Chapter 8
Causality and a Theorem of Paley and Wiener

In this chapter we turn our focus back to causal operators. In Chap. 5 we found out that material laws provide a class of causal and autonomous bounded operators. In this chapter we will present another proof of this fact, which rests on a result which characterises functions in $L_2(\mathbb{R}; H)$ with support contained in the non-negative reals; the celebrated Theorem of Paley and Wiener. With the help of this theorem, which is interesting in its own right, the proof of causality for material laws becomes very easy. At a first glance it seems that holomorphy of a material law is a rather strong assumption. In the second part of this chapter, however, we shall see that in designing autonomous and causal solution operators, there is no way of circumventing holomorphy.

In the following, let H be a Hilbert space, and we consider $L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ as the subspace of functions in $L_{2,\nu}(\mathbb{R}; H)$ vanishing on $(-\infty, 0)$.

8.1 A Theorem of Paley and Wiener

We start with the following lemma, for which we need the notion of locally integrable functions. We define

$$\begin{aligned} L_{1,\mathrm{loc}}(\mathbb{R}; H) &:= \{f\,;\ \forall\, K \subseteq \mathbb{R} \text{ compact} : \mathbb{1}_K f \in L_1(\mathbb{R}; H)\} \\ &= \left\{f\,;\ \forall\, \varphi \in C_c^\infty(\mathbb{R}) : \varphi f \in L_1(\mathbb{R}; H)\right\}. \end{aligned}$$

Lemma 8.1.1 *Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}; H)$. Then we have $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$ if and only if $f \in \bigcap_{\nu>0} L_{2,\nu}(\mathbb{R}; H)$ with $\sup_{\nu>0} \|f\|_{L_{2,\nu}(\mathbb{R};H)} < \infty$. In the latter case we have that*

$$\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)} = \lim_{\nu\to 0+} \|f\|_{L_{2,\nu}(\mathbb{R};H)} = \sup_{\nu>0} \|f\|_{L_{2,\nu}(\mathbb{R};H)}\,.$$

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Proof Let $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$ and $\nu > 0$. Then we estimate

$$\int_{\mathbb{R}} \|f(t)\|_H^2 \,\mathrm{e}^{-2\nu t}\,\mathrm{d}t = \int_{\mathbb{R}_{\geqslant 0}} \|f(t)\|_H^2 \,\mathrm{e}^{-2\nu t}\,\mathrm{d}t \leqslant \int_{\mathbb{R}_{\geqslant 0}} \|f(t)\|_H^2 \,\mathrm{d}t = \|f\|_{L_2(\mathbb{R}_{\geqslant 0}; H)}^2,$$

which proves that $f \in L_{2,\nu}(\mathbb{R}; H)$ with $\|f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)}$ for each $\nu > 0$. Moreover, $\|f\|_{L_{2,\nu}(\mathbb{R};H)} \to \|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)}$ as $\nu \to 0$ by monotone convergence and since clearly $\|f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \|f\|_{L_{2,\mu}(\mathbb{R};H)}$ for $0 < \mu \leqslant \nu$ we obtain

$$\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)} = \lim_{\nu \to 0+} \|f\|_{L_{2,\nu}(\mathbb{R};H)} = \sup_{\nu > 0} \|f\|_{L_{2,\nu}(\mathbb{R};H)}.$$

Assume now that $f \in \bigcap_{\nu > 0} L_{2,\nu}(\mathbb{R}; H)$ with $C := \sup_{\nu > 0} \|f\|_{L_{2,\nu}(\mathbb{R};H)} < \infty$. This inequality yields

$$\sup_{\nu \in (0,\infty)} \int_{(-\infty,0)} \|f(t)\|^2 \,\mathrm{e}^{-2\nu t}\,\mathrm{d}t \leqslant C^2.$$

Hence, the monotone convergence theorem yields that $g(t) := \lim_{\nu \to \infty} \|f(t)\|^2 \mathrm{e}^{-2\nu t}$ for $t \in (-\infty, 0)$ defines a function $g \in L_1(-\infty, 0)$. Thus, $[g = \infty]$ is a set of measure zero and thus $[f = 0] \cap (-\infty, 0) = (-\infty, 0) \setminus [g = \infty]$ has full measure in $(-\infty, 0)$ implying that $\operatorname{spt} f \subseteq \mathbb{R}_{\geqslant 0}$.

Finally, from

$$\sup_{\nu \in (0,\infty)} \int_{(0,\infty)} \|f(t)\|^2 \,\mathrm{e}^{-2\nu t}\,\mathrm{d}t \leqslant C^2.$$

we infer again by the monotone convergence theorem that $t \mapsto \lim_{\nu \to 0} \|f(t)\|^2 \mathrm{e}^{-2\nu t} = \|f(t)\|^2$ defines a function in $L_1(0, \infty)$, showing the remaining assertion. □

For the proof of the Paley–Wiener theorem we need a suitable space of holomorphic functions on the right half-plane, the so-called *Hardy space* $\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$, which we introduce in the following.

