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Circumference of essentially 4-connected planar triangulations

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Abstract. A 3-connected graph G is essentially 4-connected if, for any 3-cut $S \subseteq V(G)$ of G, at most one component of $G - S$ contains at least two vertices. We prove that every essentially 4-connected maximal planar graph G on n vertices contains a cycle of length at least $\frac{2}{3}(n+4)$; moreover, this bound is sharp.

Keywords: circumference, long cycle, triangulation, essentially 4-connected, planar graph 2010 *MSC:* 05C38, 05C10

1 Introduction and Preliminaries

We consider finite, simple, and undirected graphs. The *circumference* circ(G) of a graph G is the length of a longest cycle of G. A cycle C of G is an *outer independent cycle* of G if the set $V(G) \setminus V(C)$ is independent. (Note that an outer independent cycle is sometimes called a dominating cycle ([3]), although this is in contrast to the more commonly used definition of a dominating subgraph H of G, where $V(H)$ dominates $V(G)$ in the usual sense.) A set $S \subseteq V(G)$ ($S \subseteq E(G)$) is a k-*cut* (a k-*edge-cut*) of G if $|S| = k$ and $G - S$ is disconnected. A 3-cut (a 3-edge-cut) S of a 3-connected (3-edge-connected) graph G is *trivial* if at most one component of $G - S$ contains at least two vertices and the graph G is *essentially* 4-*connected* (*essentially* 4-*edge-connected*) if every 3-cut (3-edge-cut) of G is trivial. A 3-edge-connected graph G is *cyclically* 4-*edge-connected* if for every 3-edge-cut S of G, at most one component of $G - S$ contains a cycle.

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It is well-known that for (3-connected) cubic graphs different from the triangular prism $K_3 \times K_2$ (which is essentially 4-connected only) these three notions coincide (see e.g. [6] and [16]). Obviously, the line graph $H = L(G)$ of a 3-connected graph G is 4-connected if and only if G is essentially 4-edge-connected. These two observations are reasons for the quite great interest in studying all these three concepts of connectedness of graphs intensively.

Zhan [17] proved that every 4-edge-connected graph has a Hamiltonian line graph. Broersma [3] conjectured that even every essentially 4-edge-connected graph has a Hamiltonian line graph and showed that this is equivalent to the conjecture of Thomassen [14] stating that every 4-connected line graph is Hamiltonian (which is known to be equivalent to the conjecture by Matthews and Sumner [12] stating that every 4-connected claw-free graph is Hamiltonian, as shown by Ryjáček [13]). Among others, the subclass of essentially 4-edge-connected cubic graphs is interesting due to a conjecture of Fleischner and Jackson [6] stating that every essentially 4-edge-connected cubic graph has an outer independent cycle which is equivalent to the previous three conjectures.

Regarding to the existence of long cycles in essentially 4-connected graphs we mention the following

Conjecture 1 (Bondy, see [8]) *There exists a constant c, $0 < c < 1$, such that for every essentially 4-connected cubic graph on n vertices,* $\mathrm{circ}(G) \geq cn$*.*

Note that the conjecture of Fleischner and Jackson implies Conjecture 1 with $c = \frac{3}{4}$. Bondy's conjecture was later extended to all cyclically 4-edge-connected graphs (see [6]). Máčajová and Mazák [11] constructed essentially 4-connected cubic graphs on $n = 8m$ vertices with circumference $7m + 2$. We remark that the conjecture of Fleischner and Jackson and, therefore, also Bondy's Conjecture with $c = \frac{3}{4}$ (this is the result of Grünbaum and Malkevitch [7]) are true for planar graphs, which can be seen easily by the forthcoming Lemma 1. Many results concerning the circumference of essentially 4-connected planar graphs G can be found in the literature.

For the class of essentially 4-connected cubic planar graphs, Tutte [15] showed that it contains a non-Hamiltonian graph, Aldred, Bau, Holton, and McKay [1] found a smallest non-Hamiltonian graph on 42 vertices, and Van Cleemput and Zamfirescu [16] constructed a non-Hamiltonian graph on n vertices for all even $n \geq 42$. As already mentioned, Grünbaum and Malkevitch [7] proved that $\mathrm{circ}(G) \geq \frac{3}{4}n$ for any essentially 4-connected cubic planar graph G on n vertices and Zhang [18] (using the theory of Tutte paths) improved this lower bound on the circumference by 1. Recently, in [10], an infinite family of essentially 4-connected cubic planar graphs on n vertices with circumference $\frac{359}{366}n$ was constructed.

In [9], Jackson and Wormald extended the problem to find lower bounds on the circumference to the class of arbitrary essentially 4-connected planar graphs. Their result $\mathrm{circ}(G) \geq \frac{2n+4}{5}$ was improved in [5] to $\mathrm{circ}(G) \geq \frac{5}{8}(n+2)$ for every essentially 4-connected planar graph G on n vertices. On the other side, there are infinitely many essentially 4-connected maximal planar graphs G with $\mathrm{circ}(G) = \frac{2}{3}(n+4)$ ([9]). To see this, let G' be a 4-connected maximal planar graph on $n' \geq 6$ vertices and let G be obtained from G' by inserting a new vertex into each face of G' and connecting it with all three boundary vertices of that face. Then G is an essentially 4-connected maximal planar graph on $n = 3n' - 4$ vertices and, since G' is Hamiltonian, it is easy to see that $\mathrm{circ}(G) = 2n' = \frac{2}{3}(n+4)$. It is still open whether there is an essentially 4-connected planar graph G that satisfies $\mathrm{circ}(G) < \frac{2}{3}(n+4)$. Indeed, we pose the following (to our knowledge so far unstated) Conjecture 2, which has been the driving force in that area for over a decade.

