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Criteria for Poisson process convergence with applications to inhomogeneous Poisson–Voronoi tessellations[☆]

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Abstract

This article employs the relation between probabilities of two consecutive values of a Poisson random variable to derive conditions for the weak convergence of point processes to a Poisson process. As applications, we consider the starting points of k-runs in a sequence of Bernoulli random variables, point processes constructed using inradii and circumscribed radii of inhomogeneous Poisson–Voronoi tessellations and large nearest neighbor distances in a Boolean model of disks.

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1. Introduction and main results

Let X be a random variable taking values in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\lambda > 0$. It is well-known that

$$k\mathbb{P}(X = k) = \lambda\mathbb{P}(X = k - 1), \quad k \in \mathbb{N}, \tag{1}$$

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if and only if X follows a Poisson distribution with parameter λ. We use this observation to establish weak convergence to a Poisson process. Indeed, we will prove that a tight sequence of point processes ξ_n, $n \in \mathbb{N}$, satisfies

$$\lim_{n\to\infty} k\mathbb{P}(\xi_n(B) = k) - \lambda(B)\mathbb{P}(\xi_n(B) = k-1) = 0, \quad k \in \mathbb{N},$$

for any B in a certain family of sets and some locally finite measure λ, if and only if ξ_n converges in distribution to a Poisson process with intensity measure λ. Many different methods to investigate Poisson process convergence are available in the literature; we refer to surveys and classical results [21,26,27]. Using Stein's method, one can even derive quantitative bounds for the Poisson process approximation; see e.g. [1,3–5,7,10,11,15,28,34–36] and the references therein. In contrast to these results, our findings are purely qualitative and do not provide rates of convergence, but they have the advantage that the underlying conditions are easy to verify. This is demonstrated in Sections 3.2–3.4, where weak convergence of point processes constructed using inradii and circumscribed radii of inhomogeneous Poisson–Voronoi tessellations is established and large nearest neighbor distances in a Boolean model of disks are considered.

The proof of our abstract criterion for Poisson process convergence relies on characterizations of point process convergence from [20,21] and the characterizing Eq. (1) for the Poisson distribution.

Let us now give a precise formulation of our results. Let S be a locally compact second countable Hausdorff space (lcscH space) with Borel σ-field $\mathcal{S}$. A non-empty class $\mathcal{U}$ of subsets of S is called a ring if it is closed under finite unions and intersections, as well as under proper differences. Let $\widehat{\mathcal{S}}$ denote the class of relatively compact sets of $\mathcal{S}$. We say that a measure λ on S is non-atomic if $\lambda(\{x\}) = 0$ for all $x \in S$, and we define

$$\widehat{\mathcal{S}}_\lambda = \{B \in \widehat{\mathcal{S}} : \lambda(\partial B) = 0\},$$

where ∂B indicates the boundary of B.

Let $\mathcal{M}(S)$ be the space of all locally finite measures on S, endowed with the vague topology induced by the mappings $\pi_f : \mu \mapsto \mu(f) = \int f d\mu$, $f \in C_K^+(S)$, where $C_K^+(S)$ denotes the set of non-negative and continuous functions with compact support. Note that $\mathcal{M}(S)$ is a Polish space (see e.g. [20, Theorem A2.3]). Let $\mathcal{N}(S) \subset \mathcal{M}(S)$ denote the set of all locally finite counting measures. A random measure ξ on S is a random element in $\mathcal{M}(S)$ measurable with respect to the σ-field generated by the vague topology, and it is a point process if it takes values in $\mathcal{N}(S)$.

Our first main result provides a characterization of weak convergence to a Poisson process.

Theorem 1.1. *Let ξ_n, $n \in \mathbb{N}$, be a sequence of point processes, and let λ be a non-atomic locally finite measure on S. Let $\mathcal{U} \subseteq \widehat{\mathcal{S}}_\lambda$ be a ring containing a countable topological basis of S. Then the following statements are equivalent:*

(i) For all open sets $B \in \mathcal{U}$ and $k \in \mathbb{N}$, $\xi_n(B)$, $n \in \mathbb{N}$, is tight and

$$\lim_{n\to\infty} k\mathbb{P}(\xi_n(B) = k) - \lambda(B)\mathbb{P}(\xi_n(B) = k-1) = 0. \tag{2}$$

(ii) ξ_n, $n \in \mathbb{N}$, converges in distribution to a Poisson process with intensity measure λ.

Remark 1.2. Note that the sequence $\xi_n(B)$, $n \in \mathbb{N}$, in Theorem 1.1 is tight by the Markov inequality if $\mathbb{E}[\xi_n(B)] \to \lambda(B)$ as $n \to \infty$.

Remark 1.3. For a point process ϱ, the function $f : S \times \mathcal{N}(S) \to [0, \infty)$ defined as

$$f(x, \mu) = \mathbf{1}_B(x)\mathbf{1}\{\mu(B) = k\} \tag{3}$$

with $k \in \mathbb{N}$ and $B \in \mathcal{U}$ satisfies

$$\mathbb{E}\sum_{x\in\varrho} f(x,\varrho) - \int_S \mathbb{E}\big[f(x,\varrho+\delta_x)\big]d\lambda(x) \\ = k\mathbb{P}(\varrho(B) = k) - \lambda(B)\mathbb{P}(\varrho(B) = k-1), \tag{4}$$

where δ_x denotes the Dirac measure centered at $x \in S$. By the Mecke formula, the left-hand side of (4) equals zero for all integrable functions $f : S \times \mathcal{N}(S) \to [0, \infty)$ if and only if ϱ is a Poisson process with intensity measure λ (see e.g. [23, Theorem 4.1]). Theorem 1.1 shows that, if we replace ϱ by ξ_n, $n \in \mathbb{N}$, satisfying a tightness assumption, then the left-hand side of (4) vanishes as $n \to \infty$ for all f of the form (3) if and only if ξ_n, $n \in \mathbb{N}$, converges weakly to a Poisson process with intensity measure λ.

Next we apply Theorem 1.1 to investigate point processes on S that are constructed from an underlying Poisson or binomial point process on a measurable space $(Y, \mathcal{Y})$. By $\mathcal{N}_\sigma(Y)$ we denote the set of all σ-finite counting measures on Y, which is equipped with the σ-field generated by the sets

$$\{\mu \in \mathcal{N}_\sigma(Y) : \mu(B) = k\}, \quad k \in \mathbb{N}_0, B \in \mathcal{Y}.$$

For $t \geq 1$ let η_t be a Poisson process on Y with a σ-finite intensity measure P_t (i.e. η_t is a random element in $\mathcal{N}_\sigma(Y)$), while β_n is a binomial point process of $n \in \mathbb{N}$ independent points in Y which are distributed according to a probability measure Q_n. For a family of measurable functions $h_t : V_t \times \mathcal{N}_\sigma(Y) \to S$ with $V_t \in \mathcal{Y}$, $t \geq 1$, we are interested in the point processes

$$\sum_{x\in\eta_t\cap V_t} \delta_{h_t(x,\eta_t)}, \quad t \geq 1, \quad \text{and} \quad \sum_{x\in\beta_n\cap V_n} \delta_{h_n(x,\beta_n)}, \quad n \in \mathbb{N}.$$

In order to deal with both situations simultaneously, we introduce a joint notation. In the sequel, we study the point processes

$$\xi_t = \sum_{x\in\zeta_t\cap U_t} \delta_{g_t(x,\zeta_t)}, \quad t \geq 1, \tag{5}$$

where $\zeta_t = \eta_t$, $g_t = h_t$ and $U_t = V_t$ in the Poisson case, while $\zeta_t = \beta_{\lfloor t\rfloor}$, $g_t = h_{\lfloor t\rfloor}$ and $U_t = V_{\lfloor t\rfloor}$ in the binomial case. We assume

$$\mathbb{P}(\xi_t(B) < \infty) = 1 \quad \text{for all } B \in \widehat{\mathcal{S}}$$

so that ξ_t is locally finite. Let M_t be the intensity measure of ξ_t. By K_t we denote the intensity measure of ζ_t, i.e. $K_t = P_t$ if $\zeta_t = \eta_t$ and $K_t = \lfloor t\rfloor\, Q_{\lfloor t\rfloor}$ if $\zeta_t = \beta_{\lfloor t\rfloor}$. Moreover, we define $\hat{\zeta}_t = \eta_t$ in the Poisson case and $\hat{\zeta}_t = \beta_{\lfloor t\rfloor -1}$ in the binomial case. From Theorem 1.1 we derive the following criterion for convergence of ξ_t, $t \geq 1$, to a Poisson process.

Theorem 1.4. *Let $\xi_t, t \geq 1$, be a family of point processes on S given by (5) and let M be a non-atomic locally finite measure on S. Fix a ring $\mathcal{U} \subset \widehat{\mathcal{S}}_M$ containing a countable topological basis, and assume that*

$$\lim_{t\to\infty} M_t(B) = M(B) \tag{6}$$

for all open sets $B \in \mathcal{U}$. Then,

$$\lim_{t\to\infty} \int_{U_t} \mathbb{E}\Big[\mathbf{1}\{g_t(x, \hat{\zeta}_t + \delta_x) \in B\}\mathbf{1}\Big\{ \sum_{y\in\hat{\zeta}_t\cap U_t} \delta_{g_t(y,\hat{\zeta}_t+\delta_x)}(B) = m\Big\}\Big] dK_t(x) \\ - M(B)\mathbb{P}(\xi_t(B) = m) = 0 \tag{7}$$

for all open sets $B \in \mathcal{U}$ and $m \in \mathbb{N}_0$, if and only if $\xi_t, t \geq 1$, converges weakly to a Poisson process with intensity measure M.

Remark 1.5. One is often interested in Poisson process convergence for $S = \mathbb{R}^d$, $d \geq 1$, and for the situation that the intensity measure of the Poisson process is absolutely continuous (with respect to the Lebesgue measure). In this case, we can apply Theorem 1.1 and Theorem 1.4 in the following way. The family $\mathcal{R}^d$ of sets in $\mathbb{R}^d$ that are finite unions of Cartesian products of bounded intervals is a ring contained in the relatively compact sets of $\mathbb{R}^d$. For any absolutely continuous measure the boundaries of sets from $\mathcal{R}^d$ have zero measure. By $\mathcal{I}^d$ we denote the subset of open sets of $\mathcal{R}^d$, which contains a countable topological basis of $\mathbb{R}^d$. Note that the sets of $\mathcal{I}^d$ are finite unions of Cartesian products of bounded open intervals, which can be assumed to be disjoint for $d = 1$ but not for $d \geq 2$ (e.g. $((0, 2) \times (0, 1)) \cup ((0, 1) \times [1, 2))$ is a counterexample). Thus, we prove weak convergence for sequences of point processes on $\mathbb{R}^d$ to Poisson processes with absolutely continuous locally finite intensity measures by showing (i) in Theorem 1.1 or (6) and (7) for all sets from $\mathcal{I}^d$. For $d = 1$ we use the convention $\mathcal{I} = \mathcal{I}^1$.

Theorem 1.4 says that in order to establish Poisson process convergence for point processes of the form (5), one has to deal with the dependence between

$$\mathbf{1}\{g_t(x, \hat{\zeta}_t + \delta_x) \in B\} \quad \text{and} \quad \mathbf{1}\Big\{ \sum_{y\in\hat{\zeta}_t\cap U_t} \mathbf{1}\{g_t(y, \hat{\zeta}_t + \delta_x) \in B\} = m\Big\}.$$

We say that a statistic is locally dependent if its value at a given point depends only on a local and deterministic neighborhood. That is, for any fixed $x \in Y$, there exists a set $A_{t,x} \in \mathcal{Y}$ with $x \in A_{t,x}$ such that

$$\mathbf{1}\{g_t(x, \hat{\zeta}_t + \delta_x) \in B\} = \mathbf{1}\{g_t(x, \hat{\zeta}_t|_{A_{t,x}} + \delta_x) \in B\}. \tag{8}$$

Here, we denote by $\mu|_A$ the restriction of a measure μ to a set A. For further notions of local dependence in the context of point processes we refer to [4,10,11]. Next we describe heuristically how (8) can be applied to show (7) in Theorem 1.4 for $\zeta_t = \eta_t$ if

$$\mathbf{1}\Big\{ \sum_{y\in\eta_t\cap U_t} \delta_{g_t(y,\eta_t+\delta_x)}(B) = m\Big\} \approx \mathbf{1}\Big\{ \sum_{y\in\eta_t\cap A^c_{t,x}\cap U_t} \delta_{g_t(y,\eta_t|_{A^c_{t,x}})}(B) = m\Big\} \\ \approx \mathbf{1}\Big\{ \sum_{y\in\eta_t\cap U_t} \delta_{g_t(y,\eta_t)}(B) = m\Big\} \tag{9}$$

for $x \in Y$, where $\approx$ means that the probability that the indicators are equal converges to one as $t \to \infty$. Roughly speaking, (9) means that the number of points counted in $\xi_t(B)$ is with high probability not affected by the local perturbations of adding the point x or removing all points of η_t in a neighborhood of x. Under the assumption (8), the integral in (7) coincides with

$$\int_{U_t} \mathbb{E}\Big[\mathbf{1}\{g_t(x, \eta_t|_{A_{t,x}} + \delta_x) \in B\}\mathbf{1}\Big\{ \sum_{y\in\eta_t\cap U_t} \delta_{g_t(y,\eta_t+\delta_x)}(B) = m\Big\}\Big] dK_t(x).$$

By (9), the last expression can be approximated by

$$\int_{U_t} \mathbb{E}\Big[\mathbf{1}\{g_t(x,\eta_t|_{A_{t,x}}+\delta_x)\in B\}\mathbf{1}\Big\{\sum_{y\in\eta_t\cap A_{t,x}^c\cap U_t}\delta_{g_t(y,\eta_t|_{A_{t,x}^c})}(B)=m\Big\}\Big]dK_t(x).$$

Due to the independence of $\eta_t|_{A_{t,x}}$ and $\eta_t|_{A_{t,x}^c}$, this can be rewritten as

$$\int_{U_t} \mathbb{P}\Big(\sum_{y\in\eta_t\cap A_{t,x}^c\cap U_t}\delta_{g_t(y,\eta_t|_{A_{t,x}^c})}(B)=m\Big)\mathbb{E}\big[\mathbf{1}\{g_t(x,\eta_t|_{A_{t,x}}+\delta_x)\in B\}\big]dK_t(x).$$

Using once more (8) and (9), the previous term can be approximated by

$$\mathbb{P}(\xi_t(B)=m)\int_{U_t}\mathbb{E}\big[\mathbf{1}\{g_t(x,\eta_t+\delta_x)\in B\}\big]dK_t(x)=\mathbb{P}(\xi_t(B)=m)M_t(B),$$

where the last equality follows from the Mecke formula. Consequently, the expression on the left-hand side of (7) becomes small if the approximations in (9) are good.

In Section 3, we provide examples for applying our abstract main results Theorem 1.1 and Theorem 1.4. Our first example in Section 3.1 are k-runs, i.e. at least k successes in a row in a sequence of Bernoulli random variables. For the situation that the success probabilities converge to zero, we show that the rescaled starting points of the k-runs behave like a Poisson process if some independence assumptions on the underlying Bernoulli random variables are satisfied.

As the second and third example, we consider statistics related to inradii and circumscribed radii of inhomogeneous Poisson–Voronoi tessellations. We study the Voronoi tessellation generated by a Poisson process η_t, $t>0$, on $\mathbb{R}^d$ with intensity measure $t\mu$, where μ is a locally finite and absolutely continuous measure with density f. In Section 3.2, for any cell with the nucleus in a compact set, we take the μ-measure of the ball centered at the nucleus and with twice the inradius as the radius. We prove that the point process formed by these statistics converges in distribution after a transformation depending on t to a Poisson process as $t\to\infty$ under some minor assumptions on the density f. Our transformation allows us to describe the behavior of the balls with large μ-measures. In Section 3.3, we consider for each cell with the nucleus in a compact convex set the μ-measure of the ball around the nucleus with the circumscribed radius as radius and establish, after rescaling with a power of t, convergence in distribution to a Poisson process for $t\to\infty$. This result requires continuity of f, but under weaker assumptions on f, we provide lower and upper bounds for the tail distribution of the minimal μ-measure of these balls having the circumscribed radii as radii.

In [9], the limiting distributions of the maximal inradius and the minimal circumscribed radius of a stationary Poisson–Voronoi tessellation were derived. In our work, we extend these results in two directions. First, our findings imply Poisson process convergence of the transformed inradii and circumscribed radii for the stationary case. This implies the mentioned results from [9] and allows to deal with the mth largest (or smallest) value or combinations of several order statistics. Second, we deal with inhomogeneous Poisson–Voronoi tessellations. In [12] some general results for the extremes of stationary tessellations were deduced, but they cannot be applied to inhomogeneous Poisson–Voronoi tessellations. For stationary Poisson–Voronoi tessellations the convergence of the nuclei of extreme cells to a compound Poisson process was studied in [13].

As our Theorem 1.4 deals with underlying Poisson and binomial point processes, we expect that one can extend our results on inradii and circumscribed radii of Poisson–Voronoi tessellations to Voronoi tessellations constructed from an underlying binomial point process.