Definition For $\nu \in \mathbb{R}$ we define the *Hardy space*

$$\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H) := \left\{ g\colon \mathbb{C}_{\mathrm{Re}>\nu} \to H\,;\ g \text{ holomorphic},\ \sup_{\rho > \nu} \int_{\mathbb{R}} \|g(\mathrm{i}t + \rho)\|_H^2 \,\mathrm{d}t < \infty \right\}$$

and equip it with the norm $\|\cdot\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)}$ defined by

$$\|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)} \coloneqq \sup_{\rho>\nu}\left(\int_{\mathbb{R}} \|g(\mathrm{i}t+\rho)\|_H^2 \,\mathrm{d}t\right)^{\frac{1}{2}}.$$

We motivate the Theorem of Paley–Wiener first. For this, let $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ and define its *Laplace transform* as

$$\mathbb{C}_{\mathrm{Re}>\nu} \ni z \mapsto \mathcal{L}f(z) \coloneqq \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)\mathrm{e}^{-zt}\,\mathrm{d}t. \tag{8.1}$$

Note that $\mathcal{L}f(z) = \mathcal{L}_{\mathrm{Re}\,z}f(\mathrm{Im}\,z)$ for all $z \in \mathbb{C}_{\mathrm{Re}>\nu}$ due to the support constraint on f. Moreover, it is not difficult to see that the integral on the right-hand side of (8.1) exists as $\left(t \mapsto \mathrm{e}^{-\rho t}f(t)\right) \in L_1(\mathbb{R}_{\geqslant 0}; H) \cap L_2(\mathbb{R}_{\geqslant 0}; H)$ for all $\rho > \nu$. Hence, $\mathcal{L}f\colon \mathbb{C}_{\mathrm{Re}>\nu} \to H$ is holomorphic (cf. Exercise 5.6). Moreover, by Lemma 8.1.1

$$\begin{aligned}\sup_{\rho>\nu}\|\mathcal{L}f(\mathrm{i}\cdot+\rho)\|_{L_2(\mathbb{R};H)} &= \sup_{\rho>\nu}\left\|\mathcal{L}_\rho f\right\|_{L_2(\mathbb{R};H)} = \sup_{\rho>\nu}\|f\|_{L_{2,\rho}(\mathbb{R};H)}\\ &= \sup_{\rho>0}\left\|\mathrm{e}^{-\nu\cdot}f\right\|_{L_{2,\rho}(\mathbb{R};H)}\\ &= \left\|\mathrm{e}^{-\nu\cdot}f\right\|_{L_2(\mathbb{R};H)} = \|f\|_{L_{2,\nu}(\mathbb{R};H)},\end{aligned}$$

which proves that

$$\begin{aligned}\mathcal{L}\colon L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H) &\to \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)\\ f &\mapsto \left(z \mapsto (\mathcal{L}_{\mathrm{Re}\,z}f)(\mathrm{Im}\,z)\right)\end{aligned}$$

is well-defined and isometric. It turns out that $\mathcal{L}$ is actually surjective, see Corollary 8.1.3 below. The surjectivity statement is contained in the following Theorem of Paley–Wiener, [78]. We mainly follow the proof given in [101, 19.2 Theorem].

Theorem 8.1.2 (Paley–Wiener) *Let $g \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0}; H)$. Then there exists an $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$ such that*

$$\mathcal{L}_\nu f = g(\mathrm{i}\cdot+\nu) \quad (\nu > 0).$$

Proof For $\nu > 0$ we set $g_\nu \coloneqq g(\mathrm{i}\cdot+\nu) \in L_2(\mathbb{R}; H)$ and $f_\nu \coloneqq \mathcal{F}^*g_\nu \in L_2(\mathbb{R}; H)$. Moreover, we set $f \coloneqq \mathrm{e}^{(\cdot)}f_1$. We first prove that $f \in \bigcap_{\nu>0} L_{2,\nu}(\mathbb{R}; H)$ with $\sup_{\nu>0}\|f\|_{L_{2,\nu}(\mathbb{R};H)} < \infty$. For doing so, let $a > 0$, $\rho > 0$ and $x \in \mathbb{R}$. Applying

