \\
&\quad = \quad \left(\sqrt{\frac{1}{2}}(\left|0\right\rangle + \left|1\right\rangle)\right) \otimes \left(\sqrt{\frac{1}{2}}(\left|0\right\rangle - \left|1\right\rangle)\right) \otimes \left(\sqrt{\frac{1}{2}}(\left|0\right\rangle - \left|1\right\rangle)\right) \\
&\quad = \quad \frac{1}{2\sqrt{2}}\left((\left|0\right\rangle + \left|1\right\rangle) \otimes (\left|0\right\rangle - \left|1\right\rangle) \otimes (\left|0\right\rangle - \left|1\right\rangle)\right) \\
&\quad = \quad \frac{1}{2\sqrt{2}}\left(\left|000\right\rangle_3 - \left|001\right\rangle_3 - \left|010\right\rangle_3 + \left|011\right\rangle_3\right. \\
&\qquad\qquad \left. + \left|100\right\rangle_3 - \left|101\right\rangle_3 - \left|110\right\rangle_3 + \left|111\right\rangle_3\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&h(000,011) = 0,\ h(001,011) = 1,\ h(010,011) = 1,\ h(011,011) = 0, \\
&h(100,011) = 0,\ h(101,011) = 1,\ h(110,011) = 1,\ h(111,011) = 0.
\end{aligned}
$$

Gates for Boolean Functions

Let $\mathbb{B} = \{0, 1\}$ and $n \geq 0$ integer.

- An *n-ary Boolean function* is a mapping $f : \mathbb{B}^n \to \mathbb{B}$.
- Identify the domain $\mathbb{B}^n$ with the standard basis of the Hilbert space $\mathcal{H}^{\otimes n}$; i.e.,

$$(x_{n-1}, \ldots, x_1, x_0) \mapsto |x_{n-1} \ldots x_1 x_0\rangle_n \,, \tag{431}$$

where

$$|x_{n-1} \ldots x_1 x_0\rangle_n = |x_{n-1}\rangle \otimes \ldots \otimes |x_1\rangle \otimes |x_0\rangle \,. \tag{432}$$

Gates for Boolean Functions

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$ and $f : \mathbb{B}^n \to \mathbb{B}$ be a Boolean function.

The $\mathbf{U}_f$ gate is the unitary operator

$$\mathbf{U}_f : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes(n+1)} \tag{433}$$

defined by

$$\mathbf{U}_f \, |x\rangle_n \, |y\rangle_1 = |x\rangle_n \, |y \oplus f(x)\rangle_1 \tag{434}$$

for all $x \in \mathbb{B}^n$ and $y \in \mathbb{B}$, where $\oplus$ denotes the binary addition.

Gates for Boolean Functions

The gate $\mathbf{U}_f$ has the property

$$\mathbf{U}_f^{-1} = \mathbf{U}_f. \tag{435}$$

Proof.

We have

$$\begin{aligned} \mathbf{U}_f \mathbf{U}_f \left(|x\rangle_n |y\rangle_1 \right) &= \mathbf{U}_f \left(|x\rangle_n |y \oplus f(x)\rangle_1 \right) \\ &= |x\rangle_n |y \oplus f(x) \oplus f(x)\rangle_1 \\ &= |x\rangle_n |y \oplus 0\rangle_1 \\ &= |x\rangle_n |y\rangle_1 , \end{aligned}$$

since $z \oplus z = 0$ and $z \oplus 0 = z$ for all $z \in \mathbb{B}$. □

Gates for Boolean Functions – Example

Consider the unary NOT function

$$f : \mathbb{B} \to \mathbb{B} : x \mapsto \neg x. \tag{436}$$

The gate

$$\mathbf{U}_f : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2} \tag{437}$$

is given by

$$\mathbf{U}_f \left|x\right\rangle_1 \left|y\right\rangle_1 = \left|x\right\rangle_1 \left|y \oplus \neg x\right\rangle_1. \tag{438}$$

Value table:

$\left\|x\right\rangle \left\|y\right\rangle$	$\mathbf{U}_f \left\|x\right\rangle \left\|y\right\rangle$
$\left\|0\right\rangle \left\|0\right\rangle$	$\left\|0\right\rangle \left\|0 \oplus \neg 0\right\rangle = \left\|0\right\rangle \left\|1\right\rangle$
$\left\|1\right\rangle \left\|0\right\rangle$	$\left\|1\right\rangle \left\|0 \oplus \neg 1\right\rangle = \left\|1\right\rangle \left\|0\right\rangle$
$\left\|0\right\rangle \left\|1\right\rangle$	$\left\|0\right\rangle \left\|1 \oplus \neg 0\right\rangle = \left\|0\right\rangle \left\|0\right\rangle$
$\left\|1\right\rangle \left\|1\right\rangle$	$\left\|1\right\rangle \left\|1 \oplus \neg 1\right\rangle = \left\|1\right\rangle \left\|1\right\rangle$

Gates for Boolean Functions – Example

Consider the binary AND function

$$f : \mathbb{B}^2 \to \mathbb{B} : (x_1, x_0) \mapsto x_1 \wedge x_0. \tag{439}$$

The gate

$$\mathbf{U}_f : \mathcal{H}^{\otimes 3} \to \mathcal{H}^{\otimes 3} \tag{440}$$

is given by

$$\mathbf{U}_f \left|x_1 x_0\right\rangle_2 \left|y\right\rangle_1 = \left|x_1 x_0\right\rangle_2 \left|y \oplus (x_1 \wedge x_0)\right\rangle_1. \tag{441}$$

Value table:

$\left\|x_1 x_0\right\rangle \left\|y\right\rangle$	$\mathbf{U}_f \left\|x_1 x_0\right\rangle \left\|y\right\rangle$
$\left\|00\right\rangle \left\|0\right\rangle$	$\left\|00\right\rangle \left\|0 \oplus (0 \wedge 0)\right\rangle = \left\|00\right\rangle \left\|0\right\rangle$
$\left\|00\right\rangle \left\|1\right\rangle$	$\left\|00\right\rangle \left\|1 \oplus (0 \wedge 0)\right\rangle = \left\|00\right\rangle \left\|1\right\rangle$
$\left\|01\right\rangle \left\|0\right\rangle$	$\left\|01\right\rangle \left\|0 \oplus (0 \wedge 1)\right\rangle = \left\|01\right\rangle \left\|0\right\rangle$
$\left\|01\right\rangle \left\|1\right\rangle$	$\left\|01\right\rangle \left\|1 \oplus (0 \wedge 1)\right\rangle = \left\|01\right\rangle \left\|1\right\rangle$
$\left\|10\right\rangle \left\|0\right\rangle$	$\left\|10\right\rangle \left\|0 \oplus (1 \wedge 0)\right\rangle = \left\|10\right\rangle \left\|0\right\rangle$
$\left\|10\right\rangle \left\|1\right\rangle$	$\left\|10\right\rangle \left\|1 \oplus (1 \wedge 0)\right\rangle = \left\|10\right\rangle \left\|1\right\rangle$
$\left\|11\right\rangle \left\|0\right\rangle$	$\left\|11\right\rangle \left\|0 \oplus (1 \wedge 1)\right\rangle = \left\|11\right\rangle \left\|1\right\rangle$
$\left\|11\right\rangle \left\|1\right\rangle$	$\left\|11\right\rangle \left\|1 \oplus (1 \wedge 1)\right\rangle = \left\|11\right\rangle \left\|0\right\rangle$

Gates for Boolean Functions

Application of gate $\mathbf{U}_f$ to $\mathbf{H}_0^{(n+1)} |x\rangle_n |1\rangle$,

$$\begin{aligned}
\mathbf{U}_f \mathbf{H}_0^{(n+1)} |x\rangle_n |1\rangle &= \mathbf{U}_f (\mathbf{I} \otimes \ldots \otimes \mathbf{I} \otimes \mathbf{H}) |x\rangle_n |1\rangle \qquad (442)\\
&= \mathbf{U}_f \left(|x\rangle_n \otimes \left(\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right) \right)\\
&= |x\rangle_n \otimes \frac{1}{\sqrt{2}} (|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle)\\
&= |x\rangle_n \otimes \frac{1}{\sqrt{2}} (|f(x)\rangle - |1 \oplus f(x)\rangle)\\
&= |x\rangle_n \otimes \frac{1}{\sqrt{2}} (-1)^{f(x)} (|0\rangle - |1\rangle)\\
&= (-1)^{f(x)} \cdot |x\rangle_n \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) .
\end{aligned}$$

Quantum Fourier Transform

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space with basis $\{|0\rangle, |1\rangle\}$ and let $n \geq 1$ be an integer.

The $\mathbf{F}^{\otimes n}$ gate or *Fourier transform* is the unitary operator

$$\mathbf{F}^{\otimes n} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n} \tag{443}$$

defined by

$$\mathbf{F}^{\otimes n} |x\rangle_n = 2^{-n/2} \sum_{y=0}^{2^n-1} \exp\left(2\pi i \frac{xy}{2^n}\right) |y\rangle_n \tag{444}$$

for all $0 \leq x < 2^n$, where the $\exp\left(2\pi i \frac{xy}{2^n}\right)$ are 2^nth roots of unity.

Quantum Fourier Transform – Example

Case $n = 2$:

$$\begin{aligned}
\mathbf{F}^{\otimes 2} |0\rangle_2 &= \frac{1}{4} \left(|0\rangle_2 + |1\rangle_2 + |2\rangle_2 + |3\rangle_2 \right), \\
\mathbf{F}^{\otimes 2} |1\rangle_2 &= \frac{1}{4} \left(|0\rangle_2 + i\,|1\rangle_2 - |2\rangle_2 - i\,|3\rangle_2 \right), \\
\mathbf{F}^{\otimes 2} |2\rangle_2 &= \frac{1}{4} \left(|0\rangle_2 - |1\rangle_2 + |2\rangle_2 - |3\rangle_2 \right), \\
\mathbf{F}^{\otimes 2} |3\rangle_2 &= \frac{1}{4} \left(|0\rangle_2 - i\,|1\rangle_2 - |2\rangle_2 + i\,|3\rangle_2 \right).
\end{aligned}$$

Matrix representation w.r.t. standard basis:

$$M_{\mathbf{F}^{\otimes 2}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

Quantum Fourier Transform

The inverse Fourier transform of $\mathbf{F}^{\otimes n}$ is the adjoint operator $\left(\mathbf{F}^{\otimes n}\right)^*$ given by

$$\left(\mathbf{F}^{\otimes n}\right)^* |x\rangle_n = 2^{-n/2} \sum_{y=0}^{2^n-1} \exp\left(-2\pi i \frac{xy}{2^n}\right) |y\rangle_n \tag{445}$$

for all $0 \leq x < 2^n$, where the $\exp\left(-2\pi i \frac{xy}{2^n}\right)$ are 2^nth roots of unity.

Quantum Fourier Transform – Example

Case $n = 2$:

$$\begin{aligned}
(\mathbf{F}^{\otimes 2})^* \, |0\rangle_2 &= \frac{1}{4} \left(|0\rangle_2 + |1\rangle_2 + |2\rangle_2 + |3\rangle_2 \right), \\
(\mathbf{F}^{\otimes 2})^* \, |1\rangle_2 &= \frac{1}{4} \left(|0\rangle_2 - i\,|1\rangle_2 - |2\rangle_2 + i\,|3\rangle_2 \right), \\
(\mathbf{F}^{\otimes 2})^* \, |2\rangle_2 &= \frac{1}{4} \left(|0\rangle_2 - |1\rangle_2 + |2\rangle_2 - |3\rangle_2 \right), \\
(\mathbf{F}^{\otimes 2})^* \, |3\rangle_2 &= \frac{1}{4} \left(|0\rangle_2 + i\,|1\rangle_2 - |2\rangle_2 - i\,|3\rangle_2 \right).
\end{aligned}$$

Matrix representation w.r.t. standard basis:

$$M_{(\mathbf{F}^{\otimes 2})^*} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.$$

Proof.

Consider the linear operator $\mathbf{U} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ defined by

$$\mathbf{U} \left| z \right\rangle_n = 2^{-n/2} \sum_{y=0}^{2^n-1} \exp\left(-2\pi i \frac{zy}{2^n}\right) \left| y \right\rangle_n \tag{446}$$

for all $0 \leq z < 2^n$.

Claim that $\mathbf{U} = (\mathbf{F}^{\otimes n})^*$.

Proof (cont'd).

Indeed,

$$\begin{aligned}
\langle \mathbf{U} \left|z\right\rangle_n, \left|x\right\rangle_n \rangle &= \left\langle 2^{-n/2} \sum_{y=0}^{2^n-1} \exp\left(-2\pi i \frac{zy}{2^n}\right) \left|y\right\rangle_n, \left|x\right\rangle_n \right\rangle \\
&= 2^{-n/2} \sum_{y=0}^{2^n-1} \overline{\exp\left(-2\pi i \frac{zy}{2^n}\right)} \langle \left|y\right\rangle_n, \left|x\right\rangle_n \rangle \\
&= 2^{-n/2} \exp\left(2\pi i \frac{zx}{2^n}\right) \\
&= 2^{-n/2} \sum_{y=0}^{2^n-1} \exp\left(2\pi i \frac{yx}{2^n}\right) \langle \left|z\right\rangle_n, \left|y\right\rangle_n \rangle \\
&= \left\langle \left|z\right\rangle_n, 2^{-n/2} \sum_{y=0}^{2^n-1} \exp\left(2\pi i \frac{yx}{2^n}\right) \left|y\right\rangle_n \right\rangle \\
&= \langle \left|z\right\rangle_n, \mathbf{F}^{\otimes n} \left|x\right\rangle_n \rangle.
\end{aligned}$$

Proof (cont'd).

Moreover,

$$
\begin{aligned}
&\mathbf{UF}^{\otimes n} \left|x\right\rangle_n \\
&= \mathbf{U}\left(2^{-n/2}\sum_{y=0}^{2^n-1}\exp\left(2\pi i\frac{xy}{2^n}\right)\left|y\right\rangle_n\right) \\
&= 2^{-n/2}\sum_{y=0}^{2^n-1}\exp\left(2\pi i\frac{xy}{2^n}\right)\mathbf{U}\left|y\right\rangle_n \\
&= 2^{-n/2}\sum_{y=0}^{2^n-1}\exp\left(2\pi i\frac{xy}{2^n}\right)2^{-n/2}\sum_{z=0}^{2^n-1}\exp\left(-2\pi i\frac{yz}{2^n}\right)\left|z\right\rangle_n \\
&= 2^{-n}\sum_{y=0}^{2^n-1}\sum_{z=0}^{2^n-1}\exp\left(2\pi i\frac{y(x-z)}{2^n}\right)\left|z\right\rangle_n \\
&= 2^{-n}\sum_{z=0}^{2^n-1}\left(\sum_{y=0}^{2^n-1}\exp\left(2\pi i\frac{x-z}{2^n}\right)^y\right)\left|z\right\rangle_n \stackrel{!}{=} \left|x\right\rangle_n .
\end{aligned}
$$

Proof (cont'd).

For each number $r \neq 1$, the sum of the first n terms of a geometric series is

$$a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right). \tag{447}$$

Thus for $x \neq z$,

$$\begin{aligned}
\sum_{y=0}^{2^n-1} \exp\left(2\pi i \frac{x-z}{2^n}\right)^y &= \frac{1-\exp\left(2\pi i \frac{x-z}{2^n}\right)^{2^n}}{1-\exp\left(2\pi i \frac{x-z}{2^n}\right)} \\
&= \frac{1-\exp\left(2\pi i (x-z)\right)}{1-\exp\left(2\pi i \frac{x-z}{2^n}\right)} \\
&= \frac{1-1}{1-\exp\left(2\pi i \frac{x-z}{2^n}\right)} \\
&= 0.
\end{aligned}$$

So for $x \neq z$, the inner sum is 0, and for $x = z$, the inner sum is 2^n. □

Quantum Fourier Transform

The **F**-gate has the representations

$$\begin{aligned} & \mathbf{F}^{\otimes n} \left|x\right\rangle_n \\ = \quad & 2^{-n/2} \bigotimes_{m=1}^{n} \left(\left|0\right\rangle + \exp\left(2\pi i \frac{x}{2^m}\right) \left|1\right\rangle\right) \end{aligned} \tag{448}$$

for all $0 \leq x < 2^n$, or

$$\begin{aligned} & \mathbf{F}^{\otimes n} \left|x_{n-1} \ldots x_1 x_0\right\rangle \\ = \quad & 2^{-n/2} \bigotimes_{m=1}^{n} \left(\left|0\right\rangle + \exp\left(2\pi i \sum_{j=0}^{m-1} \frac{x_j}{2^{m-j}}\right) \left|1\right\rangle\right) \end{aligned} \tag{449}$$

for all $(x_{n-1}, \ldots, x_1, x_0) \in \mathbb{B}^n$.

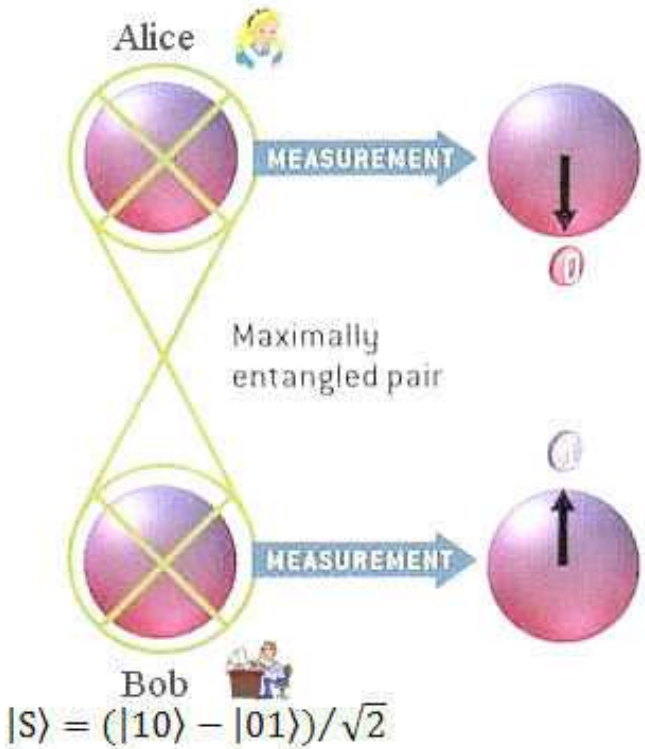
Proof.

The first equality follows by induction on n.
In the second equality, the exponential term of the first equation is transformed as follows,

$$\begin{aligned}
\exp\left(2\pi i\frac{x}{2^m}\right) &= \exp\left(2\pi i\sum_{j=0}^{n-1}\frac{x_j 2^j}{2^m}\right)\\
&= \exp\left(2\pi i\sum_{j=0}^{m-1}\frac{x_j}{2^{m-j}} + 2\pi i\sum_{j=m}^{n-1} x_j 2^{j-m}\right)\\
&= \exp\left(2\pi i\sum_{j=0}^{m-1}\frac{x_j}{2^{m-j}}\right)\prod_{j=m}^{n-1}\exp\left(2\pi i x_j 2^{j-m}\right)\\
&= \exp\left(2\pi i\sum_{j=0}^{m-1}\frac{x_j}{2^{m-j}}\right).
\end{aligned}$$

□

Measurement



Measurement

- Qubit measurement
- Measurement operators
- Multi-qubit measurement
- Partial measurement

Qubit Measurement

- Measurement is an operation to gain information about the state of a qubit

$$v = \lambda_0 |0\rangle + \lambda_1 |1\rangle , \tag{450}$$

where

$$\|v\|^2 = |\lambda_0|^2 + |\lambda_1|^2 = 1. \tag{451}$$

- The result of measurement is either $|0\rangle$ with probability $p_0 = |\lambda_0|^2$ or $|1\rangle$ with probability $p_1 = |\lambda_1|^2$.
- After measurement, the qubit is in the pure state $\frac{\lambda_0}{|\lambda_0|} |0\rangle$ or $\frac{\lambda_1}{|\lambda_1|} |1\rangle$ (classical bit).
- The measurement of the original qubit v cannot be repeated.

Qubit Measurement

The measurement of a qubit $v = \lambda_0 |0\rangle + \lambda_1 |1\rangle$ is the realization of a random variable X^v with

$$\mathbb{P}(X^v = 0) = p_0 = |\lambda_0|^2 \tag{452}$$

and

$$\mathbb{P}(X^v = 1) = p_1 = |\lambda_1|^2. \tag{453}$$

Measurement Operators

Let $n \geq 1$. A collection of *measurement operators* $\{M_0, \ldots, M_{m-1}\}$ on $\mathcal{H}^{\otimes n}$ consists of linear operators

$$M_j : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}, \quad 0 \leq j \leq m-1, \tag{454}$$

such that

$$\sum_{j=0}^{m-1} M_j^* M_j = I, \tag{455}$$

where M_j^* denotes the *adjoint operator* of M_j.

Measurement Operators

For each collection of measurement operators $\{M_0, \ldots, M_{m-1}\}$ on $\mathcal{H}^{\otimes n}$ and each n-qubit $v \in \mathcal{S}_{\mathcal{H}^{\otimes n}}$,

$$\begin{aligned}
\sum_{j=0}^{m-1} \|M_j v\|^2 &= \sum_{j=0}^{m-1} \langle M_j v, M_j v \rangle \qquad (456)\\
&= \sum_{j=0}^{m-1} \langle v, M_j^* M_j v \rangle \\
&= \left\langle v, \sum_{j=0}^{m-1} M_j^* M_j v \right\rangle \\
&= \langle v, Iv \rangle = \langle v, v \rangle \\
&= \|v\|^2 = 1.
\end{aligned}$$

Multi-Qubit Measurement

The *measurement* of an n-qubit $v \in \mathcal{S}_{\mathcal{H}^{\otimes n}}$ by the collection of measurement operators $\{M_0, \ldots, M_{m-1}\}$ on $\mathcal{H}^{\otimes n}$ is the realization of a random variable $X^v_{M_0,\ldots,M_{m-1}}$ with

$$\mathbb{P}(X^v_{M_0,\ldots,M_{m-1}} = j) = p_j = \|M_j v\|^2. \tag{457}$$

Well-definedness follows from

$$\sum_{j=0}^{m-1} \|M_j v\|^2 = 1. \tag{458}$$

After measurement, the n-qubit v becomes the n-qubit

$$\frac{M_j v}{\|M_j v\|} = \frac{1}{\sqrt{p_j}} M_j v. \tag{459}$$

Dual Space

Consider the Hilbert space $\mathcal{H}^{\otimes n}$ with standard basis

$$\{|0\rangle_n, |1\rangle_n, \ldots, |2^n - 1\rangle_n\}. \tag{460}$$

- For each basis element $|j\rangle_n \in \mathcal{H}^{\otimes n}$, $0 \leq j \leq 2^n - 1$, define the linear functional $\langle j|_n : \mathcal{H}^{\otimes n} \to \mathbb{C}$ given by (bra notation)

$$\langle j|_n |k\rangle_n := \langle j|_n (|k\rangle_n) = \langle |j\rangle_n, |k\rangle_n \rangle = \delta_{j,k}, \tag{461}$$

  where $\delta_{j,k} = 1$ if $j = k$ and 0 if $j \neq k$ (Kronecker delta).

- The set

$$\{\langle 0|_n, \langle 1|_n, \ldots, \langle 2^n - 1|_n\} \tag{462}$$

  forms a basis of the dual space of $\mathcal{H}^{\otimes n}$.

Dual Space

The linear operator $M_j = |j\rangle_n \langle j|_n$ given by

$$M_j |k\rangle_n = |j\rangle_n \langle j|_n |k\rangle_n \tag{463}$$

is a projection of $\mathcal{H}^{\otimes n}$ onto the one-dimensional subspace spanned by $|j\rangle_n$, since

$$\begin{aligned} M_j \sum_{k=0}^{2^n-1} \lambda_k |k\rangle_n &= \sum_{k=0}^{2^n-1} \lambda_k M_j |k\rangle_n \\ &= \sum_{k=0}^{2^n-1} \lambda_k |j\rangle_n \langle j|_n |k\rangle_n \\ &= \lambda_j |j\rangle_n , \quad \text{by (461).} \end{aligned} \tag{464}$$

Measurement Operators

The measurement of an n-qubit $v = \sum_{k=0}^{2^n-1} \lambda_k \left|k\right\rangle_n \in \mathcal{S}_{\mathcal{H}^{\otimes n}}$ by the collection of measurement operators $\{M_0, \ldots, M_{2^n-1}\}$ on $\mathcal{H}^{\otimes n}$ with

$$M_j = \left|j\right\rangle_n \left\langle j\right|_n, \quad 0 \leq j \leq 2^n - 1, \tag{465}$$

is the realization of a random variable X_n^v with

$$\mathbb{P}(X_n^v = j) = p_j = |\lambda_j|^2. \tag{466}$$

After measurement, the n-qubit v becomes the pure n-qubit

$$\frac{\lambda_j}{|\lambda_j|} \left|j\right\rangle_n. \tag{467}$$

Proof.

The operator M_j is self-adjoint. Indeed, let $v, w \in \mathbf{H}^{\otimes n}$. Then

$$\begin{aligned}
\langle M_j v, w\rangle &= \langle M_j \sum_{k=0}^{2^n-1} \lambda_k \left|k\right\rangle, w\rangle \\
&= \langle \lambda_j \left|j\right\rangle, w\rangle, \quad \text{by (464)}, \\
&= \bar{\lambda}_j \langle \left|j\right\rangle, \sum_{k=0}^{2^n-1} \mu_k \left|k\right\rangle\rangle = \bar{\lambda}_j \mu_j.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle v, M_j w\rangle &= \langle v, M_j \sum_{k=0}^{2^n-1} \mu_k \left|k\right\rangle\rangle \\
&= \langle v, \mu_j \left|j\right\rangle\rangle, \quad \text{by (464)}, \\
&= \mu_j \langle \sum_{k=0}^{2^n-1} \lambda_k \left|k\right\rangle, \left|j\right\rangle\rangle = \mu_j \bar{\lambda}_j.
\end{aligned}$$

Proof (cont'd)

Moreover, we have

$$\begin{aligned} M_j^* M_j &= (|j\rangle_n \langle j|_n)^* |j\rangle_n \langle j|_n = |j\rangle_n \langle j|_n |j\rangle_n \langle j|_n \\ &= |j\rangle_n (\langle j|_n |j\rangle_n) \langle j|_n = |j\rangle_n \delta_{j,j} \langle j|_n \\ &= |j\rangle_n \langle j|_n = M_j. \end{aligned}$$

Thus for each n-qubit $v = \sum_{k=0}^{2^n-1} \lambda_k |k\rangle_n$,

$$\sum_{j=0}^{2^n-1} M_j^* M_j v = \sum_{j=0}^{2^n-1} M_j v = \sum_{j=0}^{2^n-1} \lambda_j |j\rangle_n = v$$

and so

$$\sum_{j=0}^{2^n-1} M_j^* M_j = I.$$

Hence, $\{M_0, \ldots, M_{2^n-1}\}$ is a set of measurement operators.