In our applications to k-runs and Poisson–Voronoi tessellations we have local dependence. For example, the events that the considered statistics of a Voronoi cell belong to bounded intervals depend only on the Poisson process within balls around the nucleus. This allows to employ Poisson and Poisson process approximation results based on local dependence assumptions. Here, we mention, in particular, [34, Theorem 2.1], which can be also used to derive our main results in Sections 3.1–3.3 by controlling similar expressions as in our proofs. In order to apply [34, Theorem 2.1] to our statistics of Poisson–Voronoi tessellations, one can cover the observation window by N_t small disjoint half-open cubes and write the point process ξ_t as $\xi_t = \sum_{i=1}^{N_t} \xi_{t,i}$ where $\xi_{t,i}$ is the point process of statistics of Voronoi cells with nucleus in the intersection of the ith cube and the observation window. For the right cube-size, the local dependence property yields that $\xi_{t,i}$ and $\xi_{t,j}$ become independent for non-neighboring cubes i and j, whence several terms in [34, Theorem 2.1] vanish. By combining the representation $\xi_t = \sum_{i=1}^{N_t} \xi_{t,i}$ and the independence with our Theorem 1.1 one obtains the same expressions to control as from [34, Theorem 2.1]. In contrast to our results [34, Theorem 2.1] is proven by Stein's method and provides some quantitative bounds for the accuracy of the Poisson process approximation, but for some problems such as the circumscribed radii of inhomogeneous Poisson–Voronoi tessellations it might be difficult to deduce explicit rates of convergence this way.

In order to demonstrate that our main results are also applicable in situations without local dependence, we study in Section 3.4 large nearest neighbor distances in a Boolean model of disks. The considered Boolean model consists of disks whose centers are given by a stationary Poisson process and whose radii are marks associated to the points of the Poisson process. Since the radii can become arbitrarily large, one does not have local dependence.

Apart from the k-runs with underlying Bernoulli random variables, we study in our applications point processes constructed from underlying Poisson processes. Our general criterion for convergence to a Poisson process, Theorem 1.1, does not require such a structure and can be applied to general point processes. For example, quantitative bounds for the Poisson process approximation of point processes with Papangelou intensity are derived in [30, Theorem 3.3] by evaluating generalizations of the expression in Theorem 1.1. Thus, Theorem 1.1 can be used to show similarly that a sequence of point processes converges to a Poisson process if their Papangelou intensities converge to the density of the intensity measure of the Poisson process.

Before we discuss our applications in Section 3, we prove our main results in the next section.

2. Proofs of the main results

Recall that S is a locally compact second countable Hausdorff space, which is abbreviated as lcscH space. A topological space is second countable if its topology has a countable basis, and it is locally compact if every point has an open neighborhood whose topological closure is compact. A family of sets $\mathcal{C} \subset \widehat{\mathcal{S}}$ is called dissecting if

(i) every open set $G \subset S$ can be written as a countable union of sets in $\mathcal{C}$,
(ii) every relatively compact set $B \in \widehat{\mathcal{S}}$ is covered by finitely many sets in $\mathcal{C}$.

Lemma 2.1. *A countable topological basis $\mathcal{T}$ of S is dissecting.*

Proof. By the definition of a countable topological basis $\mathcal{T}$ has property (i) of a dissecting family of sets. Since, for any $B \in \widehat{\mathcal{S}}$, $\cup_{T\in\mathcal{T}} T = S \supset \overline{B}$, the compactness of $\overline{B}$ implies that (ii) is satisfied. □

Let us now state a consequence of [21, Theorem 4.15] and [20, Theorem 16.16] or [21, Theorem 4.11]. This result will be used in the proof of Theorem 1.1. We write $\xrightarrow{d}$ to denote convergence in distribution.

Lemma 2.2. *Let $\xi_n, n \in \mathbb{N}$, be a sequence of point processes on S, and let γ be a Poisson process on S with a non-atomic locally finite intensity measure λ. Let $\mathcal{U} \subset \widehat{\mathcal{S}}_\lambda$ be a ring containing a countable topological basis. Then the following statements are equivalent:*

(i) $\xi_n \xrightarrow{d} \gamma$.
(ii) $\xi_n(B) \xrightarrow{d} \gamma(B)$ *for all open sets* $B \in \mathcal{U}$.

Proof. Observe that [23, Theorem 3.6] ensures the existence of a Poisson process γ with intensity measure λ. Since λ has no atoms, from [23, Proposition 6.9] it follows that γ is a simple point process (i.e. $\mathbb{P}(\gamma(\{x\}) \leq 1$ for all $x \in S) = 1$). Elementary arguments also yield $\widehat{\mathcal{S}}_\lambda = \{B \in \widehat{\mathcal{S}} : \gamma(\partial B) = 0 \text{ a.s.}\} =: \widehat{\mathcal{S}}_\gamma$.

It follows from Lemma 2.1 that $\mathcal{U}$ is dissecting. By [20, Theorem 16.16 (ii)] or [21, Theorem 4.11] with $\mathcal{U}$ as dissecting semi-ring (see e.g. [21, p. 19] for a definition), we obtain that (i) implies (ii).

Conversely, if $\xi_n(U) \xrightarrow{d} \xi(U)$ for all $U \in \mathcal{U}$, the desired result follows from [21, Theorem 4.15], whose conditions are satisfied with $\mathcal{U}$ as dissecting ring and semi-ring. Thus, it is enough to show that (ii) implies $\xi_n(U) \xrightarrow{d} \xi(U)$ for all $U \in \mathcal{U}$.

For any $U \in \mathcal{U}$ there exists a sequence of open sets A_j, $j \in \mathbb{N}$, such that

$$U \subset A_j, \quad A_{j+1} \subset A_j \quad \text{and} \quad \overline{U} = \cap_{j\in\mathbb{N}} A_j.$$

Since $\mathcal{U}$ contains a countable topological basis, for any A_j one can find a countable family of open sets $B_\ell^{(j)}, \ell \in \mathbb{N}$, in $\mathcal{U}$ such that $\cup_{\ell\in\mathbb{N}} B_\ell^{(j)} = A_j$. In particular, they cover the compact set $\overline{U}$. So there exists a finite subcover of elements from $B_\ell^{(j)}, \ell \in \mathbb{N}$, that covers $\overline{U}$. Since $\mathcal{U}$ is a ring, the union of the elements belonging to this subcover of $\overline{U}$ is in $\mathcal{U}$ for each $j \in \mathbb{N}$. Because $\mathcal{U}$ is closed under finite intersections, we can make this family of sets from $\mathcal{U}$ that contain $\overline{U}$ monotonously decreasing in j. Thus, without loss of generality, we may assume $A_j \in \mathcal{U}$ for all $j \in \mathbb{N}$.

Since $\mathcal{U}$ is a ring and contains a countable topological basis, for the interior $\text{int}(U)$ of U there exists a sequence of open sets $B_j \in \mathcal{U}$, $j \in \mathbb{N}$, such that

$$B_j \subset U, \quad B_j \subset B_{j+1} \quad \text{and} \quad \text{int}(U) = \cup_{j\in\mathbb{N}} B_j.$$

For a fixed $m \in \mathbb{N}$, we have

$$\mathbb{P}(\xi_n(B_j) \geq m) \leq \mathbb{P}(\xi_n(U) \geq m) \leq \mathbb{P}(\xi_n(A_j) \geq m)$$

for all $n \in \mathbb{N}$. By $\xi_n(U') \xrightarrow{d} \gamma(U')$ for all open sets $U' \in \mathcal{U}$, we obtain

$$\mathbb{P}(\gamma(B_j) \geq m) \leq \liminf_{n\to\infty} \mathbb{P}(\xi_n(U) \geq m) \leq \limsup_{n\to\infty} \mathbb{P}(\xi_n(U) \geq m) \leq \mathbb{P}(\gamma(A_j) \geq m). \quad (10)$$

Moreover, from $U \in \widehat{\mathcal{S}}_\lambda$, whence $\lambda(\partial U) = 0$, it follows that $\lambda(B_j) \to \lambda(\text{int}(U)) = \lambda(U)$ and $\lambda(A_j) \to \lambda(\overline{U}) = \lambda(U)$ as $j \to \infty$. Thus, letting $j \to \infty$ in (10) and using that γ is a Poisson process lead to

$$\lim_{n\to\infty} \mathbb{P}(\xi_n(U) \geq m) = \mathbb{P}(\gamma(U) \geq m).$$

This establishes $\xi_n(U) \xrightarrow{d} \xi(U)$ and concludes the proof. $\square$

Remark 2.3. Note that in Lemma 2.2 one can replace the Poisson process γ by any simple point process γ with a locally finite intensity measure λ. The assumption that γ is a Poisson process is only used in the last step of the proof. Alternatively, one can argue here that

$$\mathbb{P}(\gamma(B_j) \neq \gamma(A_j)) = \mathbb{P}(\gamma(B_j) < \gamma(A_j)) \leq \mathbb{E}[\gamma(A_j) - \gamma(B_j)] = \lambda(A_j) - \lambda(B_j) \to 0$$

as $j \to \infty$.

We are now in the position to prove the first main result of this paper.

Proof of Theorem 1.1. Let us show (i) implies (ii). By Lemma 2.2 it is enough to prove that $\xi_n(B) \xrightarrow{d} \gamma(B)$ for all open sets $B \in \mathcal{U}$. Since $\mathbb{P}(\xi_n(B) = 0)$, $n \in \mathbb{N}$, is a bounded sequence in $[0, 1]$, there exists a subsequence such that $\lim_{j\to\infty} \mathbb{P}(\xi_{n_j}(B) = 0)$ exists; then repeated applications of (2) yield for $k \in \mathbb{N}$ that

$$\lim_{j\to\infty} \mathbb{P}(\xi_{n_j}(B) = k) = \frac{\lambda(B)^k}{k!} \lim_{j\to\infty} \mathbb{P}(\xi_{n_j}(B) = 0). \tag{11}$$

Consequently we have for any $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{N} \lim_{j\to\infty} \mathbb{P}(\xi_{n_j}(B) = k) &= \lim_{j\to\infty} \mathbb{P}(\xi_{n_j}(B) \in \{0, \dots, N\}) \\ &= 1 - \lim_{j\to\infty} \mathbb{P}(\xi_{n_j}(B) \in \{N+1, N+2, \dots\}).\end{aligned}$$

By tightness of $\xi_{n_j}(B)$, $j \in \mathbb{N}$, the right-hand side of the equation converges to 1 as $N \to \infty$ so that

$$\sum_{k\in\mathbb{N}_0} \lim_{j\to\infty} \mathbb{P}(\xi_{n_j}(B) = k) = 1.$$

Thus, from (11) we deduce $\lim_{j\to\infty} \mathbb{P}(\xi_{n_j}(B) = 0) = e^{-\lambda(B)}$. Together with (11), this proves that

$$\lim_{j\to\infty} \mathbb{P}(\xi_{n_j}(B) = k) = \frac{\lambda(B)}{k!} e^{-\lambda(B)}$$

for all $k \in \mathbb{N}_0$. In conclusion, since for any subsequence $(n_\ell)_{\ell\in\mathbb{N}}$ there exists a further subsequence $(n_{\ell_i})_{i\in\mathbb{N}}$ such that $\mathbb{P}(\xi_{n_{\ell_i}}(B) = 0)$, $i \in \mathbb{N}$, converges to $e^{-\lambda(B)}$, we obtain

$$\lim_{n\to\infty} \mathbb{P}(\xi_n(B) = k) = \frac{\lambda(B)}{k!} e^{-\lambda(B)}$$

for all $k \in \mathbb{N}_0$. Now (ii) follows by applying Lemma 2.2.

Conversely, let us assume $\xi_n \xrightarrow{d} \gamma$ for some Poisson process γ with intensity measure λ. It follows from Lemma 2.2 that, for any open set $B \in \mathcal{U}$, $\xi_n(B) \xrightarrow{d} \gamma(B)$ so that $\xi_n(B), n \in \mathbb{N}$, is tight and

$$\begin{aligned}0 &= k\,\mathbb{P}(\gamma(B) = k) - \lambda(B)\mathbb{P}(\gamma(B) = k-1) \\ &= \lim_{n\to\infty} k\,\mathbb{P}(\xi_n(B) = k) - \lambda(B)\mathbb{P}(\xi_n(B) = k-1)\end{aligned}$$

for $k \in \mathbb{N}$, which shows (i). □

Finally, we derive Theorem 1.4 from Theorem 1.1.

Proof of Theorem 1.4. By (6) and the Markov inequality we deduce that $\xi_t(B), t \geq 1$, is tight for all open $B \in \mathcal{U}$. Let $f : S \times \mathcal{N}(S) \to [0, \infty)$ be the function given by

$$f(x, \mu) = \mathbf{1}_B(x)\mathbf{1}\{\mu(B) = k\}$$

for $k \in \mathbb{N}$ and $B \in \mathcal{U}$. Then, by applying the Mecke equation (if $\zeta_t = \eta_t$) and the identity

$$\mathbb{E} \sum_{x \in \beta_n} u(x, \beta_n) = n \int_Y \mathbb{E}[u(x, \beta_{n-1} + \delta_x)] dQ_n(x)$$

for any measurable function $u : Y \times \mathcal{N}_\sigma(Y) \to [0, \infty)$ (if $\zeta_t = \beta_{\lfloor t \rfloor}$), we obtain

$$k\mathbb{P}(\xi_t(B) = k) = \mathbb{E} \sum_{z \in \xi_t} f(z, \xi_t) = \mathbb{E} \sum_{x \in \zeta_t \cap U_t} f(g_t(x, \zeta_t), \xi_t)$$

$$= \int_{U_t} \mathbb{E}\Big[\mathbf{1}\{g_t(x, \hat{\zeta}_t + \delta_x) \in B\}\mathbf{1}\Big\{ \sum_{y \in \hat{\zeta}_t \cap U_t} \delta_{g_t(y, \hat{\zeta}_t + \delta_x)}(B) = k - 1\Big\}\Big] dK_t(x).$$

Thus, Theorem 1.1 yields the equivalence between (7) and the convergence in distribution of $\xi_t, t \geq 1$, to a Poisson process with intensity measure M. □

3. Applications

The first three subsections throughout this section concern point processes on $\mathbb{R}$. By Remark 1.5, it is sufficient for the convergence of such point processes to a Poisson process on $\mathbb{R}$ with absolutely continuous locally finite intensity measure to show (i) in Theorem 1.1 or (6) and (7) for all sets from $\mathcal{I}$, i.e. for all finite unions of open and bounded intervals.

3.1. *Long head runs*

Consider a sequence of Bernoulli random variables. A k-head run is defined as an uninterrupted sequence of k successes, where k is a positive integer. For example, for $k = 1$, one simply studies the successes, while for $k = 2$ one considers the occurrence of two consecutive successes in a row. Several authors have investigated the number of k-head runs in a sequence of Bernoulli random variables; for an overview on this topic, we refer to [2]. Let the starting point of a k-head run be the index of its first success. Our goal is to find explicit conditions under which the point process of rescaled starting points of the k-head runs converges weakly to a Poisson process. Our investigation relies on two assumptions: the probability of having a k-head run is the same for all k consecutive elements of the sequence, and the Bernoulli random variables are independent if *far away*. We will see that if these conditions are satisfied and if the probability of having a k-head run goes to 0 slower than the probability of having a k-head run with at least another k-head run *nearby*, then the aforementioned point process converges in distribution to a Poisson process.

Let us now give a precise formulation of our result. Let $X_i^{(n)}$, $i, n \in \mathbb{N}$, be an array of Bernoulli distributed random variables and let $k \in \mathbb{N}$. Assume that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $q, n \in \mathbb{N}$ the random variable $X_q^{(n)}$ is independent of $\{X_\ell^{(n)} : |q - \ell| \geq f(n),\ \ell \in \mathbb{N}\}$ and that

$$y_n := \mathbb{P}\big(X_q^{(n)} = 1, \dots, X_{q+k-1}^{(n)} = 1\big) > 0$$

does not depend on q. If $X_i^{(n)}$, $i \in \mathbb{N}$, are i.i.d. for $n \in \mathbb{N}$, then $y_n = p_n^k$ with $p_n := \mathbb{P}\big(X_1^{(n)} = 1\big)$. Define

$$I_i^{(n)} = \mathbf{1}\big\{X_i^{(n)} = 1, \dots, X_{i+k-1}^{(n)} = 1\big\}, \quad i \in \mathbb{N}.$$

Let ξ_n be the point process of the k-head runs for $X_i^{(n)}, i \in \mathbb{N}$, that is

$$\xi_n = \sum_{i=1}^{\infty} I_i^{(n)} \delta_{i y_n}. \tag{12}$$

For any $i_0 \in \mathbb{N}$, let

$$W_{i_0}^{(n)} = \sum_{j \in \mathbb{N} : 1 \le |j - i_0| \le f(n)+k-2} I_j^{(n)}.$$

We denote by λ_1 the restriction of the Lebesgue measure to $[0, \infty)$.