Conjecture 2 *For every essentially 4-connected planar graph on n vertices,* $\mathrm{circ}(G) \geq \frac{2}{3}(n+4)$*.*

By the forthcoming Theorem 1, Conjecture 2 is shown to be true for essentially 4-connected maximal planar graphs.

We remark that $G - S$ has exactly two components for every 3-connected planar graph G and every 3-cut S of G. Thus, in this case, G is essentially 4-connected if and only if S forms the neighborhood of a vertex of degree 3 of G for every 3-cut S of G. This property will be used frequently in the proof of Theorem 1.

A cycle C of G is a *good cycle* of G if C is outer independent and $\deg_G(x) = 3$ for all $x \in V(G) \setminus V(C)$. An edge xy of a good cycle C is *extendable* if x and y have a common neighbor $z \in V(G) \setminus V(C)$. In this case, the cycle C' of G, obtained from C by replacing the edge xy with the path (x, z, y) is again good (and longer than C). The forthcoming Lemma 1 is an essential tool in the proof of Theorem 1 (an implicit proof for cubic essentially 4-connected planar graphs can be found in [7], the general case is proved in [4]).

Lemma 1 *Every essentially* 4*-connected planar graph on* $n \geq 11$ *vertices contains a good cycle.*

Theorem 1 *For every essentially* 4*-connected maximal planar graph* G *on* $n \geq 8$ *vertices,*

$$\mathrm{circ}(G) \geq \frac{2}{3}(n + 4).$$

2 Proof of Theorem 1

Suppose $n \geq 11$, as for $n \in \{8, 9, 10\}$, Theorem 1 follows from the fact that G is Hamiltonian ([2]). Using Lemma 1, let $C = [v_1, v_2, \ldots, v_k]$ (indices of vertices of C are taken modulo k in the whole paper) be a longest good cycle of length k of G (i.e., $\mathrm{circ}(G) \geq k$) and let $H = G[V(C)]$ be the graph obtained from G by removing all vertices of degree 3 which do not belong to C. Obviously, H is maximal planar and C is a Hamiltonian cycle of H. A face φ of H is an *empty face* of H if φ is also a face of G, otherwise φ is a *non-empty face* of H. Denote by $F_{\mathrm{e}}(H)$ the set of empty faces of H and let $f_{\mathrm{e}}(H) = |F_{\mathrm{e}}(H)|$. Note that every face of G has at least two (of three) vertices on C. The three neighbors of a vertex of $V(G) \setminus V(C)$ induce a separating 3-cycle of G creating the boundary of a non-empty face of H, which has no edge in common with C because otherwise such an edge would be an extendable edge of C in G.

Let H_1 and H_2 be the spanning subgraphs of H consisting of the cycle C and of its chords lying in the interior and in the exterior of C, respectively. Note that $E(H_1) \cap E(H_2) = E(C)$ and H_1 and H_2 are maximal outerplanar graphs, both having k-gonal outer face and $k - 2$ triangular faces. Let T_i be the weak dual of H_i, $i \in \{1, 2\}$, which is the graph having all triangular faces of H_i as vertex set such that two vertices of T_i are adjacent if the triangular faces share an edge in H_i. Obviously, T_i is a tree of maximum degree at most three.

A face φ of H is a *j-face* if exactly j of its three incident edges belong to $E(C)$. Since $n \geq 11$, there is no 3-face in H and each face of H is a j-face with $j \in \{0, 1, 2\}$. Denote by $f_j(H_i)$ the number of empty j-faces of H_i. Since C does not contain any extendable edge, the following claim is obvious.

Claim 1 *Each face of* H *incident with an edge of any longest good cycle (in particular, each* 1*- or* 2*-face) is empty.*

An edge e of C incident with a j-face φ and an ℓ-face ψ, where $j, \ell \in \{1, 2\}$, is a *(j, ℓ)-edge*. Let φ be a 2-face of H_i. The sequence $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$, $r \geq 2$, is the *φ-branch* if $\varphi_2, \ldots, \varphi_{r-1}$ are 1-faces of H_i, φ_r is a 0-face of H_i, and φ_j, φ_{j+1} $(1 \leq j \leq r - 1)$ are adjacent (i.e. B_φ is a minimal path in T_i with end vertices of degree 1 and 3). The *rim* $R(B_\varphi)$ of the φ-branch B_φ is the subgraph of C induced by all edges of C that are incident with an element of B_φ. Hence, it is easy to see:

Claim 2 *The rim of a φ-branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ is a path of length r.*

Claim 3 *Let $\varphi = [v_1, v_2, v_3]$ be a 2-face of H_i, let $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$, $r \geq 2$, be the φ-branch of H_i, and let $v_0v_2 \in E(H_{3-i})$. If*

(a) *$R(B_\varphi) = (v_1, v_2, \ldots, v_{r+1})$ is the rim of B_φ or*

(b) *$R(B_\varphi) = (v_0, v_1, \ldots, v_r)$ is the rim of B_φ and $v_{-1}v_2 \in E(H_{3-i})$, or*

(c) *$R(B_\varphi) = (v_{3-r}, \ldots, v_2, v_3)$ is the rim of B_φ and $v_{-1}v_2 \in E(H_{3-i})$,*

then φ_r is empty.