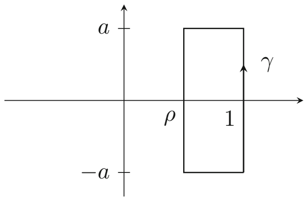

Fig. 8.1 Curve γ

Cauchy's integral theorem to the function $z \mapsto \mathrm{e}^{zx} g(z)$ and the curve γ, as indicated in Fig. 8.1, we obtain

$$
\begin{aligned}
0 = \mathrm{i}\int_{-a}^{a} \mathrm{e}^{(\mathrm{i}t+1)x} g(\mathrm{i}t+1)\,\mathrm{d}t - \int_{\rho}^{1} \mathrm{e}^{(\mathrm{i}a+\kappa)x} g(\mathrm{i}a+\kappa)\,\mathrm{d}\kappa \\
-\mathrm{i}\int_{-a}^{a} \mathrm{e}^{(\mathrm{i}t+\rho)x} g(\mathrm{i}t+\rho)\,\mathrm{d}t + \int_{\rho}^{1} \mathrm{e}^{(-\mathrm{i}a+\kappa)x} g(-\mathrm{i}a+\kappa)\,\mathrm{d}\kappa.
\end{aligned}
\tag{8.2}
$$

Moreover, since

$$
\begin{aligned}
\int_{\mathbb{R}} \left\| \int_{\rho}^{1} \mathrm{e}^{(\pm\mathrm{i}a+\kappa)x} g(\pm\mathrm{i}a+\kappa)\,\mathrm{d}\kappa \right\|_H^2 \mathrm{d}a &\leqslant \int_{\mathbb{R}} \left| \int_{\rho}^{1} \left|\mathrm{e}^{(\pm\mathrm{i}a+\kappa)x}\right|^2 \mathrm{d}\kappa \int_{\rho}^{1} \|g(\pm\mathrm{i}a+\kappa)\|_H^2 \,\mathrm{d}\kappa \right| \mathrm{d}a \\
&\leqslant \left| \int_{\rho}^{1} \mathrm{e}^{2\kappa x}\,\mathrm{d}\kappa \right| \left| \int_{\rho}^{1} \int_{\mathbb{R}} \|g(\pm\mathrm{i}a+\kappa)\|_H^2 \,\mathrm{d}a\,\mathrm{d}\kappa \right| \\
&\leqslant \left| \int_{\rho}^{1} \mathrm{e}^{2\kappa x}\,\mathrm{d}\kappa \right| |1-\rho|\, \|g\|^2_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)} < \infty,
\end{aligned}
$$

we infer that $\left(a \mapsto \int_{\rho}^{1} \mathrm{e}^{(\pm\mathrm{i}a+\kappa)x} g(\pm\mathrm{i}a+\kappa)\,\mathrm{d}\kappa\right) \in L_2(\mathbb{R}; H)$ and thus, we find a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $a_n \to \infty$ and

$$
\int_{\rho}^{1} \mathrm{e}^{(\pm\mathrm{i}a_n+\kappa)x} g(\pm\mathrm{i}a_n+\kappa)\,\mathrm{d}\kappa \to 0
$$

as $n \to \infty$. Hence, using (8.2) with a replaced by a_n and letting n tend to infinity, we derive that

$$
\int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+1)x} g(\mathrm{i}t+1)\,\mathrm{d}t - \int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+\rho)x} g(\mathrm{i}t+\rho)\,\mathrm{d}t \to 0 \quad (n \to \infty).
$$

Noting that for each $\mu > 0$ we have

$$\int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+\mu)x} g(\mathrm{i}t + \mu)\,\mathrm{d}t = \sqrt{2\pi}\mathrm{e}^{\mu x}\mathcal{F}^*(\mathbb{1}_{[-a_n,a_n]} g_\mu)(x) \quad (x \in \mathbb{R})$$

and that $\mathbb{1}_{[-a_n,a_n]} g_\mu \to g_\mu$ in $L_2(\mathbb{R}; H)$ as $n \to \infty$, we may choose a subsequence (again denoted by $(a_n)_n$) such that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left(\int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+1)x} g(\mathrm{i}t + 1)\,\mathrm{d}t - \int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+\rho)x} g(\mathrm{i}t + \rho)\,\mathrm{d}t \right) \\ &= \lim_{n\to\infty} \left(\sqrt{2\pi}\mathrm{e}^{x}\mathcal{F}^*(\mathbb{1}_{[-a_n,a_n]} g_1)(x) - \sqrt{2\pi}\mathrm{e}^{\rho x}\mathcal{F}^*(1_{[-a_n,a_n]} g_\rho)(x) \right) \\ &= \sqrt{2\pi}\left(\mathrm{e}^{x} f_1(x) - \mathrm{e}^{\rho x} f_\rho(x)\right) \end{aligned}$$

for almost every $x \in \mathbb{R}$. Hence, $f = \mathrm{e}^{(\cdot)} f_1 = \exp(\rho \mathrm{m}) f_\rho$ for each $\rho > 0$ and thus,

$$\int_{\mathbb{R}} \|f(t)\|_H^2\, \mathrm{e}^{-2\rho t}\,\mathrm{d}t = \int_{\mathbb{R}} \left\|f_\rho(t)\right\|_H^2 \mathrm{d}t < \infty$$

which shows $f \in \bigcap_{\rho>0} L_{2,\rho}(\mathbb{R}; H)$ with

$$\sup_{\rho>0} \|f\|_{L_{2,\rho}(\mathbb{R};H)} = \sup_{\rho>0} \left\|f_\rho\right\|_{L_2(\mathbb{R};H)} = \sup_{\rho>0} \left\|g_\rho\right\|_{L_2(\mathbb{R};H)} = \|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)}\,.$$

