Reduction of Dynamic Cable Stiffness to Linear Matrix Polynomial

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Abstract: For the dynamic stiffness of a sagging cable subject to harmonic boundary displacements, frequency-dependent closed-form analytic functions can be derived from the corresponding continuum equations. When considering such functions in stiffness matrices of composed structures, however, these matrices become frequency dependent, too—a troublesome fact, especially in regards to the eigenvalue problem, which becomes nonlinear. In this paper, a method for avoiding such difficulties is described whereby an analytic dynamic stiffness function is reduced to a linear matrix polynomial; the matrices of this polynomial are of any desired order. The reduction corresponds to a mathematically performed transition from a continuum to a discrete-coordinate vibrating system. In structural dynamic applications (dynamic cable stiffness), the two resultant matrices correspond to a static stiffness matrix and a mass matrix. Beyond the particular problem focused on, the method may be applied to all kinds of analytic impedance functions. In every case, the resultant matrices can easily be considered within the scope of a linear matrix-eigenvalue problem.

INTRODUCTION

In the investigation of the vibratory behavior of systems, such as cable-stayed bridges or guyed masts, increased attention has recently been focused on the influence of cable dynamics. It became apparent that the dynamic interaction between cables and beams is important. In the works concerned, cables were modeled in different ways. Abdel-Ghaffar and Khalifa (1991) discretized each cable into a number of finite truss elements. Kutterer and Starossek (1992) used an analytically found dynamic stiffness matrix. This matrix had been derived by Starossek (1991a, b) from differential continuum equations; it describes the linear steady-state response of a cable subject to harmonic boundary displacements.

From the mechanical point of view, the continuum approach seems to be particularly suitable. The cable is described by relating only to the cable boundaries; no additional degrees of freedom need to be introduced. The elements

$$K_{ij} = K_{ij}(\omega) = \frac{F_i}{\Delta_j} \quad (1)$$

of the dynamic stiffness matrix $\mathbf{K}(\omega)$ are closed-form analytic functions of the frequency of motion ω; influence of all cable parameters can easily be investigated. Moreover, the closed-form solution is valid for a wide range of cable parameters; hence, singularity problems do not occur as it may be in the case of a finite-element approach, when the sag-to-span ratio of a cable approximates zero.

However, when analytic dynamic stiffness functions are added to the

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overall stiffness matrices of composed systems, these matrices become frequency dependent, too—in contrast to an overall modeling with finite elements. This may cause some computational difficulty with regard to the eigenvalue problem because it becomes nonlinear.

The approach given here to overcome that difficulty was intended to be applied to dynamic cable stiffness, although it is neither limited to cable dynamics nor to structural dynamic problems. The analytic dynamic stiffness (or impedance) functions are linearized. This means an approximative reduction of any element $K_{ij}(\omega)$ or, when omitting indices for sake of simplicity, $K(\omega)$ of a dynamic stiffness matrix to a polynomial

$$\mathbf{S} = \mathbf{S}(\omega^2) = \mathbf{P} - \omega^2\mathbf{Q} \qquad (2)$$

that is linear relative to ω^2, and whose coefficients **P** and **Q** are constant square matrices of order n. The expression $\mathbf{S}(\omega^2)$ can easily be considered within the scope of a linear eigenvalue problem. In structural dynamic applications, such as in the case of dynamic cable stiffness, the two resultant matrices **P** and **Q** correspond to a static stiffness matrix and a mass matrix; they are assigned to the constant system matrices, respectively. Reduction of analytic impedance functions to matrix polynomials corresponds to a mathematically performed transition from a continuum to a discrete-coordinate vibrating system.

A method for carrying out that reduction for any desired order n (and thereby any required degree of accuracy) will be described in the following paragraphs. In this paper, cable damping is not being taken into account; dynamic stiffness functions and matrix polynomial are real. Without any major difficulty, it is possible to generalize the given algorithm to damped vibrations and, consequently, to complex analysis [see Starossek (1991a)].