Proof (cont'd)

Therefore, $\mathbb{P}(X_n^v = j) = p_j$ becomes

$$\begin{aligned} \|M_j v\|^2 &= \langle M_j v, M_j v\rangle = \langle v, M_j^* M_j v\rangle \\ &= \langle v, M_j v\rangle = \langle v, \lambda_j \left|j\right\rangle_n\rangle \\ &= \langle \lambda_j \left|j\right\rangle_n, \lambda_j \left|j\right\rangle_n\rangle \\ &= \overline{\lambda_j}\lambda_j = |\lambda_j|^2. \end{aligned}$$

The state after measurement is

$$\frac{1}{\sqrt{p_j}} M_j v = \frac{1}{\sqrt{|\lambda_j|^2}} \lambda_j \left|j\right\rangle_n = \frac{\lambda_j}{|\lambda_j|} \left|j\right\rangle_n .$$

□

Multi-Qubit Measurement – Example

Consider the bi-qubit

$$v = \frac{1}{\sqrt{2}} |00\rangle_2 - \frac{1}{2} |01\rangle_2 + \frac{1}{2} |10\rangle_2 .$$

Measurement:

Resulting State	Probability
$\|00\rangle$	$\frac{1}{2}$
$-\|01\rangle$	$\frac{1}{4}$
$\|10\rangle$	$\frac{1}{4}$
$\|11\rangle$	0

Partial Multi-Qubit Measurement

Let $n \geq 1$. Let p be an integer with $1 \leq p < n$ and $q = n - p$. Take the collection of measurement operators $\{M_0, \ldots, M_{2^p-1}\}$ on $\mathcal{H}^{\otimes n}$ with

$$M_j = \sum_{r=0}^{2^q-1} \left(|j\rangle_p \otimes |r\rangle_q\right) \left(|j\rangle_p \otimes |r\rangle_q\right)^*, \ 0 \leq j \leq 2^p - 1. \quad (468)$$

The *partial measurement* of the first p qubits of an n-qubit

$$v = \sum_{k=0}^{2^n-1} \lambda_k |k\rangle_n \quad (469)$$

by $\{M_0, \ldots, M_{2^p-1}\}$ is the realization of a random variable X_p^v with

$$\mathbb{P}(X_p^v = j) = p_j = \sum_{r=0}^{2^q-1} |\lambda_{j \cdot 2^q + r}|^2 \quad (470)$$

where $0 \leq j \leq 2^p - 1$.

Partial Multi-Qubit Measurement

After measurement, the n-qubit v becomes the n-qubit

$$|j\rangle_p \otimes \sum_{r=0}^{2^q-1} \frac{\lambda_{j\cdot 2^q+r}}{\sqrt{p_j}} |r\rangle_q \,. \tag{471}$$

Proof.

Above we have seen that

$$M_j^* M_j = M_j, \quad 0 \leq j \leq 2^p - 1.$$

For each n-qubit $v = \sum_{k=0}^{2^n-1} \lambda_k \left|k\right\rangle_n$, write

$$v = \sum_{j=0}^{2^p-1} \sum_{r=0}^{2^q-1} \lambda_{j,r} \left|j\right\rangle_p \otimes \left|r\right\rangle_q$$

where $\lambda_{j,r} = \lambda_{j \cdot 2^q + r}$ and $\left|j\right\rangle_p \otimes \left|r\right\rangle_q = \left|j \cdot 2^q + r\right\rangle_n$.

Proof (cont'd)

Therefore,

$$
\begin{aligned}
& M_j v \\
&= \sum_{r=0}^{2^q-1} \sum_{k=0}^{2^n-1} \lambda_k \left(|j\rangle_p \otimes |r\rangle_q \right) \left(|j\rangle_p \otimes |r\rangle_q \right)^* |k\rangle_n \\
&= \sum_{r=0}^{2^q-1} \sum_{i=0}^{2^p-1} \sum_{l=0}^{2^q-1} \lambda_{i\cdot 2^q+l} \left(|j\rangle_p \otimes |r\rangle_q \right) \left(|j\rangle_p \otimes |r\rangle_q \right)^* |i\rangle_p \otimes |l\rangle_q \\
&= \sum_{r=0}^{2^q-1} \lambda_{j\cdot 2^q+r} \left(|j\rangle_p \otimes |r\rangle_q \right) \\
&= |j\rangle_p \otimes \sum_{r=0}^{2^q-1} \lambda_{j\cdot 2^q+r} |r\rangle_q \,.
\end{aligned}
$$

Proof (cont'd)

It follows that

$$\begin{aligned}\sum_{j=0}^{2^p-1} M_j^* M_j v &= \sum_{j=0}^{2^p-1} M_j v \\ &= \sum_{j=0}^{2^p-1} |j\rangle_p \otimes \sum_{r=0}^{2^q-1} \lambda_{j\cdot 2^q+r} |r\rangle_q \\ &= v.\end{aligned}$$

Thus

$$\sum_{j=0}^{2^p-1} M_j^* M_j = I$$

and hence $\{M_0, \ldots, M_{2^p-1}\}$ is a set of measurement operators.

Proof (cont'd)

Therefore, $\mathbb{P}(X_p^v = j) = p_j$ becomes

$$\begin{aligned} \|M_j v\|^2 &= \langle M_j v, M_j v\rangle = \langle v, M_j^* M_j v\rangle \\ &= \langle v, M_j v\rangle \\ &= \sum_{r=0}^{2^q-1} \overline{\lambda_{j\cdot 2^q+r}} \lambda_{j\cdot 2^q+r} \\ &= \sum_{r=0}^{2^q-1} |\lambda_{j\cdot 2^q+r}|^2. \end{aligned}$$

The state after the measurement is

$$\frac{1}{\sqrt{p_j}} M_j v = |j\rangle_p \otimes \sum_{r=0}^{2^q-1} \frac{\lambda_{j\cdot 2^q+r}}{\sqrt{p_j}} |r\rangle_q .$$

□

Partial Multi-Qubit Measurement – Example

Let $n = 3$, $p = 1$, and $q = n - p = 2$. Then

$$\begin{aligned}
|000\rangle_3 &= |0\rangle_1 \otimes |00\rangle_2 , \\
|001\rangle_3 &= |0\rangle_1 \otimes |01\rangle_2 , \\
|010\rangle_3 &= |0\rangle_1 \otimes |10\rangle_2 , \\
|011\rangle_3 &= |0\rangle_1 \otimes |11\rangle_2 , \\
|100\rangle_3 &= |1\rangle_1 \otimes |00\rangle_2 , \\
|101\rangle_3 &= |1\rangle_1 \otimes |01\rangle_2 , \\
|110\rangle_3 &= |1\rangle_1 \otimes |10\rangle_2 , \\
|111\rangle_3 &= |1\rangle_1 \otimes |11\rangle_2 .
\end{aligned}$$

Partial Multi-Qubit Measurement – Example (cont'd)

Take the 3-qubit

$$\begin{aligned} v &= \frac{1}{\sqrt{3}}\left(|011\rangle_3 + |100\rangle_3 + |101\rangle_3\right) \\ &= \frac{1}{\sqrt{3}}|0\rangle_1 \otimes |11\rangle_2 + \frac{1}{\sqrt{3}}\left(|1\rangle_1 \otimes \left(|00\rangle_2 + |01\rangle_2\right)\right). \end{aligned}$$

Measurement of the first qubit gives the state

$$|0\rangle_1 \otimes |11\rangle_2$$

with probability $p_0 = \frac{1}{3}$ and the state

$$|1\rangle_1 \otimes \frac{1}{\sqrt{2}}(|00\rangle_2 + |01\rangle_2)$$

with probability $p_1 = \frac{2}{3}$.

Partial Multi-Qubit Measurement – Separability

Let $n \geq 1$. Let p be an integer with $1 \leq p < n$ and $q = n - p$. Take the collection of measurement operators $\{M_0, \ldots, M_{2^p-1}\}$ on $\mathcal{H}^{\otimes n}$ with

$$M_j = \sum_{r=0}^{2^q-1} \left(|j\rangle_p \otimes |r\rangle_q \right) \left(|j\rangle_p \otimes |r\rangle_q \right)^*, \ 0 \leq j \leq 2^p - 1. \quad (472)$$

The partial measurement of the first p qubits of a separable n-qubit

$$v = x \otimes y = \sum_{k=0}^{2^p-1} \lambda_k |k\rangle_p \otimes \sum_{r=0}^{2^q-1} \mu_r |r\rangle_q \quad (473)$$

by $\{M_0, \ldots, M_{2^p-1}\}$ is the realization of a random variable X_p^v with

$$\mathbb{P}(X_p^v = j) = p_j = |\lambda_j|^2 \quad (474)$$

where $0 \leq j \leq 2^p - 1$.

Partial Multi-Qubit Measurement – Separability

After measurement, the n-qubit v becomes the n-qubit

$$\left(\frac{\lambda_j}{|\lambda_j|} \, |j\rangle_p \right) \otimes y. \tag{475}$$

Multi-Qubit Measurement – Example

Consider the separable bi-qubit

$$\begin{aligned} v &= \frac{1}{\sqrt{6}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle \\ &= \left(\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right). \end{aligned}$$

The measurement of the first qubit shows the following:

Result	Resulting State	Probability
0	$\|0\rangle \otimes \left(\frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle\right)$	$\frac{1}{3}$
1	$\|1\rangle \otimes \left(\frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle\right)$	$\frac{2}{3}$

Part IX

Quantum Algorithms

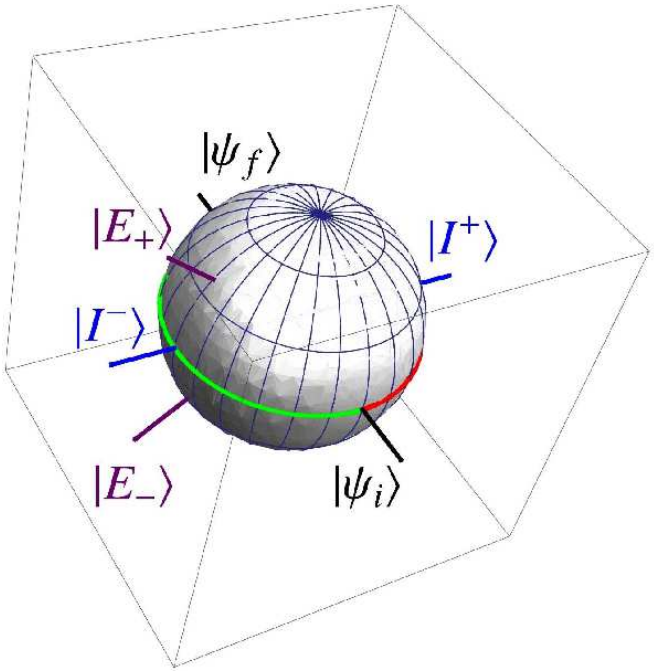
Quantum Algorithms

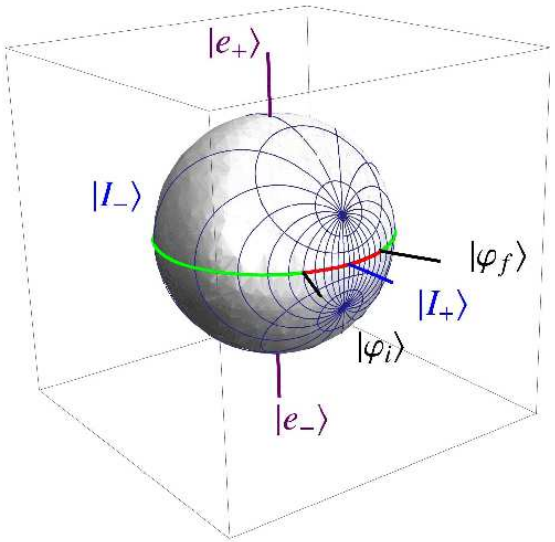
- Quantum computing is computing using quantum-mechanical phenomena.
- List of available QC algorithms:

  `www.quantumalgorithmzoo.org`

  maintained by St. Jordan (Microsoft Quantum).
- Large-scale quantum computers may be able to efficiently solve problems which are not practically feasible on classical computers.
- Post-quantum cryptography aims at cryptographic algorithms secure against QC attacks. Theory is in its infancy.
- Announcements (2017/2018): 50-qubit QC (IBM), 49-qubit QC (Intel), 72-qubit QC (Google).

Quantum Algorithms





Quantum Algorithms

- General quantum algorithms
- Deutsch algorithm
- Deutsch-Jozsa algorithm
- Grover algorithm
- Shor algorithm

Quantum Algorithms

A quantum algorithm consists of three parts:

1. Input: classical n-bit tuple, conversion into n-qubit.
2. Processing: application of a finite sequence of unitary operators on the given n-qubit.
   - Processing is deterministic and reversible (by using the inverse unitary operators).
3. Output: measurement gives classical n-bit tuple.
   - Measurement is probabilistic and irreversible.

Quantum Algorithms – Formulation

- Initial state: n-qubit $v \in \mathcal{S}_{\mathcal{H}^{\otimes n}}$.
- Computation: unitary operator $U : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ or associated representation matrix $M_U \in \mathbb{C}^{2^n \times 2^n}$.
- Final state:

$$w = U(v) = \sum_{j=0}^{2^n-1} w_j \, |j\rangle_n \in \mathcal{S}_{\mathcal{H}^{\otimes n}}. \tag{476}$$

- Measurement: random variable $Y = X_n^w$, where

$$\mathbb{P}(Y = j) = |w_j|^2, \quad 0 \leq j \leq 2^n - 1. \tag{477}$$

- Complexity of classical computation:
  - runtime $O(2^{2n})$, size of unitary matrix M_U,
  - storage $O(2^n)$, size of Hilbert space $\mathcal{H}^{\otimes n}$.

Quantum Algorithms – Full Separability

- A fully separable n-qubit $v \in \mathcal{H}^{\otimes n}$ has the form

$$v = \bigotimes_{m=1}^{n} (\alpha_m |0\rangle + \beta_m |1\rangle), \tag{478}$$

where $\alpha_m, \beta_m \in \mathbb{C}$, $1 \leq m \leq n$.

- The n-qubit v has storage complexity $O(n)$.
- Each n-qubit $|x\rangle_n$, $0 \leq x \leq 2^n - 1$, has a fully separated representation

$$|x\rangle_n = \bigotimes_{m=1}^{n} ((1 - x_{n-m}) |0\rangle + x_{n-m} |1\rangle), \tag{479}$$

where $x = \sum_{j=0}^{n-1} x_j 2^j$, $x_j \in \mathbb{B}$, $0 \leq j \leq n - 1$.

Proof.

For each $x \in \{0,1\}$,

$$|x\rangle_1 = (1-x)\,|0\rangle_1 + x\,|1\rangle_1\,.$$

Thus for $x = x_{n-1}\ldots x_0 \in \{0,\ldots,2^n-1\}$,

$$\begin{aligned} |x\rangle_n &= |x_{n-1}\rangle_1 \otimes \ldots \otimes |x_0\rangle_1 \\ &= \bigotimes_{m=1}^{n} ((1-x_{n-m})\,|0\rangle_1 + x_{n-m}\,|1\rangle_1). \end{aligned}$$

Example

$$\begin{aligned} |011\rangle_3 &= \bigotimes_{m=1}^{3} ((1-x_{3-m})\,|0\rangle + x_{3-m}\,|1\rangle) \\ &= ((1-x_2)\,|0\rangle + x_2\,|1\rangle) \otimes ((1-x_1)\,|0\rangle + x_1\,|1\rangle) \otimes ((1-x_0)\,|0\rangle + x_0\,|1\rangle) \\ &= |0\rangle \otimes |1\rangle \otimes |1\rangle\,. \end{aligned}$$

Quantum Algorithms – Hadamard Gate

$$\mathbf{H}^{\otimes n}\left(\bigotimes_{m=1}^{n}(\alpha_m|0\rangle+\beta_m|1\rangle)\right)=\bigotimes_{m=1}^{n}(\hat{\alpha}_m|0\rangle+\hat{\beta}_m|1\rangle), \quad (480)$$

where

$$\hat{\alpha}_m=\frac{\alpha_m+\beta_m}{\sqrt{2}}, \quad \hat{\beta}_m=\frac{\alpha_m-\beta_m}{\sqrt{2}}, \quad 1\leq m\leq n. \quad (481)$$

Runtime complexity $O(n)$.

Proof.

$$
\begin{aligned}
\mathbf{H}^{\otimes n} \left(\bigotimes_{m=1}^{n} (\alpha_m |0\rangle + \beta_m |1\rangle) \right) & \\
= \quad \bigotimes_{m=1}^{n} (\mathbf{H}(\alpha_m |0\rangle + \beta_m |1\rangle)) & \\
= \quad \bigotimes_{m=1}^{n} (\alpha_m \mathbf{H} |0\rangle + \beta_m \mathbf{H} |1\rangle)) & \\
= \quad \bigotimes_{m=1}^{n} \left(\alpha_m \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta_m \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) & \\
= \quad \bigotimes_{m=1}^{n} \left(\frac{\alpha_m + \beta_m}{\sqrt{2}} |0\rangle + \frac{\alpha_m - \beta_m}{\sqrt{2}} |1\rangle \right) . &
\end{aligned}
$$

Quantum Algorithms – Hadamard Gate

By (480),

$$\mathbf{H}^{\otimes n} |0\rangle_n = \bigotimes_{m=1}^{n} \left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \tag{482}$$

and

$$\mathbf{H}^{\otimes n} |1\rangle_n = \bigotimes_{m=1}^{n} \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right). \tag{483}$$

Quantum Algorithms – Fourier Transform

$$\mathbf{F}^{\otimes n}\left(\bigotimes_{m=1}^{n}(\alpha_m|0\rangle+\beta_m|1\rangle)\right) \tag{484}$$

$$=\sum_{x=0}^{2^n-1}\bigotimes_{m=1}^{n}\left(\frac{\alpha_m(1-x_{n-m})+\beta_m x_{n-m}}{\sqrt{2}}|0\rangle\right.$$

$$\left.+\frac{\alpha_m(1-x_{n-m})+\beta_m x_{n-m}}{\sqrt{2}}\exp\left(2\pi i\sum_{j=0}^{m-1}\frac{x_j}{2^{m-j}}\right)|1\rangle\right).$$

Runtime complexity $O(2^n)$.

Proof.

$$
\begin{aligned}
&\mathbf{F}^{\otimes n}\left(\bigotimes_{m=1}^{n}\left(\alpha_m\,|0\rangle+\beta_m\,|1\rangle\right)\right)\\
&\quad=\quad \mathbf{F}^{\otimes n}\sum_{x=0}^{2^n-1}\left(\prod_{m=1}^{n}\left(\alpha_m(1-x_{n-m})+\beta_m x_{n-m}\right)\right)|x\rangle_n\\
&\quad=\quad \sum_{x=0}^{2^n-1}\left(\prod_{m=1}^{n}\left(\alpha_m(1-x_{n-m})+\beta_m x_{n-m}\right)\right)\mathbf{F}^{\otimes n}\,|x\rangle_n\\
&\quad=\quad \sum_{x=0}^{2^n-1}\left(\prod_{m=1}^{n}\left(\alpha_m(1-x_{n-m})+\beta_m x_{n-m}\right)\right)\\
&\qquad\bigotimes_{m=1}^{n}\left(\frac{1}{\sqrt{2}}\,|0\rangle+\frac{1}{\sqrt{2}}\exp\left(2\pi i\sum_{j=0}^{m-1}\frac{x_j}{2^{m-j}}\,|1\rangle\right)\right),\ \text{by (449)},\\
&\quad=\quad \sum_{x=0}^{2^n-1}\bigotimes_{m=1}^{n}\left(\frac{\alpha_m(1-x_{n-m})+\beta_m x_{n-m}}{\sqrt{2}}\,|0\rangle\right.\\
&\qquad\left.+\frac{\alpha_m(1-x_{n-m})+\beta_m x_{n-m}}{\sqrt{2}}\exp\left(2\pi i\sum_{j=0}^{m-1}\frac{x_j}{2^{m-j}}\right)|1\rangle\right).
\end{aligned}
$$

Note: $(a_1a_2)(x_1\otimes x_2)=a_1x_1\otimes a_2x_2$. □

Measurement – Classical Computation

Measurement of n-qubit $w = \sum_{j=0}^{2^n-1} w_j \, |j\rangle_n \in \mathcal{S}_{\mathcal{H}^{\otimes n}}$ by classical computation:

$$\alpha_0 = 0, \quad \alpha_k = \sum_{j=0}^{k} |w_j|^2, \quad 1 \leq k \leq 2^n. \tag{485}$$

Choose value $r \in [0, 1]$ uniformly at random such that if $r \in [\alpha_k, \alpha_{k+1})$, the result is $Y = k$.

Example

Let

$$(p_0, p_1, p_2, p_3) = (0.1, 0.2, 0.4, 0.3).$$

Then

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.0, 0.1, 0.3, 0.7, 1.0)$$

Choose $r \in [0, 1]$ uniformly at random.

- If $r \in [0.0, 0.1)$, then $Y = 0$.
- If $r \in [0.1, 0.3)$, then $Y = 1$.
- If $r \in [0.3, 0.7)$, then $Y = 2$.
- If $r \in [0.7, 1.0]$, then $Y = 3$.

Deutsch Algorithm (1985)

Require: Boolean function $f : \mathbb{B} \to \mathbb{B}$, Hilbert space $\mathcal{H}^{\otimes 2}$ with basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

Ensure: $Y = 1$ if f is balanced, i.e., $f(0) \neq f(1)$, and $Y = 0$ if f is constant, i.e., $f(0) = f(1)$.

$\psi_0 \leftarrow |01\rangle_2$

$\psi_1 \leftarrow \mathbf{H}^{\otimes 2}\psi_0$

$\psi_2 \leftarrow \mathbf{U}_f\psi_1$

$\psi_3 \leftarrow \mathbf{H}_1^{(2)}\psi_2$

$Y \leftarrow X_1^{\psi_3}$ {Partial measurement of first qubit}

Unary Boolean Functions

- The number of n-ary Boolean functions $f : \mathbb{B}^n \to \mathbb{B}$ is 2^{2^n}, $n \geq 0$.
- The unary Boolean functions $f : \mathbb{B} \to \mathbb{B}$ are

x	$f_0(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$
0	0	0	1	1
1	0	1	0	1

The functions f_0, f_3 are constant, and the functions $f_1 = \mathrm{id}, f_2 = \mathrm{not}$ are balanced.

Proof.

First statement:

$$\begin{aligned}
\psi_1 &= \mathbf{H}^{\otimes 2} |01\rangle_2 \\
&= (\mathbf{H} |0\rangle) \otimes (\mathbf{H} |1\rangle) \\
&= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} |0\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} + \frac{1}{\sqrt{2}} |1\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} \sum_{x=0}^{1} |x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}.
\end{aligned} \tag{486}$$

Proof (cont'd).

Second statement:

$$\begin{aligned}
\psi_2 &= \mathbf{U}_f \psi_1 = \mathbf{U}_f \left(\frac{1}{\sqrt{2}} \sum_{x=0}^{1} |x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \qquad (487) \\
&= \frac{1}{\sqrt{2}} \sum_{x=0}^{1} \mathbf{U}_f \left(|x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{\sqrt{2}} \sum_{x=0}^{1} \left(|x\rangle \otimes \frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{\sqrt{2}} \sum_{x=0}^{1} \left(|x\rangle \otimes \frac{(-1)^{f(x)}(|0\rangle - |1\rangle)}{\sqrt{2}} \right) \\
&= \frac{1}{\sqrt{2}} \sum_{x=0}^{1} \left((-1)^{f(x)} \cdot |x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{\sqrt{2}} \left((-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle \right) \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}.
\end{aligned}$$

Proof (cont'd).

Third and fourth statements:

$$
\begin{aligned}
\psi_3 &= \mathbf{H}_1^{(2)}\psi_2 = (\mathbf{H}\otimes\mathbf{I})\psi_2 \qquad (488)\\
&= (\mathbf{H}\otimes\mathbf{I})\left(\frac{1}{\sqrt{2}}\left((-1)^{f(0)}\,|0\rangle + (-1)^{f(1)}\,|1\rangle\right)\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\\
&= \frac{1}{\sqrt{2}}\left((-1)^{f(0)}\mathbf{H}(|0\rangle) + (-1)^{f(1)}\mathbf{H}(|1\rangle)\right)\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
&= \frac{1}{\sqrt{2}}\left((-1)^{f(0)}\frac{|0\rangle+|1\rangle}{\sqrt{2}} + (-1)^{f(1)}\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
&= \left(\frac{(-1)^{f(0)}+(-1)^{f(1)}}{2}\,|0\rangle + \frac{(-1)^{f(0)}-(-1)^{f(1)}}{2}\,|1\rangle\right)\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
&= \pm\,|f(0)\oplus f(1)\rangle\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
&= \begin{cases} \pm\,|0\rangle\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0)=f(1),\\ \pm\,|1\rangle\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} & \text{if } f(0)\neq f(1). \end{cases}
\end{aligned}
$$

Thus the partial measurement of the first qubit results in $|0\rangle$ with probability 1 (almost sure) if f is constant, and otherwise in $|1\rangle$ with probability 1 (almost sure). □

Deutsch-Jozsa Algorithm (1992)

Require: Boolean function $f : \mathbb{B}^n \to \mathbb{B}$ constant or balanced, Hilbert space $\mathcal{H}^{\otimes(n+1)}$

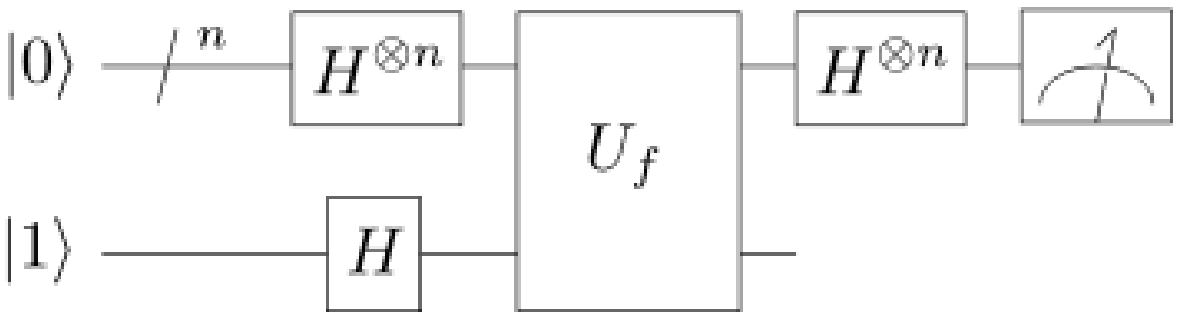
Ensure: $Y = 0$ if f is constant with probability 1; otherwise, f is balanced.

