



Brief paper

Robust targeted exploration for systems with non-stochastic disturbances[☆]Janani Venkatasubramanian^{a,b,*}, Johannes Köhler^{c,d}, Mark Cannon^e, Frank Allgöwer^a^a Institute for Systems Theory and Automatic Control, University of Stuttgart, Stuttgart, Germany^b Institute of Control Systems, Hamburg University of Technology, Hamburg, Germany^c Institute for Dynamic Systems and Control, ETH Zürich, Zürich, Switzerland^d Department of Mechanical Engineering, Imperial College London, London, United Kingdom^e Control Group, University of Oxford, Parks Road, Oxford, United Kingdom

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ABSTRACT

In this paper, we introduce a novel targeted exploration strategy designed specifically for uncertain linear time-invariant systems with energy-bounded disturbances, i.e., without any assumptions on the distribution of the disturbances. We use classical results characterizing the set of non-falsified parameters consistent with energy-bounded disturbances. We derive a semidefinite program which computes an exploration strategy that guarantees a desired accuracy of the parameter estimate. This design is based on sufficient conditions on the spectral content of the exploration data that robustly account for initial parametric uncertainty. Finally, we highlight the applicability of the exploration strategy through a numerical example involving a nonlinear system.

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1. Introduction

Designing reliable controllers for unknown dynamical systems requires accurate knowledge of the model parameters, which can be obtained from data (Gevers, 2005). The accuracy of the parameters significantly depends on the quality of the data used for system identification. Informative data can be strategically obtained from an experiment through the process of *targeted exploration* or *optimal experiment design* (Gevers & Ljung, 1986; Pronzato, 2008). Specifically, targeted exploration inputs are tailored to reduce model uncertainty, thereby ensuring the attainment of a desired accuracy in the identified model (Bombois et al., 2021; Jansson & Hjalmarsson, 2005), or the feasibility of

robust control design (Barenthin & Hjalmarsson, 2008; Ferizbegovic et al., 2019; Umenberger et al., 2019; Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024). In this paper, we develop a targeted exploration strategy for uncertain linear systems subject to energy-bounded disturbances.

Targeted exploration is typically studied in a stochastic setup, for example, with independent and identically distributed (i.i.d.) disturbances with zero mean. In this case, one can construct a data-based confidence ellipsoid for the parameters (Ljung, 1999) that can be approximately predicted and optimized. This classical asymptotic result has been utilized to design targeted exploration strategies for dual control methods (Barenthin & Hjalmarsson, 2008; Iannelli et al., 2020; Umenberger et al., 2019; Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024). Tight confidence regions for the parameters are constructed by Dean et al. (2019), assuming the data is independent, and hence it is not applicable to correlated time-series data from a single trajectory. Nevertheless, this bound has been utilized in the design of targeted exploration for dual control by Ferizbegovic et al. (2019). A common feature among all of the discussed targeted exploration approaches is that they consider linear systems subject to i.i.d. zero mean stochastic disturbances. However, real-world systems often exhibit nonlinear behavior or unmodeled dynamics which introduce additional deterministic model mismatch and cannot be explained by independent stochastic noise (Sarker et al., 2023). Instead, such effects can be modeled as bounded disturbances, assuming that the disturbances belong to a known bounded set, e.g., point-wise bounds or energy bounds. To quantify models

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under such disturbances, set-membership estimation methods have been developed, and various identification results in this direction can be found in the works of Bisoffi et al. (2021), Fogel (1979), Milanese (1989), Sarker et al. (2023) and Shakouri et al. (2024). Although these methods have recently gained popularity for data-driven robust control design (Berberich et al., 2023; van Waarde et al., 2022), a principled method for optimal experiment design in the presence of bounded disturbances is still lacking.

In this paper, we design a targeted exploration strategy that ensures a desired error bound on the estimated parameters by utilizing a data-dependent uncertainty bound based on energy-bounded disturbances (Fogel, 1979). Unlike existing experiment design methods, which often rely on stochastic disturbance assumptions (Ferizbegovic et al., 2019; Umenberger et al., 2019; Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024), our approach provides robustness against non-stochastic, adversarial disturbances and is applicable to systems with bounded nonlinearities. We consider multi-sine exploration inputs of specific frequencies and optimized amplitudes to explicitly shape and reduce uncertainty in a targeted manner. As a main contribution, we derive sufficient conditions on the spectral content of the exploration data that asymptotically guarantees a desired error-bound on the parameters estimated through exploration. We utilize these sufficient conditions to derive LMIs for exploration which ensure the desired error bound on the parameters. This approach gives rise to a targeted exploration design with minimal input energy based on a semidefinite program (SDP). Unlike existing methods for stochastic disturbances (Ferizbegovic et al., 2019; Umenberger et al., 2019; Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024), the non-stochastic identification results considered in this work yield structurally different requirements on excitation, which require new worst-case robust guarantees rather than high-probability guarantees. To this end, we formulate the problem in Section 2, and review data-dependent uncertainty bounds in Section 3. In Section 4, we derive the exploration strategy by leveraging a sufficient conditions on time-series exploration data, as proposed in our previous work (Venkatasubramanian et al., 2024b), while additionally robustly accounting for parametric uncertainty and error due to disturbances with suitable bounds. In Section 5, we provide a numerical example to highlight the applicability of the proposed exploration strategy to nonlinear systems. We remark that the proofs of Lemma 9, Propositions 11 and 12, and Theorem 13 are not included in this brief paper, but can be found in the technical report (Venkatasubramanian et al., 2024a).

2. Problem statement

Notation: For a matrix $A \in \mathbb{R}^{n \times m}$, $A^\top$ denotes the transpose. For a matrix $A \in \mathbb{C}^{n \times m}$, A^H denotes the conjugate transpose. The positive (semi-)definiteness of a matrix $A \in \mathbb{C}^{n \times n}$ is denoted by $(A = A^H \geq 0)$ $A = A^H \succ 0$. The operator $\text{vec}(A)$ stacks the columns of A to form a vector. The operator $\text{diag}(A_1, \dots, A_n)$ creates a block diagonal matrix by aligning the matrices $A_1, \dots, A_n$ along the diagonal with A_1 in the upper left corner. The Kronecker product is denoted by $\otimes$. The Euclidean norm and the weighted Euclidean norm for a vector $x \in \mathbb{R}^n$ and a matrix $P \succ 0$ are denoted by $\|x\| = \sqrt{x^\top x}$ and $\|x\|_P = \sqrt{x^\top P x}$, respectively. The largest singular value of a matrix $A \in \mathbb{C}^{m \times n}$ is $\|A\|$. Furthermore, given a matrix $M \geq 0$, $\|A\|_M = \|M^{1/2}A\|$ where $M^{1/2}$ is the symmetric square root matrix of M . The identity matrix of size n is denoted by I_n . A vector of ones of size n is denoted by $\mathbf{1}_n \in \mathbb{R}^{n \times 1}$.

Amplitude of a spectral line (Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024, Def. 2): The amplitude of the spectral line of

a sequence $\{\phi_k\}_{k=0}^{T-1}$ at a frequency $\omega \in \Omega_T := \{0, 1/T, \dots, (T-1)/T\}$ is given by

$$\bar{\phi}(\omega) := \frac{1}{T} \sum_{k=0}^{T-1} \phi_k e^{-j2\pi \omega k}. \quad (1)$$

2.1. Setting

Consider a discrete-time, linear time-invariant system of the form

$$x_{k+1} = A_{\text{tr}} x_k + B_{\text{tr}} u_k + w_k, \quad (2)$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, and $w_k \in \mathbb{R}^{n_x}$ is the disturbance. In our setting, the true system parameters $A_{\text{tr}}, B_{\text{tr}}$ are uncertain. Hence, it is necessary to collect informative data from an optimal experiment for a fixed $T \in \mathbb{N}$ time steps to improve parameter accuracy. To simplify the exposition, we assume that the initial state is at the origin, i.e., $x_0 = 0$. Furthermore, we assume that the state can be measured and the disturbances are energy-bounded.