REDUCTION OF DYNAMIC STIFFNESS FUNCTION

By making the transition to the new overall system, component $K(\omega)$ is replaced by element s_{11} of matrix **S**. The remaining elements of **S** are arranged in additional rows and columns of the new system matrices (with these elements being the only nonzero elements and their mutual relationship kept unchanged). The second to nth columns of **S** include the effect of new, fictitious degrees of freedom; the second to nth rows yield additional, homogeneous equations.

Reduction of K to **S** is effected via a rational function $\tilde{K}$ whose numerator and denominator are polynomials in ω^2; that kind of function is typical for the dynamic stiffness of a discrete-coordinate vibrating system.

Derivation of **S** proceeds from

$$\mathbf{S}\begin{pmatrix}\Delta\\ \mathbf{r}\end{pmatrix} = \begin{pmatrix}\tilde{F}\\ \mathbf{0}\end{pmatrix} \qquad (3)$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_{n-1})^T$ = vector of the new, fictitious degrees of freedom r_j; and $\tilde{F}$ approximately equals the force F between cable and remainder of the system due to a displacement Δ. The new dynamic cable stiffness (that is conveyed by **S**) is

$$\tilde{K} = \tilde{K}(\omega) = \frac{\tilde{F}}{\Delta} \qquad (4)$$

S has to be determined so that

$$\tilde{K} \simeq K \quad (5)$$

From (3), it follows for nonsingular matrix **S** that

$$\tilde{K} = [(\mathbf{S}^{-1})_{11}]^{-1} = \frac{\det\mathbf{S}}{S_{11}} \quad (6)$$

where S_{11} = cofactor of s_{11} in det **S** (minor of det **S**).

The limiting value $\tilde{K}_\infty = \tilde{K}|_{\omega\to\infty}$ is assumed to be finite. Hence, the numerator and denominator polynomials of $\tilde{K}$ must be of the same degree. For sake of simplicity, the inner structure of **Q** is furthermore assumed to be of such kind that the cofactor Q_{11} is nonzero. With the adoption of approach (2), (6) leads to the rational function

$$\tilde{K} = \tilde{K}(\omega) = \tilde{K}_\infty \prod_{j=1}^{n-1} \frac{\omega^2 - \tilde{\omega}_{0j}^2}{\omega^2 - \tilde{\omega}_{\infty j}^2} \quad (7)$$

This function has $n - 1$ roots $\tilde{\omega}_{0j}^2$ (eigenvalues of **S**); for $n - 1$ values $\tilde{\omega}_{\infty j}^2$ (eigenvalues of $\mathbf{S}_{22}$, that is, **S** reduced by first row and column), $\tilde{K}$ becomes infinite. Furthermore, it is valid that

$$\frac{\tilde{K}_\infty}{\tilde{K}_0} = \prod_{j=1}^{n-1} \frac{\tilde{\omega}_{\infty j}^2}{\tilde{\omega}_{0j}^2} \quad (8)$$

where $\tilde{K}_0 = \tilde{K}|_{\omega=0}$. Function (7) is an intermediate term conveying the transition from K to **S**; this transition is split into two steps that are defined by (5) and (6), respectively.

APPROXIMATION OF K BY $\tilde{K}$

By means of suitable algorithms, the unknown parameters $\tilde{K}_0$, $\tilde{K}_\infty$, $\tilde{\omega}_{0j}$, and $\tilde{\omega}_{\infty j}$ have to be established such that the analytic function $K(\omega)$ is approximated by the rational function $\tilde{K}(\omega)$ [as formally described by (7)] in the best possible manner. This will normally be accomplished in an iterative process. Suitable initial values for this process are the parameters K_0, K_∞, ω_{0j}, and $\omega_{\infty j}$, which can be obtained by investigation of $K(\omega)$.