$\psi_0 \leftarrow |0\rangle_n |1\rangle_1$
$\psi_1 \leftarrow \mathbf{H}^{\otimes(n+1)} \psi_0$
$\psi_2 \leftarrow \mathbf{U}_f \psi_1$
$\psi_3 \leftarrow (\mathbf{H}^{\otimes n} \otimes \mathbf{I}) \psi_2$
$Y \leftarrow X_n^{\psi_3}$ {Partial measurement of first n qubits}

Deutsch-Jozsa Algorithm

- If an n-ary Boolean function f is assumed to be either constant or balanced, a conventional deterministic algorithm requires $2^{n-1}+1$ evaluations for the decision.
- The DJ algorithm produces an answer that is always correct with a single evaluation of f.
- Straightforward extension of Deutsch algorithm.
- Inspiration for two of the most revolutionary quantum algorithms of Grover and Shor.

Deutsch-Jozsa Quantum Circuit



Proof.

First statement:

$$\begin{aligned}
\psi_1 &= \mathbf{H}^{\otimes(n+1)}(|0\rangle_n |1\rangle) \qquad (489)\\
&= \mathbf{H}^{\otimes n} |0\rangle_n \otimes \mathbf{H} |1\rangle\\
&= (\mathbf{H} |0\rangle \otimes \ldots \otimes \mathbf{H} |0\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\\
&= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \ldots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\\
&= 2^{-n/2} \sum_{x=0}^{2^n-1} |x\rangle_n \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \text{ by (425).}
\end{aligned}$$

Proof (cont'd).

Second statement:

$$
\begin{aligned}
\psi_2 &= \mathbf{U}_f \psi_1 \qquad (490)\\
&= 2^{-n/2} \sum_{x=0}^{2^n-1} \mathbf{U}_f \left(|x\rangle_n \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)\\
&= 2^{-n/2} \sum_{x=0}^{2^n-1} \left(|x\rangle_n \otimes \frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right)\\
&= 2^{-n/2} \sum_{x=0}^{2^n-1} \left(|x\rangle_n \otimes \frac{(-1)^{f(x)}(|0\rangle - |1\rangle)}{\sqrt{2}} \right)\\
&= 2^{-n/2} \sum_{x=0}^{2^n-1} (-1)^{f(x)} \cdot |x\rangle_n \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}.
\end{aligned}
$$

Proof (cont'd).

Third statement:

$$
\begin{aligned}
\psi_3 &= (\mathbf{H}^{\otimes n} \otimes \mathbf{I})\psi_2 \qquad (491)\\
&= (\mathbf{H}^{\otimes n} \otimes \mathbf{I}) \left(2^{-n/2} \sum_{x=0}^{2^n-1} (-1)^{f(x)} \cdot |x\rangle_n \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)\\
&= 2^{-n/2} \sum_{x=0}^{2^n-1} (-1)^{f(x)} \cdot \mathbf{H}^{\otimes n} |x\rangle_n \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\\
&= \frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^{f(x)} \cdot \sum_{y=0}^{2^n-1} (-1)^{h(x,y)} |y\rangle_n \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \quad \text{by (429)},\\
&= \frac{1}{2^n} \sum_{y=0}^{2^n-1} \left(\sum_{x=0}^{2^n-1} (-1)^{f(x)+h(x,y)} \right) |y\rangle_n \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\\
&= \left[\underbrace{\frac{1}{2^n} \left(\sum_{x=0}^{2^n-1} (-1)^{f(x)} \right)}_{a_0} |0\rangle_n + \frac{1}{2^n} \sum_{y=1}^{2^n-1} \left(\sum_{x=0}^{2^n-1} (-1)^{f(x)+h(x,y)} \right) |y\rangle_n \right]\\
&\quad \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \quad \text{by definition of } h.
\end{aligned}
$$

Proof (cont'd).

If f is contant, then

$$a_0 = \frac{1}{2^n}\left(\sum_{x=0}^{2^n-1}(-1)^{f(x)}\right) \tag{492}$$

has the value $a_0 = \pm 1$. But ψ_3 has norm 1 and so

$$\psi_3 = \pm |0\rangle_n \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{493}$$

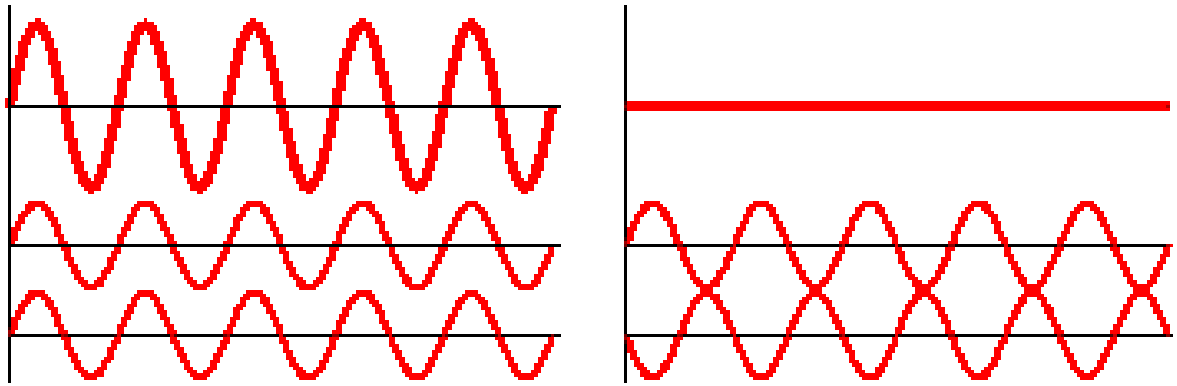
Partial measurement of the first n qubits gives $|0\rangle_n$ with probability 1 (constructive interference).

If f is balanced, then $a_0 = 0$. Partial measurement of the first n qubits does not result in $|0\rangle_n$ with probability 1 (destructive interference). □

Wave Inference

Inference means superposition of two waves to generate a wave of greater, lower, or the same amplitude.

- When in phase, constructive inference results in a wave of greater amplitude.
- When 180^o out of phase, destructive inference results.



Grover Algorithm (1996)

Require: Boolean function $f : \mathbb{B}^n \to \mathbb{B}$, $M = |f^{-1}(1)|$, Hilbert space $\mathcal{H}^{\otimes(n+1)}$, $R = \left\lceil \frac{\pi}{4}\sqrt{2^n/M} \right\rceil$.

Ensure: Measurement yields value x with $f(x) = 1$ with high probability.

$\psi_0 \leftarrow |0\rangle_n |1\rangle_1$

$\psi_1 \leftarrow \mathbf{H}^{\otimes(n+1)} \psi_0$

for r from 1 to R **do**

$\hat{\psi}_{r+1} \leftarrow \mathbf{U}_f \psi_r$

$\psi_{r+1} \leftarrow ((\mathbf{H}^{\otimes n} (2 |0\rangle_n \langle 0|_n - \mathbf{I}^{\otimes n}) \mathbf{H}^{\otimes n}) \otimes \mathbf{I}) \hat{\psi}_{r+1}$

end for

$Y \leftarrow X_n^{\psi_{R+1}}$ {Partial measurement of first n qubits; correct with high probability.}

Time complexity $O(R)$.

Grover Algorithm

- *Satisfiability problem:* Given Boolean function $f : \mathbb{B}^n \to \mathbb{B}$. Find an argument $y \in \mathbb{B}^n$ with $f(y) = 1$.
- The satisfiability problem is NP-hard; i.e., a conventional deterministic algorithm for finding $y \in \mathbb{B}^n$ with $f(y) = 1$ requires $O(2^n)$ steps.
- The Grover algorithm produces an answer in $O(\sqrt{2^n})$ steps and so has quadratic speedup.
- Thus Grover's method could brute-force a 128-bit symmetric cryptographic key in about 2^{64} iterations.
- Hence, symmetric key lengths need to be doubled for protection against the Grover quantum attack.

Proof.

First statement:

$$\begin{aligned}
\psi_1 &= \mathbf{H}^{\otimes(n+1)}(|0\rangle_n |1\rangle) = \mathbf{H}^{\otimes n} |0\rangle_n \otimes \mathbf{H} |1\rangle \qquad (494)\\
&= (\mathbf{H} |0\rangle \otimes \ldots \otimes \mathbf{H} |0\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\\
&= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \ldots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\\
&= 2^{-n/2} \sum_{x=0}^{2^n-1} |x\rangle_n \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \text{ by (425).}
\end{aligned}$$

Proof (cont'd).

Loop statement: Define the sequences $(\alpha_{r,0})_{r\in\mathbb{N}}$ and $(\alpha_{r,1})_{r\in\mathbb{N}}$.

- Initialize

$$\alpha_{1,0} = 2^{-n/2} \quad \text{and} \quad \alpha_{1,1} = 2^{-n/2}. \tag{495}$$

- Claim that by induction,

$$\psi_r = \sum_{x=0}^{2^n-1} \alpha_{r,f(x)} \left|x\right\rangle_n \otimes \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle), \quad r \geq 1. \tag{496}$$

Induction base: case $r = 1$ is clear by (494).

Proof (cont'd) - Induction step

Loop statement:

$$\begin{aligned}
\hat{\psi}_{r+1} &= \mathbf{U}_f \psi_r && (497)\\
&= \sum_{x=0}^{2^n-1} \alpha_{r,f(x)} \mathbf{U}_f \left(|x\rangle_n \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right)\\
&= \sum_{x=0}^{2^n-1} (-1)^{f(x)} \cdot \alpha_{r,f(x)} \cdot |x\rangle_n \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \text{ by (442).}
\end{aligned}$$

Proof (cont'd).

The operator $2\left|0\right\rangle_n\left\langle 0\right|_n - \mathbf{I}^{\otimes n}$ is a *Householder reflection*,

$$\left(2\left|0\right\rangle_n\left\langle 0\right|_n - \mathbf{I}^{\otimes n}\right)\left|x\right\rangle_n = \begin{cases} \left|0\right\rangle_n & \text{if } x = 0, \\ -\left|x\right\rangle_n & \text{if } x \neq 0. \end{cases} \tag{498}$$

It fixes base vector $\left|0\right\rangle_n$ and reflects all other base vectors. Indeed,

$$2\left|0\right\rangle_n\left\langle 0\right|_n\left(\left|x\right\rangle_n\right) = \begin{cases} 2\left|0\right\rangle_n & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases}$$

and so

$$\left(2\left|0\right\rangle_n\left\langle 0\right|_n - \mathbf{I}^{\otimes n}\right)\left|x\right\rangle_n = 2\left|0\right\rangle_n\left\langle 0\right|_n\left|x\right\rangle_n - \mathbf{I}^{\otimes n}\left|x\right\rangle_n$$

has the required form.

Proof (cont'd).

Grover's diffusion operator:

$$\begin{aligned}
\hat{G} &= \mathbf{H}^{\otimes n}\left(2\left|0\right\rangle_n\left\langle 0\right|_n - \mathbf{I}^{\otimes n}\right)\mathbf{H}^{\otimes n} \qquad (499)\\
&= 2\left(\mathbf{H}^{\otimes n}\left|0\right\rangle_n\right)\left(\left\langle 0\right|_n\mathbf{H}^{\otimes n}\right) - \mathbf{H}^{\otimes n}\mathbf{I}^{\otimes n}\mathbf{H}^{\otimes n}\\
&= 2\left(2^{-n/2}\sum_{z=0}^{2^n-1}\left|z\right\rangle_n\right)\left(2^{-n/2}\sum_{y=0}^{2^n-1}\left\langle y\right|_n\right) - \mathbf{I}^{\otimes n} \text{ by (425)}\\
&= 2^{1-n}\sum_{z=0}^{2^n-1}\sum_{y=0}^{2^n-1}\left|z\right\rangle_n\left\langle y\right|_n - \mathbf{I}^{\otimes n}.
\end{aligned}$$

Note that by function composition and (400),

$$\mathbf{H}^{\otimes n}\mathbf{I}^{\otimes n}\mathbf{H}^{\otimes n} = (\mathbf{HIH})^{\otimes n} = (\mathbf{HH})^{\otimes n} = \mathbf{I}^{\otimes n}.$$

Proof (cont'd).

Grover's diffusion operator:

$$\begin{aligned}
\hat{G} \sum_{x=0}^{2^n-1} a_x \left|x\right\rangle_n &= \sum_{x=0}^{2^n-1} a_x \hat{G} \left|x\right\rangle_n , \quad a_x \in \mathbb{C}, && (500)\\
&= \sum_{x=0}^{2^n-1} a_x \left(2^{1-n} \sum_{z=0}^{2^n-1} \sum_{y=0}^{2^n-1} \left|z\right\rangle_n \left\langle y\right|_n \left|x\right\rangle_n - \mathbf{I}^{\otimes n} \left|x\right\rangle_n \right) && \text{by (499)}\\
&= \sum_{x=0}^{2^n-1} a_x \left(2^{1-n} \sum_{z=0}^{2^n-1} \left|z\right\rangle_n - \left|x\right\rangle_n \right)\\
&= \sum_{x=0}^{2^n-1} \left(2^{1-n} \sum_{z=0}^{2^n-1} a_z - a_x \right) \left|x\right\rangle_n\\
&= \sum_{x=0}^{2^n-1} \left(2 \left(\frac{1}{2^n} \sum_{z=0}^{2^n-1} a_z \right) - a_x \right) \left|x\right\rangle_n .
\end{aligned}$$

Proof (cont'd).

Setting

$$a_x = (-1)^{f(x)} \cdot \alpha_{r,f(x)} \tag{501}$$

in (500) gives

$$\begin{aligned} \frac{1}{2^n} \sum_{z=0}^{2^n-1} a_z &= \frac{1}{2^n} \sum_{z=0}^{2^n-1} (-1)^{f(z)} \cdot \alpha_{r,f(z)} \qquad (502) \\ &= \frac{1}{2^n} \cdot M \cdot (-1)^1 \cdot \alpha_{r,1} + \frac{1}{2^n} \cdot (2^n - M) \cdot (-1)^0 \cdot \alpha_{r,0} \\ &= -\frac{M}{2^n} \cdot \alpha_{r,1} + \left(1 - \frac{M}{2^n}\right) \cdot \alpha_{r,0}, \end{aligned}$$

where $M = |f^{-1}(1)| = |\{x \in \mathbb{B}^n \mid f(x) = 1\}|$.

Proof (cont'd).

Using (497), (500) and (502),

$$\begin{aligned}
\psi_{r+1} &= \left(\hat{G}\otimes\mathbf{I}\right)\hat{\psi}_{r+1} \qquad (503)\\
&= \left(\hat{G}\otimes\mathbf{I}\right)\sum_{x=0}^{2^n-1}(-1)^{f(x)}\cdot\alpha_{r,f(x)}\cdot|x\rangle_n\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\\
&= \hat{G}\sum_{x=0}^{2^n-1}(-1)^{f(x)}\cdot\alpha_{r,f(x)}\cdot|x\rangle_n\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\\
&\stackrel{!}{=} \sum_{x=0}^{2^n-1}\alpha_{r+1,f(x)}\,|x\rangle_n\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle),
\end{aligned}$$

when the coefficients $\alpha_{r+1,0}$ and $\alpha_{r+1,1}$ are defined accordingly.

Proof (cont'd).

Using the above setting $a_x = (-1)^{f(x)} \cdot \alpha_{r,f(x)}$, a comparison of coefficients in (500), (502) and (503) gives

$$\begin{aligned} \alpha_{r+1,0} &= 2\left(\frac{1}{2^n}\sum_{z=0}^{2^n-1} a_z\right) - (-1)^0 \cdot \alpha_{r,0} \qquad (504)\\ &= 2\left(-\frac{M}{2^n}\cdot\alpha_{r,1} + \left(1-\frac{M}{2^n}\right)\cdot\alpha_{r,0}\right) - \alpha_{r,0}\\ &= \left(1-\frac{M}{2^{n-1}}\right)\cdot\alpha_{r,0} - \frac{M}{2^{n-1}}\cdot\alpha_{r,1}\\ &= \alpha_{r,0} - \frac{M}{2^{n-1}}(\alpha_{r,0}+\alpha_{r,1}). \end{aligned}$$

Proof (cont'd).

Similarly,

$$\begin{aligned}
\alpha_{r+1,1} &= 2\left(\frac{1}{2^n}\sum_{z=0}^{2^n-1} a_z\right) - (-1)^1 \cdot \alpha_{r,1} \qquad (505)\\
&= 2\left(-\frac{M}{2^n}\cdot\alpha_{r,1} + \left(1-\frac{M}{2^n}\right)\cdot\alpha_{r,0}\right) + \alpha_{r,1}\\
&= \left(2-\frac{M}{2^{n-1}}\right)\cdot\alpha_{r,0} + \left(1-\frac{M}{2^{n-1}}\right)\cdot\alpha_{r,1}\\
&= 2\alpha_{r,0} + \alpha_{r,1} - \frac{M}{2^{n-1}}(\alpha_{r,0}+\alpha_{r,1}).
\end{aligned}$$

Thus we have defined $\alpha_{r+1,0}$ and $\alpha_{r+1,1}$ in terms of $\alpha_{r,0}$ and $\alpha_{r,1}$.

Proof (cont'd).

- The sequences $(\alpha_{r,0})_r$ and $(\alpha_{r,1})_r$ are real-valued.
- By the normalization of ψ_r in (496),

$$\begin{aligned} \|\psi_r\| &= \sum_{x=0}^{2^n-1} |\alpha_{r,f(x)}|^2 \\ &= (2^n - M)\alpha_{r,0}^2 + M\alpha_{r,1}^2 = 1 \end{aligned} \tag{506}$$

  for each $r \in \mathbb{N}_0$.

Explicit representations of $(\alpha_{r,0})_r$ and $(\alpha_{r,1})_r$ are derived next.

Proof (cont'd).

Use the Ansatz

$$\alpha_{1,0} = \frac{1}{\sqrt{2^n - M}} \cos\frac{\beta}{2} \quad \text{and} \quad \alpha_{1,1} = \frac{1}{\sqrt{M}} \sin\frac{\beta}{2}. \tag{507}$$

Then by (495),

$$\frac{1}{\sqrt{2^n - M}} \cos\frac{\beta}{2} = \frac{1}{\sqrt{2^n}} \quad \text{and} \quad \frac{1}{\sqrt{M}} \sin\frac{\beta}{2} = \frac{1}{\sqrt{2^n}}. \tag{508}$$

Thus

$$\cos\frac{\beta}{2} = \sqrt{\frac{2^n - M}{2^n}} \quad \text{and} \quad \sin\frac{\beta}{2} = \sqrt{\frac{M}{2^n}} \tag{509}$$

and hence

$$\beta = 2 \arcsin\sqrt{\frac{M}{2^n}}. \tag{510}$$

Proof (cont'd).

For each $r \geq 1$, put

$$\alpha_{r,0} = \frac{1}{\sqrt{2^n - M}} \cos\varphi_r \quad \text{and} \quad \alpha_{r,1} = \frac{1}{\sqrt{M}} \sin\varphi_r. \tag{511}$$

Then by (510),

$$\begin{aligned}
\cos\varphi_{r+1} &= \sqrt{2^n - M} \cdot \alpha_{r+1,0} \qquad (512)\\
&= \sqrt{2^n - M} \cdot \left(\left(1 - \frac{M}{2^{n-1}} \right) \cdot \alpha_{r,0} - \frac{M}{2^{n-1}} \cdot \alpha_{r,1} \right)\\
&= \left(1 - \frac{M}{2^{n-1}} \right) \cos\varphi_r - \sqrt{2^n - M} \cdot \frac{\sqrt{M}}{2^{n-1}} \sin\varphi_r\\
&= \left(1 - 2\frac{M}{2^n} \right) \cos\varphi_r - 2\sqrt{\frac{2^n - M}{2^n}} \cdot \sqrt{\frac{M}{2^n}} \sin\varphi_r\\
&= \left(1 - 2\sin^2\frac{\beta}{2} \right) \cos\varphi_r - 2\cos\frac{\beta}{2} \cdot \sin\frac{\beta}{2} \cdot \sin\varphi_r\\
&= \cos\beta \cdot \cos\varphi_r - \sin\beta \cdot \sin\varphi_r = \cos(\varphi_r + \beta).
\end{aligned}$$

Proof (cont'd).

Put

$$\varphi_r = \frac{2r-1}{2}\beta, \quad r \geq 1. \tag{513}$$

Then

$$\varphi_1 = \frac{\beta}{2} \tag{514}$$

and

$$\varphi_{r+1} = \frac{2(r+1)-1}{2}\beta = \frac{2r-1}{2}\beta + \beta = \varphi_r + \beta. \tag{515}$$

This yields the explicit representation of the sequences of coefficients $(\alpha_{r,0})_r$ and $(\alpha_{r,1})_r$ in (511), where

$$\beta = 2 \arcsin \sqrt{\frac{M}{2^n}}. \tag{516}$$

Proof (cont'd).

The probability that the measurement yields an argument y with $f(y) = 1$ is given by

$$\begin{aligned} \mathbb{P}(f(Y) = 1) &= M\alpha_{R+1,1}^2 \text{ by (496)}, \\ &= \sin^2 \varphi_{R+1} \text{ by (511)}, \\ &= \left(\sin\left(\frac{2R+1}{2}\beta\right)\right)^2 \text{ by (513)}. \end{aligned} \tag{517}$$

The optimal number of iterations R is given by (i.e., maximal sine)

$$\frac{2R+1}{2}\beta \approx \frac{\pi}{2}. \tag{518}$$

Proof (cont'd).

Thus

$$\frac{2R+1}{2} \approx \frac{\pi}{2} \cdot \frac{1}{\frac{\beta}{2}} \tag{519}$$

and so

$$R \approx \frac{\pi}{4} \cdot \frac{1}{\frac{\beta}{2}} - \frac{1}{2} \tag{520}$$

Hence by (516),

$$R \approx \frac{\pi}{4} \cdot \frac{1}{\arcsin \sqrt{\frac{M}{2^n}}} - \frac{1}{2}. \tag{521}$$

Proof (cont'd).

Using the power series expansion

$$\arcsin(z) = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \ldots, \tag{522}$$

we obtain (use only first term)

$$R \approx \frac{\pi}{4}\cdot\frac{1}{\arcsin\sqrt{\frac{M}{2^n}}} - \frac{1}{2} \leq \frac{\pi}{4}\sqrt{\frac{2^n}{M}}. \tag{523}$$

□

Example – Maple

For $n = 30$ the input size is

$$2^{30} = 1,073,741,824.$$

If there is only one solution, i.e., $M = 1$, the number of iterations (523) is

$$R \approx 25736.$$

Note that $2^{15} = 32768$.
The probability (517) that measurement with R iterations yields an argument y with $f(y) = 1$ is nearly 1:

$$0.9999999990.$$

Shor Algorithm

- Quantum algorithm for integer factorization (Peter Shor, 1994).
- Factoring of integer n in polynomial time depending on input size $\log_2 n$.
- Could break public-key cryptographic schemes like RSA (factoring large integers is intractable on classical computers).
- Research on new cryptosystems secure from quantum computers (post-quantum cryptography).

Shor Algorithm (1994)

Require: Integer $n = p \cdot q$, distinct odd primes p and q.
Ensure: Factorization of n with success probability (i.e., finding a) $\geq 1 - \frac{1}{2^K}$ if the loop is repeated K times.

$L \leftarrow \lceil \log_2(n+1) \rceil$
while (true) **do**
  Choose $a \in \{2, \ldots, n-1\}$ uniformly at random
  if $(a, n) > 1$ **then**
    return $d \leftarrow (a, n)$ {Factor found}
  end if
  $r \leftarrow$ order of $a \mod n$ {Quantum part}
  if r is even and $a^{r/2} + 1 \not\equiv 0 \mod n$ **then**
    $p \leftarrow (a^{r/2} - 1, n)$
    $q \leftarrow (a^{r/2} + 1, n)$
    return p, q {Factors of n}
  end if
end while

Shor Algorithm

- The binary representation of $n \in \mathbb{N}$ has

$$L = \lceil \log_2(n+1) \rceil = \lfloor \log_2 n \rfloor + 1 \tag{524}$$

  binary digits.

- Shor's algorithm has complexity $O(KL^3)$ without the quantum part:
  - Euclidean algorithm has complexity $O(L^3)$.
  - Modular exponentiation has complexity $O(M(L) \cdot L)$, where $M(L)$ is the complexity of multiplication of two numbers with L bits each, say schoolbook:

$$M(L) = O(L^2).$$

  - Repetition K times.

- Quantum part has $O(L^3)$ elementary quantum gates; counted as operations on a classical computer, the algorithm has polynomial runtime.

Shor Algorithm – Non-Quantum Part

Given integers a, n with $1 \leq a < n$ and $(a, n) = 1$. Let r be the order of a mod n, i.e., $r \geq 1$ minimal with $a^r \equiv 1 \mod n$.

If r is even and $a^{r/2} + 1 \not\equiv 0 \mod n$, then

$$(a^{r/2} - 1, n) \quad \text{and} \quad (a^{r/2} + 1, n) \tag{525}$$

are two nontrivial factors of n.

Proof.

Since r is the order of $a \mod n$,

$$(a^{r/2}-1)(a^{r/2}+1) = a^r - 1 \equiv 0 \mod n.$$

Thus n divides $a^r - 1$ and so n has a common factor with $a^{r/2}-1$ or $a^{r/2}+1$.

Since a has order r modulo n, $a^{r/2}-1 \not\equiv 0 \mod n$. Moreover, by hypothesis, $a^{r/2}+1 \not\equiv 0 \mod n$. Thus n has a common factor with both, $a^{r/2}+1$ and $a^{r/2}-1$. These factors are given by the gcd's. □

Example

Let $n = 15$.