Assumption 1. The disturbances w are energy-bounded; there exists a known constant $\gamma_w > 0$ such that

$$\sum_{k=0}^{T-1} \|w_k\|^2 \leq \gamma_w. \quad (3)$$

This assumption allows the system to exhibit nonlinear behavior, as long as the nonlinearities can be bounded in energy (as discussed in Section 5). Importantly, this assumption distinguishes our approach from existing experiment design methods which are typically restricted to i.i.d. Gaussian disturbances, e.g., Ferizbegovic et al. (2019), Umenberger et al. (2019) and Venkatasubramanian, Köhler, Berberich and Allgöwer (2024).

Exploration goal: Since the true system parameters $\theta_{\text{tr}} = \text{vec}([A_{\text{tr}}, B_{\text{tr}}]) \in \mathbb{R}^{n_\theta}$, with $n_\theta = n_x(n_x + n_u)$, are not precisely known, exploratory inputs should be designed to excite the system to gather informative data. Specifically, our objective is to design inputs that excite the system in a manner as to obtain an estimate $\hat{\theta}_T = \text{vec}([\hat{A}_T, \hat{B}_T])$ that satisfies

$$(\theta_{\text{tr}} - \hat{\theta}_T)^\top (D_{\text{des}} \otimes I_{n_x}) (\theta_{\text{tr}} - \hat{\theta}_T) \leq 1, \quad (4)$$

where $D_{\text{des}} \succ 0$ is a user-defined matrix characterizing closeness of $\hat{\theta}_T$ to θ_{tr} . We assume that we have some prior knowledge about the system dynamics.

Assumption 2. The unknown parameters $\theta_{\text{tr}} = \text{vec}([A_{\text{tr}}, B_{\text{tr}}])$ lie in a known set Θ_0 , i.e., $\theta_{\text{tr}} \in \Theta_0$, where

$$\Theta_0 := \left\{ \theta : (\hat{\theta}_0 - \theta)^\top (D_0 \otimes I_{n_x}) (\hat{\theta}_0 - \theta) \leq 1 \right\}, \quad (5)$$

with an estimate $\hat{\theta}_0 = \text{vec}([\hat{A}_0, \hat{B}_0])$ for some $D_0 \succ 0$.

Remark 3. An initial estimate $\hat{\theta}_0$ and set Θ_0 as in Assumption 2 result naturally from a finite-horizon experiment under Assumption 1 (cf. Lemma 6).

2.2. Exploration strategy

The exploration input sequence takes the form

$$u_k = \sum_{i=1}^L \tilde{u}(\omega_i) \cos(2\pi \omega_i k), \quad k = 0, \dots, T-1 \quad (6)$$

where T is the exploration time and $\tilde{u}(\omega_i) \in \mathbb{R}^{n_u}$ are the amplitudes of the sinusoidal inputs at $L \in \mathbb{N}$ distinct selected

frequencies $\omega_i \in \Omega_T$ with $n_x + n_u \leq L \leq T$. In practice, frequencies may be selected based on prior information (Mehra, 1974). Since the input signal is deterministic and sinusoidal, the amplitude of the spectral line of the sequence $\{u_k\}_{k=0}^{T-1}$ at frequency ω_i is $\tilde{u}(\omega_i)$. Denote $U_e = \text{diag}(\tilde{u}(\omega_1), \dots, \tilde{u}(\omega_L)) \in \mathbb{R}^{L n_u \times L}$. The exploration input is computed such that it excites the system sufficiently with minimal input energy, based on the initial parameter estimates. Bounding the input energy by a constant γ_e^2 is equivalent to $\sum_{i=1}^L \|\tilde{u}(\omega_i)\|^2 = \mathbf{1}_L^T U_e^T U_e \mathbf{1}_L \leq \gamma_e^2$ where $\mathbf{1}_L \in \mathbb{R}^{L \times 1}$ is a vector of ones, and the bound $\gamma_e \geq 0$ is desired to be small. Using the Schur complement, this criterion is equivalent to

$$S_{\text{energy-bound}}(\gamma_e, U_e) := \begin{bmatrix} \gamma_e & \mathbf{1}_L^T U_e^T \\ U_e \mathbf{1}_L & \gamma_e I \end{bmatrix} \geq 0. \quad (7)$$

In order to design the exploration inputs, we make the following assumption regarding the system dynamics.

Assumption 4. The system matrix A_{tr} is Schur stable.

Remark 5. A_{tr} is required to be Schur stable since we consider only open-loop inputs in our exploration strategy (6). Assumption 4 could be relaxed by augmenting the inputs in (6) with an additional linear feedback, i.e., $v_k = u_k + Kx_k$, ensuring robust stability for all $\theta \in \Theta_0$ (5).

To achieve the exploration goal, the amplitudes of the sinusoidal exploration inputs need to be optimized such that by applying the exploration inputs, the obtained estimate satisfies the desired uncertainty bound (4).

3. Preliminaries on data-driven uncertainty quantification

In this section, we discuss a data-dependent uncertainty bound on the parameter estimates in the presence of energy-bounded noise (Fogel, 1979). Given observed data $\mathcal{D}_{T+1} = \{x_k, u_k\}_{k=0}^T$ of length $T+1$, the objective is to quantify the uncertainty associated with the unknown parameters θ_{tr} . Henceforth, we denote $\phi_k = [x_k^T u_k^T]^T \in \mathbb{R}^{n_\phi}$ where $n_\phi = n_x + n_u$. The system (2) can be re-written in terms of parameter $\theta_{\text{tr}} = \text{vec}([A_{\text{tr}}, B_{\text{tr}}])$ as

$$x_{k+1} = (\phi_k^T \otimes I_{n_x}) \theta_{\text{tr}} + w_k.$$

In order to simplify the exposition, we denote

$$\Phi = [\phi_0, \dots, \phi_{T-1}] \in \mathbb{R}^{n_\phi \times T}, \\ X^T = [x_1^T, \dots, x_T^T] \in \mathbb{R}^{1 \times T n_x}.$$

We obtain the following expressions for the mean $\hat{\theta}_T = \text{vec}([\hat{A}_T, \hat{B}_T])$ and covariance P of the parameters from the standard least squares formulation (Ljung, 1999, Section 1.3):

$$\hat{\theta}_T = P \sum_{k=0}^{T-1} (\phi_k^T \otimes I_{n_x})^T x_{k+1} = P(\Phi \otimes I_{n_x})X, \quad (8)$$

$$P = \left(\sum_{k=0}^{T-1} \phi_k \phi_k^T \right)^{-1} \otimes I_{n_x} = (\Phi \Phi^T)^{-1} \otimes I_{n_x}. \quad (9)$$

The non-falsified region for the uncertain parameters θ is provided in the following lemma.

Lemma 6 (Fogel, 1979). Let Assumption 1 hold. Given data set $\mathcal{D}_{T+1}$, the set of non-falsified parameters θ is given by

$$\Theta_T := \left\{ \theta : (\theta - \hat{\theta}_T)^T P^{-1} (\theta - \hat{\theta}_T) \leq G \right\} \quad (10)$$

where

$$G = \gamma_w + \|\hat{\theta}_T\|_{p-1}^2 - X^T X. \quad (11)$$

Under Assumption 1, the non-falsified set Θ_T provides an exact characterization of the set of parameters explaining the data. The ellipsoid (10), derived from energy-bounded constraints, is characterized by $\hat{\theta}_T$ and P , which coincide with the mean and covariance of the least squares estimator under Gaussian disturbances (Umenberger et al., 2019, Prop. 2.1). However, unlike the Gaussian case, the scaling G (11) of the bounding ellipsoid is also data-dependent in the case of energy-bounded disturbances, which significantly affects optimal experiment design. Furthermore, for $\gamma_w \propto T$, G scales at most linearly with T as $T \rightarrow \infty$ since $G \leq \gamma_w$ (Fogel, 1979, Lemma 4). In contrast, the scaling of the confidence ellipsoid in the Gaussian case does not depend on T (Ljung, 1999; Umenberger et al., 2019). Since P^{-1} increases linearly with T , the confidence ellipsoid in the Gaussian case shrinks with T (Ljung, 1999; Umenberger et al., 2019). However, in the considered case of energy-bounded disturbances, the size of the non-falsified set Θ_T , in general, does not decrease as $T \rightarrow \infty$.