Thus, $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$ with $\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)} = \|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)}$ by Lemma 8.1.1. Moreover,

$$\mathcal{L}_\nu f = \mathcal{F}\exp(-\nu \mathrm{m}) f = \mathcal{F}\exp(-\nu m)\exp(\nu \mathrm{m}) f_\nu = \mathcal{F} f_\nu = g_\nu = g(\mathrm{i}\cdot + \nu)$$

for each $\nu > 0$, which shows the representation formula for g. □

Summarising the results of Theorem 8.1.2 and the arguments carried out just before Theorem 8.1.2, we obtain the following statement.

Corollary 8.1.3 *Let $\nu \in \mathbb{R}$. Then the mapping*

$$\begin{aligned} \mathcal{L}\colon L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H) &\to \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H) \\ f &\mapsto \left(z \mapsto (\mathcal{L}_{\mathrm{Re}\,z} f)(\mathrm{Im}\,z)\right) \end{aligned}$$

is an isometric isomorphism. In particular, $\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ is a Hilbert space.

Proof We have argued already that $\mathcal{L}$ is well-defined and isometric. Thus, we show that $\mathcal{L}$ is onto, next. For this, let $g \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ and define $\widetilde{g}(z) := g(z+\nu)$ for $z \in \mathbb{C}_{\mathrm{Re}>0}$. Then $\widetilde{g} \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0}; H)$ and thus, Theorem 8.1.2 yields the existence of $\widetilde{f} \in L_2(\mathbb{R}_{\geqslant 0}; H)$ with

$$g(\mathrm{i}\cdot+\rho) = \widetilde{g}(\mathrm{i}\cdot+\rho-\nu) = \mathcal{L}_{\rho-\nu}\widetilde{f} = \mathcal{L}_\rho\big(\mathrm{e}^{\nu\cdot}\widetilde{f}\big) \quad (\rho > \nu).$$

Hence, setting $f := \mathrm{e}^{\nu\cdot}\widetilde{f} \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$, we obtain $\mathcal{L}f = g$. □

We can now provide an alternative proof of Theorem 5.3.6 by proving causality with the help of the Theorem of Paley–Wiener.

Proposition 8.1.4 *Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Then for $\nu > \mathrm{s_b}(M)$ we have $M(\partial_{t,\nu}) \in L(L_{2,\nu}(\mathbb{R}; H))$ and $M(\partial_{t,\nu})$ is causal and autonomous (see Exercise 5.7).*

Proof Let $\nu > \mathrm{s_b}(M)$. Then $M\colon \mathbb{C}_{\mathrm{Re}\geqslant\nu} \to L(H)$ is bounded and holomorphic on $\mathbb{C}_{\mathrm{Re}>\nu}$. Hence, by unitary equivalence, $M(\partial_{t,\nu}) \in L(L_{2,\nu}(\mathbb{R}; H))$. Moreover, $M(\partial_{t,\nu})$ is autonomous by Exercise 5.7. Thus, for causality it suffices to check that $\operatorname{spt} M(\partial_{t,\nu})f \subseteq \mathbb{R}_{\geqslant 0}$ whenever $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$. So let $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$. Then $\mathcal{L}f \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ by Corollary 8.1.3 and since M is bounded and holomorphic on $\mathbb{C}_{\mathrm{Re}>\nu}$, we infer also that

$$\big(z \mapsto M(z)\,(\mathcal{L}f)\,(z)\big) \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H).$$

Again by Corollary 8.1.3 there exists $g \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ such that

$$\mathcal{L}g(z) = M(z)\,(\mathcal{L}f)\,(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu}).$$

Thus, in particular

$$\mathcal{L}_\rho g = M(\mathrm{i}m+\rho)\mathcal{L}_\rho f \quad (\rho > \nu).$$