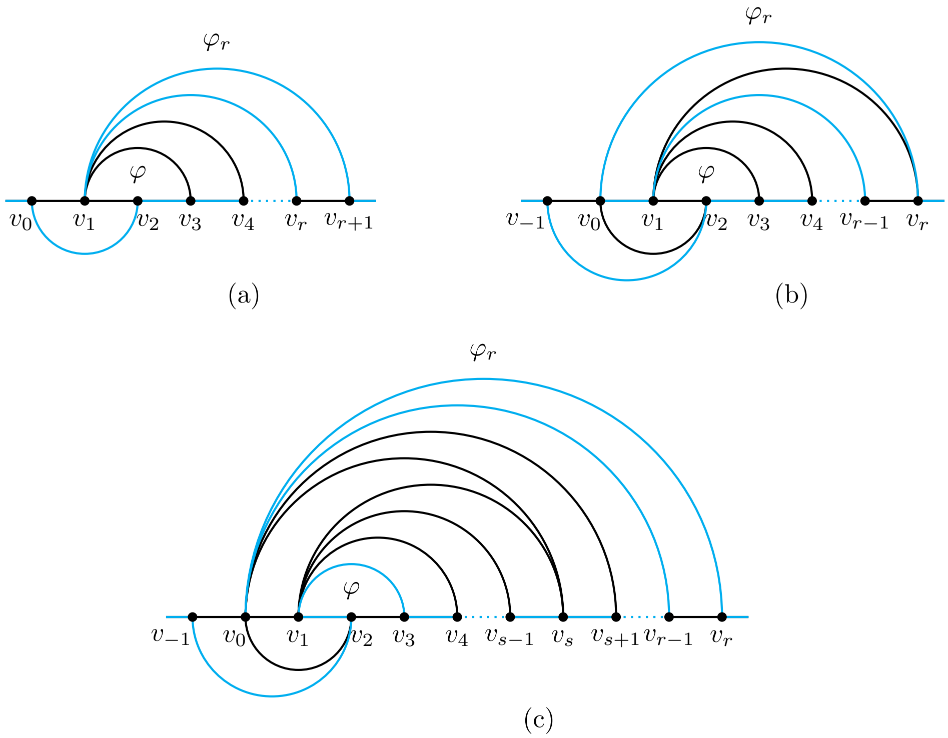

Fig. 1. A longest good cycle (cyan) sharing an edge with φ_r.

Proof.

(a) The cycle C' obtained from C by replacing the path $(v_0, v_1, \ldots, v_{r+1})$ with the path $(v_0, v_2, \ldots, v_r, v_1, v_{r+1})$ (Fig. 1(a)) is another longest good cycle of G and contains the edge v_1v_{r+1} incident with φ_r, thus φ_r is empty (by Claim 1).

(b) Let $\varphi_s = [v_0, v_1, v_s]$, for some s with $3 \leq s \leq r$, be a 1-face of H_i. The cycle C' obtained from C by replacing the path $(v_{-1}, v_0, \ldots, v_r)$ by the path $(v_{-1}, v_2, \ldots, v_{r-1}, v_1, v_0, v_r)$, for $s = r$

(Fig. 1(b)), or by the path $(v_{-1}, v_2, v_1, v_3, \dots, v_{r-1}, v_0, v_r)$, for $s \le r-1$ (Fig. 1(c)), is a longest good cycle of G and contains the edge v_0v_r incident with φ_r, thus φ_r is empty (by Claim 1).

(c) If $r \le 3$, then φ_r is empty by (a) or (b). If $r \ge 4$, then $v_0v_3, v_{-1}v_3 \in E(H_i)$, thus $\{v_{-1}, v_2, v_3\}$ is a non-trivial 3-cut, a contradiction. □

These tools will be used continuously in the following; we continue with the proof of Theorem 1. Hereby, we consider two cases. In the first case, both subgraphs H_1 and H_2 have some 0-faces. By using a customized discharging method, we distribute some weights from edges to faces to prove that sufficiently many faces are empty (each empty face will finally contain weight at most $\frac{2}{3}$). In the second case, there are only empty faces on one side of C, so that all vertices not in C are located on the other side of C. We have to prove that there are some additional empty faces on this side.

CASE 1. Let H_1 and H_2 both contain at least two 0-faces or one non-empty 0-face.

For every edge e of C we define the weight $w_0(e) = 1$. Obviously, $\sum_{e \in E(C)} w_0(e) = |E(C)| = k$.

First redistribution of weights.

Each edge of C sends weight to both incident faces as follows

Rule R1. A (1,1)-edge sends $\frac{1}{2}$ to both incident 1-faces.
Rule R2. A (1,2)-edge sends $\frac{2}{3}$ to the incident 1-face and $\frac{1}{3}$ to the incident 2-face.
Rule R3. A (2,2)-edge sends $\frac{1}{2}$ to both incident 2-faces.

The edges of C completely redistribute their weights to incident 1- and 2-faces. For an empty face φ, let $w_1(\varphi)$ be the total weight obtained by φ (in first redistribution). Obviously, for an empty face φ, it is

$$w_1(\varphi) = \begin{cases} 1, & \text{if } \varphi \text{ is a 2-face incident with two (2,2)-edges,} \\ \frac{5}{6}, & \text{if } \varphi \text{ is a 2-face incident with a (1,2)-edge and a (2,2)-edge,} \\ \frac{2}{3}, & \text{if } \varphi \text{ is a 2-face incident with two (1,2)-edges,} \\ \frac{2}{3}, & \text{if } \varphi \text{ is a 1-face incident with a (1,2)-edge,} \\ \frac{1}{2}, & \text{if } \varphi \text{ is a 1-face incident with a (1,1)-edge,} \\ 0, & \text{if } \varphi \text{ is a 0-face.} \end{cases}$$

Moreover, $\sum_{\varphi \in F_e(H)} w_1(\varphi) = |E(C)| = k$.

Second redistribution of weights.

The weight of 2-faces of H exceeding $\frac{2}{3}$ will be redistributed to 1-faces and empty 0-faces of H by the following rules. Let φ be a 2-face of H_i with $w_1(\varphi) > \frac{2}{3}$ (i.e. incident with at least one (2,2)-edge) and let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \ge 2$, be the φ-branch. Moreover, let α be a 2-face of H_{3-i} adjacent to φ and let α_2 be the face of H_{3-i} adjacent to α.