4. Targeted exploration

In this section, we propose a targeted exploration strategy based on the data-dependent uncertainty bound in Lemma 6. The exploration strategy builds upon sufficient conditions on the exploration data outlined in Section 4.1. In particular, we derive bounds on the exploration data using the spectral information of the exploration inputs in Section 4.2. Since these bounds depend on the uncertain model parameters, bounds on the effect of model uncertainty are derived in Section 4.3. We utilize these bounds to derive sufficient conditions on the spectral information of the exploration inputs in Section 4.4. However, since the derived sufficient conditions are non-convex in the decision variables, a convex relaxation procedure is carried out in Section 4.5. Finally, in Section 4.6, the exploration problem is reduced to a set of LMIs that provide us exploration inputs that ensure the exploration goal.

4.1. Sufficient conditions for exploration

Given the form of the exploration inputs in (6), the exploration goal (4), and the data-dependent uncertainty bound in Lemma 6, in what follows, we provide conditions that the exploration data have to satisfy to achieve the exploration goal. Denote the Cholesky decomposition of D_{des} as $D_{\text{des}} = D_{\text{des}}^{\frac{1}{2}T} D_{\text{des}}^{\frac{1}{2}}$. The following proposition presents a sufficient condition to ensure that the exploration goal is achieved.

Theorem 7 (Venkatasubramanian et al., 2024b, Theorem 4). Suppose Φ and X satisfy

$$\begin{bmatrix} \Phi \Phi^T - \gamma_w D_{\text{des}} & 0 \\ 0 & 0 \end{bmatrix} + \underbrace{\begin{bmatrix} D_{\text{des}}^{\frac{1}{2}T} (X^T \otimes I_{n_\phi}) \\ (\Phi \otimes I_{n_x}) \otimes I_{n_\phi} \end{bmatrix}}_{=: Z} \begin{bmatrix} D_{\text{des}}^{\frac{1}{2}T} (X^T \otimes I_{n_\phi}) \\ (\Phi \otimes I_{n_x}) \otimes I_{n_\phi} \end{bmatrix}^T \geq 0. \quad (12)$$

Then, the estimate $\hat{\theta}_T$ computed as in (8) satisfies the exploration goal (4).

To compute the amplitudes of the exploration inputs U_e , a few challenges need to be addressed. Note that Inequality (12) depends on X and Φ quadratically, which further depend on the amplitudes of the exploration inputs U_e (6) and the disturbance w . Furthermore, due to uncertain true dynamics $A_{\text{tr}}, B_{\text{tr}}$, the linear mapping from the input sequence to the state sequence is not known.

4.2. Bounds based on the theory of spectral lines

In what follows, we address the aforementioned issues by determining sufficient conditions for targeted exploration in terms of the spectral content of X and Φ based on the theory of spectral lines. Given u_k as in (6), x_k has L spectral lines from 0 to $T - 1$ at distinct frequencies $\omega_i \in \Omega_T$, $i = 1, \dots, L$ with amplitudes (Sarker et al., 2023, Lemma 1):

$$\bar{x}(\omega_i) = \underbrace{(e^{j2\pi\omega_i}I - A_{tr})^{-1}B_{tr}}_{=:V_{x,i}} \bar{u}(\omega_i) + \underbrace{(e^{j2\pi\omega_i}I - A_{tr})^{-1}\bar{w}(\omega_i)}_{=:Y_{x,i}} + \bar{x}_{err}(\omega_i). \quad (13)$$

The transient error in the amplitude of a spectral line $\bar{x}_{err}(\omega_i)$ decays uniformly (with rate $\frac{1}{\sqrt{T}}$) to 0 as $T \rightarrow \infty$ (cf. Assumption 4, (Ljung, 1999, Theorem 2.1)). To simplify the exposition, we will assume that it can be neglected (Sarker et al., 2023).

Assumption 8. The transient error satisfies $\bar{x}_{err}(\omega_i) = 0$ for all $\omega_i \in \Omega_T$.

Note that this assumption holds naturally if we let $T \rightarrow \infty$. We refer the reader to the work by Venkatasubramanian et al. (2025) for a rigorous treatment of the transient error term. More compactly, let us define

$$\bar{X} = [\bar{x}(\omega_1)^T, \dots, \bar{x}(\omega_L)^T]^T \in \mathbb{C}^{n_x L} \quad (14)$$

which satisfies

$$\bar{X} = \underbrace{V_{x,tr}U_e \mathbf{1}_L}_{=: \bar{X}_u} + \underbrace{Y_{x,tr}W \mathbf{1}_L}_{=: \bar{X}_w}, \quad (15)$$

with

$$\begin{aligned} V_{x,tr} &:= \text{diag}(V_{x,1}, \dots, V_{x,L}) \in \mathbb{C}^{n_x L \times n_u L}, \\ Y_{x,tr} &:= \text{diag}(Y_{x,1}, \dots, Y_{x,L}) \in \mathbb{C}^{n_x L \times n_x L}, \\ W &:= \text{diag}(\bar{w}(\omega_1), \dots, \bar{w}(\omega_L)) \in \mathbb{C}^{n_x L \times L}. \end{aligned} \quad (16)$$

Furthermore, ϕ_k has L spectral lines from 0 to $T - 1$ at distinct frequencies $\omega_i \in \Omega_T$, $i = 1, \dots, L$ with amplitudes

$$\bar{\phi}(\omega_i) := \underbrace{\begin{bmatrix} V_{\phi,i} \\ I_{n_u} \end{bmatrix}}_{=:V_{\phi,i}} \bar{u}(\omega_i) + \underbrace{\begin{bmatrix} Y_{\phi,i} \\ 0 \end{bmatrix}}_{=:Y_{\phi,i}} \bar{w}(\omega_i). \quad (17)$$

We compactly define

$$\bar{\Phi} = [\bar{\phi}(\omega_1), \dots, \bar{\phi}(\omega_L)] \in \mathbb{C}^{n_\phi \times L}, \quad (18)$$

which satisfies

$$\bar{\Phi} = \underbrace{V_{\phi,tr}U_e}_{=: \bar{\Phi}_u} + \underbrace{Y_{\phi,tr}W}_{=: \bar{\Phi}_w}, \quad (19)$$

with

$$\begin{aligned} V_{\phi,tr} &:= [V_{\phi,1}, \dots, V_{\phi,L}] \in \mathbb{C}^{n_\phi \times n_u L}, \\ Y_{\phi,tr} &:= [Y_{\phi,1}, \dots, Y_{\phi,L}] \in \mathbb{C}^{n_\phi \times n_x L}. \end{aligned} \quad (20)$$

The spectral content of Z (12) is denoted by $\bar{Z} \in \mathbb{C}^{(n_\phi + n_x n_\phi^2) \times (n_x n_\phi L)}$:

$$\begin{aligned} \bar{Z} &= \begin{bmatrix} D_{des}^{\frac{1}{2}T} ((\bar{X}_u + \bar{X}_w)^H \otimes I_{n_\phi}) \\ ((\bar{\Phi}_u + \bar{\Phi}_w) \otimes I_{n_x}) \otimes I_{n_\phi} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} D_{des}^{\frac{1}{2}T} ((\mathbf{1}_L^T U_e^T V_{x,tr}^H) \otimes I_{n_\phi}) \\ (V_{\phi,tr} U_e) \otimes I_{n_x n_\phi} \end{bmatrix}}_{=: \bar{Z}_u} + \underbrace{\begin{bmatrix} D_{des}^{\frac{1}{2}T} (\bar{X}_w^H \otimes I_{n_\phi}) \\ \bar{\Phi}_w \otimes I_{n_x n_\phi} \end{bmatrix}}_{=: \bar{Z}_w}. \end{aligned} \quad (21)$$

The following lemma provides lower bounds on $\Phi \Phi^T$ and ZZ^T (12) using the spectral content of the signals x_k and ϕ_k .