Theorem 3.1. *Let ξ_n, $n \in \mathbb{N}$, be the sequence of point processes given by (12). Assume that $f(n)y_n \to 0$ as $n \to \infty$ and that*

$$\lim_{n\to\infty} \sup_{i \in \mathbb{N}} y_n^{-1} \mathbb{E}\big[I_i^{(n)} \mathbf{1}\{W_i^{(n)} > 0\}\big] = 0. \tag{13}$$

Then ξ_n converges weakly to a Poisson process with intensity measure λ_1.

For underlying independent Bernoulli random variables, the Poisson approximation of the random variable $\xi_n((0, u))$, $u > 0$, is considered in e.g. [1,5,16,22] and the Poisson process convergence follows from the results of [1]. Quantitative bounds for the Poisson process approximation of 2-runs in the i.i.d. case were derived in [35, Proposition 3.C] and [36, Theorem 6.3]; see also [10, Subsection 3.5], where the Poisson process approximation for the more general problem of counting rare words is considered.

In Theorem 3.1 we can think of $f(n)y_n \to 0$ as $n \to \infty$ as a global condition, which implies that, for Borel sets $A, B \subset [0, \infty)$ with $\inf_{x\in A, y\in B} \|x - y\| > 0$ and for n sufficiently large, $\xi_n(A)$ and $\xi_n(B)$ are independent. The local condition (13) ensures that for each $u > 0$ the probability that ξ_n has clusters (i.e. points close together) in $[0, u]$ goes to zero as $n \to \infty$. This way, our assumptions are similar to the classical $D(u_n)$ and $D'(u_n)$ conditions introduced in the context of maxima of stationary sequences in [24].

As a consequence of Theorem 3.1, we can study the limiting distribution of

$$T_n = \min\{i \in \mathbb{N} : I_i^{(n)} = 1\},$$

which gives the first arrival time of a k-head run for a sequence of Bernoulli random variables.

Corollary 3.2. *If the assumptions of Theorem 3.1 are satisfied, then $y_n T_n$ converges in distribution to an exponentially distributed random variable with parameter* 1.

Clearly, in the case when the Bernoulli random variables $(X_i^{(n)})_{i\in\mathbb{N}}$ are i.i.d. with parameter $p_n > 0$, if p_n converges to 0, the assumptions of Theorem 3.1 are fulfilled with $f(n) \equiv 1$, and so ξ_n converges in distribution to a Poisson process. Other conditions for weak convergence are given in the following corollary.

Corollary 3.3. *Let ξ_n, $n \in \mathbb{N}$, be the sequence of point processes given by (12). Let us assume that $f(n)y_n \to 0$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \sup_{i \in \mathbb{N}} y_n^{-1} \sum_{j \in \mathbb{N} : 1 \le |i-j| \le f(n)+k-2} \mathbb{E}[I_i^{(n)} I_j^{(n)}] = 0.$$

Then ξ_n converges weakly to a Poisson process with intensity measure λ_1.

Let us now prove the main result of this section, Theorem 3.1.

Proof of Theorem 3.1. For any bounded interval $A \subset [0, \infty)$, the assumptions on $X_i^{(n)}$, $i \in \mathbb{N}$, imply that

$$\mathbb{E}[\xi_n(A)] = y_n \sum_{i=1}^{\infty} \delta_{iy_n}(A) = (\sup(A) y_n^{-1} + b_n) y_n - (\inf(A) y_n^{-1} + a_n) y_n$$

for some $a_n, b_n \in [-1, 1]$. By $y_n \to 0$, we have $\mathbb{E}[\xi_n(A)] \to \lambda_1(A)$ and, consequently, $\mathbb{E}[\xi_n(B)] \to \lambda_1(B)$ for all $B \in \mathcal{I}$. Moreover, $\xi_n(B)$, $n \in \mathbb{N}$, is tight (see Remark 1.2). Then, we can write $\xi_n(B)$ as

$$\xi_n(B) = \sum_{i \in \mathcal{A}_n} I_i^{(n)}$$

with $\mathcal{A}_n = \{i \in \mathbb{N} : i y_n \in B\}$.

For $i_0 \in \mathcal{A}_n$, we have for any $m \in \mathbb{N}$ that

$$\begin{aligned}&\big|\mathbb{E}\big[I_{i_0}^{(n)} \mathbf{1}\big\{\xi_n(B) - I_{i_0}^{(n)} = m-1\big\}\big] - \mathbb{E}\big[I_{i_0}^{(n)} \mathbf{1}\big\{\xi_n(B) - W_{i_0}^{(n)} - I_{i_0}^{(n)} = m-1\big\}\big]\big| \\ &\le \mathbb{E}\big[I_{i_0}^{(n)} \mathbf{1}\{W_{i_0}^{(n)} > 0\}\big].\end{aligned}$$

Together with $\mathbb{E}\big[\xi_n(B)\big] = |\mathcal{A}_n| y_n$, this yields

$$\begin{aligned}H_n &:= \Big| \sum_{i \in \mathcal{A}_n} \mathbb{E}\big[I_i^{(n)} \mathbf{1}\big\{\xi_n(B) - I_i^{(n)} = m-1\big\}\big] \\ &\qquad - \sum_{i \in \mathcal{A}_n} \mathbb{E}\big[I_i^{(n)} \mathbf{1}\big\{\xi_n(B) - W_i^{(n)} - I_i^{(n)} = m-1\big\}\big]\Big| \\ &\le \sum_{i \in \mathcal{A}_n} \mathbb{E}\big[I_i^{(n)} \mathbf{1}\{W_i^{(n)} > 0\}\big] \le \Big(\sup_{i \in \mathbb{N}} y_n^{-1} \mathbb{E}\big[I_i^{(n)} \mathbf{1}\{W_i^{(n)} > 0\}\big]\Big) \mathbb{E}[\xi_n(B)].\end{aligned}$$

Therefore from (13), we obtain $H_n \to 0$. From the independence of $I_{i_0}^{(n)}$ and $\xi_n(B) - W_{i_0}^{(n)} - I_{i_0}^{(n)}$ for $i_0 \in \mathcal{A}_n$, it follows that

$$\mathbb{E}\big[I_{i_0}^{(n)} \mathbf{1}\big\{\xi_n(B) - W_{i_0}^{(n)} - I_{i_0}^{(n)} = m-1\big\}\big] = \mathbb{E}\big[I_{i_0}^{(n)}\big] \mathbb{P}\big(\xi_n(B) - W_{i_0}^{(n)} - I_{i_0}^{(n)} = m-1\big).$$

Combining the previous arguments implies for $m \in \mathbb{N}$ that

$$\begin{aligned}&\limsup_{n\to\infty} \big| m \mathbb{P}(\xi_n(B) = m) - \lambda_1(B) \mathbb{P}(\xi_n(B) = m-1)\big| \\ &= \limsup_{n\to\infty} \Big| \sum_{i \in \mathcal{A}_n} \mathbb{E}\big[I_i^{(n)} \mathbf{1}\big\{\xi_n(B) - I_i^{(n)} = m-1\big\}\big] - \lambda_1(B) \mathbb{P}(\xi_n(B) = m-1)\Big| \\ &= \limsup_{n\to\infty} \Big| \sum_{i \in \mathcal{A}_n} \mathbb{E}\big[I_i^{(n)} \mathbf{1}\big\{\xi_n(B) - W_i^{(n)} - I_i^{(n)} = m-1\big\}\big] - \mathbb{E}[\xi_n(B)] \mathbb{P}(\xi_n(B) = m-1)\Big| \\ &= \limsup_{n\to\infty} \Big| \sum_{i \in \mathcal{A}_n} \mathbb{E}[I_i^{(n)}] \mathbb{P}\big(\xi_n(B) - W_i^{(n)} - I_i^{(n)} = m-1\big) - \sum_{i \in \mathcal{A}_n} \mathbb{E}[I_i^{(n)}] \mathbb{P}(\xi_n(B) = m-1)\Big| \\ &\le \limsup_{n\to\infty} \sum_{i \in \mathcal{A}_n} \mathbb{E}[I_i^{(n)}] \mathbb{P}\big(W_i^{(n)} + I_i^{(n)} > 0\big) \le \lambda_1(B) \limsup_{n\to\infty} \sup_{i \in \mathbb{N}} \mathbb{P}\big(W_i^{(n)} + I_i^{(n)} > 0\big).\end{aligned}$$

Finally, the inequality

$$\mathbb{P}\big(W_i^{(n)} + I_i^{(n)} > 0\big) \le (2k + 2f(n) - 3) y_n, \quad i \in \mathbb{N},$$

and the assumption $f(n)y_n \to 0$ lead to

$$\lim_{n\to\infty} \left| m\mathbb{P}(\xi_n(B) = m) - \lambda_1(B)\mathbb{P}(\xi_n(B) = m-1) \right| = 0.$$

The result follows by applying Theorem 1.1. □

Proof of Corollary 3.3. This follows directly from Theorem 3.1 and

$$\mathbb{E}\big[I_i^{(n)}\mathbf{1}\{W_i^{(n)} > 0\}\big] \le \mathbb{E}\big[I_i^{(n)}W_i^{(n)}\big] = \sum_{j\in\mathbb{N}:\,1\le|i-j|\le f(n)+k-2} \mathbb{E}\big[I_i^{(n)}I_j^{(n)}\big]$$

for any $i \in \mathbb{N}$. □

3.2. Inradii of an inhomogeneous Poisson–Voronoi tessellation

In this section, we consider the inradii of an inhomogeneous Voronoi tessellation generated by a Poisson process with a certain intensity measure $t\mu$, $t > 0$; recall that the inradius of a cell is the largest radius for which the ball centered at the nucleus is contained in the cell. We study the point process on $\mathbb{R}$ constructed by taking for any cell with the nucleus in a compact set, a transform of the μ-measure of the ball centered at the nucleus and with twice the inradius as the radius. The aim is to continue the work started in [9] by extending the result on the largest inradius to inhomogeneous Poisson–Voronoi tessellations and proving weak convergence of the aforementioned point process to a Poisson process.

For any locally finite counting measure ν on $\mathbb{R}^d$, we denote by $N(x,\nu)$ the Voronoi cell with nucleus $x \in \mathbb{R}^d$ generated by $\nu + \delta_x$, that is

$$N(x,\nu) = \left\{ y \in \mathbb{R}^d \,:\, \|x-y\| \le \|y-x'\|, x \ne x' \in \nu \right\},$$

where $\|\cdot\|$ denotes the Euclidean norm. For $x \in \nu$ we have $N(x,\nu) = N(x,\nu-\delta_x)$. Voronoi tessellations, i.e. tessellations consisting of Voronoi cells $N(x,\nu)$, $x \in \nu$, arise in different fields such as biology [31], astrophysics [32] and communication networks [6]. For more details on Poisson–Voronoi tessellations, i.e. Voronoi tessellations generated by an underlying Poisson process, we refer the reader to e.g. [8,25,33]. The inradius of the Voronoi cell $N(x,\nu)$ is given by

$$c(x,\nu) = \sup\{R \ge 0 \,:\, \mathbf{B}(x,R) \subset N(x,\nu)\},$$

where $\mathbf{B}(x,r)$ denotes the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$.

Let η_t, $t > 0$, be a Poisson process on $\mathbb{R}^d$ with intensity measure $t\mu$, where μ is a locally finite measure on $\mathbb{R}^d$ with density $f : \mathbb{R}^d \to [0,\infty)$. Consider a compact set $W \subset \mathbb{R}^d$ with $\mu(W) = 1$, and assume that there exists a bounded open set $A \subset \mathbb{R}^d$ with $W \subset A$ such that $f_{min} := \inf_{x\in A} f(x) > 0$ and $f_{max} := \sup_{x\in A} f(x) < \infty$. For any Voronoi cell $N(x,\eta_t)$ with $x \in \eta_t$, we take the μ-measure of the ball around x with twice the inradius as radius, and we define the point process ξ_t on $\mathbb{R}$ as

$$\xi_t = \xi_t(\eta_t) = \sum_{x\in\eta_t\cap W} \delta_{t\mu(\mathbf{B}(x,2c(x,\eta_t)))-\log(t)}. \tag{14}$$

Let M be the measure on $\mathbb{R}$ given by $M([u,\infty)) = e^{-u}$ for $u \in \mathbb{R}$.

Theorem 3.4. *Let ξ_t, $t > 0$, be the family of point processes on $\mathbb{R}$ given by (14). Then ξ_t converges in distribution to a Poisson process with intensity measure M.*

For an underlying homogeneous Poisson or binomial point process on $[0,1]^d$ with $d \in \{1,2\}$ or on the torus, the statement of Theorem 3.4 was established in [29, Theorem 2].

The next theorem shows that if the density function f is Hölder continuous, it is possible to take out the factor 2 from $\mu(\mathbf{B}(x, 2c(x,\eta_t)))$ and to consider $2^d\mu(\mathbf{B}(x, c(x,\eta_t)))$. Recall that a function $h : \mathbb{R}^d \to \mathbb{R}$ is Hölder continuous with exponent $b > 0$ if there exists a constant $C > 0$ such that

$$|h(x) - h(y)| \leq C\|x - y\|^b$$

for all $x, y \in \mathbb{R}^d$. We define the point process $\widehat{\xi}_t$ as

$$\widehat{\xi}_t = \widehat{\xi}_t(\eta_t) = \sum_{x\in\eta_t\cap W} \delta_{2^d t\mu(\mathbf{B}(x,c(x,\eta_t)))-\log(t)}.$$

Theorem 3.5. *Let f be Hölder continuous. Then $\widehat{\xi}_t, t > 0$, converges in distribution to a Poisson process with intensity measure M.*

As a corollary of the previous theorems, we have the following generalization to the inhomogeneous case of the result obtained in [9, Theorem 1, Equation (2a)] for the stationary case; see also [12, Section 5] for the maximal inradius of a stationary Poisson–Voronoi tessellation and of a stationary Gauss–Poisson–Voronoi tessellation.

Corollary 3.6. *For $u \in \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{P}\Big(\max_{x\in\eta_t\cap W} t\mu(\mathbf{B}(x, 2c(x,\eta_t))) - \log(t) \leq u\Big) = e^{-e^{-u}}. \tag{15}$$

Moreover, if f is Hölder continuous,

$$\lim_{t\to\infty} \mathbb{P}\Big(\max_{x\in\eta_t\cap W} 2^d t\mu(\mathbf{B}(x, c(x,\eta_t))) - \log(t) \leq u\Big) = e^{-e^{-u}}. \tag{16}$$

Under the assumption that f is continuous and for the situation that the Poisson process is restricted to the unit cube, (15) is established in [28].

For an underlying binomial point process, (15) was shown under similar assumptions in [17]. The related problem of maximal weighted rth nearest neighbor distances for the points of a binomial point process was studied in [18]; see also [19]. For results on large rth nearest neighbor balls of Poisson or binomial point processes on the unit cube or torus we refer to [7,14,28,29].

For the proofs of Theorem 3.4 and Theorem 3.5, we will use the quantities $v_t(x,u)$ and $q_t(x,u)$, which are introduced in the next lemma.

Lemma 3.7. *For any $u \in \mathbb{R}$ there exists $t_0 > 0$ such that for all $x \in W$ and $t > t_0$ the equations*

$$t\mu(\mathbf{B}(x, 2v_t(x,u))) = u + \log(t) \quad \textit{and} \quad 2^d t\mu(\mathbf{B}(x, q_t(x,u))) = u + \log(t) \tag{17}$$

have unique solutions $v_t(x,u)$ and $q_t(x,u)$, respectively, which satisfy

$$\max\{v_t(x,u), q_t(x,u)\} \leq \Big(\frac{u + \log(t)}{2^d f_{min} k_d t}\Big)^{1/d} \tag{18}$$

for all $x \in W$ and $t > t_0$, where k_d is the volume of the d-dimensional unit ball.