Rule R4. φ sends $w_1(\varphi) - \frac{2}{3}$ to φ_r if φ_r is empty and $r \leq 3$.

Rule R5. φ sends $\frac{1}{6}$ to φ_j if φ_j $(2 \leq j \leq r-1)$ is a 1-face incident with a (1,1)-edge.

Rule R6. φ sends $\frac{1}{6}$ to φ_r if φ_r is empty and $r \geq 4$.

Rule R7. φ sends $\frac{1}{6}$ to α_2 if α is incident with a (1,2)-edge and α_2 is an empty 0-face.

Rule R8. φ sends $\frac{1}{6}$ to β_2, where β is a 2-face of H_{3-i} having exactly one common vertex with φ and incident with two (1,2)-edges and β_2 is an empty 0-face of H_{3-i} adjacent to β.

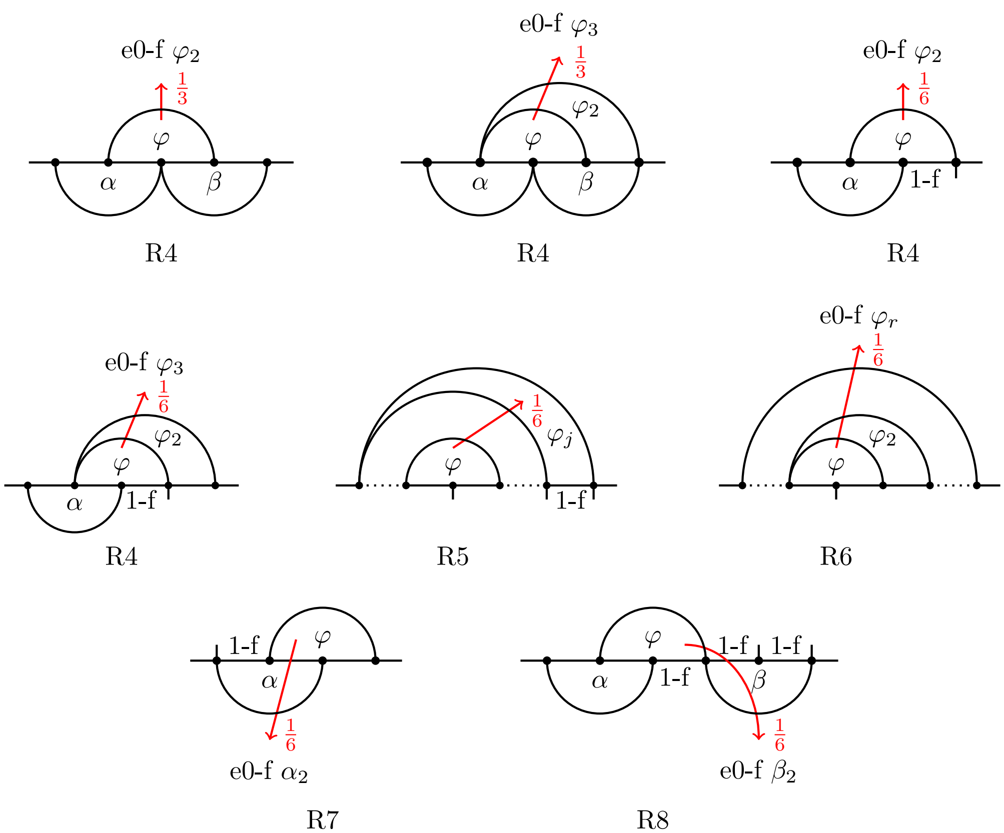

Fig. 2. Redistribution rules R4–8 (1-f is a 1-face and e0-f is an empty 0-face).

For an empty face φ, let $w_2(\varphi)$ be the total weight obtained by φ (after second redistribution). Obviously, $\sum_{\varphi \in F_e(H)} w_2(\varphi) = |E(C)| = k$ (as non-empty faces do not obtain any weight). In the following, we will show that the weight $w_2(\varphi)$ of each (empty) face φ does not exceed $\frac{2}{3}$ which will mean $k = \sum_{\varphi \in F_e(H)} w_2(\varphi) \leq \frac{2}{3} f_e(H)$. The maximal planar graph G has exactly $2n - 4$ faces. Each of $f_e(H) \geq \frac{3}{2}k$ empty faces of H is a face of G as well, and each of $n - k$ (pairwise non-adjacent) vertices of G not belonging to C (whose removal has created a non-empty face of H) is incident with three ("private") faces of G. Hence $2n - 4 = |F(G)| = f_e(H) + 3(n - k) \geq \frac{3}{2}k + 3n - 3k$ and finally $k \geq \frac{2}{3}(n + 4)$ will follow.

Weight of a 2-face.

Let $\varphi = [v_1, v_2, v_3]$ be a 2-face of H_i and let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \geq 2$, be the φ-branch. As already mentioned, $\frac{2}{3} \leq w_1(\varphi) \leq 1$. We check that the weight of φ exceeding $\frac{2}{3}$ will be shifted in the second redistribution.

1. Let φ be incident with two (2,2)-edges (note that $w_1(\varphi) = 1$). Denote $\alpha = [v_0, v_1, v_2]$ and $\beta = [v_2, v_3, v_4]$ the 2-faces of H_{3-i} adjacent to φ. Let α_2 and β_2 be the face of H_{3-i} adjacent to α and β, respectively. Each of the faces φ_2, α_2, and β_2 is either a 1-face or empty 0-face (by Claim 3a).