Lemma 9. Let Assumptions 4 and 8 hold. For any $\epsilon \in (0, 1)$, ϕ_k and Z satisfy

$$\Phi \Phi^T \succeq T \left((1 - \epsilon) \bar{\Phi}_u \bar{\Phi}_u^H - \left(\frac{1 - \epsilon}{\epsilon} \right) \bar{\Phi}_w \bar{\Phi}_w^H \right), \quad (22)$$

and

$$ZZ^T \succeq T \left((1 - \epsilon) \bar{Z}_u \bar{Z}_u^H - \left(\frac{1 - \epsilon}{\epsilon} \right) \bar{Z}_w \bar{Z}_w^H \right), \quad (23)$$

respectively.

The proof of Lemma 9 may be derived using the Parseval-Plancher identity; cf. (Venkatasubramanian et al., 2024a, Appendix A). The matrices $\bar{\Phi}_u$, $\bar{\Phi}_w$, $\bar{Z}_u$ and $\bar{Z}_w$ in Lemma 9 depend on the transfer matrices $V_{x,tr}$, $V_{\phi,tr}$, $Y_{x,tr}$, and $Y_{\phi,tr}$ which, in turn, are dependent on the true dynamics A_{tr} , B_{tr} , and hence, uncertain. Therefore, in what follows, suitable bounds are derived.

4.3. Bounds on transfer matrices

Denote

$$\tilde{V}_\phi = V_{\phi,tr} - \hat{V}_\phi, \quad \tilde{V}_x = V_{x,tr} - \hat{V}_x, \quad (24)$$

where the estimates

$$\begin{aligned} \hat{V}_\phi &= [\hat{V}_{\phi,1}, \dots, \hat{V}_{\phi,L}] \in \mathbb{C}^{n_\phi \times L n_u}, \\ \hat{V}_x &= \text{diag}(\hat{V}_{x,1}, \dots, \hat{V}_{x,L}) \in \mathbb{C}^{n_x L \times n_u L} \end{aligned} \quad (25)$$

are computed using the initial estimates $\hat{\theta}_0 = \text{vec}([\hat{A}_0, \hat{B}_0])$ (cf. Assumption 2). We can compute matrices $\tilde{\Gamma}_\phi$, $\tilde{\Gamma}_x$, Γ_ϕ and Γ_x such that

$$\tilde{V}_\phi \tilde{V}_\phi^H \preceq \tilde{\Gamma}_\phi, \quad \tilde{V}_x \tilde{V}_x^H \preceq \tilde{\Gamma}_x, \quad (26)$$

$$Y_{\phi,tr} Y_{\phi,tr}^H \preceq \Gamma_\phi, \quad Y_{x,tr} Y_{x,tr}^H \preceq \Gamma_x \quad (27)$$

using $\theta_{tr} \in \Theta_0$ (cf. Assumption 2). Conditions (26) and (27) are LMIs, and hence matrices $\tilde{\Gamma}_x$, $\tilde{\Gamma}_\phi \succ 0$ and Γ_x , $\Gamma_\phi \succ 0$ may be computed using robust control methods as shown in (Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024, Appendices A-B), or scenario methods as shown in (Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024, Appendix C). Utilizing (27), we derive bounds on $\bar{\Phi}_w$, $\bar{X}_w$ and $\bar{Z}_w$ in the following lemma.

Lemma 10. Let Assumptions 1, 2, 4 and 8 hold. Given the bounds on $Y_{\phi,tr}$ and $Y_{x,tr}$ (27), we have

$$\bar{\Phi}_w \bar{\Phi}_w^H \preceq \bar{W}_\phi := \frac{\gamma_w}{T} \Gamma_\phi \quad (28)$$

and

$$\bar{Z}_w \bar{Z}_w^H \preceq \underbrace{\left(\frac{\gamma_w}{T} \|\Gamma_x\| \|D_{des}\| + \frac{\gamma_w}{T} \|\Gamma_\phi\| \right) I_{(n_\phi + n_x n_\phi^2)}}_{\bar{W}_Z}. \quad (29)$$

Lemma 10 uses properties of the induced matrix norm; cf. (Venkatasubramanian et al., 2024a, Appendix B). Given the bounds in Lemma 10, we next derive spectral lines-based sufficient conditions for targeted exploration.

4.4. Sufficient conditions for targeted exploration based on the theory of spectral lines

The following proposition provides a condition in terms of the spectral content of ϕ , which, if satisfied, ensures that the exploration goal (4) is achieved.

Proposition 11. Let *Assumptions 1 and 2* hold. Suppose the matrices $\bar{\Phi}_u$ and $\bar{Z}_u$ satisfy

$$\begin{aligned} & \begin{bmatrix} T((1-\epsilon)\bar{\Phi}_u\bar{\Phi}_u^H - (\frac{1-\epsilon}{\epsilon})\bar{W}_\phi) - \gamma_w D_{\text{des}} & 0 \\ 0 & 0 \end{bmatrix} \\ & + T\left((1-\epsilon)\bar{Z}_u\bar{Z}_u^H - \left(\frac{1-\epsilon}{\epsilon}\right)\bar{W}_Z\right) \succeq 0. \end{aligned} \quad (30)$$

Then, the estimate $\hat{\theta}_T$ computed as in (8) satisfies the exploration goal (4).

The proof of Proposition 11 can be derived by substituting the terms in Theorem 7 with bounds derived in Lemmas 9 and 10; cf. (Venkatasubramanian et al., 2024a, Appendix C). Note that $\bar{\Phi}_u$ and $\bar{Z}_u$ depend linearly on the decision variable U_e . Determining a lower bound based on the Inequality (30) results in non-convex constraints in U_e . To overcome this problem, we utilize a convex relaxation procedure.

4.5. Convex relaxation

The following proposition provides a sufficient condition linear in $\bar{Z}_u$ which, ensures the exploration goal (4).

Proposition 12. Let *Assumptions 1 and 2* hold. Suppose the matrices $\bar{\Phi}_u$, $\bar{Z}_u$ and $\hat{Z} \in \mathbb{C}^{(n_\phi + n_x n_\phi^2) \times L n_x n_\phi}$ satisfy

$$\begin{aligned} & \begin{bmatrix} ((1-\epsilon)\bar{\Phi}_u\bar{\Phi}_u^H - (\frac{1-\epsilon}{\epsilon})\bar{W}_\phi) - \frac{\gamma_w}{T} D_{\text{des}} & 0 \\ 0 & 0 \end{bmatrix} \\ & + ((1-\epsilon)(\bar{Z}_u\hat{Z}^H + \hat{Z}\bar{Z}_u^H - \hat{Z}\hat{Z}^H) - (\frac{1-\epsilon}{\epsilon})\bar{W}_Z) \succeq 0, \end{aligned} \quad (31)$$

then an estimate $\hat{\theta}_T$ computed as in (8) satisfies the exploration goal (4).