Proof. Let $u \in \mathbb{R}$ be fixed and set $m = \inf\{\|x - y\| : x \in \partial W, y \in \partial A\} \in (0, \infty)$. Note that $B(x, m) \subset A$ for all $x \in W$. Choose $t_0 > 0$ such that

$$\frac{u + \log(t)}{t} < f_{min} k_d m^d$$

for $t > t_0$. For $x \in W$ and $t > t_0$ this implies that

$$2^d t\mu(\mathbf{B}(x, m)) \geq t\mu(\mathbf{B}(x, m)) \geq t f_{min} k_d m^d > u + \log(t)$$

and, obviously, $t\mu(\mathbf{B}(x, 0)) = 0$. Since the function $[0, m] \ni a \to \mu(\mathbf{B}(x, a))$ is continuous and strictly increasing because of $f_{min} > 0$, by the intermediate value theorem, the equations in (17) have unique solutions $v_t(x, u)$ and $q_t(x, u)$. Since $\max\{2v_t(x, u), q_t(x, u)\} < m$, we obtain

$$\frac{u + \log(t)}{t} = \mu(\mathbf{B}(x, 2v_t(x, u))) \geq 2^d f_{min} k_d v_t(x, u)^d$$

and

$$\frac{u + \log(t)}{t} = 2^d \mu(\mathbf{B}(x, q_t(x, u))) \geq 2^d f_{min} k_d q_t(x, u)^d,$$

which prove (18). □

Let M_t be the intensity measure of ξ_t. Then from the Mecke formula and Lemma 3.7 it follows that for any $u \in \mathbb{R}$ there exists $t_0 > 0$ such that for $t > t_0$,

$$\begin{aligned} M_t([u, \infty)) &= t \int_W \mathbb{P}\big(t\mu(\mathbf{B}(x, 2c(x, \eta_t + \delta_x))) \geq u + \log(t)\big) f(x) dx \\ &= t \int_W \mathbb{P}(c(x, \eta_t + \delta_x) \geq v_t(x, u)) f(x) dx \\ &= t \int_W \mathbb{P}\big(\eta_t(\mathbf{B}(x, 2v_t(x, u))) = 0\big) f(x) dx \\ &= t \int_W e^{-t\mu(\mathbf{B}(x, 2v_t(x, u)))} f(x) dx = t e^{-u - \log(t)} \mu(W) = e^{-u} = M([u, \infty)), \end{aligned} \tag{19}$$

where we used (17) and $\mu(W) = 1$ in the last steps. For any $y \in \mathbb{R}$ and point configuration ν on $\mathbb{R}^d$ with $y \in \nu$, we denote by $h_t(y, \nu)$ the quantity

$$h_t(y, \nu) = t\mu(\mathbf{B}(y, 2c(y, \nu))) - \log(t), \tag{20}$$

where $c(y, \nu)$ is the inradius of the Voronoi cell with nucleus y generated by ν. So we can rewrite ξ_t as

$$\xi_t = \sum_{x \in \eta_t \cap W} \delta_{h_t(x, \eta_t)}.$$

Proof of Theorem 3.4. From Theorem 1.4 and (19) it follows that it is enough to show that

$$\begin{aligned} &\lim_{t\to\infty} t \int_W \mathbb{E}\Big[\mathbf{1}\{h_t(x, \eta_t + \delta_x) \in B\} \mathbf{1}\Big\{\sum_{y \in \eta_t \cap W} \delta_{h_t(y, \eta_t + \delta_x)}(B) = m\Big\}\Big] f(x) dx \\ &\quad - M(B)\mathbb{P}(\xi_t(B) = m) = 0 \end{aligned} \tag{21}$$

for any $m \in \mathbb{N}_0$ and $B \in \mathcal{I}$. Let $B = \bigcup_{j=1}^{\ell} (u_{2j-1}, u_{2j})$ with $u_1 < u_2 < \cdots < u_{2\ell}$ and $\ell \in \mathbb{N}$. By Lemma 3.7 there is a $t_0 > 0$ such that $v_t(x, u_k)$ exists for all $k = 1, \ldots, 2\ell$, $x \in W$ and

$t > t_0$. Assume $t > t_0$ in the following. Elementary arguments imply that

$$t \int_W \mathbb{E}\Big[\mathbf{1}\{h_t(x, \eta_t + \delta_x) \in B\}\mathbf{1}\Big\{\sum_{y \in \eta_t \cap W} \delta_{h_t(y,\eta_t+\delta_x)}(B) = m\Big\}\Big] f(x)dx$$
$$= \sum_{j=1}^{\ell} t \int_W \mathbb{E}\Big[\mathbf{1}\{h_t(x, \eta_t + \delta_x) \in (u_{2j-1}, u_{2j})\}\mathbf{1}\Big\{\sum_{y \in \eta_t \cap W} \delta_{h_t(y,\eta_t+\delta_x)}(B) = m\Big\}\Big] f(x)dx. \tag{22}$$

For each $k = 1, \ldots, 2\ell$, set $w_{k,t,x} = 2v_t(x, u_k)$. Since $h_t(x, \eta_t + \delta_x) \in (u_{2j-1}, u_{2j})$ if and only if $c(x, \eta_t + \delta_x) \in (v_t(x, u_{2j-1}), v_t(x, u_{2j}))$, or equivalently, $\eta_t(\mathbf{B}(x, w_{2j-1,t,x})) = 0$ and $\eta_t(\mathbf{B}(x, w_{2j,t,x})) > 0$, we obtain that

$$S_j := t \int_W \mathbb{E}\Big[\mathbf{1}\{h_t(x, \eta_t + \delta_x) \in (u_{2j-1}, u_{2j})\}\mathbf{1}\Big\{\sum_{y \in \eta_t \cap W} \delta_{h_t(y,\eta_t+\delta_x)}(B) = m\Big\}\Big] f(x)dx$$
$$= t \int_W \mathbb{E}\Big[\mathbf{1}\left\{\eta_t\big(\mathbf{B}(x, w_{2j-1,t,x})\big) = 0\right\}$$
$$\times \mathbf{1}\Big\{\eta_t\big(\mathbf{B}(x, w_{2j,t,x})\big) > 0, \sum_{y \in \eta_t \cap W} \delta_{h_t(y,\eta_t+\delta_x)}(B) = m\Big\}\Big] f(x)dx.$$

For any point configuration ν on $\mathbb{R}^d$ and $x \in W$, let $\xi_{t,x}(\nu)$ be the counting measure given by $\xi_{t,x}(\nu) = \sum_{y \in \nu \cap W} \delta_{h_t(y,\nu+\delta_x)}$ so that

$$S_j = t \int_W \mathbb{E}\big[\mathbf{1}\left\{\eta_t\big(\mathbf{B}(x, w_{2j-1,t,x})\big) = 0\right\}$$
$$\times \mathbf{1}\left\{\eta_t\big(\mathbf{B}(x, w_{2j,t,x})\big) > 0, \xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{2j-1,t,x})^c}\big)(B) = m\right\}\big] f(x)dx$$
$$= t \int_W \mathbb{P}\left(\eta_t\big(\mathbf{B}(x, w_{2j-1,t,x})\big) = 0\right)$$
$$\times \mathbb{P}\big(\eta_t\big(\mathbf{B}(x, w_{2j,t,x}) \setminus \mathbf{B}(x, w_{2j-1,t,x})\big) > 0, \xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{2j-1,t,x})^c}\big)(B) = m\big) f(x)dx.$$

Similar arguments as used to compute $M_t([u, \infty))$ for $u \in \mathbb{R}$ imply for $x \in W$ that

$$t\mathbb{P}\left(\eta_t\big(\mathbf{B}(x, w_{2j-1,t,x})\big) = 0\right) = e^{-u_{2j-1}},$$

and so we deduce that

$$S_j = e^{-u_{2j-1}} \int_W \mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{2j-1,t,x})^c}\big)(B) = m\right) f(x)dx$$
$$- e^{-u_{2j-1}} \int_W \mathbb{P}\big(\eta_t\big(\mathbf{B}(x, w_{2j,t,x}) \setminus \mathbf{B}(x, w_{2j-1,t,x})\big) = 0,$$
$$\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{2j-1,t,x})^c}\big)(B) = m\big) f(x)dx. \tag{23}$$

Furthermore, we can rewrite the second integral as

$$\int_W \mathbb{P}\left(\eta_t\big(\mathbf{B}(x, w_{2j,t,x}) \setminus \mathbf{B}(x, w_{2j-1,t,x})\big) = 0\right) \mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{2j,t,x})^c}\big)(B) = m\right) f(x)dx$$
$$= \int_W e^{-t\mu(\mathbf{B}(x,w_{2j,t,x}))+t\mu(\mathbf{B}(x,w_{2j-1,t,x}))} \mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{2j,t,x})^c}\big)(B) = m\right) f(x)dx$$
$$= e^{-u_{2j}+u_{2j-1}} \int_W \mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{2j,t,x})^c}\big)(B) = m\right) f(x)dx.$$

Combining this and (23) yields

$$S_j = e^{-u_{2j-1}} \int_W \mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{2j-1,t,x})^c}\big)(B) = m\right) f(x)dx$$
$$- e^{-u_{2j}} \int_W \mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{2j,t,x})^c}\big)(B) = m\right) f(x)dx.$$

Substituting this into (22) implies that to prove (21) and to complete the proof, it is enough to show for all $x \in W$ and $k = 1, \dots, 2\ell$ that

$$\lim_{t\to\infty} \mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{k,t,x})^c}\big)(B) = m\right) - \mathbb{P}(\xi_t(B) = m) = 0. \tag{24}$$

Let $x \in W$, $k \in \{1, \dots, 2\ell\}$ and $\varepsilon > 0$ be fixed. Set

$$a_t = 2\Big(\frac{u_{2\ell} + \log(t)}{2^d f_{min} k_d t}\Big)^{1/d}.$$

From the application of Lemma 3.7 at the beginning of the proof it follows that $w_{k,t,y} \leq w_{2\ell,t,y} \leq a_t$ for all $y \in W$ and $t > t_0$. Without loss of generality we may assume that $2a_t < \min\{\|z_1 - z_2\| : z_1 \in \partial W, z_2 \in \partial A\}$. Therefore the observation

$$h_t(y, \nu) \in B \quad \text{if and only if} \quad h_t(y, \nu|_{\mathbf{B}(y,w_{2\ell,t,y})}) \in B$$

for any point configuration ν on $\mathbb{R}^d$ and $y \in \nu \cap W$ leads to

$$\begin{aligned}
&\left|\mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{k,t,x})^c}\big)(B) = m\right) - \mathbb{P}\left(\xi_t(B) = m\right)\right| \leq \mathbb{E}\big[|\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{k,t,x})^c}\big)(B) - \xi_t(B)|\big]\\
&\leq \mathbb{E} \sum_{y\in\eta_t\cap\mathbf{B}(x,2a_t)\cap\mathbf{B}(x,w_{k,t,x})^c\cap W} \mathbf{1}\{h_t(y, \eta_t|_{\mathbf{B}(x,w_{k,t,x})^c} + \delta_x) \in B\}\\
&\quad + \mathbb{E} \sum_{y\in\eta_t\cap\mathbf{B}(x,2a_t)\cap W} \mathbf{1}\{h_t(y, \eta_t) \in B\}\\
&\leq \mathbb{E} \sum_{y\in\eta_t\cap\mathbf{B}(x,2a_t)\cap\mathbf{B}(x,w_{k,t,x})^c\cap W} \mathbf{1}\{h_t(y, \eta_t|_{\mathbf{B}(x,w_{k,t,x})^c} + \delta_x) > u_1\}\\
&\quad + \mathbb{E} \sum_{y\in\eta_t\cap\mathbf{B}(x,2a_t)\cap W} \mathbf{1}\{h_t(y, \eta_t) > u_1\}.
\end{aligned}$$

Then, the Mecke formula and (20) imply that

$$\begin{aligned}
&\left|\mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{k,t,x})^c}\big)(B) = m\right) - \mathbb{P}\left(\xi_t(B) = m\right)\right|\\
&\leq t \int_{\mathbf{B}(x,2a_t)\cap\mathbf{B}(x,w_{k,t,x})^c\cap W} \mathbb{P}(h_t(y, \eta_t|_{\mathbf{B}(x,w_{k,t,x})^c} + \delta_x + \delta_y) > u_1) f(y)dy\\
&\quad + t \int_{\mathbf{B}(x,2a_t)\cap W} \mathbb{P}(h_t(y, \eta_t + \delta_y) > u_1) f(y)dy\\
&= t \int_{\mathbf{B}(x,2a_t)\cap\mathbf{B}(x,w_{k,t,x})^c\cap W} \mathbb{P}\big(t\mu(\mathbf{B}(y, 2c(y, \eta_t|_{\mathbf{B}(x,w_{k,t,x})^c} + \delta_x + \delta_y))) > u_1 + \log(t)\big) f(y)dy\\
&\quad + t \int_{\mathbf{B}(x,2a_t)\cap W} \mathbb{P}\big(t\mu(\mathbf{B}(y, 2c(y, \eta_t + \delta_y))) > u_1 + \log(t)\big) f(y)dy.
\end{aligned}$$

Since $c(y, \nu + \delta_y + \delta_x) > v_t(y, u_1)$ only if $c(y, \nu + \delta_y) > v_t(y, u_1)$ for any point configuration ν on $\mathbb{R}^d$ and $x, y \in W$, it follows for $x \in W$ and $y \in \mathbf{B}(x, 2a_t) \cap \mathbf{B}(x, w_{k,t,x})^c \cap W$ that

$$\begin{aligned}
&\mathbb{P}\big(t\mu(\mathbf{B}(y, 2c(y, \eta_t|_{\mathbf{B}(x,w_{k,t,x})^c} + \delta_x + \delta_y))) > u_1 + \log(t)\big)\\
&= \mathbb{P}\big(c(y, \eta_t|_{\mathbf{B}(x,w_{k,t,x})^c} + \delta_x + \delta_y) > v_t(y, u_1)\big)
\end{aligned}$$

$$\le \mathbb{P}\big(c(y, \eta_t|_{\mathbf{B}(x,w_{k,t,x})^c} + \delta_y) > v_t(y, u_1)\big)$$
$$= \mathbb{P}\big(\eta_t|_{\mathbf{B}(x,w_{k,t,x})^c}(\mathbf{B}(y, 2v_t(y, u_1))) = 0\big) = \exp(-t\mu(\mathbf{B}(y, 2v_t(y, u_1)) \cap \mathbf{B}(x, w_{k,t,x})^c)).$$

Let λ_d denote the Lebesgue measure on $\mathbb{R}^d$. For $A_1, A_2 \in \mathcal{B}(\mathbb{R}^d)$ with $A_1, A_2 \subset A$ and $\lambda_d(A_2) > 0$ we obtain

$$\frac{\mu(A_1)}{\mu(A_2)} \ge \frac{f_{min}}{f_{max}} \frac{\lambda_d(A_1)}{\lambda_d(A_2)}.$$

With $\tau := f_{min}/f_{max} \in (0, 1]$,

$$A_1 = \mathbf{B}(y, 2v_t(y, u_1)) \cap \mathbf{B}(x, w_{k,t,x})^c \quad \textit{and} \quad A_2 = \mathbf{B}(y, 2v_t(y, u_1)),$$

this implies for $x \in W$ and $y \in \mathbf{B}(x, 2a_t) \cap \mathbf{B}(x, w_{k,t,x})^c \cap W$ that

$$t\mu(\mathbf{B}(y, 2v_t(y, u_1)) \cap \mathbf{B}(x, w_{k,t,x})^c) \ge \frac{\tau}{2} t\mu(\mathbf{B}(y, 2v_t(y, u_1))) = \frac{\tau}{2}(u_1 + \log(t)).$$

Moreover, we have

$$\mathbb{P}\big(t\mu(\mathbf{B}(y, 2c(y, \eta_t + \delta_y))) > u_1 + \log(t)\big) = \mathbb{P}\big(\eta_t(\mathbf{B}(y, 2v_t(y, u_1))) = 0\big) = e^{-u_1 - \log(t)}.$$

In conclusion, combining the previous bounds leads to

$$\Big|\mathbb{P}\left(\xi_{t,x}\big(\eta_t|_{\mathbf{B}(x,w_{k,t,x})^c}\big)(B) = m\right) - \mathbb{P}\left(\xi_t(B) = m\right)\Big|$$
$$\le t^{1-\tau/2} e^{-\tau u_1/2} \mu(\mathbf{B}(x, 2a_t)) + e^{-u_1}\mu(\mathbf{B}(x, 2a_t)) \le (2a_t)^d k_d f_{max}(t^{1-\tau/2} e^{-\tau u_1/2} + e^{-u_1}),$$

where in the last step we used the fact that f is bounded by f_{max} in A and, by the choice of a_t, $\mathbf{B}(x, 2a_t) \subset A$. Again, from the definition of a_t it follows that the right-hand side converges to 0 as $t \to \infty$. This shows (24) and concludes the proof. □

Next, we derive Theorem 3.5 from Theorem 3.4.