1.1. Let α_2 and β_2 be 0-faces (possibly $\alpha_2 = \beta_2$).
1.1.1. If edges v_0v_1 and v_3v_4 of C do not belong to the rim $R(B_\varphi)$ of B_φ, then $r = 2$, thus φ sends $\frac{1}{3}$ to empty 0-face φ_2 (by R4).
1.1.2. If v_0v_1 belongs to the rim $R(B_\varphi)$ and v_3v_4 does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_0, v_1, v_3]$ is a 1-face and φ_r is empty (by Claim 3a). Thus φ sends weight $\geq \frac{1}{6}$ to φ_r (by R4 or R6) and $\frac{1}{6}$ to α_2 (by R7). (Similarly if v_0v_1 does not belong to $R(B_\varphi)$ and v_3v_4 belongs to $R(B_\varphi)$.)
1.1.3. If edges v_0v_1 and v_3v_4 belong to the rim $R(B_\varphi)$, then both are (1,2)-edges. Thus φ sends $\frac{1}{6}$ to α_2 and $\frac{1}{6}$ to β_2 (by R7).

1.2. Let $\alpha_2 = [v_{-1}, v_0, v_2]$ be a 1-face and β_2 be a 0-face. (Similarly if α_2 is a 0-face and β_2 is a 1-face.)
1.2.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r \leq 3$ and φ_r is empty (by proof of Claim 3c). Thus φ sends $\frac{1}{3}$ to φ_r (by R4).
1.2.2. If v_3v_4 belongs to the rim $R(B_\varphi)$ and v_0v_1 does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_1, v_3, v_4]$ is a 1-face and φ_r is empty (by Claim 3a). Thus φ sends weight $\geq \frac{1}{6}$ to φ_r (by R4 or R6) and $\frac{1}{6}$ to β_2 (by R7).
1.2.3. Let edges v_3v_4 and v_0v_1 belong to the rim $R(B_\varphi)$, then both are (1,2)-edges. If v_0v_1 and v_3v_4 are incident with φ_2 and φ_3, then $\{v_0, v_2, v_4\}$ is a non-trivial 3-cut, a contradiction. If $\varphi_2 = [v_0, v_1, v_3]$ and $\varphi_3 = [v_{-1}, v_0, v_3]$, then $\{v_{-1}, v_2, v_3\}$ is a non-trivial 3-cut, a contradiction as well. Thus $\varphi_2 = [v_1, v_3, v_4]$ and $\varphi_3 = [v_1, v_4, v_5]$.
1.2.3.1. If $v_{-1}v_0$ does not belong to the rim $R(B_\varphi)$, then φ_r is empty (by Claim 3b). Thus φ sends $\frac{1}{6}$ to φ_r (by R6) and $\frac{1}{6}$ to β_2 (by R7).
1.2.3.2. If $v_{-1}v_0$ belongs to the rim $R(B_\varphi)$, then $v_{-1}v_0$ is a (1,1)-edge. Thus φ sends $\frac{1}{6}$ to φ_j, a 1-face of B_φ incident with $v_{-1}v_0$ (by R5) and $\frac{1}{6}$ to β_2 (by R7).

1.3. Let $\alpha_2 = [v_{-1}, v_0, v_2]$ and $\beta_2 = [v_2, v_4, v_5]$ be 1-faces.
1.3.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r \leq 3$ and φ_r is empty (by proof of Claim 3c). Thus φ sends $\frac{1}{3}$ to φ_r (by R4). (Similarly if v_0v_1 does not belong to $R(B_\varphi)$.)
1.3.2. Let edges v_0v_1 and v_3v_4 belong to the rim $R(B_\varphi)$, then both are (1,2)-edges. If v_0v_1 and v_3v_4 are incident with φ_2 and φ_3, then $\{v_0, v_2, v_4\}$ is a non-trivial 3-cut, a contradiction. If $\varphi_2 = [v_0, v_1, v_3]$ and $\varphi_3 = [v_{-1}, v_0, v_3]$, then $\{v_{-1}, v_2, v_3\}$ is a non-trivial 3-cut, a contradiction as well. (Similarly if $\varphi_2 = [v_1, v_3, v_4]$ and $\varphi_3 = [v_1, v_4, v_5]$.)

2. Let φ be incident with (2,2)-edge v_1v_2 and (1,2)-edge v_2v_3 (note that $w_1(\varphi) = \frac{5}{6}$). Denote $\alpha = [v_0, v_1, v_2]$ the 2-face of H_{3-i} adjacent to φ and let α_2 be the face of H_{3-i} adjacent to α. Each of the faces φ_2 and α_2 is either a 1-face or empty 0-face (by Claim 3a).

2.1. Let α_2 be 0-face.
2.1.1. If v_0v_1 does not belong to the rim $R(B\varphi)$, then φ_r is empty (by Claim 3a). Thus φ sends $\frac{1}{6}$ to φ_r (by R4 or R6).
2.1.2. If v_0v_1 belongs to the rim $R(B\varphi)$, then v_0v_1 is a (1,2)-edge. Thus φ sends $\frac{1}{6}$ to α_2 (by R7).

2.2. Let α_2 be a 1-face incident with $v_{-1}v_0$ (i.e. $\alpha_2 = [v_{-1}, v_0, v_2]$).
2.2.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r \le 3$ and φ_r is empty (by proof of Claim 3c). Thus φ sends $\frac{1}{6}$ to φ_r (by R4).
2.2.2. If v_3v_4 belongs to the rim $R(B_\varphi)$ and v_0v_1 does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_1, v_3, v_4]$ is a 1-face and φ_r is empty (by Claim 3a). Thus φ sends $\frac{1}{6}$ to φ_r (by R4 or R6).
2.2.3. Let edges v_3v_4 and v_0v_1 belong to the rim $R(B_\varphi)$. If $v_{-1}v_0$ does not belong to $R(B_\varphi)$, then φ_r is empty (by Claim 3b). Thus φ sends $\frac{1}{6}$ to φ_r (by R6). Otherwise $v_{-1}v_0$ belongs to $R(B_\varphi)$, thus it is a (1,1)-edge incident with a 1-face φ_j of B_φ. Hence φ sends $\frac{1}{6}$ to φ_j (by R5).