The proof of the proposition can be derived by substituting the quadratic term $\bar{Z}_u\bar{Z}_u^H$ in Proposition 11 with a lower bound linear in $\bar{Z}_u$ obtained by completing the square; cf. (Venkatasubramanian et al., 2024a, Appendix D). The bound derived in (31) is tight if $\hat{Z} = \bar{Z}_u$. Since $\bar{Z}_u$ comprises uncertain elements $V_{x,\text{tr}}$, $V_{\phi,\text{tr}}$ and the unknown decision variable U_e , we consider a candidate $\hat{Z}$. Later, this relaxation is embedded an iterative process to reduce conservatism. In what follows, we utilize Inequality (31) to derive a condition linear in the decision variable U_e that ensures the exploration goal (4).

4.6. Exploration SDP

In this section, we provide a sufficient condition that ensure the exploration goal (4) using Proposition 12. In (21), $\bar{Z}_u$ can be written as $\bar{Z}_u = \bar{Z}_{u,1} + \bar{Z}_{u,2}$, where

$$\begin{aligned} \bar{Z}_{u,1} &= \begin{bmatrix} D_{\text{des}}^{\frac{1}{2}\top} (\mathbf{1}_L^\top U_e^\top \otimes I_{n_\phi}) \\ 0 \end{bmatrix} (V_{x,\text{tr}}^H \otimes I_{n_\phi}), \\ \bar{Z}_{u,2} &= \begin{bmatrix} 0 \\ (V_{\phi,\text{tr}} \otimes I_{n_x n_\phi}) \end{bmatrix} (U_e \otimes I_{n_x n_\phi}). \end{aligned} \quad (32)$$

In order to robustly account for uncertainties in Inequality (31), we split Inequality (31) into three inequalities that are handled separately. The following inequalities are equivalent to Inequality (31) if $\bar{D}_1 + \bar{D}_2 + \bar{D}_3 \succeq 0$:

$$(1-\epsilon)(\bar{Z}_{u,1}\hat{Z}^H + \hat{Z}\bar{Z}_{u,1}^H) - \bar{D}_1 \succeq 0, \quad (33a)$$

$$(1-\epsilon)(\bar{Z}_{u,2}\hat{Z}^H + \hat{Z}\bar{Z}_{u,2}^H - \hat{Z}\hat{Z}^H) - (\frac{1-\epsilon}{\epsilon})\bar{W}_Z - \bar{D}_2 \succeq 0, \quad (33b)$$

$$\begin{bmatrix} ((1-\epsilon)\bar{\Phi}_u\bar{\Phi}_u^H - (\frac{1-\epsilon}{\epsilon})\bar{W}_\phi) - \frac{\gamma_w}{T} D_{\text{des}} & 0 \\ 0 & 0 \end{bmatrix} - \bar{D}_3 \succeq 0. \quad (33c)$$

The following theorem provides a sufficient condition linear in U_e which ensures the exploration goal (4).

Theorem 13. Let *Assumptions 1, 2, 4 and 8* hold. Suppose there exist matrices U_e , $\bar{D}_1$, $\bar{D}_2$ and $\bar{D}_3$, and scalars $\tau_1 \geq 0$, $\tau_2 \geq 0$, and $\tau_3 \geq 0$ such that

$$\begin{aligned} S_{\text{exp-1}}(\epsilon, \tau_1, U_e, \hat{Z}, \hat{V}_x, \tilde{F}_x, D_{\text{des}}, \bar{D}_1) &\geq 0, \\ S_{\text{exp-2}}(\epsilon, \tau_2, U_e, \hat{Z}, \bar{W}_Z, \hat{V}_\phi, \tilde{F}_\phi, D_{\text{des}}, \bar{D}_2) &\geq 0, \\ S_{\text{exp-3}}(\epsilon, \tau_3, U_e, \hat{U}, \bar{W}_\phi, \hat{V}_\phi, \tilde{F}_\phi, \gamma_w, D_{\text{des}}, \bar{D}_3) &\geq 0, \\ \bar{D}_1 + \bar{D}_2 + \bar{D}_3 &\succeq 0, \end{aligned} \quad (35)$$

where $S_{\text{exp-1}}$, $S_{\text{exp-2}}$, and $S_{\text{exp-3}}$ are defined in (34a), (34b) and (34c), respectively (see Box 1). Then, an estimate $\hat{\theta}_T$ computed as in (8) upon the application of the input (6) satisfies the exploration goal (4).

The proof of Theorem 13 is provided in (Venkatasubramanian et al., 2024a, Appendix E). The key idea of the proof is the application of robust control tools, in particular, the matrix S-lemma (Boyd & Vandenberghe, 2004; van Waarde et al., 2022), to account for parametric uncertainty. In particular, feasibility of LMIs (34b) and (34c) requires $(\tilde{F}_\phi - \hat{V}_\phi \hat{V}_\phi^H) \prec 0$, which holds if the initial uncertainty is sufficiently small. Consequently, we can pose the exploration problem of designing exploration inputs that excite the system with minimal energy to obtain an estimate $\hat{\theta}_T$ that satisfies (4) using the following SDP:

$$\begin{aligned} & \inf_{\substack{U_e, \gamma_e, \\ \tau_1 \geq 0, \tau_2 \geq 0, \\ \tau_3 \geq 0}} \gamma_e \\ & \text{s.t. } S_{\text{energy-bound}}(\gamma_e, U_e) \geq 0 \\ & S_{\text{exp-1}}(\epsilon, \tau_1, U_e, \hat{Z}, \hat{V}_x, \tilde{F}_x, D_{\text{des}}, \bar{D}_1) \geq 0 \\ & S_{\text{exp-2}}(\epsilon, \tau_2, U_e, \hat{Z}, \bar{W}_Z, \hat{V}_\phi, \tilde{F}_\phi, D_{\text{des}}, \bar{D}_2) \geq 0 \\ & S_{\text{exp-3}}(\epsilon, \tau_3, U_e, \hat{U}, \bar{W}_\phi, \hat{V}_\phi, \tilde{F}_\phi, \gamma_w, D_{\text{des}}, \bar{D}_3) \geq 0 \\ & \bar{D}_1 + \bar{D}_2 + \bar{D}_3 \succeq 0. \end{aligned} \quad (36)$$

A solution of (36) gives us the parameters required for the implementation of the exploration input, i.e., $U_e = \text{diag}(\bar{u}(\omega_1), \dots, \bar{u}(\omega_L))$, which guarantees the desired uncertainty bound D_{des} (4). The suboptimality introduced by the convex relaxation procedure can be reduced by iterating Problem (36) multiple times until γ_e does not change by re-computing $\hat{U}$ and $\hat{Z}$ for the next iteration as

$$\hat{U} = U_e^*, \quad \hat{Z} = \begin{bmatrix} D_{\text{des}}^{\frac{1}{2}\top} \left((\mathbf{1}_L^\top U_e^{*\top} \hat{V}_x^H) \otimes I_{n_\phi} \right) \\ (\hat{V}_\phi U_e^*) \otimes I_{n_x n_\phi} \end{bmatrix} \quad (37)$$

wherein U_e^* is the solution from the previous iteration.

Remark 14 (Efficient Implementation and Conservatism). If D_{des} is a scaled identity matrix, i.e., $D_{\text{des}} = cI_{n_\phi}$ with $c > 0$, the Kronecker product with I_{n_ϕ} in LMIs (34a) and (34b) can be factored out. Hence, the dimensions of LMIs (34a) and (34b) can be reduced by a factor of n_ϕ , i.e., to $(L n_u + 1 + n_x n_\phi)$, and to $(L n_u n_x + 1 + n_x n_\phi)$, respectively. The resulting LMIs are equivalent to the LMIs (34a) and (34b) before factoring. Furthermore, LMIs (34a)–(34c) are derived by invoking the matrix S-lemma (van Waarde et al., 2022). LMIs (34a)–(34c) are sufficient conditions for (33a)–(33c) under conditions (24) and (26). However, due to the imposed structural constraints due to Kronecker products, the LMIs (34a)–(34c) may, in general, be conservative.