Proof of Theorem 3.5. Assume that f is Hölder continuous with exponent $b > 0$. From Lemma 2.2, Theorem 3.4 and Remark 1.5 we obtain that it is enough to show that $\mathbb{E}[|\xi_t(B) - \widehat{\xi}_t(B)|] \to 0$ as $t \to \infty$ for all $B \in \mathcal{I}$. By Lemma 3.7, for any $u \in \mathbb{R}$ there exists $t_0 > 0$ such that

$$\mu(\mathbf{B}(x, 2v_t(x, u))) = 2^d \mu(\mathbf{B}(x, q_t(x, u)))$$
$$= 2^d k_d f(x) q_t(x, u)^d + 2^d \int_{\mathbf{B}(x,q_t(x,u))} \big(f(y) - f(x)\big) dy$$
$$= \mu(\mathbf{B}(x, 2q_t(x, u))) - \int_{\mathbf{B}(x,2q_t(x,u))} \big(f(y) - f(x)\big) dy + 2^d \int_{\mathbf{B}(x,q_t(x,u))} \big(f(y) - f(x)\big) dy$$

for all $x \in W, t > t_0$, where

$$q_t(x, u) \le \left(\frac{u + \log(t)}{2^d f_{min} k_d t}\right)^{1/d}.$$

Thus, the Hölder continuity of f and elementary arguments establish that

$$|\mu(\mathbf{B}(x, 2v_t(x, u))) - \mu(\mathbf{B}(x, 2q_t(x, u)))| \le C\Big(\frac{u + \log(t)}{t}\Big)^{1+b/d}, \quad x \in W, t > t_0, \tag{25}$$

for some $C > 0$. In particular, from the definition of $v_t(x, u)$ it follows that

$$\begin{aligned} \mu(\mathbf{B}(x, 2v_t(x, u))) &= \frac{u + \log(t)}{t} \\ \mu(\mathbf{B}(x, 2q_t(x, u))) &\ge \frac{u + \log(t)}{t} - C\Big(\frac{u + \log(t)}{t}\Big)^{1+b/d} \end{aligned} \tag{26}$$

for $t > t_0$. Next, we write $B = \bigcup_{j=1}^{\ell}(u_{2j-1}, u_{2j})$ for some $\ell \in \mathbb{N}$ and $u_1 < \cdots < u_{2\ell}$. The triangle inequality yields

$$\begin{aligned}\mathbb{E}[|\xi_t(B) - \widehat{\xi}_t(B)|] &\leq \sum_{j=1}^{\ell} \mathbb{E}[|\xi_t((u_{2j-1}, u_{2j})) - \widehat{\xi}_t((u_{2j-1}, u_{2j}))|] \\ &\leq \sum_{j=1}^{\ell} \mathbb{E}[|\xi_t((u_{2j-1}, \infty)) - \widehat{\xi}_t((u_{2j-1}, \infty))|] + \mathbb{E}[|\xi_t([u_{2j}, \infty)) - \widehat{\xi}_t([u_{2j}, \infty))|].\end{aligned} \tag{27}$$

Moreover, the Mecke formula establishes for $u \in \mathbb{R}$ that

$$\begin{aligned}&\mathbb{E}[|\xi_t((u, \infty)) - \widehat{\xi}_t((u, \infty))|] \\ &\leq \mathbb{E} \sum_{x \in \eta_t \cap W} |\mathbf{1}\{c(x, \eta_t + \delta_x) > v_t(x, u)\} - \mathbf{1}\{c(x, \eta_t + \delta_x) > q_t(x, u)\}| \\ &= t \int_W \mathbb{P}\big(\eta_t(\mathbf{B}(x, 2v_t(x, u))) = 0, \eta_t(\mathbf{B}(x, 2q_t(x, u))) > 0\big) f(x) dx \\ &\quad + t \int_W \mathbb{P}\big(\eta_t(\mathbf{B}(x, 2v_t(x, u))) > 0, \eta_t(\mathbf{B}(x, 2q_t(x, u))) = 0\big) f(x) dx \\ &\leq f_{max} t \int_W \big[\exp\big(-t\mu(\mathbf{B}(x, 2v_t(x, u)))\big) + \exp\big(-t\mu(\mathbf{B}(x, 2q_t(x, u)))\big)\big] \\ &\quad \times \big[1 - \exp\big(-t|\mu(\mathbf{B}(x, 2q_t(x, u))) - \mu(\mathbf{B}(x, 2v_t(x, u)))|\big)\big] dx.\end{aligned}$$

Therefore, from (25) and (26), it follows that

$$\lim_{t \to \infty} \mathbb{E}[|\xi_t((u, \infty)) - \widehat{\xi}_t((u, \infty))|] = 0. \tag{28}$$

Together with (27) and a similar computation for the half-closed intervals on the right-hand side of (27), this concludes the proof. □

Proof of Corollary 3.6. Let $u \in \mathbb{R}$ be fixed. By Markov's inequality we have for $u_0 > u$ that

$$\begin{aligned}\mathbb{P}(\xi_t((u, u_0)) > 0) \leq \mathbb{P}(\xi_t((u, \infty)) > 0) &= \mathbb{P}\big(\max_{x \in \eta_t \cap W} t\mu(\mathbf{B}(x, 2c(x, \eta_t))) - \log(t) > u\big) \\ &\leq \mathbb{P}(\xi_t((u, u_0)) > 0) + \mathbb{E}[\xi_t([u_0, \infty))].\end{aligned}$$

Thus, Theorem 3.4 and (19) yield

$$\limsup_{t \to \infty} |\mathbb{P}\big(\max_{x \in \eta_t \cap W} t\mu(\mathbf{B}(x, 2c(x, \eta_t))) - \log(t) > u\big) - 1 + e^{-M((u, u_0))}| \leq e^{-u_0}.$$

Then, letting $u_0 \to \infty$ leads to (15). Since, for $u > 0$,

$$\big|\mathbb{P}(\xi_t((u, \infty)) > 0) - \mathbb{P}(\widehat{\xi}_t((u, \infty)) > 0)\big| \leq \mathbb{E}[|\xi_t((u, \infty)) - \widehat{\xi}_t((u, \infty))|],$$

(15) and (28) imply (16). □

3.3. Circumscribed radii of an inhomogeneous Poisson–Voronoi tessellation

In this section, we consider the circumscribed radii of an inhomogeneous Voronoi tessellation generated by a Poisson process with a certain intensity measure $t\mu$, $t > 0$; recall that the circumscribed radius of a cell is the smallest radius for which the ball centered at the nucleus contains the cell. We study the point process on the non-negative real line constructed by taking for any cell with the nucleus in a compact convex set, a transform of the μ-measure of the ball centered at the nucleus and with the circumscribed radius as the radius. The aim is to continue the work started in [9] by extending the result on the smallest circumscribed radius to inhomogeneous Poisson–Voronoi tessellations and by proving weak convergence of the aforementioned point process to a Poisson process.

More precisely, let μ be an absolutely continuous measure on $\mathbb{R}^d$ with continuous density $f : \mathbb{R}^d \to [0, \infty)$. Consider a Poisson process η_t with intensity measure $t\mu$, $t > 0$. The circumscribed radius of the Voronoi cell $N(x, \eta_t)$ with $x \in \eta_t$ is given by

$$C(x, \eta_t) = \inf\{R \geq 0 : \mathbf{B}(x, R) \supset N(x, \eta_t)\},$$

with the convention $\inf \emptyset = \infty$; see Section 3.2 for more details on Voronoi tessellations.

Let $W \subset \mathbb{R}^d$ be a compact convex set with $f > 0$ on W. The convexity assumption allows us to avoid some technical problems. We consider the point process

$$\xi_t = \sum_{x \in \eta_t \cap W} \delta_{\alpha_2 t^{(d+2)/(d+1)} \mu(\mathbf{B}(x, C(x,\eta_t)))}. \tag{29}$$

Here the positive constant α_2 is given by

$$\alpha_2 = \left(\frac{2^{d(d+1)}}{(d+1)!} p_{d+1} \right)^{1/(d+1)}$$

with

$$p_{d+1} := \mathbb{P}\Big(N\Big(0, \sum_{j=1}^{d+1} \delta_{Y_j}\Big) \subseteq \mathbf{B}(0, 1) \Big),$$

where $Y_1, \dots, Y_{d+1}$ are independent and uniformly distributed random points in $\mathbf{B}(0, 2)$. We write M for the measure on $[0, \infty)$ satisfying $M([0, u]) = \mu(W) u^{d+1}$ for $u \geq 0$.

Theorem 3.8. *Let ξ_t, $t > 0$, be the family of point processes on $[0, \infty)$ given by (29). Then ξ_t converges in distribution to a Poisson process with intensity measure M.*

An immediate consequence of this theorem is that a transform of the minimal μ-measure of the balls, having circumscribed radii and nuclei of the Voronoi cells as radii and centers respectively, converges to a Weibull distributed random variable. This generalizes [9, Theorem 1, Equation (2d)]. For the situation that, in contrast to Theorem 3.8, the density of the intensity measure of the underlying Poisson process is not continuous, we can still derive some upper and lower bounds.

Theorem 3.9. *Let ζ_t be a Poisson process with intensity measure $t\vartheta$, where $t > 0$ and ϑ is an absolutely continuous measure on $\mathbb{R}^d$ with density ϕ. Let $f_1, f_2 : \mathbb{R}^d \to [0, \infty)$ be continuous and $f_1, f_2 > 0$ on W.*

(i) If there exists $s \in (0, 1]$ such that $s\phi \leq f_1 \leq \phi$, then

$$\limsup_{t \to \infty} \mathbb{P}\Big(s\alpha_2 t^{(d+2)/(d+1)} \min_{x \in \zeta_t \cap W} \vartheta(\mathbf{B}(x, C(x, \zeta_t))) > u \Big) \leq \exp\big(-s\vartheta(W) u^{d+1}\big)$$

for $u \geq 0$.

(ii) If there exists $r \geq 1$ such that $\phi \leq f_2 \leq r\phi$, then

$$\liminf_{t \to \infty} \mathbb{P}\Big(r\alpha_2 t^{(d+2)/(d+1)} \min_{x \in \zeta_t \cap W} \vartheta(\mathbf{B}(x, C(x, \zeta_t))) > u \Big) \geq \exp\big(-r\vartheta(W) u^{d+1}\big)$$

for $u \geq 0$.

Let us now prepare the proof of Theorem 3.8. We first have to study the distribution of $C(x, \eta_t + \delta_x)$, which is defined as the circumscribed radius of the Voronoi cell with nucleus $x \in \mathbb{R}^d$ generated by $\eta_t + \delta_x$. To this end, we define $g : W \times T \to [0, \infty)$ by the equation

$$\mu(\mathbf{B}(x, g(x, u))) = u \tag{30}$$

for $T := [0, \mu(W)]$. Since W is compact and convex and $f > 0$ on W, we have that (30) admits a unique solution $g(x, u)$ for all $(x, u) \in W \times T$. As this is the only place where we use the convexity of W, we believe that one can omit this assumption. However, we refrained from doing so in order to not further increase the complexity of the proof. Set

$$s_t = \alpha_2 t^{(d+2)/(d+1)}.$$

Thus, we may write

$$\mathbb{P}(s_t \mu(\mathbf{B}(x, C(x, \eta_t + \delta_x))) \le u) = \mathbb{P}(C(x, \eta_t + \delta_x) \le g(x, u/s_t)), \quad u/s_t \in T. \tag{31}$$

Lemma 3.10. *For any $u \in T$, $g(\cdot, u) : W \to \mathbb{R}$ is continuous and*

$$\lim_{u \to 0^+} \sup_{x \in W} |g(x, u)| = 0.$$

Proof. First we show that $g(\cdot, u)$ is continuous for any fixed $u \in T$. For $u = 0$, we obtain $g(x, u) = 0$ for all $x \in W$. Assume $u > 0$ and let $x_0 \in W$ and $\varepsilon > 0$. Then for all $x \in \mathbf{B}(x_0, \varepsilon')$ with $\varepsilon' := \min\{g(x_0, u)/2, \varepsilon\}$, we have

$$\mathbf{B}(x_0, g(x_0, u)) \subset \mathbf{B}(x, g(x_0, u) + \varepsilon') \quad \text{and} \quad \mathbf{B}(x, g(x_0, u) - \varepsilon') \subset \mathbf{B}(x_0, g(x_0, u)).$$

Together with (30), this leads to

$$\mu(\mathbf{B}(x, g(x_0, u) + \varepsilon')) \ge u \quad \text{and} \quad \mu(\mathbf{B}(x, g(x_0, u) - \varepsilon')) \le u.$$

Now it follows from the definition of g that

$$g(x, u) \le g(x_0, u) + \varepsilon' \quad \text{and} \quad g(x, u) \ge g(x_0, u) - \varepsilon'.$$

This yields

$$|g(x, u) - g(x_0, u)| \le \varepsilon' \le \varepsilon$$

for all $x \in \mathbf{B}(x_0, \varepsilon')$ so that $g(\cdot, u)$ is continuous at x_0. In conclusion since

$$\lim_{u \to 0^+} g(x, u) = 0$$

and $g(x, u_1) < g(x, u_2)$ for all $x \in W$ and $0 \le u_1 < u_2$, Dini's theorem implies that $\sup_{x \in W} |g(x, u)| \to 0$ as $u \to 0$. □

We define

$$\beta = \min_{x \in W} f(x) > 0.$$

Lemma 3.11. *There exists $u_0 \in T$ such that*

$$g(x, u) \le \left(\frac{2u}{\beta k_d}\right)^{1/d} \tag{32}$$

for all $u \in [0, u_0]$ and $x \in W$.

Proof. Since f is continuous and $f > 0$ on W, it follows that

$$\min_{x\in W+\overline{\mathbf{B}(0,\delta)}} f(x) > \frac{\beta}{2}$$

for some $\delta > 0$. Furthermore, by Lemma 3.10 we obtain that there exists $u_0 \in T$ such that $g(x,u) \le \delta$ for all $u \in [0,u_0]$ and $x \in W$. Then, we obtain

$$u = \mu(\mathbf{B}(x, g(x,u))) = \int_{\mathbf{B}(x,g(x,u))} f(y)dy \ge \frac{\beta}{2} k_d g(x,u)^d$$

for all $x \in W$ and $u \in [0,u_0]$, which shows (32). □

For $x \in W$ and $u > 0$, we consider a sequence of independent and identically distributed random points $(X_i^{(x,u)})_{i\in\mathbb{N}}$ in $\mathbb{R}^d$ with distribution

$$\mathbb{P}\big(X_i^{(x,u)} \in E\big) = \frac{\mu(\mathbf{B}(x,2u)\cap E)}{\mu(\mathbf{B}(x,2u))}, \quad i \in \mathbb{N}, E \in \mathcal{B}(\mathbb{R}^d).$$

Recall that, for $k \ge d+1$, $N\Big(x, \sum_{j=1}^k \delta_{X_j^{(x,u)}}\Big)$ denotes the Voronoi cell with nucleus x generated by $X_1^{(x,u)}, \dots, X_k^{(x,u)}$ and x. Then the distribution function of $C(x,\eta_t+\delta_x)$ is equal to

$$\mathbb{P}\big(C(x,\eta_t+\delta_x) \le u\big) = \sum_{k=d+1}^{\infty} \mathbb{P}\big(\eta_t(\mathbf{B}(x,2u)) = k\big) p_k(x,u) \tag{33}$$

for $u \ge 0$, where $p_k(x,u)$ is defined as

$$p_k(x,u) = \mathbb{P}\Big(N\Big(x, \sum_{j=1}^k \delta_{X_j^{(x,u)}}\Big) \subseteq \mathbf{B}(x,u)\Big)$$

for $u > 0$ and $p_k(x,0) = 0$ for all $x \in W$, see also [8, Subsection 5.2.3] for more details on $p_k(x,u)$ in the case that η_t is a stationary Poisson process. Combining (31) and (33) establishes for $u/s_t \in T$ that

$$\begin{aligned}&\mathbb{P}(s_t\mu(\mathbf{B}(x, C(x,\eta_t+\delta_x))) \le u)\\ &= \sum_{k=d+1}^{\infty} \mathbb{P}\big(\eta_t(\mathbf{B}(x, 2g(x,u/s_t))) = k\big) p_k(x, g(x,u/s_t)).\end{aligned} \tag{34}$$

For $k \in \mathbb{N}$ with $k \ge d+1$, we define the probability

$$p_k = \mathbb{P}\Big(N\Big(0, \sum_{j=1}^k \delta_{Y_j}\Big) \subseteq \mathbf{B}(0,1)\Big),$$

where $Y_1,\dots,Y_k$ are independent and uniformly distributed random points in $\mathbf{B}(0,2)$. As discussed in [9, Section 3], one can reinterpret p_k as the probability to cover the unit sphere with k independent spherical caps with random radii. In the next lemma, we prove that $p_k(x,r) \to p_k$ as $r \to 0$ for all $x \in W$ and $k \ge d+1$, which together with Lemma 3.11 yields $p_k(x, g(u/s_t)) \to p_k$ as $t \to \infty$.