- $a = 2$ has order $r = 4$ mod 15, since $2^4 \equiv 1 \mod 15$. Moreover, $2^2 + 1 = 5 \not\equiv 0 \mod 15$. Thus two factors of 15 are
$$(2^2 + 1, 15) = 5 \quad \text{and} \quad (2^2 - 1, 15) = 3.$$
- $a = 11$ has order $r = 2$ mod 15, since $11^2 \equiv 1 \mod 15$. Moreover, $11^1 + 1 = 12 \not\equiv 0 \mod 15$. Thus two factors of 15 are
$$(11 + 1, 15) = 3 \quad \text{and} \quad (11 - 1, 15) = 5.$$

Shor Algorithm – Success Probability

Let $n = p \cdot q$ with distinct odd primes p and q.

Choose integer a with $1 \leq a \leq n-1$ and $(a, n) = 1$ uniformly at random. Let r be the order of a mod n. Then

$$\mathbb{P}(\{r \text{ even with } a^{r/2} + 1 \not\equiv 0 \bmod n\}) \geq \frac{1}{2}. \tag{526}$$

Repeating this experiment K times (while-loop) gives success probability

$$\geq 1 - \frac{1}{2^K}. \tag{527}$$

(M.A. Nielsen, I.L. Chuang: *Quantum Computation and Quantum Information*, Cambridge Univ. Press, Cambridge, 2000.)

Shor Algorithm – Quantum Part

Require: Integer $1 \leq a \leq n-1$, small $\epsilon > 0$, $L = \lfloor \log_2 n \rfloor + 1$, Hilbert space $\mathcal{H}^{\otimes(m+L)}$.

Ensure: Measurement $0 \leq y \leq 2^m - 1$ yields order r of $a \mod n$ if possible.

$m \leftarrow 2L + 1 + \lceil \log_2(1 + 1/(2\epsilon)) \rceil$

$\psi_0 \leftarrow |0\rangle_m |1\rangle_L$

$\psi_1 \leftarrow \left(\mathbf{H}^{\otimes m} \otimes \mathbf{I}^{\otimes L}\right) \psi_0$

$\psi_2 \leftarrow \mathbf{E}_a \psi_1$ {Exponential gate}

$\psi_3 \leftarrow \left(\mathbf{F}^{\otimes m} \otimes \mathbf{I}^{\otimes L}\right) \psi_2$ {Fourier gate}

$Y \leftarrow X_m^{\psi_3}$ {Partial measurement of first m qubits: $Y = y$}

Write $\frac{y}{2^m} = [a_0, \ldots, a_K]$ {Continued fraction}

for k from 0 to K **do**

  Write $\frac{s}{r} = [a_0, \ldots, a_k]$ {Convergent of $\frac{y}{2^m}$}

  if r is the order of a **then**

    return r

  end if

end for

Loop with next value of a {Non-quantum part}

Exponential Gate

Let m, n, a, L be positive integers, $1 < n < 2^L$ and $1 \leq a < n$.

The $\mathbf{E}_a$-gate or *exponential gate* is the linear operator

$$\mathbf{E}_a : \mathcal{H}^{\otimes(m+L)} \to \mathcal{H}^{\otimes(m+L)} \tag{528}$$

defined by

$$\mathbf{E}_a \left|x\right\rangle_m \left|y\right\rangle_L = \tag{529}$$
$$\begin{cases} \left|x\right\rangle_m \left|a^x y \mod n\right\rangle_L & \text{if } 0 \leq y \leq n-1, \\ \left|x\right\rangle_m \left|y\right\rangle_L & \text{if } n \leq y \leq 2^L - 1. \end{cases}$$

Exponential Gate

The $\mathbf{E}_a$-gate is a unitary operator.

Proof.

If $y < n$ and $a \mod n$ has order r, then

$$\begin{aligned} \mathbf{E}_a^r \left|x\right\rangle_m \left|y\right\rangle_L &= \left|x\right\rangle_m \left|a^{rx} y \mod n\right\rangle_L \\ &= \left|x\right\rangle_m \left|y \mod n\right\rangle_L \\ &= \left|x\right\rangle_m \left|y\right\rangle_L . \end{aligned}$$

If $y \geq n$,

$$\mathbf{E}_a \left|x\right\rangle_m \left|y\right\rangle_L = \left|x\right\rangle_m \left|y\right\rangle_L .$$

Thus $\mathbf{E}_a^r$ is the identity gate and hence

$$\mathbf{E}_a^{-1} = \mathbf{E}_a^{r-1}. \tag{530}$$

□

Shor Algorithm – Quantum Part

First statement:

$$\begin{aligned}
\psi_1 &= \left(\mathbf{H}^{\otimes m} \otimes \mathbf{I}^{\otimes L}\right)\left(|0\rangle_m |1\rangle_L\right) \qquad (531)\\
&= \mathbf{H}^{\otimes m}|0\rangle_m \otimes \mathbf{I}^{\otimes L}|1\rangle_L\\
&= (\mathbf{H}|0\rangle \otimes \ldots \otimes \mathbf{H}|0\rangle) \otimes |1\rangle_L\\
&= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \ldots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes |1\rangle_L\\
&= 2^{-m/2} \sum_{x=0}^{2^m-1} |x\rangle_m |1\rangle_L \text{ by (425).}
\end{aligned}$$

Shor Algorithm – Quantum Part

Second statement:

$$\begin{aligned} \psi_2 &= \mathbf{E}_a \psi_1 \\ &= 2^{-m/2} \sum_{x=0}^{2^m-1} \mathbf{E}_a \left(|x\rangle_m |1\rangle_L \right) \\ &= 2^{-m/2} \sum_{x=0}^{2^m-1} |x\rangle_m |a^x \mod n\rangle_L . \end{aligned} \tag{532}$$

Shor Algorithm – Quantum Part

Third statement:

$$\begin{aligned}
\psi_3 &= \left(\mathbf{F}^{\otimes m} \otimes \mathbf{I}^{\otimes L}\right) \psi_2 \qquad (533)\\
&= 2^{-m/2} \sum_{x=0}^{2^m-1} \left(\mathbf{F}^{\otimes m} \left|x\right\rangle_m\right) \otimes \left(\mathbf{I}^{\otimes L} \left|a^x \mod n\right\rangle_L\right)\\
&= 2^{-m/2} \sum_{x=0}^{2^m-1} \left(2^{-m/2} \sum_{y=0}^{2^m-1} \exp\left(2\pi i \frac{xy}{2^m}\right) \left|y\right\rangle_m\right) \left|a^x \mod n\right\rangle_L\\
&= \frac{1}{2^m} \sum_{y=0}^{2^m-1} \sum_{x=0}^{2^m-1} \exp\left(2\pi i \frac{xy}{2^m}\right) \left|y\right\rangle_m \left|a^x \mod n\right\rangle_L
\end{aligned}$$

by the Fourier gate (444).

Shor Algorithm – Quantum Part

Since r is the order of $a \mod n$, we have

$$|a^r \mod n\rangle_L = |1\rangle_L . \tag{534}$$

For each x with $0 \leq x < 2^m$ write

$$x = j \cdot r + k, \tag{535}$$

where $0 \leq k \leq r-1$ (residue) and $0 \leq j \leq \left\lfloor \frac{2^m - 1 - k}{r} \right\rfloor$ (quotient).

Then

$$|a^x \mod n\rangle_L = |a^k\rangle_L . \tag{536}$$

Put

$$b_k = \left\lfloor \frac{2^m - 1 - k}{r} \right\rfloor + 1. \tag{537}$$

Shor Algorithm – Quantum Part

Then

$$
\begin{aligned}
&\psi_3 && (538)\\
&= \frac{1}{2^m}\sum_{y=0}^{2^m-1}\sum_{x=0}^{2^m-1}\exp\left(2\pi i\frac{xy}{2^m}\right)|y\rangle_m\,|a^x \mod n\rangle_L\\
&= \frac{1}{2^m}\sum_{y=0}^{2^m-1}\sum_{k=0}^{r-1}\sum_{j=0}^{b_k-1}\exp\left(2\pi i\frac{(j\cdot r+k)y}{2^m}\right)|y\rangle_m\,|a^k \mod n\rangle_L\\
&= \sum_{y=0}^{2^m-1}|y\rangle_m\\
&\quad\otimes\left(\sum_{k=0}^{r-1}\frac{\exp\left(2\pi i\frac{ky}{2^m}\right)}{2^m}\left(\sum_{j=0}^{b_k-1}\exp\left(2\pi i\frac{jry}{2^m}\right)\right)|a^k \bmod n\rangle_L\right).
\end{aligned}
$$

Shor Algorithm – Quantum Part

Partial measurement of the first m qubits (470):

$$\begin{aligned}\mathbb{P}(Y=y) &= p_y \\ &= \sum_{k=0}^{r-1}\left|\frac{\exp\left(2\pi i\frac{ky}{2^m}\right)}{2^m}\left(\sum_{j=0}^{b_k-1}\exp\left(2\pi i\frac{jry}{2^m}\right)\right)\right|^2 \\ &= \frac{1}{2^{2m}}\sum_{k=0}^{r-1}\left|\sum_{j=0}^{b_k-1}\left(\exp\left(2\pi i\frac{ry}{2^m}\right)\right)^j\right|^2,\end{aligned} \tag{539}$$

where $y\in\{0,\ldots,2^m-1\}$.

Measurement Approximation

- Given $Y = y$. Find good approximation of $\frac{y}{2^m}$ via fraction $\frac{s}{r}$ with $0 \leq s \leq r$.
- Good measurement set

$$E = \left\{ y \mid 0 \leq y < 2^m, \left| \frac{y}{2^m} - \frac{s}{r} \right| \leq \frac{1}{2r^2} \text{ for some } 0 \leq s \leq r \right\}. \tag{540}$$

- If $y \notin E$, then

$$\left| \frac{y}{2^m} - \frac{s}{r} \right| > \frac{1}{2r^2} \text{ for all } 0 \leq s \leq r \tag{541}$$

$$\iff \left| \frac{ry}{2^m} - s \right| > \frac{1}{2r} \text{ for all } 0 \leq s \leq r$$

$$\iff \frac{ry}{2^m} = s + \Delta y \text{ for some } 0 \leq s \leq r \text{ and } \frac{1}{2r} < |\Delta y| \leq \frac{1}{2}.$$

Measurement Approximation

Probability of $y \notin E$:

$$
\begin{aligned}
p_y &= \frac{1}{2^{2m}} \sum_{k=0}^{r-1} \left| \left(\sum_{j=0}^{b_k-1} \exp\left(2\pi i \frac{ry}{2^m}\right) \right)^j \right|^2 \qquad (542) \\
&= \frac{1}{2^{2m}} \sum_{k=0}^{r-1} \left| \sum_{j=0}^{b_k-1} \left(\exp\left(2\pi i(s+\Delta y)\right)\right)^j \right|^2 \\
&= \frac{1}{2^{2m}} \sum_{k=0}^{r-1} \left| \sum_{j=0}^{b_k-1} \left(\exp\left(2\pi i \Delta y\right)\right)^j \right|^2 \\
&= \frac{1}{2^{2m}} \sum_{k=0}^{r-1} \left| \frac{1-\exp(2\pi i b_k \Delta y)}{1-\exp(2\pi i \Delta y)} \right|^2
\end{aligned}
$$

by using the geometric series (447).

Measurement Approximation

- We have

$$\begin{aligned} |1-\exp(i\varphi)|^2 &= |1-\cos(\varphi)-i\sin(\varphi)|^2 \\ &= (1-\cos(\varphi))^2+\sin^2(\varphi) \\ &= 2-2\cos(\varphi)=4\sin^2\frac{\varphi}{2}. \end{aligned} \tag{543}$$

- For $-\pi \leq \varphi \leq \pi$,

$$|1-\exp(i\varphi)|^2 = 4\sin^2\frac{\varphi}{2} \geq 4\left(\frac{\varphi}{\pi}\right)^2 = \frac{4}{\pi^2}\varphi^2. \tag{544}$$

- Since $|2\pi\Delta y| \leq \pi$,

$$p_y \leq \frac{1}{2^{2m}}\sum_{k=0}^{r-1}\frac{4}{\frac{4}{\pi^2}(2\pi\Delta y)^2} = \frac{1}{2^{2m}}\frac{r}{4(\Delta y)^2}. \tag{545}$$

Measurement Approximation

Put

$$A = \frac{2^m}{2r^2}. \tag{546}$$

Then for some $0 \leq s \leq r$,

$$\begin{aligned} p_y &\leq \frac{1}{2^{2m}} \frac{r}{4(\Delta y)^2} \\ &= \frac{1}{2^{2m}} \frac{r}{4\left(\frac{ry}{2^m} - s\right)^2}, \text{ by (541)}, \\ &= \frac{1}{4r^4A^2} \cdot \frac{r}{4\left(\frac{y}{2rA} - s\right)^2} \\ &= \frac{1}{4r\left(y - 2srA\right)^2}. \end{aligned} \tag{547}$$

Measurement Approximation

If $y \notin E$, there exists $0 \leq s \leq r$ such that by (541),

$$\begin{aligned} \frac{1}{2r} < |\Delta y| \leq \frac{1}{2} &\iff \frac{1}{2r} < |\frac{ry}{2^m} - s| \leq \frac{1}{2}, \text{ mult. by } \frac{2^m}{r}, \\ &\iff \frac{2^m}{2r^2} < |y - s\frac{2^m}{r}| \leq \frac{2^m}{2r} \\ &\iff A < |y - 2rsA| \leq rA. \end{aligned} \tag{548}$$

Thus

$$2srA + A < y \leq 2srA + rA \tag{549}$$

or

$$2srA - rA \leq y < 2srA - A, \tag{550}$$

where $y \in \{0, \ldots, 2^m - 1\}$.

Measurement Approximation

$$\mathbb{P}(Y \notin E) \leq \frac{1}{2(A-1)}. \tag{551}$$

Proof.

By (547),

$$\begin{aligned}
\mathbb{P}(Y \notin E) &\leq \frac{1}{4r}\left[\sum_{A<y\leq rA}\frac{1}{y^2} \quad (s=0)\right.\\
&\quad + \sum_{s=1}^{r-1}\sum_{A<|y-2srA|\leq rA}\frac{1}{(y-2srA)^2}\\
&\quad \left. + \sum_{2^m-rA\leq y<2^m-A}\frac{1}{(y-2^m)^2}\right] \quad (s=r,\ 2^m=2r^2A)\\
&\leq \frac{1}{4r}\left[\int_{A-1}^{rA}\frac{1}{y^2}dy + \sum_{s=1}^{r-1}2\int_{A-1}^{rA}\frac{1}{y^2}dy + \int_{A-1}^{rA}\frac{1}{y^2}dy\right]\\
&\leq \frac{1}{4r}2r\int_{A-1}^{rA}\frac{1}{y^2}dy = \frac{1}{2}\left[-\frac{1}{y}\right]_{A-1}^{rA}\\
&= \frac{1}{2}\left(\frac{1}{A-1}-\frac{1}{rA}\right) \leq \frac{1}{2(A-1)}.
\end{aligned}$$

Measurement Approximation

Given small, ugly number ϵ with $0 < \epsilon < 1$ and

$$\mathbb{P}(Y \notin E) \leq \frac{1}{2(A-1)} \overset{!}{\leq} \epsilon. \tag{552}$$

Then

$$\begin{aligned} \frac{1}{2(A-1)} \leq \epsilon \quad &\Longleftrightarrow \quad A \geq 1 + \frac{1}{2\epsilon} \\ &\Longleftrightarrow \quad \frac{2^m}{2r^2} \geq 1 + \frac{1}{2\epsilon} \\ &\Longleftrightarrow \quad m - 1 - 2\log_2 r \geq \log_2\left(1 + \frac{1}{2\epsilon}\right) \\ &\Longleftrightarrow \quad m \geq 2\log_2 r + 1 + \log_2\left(1 + \frac{1}{2\epsilon}\right). \end{aligned} \tag{553}$$

Measurement Approximation

Choose the integer m such that

$$m \geq 2L + 1 + \left\lceil \log_2 \left(1 + \frac{1}{2\epsilon} \right) \right\rceil, \tag{554}$$

where $L = \lceil \log_2(n+1) \rceil = \lfloor \log_2 n \rfloor + 1$ and $1 \leq r \leq n$.

Then for each measurement result $y \in E$ there is with probability $\geq 1 - \epsilon$ an integer $0 \leq s \leq r$ such that

$$\left| \frac{y}{2^m} - \frac{s}{r} \right| \leq \frac{1}{2r^2}. \tag{555}$$

For instance, if $\epsilon = 10^{-7}$, then

$$m \geq 2L + 1 + \left\lceil \log_2 \left(1 + \frac{1}{2 \cdot 10^{-7}} \right) \right\rceil = 2L + 24.$$

In comparison with the eventually large number L the effort to obtain good measurement plays no significant role.

Measurement Approximation

Take

$$m \geq 2L + 1 + \left\lceil \log_2 \left(1 + \frac{1}{2 \cdot \epsilon} \right) \right\rceil ,$$

where

ϵ	$\left\lceil \log_2 \left(1 + \frac{1}{2 \cdot \epsilon} \right) \right\rceil$
10^{-1}	3
10^{-2}	6
10^{-3}	9
10^{-4}	13
10^{-5}	16
10^{-6}	19
10^{-7}	23
10^{-8}	26
10^{-9}	29
10^{-10}	33

Measurement Approximation

Let $m, r \geq 1$ be integers with $r^2 \leq 2^m$, and let $0 \leq y < 2^m$ and $0 \leq s \leq r$. If

$$\left| \frac{y}{2^m} - \frac{s}{r} \right| \leq \frac{1}{2r^2}, \tag{556}$$

then $\frac{s}{r}$ is a convergent of the continuous fraction $\frac{y}{2^m}$.

(M.A. Nielsen, I.L. Chuang: *Quantum Computation and Quantum Information*, Cambridge Univ. Press, Cambridge, 2000.)

Measurement Approximation

Find the order r of a mod n.

- Compute the (finite) continuous fraction

$$\frac{y}{2^m} = [a_0, \ldots, a_K]. \tag{557}$$

- For each k, $0 \leq k \leq K$, take the convergent

$$\frac{s_k}{r_k} = [a_0, \ldots, a_k]. \tag{558}$$

  and check if r_k is the order of a mod n.

- Computation of r fails if $y \notin E$ or $\frac{y}{2^m}$ is not a good appoximation. Then a new value for a is chosen.

Example

Let $n = 33$.

- We have
$$L = \lceil \log_2(n+1) \rceil = 6.$$
- Let $\epsilon = 10^{-1}$. Then
$$m = 2L + 1 + \left\lceil \log_2(1 + \frac{1}{2\epsilon}) \right\rceil = 16.$$
- With probability $\geq 1 - \epsilon = 90\%$ we obtain $Y = y$ with
$$\left| \frac{y}{2^m} - \frac{s}{r} \right| \leq \frac{1}{2r^2}.$$
- Here $r = 10$ (yet unknown), $2^m = 2^{16} = 65536$ and
$$\left| \frac{y}{65536} - \frac{s}{10} \right| \leq \frac{1}{200} = 0.005.$$

Example (cont'd)

Suppose $y = 19412$ and choose $a = 5$, where $(33, 5) = 1$.

- The Euclidean algorithm yields

$$\begin{aligned}
19412 &= 0 \cdot 65536 + 19412,\\
65536 &= 3 \cdot 19412 + 7300,\\
19412 &= 2 \cdot 7300 + 4812,\\
7300 &= 1 \cdot 4812 + 2488,\\
4812 &= 1 \cdot 2488 + 2324,\\
2488 &= 1 \cdot 2324 + 164,\\
2324 &= 14 \cdot 164 + 28,\\
164 &= 5 \cdot 28 + 24,\\
28 &= 1 \cdot 24 + 4,\\
24 &= 6 \cdot 4 + 0.
\end{aligned}$$

- This gives the continuous fraction

$$\frac{19412}{65536} = [0, 3, 2, 1, 1, 1, 14, 5, 1, 6].$$

Example (cont'd)

- Consider successively the convergents:

$$\begin{aligned} [0,3] &= \frac{1}{3}, \\ [0,3,2] &= \frac{1}{3+\frac{1}{2}} = \frac{2}{7}, \\ [0,3,2,1] &= \frac{1}{3+\frac{1}{2+\frac{1}{1}}} = \frac{3}{10} \left(= \frac{s}{r}\right). \end{aligned}$$

  The last fraction provides the order $r = 10$ of $a = 5$ mod 33; i.e.,

$$5^{10} \equiv 1 \mod 33.$$

- Since the order $r = 10$ is even and $5^5 + 1 = 3126 = 24 \not\equiv 0 \mod 33$, nontrivial factors of 33 are

$$(5^5 + 1, 33) = 3 \quad \text{and} \quad (5^5 - 1, 33) = 11.$$

Part X

Basic Number Theory (Appendix)

Basic Number Theory

> It would not be an exaggeration to state that abstract cryptography is identical with abstract mathematics.

A. Adrian Albert, 1941

Basic Number Theory

- Unique factorization
- Euclidean algorithm
- Continued fractions
- Lagrange's theorem
- Chinese remainder theorem
- Euler's totient function
- Fast modular exponentiation
- Prime numbers
- Quadratic reciprocity
- Complex numbers

Unique Factorization Theorem

Any non-zero integer can be factored into a product of primes in a unique way, up to the order of the primes.

Example

```
> ifactor(123456789000);
```

$$2^2 \cdot 3^2 \cdot 5^3 \cdot 3803 \cdot 3607$$

Euclidean Algorithm

Let a and b be integers with $a > b > 0$. Successive division with remainder,

$$\begin{aligned}
a &= q_1 b + r_1, & 0 < r_1 < b,\\
b &= q_2 r_1 + r_2, & 0 < r_2 < r_1,\\
r_1 &= q_3 r_2 + r_3, & 0 < r_3 < r_2,\\
&\vdots\\
r_{n-2} &= q_n r_{n-1} + r_n, & 0 < r_n < r_{n-1},\\
r_{n-1} &= q_{n+1} r_n + r_{n+1}, & r_{n+1} = 0,
\end{aligned}$$

computes the gcd $(a, b) = r_n$.

Euclidean Algorithm – Complexity

Let (F_n) denote the sequence of Fibonacci numbers with $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Thus $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, and so on.
The Euclidean algorithm takes n divisions for consecutive Fibonacci numbers F_{n+2} and F_{n+1}:

$$\begin{aligned} F_{n+2} &= 1 \cdot F_{n+1} + F_n, \\ F_{n+1} &= 1 \cdot F_n + F_{n-1}, \\ &\vdots \\ F_4 &= 1 \cdot F_3 + F_2, \\ F_3 &= 2 \cdot F_2 + 0. \end{aligned}$$

Euclidean Algorithm – Complexity

- Suppose $F_{n+2} > b \geq F_{n+1}$ for some $n \geq 0$.
- We have $F_{n+1} > \varphi^{n-1}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Thus $\log_{10} b > (n-1) \log_{10} \varphi$.
- Since $\log_{10} \varphi = 0.48121 > 2/5$, $(n-1) < 5/2 \log_{10} b$.
- Thus $n \leq 5/2 \log_{10} b + 1$ and hence the number of divisions is $O(h)$, where h is the number of decimal digits in b.

Extended Euclidean Algorithm

Let a and b be integers with $a > b > 0$. There are integers s and t such that

$$(a, b) = s \cdot a + t \cdot b.$$

Example

```
> igcdex(25, 32, 's', 't');
1
> s, t;
9, -7
```

Extended Euclidean Algorithm

$$\begin{array}{rcll} r_1 & = & a - bq_0 & = & as_1 + bt_1, \\ r_2 & = & b - r_1 q_1 & = & as_2 + bt_2, \\ r_3 & = & r_1 - r_2 q_2 & = & as_3 + bt_3, \\ & \vdots & & & \\ r_n & = & r_{n-2} - r_{n-1} q_{n-1} & = & as_n + bt_n. \end{array}$$

The number of steps is $O(h)$, where h is the number of digits in b.

Continued Fractions

The Euclidean algorithm

$$\begin{aligned} 67 &= 2 \cdot 24 + 19, \\ 24 &= 1 \cdot 19 + 5, \\ 19 &= 3 \cdot 5 + 4, \\ 5 &= 1 \cdot 4 + 1, \end{aligned}$$

leads to a system of fractional equations

$$\begin{aligned} \frac{67}{24} &= 2 + \frac{19}{24}, \\ \frac{24}{19} &= 1 + \frac{5}{19}, \\ \frac{19}{5} &= 3 + \frac{4}{5}, \\ \frac{5}{4} &= 1 + \frac{1}{4}. \end{aligned}$$

Continued Fractions

This gives the *continued fraction*

$$\begin{aligned}
\frac{67}{24} &= 2+\frac{19}{24} = 2+\frac{1}{\frac{24}{19}} \\
&= 2+\frac{1}{1+\frac{5}{19}} = 2+\frac{1}{1+\frac{1}{\frac{19}{5}}} \\
&= 2+\frac{1}{1+\frac{1}{3+\frac{4}{5}}} = 2+\frac{1}{1+\frac{1}{3+\frac{1}{\frac{5}{4}}}} \\
&= 2+\frac{1}{1+\frac{1}{3+\frac{1}{1+\frac{1}{4}}}} \\
&= 2+\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{4}.
\end{aligned}$$

Partial quotients: $2,1,3,1,4$, *complete quotients:* $\frac{67}{24}, \frac{24}{19}, \frac{19}{5}, \frac{5}{4}$.

Continued Fractions

- Each rational number $\frac{b}{c} > 1$ can be represented as continued fraction
$$\frac{b}{c} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_n},$$
where $a_0, \ldots, a_n \geq 1$ are integers and $a_n > 1$.
- Each rational number has exactly one representation as continued fraction. Indeed, write
$$a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots = a_0' + \frac{1}{a_1'+}\frac{1}{a_2'+}\cdots.$$
Since a_0, a_0' are the integral parts, $a_0 = a_0'$. Cancelling a_0 and inverting gives
$$a_1 + \frac{1}{a_2+}\cdots = a_1' + \frac{1}{a_2'+}\cdots.$$
Continuing this way implies uniqueness.

Continued Fractions

Euler's rule:

> *Consider the sequence* $(a_0, a_1, \ldots, a_n)$*: First take the product of all the terms. Then take every product that can be obtained by omitting any pair of consecutive terms. Then take every product that can be obtained by omitting any two separate pairs of terms, and so on. The sum of all such products gives the value of* $[a_0, a_1, \ldots, a_n]$.