Remark 15 (Computational Effort). The computational effort of Problem (36) scales polynomially with the state dimension n_x , control input dimension n_u , and the number of frequencies L .

$$S_{\text{exp-1}}(\epsilon, \tau_1, U_e, \hat{Z}, \hat{V}_x, \tilde{r}_x, D_{\text{des}}, \bar{D}_1) := \quad (34a)$$

$$\begin{bmatrix} 0 & \begin{bmatrix} (1-\epsilon)D_{\text{des}}^{\frac{1}{2}\top} (\mathbf{1}_L^\top U_e^\top \otimes I_{n_\phi}) \end{bmatrix}^\top \\ \begin{bmatrix} (1-\epsilon)D_{\text{des}}^{\frac{1}{2}\top} (\mathbf{1}_L^\top U_e^\top \otimes I_{n_\phi}) \\ 0 \end{bmatrix} & -\bar{D}_1 \end{bmatrix} - \tau_1 \begin{bmatrix} -I & (\hat{V}_x^\top \otimes I_{n_\phi}) \hat{Z}^\top \\ \hat{Z}(\hat{V}_x \otimes I_{n_\phi}) & \hat{Z}((\tilde{r}_x - \hat{V}_x \hat{V}_x^\top) \otimes I_{n_\phi}) \hat{Z}^\top \end{bmatrix} \succeq 0$$

$$S_{\text{exp-2}}(\epsilon, \tau_2, U_e, \hat{Z}, \bar{W}_Z, \hat{V}_\phi, \tilde{r}_\phi, D_{\text{des}}, \bar{D}_2) := \quad (34b)$$

$$\begin{bmatrix} 0 & (1-\epsilon)(U_e \otimes I_{n_x n_\phi}) \hat{Z}^\top \\ (1-\epsilon) \hat{Z}(U_e^\top \otimes I_{n_x n_\phi}) & -(1-\epsilon) \hat{Z} \hat{Z}^\top - \left(\frac{1-\epsilon}{\epsilon}\right) \bar{W}_Z - \bar{D}_2 \end{bmatrix} - \tau_2 \begin{bmatrix} -I & \begin{bmatrix} 0 \\ \hat{V}_\phi \otimes I_{n_x n_\phi} \end{bmatrix}^\top \\ \begin{bmatrix} \hat{V}_\phi \otimes I_{n_x n_\phi} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ (\tilde{r}_\phi - \hat{V}_\phi \hat{V}_\phi^\top) \otimes I_{n_x n_\phi} \end{bmatrix} \end{bmatrix} \succeq 0$$

$$S_{\text{exp-3}}(\epsilon, \tau_3, U_e, \hat{U}, \bar{W}_\phi, \hat{V}_\phi, \tilde{r}_\phi, \gamma_w, D_{\text{des}}, \bar{D}_3) := \quad (34c)$$

$$\begin{bmatrix} (1-\epsilon)(U_e \hat{U}^\top + \hat{U} U_e^\top - \hat{U} \hat{U}^\top) & 0 \\ 0 & \begin{bmatrix} -\left(\frac{1-\epsilon}{\epsilon}\right) \bar{W}_\phi - \frac{\gamma_w}{T} D_{\text{des}} & 0 \\ 0 & 0 \end{bmatrix} - \bar{D}_3 \end{bmatrix} - \tau_3 \begin{bmatrix} -I & \begin{bmatrix} \hat{V}_\phi \\ 0 \end{bmatrix}^\top \\ \begin{bmatrix} \hat{V}_\phi \\ 0 \end{bmatrix} & \begin{bmatrix} \tilde{r}_\phi - \hat{V}_\phi \hat{V}_\phi^\top & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \succeq 0$$

Box 1.

While this makes moderately-sized problems tractable, large-scale problems may become computationally demanding. In such cases, problem-specific structures and efficient solvers may be utilized to reduce computational demand (Majumdar et al., 2020).

The overall exploration strategy is summarized in Algorithm 1. The resulting strategy optimally excites the system with inputs (6) to determine model parameters up to a user defined closeness D_{des} .

Algorithm 1 Targeted exploration

- 1: Specify exploration length T , frequencies ω_i , $i = 1, \dots, L$, energy bound γ_w , initial estimates $\hat{A}_0$, $\hat{B}_0$, desired accuracy of parameters D_{des} .
- 2: Compute $\hat{V}_\phi$ and $\hat{V}_x$ (25) using the initial estimates.
- 3: Compute bounds $\tilde{r}_\phi$, $\tilde{r}_x$ (26), and Γ_ϕ , Γ_x (27) via methods described in (Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024, Appendices A-C).
- 4: Compute matrices $\bar{W}_\phi$ (28) and $\bar{W}_Z$ (29).
- 5: Select initial candidates $\hat{Z}$ and $\hat{U}$ (37).
- 6: Set tolerance $\text{tol} > 0$.
- 7: **while** $|\frac{\gamma_e - \gamma_e^*}{\gamma_e}| \geq \text{tol}$ **do**
- 8: Solve the optimization problem (36).
- 9: Update $\hat{Z}$ and $\hat{U}$ (37).
- 10: **end while**
- 11: Apply the exploration input (6) for $k = 0, \dots, T - 1$.
- 12: Compute parameter estimate $\hat{\theta}_T$ (8); compute parameter set Θ_T (10).

4.7. Discussion

In what follows, we examine the key features of the proposed work and discuss its connections to the state-of-the-art.

Summary - proposed approach: The proposed targeted exploration strategy outlined in Algorithm 1 yields a multi-sine exploration

input with minimal input energy to generate data from which estimates of uncertain parameters can be derived with a desired error bound (4). The frequencies ω_i of the multi-sine input (6) are predetermined, enabling intuitive tuning based on prior knowledge about the system. The proposed approach assumes energy-bounded disturbances, as commonly considered for data-driven models (Berberich et al., 2023; Fogel, 1979; van Waarde et al., 2022). As the main result, the data-dependent uncertainty bound in Lemma 6 is utilized to derive sufficient conditions in the spectral content of the exploration inputs. The proposed exploration is *targeted*, as the optimized amplitudes at different frequencies impact both the magnitude and the shape/orientation of the remaining uncertainty after exploration. Furthermore, the proposed exploration strategy is robust, i.e., parametric uncertainty is accounted for by using robust control tools.

Limitations: The proposed exploration strategy relies on the energy-bound of the disturbances γ_w . The strategy requires solving an SDP iteratively to mitigate suboptimality arising from the convex relaxation procedure. Additionally, the proposed robust exploration strategy is more conservative for large initial uncertainty.

Related works: The derived targeted exploration strategy is similar to, and inspired by the works of Barenthin and Hjalmarsson (2008), Ferizbegovic et al. (2019), Umenberger et al. (2019) and Venkatasubramanian, Köhler, Berberich and Allgöwer (2024), however, with a few crucial differences. A targeted exploration method is proposed by Barenthin and Hjalmarsson (2008) to identify parameters up to a desired accuracy. However, the conditions are not robust to uncertainty, requiring iterative experiments in practice. In Ferizbegovic et al. (2019), the proposed targeted exploration strategy assumes independent data, and hence lacks applicability to correlated time-series data from a single trajectory. Furthermore, the methods in Ferizbegovic et al. (2019) and Umenberger et al. (2019) do not yield any guarantees for exploration since the uncertainty bounds are heuristically approximated. The method in Venkatasubramanian, Köhler, Berberich

and Allgöwer (2024) robustly accounts for parametric uncertainties and provides an *a priori* guaranteed bound on the uncertainty after exploration. However, all the methods (Barenthin & Hjalmarsson, 2008; Ferizbegovic et al., 2019; Umenberger et al., 2019; Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024) assume i.i.d. Gaussian disturbances. In contrast to these methods, we consider energy-bounded disturbances without assumptions on the distribution or independence of the disturbances. This encompasses a broader class of uncertainties, including those arising from unmodeled dynamics or nonlinearities, and enables the development of a targeted exploration strategy with guarantees. In our proposed targeted exploration method, we quantify and guarantee *a priori* uncertainty bounds on the parameters obtained from a single experiment by robustly accounting for the impact of the uncertain model parameters and disturbances, similar to Venkatasubramanian, Köhler, Berberich and Allgöwer (2024). A notable difference to the theoretical analysis in Venkatasubramanian, Köhler, Berberich and Allgöwer (2024) is the appearance of the term ZZ^T in Theorem 7, which cannot be directly addressed with standard robust control tools and presented significant additional challenges in this work.