Lemma 3.12. *For any $k \ge d+1$ and $x \in W$,*

$$\lim_{r\to 0^+} p_k(x,r) = p_k.$$

Proof. Let $x \in W, k \geq d+1$ and $r > 0$ be fixed. From the definition of $p_k(x,r)$ and the independence of $X_1^{(x,r)}, \dots, X_k^{(x,r)}$ it follows that

$$\begin{aligned} p_k(x,r) &= \mathbb{P}\Big(N\Big(x, \sum_{j=1}^k \delta_{X_j^{(x,r)}}\Big) \subseteq \mathbf{B}(x,r)\Big) \\ &= \frac{1}{\mu(\mathbf{B}(x,2r))^k} \int_{\mathbf{B}(x,2r)^k} \mathbf{1}\Big\{N\Big(x, \sum_{j=1}^k \delta_{z_j}\Big) \subseteq \mathbf{B}(x,r)\Big\} \prod_{j=1}^k f(z_j)\, d(z_1,\dots,z_k) \\ &= \frac{(2r)^{kd}}{\mu(\mathbf{B}(x,2r))^k} \int_{\mathbf{B}(0,1)^k} \mathbf{1}\Big\{N\Big(x, \sum_{j=1}^k \delta_{x+2rz_j}\Big) \subseteq \mathbf{B}(x,r)\Big\} \\ &\quad \times \prod_{j=1}^k f(x+2rz_j)\, d(z_1,\dots,z_k). \end{aligned}$$

Furthermore, by the definition of $N\big(x, \sum_{j=1}^k \delta_{x+2rz_j}\big)$ we deduce that

$$\mathbf{1}\Big\{N\Big(x, \sum_{j=1}^k \delta_{x+2rz_j}\Big) \subseteq \mathbf{B}(x,r)\Big\} = \mathbf{1}\Big\{N\Big(0, \sum_{j=1}^k \delta_{2z_j}\Big) \subseteq \mathbf{B}(0,1)\Big\}$$

for all $z_1,\dots,z_k \in \mathbf{B}(0,1)$, whence

$$\begin{aligned} p_k(x,r) &= \frac{(2r)^{kd}}{\mu(\mathbf{B}(x,2r))^k} \int_{\mathbf{B}(0,1)^k} \mathbf{1}\Big\{N\Big(0, \sum_{j=1}^k \delta_{2z_j}\Big) \subseteq \mathbf{B}(0,1)\Big\} \\ &\quad \times \prod_{j=1}^k f(x+2rz_j)\, d(z_1,\dots,z_k). \end{aligned}$$

Using the dominated convergence theorem for the integral, the continuity of f and

$$\lim_{r\to 0^+} \frac{(2r)^{kd}}{\mu(\mathbf{B}(x,2r))^k} = \frac{1}{k_d^k f(x)^k},$$

we obtain

$$\begin{aligned} \lim_{r\to 0^+} p_k(x,r) &= \frac{1}{k_d^k} \int_{\mathbf{B}(0,1)^k} \mathbf{1}\Big\{N\Big(0, \sum_{j=1}^k \delta_{2z_j}\Big) \subseteq \mathbf{B}(0,1)\Big\}\, d(z_1,\dots,z_k) \\ &= \frac{1}{(2^d k_d)^k} \int_{\mathbf{B}(0,2)^k} \mathbf{1}\Big\{N\Big(0, \sum_{j=1}^k \delta_{z_j}\Big) \subseteq \mathbf{B}(0,1)\Big\}\, d(z_1,\dots,z_k) = p_k, \end{aligned}$$

which concludes the proof. □

Let M_t be the intensity measure of ξ_t and let

$$\begin{aligned} \widehat{M}_t([0,u]) = t \int_W \mathbb{E}\Big[&\mathbf{1}\big\{s_t \mu(\mathbf{B}(x, C(x, \eta_t + \delta_x))) \in [0,u]\big\} \\ &\times \mathbf{1}\Big\{\eta_t\Big(\mathbf{B}\Big(x, 4\Big(\frac{2u}{\beta s_t k_d}\Big)^{1/d}\Big)\Big) = d+1\Big\}\Big] f(x) dx \end{aligned}$$

and

$$\theta_t([0,u]) = t\int_W \mathbb{E}\Big[\mathbf{1}\{s_t\mu(\mathbf{B}(x,C(x,\eta_t+\delta_x)))\in[0,u]\} \times \mathbf{1}\Big\{\eta_t\Big(\mathbf{B}\Big(x,4\Big(\frac{2u}{\beta s_t k_d}\Big)^{1/d}\Big)\Big)>d+1\Big\}\Big]f(x)dx$$

for $u\geq 0$. Observe that

$$M_t([0,u]) = \widehat{M}_t([0,u]) + \theta_t([0,u]), \quad u\geq 0. \tag{35}$$

Lemma 3.13. *For any $u\geq 0$,*

$$\lim_{t\to\infty}\widehat{M}_t([0,u]) = \mu(W)u^{d+1}$$

and

$$\theta_t([0,u]) \leq t\int_W \mathbb{P}\Big(\eta_t\Big(\mathbf{B}\Big(x,4\Big(\frac{2u}{\beta s_t k_d}\Big)^{1/d}\Big)\Big)>d+1\Big)f(x)dx \to 0 \quad \textit{as} \quad t\to\infty.$$

Proof. Let $u\geq 0$ be fixed and $u_t := u/s_t$. Without loss of generality we may assume $u_t\in T$. For $x\in W$ we deduce from (30), $g(x,u_t)\to 0$ as $t\to\infty$ and the continuity of f that

$$\lim_{t\to\infty}\frac{\mu(\mathbf{B}(x,2g(x,u_t)))}{u_t} = \lim_{t\to\infty}\frac{\mu(\mathbf{B}(x,2g(x,u_t)))}{2^d k_d g(x,u_t)^d}\frac{2^d k_d g(x,u_t)^d}{\mu(\mathbf{B}(x,g(x,u_t)))} = \frac{2^d f(x)}{f(x)} = 2^d.$$

Together with $u_t = u/s_t$ and $s_t = \alpha_2 t^{(d+2)/(d+1)}$, this leads to

$$\lim_{t\to\infty} t^{d+2}\mu\big(\mathbf{B}(x,2g(x,u_t))\big)^{d+1} = (2^d u/\alpha_2)^{d+1}. \tag{36}$$

Similarly, we obtain from Lemma 3.11 that for t sufficiently large,

$$\begin{aligned}\sup_{x\in W} t^{d+2}\mu\big(\mathbf{B}(x,2g(x,u_t))\big)^{d+1} &\leq t^{d+2}(2^{d+1}u_t/\beta)^{d+1}\sup_{x\in W}\sup_{y\in\mathbf{B}(x,2g(x,u_t))} f(y)^{d+1}\\ &\leq (2^{d+1}u/(\alpha_2\beta))^{d+1}\sup_{y\in W+\mathbf{B}(0,1)} f(y)^{d+1}.\end{aligned} \tag{37}$$

Let us now compute the limit of $\widehat{M}_t([0,u])$. By Lemma 3.11 we obtain for $\ell_t := 4\big(\frac{2u_t}{\beta k_d}\big)^{1/d}$ that there exists $t_0>0$ such that $2g(x,u_t)\leq \ell_t$ for all $t>t_0$ and $x\in W$. From (34) we deduce for $x\in W$ that $s_t\mu(\mathbf{B}(x,C(x,\eta_t+\delta_x)))\in[0,u]$ only if there are at least $d+1$ points of η_t in $\mathbf{B}\big(x,2g(x,u_t)\big)$. Then for $t>t_0$, we have

$$\begin{aligned}\widehat{M}_t([0,u]) &= t\int_W \mathbb{P}\big(\eta_t(\mathbf{B}(x,2g(x,u_t))) = d+1\big)p_{d+1}(x,g(x,u_t))\\ &\qquad\times \mathbb{P}\big(\eta_t\big(\mathbf{B}(x,\ell_t)\setminus\mathbf{B}(x,2g(x,u_t))\big)=0\big)f(x)dx\\ &= \int_W \frac{t^{d+2}\mu\big(\mathbf{B}(x,2g(x,u_t))\big)^{d+1}}{(d+1)!}e^{-t\mu(\mathbf{B}(x,\ell_t))}p_{d+1}(x,g(x,u_t))f(x)dx.\end{aligned}$$

Elementary arguments imply that

$$\lim_{t\to\infty} t\mu(\mathbf{B}(x,\ell_t)) = 0.$$

Therefore combining (36) and Lemma 3.12 yields

$$\lim_{t\to\infty} \frac{t^{d+2}\mu\big(\mathbf{B}(x, 2g(x,u_t))\big)^{d+1}}{(d+1)!} e^{-t\mu(\mathbf{B}(x,\ell_t))} p_{d+1}(x, g(x,u_t)) f(x)$$
$$= \Big(\frac{2^d u}{\alpha_2}\Big)^{d+1} \frac{p_{d+1}}{(d+1)!} f(x) = u^{d+1} f(x),$$

where we used $\alpha_2 = \big(\frac{2^{d(d+1)}}{(d+1)!} p_{d+1}\big)^{1/(d+1)}$ in the last step. Thus, by (37) and the dominated convergence theorem we obtain

$$\lim_{t\to\infty} \widehat{M}_t([0,u]) = u^{d+1} \int_W f(x)dx = \mu(W)u^{d+1}.$$

Finally, let us compute the limit of $\theta_t([0,u])$. For a Poisson distributed random variable Z with parameter $v > 0$ we have

$$\mathbb{P}(Z \ge d+2) = \sum_{k=d+2}^{\infty} \frac{v^k}{k!} e^{-v} \le v^{d+2} \sum_{k=0}^{\infty} \frac{v^k}{k!} e^{-v} = v^{d+2}.$$

This implies that

$$\begin{aligned}
\theta_t([0,u]) &\le t \int_W \mathbb{P}\Big(\eta_t\Big(\mathbf{B}\Big(x, 4\Big(\frac{2u_t}{\beta k_d}\Big)^{1/d}\Big)\Big) > d+1\Big) f(x)dx \\
&\le t^{d+3} \int_W \mu\Big(\mathbf{B}\Big(x, 4\Big(\frac{2u_t}{\beta k_d}\Big)^{1/d}\Big)\Big)^{d+2} f(x)dx \\
&\le \sup_{y\in W+\mathbf{B}\big(0,4\big(\frac{2u_t}{\beta k_d}\big)^{1/d}\big)} f(y)^{d+2} \int_W f(x)dx \frac{2^{2d^2+5d+2}}{\beta^{d+2}} t^{d+3} u_t^{d+2} \\
&= \sup_{y\in W+\mathbf{B}\big(0,4\big(\frac{2u_t}{\beta k_d}\big)^{1/d}\big)} f(y)^{d+2} \mu(W) \frac{2^{2d^2+5d+2}}{\beta^{d+2}} \frac{1}{\alpha_2^{d+2}} t^{-\frac{1}{d+1}} u^{d+2}.
\end{aligned}$$

Here, the supremum converges to a constant as $t \to \infty$ so that the second inequality in the assertion is proven. □

In the next lemma, we show a technical result, which will be needed in the proof of Theorem 3.8. For $A \subset \mathbb{R}^d$, let conv(A) denote the convex hull of A.

Lemma 3.14. *Let $x_0, \dots, x_{d+1} \in \mathbb{R}^d$ be in general position (i.e. no k-dimensional affine subspace of $\mathbb{R}^d$ with $k \in \{0, \dots, d-1\}$ contains more than $k+1$ of the points) and assume that $N(x_0, \sum_{j=0}^{d+1} \delta_{x_j})$ is bounded. Then,*

(a) $x_0 \in \operatorname{int}(\operatorname{conv}(\{x_1, \dots, x_{d+1}\}))$;
(b) $N(x_i, \sum_{j=0}^{d+1} \delta_{x_j})$ is unbounded for any $i \in \{1, \dots, d+1\}$.

Proof. Assume that $x_0 \notin \operatorname{int}(\operatorname{conv}(\{x_1, \dots, x_{d+1}\}))$. By the hyperplane separation theorem for convex sets there exists a hyperplane through x_0 with a normal vector $u \in \mathbb{R}^d$ such that $\langle u, x_i\rangle \le \langle u, x_0\rangle$ for all $i \in \{1, \dots, d+1\}$, where $\langle \cdot, \cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^d$. Define the set $R = \{x_0 + ru : r \in [0, \infty)\}$. For any $y \in R$, x_0 is the closest point to y

out of $\{x_0, \dots, x_{d+1}\}$, whence $R \subset N(x_0, \sum_{j=0}^{d+1} \delta_{x_j})$ and $N(x_0, \sum_{j=0}^{d+1} \delta_{x_j})$ is unbounded. This gives us a contradiction and, thus, proves part (a).

Let $i \in \{1, \dots, d+1\}$ and assume that $N(x_i, \sum_{j=0}^{d+1} \delta_{x_j})$ is bounded. It follows from part (a) that $x_i \in \operatorname{int}(\operatorname{conv}(\{x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{d+1}\}))$. On the other hand, again by part (a), we have $x_0 \in \operatorname{int}(\operatorname{conv}(\{x_1, \dots, x_{d+1}\}))$. This implies that

$$\operatorname{conv}(\{x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{d+1}\}) = \operatorname{conv}(\{x_0, \dots, x_{d+1}\}) = \operatorname{conv}(\{x_1, \dots, x_{d+1}\}),$$

and, thus, either $x_i, x_0 \in \operatorname{conv}(\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{d+1}\})$ or $x_i = x_0$. This gives us a contradiction and concludes the proof of part (b). □

Finally, we are in position to prove the main result of this section.

Proof of Theorem 3.8. From Lemma 3.13 and (35) we deduce that $M_t(B) \to M(B)$ as $t \to \infty$ for all $B \in \mathcal{I}$. Then, by Theorem 1.4 it is sufficient to show

$$\lim_{t\to\infty} t \int_W \mathbb{E}\Big[\mathbf{1}\{s_t\mu(\mathbf{B}(x, C(x, \eta_t + \delta_x))) \in B\} \times \mathbf{1}\Big\{\sum_{y\in\eta_t\cap W} \delta_{s_t\mu(\mathbf{B}(y,C(y,\eta_t+\delta_x)))}(B) = m\Big\}\Big] f(x)dx - M(B)\mathbb{P}(\xi_t(B) = m) = 0$$

for $m \in \mathbb{N}_0$ and $B \in \mathcal{I}$. Put $\overline{u} = \sup(B)$ and let $\ell_t = 4\big(\frac{2\overline{u}}{\beta s_t k_d}\big)^{1/d}$. We write

$$\begin{aligned}
&t \int_W \mathbb{E}\Big[\mathbf{1}\{s_t\mu(\mathbf{B}(x, C(x, \eta_t + \delta_x))) \in B\}\mathbf{1}\Big\{\sum_{y\in\eta_t\cap W} \delta_{s_t\mu(\mathbf{B}(y,C(y,\eta_t+\delta_x)))}(B) = m\Big\}\Big] f(x)dx \\
&= t \int_W \mathbb{E}\Big[\mathbf{1}\{s_t\mu(\mathbf{B}(x, C(x, \eta_t + \delta_x))) \in B\}\mathbf{1}\{\eta_t(\mathbf{B}(x, \ell_t)) = d+1\} \\
&\qquad \times \mathbf{1}\Big\{\sum_{y\in\eta_t\cap W} \delta_{s_t\mu(\mathbf{B}(y,C(y,\eta_t+\delta_x)))}(B) = m\Big\}\Big] f(x)dx \\
&\quad + t \int_W \mathbb{E}\Big[\mathbf{1}\{s_t\mu(\mathbf{B}(x, C(x, \eta_t + \delta_x))) \in B\}\mathbf{1}\{\eta_t(\mathbf{B}(x, \ell_t)) > d+1\} \\
&\qquad \times \mathbf{1}\Big\{\sum_{y\in\eta_t\cap W} \delta_{s_t\mu(\mathbf{B}(y,C(y,\eta_t+\delta_x)))}(B) = m\Big\}\Big] f(x)dx \\
&=: A_t + R_t.
\end{aligned}$$

By Lemma 3.13, we obtain $R_t \to 0$ as $t \to \infty$. Let us study A_t. From Lemma 3.11 it follows that there exists $t_0 > 0$ such that $\overline{u}/s_t \in T$ and $\ell_t \geq 4g(y, \overline{u}/s_t)$ for all $y \in W$ and $t > t_0$. Assume $t > t_0$. In case there are only $d+1$ points of η_t in $\mathbf{B}(x, \ell_t)$, we deduce that $s_t\mu(\mathbf{B}(x, C(x, \eta_t + \delta_x))) \in B$ only if the $d+1$ points belong to $\mathbf{B}(x, 2g(x, \overline{u}/s_t))$. Then, by $\ell_t \geq 4g(x, \overline{u}/s_t)$ we obtain

$$\begin{aligned}
A_t = t \int_W \mathbb{E}\Big[&\mathbf{1}\{s_t\mu(\mathbf{B}(x, C(x, \eta_t + \delta_x))) \in B\} \\
&\times \mathbf{1}\{\eta_t(\mathbf{B}(x, \ell_t) \setminus \mathbf{B}(x, \ell_t/2)) = 0, \eta_t(\mathbf{B}(x, \ell_t/2)) = d+1\} \\
&\times \mathbf{1}\Big\{\sum_{y\in\eta_t\cap W} \delta_{s_t\mu(\mathbf{B}(y,C(y,\eta_t+\delta_x)))}(B) = m\Big\}\Big] f(x)dx.
\end{aligned} \tag{38}$$

Furthermore, since $\ell_t \geq 4g(y, \overline{u}/s_t)$ for all $y \in W$, we have

$$\mathbf{B}(y, 2g(y, \overline{u}/s_t)) \cap \mathbf{B}(x, \ell_t/2) = \emptyset, \quad y \in \mathbf{B}(x, \ell_t)^c \cap W.$$

Now the observation that

$$s_t\mu(\mathbf{B}(y, C(y, \eta_t+\delta_x))) \in B \quad \text{if and only if} \quad s_t\mu(\mathbf{B}(y, C(y, (\eta_t+\delta_x)|_{\mathbf{B}(y,2g(y,\overline{u}/s_t))}))) \in B$$

for $y \in \eta_t$ establishes that

$$\begin{aligned} A_t = t\int_W \mathbb{E}\Big[&\mathbf{1}\big\{s_t\mu(\mathbf{B}(x, C(x, \eta_t+\delta_x))) \in B\big\} \\ &\times \mathbf{1}\big\{\eta_t(\mathbf{B}(x,\ell_t)\setminus \mathbf{B}(x,\ell_t/2)) = 0, \eta_t(\mathbf{B}(x,\ell_t/2)) = d+1\big\} \\ &\times \mathbf{1}\Big\{\xi_t(\eta_t|_{\mathbf{B}(x,\ell_t)^c})(B) + \sum_{y\in\eta_t\cap\mathbf{B}(x,\ell_t/2)\cap W} \delta_{s_t\mu(\mathbf{B}(y,C(y,\eta_t+\delta_x)))}(B) = m\Big\}\Big] f(x)dx. \end{aligned}$$