Since $f, g \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ we infer that $\mathcal{L}_\rho g \to \mathcal{L}_\nu g$ and $\mathcal{L}_\rho f \to \mathcal{L}_\nu f$ in $L_2(\mathbb{R}; H)$ as $\rho \to \nu$ by dominated convergence. Moreover, $M(\mathrm{i}m+\rho) \to M(\mathrm{i}m+\nu)$ strongly on $L_2(\mathbb{R}; H)$ as $\rho \to \nu$ (cf. Exercise 8.2). Hence, we derive

$$\mathcal{L}_\nu g = M(\mathrm{i}m+\nu)\mathcal{L}_\nu f,$$

and thus, $g = M(\partial_{t,\nu})f$ which shows causality. □

8.2 A Representation Result

In this section we argue that our solution theory needs holomorphy as a central property for the material law. There are two key properties for rendering $T \in L(L_{2,\nu_0}(\mathbb{R}; H))$ a material law operator. The first one is causality (i.e., $\mathbb{1}_{(-\infty,a]}(\mathrm{m})T\mathbb{1}_{(-\infty,a]}(\mathrm{m}) = \mathbb{1}_{(-\infty,a]}(\mathrm{m})T$ for all $a \in \mathbb{R}$) and, secondly, T needs to be autonomous (i.e., $\tau_h T = T\tau_h$ for all $h \in \mathbb{R}$ where $\tau_h f = f(\cdot + h)$). The main theorem of this section reads as follows:

Theorem 8.2.1 *Let $\nu_0 \in \mathbb{R}$ and let $T \in L(L_{2,\nu_0}(\mathbb{R}; H))$ be causal and autonomous. Then $T|_{L_{2,\nu_0} \cap L_{2,\nu}}$ has a unique extension $T_\nu \in L(L_{2,\nu}(\mathbb{R}; H))$ for each $\nu > \nu_0$ and there exists a unique $M\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to L(H)$ holomorphic and bounded such that $T_\nu = M(\partial_{t,\nu})$ for each $\nu > \nu_0$.*

We consider the following (shifted) variant of Theorem 8.2.1 first.

Theorem 8.2.2 *Let $T \in L(L_2(\mathbb{R}; H))$ be causal and autonomous. Then there exists $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$, a material law (i.e., holomorphic and bounded), such that*

$$(\mathcal{L}Tf)(z) = M(z)(\mathcal{L}f)(z) \quad (f \in L_2(\mathbb{R}_{\geqslant 0}; H), z \in \mathbb{C}_{\mathrm{Re}>0}).$$

Proof For $s > 0$ and $x \in H$ we define $f_{x,s} := \mathbb{1}_{(0,s)}x$ and compute

$$\mathcal{L}f_{x,s}(z) = \frac{1}{\sqrt{2\pi}}\int_0^s \mathrm{e}^{-zt}x\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\frac{1-\mathrm{e}^{-zs}}{z}x \quad (z \in \mathbb{C}_{\mathrm{Re}>0}). \tag{8.3}$$

We define $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ via

$$M(z)x := \frac{\sqrt{2\pi}z}{1-\mathrm{e}^{-z}}\mathcal{L}Tf_{x,1}(z),$$

which is well-defined since $\operatorname{spt} Tf_{x,1} \subseteq [0,\infty)$ (use causality of T); $M(z) \in L(H)$, since T is bounded. Also, $M(\cdot)x$ is evidently holomorphic for every $x \in H$ as a product of two holomorphic mappings and thus by Exercise 5.3, M is holomorphic itself. Next, we show that for all $z \in \mathbb{C}_{\mathrm{Re}>0}$ and $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$, we have

$$(\mathcal{L}Tf)(z) = M(z)(\mathcal{L}f)(z). \tag{8.4}$$

By definition of M, the equality is true for f replaced by $f_{x,1}$, $x \in H$. Next, observe that $\operatorname{lin}\left\{\mathbb{1}_{(a,a+1/n)}x\,;\, a \geqslant 0, n \in \mathbb{N}, x \in H\right\}$ is dense in $L_2(\mathbb{R}_{\geqslant 0}; H)$. Hence, for (8.4), it suffices to show

$$\left(\mathcal{L}T\mathbb{1}_{(a,a+1/n)}x\right)(z) = M(z)\left(\mathcal{L}\mathbb{1}_{(a,a+1/n)}x\right)(z) \tag{8.5}$$

for all $a \geqslant 0, n \in \mathbb{N}, x \in H$, and $z \in \mathbb{C}_{\mathrm{Re}>0}$. Next, using that T is autonomous in the situation of (8.5), we see $\left(T\mathbb{1}_{(a,a+1/n)}x\right) = \left(T\tau_{-a}\mathbb{1}_{(0,1/n)}x\right) = \tau_{-a}\left(T\mathbb{1}_{(0,1/n)}x\right)$ and, by a straightforward computation, $(\mathcal{L}\tau_{-a}f)(z) = \mathrm{e}^{-za}\mathcal{L}f(z)$ for all $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$. Thus,