TRANSITION FROM $\tilde{K}$ to S

Application of (6) to **S** should yield a function $\tilde{K}$ as described by (7). However, this condition is met by an infinite number of different matrix polynomials **S**; hence, it does not lead to a determined solution. In this paragraph, a particular solution is presented. It is valid for any desired order n.

Derivation of **S** is mainly based upon a decomposition of the rational function $\tilde{K}$ into partial fractions. This decomposition is always possible. If all $\tilde{\omega}_{0j}$, $\tilde{\omega}_{\infty j}$ are different, it leads to

$$\tilde{K} = \tilde{K}_\infty \left(1 + \sum_{j=1}^{n-1} \frac{A_j}{\omega^2 - \tilde{\omega}_{\infty j}^2}\right) \quad (9)$$

where

$$A_j = A_j(\tilde{\omega}_{0k}^2, \tilde{\omega}_{\infty k}^2) \in \mathbb{R}; \qquad k = 1, \ldots, n-1 \quad (10)$$

are constant parameters for which closed-form expressions can be derived

(Strubecker 1966). By utilizing (9), basic relation (4) is transformed and, by inclusion of trivial equations, expanded to the system of equations

$$\mathbf{S}\begin{pmatrix}\Delta \\ \hline \mathbf{r}\end{pmatrix} = \underbrace{\tilde{K}_\infty \left[\begin{array}{c|ccccc} 1 & 1 & 1 & \cdots & 1 \\ \hline 1 & \dfrac{\tilde{\omega}_{\infty 1}^2 - \omega^2}{A_1} & 0 & \cdots & 0 \\ 1 & 0 & \dfrac{\tilde{\omega}_{\infty 2}^2 - \omega^2}{A_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & \dfrac{\tilde{\omega}_{\infty, n-1}^2 - \omega^2}{A_{n-1}} \end{array}\right]}_{= \mathbf{S}} \begin{bmatrix} \Delta \\ \hline r_1 \\ r_2 \\ \vdots \\ r_{n-1} \end{bmatrix} = \begin{bmatrix} \tilde{F} \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{(11)}$$

Matrix **S** in this equation satisfies (3), and is a linear matrix polynomial as established by (2). Mechanically, it corresponds to a parallel system of spring-mounted lumped masses.

The described algorithm is applied to the special case $n = 2$. After decomposition of (7) in accordance with (9) and substitution into the general expression (11), the matrix polynomial

$$\mathbf{S} = \tilde{K}_\infty \begin{pmatrix} 1 & 1 \\ 1 & \dfrac{\tilde{\omega}_\infty^2}{\tilde{\omega}_\infty^2 - \tilde{\omega}_0^2} \end{pmatrix} - \omega^2 \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{\tilde{K}_\infty}{\tilde{\omega}_\infty^2 - \tilde{\omega}_0^2} \end{pmatrix} \quad \text{(12)}$$

is obtained. Mechanically, it corresponds to a two-degree-of-freedom system with one lumped mass. Such a system was proposed by Veletsos and Darbre (1983) as a simple model for a vibrating cable. Spring and mass properties of that model were directly attached to cable parameters. This, however, does not provide the best possible choice of model parameters [see Starossek (1991a)]. Especially for tightly stretched cables, utilization of the more refined method presented in this paper is preferable.

CONCLUSIONS

A method for reducing an analytic dynamic stiffness function to a linear matrix polynomial (with two constant matrices of any desired order) was presented. This reduction corresponds to a mathematically performed transition from a continuum to a discrete-coordinate vibrating system. In structural dynamic applications, such as in the case of dynamic cable stiffness, the two resultant matrices correspond to a static stiffness matrix and a mass matrix. Unlike the original analytic function, these matrices can easily be considered within the scope of a linear matrix-eigenvalue problem. Beyond the particular problem of dynamic cable stiffness focused on in this paper, the method is applicable to all kinds of analytic impedance functions.

APPENDIX. REFERENCES

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