Suppose that $s_t\mu(\mathbf{B}(x, C(x, \eta_t+\delta_x))) \in B$ and that there are exactly $d+1$ points $y_1, \dots, y_{d+1}$ of η_t in $\mathbf{B}(x,\ell_t/2)$ and $\eta_t \cap \mathbf{B}(x,\ell_t) \cap \mathbf{B}(x,\ell_t/2)^c = \emptyset$. From Lemma 3.14 it follows that $x \in \operatorname{int}(\operatorname{conv}(\{y_1,\dots,y_{d+1}\}))$ and that the Voronoi cells $N(y_i, \eta_t|_{\mathbf{B}(x,\ell_t)} + \delta_x)$, $i = 1,\dots,d+1$, are unbounded. In particular, we have

$$C(y_i, \eta_t+\delta_x) > \ell_t/4 > g(y_i, \overline{u}/s_t), \quad i = 1,\dots,d+1.$$

Together with the same arguments used to show (38), this implies that

$$\begin{aligned} A_t &= t\int_W \mathbb{E}\big[\mathbf{1}\big\{s_t\mu(\mathbf{B}(x, C(x, \eta_t+\delta_x))) \in B\big\} \\ &\qquad \times \mathbf{1}\big\{\eta_t(\mathbf{B}(x,\ell_t)\setminus \mathbf{B}(x,\ell_t/2)) = 0, \eta_t(\mathbf{B}(x,\ell_t/2)) = d+1\big\} \\ &\qquad \times \mathbf{1}\big\{\xi_t(\eta_t|_{\mathbf{B}(x,\ell_t)^c})(B) = m\big\}\big] f(x)dx \\ &= t\int_W \mathbb{E}\big[\mathbf{1}\big\{s_t\mu(\mathbf{B}(x, C(x, \eta_t+\delta_x))) \in B\big\}\mathbf{1}\big\{\eta_t(\mathbf{B}(x,\ell_t)) = d+1\big\}\big] \\ &\qquad \times \mathbb{P}\big(\xi_t(\eta_t|_{\mathbf{B}(x,\ell_t)^c})(B) = m\big) f(x)dx. \end{aligned}$$

Furthermore, we obtain

$$\big|\mathbb{P}\big(\xi_t(\eta_t|_{\mathbf{B}(x,\ell_t)^c})(B) = m\big) - \mathbb{P}\big(\xi_t(B) = m\big)\big| \le \mathbb{P}(\eta_t(\mathbf{B}(x,\ell_t)) > 0) \le t\mu(\mathbf{B}(x,\ell_t))$$

for any $x \in W$, where we used the Markov inequality in the last step. Combining the previous formulas leads to

$$\begin{aligned} &\big|A_t - M(B)\mathbb{P}(\xi_t(B) = m)\big| \\ &\le |M_t(B) - M(B)| \\ &\quad + t\int_W \mathbb{E}\big[\mathbf{1}\big\{s_t\mu(\mathbf{B}(x, C(x, \eta_t+\delta_x))) \in B\big\}\mathbf{1}\big\{\eta_t(\mathbf{B}(x,\ell_t)) > d+1\big\}\big] f(x)dx \\ &\quad + t\int_W \mathbb{E}\big[\mathbf{1}\big\{s_t\mu(\mathbf{B}(x, C(x, \eta_t+\delta_x))) \in B\big\}\mathbf{1}\big\{\eta_t(\mathbf{B}(x,\ell_t)) = d+1\big\}\big] \\ &\quad \times \big|\mathbb{P}\big(\xi_t(\eta_t|_{\mathbf{B}(x,\ell_t)^c})(B) = m\big) - \mathbb{P}(\xi_t(B) = m)\big| f(x)dx \\ &\le |M_t(B) - M(B)| + t\int_W \mathbb{P}\big(\eta_t(\mathbf{B}(x,\ell_t)) > d+1\big) f(x)dx \\ &\quad + \widehat{M}_t([0,\overline{u}]) \sup_{x\in W} t\mu(\mathbf{B}(x,\ell_t)). \end{aligned}$$

It follows from Lemma 3.13 that, as $t \to \infty$, $\widehat{M}_t([0,\overline{u}]) \to M([0,\overline{u}])$, $M_t(B) \to M(B)$ and the integral on the right-hand side vanishes. Without loss of generality we may assume $\ell_t \le 1$, and thus the continuity of f on $W + \overline{\mathbf{B}(0,1)}$ implies that

$$t\mu(\mathbf{B}(x,\ell_t)) \le k_d \max_{z\in W+\overline{\mathbf{B}(0,1)}} f(z) t\ell_t^d$$

for all $x \in W$. Now $\ell_t = 4\big(\frac{2\overline{u}}{\beta s_t k_d}\big)^{1/d}$ and $s_t = \alpha_2 t^{(d+2)/(d+1)}$ yield that the right-hand side vanishes as $t \to \infty$. Thus, we obtain

$$\lim_{t\to\infty} A_t - M(B)\mathbb{P}(\xi_t(B) = m) = 0,$$

which together with $R_t \to 0$ as $t \to \infty$ concludes the proof. □

Proof of Theorem 3.9. Let γ be a Poisson process on $\mathbb{R}^d \times [0, \infty)$ with the restriction of the Lebesgue measure as intensity measure. Let μ_1 and μ_2 denote the absolutely continuous measures with densities f_1 and f_2, respectively. Then, [23, Corollary 5.9 and Proposition 6.16] imply that

$$\varrho_t^{(1)} = \sum_{(x,y)\in\gamma} \mathbf{1}\{y \le t f_1(x)\}\delta_x, \quad \varrho_t^{(2)} = \sum_{(x,y)\in\gamma} \mathbf{1}\{y \le t f_2(x)\}\delta_x$$

and

$$\varrho_t = \sum_{(x,y)\in\gamma} \mathbf{1}\{y \le t\phi(x)\}\delta_x$$

are Poisson processes on $\mathbb{R}^d$ with intensity measures $t\mu_1$, $t\mu_2$ and $t\vartheta$, respectively. They satisfy

$$\varrho_t^{(1)}(A) \le \varrho_t(A) \le \varrho_t^{(2)}(A) \text{ a.s.} \quad \text{and} \quad \varrho_t \overset{d}{=} \zeta_t, \qquad A \subset \mathbb{R}^d,\ t > 0.$$

Therefore for any $v \ge 0$, we obtain

$$\mathbb{P}\big(\min_{x\in\zeta_t\cap W} \mu_1(\mathbf{B}(x, C(x, \zeta_t))) > v\big) \le \mathbb{P}\big(\min_{x\in\varrho_t^{(1)}\cap W} \mu_1(\mathbf{B}(x, C(x, \varrho_t^{(1)}))) > v\big) \tag{39}$$

and similarly

$$\mathbb{P}\big(\min_{x\in\zeta_t\cap W} \mu_2(\mathbf{B}(x, C(x, \zeta_t))) > v\big) \ge \mathbb{P}\big(\min_{x\in\varrho_t^{(2)}\cap W} \mu_2(\mathbf{B}(x, C(x, \varrho_t^{(2)}))) > v\big). \tag{40}$$

From Theorem 3.8, it follows for $j = 1, 2$ and $\nu(t) = u(\alpha_2 t^{(d+2)/(d+1)})^{-1}$ with $u \ge 0$, that

$$\lim_{t\to\infty} \mathbb{P}\big(\min_{x\in\varrho_t^{(j)}\cap W} \mu_j(\mathbf{B}(x, C(x, \varrho_t^{(j)}))) > \nu(t)\big) = e^{-\mu_j(W)u^{d+1}}.$$

If $s\phi \le f_1 \le \phi$ for some $s \in (0, 1]$, combining (39), the previous limit with $j = 1$, and the inequality

$$\mathbb{P}\big(\min_{x\in\zeta_t\cap W} s\vartheta(\mathbf{B}(x, C(x, \zeta_t))) > \nu(t)\big) \le \mathbb{P}\big(\min_{x\in\zeta_t\cap W} \mu_1(\mathbf{B}(x, C(x, \zeta_t))) > \nu(t)\big)$$

implies that

$$\limsup_{t\to\infty} \mathbb{P}\big(\min_{x\in\zeta_t\cap W} s\vartheta(\mathbf{B}(x, C(x, \zeta_t))) > \nu(t)\big) \le e^{-\mu_1(W)u^{d+1}}.$$

Then, $s\vartheta(W) \le \mu_1(W)$ concludes the proof of (i). Analogously, if $\phi \le f_2 \le r\phi$ for some $r \ge 1$, combining (40), the limit above with $j = 2$, the inequality

$$\mathbb{P}\big(\min_{x\in\zeta_t\cap W} r\vartheta(\mathbf{B}(x, C(x, \zeta_t))) > \nu(t)\big) \ge \mathbb{P}\big(\min_{x\in\zeta_t\cap W} \mu_2(\mathbf{B}(x, C(x, \zeta_t))) > \nu(t)\big)$$

and $\mu_2(W) \le r\vartheta(W)$ for $u \ge 0$ shows (ii). □

3.4. Large nearest neighbor distances in a Boolean model of disks

Let η be a Poisson process on $\mathbb{R}^2 \times [0, \infty)$ with intensity measure $\gamma\lambda_2 \otimes \mathbb{Q}$, where $\gamma > 0$, λ_2 is the Lebesgue measure on $\mathbb{R}^2$ and $\mathbb{Q}$ is a probability measure on $[0, \infty)$. We can think of η as a marked Poisson process on $\mathbb{R}^2$ with marks in $[0, \infty)$. By putting around each point of η a closed disk whose radius is given by the mark, we obtain the Boolean model

$$Z = \bigcup_{(x,R_x)\in\eta} \overline{\mathbf{B}}(x, R_x).$$

Here and in the following we denote by $\overline{\mathbf{B}}(y, r)$ the closed disk with center y and radius r. Boolean models, where one can allow much more general shapes than disks (or balls), are one of the most prominent models from stochastic geometry; see e.g. [23,33]. For $(x, R_x) \in \eta$ we are interested in the minimal distance between x and $\bigcup_{(y,R_y)\in\eta, y\neq x} \overline{\mathbf{B}}(y, R_y)$. Thus, we define

$$D(x, \eta) = \min\Big\{\|x - z\| : z \in \bigcup_{(y,R_y)\in\eta, y\neq x} \overline{\mathbf{B}}(y, R_y)\Big\},$$

which is the minimal distance from x to other disks of the Boolean model. We say that $\overline{\mathbf{B}}(x, R_x)$ is isolated in Z if it does not intersect other disks of Z. Obviously, $\overline{\mathbf{B}}(x, R_x)$ is isolated if $D(x, \eta) > R_x$. Then $D(x, \eta) - R_x$ is the minimal distance between the disk around x and the other disks of the Boolean model. Moreover, in this case $\overline{\mathbf{B}}(x, R_x)$ is still isolated if we increase each of the radii of the disks by less than $(D(x, \eta)- R_x)/2$. Thus, we consider in the following large $D(x, \eta)$ and the maximum of $D(x, \eta) - R_x$ with $(x, R_x) \in \eta$ such that x belongs to an observation window and study the asymptotic behavior for increasing observation windows. By R we denote a random variable distributed according to $\mathbb{Q}$ and $\xrightarrow{\mathbb{P}}$ stands for convergence in probability.

Theorem 3.15. *Assume that $\mathbb{E}[R^4] < \infty$. Let $(W_t)_{t>1}$ be a family of measurable sets in $\mathbb{R}^2$ such that $\lambda_2(W_t) = t$ for $t > 1$ and define*

$$\xi_t = \sum_{(x,R_x)\in\eta, x\in W_t} \delta_{(\gamma\pi(D(x,\eta)+\mathbb{E}[R])^2-\log(t), R_x)}$$

for $t > 1$. Then, as $t \to \infty$, ξ_t converges in distribution to a Poisson process on $\mathbb{R} \times [0, \infty)$ with intensity measure $\mu \otimes \mathbb{Q}$, where μ is the measure on $\mathbb{R}$ such that

$$\mu((u, \infty)) = \gamma \exp(\gamma\pi(\mathbb{E}[R]^2 - \mathbb{E}[R^2])) \exp(-u)$$

for $u \in \mathbb{R}$. Moreover, we have

$$\max_{(x,R_x)\in\eta, x\in W_t} D(x, \eta) - R_x - \sqrt{\frac{\log(t)}{\gamma\pi}} + \mathbb{E}[R] + r_{\min} \xrightarrow{\mathbb{P}} 0 \quad as \quad t \to \infty, \tag{41}$$

where $r_{\min}$ is the infimum of the support of $\mathbb{Q}$.

The statement in (41) says that for large t we can find with high probability a disk of the Boolean model with center in W_t whose distance to the nearest other disk of the Boolean model is close to $\sqrt{\log(t)/(\gamma\pi)} - \mathbb{E}[R] - r_{\min}$. Roughly speaking, this means that there are very isolated disks with center in W_t in the Boolean model in the sense that they are surrounded by large empty regions.

For the special case $\mathbb{P}(R = 0) = 1$, $D(x, \eta)$ becomes the distance from x to its nearest neighbor in η, which is the same as twice the inradius of the Voronoi cell around x. Thus, ξ_t is the same point process as in Section 3.2 for an underlying stationary Poisson process. For this situation point process convergence follows from Theorem 3.4 and we refer to Section 3.2 for a discussion of the existing literature. In Section 3.2 we have a local dependence structure since whether the inradius of the cell of x belongs to some bounded set depends only on the points of η in some neighborhood of x. For the Boolean model considered in this subsection this is no longer the case. Since the radii of the disks can become arbitrarily large, a point of η that is far away from x and has a large mark can still affect $D(x, \eta)$.

We expect that our results can be extended to Boolean models of balls in higher dimensions, but we refrained from doing so since then ξ_t requires a more involved transformation of the $(D(x, \eta))_{(x,R_x)\in\eta,x\in W_t}$.

Proof of Theorem 3.15. Throughout the proof we use the abbreviation

$$D_t(x, \eta) = \gamma\pi(D(x, \eta) + \mathbb{E}[R])^2 - \log(t)$$

for $t > 1$. For $u \in \mathbb{R}$ and a measurable set $A \subseteq \mathbb{R}$ it follows from the Mecke formula, the fact that $D(x, \eta + \delta_{(x,R_x)}) = D(x, \eta)$ for $(x, R_x) \in \mathbb{R}^2 \times [0, \infty)$ and the stationarity of η that

$$\begin{aligned}\mathbb{E}[\xi_t((u, \infty) \times A)] &= \mathbb{E} \sum_{(x,R_x)\in\eta,x\in W_t} \mathbf{1}\{R_x \in A\}\mathbf{1}\{D_t(x, \eta) > u\} \\ &= \gamma\lambda_2(W_t)\mathbb{P}(R \in A)\mathbb{P}(D_t(0, \eta) > u) = \gamma t\mathbb{Q}(A)\mathbb{P}(D_t(0, \eta) > u).\end{aligned}$$

For $s > 0$ we have

$$\begin{aligned}\mathbb{P}(D(0, \eta) > s) &= \mathbb{P}(\overline{\mathbf{B}}(x, R_x) \cap \overline{\mathbf{B}}(0, s) = \emptyset \text{ for all } (x, R_x) \in \eta) \\ &= \exp\big(-\gamma(\lambda_2 \otimes \mathbb{Q})(\{(x, R_x) \in \mathbb{R}^2 \times [0, \infty) : \overline{\mathbf{B}}(x, R_x) \cap \overline{\mathbf{B}}(0, s) \neq \emptyset\})\big).\end{aligned}$$

Since

$$\begin{aligned}&(\lambda_2 \otimes \mathbb{Q})(\{(x, R_x) \in \mathbb{R}^2 \times [0, \infty) : \overline{\mathbf{B}}(x, R_x) \cap \overline{\mathbf{B}}(0, s) \neq \emptyset\}) \\ &= \mathbb{E}[\pi(R + s)^2] = \pi(s^2 + 2s\mathbb{E}[R] + \mathbb{E}[R^2]) = \pi(s + \mathbb{E}[R])^2 + \pi(\mathbb{E}[R^2] - \mathbb{E}[R]^2),\end{aligned}$$

we obtain

$$\mathbb{P}(D(0, \eta) > s) = \exp(\gamma\pi(\mathbb{E}[R]^2 - \mathbb{E}[R^2])) \exp(-\gamma\pi(s + \mathbb{E}[R])^2).$$