Example:

- $[a_0] = a_0$,
- $[a_0, a_1] = a_0 a_1 + 1$,
- $[a_0, a_1, a_2] = a_0 a_1 a_2 + a_0 + a_2$,
- $[a_0, a_1, a_2, a_3] = a_0 a_1 a_2 a_3 + a_2 a_3 + a_0 a_3 + a_0 a_1 + 1$.
- $[a_0, a_1, a_2, a_3, a_4] =$
  $a_0 a_1 a_2 a_3 a_4 + a_2 a_3 a_4 + a_0 a_3 a_4 + a_0 a_1 a_4 + a_0 a_1 a_2 + a_4 + a_2 + a_0$.

Continued Fractions – Example

Case $n = 1$:

$$a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{[a_0, a_1]}{[a_1]}.$$

Case $n = 2$:

$$\begin{aligned} a_0 + \frac{1}{a_1 + \frac{1}{a_2}} &= a_0 + \frac{a_2}{a_1 a_2 + 1} = \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1} \\ &= \frac{[a_0, a_1, a_2]}{[a_1, a_2]}. \end{aligned}$$

Case $n = 3$:

$$\begin{aligned} a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3}}} &= a_0 + \frac{a_2 a_3 + 1}{a_1 a_2 a_3 + a_1 + a_3} \\ &= \frac{a_0 a_1 a_2 a_3 + a_0 a_1 + a_0 a_3 + a_2 a_3 + 1}{a_1 a_2 a_3 + a_1 + a_3} \\ &= \frac{[a_0, a_1, a_2, a_3]}{[a_1, a_2, a_3]}. \end{aligned}$$

Continued Fractions

The general continued fraction has the value

$$a_0 + \frac{1}{a_1+} \ldots \frac{1}{a_n} = \frac{[a_0, a_1, \ldots, a_n]}{[a_1, a_2, \ldots, a_n]}.$$

Continued Fractions

Recurrence relation:

$$[a_0, a_1, \ldots, a_n] = a_0[a_1, a_2, \ldots, a_n] + [a_2, a_3, \ldots, a_n].$$

Case $n = 2$:

$$\begin{aligned} [a_0, a_1, a_2] &= a_0[a_1, a_2] + [a_2] = a_0(a_1a_2 + 1) + a_2 \\ &= a_0a_1a_2 + a_0 + a_2. \end{aligned}$$

Case $n = 3$:

$$\begin{aligned} [a_0, a_1, a_2, a_3] &= a_0[a_1, a_2, a_3] + [a_2, a_3] \\ &= a_0(a_1a_2a_3 + a_1 + a_3) + (a_2a_3 + 1) \\ &= a_0a_1a_2a_3 + a_0a_1 + a_0a_3 + a_2a_3 + 1. \end{aligned}$$

Continued Fractions – Example

Example:

$$\begin{aligned} [4,2] &= 4 \cdot 2 + 1 = 9, \\ [1,4,2] &= 1 \cdot [4,2] + [2] = 9 + 2 = 11, \\ [2,1,4,2] &= 2 \cdot [1,4,2] + [4,2] = 2 \cdot 11 + 9 = 31. \end{aligned}$$

Thus

$$2 + \frac{1}{1+}\frac{1}{4+}\frac{1}{2} = \frac{[2,1,4,2]}{[1,4,2]} = \frac{31}{11}.$$

Continued Fractions

By Euler's rule, the value $[a_0, a_1, \ldots, a_n]$ is unchanged if the terms are written in reverse order,

$$[a_0, a_1, \ldots, a_n] = [a_n, a_{n-1}, \ldots, a_0].$$

Equivalent recurrence relation:

$$[a_n, a_{n-1}, \ldots, a_0] = a_n[a_{n-1}, \ldots, a_0] + [a_{n-2}, \ldots, a_0]$$

or

$$[a_0, a_1, \ldots, a_n] = a_n[a_0, a_1, \ldots, a_{n-1}] + [a_0, a_1, \ldots, a_{n-2}]$$

Example: $[2, 4, 1, 2] = [2, 1, 4, 2] = 31.$

Continued Fractions

Given a continued fraction

$$a_0 + \frac{1}{a_1+} \cdots \frac{1}{a_n}.$$

Stopping at an earlier term than a_n gives the *convergents* of the continued fraction,

$$a_0 + \frac{1}{a_1+} \cdots \frac{1}{a_m} = \frac{[a_0, a_1, \ldots, a_m]}{[a_1, a_2, \ldots, a_m]}, \quad 0 \leq m \leq n.$$

That is,

$$a_0, a_0 + \frac{1}{a_1}, a_0 + \frac{1}{a_1+} \frac{1}{a_2}, \ldots.$$

Continued Fractions – Example

Successive convergents of $\frac{67}{24}$:

$$
\begin{aligned}
&2,\\
&2+\tfrac{1}{1}=\tfrac{[2,1]}{[1]}=3,\\
&2+\tfrac{1}{1+}\tfrac{1}{3}=\tfrac{[2,1,3]}{[1,3]}=\tfrac{11}{4}=2.75,\\
&2+\tfrac{1}{1+}\tfrac{1}{3+}\tfrac{1}{1}=\tfrac{[2,1,3,1]}{[1,3,1]}=\tfrac{14}{5}=2.80,\\
&2+\tfrac{1}{1+}\tfrac{1}{3+}\tfrac{1}{1+}\tfrac{1}{4}=\tfrac{[2,1,3,1,4]}{[1,3,1,4]}=\tfrac{67}{24}=2.79.
\end{aligned}
$$

These numbers are alternately less than or greater than $\frac{67}{24}$.

Continued Fractions

- Recurrence relation:

$$[a_0, a_1, \ldots, a_n] = a_n[a_0, a_1, \ldots, a_{n-1}] + [a_0, a_1, \ldots, a_{n-2}].$$

- Setting $b_0 = a_0$, $b_1 = a_0a_1 + 1$, $c_0 = 1$, $c_1 = a_1$, and

$$\begin{aligned} b_n &= [a_0, a_1, \ldots, a_n], \\ c_n &= [a_1, a_2, \ldots, a_n], \quad n \geq 2, \end{aligned}$$

  we obtain

$$\begin{aligned} b_n &= a_n b_{n-1} + b_{n-2} \\ c_n &= a_n c_{n-1} + c_{n-2}, \quad n \geq 2. \end{aligned}$$

- Both sequences (b_n) and (c_n) are strictly increasing.

Continued Fractions

For $n \geq 1$,

$$b_n c_{n-1} - c_n b_{n-1} = (-1)^{n-1}.$$

Thus b_n and c_n are relatively prime.

Proof.

For $n = 1$,

$$b_1 c_0 - c_1 b_0 = (a_0 a_1 + 1) - a_1 a_0 = 1.$$

For $n \geq 2$,

$$\begin{aligned} & b_n c_{n-1} - c_n b_{n-1} \\ = \; & (a_n b_{n-1} + b_{n-2}) c_{n-1} - (a_n c_{n-1} + c_{n-2}) b_{n-1} \\ = \; & -(b_{n-1} c_{n-2} - c_{n-1} b_{n-2}). \end{aligned}$$

Thus consecutive expressions alternate, as required. □

Continued Fractions

For $n \geq 1$, write

$$\frac{b_n}{c_n} - \frac{b_{n-1}}{c_{n-1}} = \frac{(-1)^{n-1}}{c_{n-1}c_n}.$$

- The difference on the left is positive if n is odd, and negative if n is even.
- Since (c_n) is strictly increasing, the difference decreases as n increases (by the right-hand side).
- Thus $b_1/c_1 > b_0/c_0$ and $b_2/c_2 < b_1/c_1$ and so on.
- Since $b/c = b_n/c_n$ for some n, all even convergents are less than b/c and all odd convergents are greater than b/c.

Lagrange's Theorem

For each finite group G, the order (number of elements) of each subgroup U of G divides the order of G; i.e.,

$$|G| = [G : U] \cdot |U|,$$

where $[G : U]$ is the index of U in G.

Proof.

The left cosets $gU = \{gu \mid u \in U\}$, $g \in G$, of U in G form a partition of G by using the equivalence relation

$$g \simeq h \quad :\Longleftrightarrow \quad gh^{-1} \in U, \quad g, h \in G.$$

Moreover, for each $g \in G$, the mapping $\phi : U \to gU : u \mapsto gu$ is a bijection between U and the left coset gU. Thus the left cosets of U in G have the same size as U. □

Lagrange's Theorem

The *order* of an element g of a finite group G is the smallest integer $n \geq 1$ for which g^n is the identity element.

By Lagrange's theorem, the order of an element g in a finite group G divides the order of G.

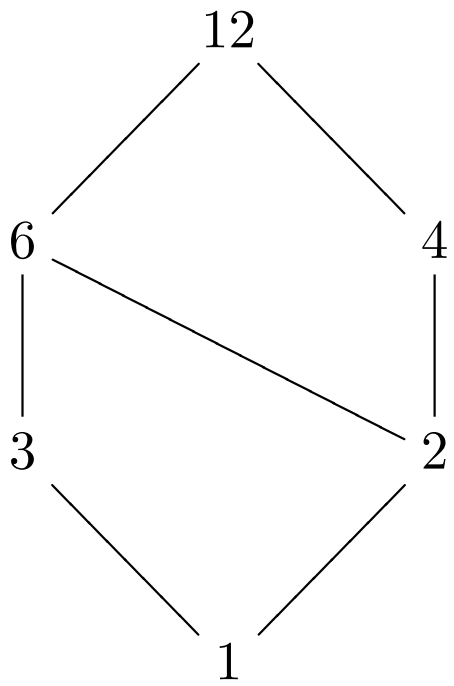
Cyclic Groups

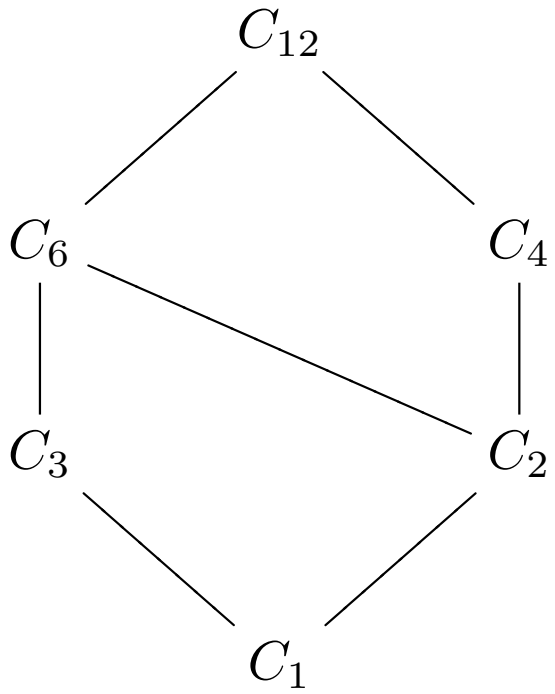
A *cyclic group* is a group which is generated by a single element.

- Each finite cyclic group of order n is isomorphic to the additive group of $\mathbb{Z}_n$.
- Each infinite cyclic group is isomorphic to the additive group of $\mathbb{Z}$.
- Each subgroup and quotient group of a cyclic group is cyclic.
- The lattice of subgroups of $\mathbb{Z}$ is isomorphic to the lattice of natural numbers ordered by divisibility.
- The lattice of subgroups of $\mathbb{Z}_n$ is isomorphic to the lattice of natural numbers dividing n ordered by divisibility.

Cyclic Groups

Lattice of subgroups of C_{12}:





Chinese Remainder Theorem

Let $n_1, \ldots, n_k$ be pairwise relatively prime numbers ≥ 2 and let $a_1, \ldots, a_k$ be integers. There is a solution x of the system of simultaneous congruences

$$\begin{aligned} x &\equiv a_1 \mod n_1, \\ &\vdots \\ x &\equiv a_k \mod n_k. \end{aligned}$$

If x and x' are two solutions, then $x \equiv x' \mod N$, where $N = n_1 \cdots n_k$.

Example

```
> chrem([1,2,1], [3,5,7]);
22
```

Euler's Totient Function

For each integer $n \geq 2$, the set of units (or invertible elements) modulo n is given by

$$\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n \mid (a, n) = 1\}.$$

Define

$$\phi(n) = |\mathbb{Z}_n^*|.$$

Example

We have $\mathbb{Z}_{15}^* = \{1, 2, 4, 7, 8, 11, 13, 14\}$ and so $\phi(15) = 8$.

Euler's Totient Function

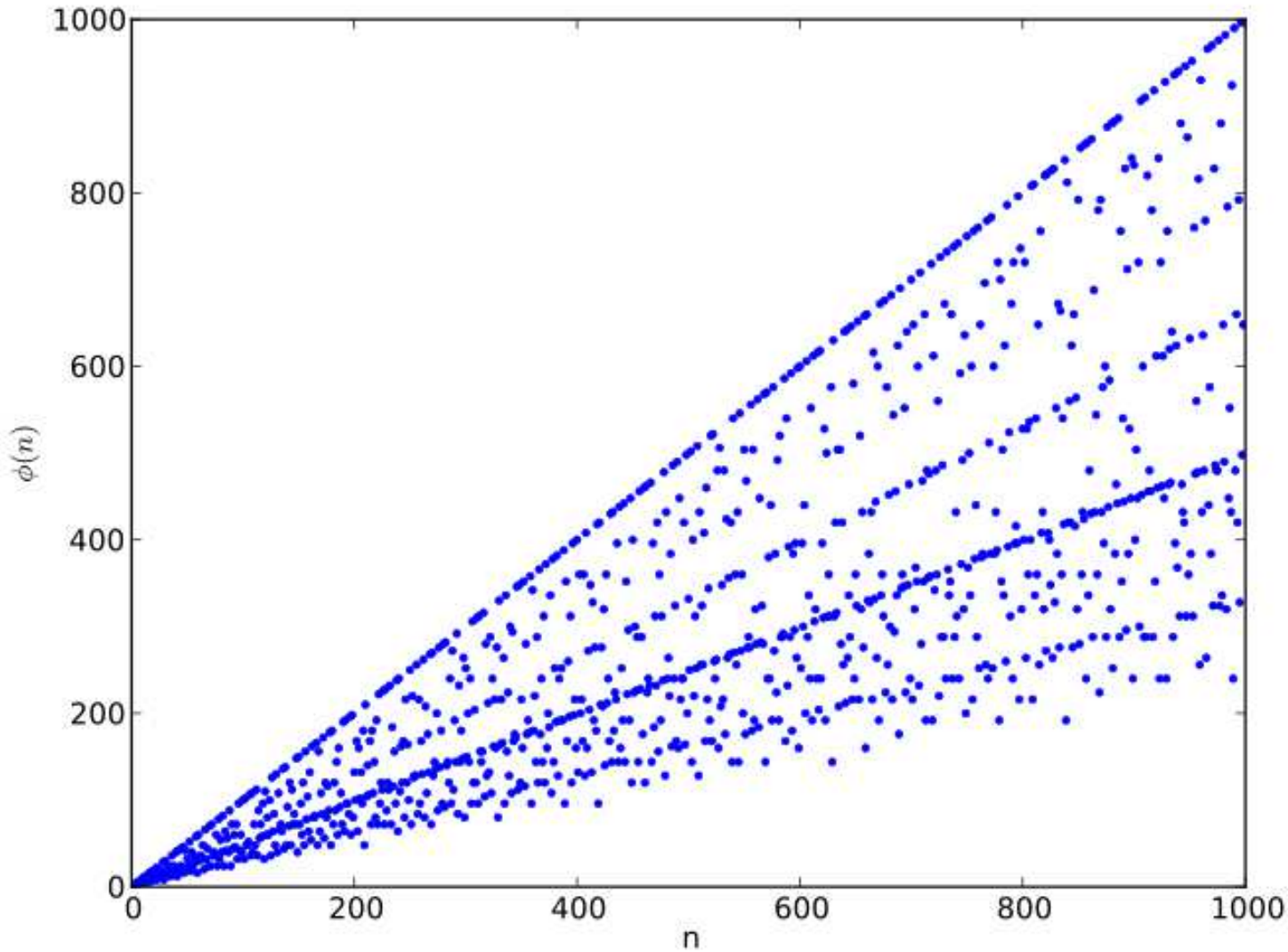
For primes p and q, and numbers $n \geq 2$ and r,

$$\begin{aligned} \phi(p) &= p-1, \\ \phi(p^r) &= p^r - p^{r-1}, \\ \phi(pq) &= (p-1)(q-1), \\ \phi(n) &= \prod_{p|n} \left(1 - \frac{1}{p}\right). \end{aligned}$$

Example

```
> with(numtheory):
> phi(123456789);
82260072
```

Euler's Totient Function



Euler's Theorem

Let $n \geq 2$ and a be integers. If $(a, n) = 1$, then

$$a^{\phi(n)} \equiv 1 \mod n.$$

This is a special case of Lagrange's theorem.

Example

- We have $\phi(15) = 8$.
- Thus for each integer a with $(a, n) = 1$, we have $a^{\phi(15)} = a^8 \equiv 1 \mod 15$.
- The remaining non-zero elements are zero divisors.

Fermat's Litte Theorem

Let p be a prime and a be an integer. Then

$$a^p \equiv a \mod p.$$

Proof.

We have $\phi(p) = p - 1$. Since p is a prime, for each integer a with $1 \leq a \leq p - 1$, we have $(a, p) = 1$. Thus by Euler's theorem,

$$a^{p-1} \equiv 1 \mod p.$$

Hence, $a^p \equiv a \mod p$. This congruence also holds for $a = 0$. □

Fast Modular Exponentiation

Require: Modulus $n \geq 2$ and positive integers x and e
Ensure: Exponentiation $x^e \mod n$

$a \leftarrow e$, $b \leftarrow 1$, $c \leftarrow x$
while $a > 0$ **do**
  if a is even **then**
    $a \leftarrow a/2$, $b \leftarrow b$, $c \leftarrow c^2 \mod n$
  else
    $a \leftarrow a - 1$, $b \leftarrow bc \mod n$, $c \leftarrow c$
  end if
end while
return b

Time complexity $O(\log_2 e)$.

Fast Modular Exponentiation

Trace for the computation of $x^{19} \mod n$:

a	b	c
19	1	x
18	x	x
9	x	x^2
8	$x \cdot x^2$	x^2
4	$x \cdot x^2$	x^4
2	$x \cdot x^2$	x^8
1	$x \cdot x^2$	x^{16}
0	$x \cdot x^2 \cdot x^{16}$	x^{19}

All operations are conducted modulo n. The computation makes use of the Horner scheme.

Prime Number Theorem

Let $\pi(x)$ be the number of primes in the range of 2 to x. Then $\pi(x)$ is approximately $x/\ln x$, i.e.,

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1.$$

Example

The number of 100-digit primes is about

$$\pi(10^{100}) - \pi(10^{99}) \approx \frac{10^{100}}{\ln 10^{100}} - \frac{10^{99}}{\ln 10^{99}} \approx 9 \cdot 10^{97}.$$

The density of the 100-digit primes among the odd 100-digit numbers is about

$$\frac{9 \cdot 10^{97}}{\frac{1}{2}(10^{100} - 10^{99})} = 0.02.$$

Sieve of Eratosthenes

Generate a list of all primes ≤ 100:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Quadratic Reciprocity

Let $p > 2$ be a prime.

- A nonzero element $x \in \mathbb{F}_p$ is a *square* or *quadratic residue* modulo p if $x = y^2$ for some $y \in \mathbb{F}_p$. The remaining nonzero elements are the *nonsquares* or *nonresidues* modulo p.
- For $p = 11$, the squares modulo 11 are $1^2 = 1$, $2^2 = 4$, $3^3 = 9$, $4^2 = 5$, and $5^2 = 3$. The nonsquares are 2, 6, 7, 8, and 10.
- If g is a generator of $\mathbb{F}_p^*$, the squares of g are exactly of the form g^j with j even. Thus $\mathbb{F}_p^*$ has $(p-1)/2$ squares g^{2j}, $1 \leq j \leq (p-1)/2$, and $(p-1)/2$ nonsquares g^{2j+1}, $1 \leq j \leq (p-1)/2$.

Legendre Symbol

Let $p > 2$ be a prime and a be an integer.
The *Legendre symbol* is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } a \text{ is a quadratic residue mod } p, \\ -1 & \text{otherwise.} \end{cases}$$

In Maple:

```
> with(numtheory):
> legendre(4, 11);
1
> legendre(6, 11);
-1
```

Property of Legendre Symbol

Let $p > 2$ be a prime and a be an integer.

$$\left(\frac{a}{p}\right) = a^{(p-1)/2} \mod p.$$

Proof.

- If $p \mid a$, both sides are 0 modulo p.
- Let $p \nmid a$.

  By Little Fermat in $\mathbb{F}_p$, the square of $a^{(p-1)/2}$ is 1. So $a^{(p-1)/2} = \pm 1$.

  Let g be a generator of $\mathbb{F}_p$ with $a = g^j$. Then a is a square iff j is even.

  And $a^{(p-1)/2} = g^{j(p-1)/2}$ is 1 iff $j(p-1)/2$ is a multiple of $p-1$ (since g generates $\mathbb{F}_p^*$), i.e., j is even. □

Properties of Legendre Symbol

- $\left(\frac{a}{p}\right)$ depends only on the residue of a modulo p.
- $\left(\frac{1}{p}\right) = 1$ and $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$.
- $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.
- If b is prime to p, then $\left(\frac{ab^2}{p}\right) = \left(\frac{a}{p}\right)$.

Proof.

Part 1 is obvious.
Part 2 follows from the above property.
Part 3 follows from the above property and $(ab)^{(p-1)/2} = a^{(p-1)/2}b^{(p-1)/2}$.
Part 4 holds since $\left(\frac{ab^2}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b^2}{p}\right) = \left(\frac{a}{p}\right)$. □

Properties of Legendre Symbol

For each odd prime p,

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \ mod\ 8, \\ -1 & \text{if } p \equiv \pm 3 \ mod\ 8. \end{cases}$$

Proof (Sketch).

Let $f(n) = (-1)^{(n^2-1)/8}$ for n odd, and $f(n) = 0$ for n even.

- $p^2 \equiv 1 \mod 8$ for any odd prime p. So $\mathbb{F}_{p^2}$ has a primitive 8-th root of unity ξ.
- Define the Gauss sum $G = \sum_{j=0}^{7} f(j)\xi^j$ in $\mathbb{F}_{p^2}$.
- We have $G^p = \left(\frac{2}{p}\right) G$ and $G^p = f(p)G$.
- Division by $G \neq 0$ gives the result. □

Law of Quadratic Reciprocity

Let p and q be odd primes.

$$\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} \left(\frac{p}{q}\right) = \begin{cases} -\left(\frac{p}{q}\right) & \text{if } p \equiv q \equiv 3 \mod 4, \\ \left(\frac{p}{q}\right) & \text{otherwise.} \end{cases}$$

Proof (Sketch).

- Take a power t of p with $p^t \equiv 1 \mod q$ (such as $t = q - 1$).
- Then the field $\mathbb{F}_{p^t}$ has a primitive q-th root of unity ξ.
- Define the Gauss sum $G = \sum_{j=0}^{q-1} \left(\frac{j}{q}\right) \xi^j$ in $\mathbb{F}_{q^t}$.
- Basic fact: $G^2 = (-1)^{(q-1)/2} q$.
- We have $G^p = (-1)^{(p-1)(q-1)/4} \left(\frac{q}{p}\right) G$ and $G^p = \left(\frac{p}{q}\right) G$.
- Division by $G \neq 0$ gives the result. □

Example

Determine whether 7411 is a square modulo 9283.

- 7411 and 9283 are both primes.
- Both numbers are $\equiv 3 \mod 4$.
- By quadratic reciprocity,

$$\left(\frac{7411}{9283}\right) = -\left(\frac{9283}{7411}\right).$$

- Taking the residue modulo 7411,

$$-\left(\frac{9283}{7411}\right) = -\left(\frac{1872}{7411}\right).$$

- Since $1872 = 2^4 \cdot 3^2 \cdot 13$,

$$-\left(\frac{1872}{7411}\right) = -\left(\frac{2^4}{7411}\right)\left(\frac{3^2}{7411}\right)\left(\frac{13}{7411}\right) = -\left(\frac{13}{7411}\right).$$

Example (cont'd)

- Since $13 \equiv 1 \mod 4$, by quadratic reciprocity

$$-\left(\frac{13}{7411}\right) = -\left(\frac{7411}{13}\right).$$

- Since $7411 \equiv 1 \mod 13$,

$$-\left(\frac{7411}{13}\right) = -\left(\frac{1}{13}\right) = -1.$$

Thus 7411 is a quadratic nonresidue modulo 9283.

Jacobi Symbol

Let a be an integer and $n > 0$ be an odd number with prime factorization $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$.
The *Jacobi symbol* is

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{\alpha_1} \cdots \left(\frac{a}{p_r}\right)^{\alpha_r}.$$

Warning

If $\left(\frac{a}{n}\right) = 1$ for n composite, it is not necessarily true that a is a square modulo n.
For instance, $\left(\frac{2}{15}\right) = \left(\frac{2}{3}\right)\left(\frac{2}{5}\right) = (-1)(-1) = 1$, but there is no integer y with $y^2 \equiv 2 \mod 15$.

Jacobi Symbol

Straightforward generalization of previous results to Jacobi symbol.

- For any positive odd number n,

$$\left(\frac{2}{n}\right) = (-1)^{(n^2-1)/8}.$$

- For any two positive odd integers m and n,

$$\left(\frac{m}{n}\right) = (-1)^{(m-1)(n-1)/4} \left(\frac{n}{m}\right).$$

Example

Determine whether 7411 is a square modulo 9283.