2.3. Let α_2 be a 1-face incident with v_2v_3 (i.e. $\alpha_2 = [v_0, v_2, v_3]$). Since $v_0v_3 \in E(H_{3-i})$, φ_2 cannot be the 1-face $[v_0, v_1, v_3]$ in H_i.
2.3.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r = 2$, thus φ sends $\frac{1}{6}$ to φ_2 (by R4).
2.3.2. If v_3v_4 belongs to the rim $R(B_\varphi)$, then $r \ge 3$ and $\varphi_2 = [v_1, v_3, v_4]$.
2.3.2.1. If v_3v_4 is incident with a 1-face of H_{3-i} (i.e., v_3v_4 is a (1,1)-edge), then φ sends $\frac{1}{6}$ to φ_2 (by R5).
2.3.2.2. Let v_3v_4 be incident with a 2-face β of H_{3-i} (necessarily, $\beta = [v_3, v_4, v_5]$). If $r = 3$, then φ_3 is empty (by Claim 3a), thus φ sends $\frac{1}{6}$ to φ_3 (by R4). If $r = 4$, then $\varphi_3 = [v_1, v_4, v_5]$ (as $\{v_0, v_3, v_4\}$ is a non-trivial 3-cut if $\varphi_3 = [v_0, v_1, v_4]$) and φ_4 is empty (by Claim 3a), thus φ sends $\frac{1}{6}$ to φ_4 (by R6). Finally, let $r \ge 5$. Necessarily $\varphi_3 = [v_1, v_4, v_5]$ (as for $r = 4$) and $\varphi_4 = [v_1, v_5, v_6]$ (as $\{v_0, v_3, v_5\}$ is a non-trivial 3-cut if $\varphi_4 = [v_0, v_1, v_5]$) are 1-faces of B_φ. If v_5v_6 is a (1,1)-edge, then φ sends $\frac{1}{6}$ to φ_4 (by R5). Otherwise v_5v_6 is a (1,2)-edge, thus it does not belong to β-branch (in H_{3-i}) and therefore β_2 is a 0-face, which is, moreover, empty (as the cycle obtained from C by replacing the path $(v_0, \ldots, v_5)$ by the path $(v_0, v_2, v_1, v_4, v_3, v_5)$ is a longest good cycle of G and contains the edge v_3v_5 incident with β_2 (Claim 1)). Hence φ sends $\frac{1}{6}$ to β_2 (by R8).

Weight of a 1-face.

To estimate the weight of a 1-face, we use the following simple observation:

Claim 4 *Each 1-face of H belongs to at most one branch.*

Let ψ be a 1-face incident with an edge e of C. If e is a (1,2)-edge, then ψ obtains weight $\frac{2}{3}$ from e (by R2) only. Otherwise e is a (1,1)-edge, thus ψ obtains $\frac{1}{2}$ from e (by R1). Furthermore, in this case, ψ can get $\frac{1}{6}$ from a 2-face φ (by R5) if ψ belongs to the φ-branch. Hence $w_2(\psi) \le \frac{2}{3}$.

Weight of an empty 0-face.

Each empty 0-face ω belongs to at most two branches (in Case 1). Let φ be a 2-face of H_i with the φ-branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ such that $\varphi_r = \omega$, and let e be the edge incident with φ_r and φ_{r-1} (where $\varphi_{r-1} = \varphi$ for $r = 2$).

If φ is adjacent to two 2-faces, then ω gets through e the weight $\frac{1}{3}$ (by R4) for $r \le 3$ or the weight $\frac{1}{6}$ (by R6) for $r \ge 4$. If φ is adjacent to one 2-face, then ω gets through e the weight $\frac{1}{6}$ (by R4) and additionally $\frac{1}{6}$ (by R7) for $r = 2$ or the weight at most $\frac{1}{6}$ (by R4) for $r = 3$ or the weight $\frac{1}{6}$ (by R6) for $r \ge 4$. Finally, if φ is adjacent to no 2-face, then ω gets through e the weight $\frac{1}{6}$ (by R6) for $r \ge 4$ or the weight at most $2 \times \frac{1}{6}$ (by R8) for $r \le 3$.

We showed that $w_2(\varphi) \le \frac{2}{3}$ for each empty face φ and completed the Case 1. Thus, we can assume that in H_i are only empty faces and among them, at most one face is a 0-face. To complete the proof, we have to show that there are some empty faces in H_{3-i} as well.

CASE 2. Let H_i contain no 0-face or exactly one 0-face which is additionally empty.

Obviously, if H_i contains no 0-face, then it contains two 2-faces α_1 and α_2 (since T_i is a path and 2-faces of H_i are leaves of T_i). Note that, (only) in this case, the branches in H_i are not defined.

Remember that $H = G[V(C)]$ has $k \geq 7$ vertices (as otherwise G with at most $k + 2 \leq 8$ vertices is Hamiltonian). If H_i contains exactly one 0-face, then it contains three 2-faces α_1, α_2 and α_3 (since T_i is a subdivision of $K_{1,3}$ and 2-faces of H_i are leaves of T_i). We assume that H_{3-i} contains at least two 0-faces as otherwise all but at most one faces of H_{3-i} are empty and G has $n \leq |V(H)| + 1 = k + 1$ vertices and Theorem 1 follows immediately (with $n \geq 11$).

Distribution of points.

To estimate the number of empty 0- and 1-faces in H_{3-i}, each 2-face α_j of H_i ($j \in \{1, 2\}$ if H_i contains no 0-face and $j \in \{1, 2, 3\}$ if H_i contains one 0-face, respectively) will distribute 1 or 2 points to faces of H_{3-i}. Let α_j be adjacent to the faces φ and ψ of H_{3-i}.