$$\left(\mathcal{L}T\mathbb{1}_{(a,a+1/n)}x\right)(z) = \mathrm{e}^{-za}\left(\mathcal{L}T\mathbb{1}_{(0,1/n)}x\right)(z),$$

which yields that it suffices to show (8.5) for $a = 0$ only, that is, for $f = f_{x,1/n}$. Furthermore, we compute for $n \in \mathbb{N}$ and $z \in \mathbb{C}_{\mathrm{Re}>0}$

$$\begin{aligned}\mathcal{L}Tf_{x,1}(z) &= \sum_{k=0}^{n-1}(\mathcal{L}T\mathbb{1}_{(k/n,(k+1)/n)}x)(z) = \sum_{k=0}^{n-1}\mathrm{e}^{-zk/n}(\mathcal{L}T\mathbb{1}_{(0,1/n)}x)(z) \\ &= \frac{1-\mathrm{e}^{-z}}{1-\mathrm{e}^{-z/n}}(\mathcal{L}Tf_{x,1/n})(z).\end{aligned}$$

Thus, using (8.3) for $s = 1/n$, we deduce from the definition of M,

$$\begin{aligned}\mathcal{L}Tf_{x,1/n}(z) &= \frac{1-\mathrm{e}^{-z/n}}{\sqrt{2\pi}z}\frac{\sqrt{2\pi}z}{1-\mathrm{e}^{-z}}\mathcal{L}Tf_{x,1}(z) = \frac{1-\mathrm{e}^{-z/n}}{\sqrt{2\pi}z}M(z)x \\ &= M(z)\mathcal{L}f_{x,1/n}(z).\end{aligned}$$

Hence, (8.4) holds for all $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$. It remains to show boundedness of M. For this, let $z \in \mathbb{C}_{\mathrm{Re}>0}$ and $x \in H$. Set $f := \mathbb{1}_{[0,\infty)}\mathrm{e}^{-z^*}x$ as well as $c := 2\operatorname{Re} z\sqrt{2\pi}$. Then

$$\mathcal{L}f(z) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-zt-z^*t}x\,\mathrm{d}t = \frac{x}{c}.$$

By virtue of (8.4), we get $\mathcal{L}Tf(z) = M(z)\mathcal{L}f(z)$ and thus $M(z)x = c\mathcal{L}Tf(z)$. This leads to

$$\begin{aligned}\|M(z)x\| &\leqslant \frac{c}{\sqrt{2\pi}}\int_0^\infty \left\|\mathrm{e}^{-zt}Tf(t)\right\|\,\mathrm{d}t \leqslant \frac{c}{\sqrt{2\pi}}\left\|\mathbb{1}_{[0,\infty)}\mathrm{e}^{-z(\cdot)}\right\|_{L_2(\mathbb{R})}\|Tf\|_{L_2(\mathbb{R})} \\ &\leqslant \frac{c}{\sqrt{2\pi}}\left\|\mathbb{1}_{[0,\infty)}\mathrm{e}^{-z(\cdot)}\right\|_{L_2(\mathbb{R})}^2\|T\|_{L(L_2(\mathbb{R};H))}\|x\|_H = \|T\|_{L(L_2(\mathbb{R};H))}\|x\|_H\,,\end{aligned}$$

where we used that $\|f\|_{L_2(\mathbb{R};H)} = \left\|\mathbb{1}_{[0,\infty)}\mathrm{e}^{-z(\cdot)}\right\|_{L_2(\mathbb{R})}\|x\|_H$. Thus, $\|M(z)\| \leqslant \|T\|$, which yields boundedness of M and the assertion of the theorem. □

We can now prove our main result of this section.

Proof of Theorem 8.2.1 We just prove the existence of a function M. The proof of its uniqueness is left as Exercise 8.3.
We first prove the assertion for $\nu_0 = 0$. So, let $T \in L(L_2(\mathbb{R}; H))$ be causal and autonomous. According to Theorem 8.2.2 we find $M \colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ holomorphic and bounded such that

$$(\mathcal{L}Tf)(z) = M(z)(\mathcal{L}f)(z) \quad (f \in L_2(\mathbb{R}_{\geqslant 0}; H), z \in \mathbb{C}_{\mathrm{Re}>0}).$$

Let now $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}; H)$ and set $a := \inf \operatorname{spt} \varphi$. Then $\tau_a \varphi \in L_2(\mathbb{R}_{\geqslant 0}; H)$, and for $\nu > 0$ we compute

$$\begin{aligned} \mathcal{L}_\nu T\varphi &= \mathcal{L}_\nu \tau_{-a} T \tau_a \varphi = \mathrm{e}^{-(\mathrm{im}+\nu)a} \mathcal{L}_\nu T \tau_a \varphi = \mathrm{e}^{-(\mathrm{im}+\nu)a} M(\mathrm{im}+\nu) \mathcal{L}_\nu \tau_a \varphi \\ &= M(\mathrm{im}+\nu)\mathcal{L}_\nu \varphi. \end{aligned} \tag{8.6}$$