- Using the Jacobi symbol, factoring is not necessary (except pulling out powers of 2)!
- Since $1872 = 16 \cdot 117$,

$$-\left(\frac{1872}{7411}\right) = -\left(\frac{2^4}{7411}\right)\left(\frac{117}{7411}\right) = -\left(\frac{117}{7411}\right).$$

- Since $117 \equiv 1 \mod 4$, by quadratic reciprocity

$$-\left(\frac{117}{7411}\right) = -\left(\frac{7411}{117}\right).$$

- Since $7411 \equiv 40 \mod 117$,

$$-\left(\frac{7411}{117}\right) = -\left(\frac{40}{117}\right).$$

Example (cont'd)

- Since $40 = 2^3 \cdot 5$,

$$-\left(\frac{40}{117}\right) = -\left(\frac{2^3}{117}\right)\left(\frac{5}{117}\right) = -\left(\frac{2}{117}\right)\left(\frac{5}{117}\right).$$

- Since $117 \equiv -3 \ mod\ 8$,

$$-\left(\frac{2}{117}\right)\left(\frac{5}{117}\right) = \left(\frac{5}{117}\right).$$

- Since $5 \equiv 1 \mod 4$,

$$\left(\frac{5}{117}\right) = \left(\frac{117}{5}\right).$$

Example (cont'd)

- Since $117 \equiv 2 \mod 5$,

$$\left(\frac{117}{5}\right) = \left(\frac{2}{5}\right).$$

- But 2 is not a square in $\mathbb{F}_5$ and so

$$\left(\frac{2}{5}\right) = -1.$$

Complex Numbers

The *field of complex numbers* is given by the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

The terms $a + bi$ are considered as formal sums, where i with $i^2 = 1$ is the *imaginary unit*.

- Addition

$$(a + bi) + (a' + b'i) = (a + a') + (b + b')i$$

- Multiplikation

$$\begin{aligned}(a + bi) \cdot (a' + b'i) &= aa' + (ab' + a'b)i + bb'i^2 \\ &= (aa' - bb') + (ab' + a'b)i.\end{aligned}$$

Complex Numbers

- Zero element $0 = 0 + 0i$, unit element $1 = 1 + 0i$.
- Additive inverse

$$-(a + bi) = (-a) + (-b)i.$$

- Multiplikative inverse

$$\begin{aligned} 1 &= (a + bi) \cdot (a' + b'i) = aa' + (ab' + a'b)i + bb'i^2 \\ &= (aa' - bb') + (ab' + a'b)i. \end{aligned}$$

Comparing coefficients

$$aa' - bb' = 1 \quad \text{and} \quad ab' + a'b = 0.$$

gives

$$(a + bi)^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

Complex Numbers

The field $\mathbb{C}$ can also be defined as set of pairs

$$\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$$

with addition

$$(a, b) + (a', b') = (a + a', b + b')$$

and multiplication

$$(a, b) \cdot (a', b') = (aa' - bb', ab' + a'b).$$

The zero element is $0 = (0, 0)$, the unit element is $1 = (1, 0)$, and the imaginary unit i is $(0, 1)$.

Complex Numbers

$M = \{(a, 0) \mid a \in \mathbb{R}\}$ is a subfield of $\mathbb{C}$ isomorphic to $\mathbb{R}$.

Proof.

The mapping $\phi : \mathbb{R} \to M : a \mapsto (a, 0)$ is bijective with

$$\phi(a + b) = (a + b, 0) = (a, 0) + (b, 0) = \phi(a) + \phi(b)$$

and

$$\phi(a \cdot b) = (a \cdot b, 0) = (a, 0) \cdot (b, 0) = \phi(a) \cdot \phi(b).$$

Complex Numbers

```
> Complex(2,3);
```

$$2 + 3i$$

```
> Complex(2,3) + Complex(-3,5);
```

$$-1 + 8i$$

```
> Complex(2,3) * Complex(-4,5);
```

$$-23 - 2i$$

```
> solve( z^2+z+1=0, z );
```

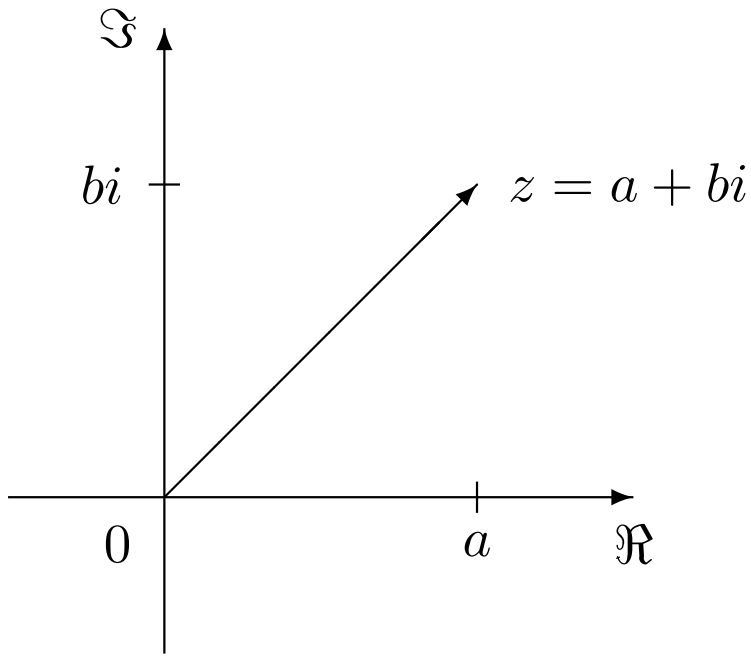
$$-\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \quad -\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$$

Complex Numbers

The representation of $z \in \mathbb{C}$ with $z = a + bi$, $a, b \in \mathbb{R}$, is called *normal form* with

$$a = \Re z \quad \text{and} \quad b = \Im z,$$

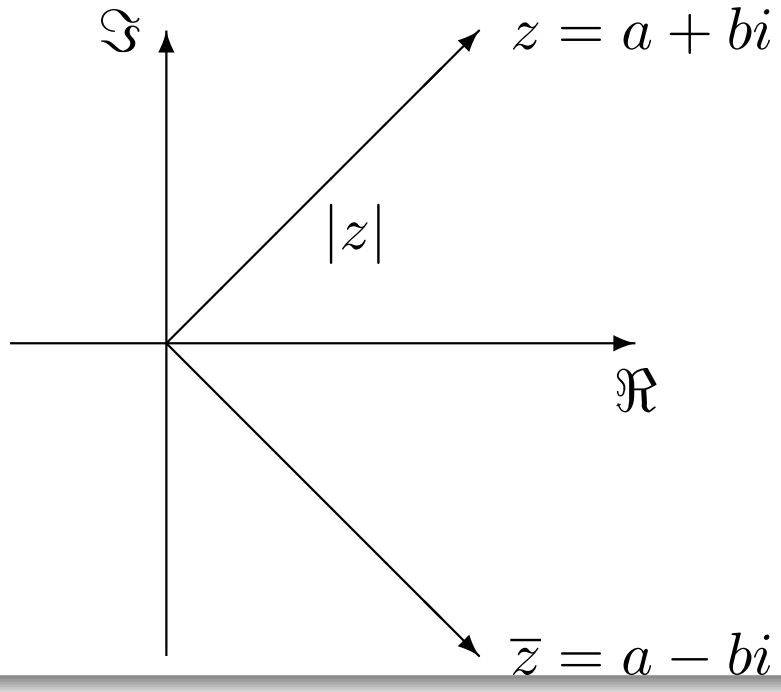
$\Re z$ *real part* and $\Im z$ *imaginary part* of z.



Complex Numbers

Let $z = a + bi \in \mathbb{C}$.

- $\overline{z} = a - bi$ is the *conjugate complex* of z.
- $|z| = \sqrt{a^2 + b^2}$ is the *absolute value* of z.



Complex Numbers

Complex conjugation

$$\mathbb{C} \to \mathbb{C} : z = a + bi \mapsto \overline{z} = a - bi$$

is an involutory automorphism

$$\overline{\overline{z}} = z, \quad \overline{y} + \overline{z} = \overline{y + z} \quad \text{and} \quad \overline{y} \cdot \overline{z} = \overline{y \cdot z}, \quad y, z \in \mathbb{C}.$$

For all $y, z \in \mathbb{C}$:

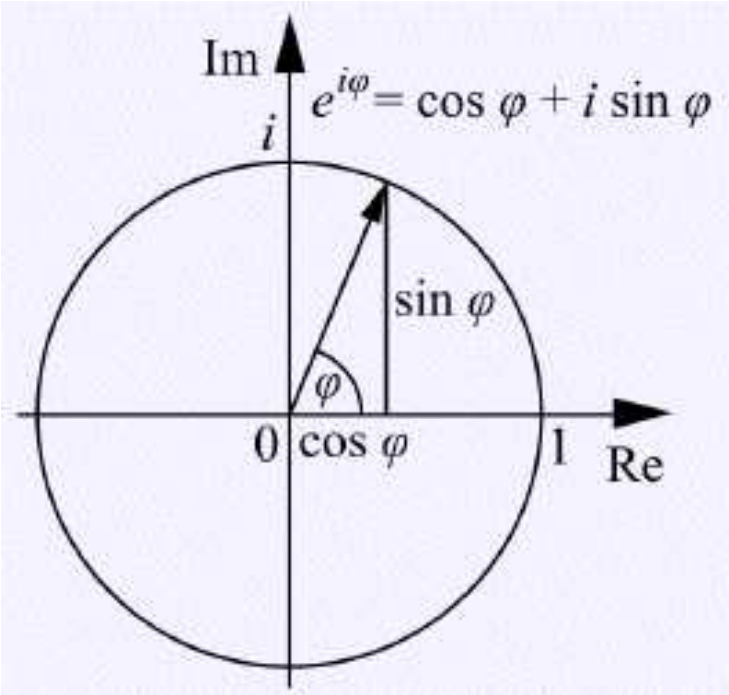
- $z = \overline{z} \iff z \in \mathbb{R}$,
- $|z| = |\overline{z}|$,
- $z \cdot \overline{z} = |z|^2$,
- $|y \cdot z| = |y| \cdot |z|$,
- $|y + z| \leq |y| + |z|$ (triangular inequality).

Complex Numbers

Each complex number z has the unique representation

$$z = r(\cos\phi + i\sin\phi)$$

with $r = |z|$ and $-\pi < \phi \leq \pi$.



Complex Numbers

The *polar representation* of $z = a + ib$, $a, b \in \mathbb{R}$, is

$$z = r(\cos\phi + i\sin\phi)$$

with $r = \sqrt{a^2 + b^2}$ and

- $a > 0, b > 0$: $\phi > 0$ and $\tan\phi = \frac{b}{a}$.
- $a < 0, b > 0$: $\phi > 0$ and $\tan(180^o - \phi) = \frac{b}{|a|}$.
- $a < 0, b < 0$: $\phi < 0$ and $\tan(180^o + \phi) = \frac{b}{a}$.
- $a < 0, b < 0$: $\phi < 0$ and $\tan(-\phi) = \frac{|b|}{a}$.

Complex Numbers

```
> polar(4+2*I);
```

$$\text{polar}(2\sqrt{5}, \arctan(\frac{1}{2}))$$

```
> r := |4+2*I|;
> a := argument(4+2*I);
```

$$r := 2\sqrt{5}, \quad a := \arctan(\frac{1}{2})$$

```
> evalf(convert(arctan(1/2),degrees));
```

$$26.56505117 \text{ degrees}$$

The polar form of $4 + 2i$ is

$$4.47(\cos\ 26.56^o + i \sin\ 26.56^o).$$

Complex Numbers

The *exponential function* $\exp : \mathbb{C} \to \mathbb{C}$ is the power series

$$e^z := \exp(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

This sum converges for all $z \in \mathbb{C}$ (radius ∞).

- The *power function* $\mathbb{C} \to \mathbb{C} : z \mapsto a^z$ to base $a \in \mathbb{R}_{>0}$ is

$$a^z := \exp(z \cdot \log a).$$

- The *sine and cosine functions* are

$$\cos\phi := \Re e^{i\phi} \quad \text{and} \quad \sin\phi := \Im e^{i\phi}, \quad \phi \in \mathbb{R}.$$

Complex Numbers

The sine and cosine functions fulfill

$$\begin{aligned} \sin\phi &= \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} \pm \dots \\ \cos\phi &= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} \pm \dots . \end{aligned}$$

Eulerian Formula

For each $\phi \in \mathbb{R}$,

$$e^{i\phi} = \cos\phi + i\sin\phi.$$

Complex Numbers

Each complex number $z \in \mathbb{C}$ has the unique representation

$$z = re^{i\phi}$$

with $r = |z|$ and $-\pi < \phi \leq \pi$.

Example

The complex number $z = 4 + 2i$ has the polar form $z = r(\cos\,\phi + i \sin\,\phi)$ with $r = 2\sqrt{5}$ and $\phi = \arctan(\frac{1}{2})$, and thus the representation $z = re^{i\phi}$.

Complex Numbers

For each $n \in \mathbb{Q}$ and $\phi \in \mathbb{R}$,

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$$

Proof.

By Euler's formula,

$$\left(e^{i\phi}\right)^n = (\cos\phi + i\sin\phi)^n$$

and

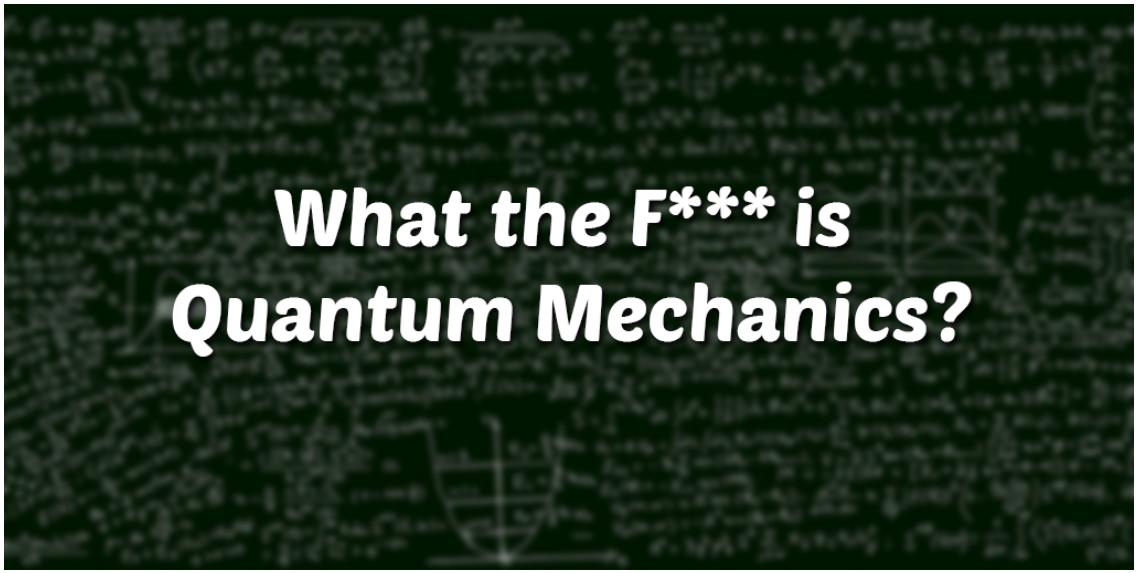
$$\left(e^{i\phi}\right)^n = e^{i(n\phi)} = \cos n\phi + i\sin n\phi.$$

□

Part XI

Quantum Mechanics (Appendix)

Quantum Mechanics



Quantum Mechanics – Contents

- Wave functions
- Harmonic oscillator
- Hydrogen atom
- Hydrogen ion
- Schrödinger equation

Quantum Mechanics – Reading

- Leon van Dommelen: *Quantum Mechanics for Engineers*, Techn. Report, Tallahassee, FL, 2012.
- David J. Griffiths: *Introduction to Quantum Mechanics*, Prentice Hall, Upper Saddle River, NJ, 1995.
- C. Jansson: *Quantum Information Theory for Engineers*, Techn. Report, TUHH, 2017.

Quantum Mechanics

- Fundamental theory describing nature at smallest scale (atoms and subatomic particles).
- Energy, momentum and other quantities of a system are restricted to discrete values (quantization), objects have dual characteristics (wave-particle duality), and there are limits to precision (uncertainty principle).
- In the modern theory, a mathematical function (wave function) provides information about the probability amplitude of position, momentum, and other physical properties of a particle.
- Classical physics describes nature at macroscopic scale and can be seen as an approximation valid at large scale.

Newtonian Physics

Newton picture for a particle with mass m:

- velocity $v_x = \frac{dx}{dt}$,
- linear momentum $p_x = mv_x$,
- force (Newton's second law) $F_x = m\frac{dv_x}{dt}$.

Numerical position, numerical velocity or linear momentum for the particle do not exist!

Quantum mechanics does not use forces, but the potential energy V. This implicitly implies

$$F_x = -\frac{\partial V}{\partial x},\ F_y = -\frac{\partial V}{\partial y},\ F_z = -\frac{\partial V}{\partial z}.$$

Quantum Physics

- What does exist is the so-called wave function $\psi(x, y, z; t)$ depending on location $r = (x, y, z)$ and time t.
- The wave function $\psi(x, y, z; t)$ gives the quantum amplitude that the particle can be found at position $r = (x, y, z)$.
- Born's statistical interpretation:

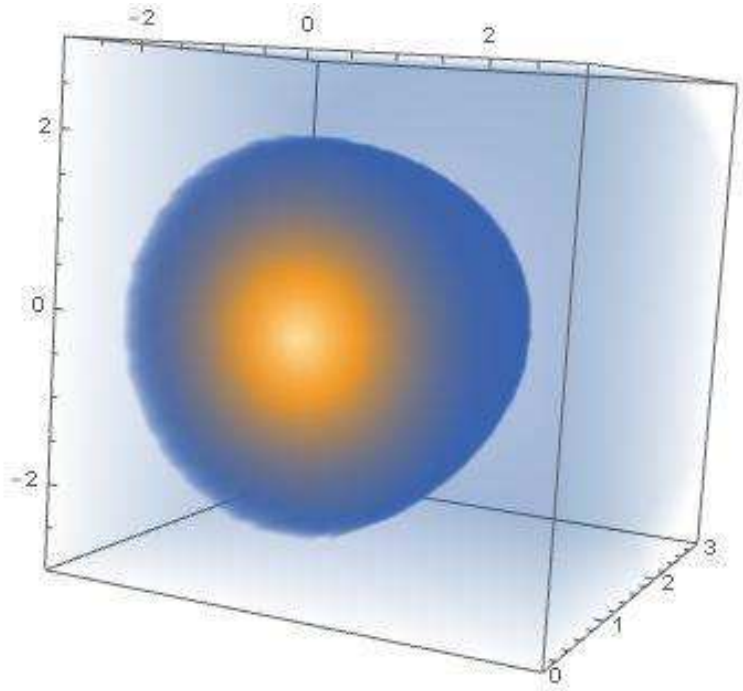
$$|\psi(r; t)|^2 d^3 r \tag{559}$$

is the probability of finding the particle within a small volume of size $d^3 r = dxdydz$ around the given location $r = (x, y, z)$; the particle can be found in a so-called blob.

- Normalization: The total probability of finding the particle somewhere is

$$\int_r |\psi(r; t)|^2 d^3 r = 1. \tag{560}$$

Visualization of Wave Function



Lighter regions are regions where the particle is more likely to be found if the location is narrowed down.

Heisenberg Uncertainty Principle

There is always a minimum combined uncertainty in position and linear momentum.

- A particle cannot have a mathematically precise position x, as this would require an infinite uncertainty in linear momentum p_x.
- A particle cannot have a mathematically precise linear momentum p_x, as this would imply an infinite uncertainty in position x.
- The area of the blob of a particle cannot be contracted to a point. The minimum area is comparable in size to the reduced Planck constant (Dirac constant),

$$\hbar = \frac{h}{2\pi} \approx 1.054 \cdot 10^{-34} \text{ J s}. \tag{561}$$

Operators of Quantum Mechanics

The following operators are the key to quantum mechanics.

- The x-position operator $\hat{x}$ maps wave function ψ to $x\psi$,

$$\psi(x,y,z;t) \mapsto x\psi(x,y,z;t). \tag{562}$$

  The operators $\hat{y}$ and $\hat{z}$ are similarly defined.

- The x-momentum operator $\hat{p}_x$ maps wave function ψ to its x-derivative

$$\psi \mapsto \hat{p}_x\psi = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi = \frac{\hbar}{i}\psi_x(x,y,z;t). \tag{563}$$

  The operators $\hat{p}_y$ and $\hat{p}_z$ are similarly defined,

$$\hat{p}_y = \frac{\hbar}{i}\frac{\partial}{\partial y} \quad \text{and} \quad \hat{p}_z = \frac{\hbar}{i}\frac{\partial}{\partial z}. \tag{564}$$

  The factor i makes them Hermitian operators.

Operators of Quantum Mechanics

- The kinetic energy operator is

$$\hat{T} = -\frac{\hbar^2}{2m}\nabla^2 \tag{565}$$

with Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{566}$$

- The total energy operator or Hamiltonian is the sum of the kinetic energy operator and the potential energy operator $V = V(x, y, z)$,

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V. \tag{567}$$

Eigenfunctions and Eigenstates

Total energy or Hamiltonian of the system:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V. \tag{568}$$

The solutions of the Hamiltonian eigenvalue problem called time-independent Schrödinger equation

$$H\psi = E\psi \tag{569}$$

are the eigenfunctions ψ with associated eigenvalues E (for energy).

Orthodox Statistical Interpretation or "Copenhagen Interpretation"

- The only measurable values (position, momentum, energy, ...) are the eigenvalues of the correponding operator (position, momentum, total energy, ...).
- E.g., if the total energy of a particle is measured, the only outcomes are the eigenvalues of the total energy Hamiltonian.
- Niels Bohr (1885-1962):

  > *According to the orthodox interpretation, measurement causes the wave function ψ to collapse into one of the eigenfunctions of the quantity being measured.*

Orthodox Statistical Interpretation

- The wave function of a system is a linear combination of eigenfunctions,

$$\psi = c_1\psi_1 + c_2\psi_2 + \cdots . \tag{570}$$

- Measurement causes the wave function to collapse into one of the eigenfunctions of the quantity being measured with the measured value being the corresponding eigenvalue; e.g., energy measurement:

$$\left.\begin{array}{l}\psi = c_1\psi_1 + c_2\psi_2 + \ldots \\ \text{energy is uncertain}\end{array}\right\} \overset{\text{energy measurement}}{\longrightarrow} \left\{\begin{array}{l}\psi = c_n\psi_n \\ \text{energy certain } c_n \text{ for some } n.\end{array}\right. \tag{571}$$

- By measurement, the wave function ψ will collapse to eigenfunction ψ_n with probability $|c_n|^2$; i.e., the square magnitudes of the coefficients of the eigenfunctions give the probabilities of the corresponding eigenvalues.

Orthodox Interpretation

- If the wave function before the measurement is (in terms of eigenfunctions)

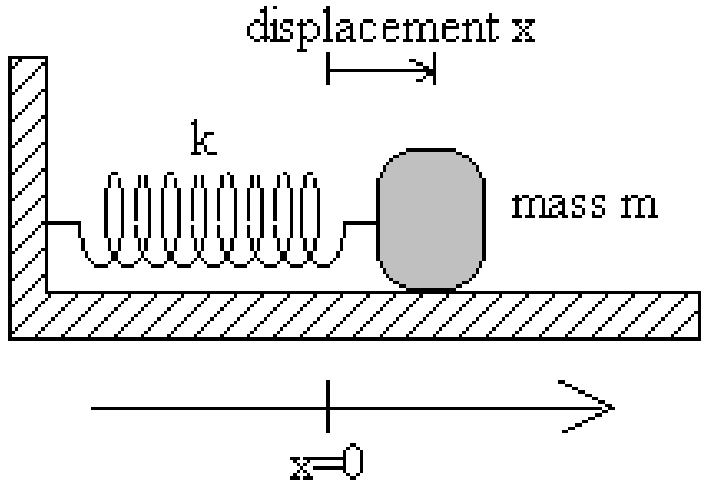
$$\psi = c_1\psi_1 + c_2\psi_2 + \cdots , \tag{572}$$

it will collapse after measurement to eigenfunction ψ_n with probability $|c_n|^2$; this requires the condition

$$\sum_n |c_n|^2 = 1. \tag{573}$$

- After real-life measurement, follow-up measurements have statistics that are consistent with a collapsed wave function $\psi = c_n\psi_n$ (with certainty).

Harmonic Oscillator



Vibrating system as a model for an atom in a trap or crystal vibrations.

Harmonic Oscillator

- Application: particle constrained to remain at approximately the same position; e.g., atom in a solid or molecule.
- Harmonic oscillator: forces that push back the particle to its nominal position are proportional to the distance the particle is away from it; particle's displacement is given by (x, y, z).
- The particle vibrates back and forth around its nominal position with frequency (adopted from classical physics)

$$\omega = \sqrt{\frac{c}{m}} \tag{574}$$

in radians per second, where c is the spring constant (measure of strength of the force).

Harmonic Oscillator

- Total energy Hamiltonian of the system:

$$H = -\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{1}{2}c(x^2 + y^2 + z^2), \quad (575)$$

  where $r = \sqrt{x^2 + y^2 + z^2}$ is the particle distance from the nominal position.

- Each eigenfunction ψ and its eigenvalue E must satisfy the time-independent Schrödinger equation

$$H\psi = E\psi. \quad (576)$$

  Energy levels $E = E_n$ are discretely quantized. The boundary condition is that ψ becomes 0 at large distance from the nominal position.

Harmonic Oscillator

By separation of variables, each eigenfunction is a product of 1D eigenfunctions,

$$\psi(x,y,z) = \psi_x(x)\psi_y(y)\psi_z(z). \tag{577}$$

This leads to three 1D eigenvalue problems

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}cx^2\right)\psi_x = E_x\psi_x \tag{578}$$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}cy^2\right)\psi_y = E_y\psi_y \tag{579}$$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{1}{2}cz^2\right)\psi_z = E_z\psi_z. \tag{580}$$

Harmonic Oscillator

The 1D eigenvalue problem

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}cx^2\right)\psi_x = E_x\psi_x \tag{581}$$

has the solutions

$$\psi_{x,n_x}(x) = h_{n_x}(x), \tag{582}$$

$$E_{x,n_x} = \frac{2n_x+1}{2}\hbar\omega, \quad n_x\in\mathbb{N}_0, \tag{583}$$

where the $h_{n_x}(x)$ come from Hermite polynomials. Similarly,

$$E_{y,n_y} = \frac{2n_y+1}{2}\hbar\omega, \quad n_y\in\mathbb{N}_0, \tag{584}$$

$$E_{z,n_z} = \frac{2n_z+1}{2}\hbar\omega, \quad n_z\in\mathbb{N}_0. \tag{585}$$

Harmonic Oscillator

Eigenfunctions

$$h_n(\xi) = \frac{1}{(\pi\ell^2)^{1/4}} \frac{H_n(\xi)}{\sqrt{2^n n!}} e^{-\xi^2/2}, \tag{586}$$

where $\omega = \sqrt{\frac{c}{m}}$, $\ell = \sqrt{\frac{\hbar}{(m\omega)}}$, $\xi = \frac{x}{\ell}$, and H_n is the nth Hermite polynomial,

$$H_n(x) = (-1)^n \sum_{k_1+k_2=n} \frac{n!}{k_1!k_2!} (-1)^{k_1+k_2} (2x)^{k_1}, \ n \geq 0, \tag{587}$$

or given by the recurrence relation

$$\begin{aligned} H_{-1}(x) &= 0, && (588)\\ H_0(x) &= 1, && (589)\\ H_n(x) &= 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x), \ n \geq 1. && (590) \end{aligned}$$

Harmonic Oscillator

- Total energy

$$E = E_x + E_y + E_z. \tag{591}$$

- Total energy eigenvalues

$$E_{n_x n_y n_z} = \frac{2n_x + 2n_y + 2n_z + 3}{2}\hbar\omega \tag{592}$$

with quantum numbers $n_x, n_y, n_z \geq 0$.