Application to dual control: The proposed targeted exploration strategy can be integrated into a robust *dual control* framework to guarantee a desired closed-loop performance after exploration (Barenthin & Hjalmarsson, 2008; Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024). This can be achieved by co-designing the targeted exploration problem (36) with a robust gain-scheduled controller that accounts for changes in uncertainty during exploration (Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024). Such a joint design allows optimizing closed-loop performance after exploration, while ensuring exploration reduces only the necessary uncertainty to achieve the desired performance within exploration energy constraints.

5. Numerical example

In this section, we demonstrate the applicability of the proposed targeted exploration strategy to a nonlinear system using a numerical example. Numerical simulations¹ were performed on MATLAB using CVX (Grant & Boyd, 2014) in conjunction with the solver MOSEK.

Problem Setup: We consider a chain of two mass–spring–damper systems. The model equations are given by

$$\begin{aligned} m_1 \ddot{p}_1 &= -(k_1 + k_2)p_1 - (d_1 + d_2)\dot{p}_1 + k_2 p_2 + d_2 \dot{p}_2 + F_1^{\text{nl}}, \\ m_2 \ddot{p}_2 &= k_2 p_1 + d_2 \dot{p}_1 - k_2 p_2 - d_2 \dot{p}_2 + F_2^{\text{nl}} + u, \end{aligned} \quad (38)$$

$$F_i^{\text{nl}} = \zeta_i \tanh(\alpha_i \dot{p}_i), \quad (39)$$

with positions p_i , velocities $\dot{p}_i$, masses $m_i > 0$, spring constants $k_i \geq 0$, damping coefficients $d_i \geq 0$, nonlinear Coulomb frictions F_i^{nl} with constants $\alpha_i, \zeta_i \geq 0, i \in \{1, 2\}$, and input $u \in \mathbb{R}$. We consider $x = [p_1, \dot{p}_1, p_2, \dot{p}_2]^T \in \mathbb{R}^4$ and an Euler discretization with sampling period $T_s = 0.5$. The resulting system dynamics correspond to our setup (2) with the disturbances $w_k = [0, \frac{F_1^{\text{nl}}(x_2)}{m_1}, 0, \frac{F_2^{\text{nl}}(x_4)}{m_2}]^T$. The disturbances satisfy Assumption 1 with $\gamma_w = T \left(\frac{\zeta_1^2}{m_1^2} + \frac{\zeta_2^2}{m_2^2} \right)$. In this example, we consider the following true values for the masses, spring constants and damping coefficients: $m_1 = 1, m_2 = 2, k_1 = 1, k_2 = 1.5, d_1 = 0.5$ and $d_2 = 1.1$ (38). We select the coefficients of Coulomb friction as $\alpha_1 = 1$ and $\alpha_2 = 1$ in (39). Later, we vary ζ_1 and ζ_2 to study the effect of the disturbance bound γ_w .

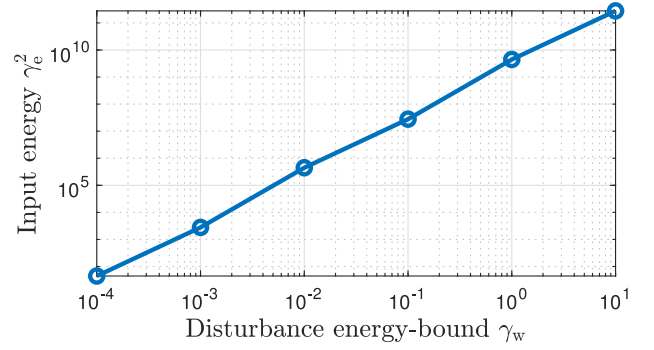


Fig. 1. Illustration of the exploration input energy γ_e^2 , in comparison with the disturbance energy bound γ_w for the initial uncertainty level $\|D_0\| = 10^2$.

The goal of the proposed targeted exploration strategy is to achieve a desired error bound $D_{\text{des}}^{-1} = 10^0 I_{n_\phi}$ on the parameters (4). Furthermore, we select $L = 20$ equally-spaced frequencies $\omega_i \in \{0, 0.05, 0.1, 0.15, \dots, 0.95\}$. We set $T = 100$ and $\epsilon = 0.5$. Throughout the example, the constants $\Gamma_\phi, \Gamma_x, \tilde{\Gamma}_\phi$ and $\tilde{\Gamma}_x$ are computed using the scenario approach (Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024, Appendix C) with confidence level $\beta = 10^{-10}$. In what follows, we analyze the effectiveness and conservatism of the proposed targeted exploration strategy.

Required input energy γ_e^2 for different energy bounds γ_w : To study the effect of the energy bound γ_w on the required input energy γ_e^2 , we select the initial uncertainty level as $D_0^{-1} = 10^{-4} I_{n_\phi}$ and the initial estimate as $\hat{\theta}_0 = \theta_{\text{tr}} + \frac{\|D_0\|^{1/2}}{\|\theta_{\text{tr}}\|} \theta_{\text{tr}}$, i.e., on the boundary of Θ_0 (cf. Assumption 2). We run five trials for the following energy bounds $\gamma_w \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^0\}$. Each trial comprises: (i) computing the corresponding constants $\Gamma_\phi, \Gamma_x, \tilde{\Gamma}_\phi$ and $\tilde{\Gamma}_x$, (ii) executing Algorithm 1 to obtain the exploration inputs (6) and the required input energy γ_e^2 . From Fig. 1, it can be observed that the input energy γ_e^2 scales roughly linearly with the disturbance energy bound γ_w . In general, $\gamma_e \rightarrow 0$ as $\gamma_w \rightarrow 0$, i.e., the exploration input energy γ_e reduces to zero as the energy of the disturbances γ_w reduce to zero.

Conservatism related to the initial uncertainty bound D_0 and comparison with naive exploration: To study the effect of the initial uncertainty bound D_0^{-1} , we select eight uncertainty levels $D_0 \in \{10^i I_{n_\phi}, 5 \times 10^i I_{n_\phi}\}, i = 4, \dots, 7$ and $\gamma_w = 1$. Based on γ_w , we compute $\zeta_1 = \zeta_2 = \frac{m_1 m_2 \sqrt{\gamma_w}}{\sqrt{T(m_1^2 + m_2^2)}}$. Furthermore, to highlight

the benefits of the proposed method, we compare the targeted exploration method with a naive exploration strategy that uses inputs of equal total energy but with non-optimized amplitudes, i.e., inputs where the energy is uniformly distributed across all frequencies. This baseline does not rely on any model information and illustrates how optimization improves input design. Each trial comprises: (i) computing the initial estimate as $\hat{\theta}_0 = \theta_{\text{tr}} + \frac{\|D_0\|^{1/2}}{\|\theta_{\text{tr}}\|} \theta_{\text{tr}}$, (ii) computing the corresponding constants $\Gamma_\phi, \Gamma_x, \tilde{\Gamma}_\phi$ and $\tilde{\Gamma}_x$, (iii) executing Algorithm 1 to obtain the exploration inputs (6), and generate a dataset by applying the targeted exploration input, (iv) computing the naive exploration inputs, and generating a corresponding dataset. From both datasets, we compute the corresponding estimate of the parameters $\hat{\theta}_T$, and $\|G \cdot P\|$, the error bound guaranteed by Lemma 6 with the data-dependent covariance matrix P (9) and scaling G (11). Fig. 2 shows

¹ The source code for the simulations is available at <https://github.com/janavenkatasubramanian/NonstochTE>.