Rule P1. If φ and ψ are 2-faces of H_{3-i} with branches $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ and $B_\psi = (\psi, \psi_2, \ldots, \psi_t)$, then φ_r and ψ_t will each receive 1 point (or 2 points if $\varphi_r = \psi_t$) from α_j.

Rule P2. If φ and ψ are 1-faces of H_{3-i}, then φ and ψ will each receive 1 point from α_j.

Rule P3. If φ is a 2-faces of H_{3-i} with φ-branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ and ψ is a 1-face of H_{3-i} not belonging to B_φ, then φ_r and ψ will each receive 1 point from α_j.

Rule P4. If φ is a 2-faces of H_{3-i} with φ-branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ and ψ is a 1-face of H_{3-i} belonging to B_φ, then only ψ will receive 1 point from α_j.

For a face φ of H_{3-i}, let $p(\varphi)$ be the total number of points carried by φ (in the distribution of points).

Claim 5 $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq \sum_{\varphi \in F_e(H_{3-i})} p(\varphi)$.

Proof. We have to prove that each 1-face of H_{3-i} gets at most 1 point and that each 0-face of H_{3-i} gets points only if it is empty and it gets at most 2 points. Consequently, Claim 5 follows by simple counting.

Let β be a 1-face of H_{3-i}. Since β can only get points if it is adjacent to some α_j and there can only be one such face then $p(\beta) \leq 1$.

Let β be a 0-face of H_{3-i}. Since β can only get points if it belongs to a branch and it belongs to at most two branches (as there are at least two 0-faces in H_{3-i}), then $p(\beta) \leq 2$. Assume first that β gets a point by P1. Then there is α_j incident with two $(2, 2)$-edges and adjacent 2-faces φ and ψ of H_{3-i}. Let $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ with $\varphi_r = \beta$ be the branch which ends in β. By Claim 3a, $\varphi_r = \beta$ is an empty 0-face.

Thus, assume that β gets a point by P3. Then there is α_j incident with a $(1, 2)$-edge with adjacent 1-face ψ in H_{3-i} and a $(2, 2)$-edge with adjacent 2-face φ such that ψ does not belong to the branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ with $\varphi_r = \beta$. Since the common edge of α_j and ψ does not belong to the rim $R(B_\varphi)$, again by Claim 3a, $\varphi_r = \beta$ is an empty 0-face. □

Claim 6 $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$.

Proof. If $\sum_{\varphi \in F_e(H_{3-i})} p(\varphi) \geq 4$, then $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$ (by Claim 5). Assume $\sum_{\varphi \in F_e(H_{3-i})} p(\varphi) \leq 3$.

1. Let H_i contains exactly one 0-face. As there are three 2-faces $\alpha_1, \alpha_2, \alpha_3$ in H_i (note, that T_i is a subdivided 3-star in this case), then $\sum_{\varphi \in F_e(H_{3-i})} p(\varphi) = 3$. Furthermore, only P4 was applied to each α_j ($j \in \{1,2,3\}$) hence there are three 1-faces with 1 point and they belong to three different branches.

Since $|V(H)| = k \geq 7$, there is $j \in \{1,2,3\}$ such that α_j is adjacent to a 1-face δ of H_i. Let φ be the adjacent 2-face of α_j in H_{3-i} and $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ be its branch.

1.1. If $r \geq 4$, then φ_2 and φ_3 are 1-faces of the same branch. Thus, at most one among φ_2 and φ_3 has a point and $f_1(H_{3-i}) \geq 4$.

1.2. If $r = 3$, then δ and φ are not adjacent (i.e. $\delta \neq \varphi_2$, since H has no multiple edges) and φ_3 is an empty 0-face (by Claim 3b), hence $f_1(H_{3-i}) + f_0(H_{3-i}) \geq 4$.

2. Let H_i contains no 0-face. Since $\sum_{\varphi \in F_e(H_{3-i})} p(\varphi) \leq 3$, there is $j \in \{1,2\}$ such that P4 was applied to α_j. Let δ be the 1-face of H_i adjacent to α_j (since $|V(H)| = k \geq 7$), let φ and ψ be the 2-face and 1-face of H_{3-i} adjacent with α_j, respectively, and let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ be the branch of φ. We may assume $\alpha_j = [v_1, v_2, v_3]$ and $\varphi = [v_2, v_3, v_4]$.

2.1. Let $r \leq 4$.
2.1.1 If $\delta = [v_0, v_1, v_3]$, then v_0v_1 does not belong to the rim $R(B_\varphi)$ (otherwise $\varphi_2 = [v_1, v_2, v_4]$, $\varphi = [v_0, v_1, v_4]$ and v_0, v_3, v_4 is a non-trivial 3-cut, a contradiction) and φ_r is an empty 0-face (by Claim 3b). By P1–4, there is a face in H_{3-i} other than ψ and φ_r with a point, thus $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$.
2.1.2 If $\delta = [v_1, v_3, v_4]$, then $\varphi_2 = [v_2, v_4, v_5]$ (since $v_1v_4 \in E(H_i)$), $\psi = \varphi_3 = [v_1, v_2, v_5]$, and $\{v_1, v_4, v_5\}$ is a non-trivial 3-cut, a contradiction.