- Corresponding eigenfunctions of the complete system

$$\psi_{n_x n_y n_z}(x, y, z) = h_{n_x}(x)h_{n_y}(y)h_{n_z}(z). \tag{593}$$

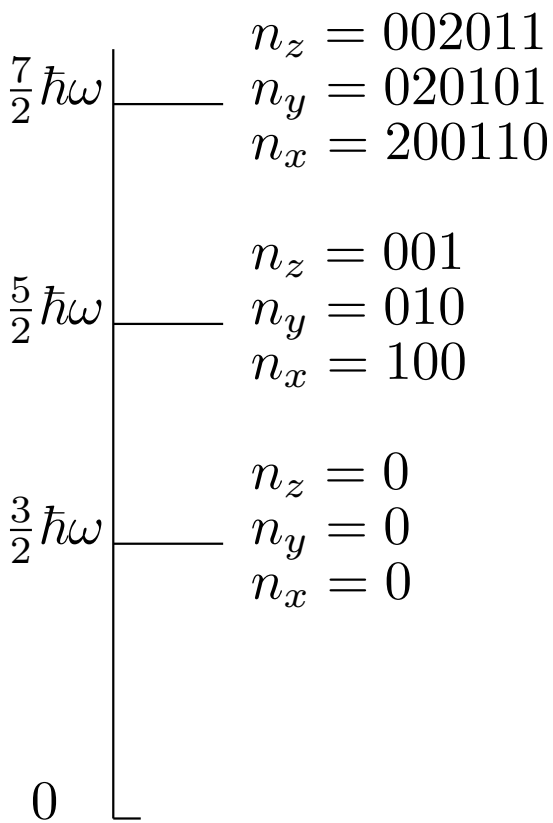
- Ground state energy

$$E_{000} = \frac{3}{2}\hbar\omega. \tag{594}$$

So even at absolute zero temperature, the particle is not completely at rest in its nominal position (by the Heisenberg uncertainty principle).

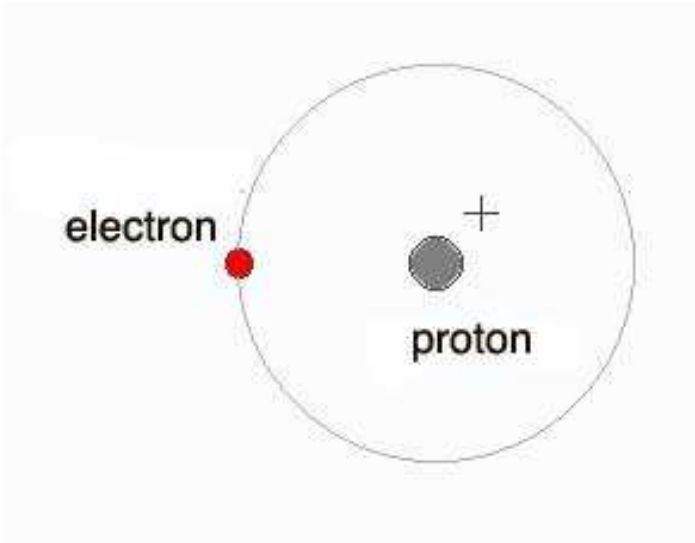
Harmonic Oscillator

Lower part of energy spectrum:



The energy values are discrete and evenly spaced with constant energy difference of $\hbar\omega$ between successive levels.

Hydrogen Atom



Hydrogen Atom

- A hydrogen atom consists of a nucleus with a single proton and an electron encircling the nucleus.
- The nucleus is much heavier than the electron; it is assumed to be stationary and only the motion of the electron is considered.
- The energy levels of the electron determine the photons that the atom will absorb or emit.

Hydrogen Atom

- Electron experiences an electrostatic Coulomb attraction to the oppositely charged nucleus.
- Potential energy of the system:

$$V = -\frac{e^2}{4\pi\epsilon_0 r}, \tag{595}$$

where r is the distance between proton and electron,

$$e = 1.6 \cdot 10^{-19}\ \mathrm{C} \tag{596}$$

is the magnitude of electric charges of electron and proton, and

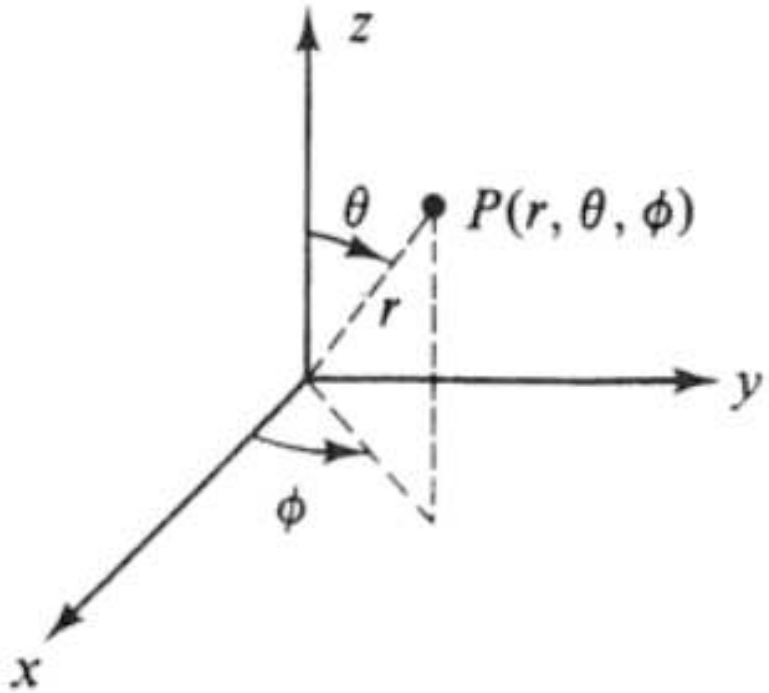
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C}^2/\mathrm{Jm} \tag{597}$$

is the permittivity of space.

Hydrogen Atom

- Potential energy V cannot be split into separate parts for Cartesian coordinates x, y, z as for the harmonic oscillator.
- Nucleus is put at the origin of the coordinate system and spherical coordinates are used: r (distance from nucleus), θ (angle from z-axis), and ϕ (angle around z-axis).

Spherical Coordinates



$$r = \sqrt{x^2 + y^2 + z^2}, \tag{598}$$

$$\theta = \arccos \frac{z}{r}, \tag{599}$$

$$\phi = \arctan \frac{y}{x}. \tag{600}$$

Hydrogen Atom

Total energy Hamiltonian of the system:

$$H = \hat{T} + V = -\frac{\hbar^2}{2m_e}\nabla^2 + V, \tag{601}$$

where

$$m_e = 9.109 \cdot 10^{-31}\ \text{kg} \tag{602}$$

is the mass of the electron.

Hydrogen Atom

Total energy Hamiltonian in spherical coordinates:

$$H = -\frac{\hbar^2}{2m_e r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}. \tag{603}$$

By separation of variables, the wave function has the form

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \tag{604}$$

and so the eigenvalue problem with $\psi = R\Theta\Phi$ becomes

$$H(R\Theta\Phi) = E(R\Theta\Phi). \tag{605}$$

Hydrogen Atom

- The eigenfunctions ψ_{nlm} are numbered as

$$n > l \geq |m| \geq 0 \tag{606}$$

with principal quantum number $n \geq 1$, azimuthal quantum number l, and magnetic quantum number m.

- Eigenfunctions

$$\psi_{nlm}(r, \theta, \phi) \quad = \quad R_{nl}(r) Y_l^m(\theta, \phi), \tag{607}$$

where the R_{nl} are radial wave functions and the Y_m^l are the spherical harmonics,

$$Y_l^m(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^m(\cos\theta) e^{im\phi}, \tag{608}$$

where P_l^m is the associated Legendre function of the first kind.

Hydrogen Atom

- Eigenfunctions

$$\psi_{nlm}(r,\theta,\phi) \tag{609}$$
$$= -\frac{2}{n^2}\sqrt{\frac{n-l-1)!}{[(n+l)!a_0]^3}}\left(\frac{2\rho}{n}\right)^l L_{n+l}^{(2l+1)} e^{-\rho/n} Y_l^m(\theta,\phi),$$

where $L_{n+l}^{(2l+1)}$ is the $2l+1$-th derivative of the associated Laguerre polynomial L_{n+l}, $\rho = r/a_0$ is the scaled radial distance from the nucleus, and a_0 is the Bohr radius

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} \approx 0.53 \cdot 10^{-10}\ \mathrm{m} \approx 0.5\ \mathring{A}. \tag{610}$$

Hydrogen Atom

- Energy eigenvalues (Bohr energies)

$$E_n = -\frac{\hbar^2}{2m_e a_0^2}\frac{1}{n^2} = \frac{1}{n^2}E_1, \quad n \geq 1, \tag{611}$$

with energy of ground state at absolute zero temperature

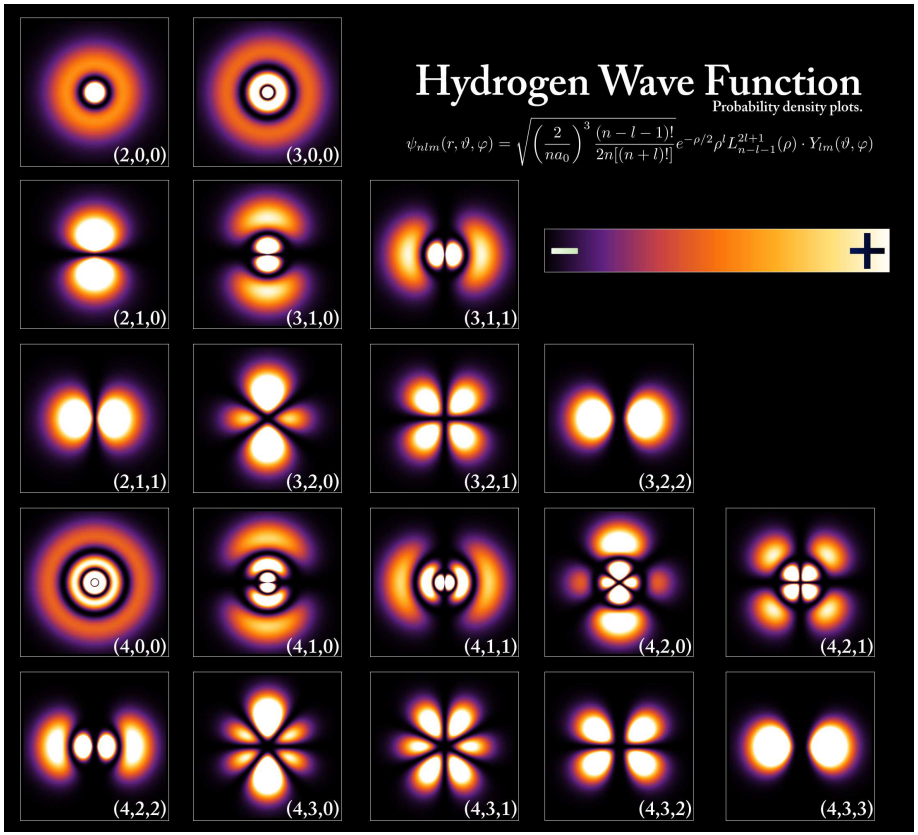
$$E_1 = -\frac{\hbar^2}{2m_e a_0^2} = -13.6\ \mathrm{eV}, \tag{612}$$

where eV stands for electron volt ($1\ \mathrm{eV} = 1.6 \cdot 10^{-19}\ \mathrm{J}$).

- The ground state of lowest energy is given by eigenfunction

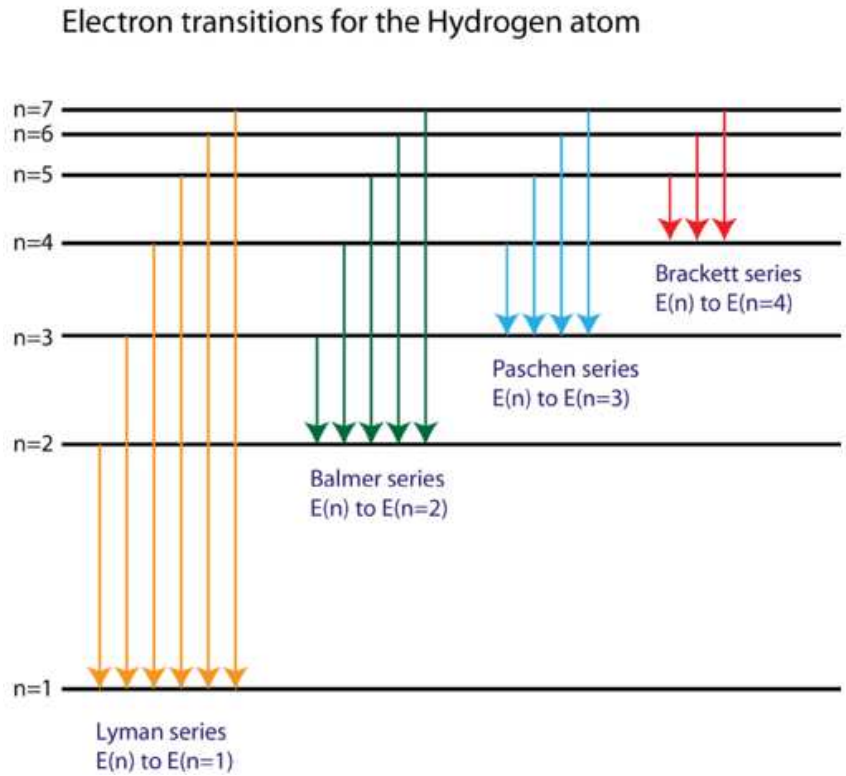
$$\psi_{100}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}. \tag{613}$$

Hydrogen Atom



Eigenfunctions of the electron in a hydrogen atom at different energy levels. The brighter areas represent a higher probability of finding the electron.

Hydrogen Atom



Spectrum of hydrogen atom.

If electron is excited from ground state to higher state, but still bound energy level, it will in time again transition back to a lower enery level.
Energy lost by the electron during transition is emitted as a photon: Lyman transition (UV light), Balmer transition (visible light), Paschen transition (infrared light).
Ionization energy is 13.6 eV (minimum energy to raise electron from ground state to state of free electron; ion is left).

Heisenberg Uncertainty

The position operator $\hat{x}$ and the linear momentum operator $\hat{p}_x$ do not commute, since

$$\begin{aligned} \hat{p}_x\hat{x}\psi &= \frac{\hbar}{i}\frac{\partial}{\partial x}(x\psi) = \frac{\hbar}{i}\psi + \frac{\hbar}{i}x\frac{\partial\psi}{\partial x} \\ &= -i\hbar\psi + \hat{x}\hat{p}_x\psi. \end{aligned} \tag{614}$$

Thus their commutator is

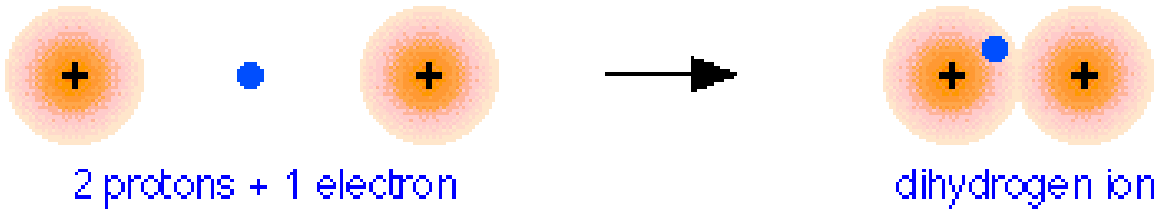
$$[\hat{x}, \hat{p}_x] = \hat{x}\hat{p}_x - \hat{p}_x\hat{x} = i\hbar. \tag{615}$$

A nonzero commutator demands a minimum amount of uncertainty in the quantities x and p_x given by their standard deviations σ_x and σ_{p_x},

$$\sigma_x\sigma_{p_x} \geq \frac{1}{2}\hbar. \tag{616}$$

This is the precise uncertainty relationship first formulated by Werner Heisenberg (1901-1976).

Hydrogen Molecular Ion H_2^+



Ion consists of two protons and a single electron circling them. A covalent bond forms in which the protons share the electron that holds the ion together.

Hydrogen Molecular Ion

- Total energy Hamiltonian of the H_2^+-ion:

$$H = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{e^2}{r\pi\epsilon_0}\left[\frac{1}{r_L} + \frac{1}{r_R}\right], \tag{617}$$

where r_L and r_R are the distances between the electron and the left and right protons, resp., i.e.,

$$r_L = |\vec{r} - \vec{r}_{Lp}| \text{ and } r_R = |\vec{r} - \vec{r}_{Rp}|, \tag{618}$$

where $\vec{r}_{Lp}$ and $\vec{r}_{Rp}$ are the positions of the left and right protons, resp.

- Born-Oppenheimer approximation: nuclei of the protons are at fixed positions.
- Two-state quantum system (single qubit system).

Hydrogen Molecular Ion – Energy when fully dissociated

- When the protons are far apart, there are two lowest energy states ψ_L and ψ_R, in which the electron is in the ground state around the left and right proton, resp. In either case, there is an hydrogen atom and a free proton.
- Wave functions

$$\psi_L = \psi_{100}(|\vec{r} - \vec{r}_{Lp}|) \text{ or } \psi_R = \psi_{100}(|\vec{r} - \vec{r}_{Rp}|). \tag{619}$$

  where $\vec{r}$, $\vec{r}_{Lp}$, and $\vec{r}_{rp}$ are the positions of electron, left proton, and right proton, resp.
- Wave function of the hydrogen atom in the ground state is given by (613),

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}. \tag{620}$$

Hydrogen Molecular Ion – Energy when sharing the electron

The protons share the electron such that there is a probability of finding the electron around either proton.

- The wave function is the linear combination

$$\psi = a\psi_L + b\psi_R, \tag{621}$$

  where the electron is shared by the protons in ways that depend on the chosen values of a and b, which are not independent.

- The functions ψ_L and ψ_R are not eigenfunctions of the system (see (633)).

Schrödinger Equation

The Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian in Cartesian coordinates.

Ψ is the wavefunction.

V is the potential.

$\hbar$ is the Planck constant divided by 2π.

The particle mass is represented by m.

Schrödinger Equation

- *Newton's 2nd law of motion* states that the linear momentum changes in time proportional to the applied force,

$$F = \frac{dp}{dt} = m\frac{dv}{dt} = ma. \tag{622}$$

- Schrödinger's equation is the equivalent in quantum mechanics and states that the time derivative of the wave function is obtained by applying the Hamiltonian on it,

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi. \tag{623}$$

Nonrelativistic version; in the relativistic case, particles may be created out of pure energy or destroyed (quantum field theory).

Solution of Schrödinger Equation

Write the wave function ψ in terms of the energy eigenfunctions $\psi_{\vec{n}}$ of the Hamiltonian,

$$\psi = \sum_{\vec{n}} c_{\vec{n}}(t)\psi_{\vec{n}}, \tag{624}$$

where the coefficients $c_{\vec{n}}$ evolve in time as complex exponentials,

$$c_{\vec{n}}(t) = c_{\vec{n}}(0)e^{-iE_{\vec{n}}t/\hbar} \tag{625}$$

for each quantum number $\vec{n}$ and with initial values $c_{\vec{n}}(0)$ given by the wave function.

Solution of Schrödinger Equation

Coefficients vary with time but their square magnitudes do not:

$$\begin{aligned} |c_{\vec{n}}(t)|^2 &= \overline{c_{\vec{n}}(t)} c_{\vec{n}}(t) \\ &= \overline{c_{\vec{n}}(0)} e^{iE_{\vec{n}}t/\hbar} c_{\vec{n}}(0) e^{-iE_{\vec{n}}t/\hbar} \\ &= |c_{\vec{n}}(0)|^2, \end{aligned} \tag{626}$$

where $\overline{z}$ is the complex conjugate of a complex number z and the square of the length is given by $|z|^2 = \overline{z} \cdot z$.
So the probability of measuring a given energy level does not vary with time.

Schrödinger Equation

Solution for harmonic oscillator:

$$\psi(x,y,z;t) = \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\sum_{n_z=0}^{\infty} c_{n_x,n_y,n_z}(0)e^{-iE_{n_x,n_y,n_z}t/\hbar}\psi_{n_x,n_y,n_z}(x,y,z), \tag{627}$$

where

$$E_{n_x,n_y,n_z} = \frac{2n_x+2n_y+2n_z+3}{2}\hbar\omega \tag{628}$$

and

$$\psi_{n_x,n_y,n_z}(x,y,z) = h_{n_x}(x)h_{n_y}(y)h_{n_z}(z). \tag{629}$$

Schrödinger Equation

Solution for hydrogen atom:

$$\psi(r,\theta,\phi;t) = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} c_{nlm}(0)e^{-iE_nt/\hbar}\psi_{nlm}(r,\theta,\phi), \quad (630)$$

where

$$E_n = -\frac{\hbar^2}{2m_e a_0^2 n^2}, \quad n \geq 1, \quad (631)$$

and

$$\psi_{mln}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi). \quad (632)$$

Schrödinger Equation

Solution for hydrogen ion:

$$\psi = c_L e^{-iE_L t/\hbar} \frac{\psi_L + \psi_R}{\sqrt{2}} + c_H e^{-iE_H t/\hbar} \frac{\psi_L - \psi_R}{\sqrt{2}}, \qquad (633)$$

where E_L is the ground state of lowest energy and E_H is the state of highest energy, and $\frac{\psi_L+\psi_R}{\sqrt{2}}$ and $\frac{\psi_L-\psi_R}{\sqrt{2}}$ are the eigenfunctions.

Further Reading

Leon van Dommelen: *Quantum Mechanics for Engineers*, Techn. Report, Tallahassee, FL, 2012.

Part XII

Hilbert Spaces (Appendix)

Hilbert Spaces

- Inner product spaces
- Normed spaces
- Convergence and completeness
- Hilbert spaces
- Tensor spaces
- Unitary operators

Vector Spaces

A $\mathbb{K}$-*vector space* is a set V with vector addition

$$+ : V \times V \to \mathbb{K} : (v, w) \mapsto v + w \tag{634}$$

and scalar multiplication

$$\cdot : \mathbb{K} \times V \to V : (\alpha, v) \mapsto \alpha \cdot v \tag{635}$$

such that $(V, +)$ is an abelian group and for all $\alpha, \beta \in \mathbb{K}$ and $v, w \in V$,

$$\begin{aligned} (\alpha + \beta)v &= \alpha v + \beta v, & (636)\\ \alpha(v + w) &= \alpha v + \alpha w, & (637)\\ (\alpha\beta)v &= \alpha(\beta v), & (638)\\ 1 \cdot v &= v. & (639) \end{aligned}$$

Inner Product Spaces

A $\mathbb{K}$-vector space V is an *inner product space* if it supports an *inner product*; i.e., a mapping

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$$

such that for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{K}$,

$$\langle v, v \rangle \neq 0 \Longleftrightarrow v \neq 0, \tag{640}$$
$$\langle v, w \rangle = \overline{\langle w, v \rangle}, \tag{641}$$
$$\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle, \tag{642}$$

where $\mathbb{K} \to \mathbb{K} : z \mapsto \overline{z}$ is an involutory field automorphism.

Inner Product Spaces – Example

If $\mathbb{K} = \mathbb{R}$, (641) becomes *symmetry*

$$\langle v, w \rangle = \langle w, v \rangle$$

and (641), (642) show that both mappings

$$v \mapsto \langle v, w \rangle \text{ and } w \mapsto \langle v, w \rangle$$

are $\mathbb{R}$-linear. Therefore, the mapping

$$(v, w) \mapsto \langle v, w \rangle$$

is $\mathbb{R}$-*bilinear*.

Field of Complex Numbers

- The *complex conjugate* of a complex number $z = x + iy$ is

$$\overline{z} = x - iy.$$

- The *length* of a complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}$$

with $|z|^2 = z\overline{z}$.

- *Complex conjugation*

$$\mathbb{C} \to \mathbb{C} : z \mapsto \overline{z}$$

is an involutory field automorphism,

$$\overline{z + w} = \overline{z} + \overline{w},\ \overline{z \cdot w} = \overline{z} \cdot \overline{w},\ \overline{\overline{z}} = z,\ |\overline{z}| = |z|,$$

and $\overline{z} = z$ iff $z \in \mathbb{R}$.

Inner Product Spaces – Example

If $\mathbb{K} = \mathbb{C}$, then (641) and (642) give

$$\begin{aligned} \langle \alpha u + \beta v, w \rangle &= \overline{\langle w, \alpha u + \beta v \rangle} \\ &= \overline{\alpha \langle w, u \rangle + \beta \langle w, v \rangle} \\ &= \overline{\alpha}\overline{\langle w, u \rangle} + \overline{\beta}\overline{\langle w, v \rangle} \\ &= \overline{\alpha}\langle u, w \rangle + \overline{\beta}\langle v, w \rangle, \end{aligned} \tag{643}$$

i.e., $v \mapsto \langle v, w \rangle$ is *skew-linear*. The inner product $\langle \cdot, \cdot \rangle$ is a *sesquilinear form*.

We have by (643)

$$\langle \alpha u, v \rangle = \overline{\alpha} \langle u, v \rangle \tag{644}$$

and by (642)

$$\langle u, \beta v \rangle = \beta \langle u, v \rangle. \tag{645}$$

Cauchy-Schwarz Inequality

Let V be an inner product space. Then for all $u, v \in V$,

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \ \langle v, v\rangle. \tag{646}$$

Equality holds iff $v = \alpha u$ for some $\alpha \in \mathbb{C}$.

Proof.