The latter implies

$$\begin{aligned} \|T\varphi\|_{L_{2,\nu}(\mathbb{R};H)} &= \|\mathcal{L}_\nu T\varphi\|_{L_2(\mathbb{R};H)} = \|M(\mathrm{im}+\nu)\mathcal{L}_\nu \varphi\|_{L_2(\mathbb{R};H)} \\ &\leqslant \|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>0}} \|\varphi\|_{L_{2,\nu}(\mathbb{R};H)} \end{aligned}$$

and hence, $T|_{C_{\mathrm{c}}^{\infty}(\mathbb{R};H)}$ has a unique continuous extension $T_\nu \in L(L_{2,\nu}(\mathbb{R}; H))$. Using (8.6) we obtain

$$T_\nu = \mathcal{L}_\nu^* M(\mathrm{im}+\nu)\mathcal{L}_\nu = M(\partial_{t,\nu})$$

by approximation.

Let now $\nu_0 \in \mathbb{R}$. Then the operator

$$\widetilde{T} := \mathrm{e}^{-\nu_0 \mathrm{m}} T \mathrm{e}^{\nu_0 \mathrm{m}} \in L(L_2(\mathbb{R}; H))$$

is causal and autonomous as well. Thus, $\widetilde{T}|_{C_{\mathrm{c}}^{\infty}(\mathbb{R};H)}$ has continuous extensions $\widetilde{T}_\rho \in L(L_{2,\rho}(\mathbb{R}; H))$ for each $\rho > 0$ and there is $\widetilde{M} \colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ holomorphic and bounded such that $\widetilde{T}_\rho = \widetilde{M}(\partial_{t,\rho})$ for each $\rho > 0$. Using $T|_{C_{\mathrm{c}}^{\infty}(\mathbb{R};H)} = \mathrm{e}^{\nu_0 \mathrm{m}} \widetilde{T}|_{C_{\mathrm{c}}^{\infty}(\mathbb{R};H)} \mathrm{e}^{-\nu_0 \mathrm{m}}$, we derive that $T|_{C_{\mathrm{c}}^{\infty}(\mathbb{R};H)}$ has the unique continuous extension $T_\nu = \mathrm{e}^{\nu_0 \mathrm{m}} \widetilde{T}_{\nu-\nu_0} \mathrm{e}^{-\nu_0 \mathrm{m}} \in L(L_{2,\nu}(\mathbb{R}; H))$ for each $\nu > \nu_0$ and

$$\begin{aligned} \mathcal{L}_\nu T_\nu &= \mathcal{L}_\nu \mathrm{e}^{\nu_0 \mathrm{m}} \widetilde{T}_{\nu-\nu_0} \mathrm{e}^{-\nu_0 \mathrm{m}} = \mathcal{L}_{\nu-\nu_0} \widetilde{T}_{\nu-\nu_0} \mathrm{e}^{-\nu_0 \mathrm{m}} = \widetilde{M}(\mathrm{im}+\nu-\nu_0)\mathcal{L}_{\nu-\nu_0} \mathrm{e}^{-\nu_0 \mathrm{m}} \\ &= \widetilde{M}(\mathrm{im}+\nu-\nu_0)\mathcal{L}_\nu. \end{aligned}$$

Hence,

$$T_\nu = M(\partial_{t,\nu})$$

for the holomorphic and bounded function M given by $M(z) := \widetilde{M}(z - \nu_0)$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. □

8.3 Comments

The stated Theorem of Paley and Wiener is of course not the only theorem characterising properties of the support of L_2-functions in terms of their Fourier or Laplace transform. For instance, a similar result holds for functions having compact support, see e.g. [101, 19.3 Theorem] and Exercise 8.7. These theorems provide a nice connection between L_2-functions and spaces of holomorphic functions in form of Hardy spaces. In this chapter we just introduced the Hardy space $\mathcal{H}_2$ and it is not surprising that there are also the Hardy spaces $\mathcal{H}_p$ for $1 \leqslant p \leqslant \infty$. We refer to [35] for this topic.

The representation result presented in the second part of this chapter was originally proved by Fourès and Segal in 1955, [41]. In this article the authors prove an analogous representation result for causal operators on $L_2(\mathbb{R}^d; H)$, where causality is defined with respect to a closed and convex cone on $\mathbb{R}^d$. The quite elementary proof of Theorem 8.2.2 for $d = 1$ presented here was kindly communicated to us by Hendrik Vogt.