For $t > 1$ so large that $\log(t) + u > 0$ this implies that

$$\begin{aligned}\mathbb{P}(D_t(0, \eta) > u) &= \mathbb{P}(\gamma\pi(D(0, \eta) + \mathbb{E}[R])^2 > \log(t) + u) \\ &= \exp(\gamma\pi(\mathbb{E}[R]^2 - \mathbb{E}[R^2])) \exp\left(-\gamma\pi\left(\sqrt{\frac{\log(t) + u}{\gamma\pi}}\right)^2\right) \qquad (42) \\ &= \exp(\gamma\pi(\mathbb{E}[R]^2 - \mathbb{E}[R^2])) \exp(-\log(t) - u),\end{aligned}$$

whence

$$\lim_{t\to\infty} \mathbb{E}[\xi_t((u, \infty) \times A)] = \mu((u, \infty))\mathbb{Q}(A). \qquad (43)$$

In the following we will apply Theorem 1.4. We choose $\mathcal{U}$ as the ring generated by

$$\{(a, b) \times (\underline{s}, \overline{s}) : a, b, \underline{s}, \overline{s} \in \mathbb{R}, a \leq b, \underline{s} \leq \overline{s}, \mathbb{Q}(\{\underline{s}\}) = \mathbb{Q}(\{\overline{s}\}) = 0\}.$$

The sets of $\mathcal{U}$ can be written as unions of Cartesian products of bounded intervals. For open sets from $\mathcal{U}$ as considered in Theorem 1.4 these intervals can be chosen open. We fix $m \in \mathbb{N}_0$ and an open set $B \in \mathcal{U}$, i.e. $B = \bigcup_{i=1}^{k} (a_i, b_i) \times (\underline{s}_i, \overline{s}_i)$ with $k \in \mathbb{N}$. In the following we show that, for $u \in \mathbb{R}$ and $\underline{s}, \overline{s} \in [0, \infty)$ with $\underline{s} \leq \overline{s}$,

$$\begin{aligned}\lim_{t\to\infty} \gamma \int_{W_t} \int_0^\infty \mathbb{E}\Big[&\mathbf{1}\{D_t(x, \eta + \delta_{(x,R_x)}) \in (u, \infty), R_x \in (\underline{s}, \overline{s})\} \\ &\times \mathbf{1}\Big\{ \sum_{(y,R_y)\in\eta, y\in W_t} \mathbf{1}\{(D_t(y, \eta + \delta_{(x,R_x)}), R_y) \in B\} = m \Big\}\Big] d\mathbb{Q}(R_x) dx \\ &- \mu((u, \infty))\mathbb{P}(R \in (\underline{s}, \overline{s}))\mathbb{P}(\xi_t(B) = m) = 0.\end{aligned} \tag{44}$$

By (43) this identity still holds if we replace (u, ∞) by $[u, \infty)$. Since, by the inclusion–exclusion principle, the left hand-side of (7) can be written in terms of expressions as on the left-hand side of (44) (or with $[u, \infty)$ instead of (u, ∞)), (44) implies (7) and, thus, proves the point process convergence.

Define $a = \min\{a_1, \ldots, a_k\}$ and $b = \max\{b_1, \ldots, b_k\}$ as well as

$$r_{t,u} = \sqrt{\frac{\log(t) + u}{\gamma\pi}} - \mathbb{E}[R], \quad \underline{r}_t = \sqrt{\frac{\log(t) + a}{\gamma\pi}} - \mathbb{E}[R] \quad \text{and} \quad \overline{r}_t = \sqrt{\frac{\log(t) + b}{\gamma\pi}} - \mathbb{E}[R].$$

In the following we always assume that t is so large that the expressions in the square roots are positive and that $r_{t,u}$, $\underline{r}_t$ and $\overline{r}_t$ are positive. Let $(x, R_x) \in \mathbb{R}^2 \times [0, \infty)$ be fixed. By $\eta_{x,t,u}$ we denote the restriction of η to

$$M_{x,t,u} := \{(y, R_y) \in \mathbb{R}^2 \times [0, \infty) : \overline{\mathbf{B}}(y, R_y) \cap \overline{\mathbf{B}}(x, r_{t,u}) = \emptyset\}.$$

Since the event $D_t(x, \eta + \delta_{(x,R_x)}) \in (u, \infty)$ is independent from $\eta_{x,t,u}$ and implies $\eta = \eta_{x,t,u}$, we see that

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}\{D_t(x, \eta + \delta_{(x,R_x)}) \in (u, \infty)\}\mathbf{1}\Big\{ \sum_{(y,R_y)\in\eta, y\in W_t} \mathbf{1}\{(D_t(y, \eta + \delta_{(x,R_x)}), R_y) \in B\} = m \Big\}\Big] \\ &= \mathbb{P}(D_t(x, \eta) > u)\mathbb{P}\Big(\sum_{(y,R_y)\in\eta_{x,t,u}, y\in W_t} \mathbf{1}\{(D_t(y, \eta_{x,t,u} + \delta_{(x,R_x)}), R_y) \in B\} = m \Big).\end{aligned} \tag{45}$$

Observe that

$$\begin{aligned}&\Big|\mathbf{1}\Big\{ \sum_{(y,R_y)\in\eta_{x,t,u}, y\in W_t} \mathbf{1}\{(D_t(y, \eta_{x,t,u} + \delta_{(x,R_x)}), R_y) \in B\} = m \Big\} \\ &\quad - \mathbf{1}\Big\{ \sum_{(y,R_y)\in\eta_{x,t,u}, y\in W_t} \mathbf{1}\{(D_t(y, \eta_{x,t,u}), R_y) \in B\} = m \Big\}\Big| \\ &\leq \sum_{(y,R_y)\in\eta_{x,t,u}, y\in W_t} \mathbf{1}\{\|x - y\| \leq R_x + \overline{r}_t\}\mathbf{1}\{D_t(y, \eta_{x,t,u}) > a\}\end{aligned}$$

since y can be only taken into account in one of the sums if $D_t(y, \eta_{x,t,u}) > a$ and it can be counted differently in the sums only if its contribution is affected by (x, R_x) (i.e.

$\|x-y\| \leq R_x + \overline{r}_t$). Similarly, we obtain that

$$\Bigg|\mathbf{1}\Bigg\{\sum_{(y,R_y)\in\eta_{x,t,u},y\in W_t}\mathbf{1}\{(D_t(y,\eta_{x,t,u}),R_y)\in B\}=m\Bigg\} - \mathbf{1}\Bigg\{\sum_{(y,R_y)\in\eta,y\in W_t}\mathbf{1}\{(D_t(y,\eta),R_y)\in B\}=m\Bigg\}\Bigg|$$
$$\leq \sum_{(y,R_y)\in\eta,y\in W_t}\mathbf{1}\{\|x-y\|\leq r_{t,u}+R_y\}\mathbf{1}\{D_t(y,\eta)>a\}$$
$$+\sum_{(y,R_y)\in\eta_{x,t,u},y\in W_t}\mathbf{1}\{\exists(z,R_z)\in\eta:\|x-z\|\leq R_z+r_{t,u},\|y-z\|\leq R_z+\overline{r}_t\}$$
$$\times\mathbf{1}\{D_t(y,\eta_{x,t,u})>a\},$$

where the first sum represents the points belonging to η but not to $\eta_{x,t,u}$ and the second sum is for the points whose contribution differs with respect to η and $\eta_{x,t,u}$. These estimates imply together with the Mecke formula that

$$\Bigg|\mathbb{P}\Bigg(\sum_{(y,R_y)\in\eta_{x,t,u},y\in W_t}\mathbf{1}\{(D_t(y,\eta_{x,t,u}+\delta_{(x,R_x)}),R_y)\in B\}=m\Bigg)-\mathbb{P}(\xi_t(B)=m)\Bigg|$$
$$\leq\gamma\int_{W_t}\mathbf{1}\{\|x-y\|\leq R_x+\overline{r}_t\}\mathbb{P}(D_t(y,\eta_{x,t,u})>a)dy$$
$$+\gamma\int_{W_t}\int_0^\infty\mathbf{1}\{\|x-y\|\leq r_{t,u}+R_y\}\mathbb{P}(D_t(y,\eta)>a)d\mathbb{Q}(R_y)dy$$
$$+\gamma\int_{M_{x,t,u}}\mathbb{P}(\exists(z,R_z)\in\eta:\|x-z\|\leq R_z+r_{t,u},\|y-z\|\leq R_z+\overline{r}_t)$$
$$\times\mathbb{P}(D_t(y,\eta_{x,t,u})>a)d(\lambda_2\otimes\mathbb{Q})((y,R_y))$$
$$=: T_1+T_2+T_3.$$

One can find constants $c, v_0 \in (0,\infty)$ independent from x and t such that

$$\gamma(\lambda_2\otimes\mathbb{Q})(\{(z,R_z)\in M_{x,t,u}:\overline{\mathbf{B}}(y,v)\cap\overline{\mathbf{B}}(z,R_z)\neq\emptyset\})\geq cv^2$$

for all $y\in\mathbb{R}^2\setminus\overline{\mathbf{B}}(x,r_{t,u})$ and $v\geq v_0$. The idea here is to construct a subset of $\overline{\mathbf{B}}(y,v)$ such that for points in this set with sufficiently small radii $\overline{\mathbf{B}}(x,r_{t,u})$ is not intersected. For t sufficiently large the previous inequality yields that

$$\mathbb{P}(D_t(y,\eta_{x,t,u})>a)=\mathbb{P}(D(y,\eta_{x,t,u})>\underline{r}_t)\leq\exp(-c\underline{r}_t^2)$$

and there exist $C,\tau\in(0,\infty)$ such that

$$\mathbb{P}(D_t(y,\eta)>a)\leq\mathbb{P}(D_t(y,\eta_{x,t,u})>a)\leq Ct^{-\tau}.$$

Thus, we obtain that

$$T_1\leq C\gamma\pi(R_x+\overline{r}_t)^2t^{-\tau}\quad\text{and}\quad T_2\leq C\gamma\pi\mathbb{E}[(R+r_{t,u})^2]t^{-\tau}$$

for t sufficiently large. Because of

$$\mathbb{P}(\exists(z,R_z)\in\eta:\|x-z\|\leq R_z+r_{t,u},\|y-z\|\leq R_z+\overline{r}_t)$$
$$\leq\gamma\int_{\mathbb{R}^2}\int_0^\infty\mathbf{1}\{\|x-z\|\leq R_z+r_{t,u},\|y-z\|\leq R_z+\overline{r}_t\}d\mathbb{Q}(R_z)dz,$$

we deduce that

$$T_3 \le C\gamma^2\pi^2\mathbb{E}[(R+r_{t,u})^2(R+\overline{r}_t)^2]t^{-\tau}.$$

Since the estimates for T_1, T_2, T_3 vanish uniformly in x for $t \to \infty$, together with (42) and (45) we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\gamma\int_{W_t}\int_0^\infty \mathbb{E}\Big[\mathbf{1}\{D_t(x,\eta+\delta_{(x,R_x)})\in(u,\infty),\, R_x\in(\underline{s},\overline{s})\}\\
&\qquad\times\mathbf{1}\Big\{\sum_{(y,R_y)\in\eta,\, y\in W_t}\mathbf{1}\{(D_t(y,\eta+\delta_{(x,R_x)}),R_y)\in B\}=m\Big\}\Big]d\mathbb{Q}(R_x)dx\\
&\qquad-\mu((u,\infty))\mathbb{P}(R\in(\underline{s},\overline{s}))\mathbb{P}(\xi_t(B)=m)\\
&=\lim_{t\to\infty}\gamma\int_{W_t}\mathbb{P}(D_t(x,\eta)>u)\mathbb{P}(R\in(\underline{s},\overline{s}))dx\ \mathbb{P}(\xi_t(B)=m)\\
&\qquad-\mu((u,\infty))\mathbb{P}(R\in(\underline{s},\overline{s}))\mathbb{P}(\xi_t(B)=m)=0,
\end{aligned}$$

which establishes (44) and, thus, proves the point process convergence of ξ_t.

Let $\varepsilon>0$ be fixed. We have

$$\begin{aligned}
&\mathbb{P}\Big(\max_{(x,R_x)\in\eta,x\in W_t} D(x,\eta)-R_x-\sqrt{\frac{\log(t)}{\gamma\pi}}+\mathbb{E}[R]+r_{\min}\ge\varepsilon\Big)\\
&=\mathbb{P}\Big(\exists(x,R_x)\in\eta: x\in W_t,\, D(x,\eta)-R_x-\sqrt{\frac{\log(t)}{\gamma\pi}}+\mathbb{E}[R]+r_{\min}\ge\varepsilon\Big)\\
&\le\mathbb{P}\Big(\exists(x,R_x)\in\eta: x\in W_t,\, D(x,\eta)+\mathbb{E}[R]\ge\sqrt{\frac{\log(t)}{\gamma\pi}}+\varepsilon\Big)\\
&=\mathbb{P}\Big(\exists(x,R_x)\in\eta: x\in W_t,\, \gamma\pi(D(x,\eta)+\mathbb{E}[R])^2-\log(t)\ge 2\varepsilon\sqrt{\gamma\pi\log(t)}+\gamma\pi\varepsilon^2\Big)\\
&\le\mathbb{E}[\xi_t([2\varepsilon\sqrt{\gamma\pi\log(t)}+\gamma\pi\varepsilon^2,\infty)\times[0,\infty))].
\end{aligned}$$

Since, for any $u>0$ and t sufficiently large, the right-hand side is dominated by $\mathbb{E}[\xi_t((u,\infty)\times[0,\infty))]$, (43) yields

$$\lim_{t\to\infty}\mathbb{P}\Big(\max_{(x,R_x)\in\eta,x\in W_t} D(x,\eta)-R_x-\sqrt{\frac{\log(t)}{\gamma\pi}}+\mathbb{E}[R]+r_{\min}\ge\varepsilon\Big)=0.$$

On the other hand, we have

$$\begin{aligned}
&\mathbb{P}\Big(\max_{(x,R_x)\in\eta,x\in W_t} D(x,\eta)-R_x-\sqrt{\frac{\log(t)}{\gamma\pi}}+\mathbb{E}[R]+r_{\min}\ge-\varepsilon\Big)\\
&=\mathbb{P}\Big(\exists(x,R_x)\in\eta: x\in W_t,\, D(x,\eta)-R_x-\sqrt{\frac{\log(t)}{\gamma\pi}}+\mathbb{E}[R]+r_{\min}\ge-\varepsilon\Big)\\
&\ge\mathbb{P}\Big(\exists(x,R_x)\in\eta: x\in W_t,\, R_x\le r_{\min}+\frac{\varepsilon}{2},\, D(x,\eta)+\mathbb{E}[R]\ge\sqrt{\frac{\log(t)}{\gamma\pi}}+R_x-r_{\min}-\varepsilon\Big)\\
&\ge\mathbb{P}\Big(\exists(x,R_x)\in\eta: x\in W_t,\, R_x\le r_{\min}+\frac{\varepsilon}{2},\, D(x,\eta)+\mathbb{E}[R]\ge\sqrt{\frac{\log(t)}{\gamma\pi}}-\frac{\varepsilon}{2}\Big)
\end{aligned}$$

$$= \mathbb{P}\Big(\exists (x, R_x) \in \eta : x \in W_t, R_x \le r_{\min} + \varepsilon/2,$$
$$\gamma\pi(D(x,\eta) + \mathbb{E}[R])^2 - \log(t) \ge -\varepsilon\sqrt{\gamma\pi\log(t)} + \gamma\pi\varepsilon^2/4\Big)$$
$$= \mathbb{P}(\xi_t([-\varepsilon\sqrt{\gamma\pi\log(t)} + \gamma\pi\varepsilon^2/4, \infty) \times [r_{\min}, r_{\min} + \varepsilon/2]) > 0).$$

For any fixed $u > 0$ and t sufficiently large we have

$$\xi_t([-\varepsilon\sqrt{\gamma\pi\log(t)} + \gamma\pi\varepsilon^2/4, \infty) \times [r_{\min}, r_{\min} + \varepsilon/2]) \ge \xi_t((-u, u) \times [r_{\min}, r_{\min} + \varepsilon/2])$$

so that the point process convergence of ξ_t and letting $u \to \infty$ yield

$$\lim_{t\to\infty} \mathbb{P}\left(\max_{(x,R_x)\in\eta, x\in W_t} D(x,\eta) - R_x - \sqrt{\frac{\log(t)}{\gamma\pi}} + \mathbb{E}[R] + r_{\min} \ge -\varepsilon\right) = 1,$$

which completes the proof of (41). □

Remark 3.16. Due to the assumption $\mathbb{E}[R^4] < \infty$ in Theorem 3.15, one can show that there exists a family of positive real numbers $(s_t)_{t>1}$ such that $\frac{s_t^2}{t} \to 0$ and

$$\mathbb{P}(\exists (x_1, R_{x_1}), (x_2, R_{x_2}) \in \eta : x_1 \in W_t, \|x_1 - x_2\| > r_{t,u} + s_t, \overline{\mathbf{B}}(x_1, r_{t,u}) \cap \overline{\mathbf{B}}(x_2, R_{x_2}) \ne \emptyset) \to 0$$

as $t \to \infty$, i.e. the probability that the restriction of ξ_t to (u, ∞) is affected by a radius greater than s_t vanishes as $t \to \infty$. Thus, it is sufficient to study the modified model where all radii are truncated at s_t. Since the truncation leads to local dependence, Poisson and Poisson process approximation results that require local dependence become available, which could provide an alternative approach to prove Theorem 3.15.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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