For $\lambda \in \mathbb{C}$ and $u, v \in V$,

$$0 \leq \langle u - \lambda v, u - \lambda v \rangle = \langle u, u \rangle - \lambda \langle u, v \rangle - \overline{\lambda} \langle v, u \rangle + |\lambda|^2 \langle v, v \rangle.$$

Let $\langle v, u \rangle = \beta e^{i\theta}$, $\beta \in \mathbb{R}$, and $\lambda = e^{-i\theta} t$, $t \in \mathbb{R}$. Then

$$\begin{aligned} 0 &\leq \langle u, u \rangle - e^{-i\theta} t \beta e^{i\theta} - e^{i\theta} t \beta e^{-i\theta} + t^2 \langle v, v \rangle \\ &= \langle u, u \rangle - 2\beta t + t^2 \langle v, v \rangle = \alpha - 2\beta t + \gamma t^2 = q(t), \end{aligned}$$

where $\alpha = \langle u, u \rangle$ and $\gamma = \langle v, v \rangle$ are real numbers. So $q(t)$ is a real-valued quadratic polynomial with $q(t) \geq 0$ for all t. Thus $q(t) = 0$ has at most one real solution t; i.e., the discriminant is non-positive:

$$D = 4\beta^2 - 4\alpha\gamma \leq 0.$$

Hence, $0 \geq 4\beta^2 - 4\alpha\gamma = |\langle u, v \rangle|^2 - \langle u, u \rangle \langle v, v \rangle$ as claimed. □

Normed Spaces

A $\mathbb{K}$-vector space V is a *normed space* if it supports a *norm*; i.e., a mapping

$$\|\cdot\| : V \to \mathbb{R}_{\geq 0},$$

such that for all $v, w \in V$ and $\alpha \in \mathbb{C}$,

$$\|v\| \neq 0 \Longleftrightarrow v \neq 0, \tag{647}$$
$$\|\alpha v\| = |\alpha| \cdot \|v\|, \tag{648}$$
$$\|v + w\| \leq \|v\| + \|w\|. \tag{649}$$

Normed Spaces

Each inner product space V is also a normed space with norm

$$\|v\| = \sqrt{\langle v, v\rangle}, \quad v \in V. \tag{650}$$

The Cauchy-Schwarz inequality (646) becomes

$$|\langle u, v\rangle| \leq \|u\|\|v\|, \quad u, v \in V. \tag{651}$$

Normed Spaces – Proof

- The mapping $\|\cdot\| : V \to \mathbb{R}_{\geq 0} : v \mapsto \|v\|$ is well-defined.
- (640) and (647) are equivalent.
- (648) follows from (641), (642):

$$\|\alpha v\|^2 = \langle \alpha v, \alpha v\rangle = \alpha\overline{\alpha}\langle v, v\rangle = |\alpha|^2 \cdot \|v\|^2.$$

- (649) follows from Cauchy-Schwarz (651),

$$\begin{aligned}
\|v+w\|^2 &= \langle v+w, v+w\rangle \\
&= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\
&= \|v\|^2 + 2\Re\langle v, w\rangle + \|w\|^2 \\
&\leq \|v\|^2 + 2|\langle v, w\rangle| + \|w\|^2 \\
&\leq \|v\|^2 + 2\|v\| \cdot \|w\| + \|w\|^2 \\
&= (\|v\| + \|w\|)^2.
\end{aligned}$$

□

Inner Product Spaces – Examples

- $\mathbb{R}^n$ ($\mathbb{R}$-vector space):

$$\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i, \tag{652}$$

$$\|v\| = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2}. \tag{653}$$

- $\mathbb{C}^n$ ($\mathbb{C}$-vector space):

$$\langle u, v\rangle = \sum_{i=1}^{n} \overline{u}_i v_i, \tag{654}$$

$$\|v\| = \left(\sum_{i=1}^{n} |v_i|^2\right)^{1/2}. \tag{655}$$

Normed Spaces – Parallelogram Law

Let V be an inner product space. Then for all $u, v \in V$,

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2. \tag{656}$$

Proof.

We have

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ \|u-v\|^2 &= \langle u-v, u-v\rangle, \\ &= \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle. \end{aligned}$$

Adding these equations gives

$$\|u+v\|^2 + \|u-v\|^2 = 2\langle u,u\rangle + 2\langle v,v\rangle = 2\|u\|^2 + 2\|v\|^2.$$

Normed Spaces

Most real and complex normed spaces do not have inner products.

- Consider the p-norm

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}}.$$

- If a normed linear space V satisfies the parallelogram law, the norm arises from an inner product.
- This particularly holds for the p-norm iff $p = 2$, the socalled *Euclidean norm*.

Convergence

Given a normed space $(V, \|\cdot\|)$.

- A sequence $(v_n)_{n\in\mathbb{N}_0} \subseteq V$ *converges* to an element $v \in V$ if $(\|v - v_n\|)_{n\in\mathbb{N}_0}$ converges to 0 in $\mathbb{R}$,

$$\lim_{n\to\infty} v_n = v \quad :\Longleftrightarrow \quad \lim_{n\to\infty} \|v - v_n\| = 0. \tag{657}$$

- A sequence $(v_n)_{n\in\mathbb{N}_0} \subseteq V$ is *Cauchy* if

$$\lim_{k,n\to\infty} \|v_k - v_n\| = 0. \tag{658}$$

Hilbert Space

A *Hilbert space* is a complete inner product space; i.e., an inner product space in which every Cauchy sequence converges.

Examples

The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are "typical" Hilbert spaces.

Hilbert Space

Given a Hilbert space $\mathcal{H}$.

- Two vectors $u, v \in \mathcal{H}$ are *orthogonal*, written $u \perp v$, if

$$\langle u, v \rangle = 0. \tag{659}$$

- The *orthogonal complement* $U^{\perp}$ of a subset $U \subseteq \mathcal{H}$ is

$$U^{\perp} = \{v \in \mathcal{H} \mid \langle u, v \rangle = 0 \; \forall u \in U\}. \tag{660}$$

Hilbert Space

Let $\mathcal{H}$ be a Hilbert space. The orthogonal complement $U^\perp$ of $U \subseteq \mathcal{H}$ is a closed linear subspace of $\mathcal{H}$ and $U \subseteq (U^\perp)^\perp$.

Proof.

- Let $v, w \in U^\perp$, $u \in U$, and $\alpha, \beta \in \mathbb{C}$. Then

$$\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle = 0$$

  and so $\alpha v + \beta v \in U^\perp$; i.e., $U^\perp$ is a subspace of $\mathcal{H}$.

- Let $(v_n) \subseteq U^\perp$ with $\lim_{n\to\infty} v_n = v$ and $u \in U$. Then

$$\begin{aligned} |\langle u, v \rangle| &= |\langle u, v \rangle - \langle u, v_n \rangle| = |\langle u, v - v_n \rangle| \\ &\leq \|u\| \cdot \|v - v_n\| \overset{n\to\infty}{\longrightarrow} 0 \quad \text{by (651)} \end{aligned}$$

  and so $v \in U^\perp$; i.e., $U^\perp$ is closed. □

Linear Functionals

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space. A *continuous linear functional* on $\mathcal{H}$ is a mapping

$$\Lambda : \mathcal{H} \to \mathbb{C} : v \mapsto \Lambda(v), \tag{661}$$

which is $\mathbb{C}$-linear,

$$\Lambda(\alpha u + \beta v) = \alpha\Lambda(u) + \beta\Lambda(v), \quad u, v \in \mathcal{H},\ \alpha, \beta \in \mathbb{C}, \tag{662}$$

and satisfies for some constant $c = c(\Lambda) \in \mathbb{R}_{>0}$,

$$|\Lambda(u - v)| \leq c \cdot \|u - v\|, \quad u, v \in \mathcal{H}. \tag{663}$$

Linear Functionals – Example

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space and $u \in \mathcal{H}$. The mapping

$$\Lambda_u : \mathcal{H} \to \mathbb{C} : v \mapsto \langle u, v \rangle \tag{664}$$

is $\mathbb{C}$-linear, since for all $v, w \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} \Lambda_u(\alpha v + \beta w) &= \langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle \\ &= \alpha \Lambda_u(v) + \beta \Lambda_u(w), \end{aligned}$$

and is continuous by Cauchy-Schwarz (651),

$$|\Lambda_u(v - w)| = |\langle u, v - w \rangle| \leq \|u\| \cdot \|v - w\|, \quad v, w \in \mathcal{H},$$

with $c(\Lambda_u) = \|u\|$.

Riesz Representation

Each continuous linear functional Λ on a $\mathbb{C}$-Hilbert space has the form (664); i.e., there is a unique $u \in \mathcal{H}$ such that $\Lambda = \Lambda_u$ with

$$\Lambda_u : \mathcal{H} \to \mathbb{C} : v \mapsto \langle u, v \rangle. \tag{665}$$

Proof.

- *Uniqueness:* Let $u, u' \in \mathcal{H}$ with $\langle u, v \rangle = \langle u', v \rangle$ for all $v \in \mathcal{H}$. Then $\langle u - u', v \rangle = 0$. In particular, taking $v = u - u'$ gives

$$\|u - u'\|^2 = \langle u - u', u - u' \rangle = 0$$

and so $u = u'$.

Proof (cont'd).

- *Existence:* Let $K = \ker \Lambda$. Since Λ is linear and continuous, K is a closed linear subspace of $\mathcal{H}$. If $K = \mathcal{H}$, take $u = 0 \in \mathcal{H}$ and so $\Lambda = 0$.

  Otherwise, take $u \in K^\perp$ with $\Lambda(u) = 1$, and let $v \in \mathcal{H}$ with $\alpha = \Lambda(v)$. Then $\Lambda(v - \alpha u) = \Lambda(v) - \alpha\Lambda(u) = 0$ and so $v - \Lambda(v)u \in K$. Therefore,

$$0 \overset{\perp}{=} \langle u, v - \Lambda(v)u \rangle = \langle u, v \rangle - \Lambda(v)\|u\|^2.$$

  So if $u_0 = u/\|u\|^2$, then

$$\Lambda(v) = \langle u, v \rangle / \|u\|^2 = \langle u_0, v \rangle.$$

□

Bounded Linear Operators

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space.

- A linear functional $\Lambda : \mathcal{H} \to \mathbb{C}$ is *bounded* if there is a constant $c > 0$ such that

$$|\Lambda(v)| \leq c \cdot \|v\|, \quad v \in \mathcal{H}. \tag{666}$$

- For a bounded linear functional Λ, define the *norm* of Λ as

$$\|\Lambda\| = \sup\{|\Lambda(v)| \mid \|v\| \leq 1,\ v \in \mathcal{H}\} < \infty. \tag{667}$$

  We have

$$\begin{aligned} \|\Lambda\| &= \sup\{|\Lambda(v)| \mid \|v\| \leq 1,\ v \in \mathcal{H}\} \\ &= \sup\{|\Lambda(v)|/\|v\| \mid 0 \neq v \in \mathcal{H}\} \\ &= \inf\{c > 0 \mid |\Lambda(v)| \leq c\|v\|,\ v \in \mathcal{H}\}. \end{aligned} \tag{668}$$

- A linear functional is bounded iff it is continuous.

Bounded Linear Operators

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space and $u \in \mathcal{H}$. The mapping

$$\Lambda_u : \mathcal{H} \to \mathbb{C} : v \mapsto \langle u, v \rangle \tag{669}$$

is a linear functional. By Cauchy-Schwarz (651),

$$|\Lambda_u(v)| = |\langle u, v \rangle| \leq \|u\| \|v\|. \tag{670}$$

So Λ_u is bounded and

$$\Lambda_u \left(\frac{u}{\|u\|} \right) = \left\langle u, \frac{u}{\|u\|} \right\rangle = \frac{1}{\|u\|} \langle u, u \rangle = \|u\|. \tag{671}$$

Hence, $\|\Lambda_u\| = \|u\|$.

Orthonormal Bases

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space.

- The *linear span* of a family $\{e_n \mid n = 1, \ldots, N\} \subseteq \mathcal{H}$, $N \in \mathbb{N} \cup \{\infty\}$, is the set of all *finite* $\mathbb{C}$-linear combinations of the e_n; i.e.,

$$\operatorname{span}\{e_1, \ldots, e_N\} = \left\{ \sum_{\substack{i \\ \text{finite}}} \alpha_i e_i \mid \alpha_i \in \mathbb{C} \right\}. \tag{672}$$

- The sequence (e_n) is a *(countable) orthonormal system* (ONS) if

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \tag{673}$$

  i.e., $\|e_i\| = 1$ and $e_i \perp e_j$ if $i \neq j$.

- An orthonormal basis of $\mathcal{H}$ is a *Hilbert basis* of $\mathcal{H}$.

Hilbert Space

Each finite-dimensional $\mathbb{C}$-inner product space V is complete.

Proof.

Let $\{e_1, \ldots, e_N\}$ be a Hilbert basis of V.
Suppose the sequence of vectors $u_n = \sum_{i=1}^N a_i^{(n)} e_i$ is Cauchy; i.e.,

$$\|u_k - u_n\|^2 = \sum_i |a_i^{(k)} - a_i^{(n)}|^2 \overset{k,n\to\infty}{\longrightarrow} 0.$$

Then for each $1 \leq i \leq N$, $(a_i^{(n)})$ is Cauchy, since $|a_i^{(k)} - a_i^{(n)}| \to 0$ as $k, n \to \infty$. But $\mathbb{C}$ is complete and so $(a_i^{(n)})$ has a limit a_i. Then (u_n) converges to $u = \sum_{i=1}^N a_i e_i$, since

$$\|u - u_n\|^2 = \sum_i |a_i - a_i^{(n)}|^2 \overset{n\to\infty}{\longrightarrow} 0.$$

Gram-Schmidt Orthonormalization Procedure

Require: Given $\mathbb{C}$-Hilbert space $\mathcal{H}$ with basis $\{v_1, \ldots, v_n\}$.
Ensure: ONS $\{e_1, \ldots, e_n\}$.

$e_1 \leftarrow v_1/\|v_1\|$.
for $k \leftarrow 1$ to $n-1$ **do**
$\hat{e}_{k+1} \leftarrow v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1}, e_i \rangle e_i$
$e_{k+1} \leftarrow \hat{e}_{k+1}/\|\hat{e}_{k+1}\|$
end for

Tensor Spaces

The *tensor product* of two $\mathbb{K}$-vector spaces V, W is a $\mathbb{K}$-vector space $V \otimes_{\mathbb{K}} W$ with a $\mathbb{K}$-bilinear mapping

$$\phi : V \times W \to V \otimes_{\mathbb{K}} W : (v, w) \to v \otimes w,$$

bilinearity means that for all $v, v' \in V$, $w, w' \in W$, and $\lambda \in \mathbb{K}$,

$$(v + v') \otimes w = v \otimes w + v' \otimes w, \tag{674}$$
$$v \otimes (w + w') = v \otimes w' + v \otimes w', \tag{675}$$
$$\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w). \tag{676}$$

*Tensor Spaces

Let V, W be $\mathbb{K}$-vector spaces.

- Let F be the $\mathbb{K}$-vector space with basis elements $v \otimes w$, where $v \in V$ and $w \in W$. The elements of F are $\mathbb{K}$-linear combinations of the form

$$\sum_{\text{finite}} \lambda_{v \otimes w} v \otimes w. \tag{677}$$

- Consider the $\mathbb{K}$-subspace R of F generated by

$$(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w, \tag{678}$$
$$v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2, \tag{679}$$
$$(\lambda v) \otimes w - v \otimes (\lambda w) \tag{680}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $\lambda \in \mathbb{K}$.

*Tensor Spaces

- Define the quotient $\mathbb{K}$-vector space

$$V \otimes_{\mathbb{K}} W = F/R = \{v + R \mid v \in F\}. \tag{681}$$

- The relations in $V \otimes_{\mathbb{K}} W$ are

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \tag{682}$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \tag{683}$$
$$(\lambda v) \otimes w = v \otimes (\lambda w) \tag{684}$$

  for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $\lambda \in \mathbb{K}$.

- Scalar multiplication in $V \otimes_{\mathbb{K}} W$ is given by

$$\lambda(v \otimes w) = (\lambda v) \otimes w \tag{685}$$

  for all $v \in V$, $w \in W$ and $\lambda \in \mathbb{K}$.

Tensor Spaces

Let V and W be $\mathbb{K}$-vector spaces with $\mathbb{K}$-bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$, resp.

- The tensor product $V \otimes_{\mathbb{K}} W$ has basis

$$\{v_i \otimes w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}. \tag{686}$$

- Let the vector space U have $\mathbb{K}$-basis $u_1, \ldots, u_{mn}$. Then the mapping $V \otimes_{\mathbb{K}} W \to U$ given by

$$v_i \otimes w_j \mapsto u_{(i-1)n+j} \tag{687}$$

  is a $\mathbb{K}$-vector space isomorphism.

- The tensor product $V \otimes_{\mathbb{K}} W$ has dimension

$$\dim_{\mathbb{K}} V \otimes_{\mathbb{K}} W = \dim_{\mathbb{K}} V \cdot \dim_{\mathbb{K}} W. \tag{688}$$

Tensor Spaces – Universal Property

Let U, V, and W be finite-dimensional $\mathbb{K}$-vector spaces and

$$\phi : V \times W \to V \otimes_{\mathbb{K}} W : (v, w) \mapsto v \otimes w.$$

For each $\mathbb{K}$-bilinear mapping

$$\psi : V \times W \to U$$

there is a unique $\mathbb{K}$-linear mapping

$$\rho : V \otimes_{\mathbb{K}} W \to U$$

such that

$$\psi = \rho \circ \phi. \tag{689}$$

Each linear mapping $\psi : V \times W \to U$ factors through $V \otimes_{\mathbb{K}} W$

Tensor Spaces – Universal Property

For finite-dimensional $\mathbb{K}$-vector spaces V and W, the tensor spaces $V \otimes_{\mathbb{K}} W$ and $W \otimes_{\mathbb{K}} V$ are isomorphic.

Proof.

Let

$$\phi : V \times W \to V \otimes_{\mathbb{K}} W : (v, w) \mapsto v \otimes w.$$

For the $\mathbb{K}$-bilinear mapping

$$\psi : V \times W \to W \otimes_{\mathbb{K}} V : (v, w) \mapsto w \otimes v$$

by (689) there is a unique $\mathbb{K}$-linear mapping

$$\rho : V \otimes_{\mathbb{K}} W \to W \otimes_{\mathbb{K}} V : v \otimes w \mapsto w \otimes v.$$

Similarly, there is a unique $\mathbb{K}$-linear mapping

$$\rho' : W \otimes_{\mathbb{K}} V \to V \otimes_{\mathbb{K}} W : w \otimes v \mapsto v \otimes w.$$

The maps ρ, ρ' are inverse to each other and so isomorphisms. □

Tensor Product Space

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space.

The Hilbert space

$$\mathcal{H}^{\otimes n} = \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H} \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathcal{H}, \quad n \text{ times},$$

is the n-fold tensor product of $\mathcal{H}$ over $\mathbb{C}$ equipped with the $\mathcal{H}$-induced scalar product

$$\langle v_1 \otimes \ldots \otimes v_n, w_1 \otimes \ldots \otimes w_n \rangle = \prod_{i=1}^{n} \langle v_i, w_i \rangle_{\mathcal{H}},$$

where $\langle , \rangle_{\mathcal{H}}$ denotes the scalar product on $\mathcal{H}$.

Unitary Operators

Let $\mathcal{H}$ be $\mathbb{C}$-Hilbert space. For each bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ there is a uniquely determined bounded linear operator $A^* : \mathcal{H} \to \mathcal{H}$ such that

$$\langle Au, v\rangle = \langle u, A^*v\rangle, \quad u, v \in \mathcal{H}. \tag{690}$$

A^* is the *adjoint operator* of A.

Proof.

For each $v \in \mathcal{H}$ define the linear functional

$$u \mapsto \langle Au, v\rangle.$$

By the Riesz representation theorem, there is a unique $w \in \mathcal{H}$ such that

$$\langle Au, v\rangle = \langle u, w\rangle.$$

Putting $A^*v = w$ gives $\langle Au, v\rangle = \langle u, A^*v\rangle$. □

Unitary Operators

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space. Let $A, B : \mathcal{H} \to \mathcal{H}$ be bounded linear operators and $\lambda \in \mathbb{C}$. Then

$$(A+B)^* = A^* + B^*, \quad (AB)^* = B^* A^*, \tag{691}$$

and

$$(\lambda A)^* = \overline{\lambda} A^*, \quad (A^*)^* = A. \tag{692}$$

Proof.

Let $u, v \in \mathcal{H}$.

$$\begin{aligned} \langle (A+B)u, v\rangle &= \langle Au, v\rangle + \langle Bu, v\rangle = \langle u, A^* v\rangle + \langle u, B^* v\rangle \\ &= \langle u, (A^* + B^*)v\rangle. \end{aligned}$$

By the uniqueness of the adjoint operator, $(A+B)^* = A^* + B^*$.

Proof (cont'd)

Let $u, v \in \mathcal{H}$.

$$\langle (AB)u, v\rangle \quad = \quad \langle A(Bu), v\rangle = \langle Bu, A^*v\rangle = \langle u, B^*(A^*v)\rangle.$$

By the uniqueness of the adjoint operator, $(AB)^* = B^*A^*$.
Let $\lambda \in \mathbb{C}$.

$$\begin{aligned}\langle (\lambda A)u, v\rangle &= \langle A(\lambda u), v\rangle = \langle \lambda u, A^*v\rangle = \overline{\lambda}\langle u, A^*v\rangle \\ &= \langle u, \overline{\lambda}A^*v\rangle.\end{aligned}$$

By the uniqueness of the adjoint operator, $(\lambda A)^* = \overline{\lambda}A^*$.
Finally,

$$\langle A^*u, v\rangle \quad = \quad \overline{\langle v, A^*u\rangle} = \overline{\langle Av, u\rangle} = \langle u, Av\rangle.$$

But $\langle A^*u, v\rangle = \langle u, (A^*)^*v\rangle$ and so by the uniqueness of the adjoint operator, $(A^*)^* = A$. □

Unitary Operators

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ be a linear operator. If A is presented by a matrix M_A, the adjoint operator A^* is represented by the conjugate transpose of M_A.

Proof.

Let $\{e_1, \ldots, e_n\}$ be a Hilbert basis of $\mathcal{H}$. Let $M_A = (l_{ij})$ and $M_{A^*} = (m_{ij})$ be representation matrices of A and A^*, resp. Then

$$Ae_i = \sum_{k=1}^{n} l_{ki} e_k \quad \text{and} \quad A^* e_i = \sum_{k=1}^{n} m_{ki} e_k.$$

Thus

$$\langle Ae_i, e_j \rangle = \sum_{k=1}^{n} \bar{l}_{ki} \langle e_k, e_j \rangle = \bar{l}_{ji}$$

and

$$\langle e_i, A^* e_j \rangle = \sum_{k=1}^{n} m_{kj} \langle e_i, e_k \rangle = m_{ij}.$$

But $\langle Ae_i, e_j \rangle = \langle e_i, A^* e_j \rangle$ and so $m_{ij} = \bar{l}_{ji}$. □

Unitary Operators

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space.

- A linear operator $A : \mathcal{H} \to \mathcal{H}$ is *unitary* if

$$A^*A = \mathrm{id}_{\mathcal{H}} = AA^*, \tag{693}$$

  where A^* is the adjoint operator of A and $\mathrm{id}_{\mathcal{H}}$ is the identity operator.

- For each unitary operator $A : \mathcal{H} \to \mathcal{H}$ and each $v \in \mathcal{H}$ with $\|v\|^2 = 1$,

$$\|Av\|^2 = \langle Av, Av\rangle = \langle v, A^*Av\rangle = \langle v, v\rangle = \|v\|^2 = 1. \tag{694}$$

- In quantum computing, unitary operators on Hilbert spaces are called *quantum gates*.

Unitary Operators

Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space.
Let

$$A_k : \mathcal{H}^{\otimes n_k} \to \mathcal{H}^{\otimes n_k}, \quad n_k \geq 1, \tag{695}$$

be a collection of linear operators, $1 \leq k \leq m$.
The *tensor product* of these operators is the linear operator

$$A_1 \otimes \ldots \otimes A_m : \mathcal{H}^{\otimes (n_1 + \ldots + n_m)} \to \mathcal{H}^{\otimes (n_1 + \ldots + n_m)} \tag{696}$$

given by

$$(A_1 \otimes \ldots \otimes A_m)(u_1 \otimes \ldots \otimes u_m) = (A_1 u_1) \otimes \ldots \otimes (A_m u_m), \tag{697}$$

where $u_k \in \mathcal{H}^{\otimes n_k}$, $1 \leq k \leq m$.

Unitary Operators

The adjoint operator of $A_1 \otimes \ldots \otimes A_m$ is

$$(A_1 \otimes \ldots \otimes A_m)^* = A_1^* \otimes \ldots \otimes A_m^*. \tag{698}$$

Proof.

Case $m = 2$:

$$\begin{aligned} & \langle (A_1 \otimes A_2)(u_1 \otimes u_2), v_1 \otimes v_2 \rangle \\ = \;& \langle A_1 u_1 \otimes A_2 u_2, v_1 \otimes v_2 \rangle = \langle A_1 u_1, v_1 \rangle \cdot \langle A_2 u_2, v_2 \rangle \\ = \;& \langle u_1, A_1^* v_1 \rangle \cdot \langle u_2, A_2^* v_2 \rangle = \langle u_1 \otimes u_2, A_1^* v_1 \otimes A_2^* v_2 \rangle \\ = \;& \langle u_1 \otimes u_2, (A_1^* \otimes A_2^*)(v_1 \otimes v_2) \rangle. \end{aligned}$$

By the uniqueness of the adjoint operator, $(A_1 \otimes A_2)^* = A_1^* \otimes A_2^*$. □

Unitary Operators

If $A_1, \ldots, A_m$ are unitary operators, then $A_1 \otimes \ldots \otimes A_m$ is also unitary.

Proof.

Case $m = 2$:

$$\begin{aligned}(A_1 \otimes A_2)^*(A_1 \otimes A_2)(u_1 \otimes u_2) &= (A_1^* \otimes A_2^*)(A_1 u_1 \otimes A_2 u_2) \\ &= A_1^* A_1 u_1 \otimes A_2^* A_2 u_2 \\ &= u_1 \otimes u_2.\end{aligned}$$

Thus $(A_1 \otimes A_2)^*(A_1 \otimes A_2)$ is the identity operator. □