Exercises

Exercise 8.1 Let $\Lambda \subseteq \mathbb{R}_{>0}$ be a set with an accumulation point in $\mathbb{R}_{>0}$. Prove that $\{(x \mapsto \mathrm{e}^{-\lambda x})\ ;\ \lambda \in \Lambda\}$ is a total set in $L_1(\mathbb{R}_{\geqslant 0})$.
Hint: Use that the set is total if and only if

$$\forall f \in L_\infty(\mathbb{R}_{\geqslant 0}) : \left(\forall \lambda \in \Lambda : \int_{\mathbb{R}_{\geqslant 0}} \mathrm{e}^{-\lambda x} f(x)\,\mathrm{d}x = 0 \Rightarrow f = 0\right).$$

Exercise 8.2 Let $M\colon \mathrm{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Moreover, let $\nu > \mathrm{s_b}(M)$. Show that $\lim_{\rho \to \nu+} M(\mathrm{im} + \rho) = M(\mathrm{im} + \nu)$ where the limit is meant in the strong operator topology on $L_2(\mathbb{R}; H)$.

Exercise 8.3 Prove the uniqueness statement in Theorem 8.2.1.

Exercise 8.4 Give an example of a continuous and bounded function $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ such that the corresponding operator $M(\partial_{t,\nu})$ is not causal for any $\nu > 0$.

Exercise 8.5 Prove the following distributional variant of the Paley–Wiener theorem: Let $\nu_0 > 0$, $k \in \mathbb{N}$, $f\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to \mathbb{C}$, and set $h(z) := \frac{1}{z^k} f(z)$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$.

We assume that $h \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu_0}; \mathbb{C})$. For $\nu > \nu_0$ we define the distribution $u\colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$u(\psi) := \left\langle \mathcal{L}_\nu^* h(\mathrm{i}\cdot+\nu), (\partial_{t,\nu}^*)^k \psi \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})} \quad (\psi \in C_c^\infty(\mathbb{R}; \mathbb{C})).$$

Prove that $\operatorname{spt} u \subseteq \mathbb{R}_{\geqslant 0}$, where

$$\operatorname{spt} u := \mathbb{R} \setminus \bigcup \left\{ U \subseteq \mathbb{R} \text{ open}\,;\ \forall\, \psi \in C_c^\infty(U; \mathbb{C}) : u(\psi) = 0 \right\}.$$

What is u if $f = \mathbb{1}_{\mathbb{C}_{\mathrm{Re}>\nu_0}}$?

Exercise 8.6 Let $g \in L_2(\mathbb{R})$, $a > 0$ such that $\operatorname{spt} g \subseteq [-a, a]$. Show that $f := \mathcal{F} g$ extends to a holomorphic function $\widetilde{f}\colon \mathbb{C} \to \mathbb{C}$ with $\widetilde{f}(\mathrm{i}t) = f(t)$ for each $t \in \mathbb{R}$ such that

$$\exists C \geqslant 0\, \forall z \in \mathbb{C} : |f(z)| \leqslant C \mathrm{e}^{a|\operatorname{Re} z|}.$$

Exercise 8.7 Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic such that

(a) $\exists C \geqslant 0,\ a > 0\, \forall z \in \mathbb{C} : |f(z)| \leqslant C \mathrm{e}^{a|\operatorname{Re} z|}$,
(b) $f(\mathrm{i}\cdot) \in L_2(\mathbb{R})$.

Prove that $g := \mathcal{F}^* f(\mathrm{i}\cdot)$ satisfies $\operatorname{spt} g \subseteq [-a, a]$.
Hint: Apply Theorem 8.1.2 to the function $h : \mathbb{C}_{\mathrm{Re}>0} \to \mathbb{C}$ given by

$$h(z) := \mathrm{e}^{-za} \frac{f(z)}{z+1} \quad (z \in \mathbb{C}_{\mathrm{Re}>0})$$

to derive that $\operatorname{spt} g \subseteq \mathbb{R}_{\geqslant -a}$.
Remark: The assertion even holds true if one replaces condition (a) by

$$\exists C \geqslant 0,\ a > 0\, \forall z \in \mathbb{C} : |f(z)| \leqslant C \mathrm{e}^{a|z|}.$$

References

35. P.L. Duren, *Theory of H^p Spaces*, vol. XII (Academic, New York, London, 1970), 258 p. 1970.
41. Y. Fourès, I. Segal, Causality and analyticity. Trans. Am. Math. Soc. **78**, 385–405 (1955)
78. R.E. Paley, N. Wiener, *Fourier Transforms in the Complex Domain*, vols. 19, VIII. American Mathematical Society Colloquium publications (American Mathematical Society, New York, 1934)
101. W. Rudin, *Real and Complex Analysis*. Mathematics Series (McGraw-Hill, New York, 1987)