2.2. Let $r = 5$. There are three 1-faces (in fact φ_2, φ_3, and φ_4) all belonging to the same branch B_φ. We may assume that P4 was applied to α_j and P2 was applied to α_{3-j}, and all three 1-faces are adjacent to α_1 or α_2 (since otherwise there is another 1-face or empty 0-face and Claim 6 follows).
2.2.1. If $\alpha_{3-j} = [v_{-1}, v_0, v_1]$, then rim $R(B_\varphi) = (v_{-1}, \dots, v_4)$, thus $\varphi_2 = [v_1, v_2, v_4]$ and $\delta = [v_1, v_3, v_4]$, a contradiction to the simplicity of H.
2.2.2. If $\alpha_{3-j} = [v_4, v_5, v_6]$ and $\delta = [v_0, v_1, v_3]$, then rim $R(B_\varphi) = (v_1, \dots, v_6)$ and φ_5 is an empty 0-face (by Claim 3b), thus $f_1(H_{3-i}) + f_0(H_{3-i}) \geq 4$.
2.2.3. If $\alpha_{3-j} = [v_4, v_5, v_6]$ and $\delta = [v_1, v_3, v_4]$, then rim $R(B_\varphi) = (v_1, \dots, v_6)$. Hence $v_1v_6 \in E(H_{3-i})$ and consequently $\{v_1, v_4, v_6\}$ is a non-trivial 3-cut, a contradiction.

2.3. If $r \geq 6$, then there are at least four 1-faces in B_φ, thus $f_1(H_{3-i}) \geq 4$. □

Remember that each j-face of H_{3-i} is incident with j ("private") edges of C, hence $2f_2(H_{3-i}) + f_1(H_{3-i}) = k$. As each of the $k-2$ triangular faces of H_i is empty, all non-empty faces of H belong to H_{3-i} and their number is $(k-2) - f_2(H_{3-i}) - f_1(H_{3-i}) - f_0(H_{3-i}) = (k-2) - \frac{1}{2}(k - f_1(H_{3-i})) - f_1(H_{3-i}) - f_0(H_{3-i}) = \frac{k}{2} - 2 - \frac{1}{2}(f_1(H_{3-i}) + 2f_0(H_{3-i})) \leq \frac{k}{2} - 4$ (by Claim 6). Finally, at most $\frac{k}{2} - 4$ vertices of G lie outside the cycle C (and exactly k vertices on C), hence $n \leq k + (\frac{k}{2} - 4)$ and $k \geq \frac{2}{3}(n+4)$ follows, which completes the proof of Theorem 1.

References

[1] R. E. L. Aldred, S. Bau, D. A. Holton, and B. D. McKay. Nonhamiltonian 3-connected cubic planar graphs. *SIAM Journal on Discrete Mathematics*, 13(1):25–32, 2000. doi:10.1137/s0895480198348665.

[2] D. Barnette and E. Jucovič. Hamiltonian circuits on 3-polytopes. *Journal of Combinatorial Theory*, 9(1):54–59, 1970. doi:10.1016/S0021-9800(70)80054-0.

[3] H. J. Broersma. On some intriguing problems in hamiltonian graph theory—a survey. *Discrete Mathematics*, 251(1-3):47–69, 2002. doi:10.1016/s0012-365x(01)00325-9.

[4] I. Fabrici, J. Harant, and S. Jendrol'. On longest cycles in essentially 4-connected planar graphs. *Discussiones Mathematicae Graph Theory*, 36(3):565–575, 2016. doi:10.7151/dmgt.1875.

[5] I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt. On the circumference of essentially 4-connected planar graphs. *Journal of Graph Algorithms and Applications*, 24(1):21–46, 2020. doi:10.7155/jgaa.00516.

[6] H. Fleischner and B. Jackson. A note concerning some conjectures on cyclically 4–edge connected 3–regular graphs. *Annals of Discrete Mathematics*, 41:171–177, 1989. doi:10.1016/s0167-5060(08)70458-8.

[7] B. Grünbaum and J. Malkevitch. Pairs of edge-disjoint Hamilton circuits. *Aequationes Mathematicae*, 14:191–196, 1976.

[8] B. Jackson. Longest cycles in 3-connected cubic graphs. *Journal of Combinatorial Theory, Series B*, 41(1):17–26, 1986. doi:10.1016/0095-8956(86)90024-9.

[9] B. Jackson and N. C. Wormald. Longest cycles in 3-connected planar graphs. *Journal of Combinatorial Theory, Series B*, 54(2):291–321, 1992. doi:10.1016/0095-8956(92)90058-6.

[10] O.-H. S. Lo and J. M. Schmidt. Longest cycles in cyclically 4-edge-connectedcubic planar graphs. *Australasian Journal of Combinatorics*, 72(1):155–162, 2018.

[11] E. Máčajová and J. Mazák. Cubic graphs with large circumference deficit. *Journal of Graph Theory*, 82(4):433–440, 2016. doi:10.1002/jgt.21911.

[12] M. M. Matthews and D. P. Sumner. Longest paths and cycles in $K_{1,3}$-free graphs. *Journal of Graph Theory*, 9(2):269–277, 1985. doi:10.1002/jgt.3190090208.

[13] Z. Ryjáček. On a closure concept in claw-free graphs. *Journal of Combinatorial Theory, Series B*, 70(2):217–224, 1997. doi:10.1006/jctb.1996.1732.

[14] C. Thomassen. Reflections on graph theory. *Journal of Graph Theory*, 10(3):309–324, 1986. doi:10.1002/jgt.3190100308.

[15] W. T. Tutte. A non-hamiltonian planar graph. *Acta Mathematica Academiae Scientiarum Hungaricae*, 11(3-4):371–375, 1960. doi:10.1007/bf02020951.

[16] N. Van Cleemput and C. T. Zamfirescu. Regular non-hamiltonian polyhedral graphs. *Applied Mathematics and Computation*, 338:192–206, 2018. doi:10.1016/j.amc.2018.05.062.

[17] S. Zhan. Hamiltonian connectedness of line graphs. *Ars Combinatoria*, 22:89–95, 1986.

[18] C.-Q. Zhang. Longest cycles and their chords. *Journal of Graph Theory*, 11(4):521–529, 1987. doi:10.1002/jgt.3190110409.