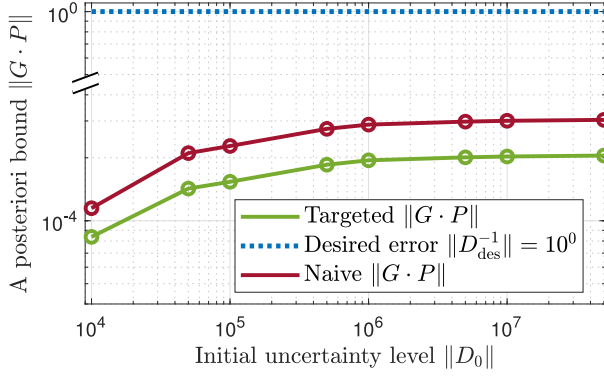


Fig. 2. Illustration of the *a posteriori* guaranteed bound on the squared error of the parameters $\|G \cdot P\|$ for both targeted and naive exploration with same input energy, in comparison with the desired bound on the squared error $\|D_{\text{des}}^{-1}\|$ for different initial uncertainty bounds D_0^{-1} .

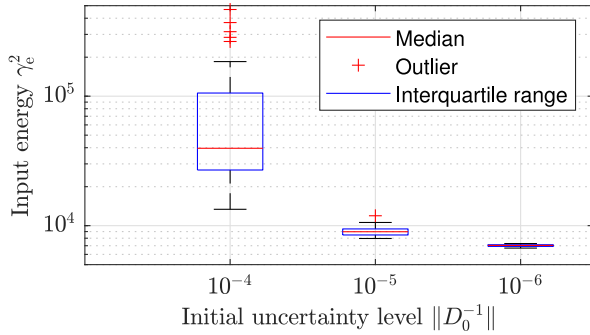


Fig. 3. Illustration of the distribution of the input energy γ_e^2 for different initial uncertainty bounds D_0^{-1} .

that the targeted exploration inputs achieve the exploration goal for all tested initial uncertainty levels D_0 and energy bound γ_w , i.e., the guaranteed error bound on the parameters $\|G \cdot P\|$ is lower than the desired error bound $\|D_{\text{des}}^{-1}\| = 10^0$. The guaranteed bound is significantly lower than the desired bound due to the inherent conservatism of the proposed strategy which is derived by utilizing worst case bounds on the transfer matrices Γ_ϕ , Γ_x , $\tilde{\Gamma}_\phi$ and $\tilde{\Gamma}_x$, and worst case bounds $\tilde{W}_\phi$, $\tilde{W}_x$. This conservatism decreases as the initial uncertainty is reduced, or equivalently, as D_0 increases, since smaller initial uncertainty results in tighter and more accurate bounds on Γ_ϕ , Γ_x , $\tilde{\Gamma}_\phi$ and $\tilde{\Gamma}_x$, yielding more efficient exploration. Furthermore, Fig. 2 shows that the proposed targeted exploration method guarantees a lower error bound $\|G \cdot P\|$, by roughly 50%, compared to naive exploration under the same energy budget, thereby highlighting the benefits of our approach.

Sensitivity related to the initial estimate $\hat{\theta}_0$: To study the sensitivity related to the initial estimate $\hat{\theta}_0$, we select three different uncertainty levels $D_0 \in \{10^i I_{n_\phi}\}$, $i = 4, 5, 6$, with $\gamma_w = 1$. For each level, we run 50 trials. Each trial comprises: (i) generating a random initial estimate $\hat{\theta}_0$ that satisfies Assumption 2, i.e., $(\hat{\theta}_0 - \theta_{\text{tr}})^\top (D_0 \otimes I_{n_x})(\hat{\theta}_0 - \theta_{\text{tr}}) \leq 1$, (ii) computing constants Γ_ϕ , Γ_x , $\tilde{\Gamma}_\phi$ and $\tilde{\Gamma}_x$, (iii) executing Algorithm 1 to obtain the exploration inputs (6) and the input energy γ_e^2 . Fig. 3 shows that the variability of input energy γ_e^2 is higher for larger initial uncertainty levels $\|D_0^{-1}\|$. This variability significantly reduces as initial uncertainty levels reduce. This behavior arises because smaller initial uncertainty

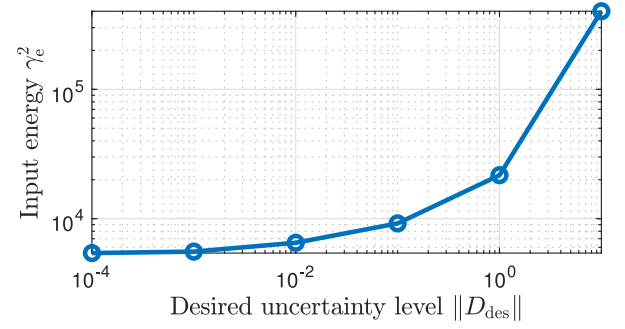


Fig. 4. Illustration of input energy γ_e^2 for different desired uncertainty levels $\|D_{\text{des}}\|$.

levels yield tighter bounds on Γ_ϕ , Γ_x , $\tilde{\Gamma}_\phi$ and $\tilde{\Gamma}_x$, resulting in more consistent input energy.

Sensitivity related to the desired error bound $\|D_{\text{des}}\|$: To study the sensitivity related to the desired uncertainty level $\|D_{\text{des}}\|$, we select six levels $D_{\text{des}} = 10^i I_{n_\phi}$, $i \in \{-4, -3, -2, -1, 0, 1\}$. We set the initial uncertainty level to $D_0 = 10^4 I_{n_\phi}$, $\gamma_w = 1$, choose $\hat{\theta}_0 = \theta_{\text{tr}} + \frac{\|D_0\|^{1/2}}{\|\theta_{\text{tr}}\|} \theta_{\text{tr}}$, and compute the corresponding constants Γ_ϕ , Γ_x , $\tilde{\Gamma}_\phi$ and $\tilde{\Gamma}_x$. For each $\|D_{\text{des}}\|$, we execute Algorithm 1 to obtain the exploration inputs (6) and the input energy γ_e^2 . Fig. 4 shows that achieving a small uncertainty (large $\|D_{\text{des}}\|$) requires higher input energy. In contrast, the required input energy scales linearly with $\|D_{\text{des}}\|$ in the stochastic disturbances setting (Venkatasubramanian, Köhler, Berberich and Allgöwer, 2024). This difference is due to the data-dependent scaling G (11) of the non-falsified set Θ_T (cf. Lemma 6).

6. Conclusion

In this article, we present a targeted exploration strategy for linear systems subject to energy-bounded disturbances. We derived LMIs that can robustly guarantee an *a priori* error-bound on the estimated parameters after exploration. The proposed strategy utilizes multi-sine inputs in selected frequencies and optimized amplitudes to shape the uncertainty bound over the parameters in a targeted manner. We demonstrate the applicability of the proposed targeted exploration strategy to systems with nonlinearities, while analyzing its benefits and inherent conservatism, with a numerical example. This is the first targeted exploration approach that robustly ensures a user-chosen accuracy on the parameters without requiring any independence conditions on the disturbances. We view addressing the limitations related to conservatism and scalability as an important direction for future work.

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