Single Machine Batch Scheduling to Minimize the Weighted Number of Tardy Jobs

Danny Hermelin^{*} Matthias Mnich[†] Simon Omlor[‡]

Abstract

The $1|B, r_j| \sum w_j U_j$ scheduling problem takes as input a batch setup time Δ and a set of n jobs, each having a processing time, a release date, a weight, and a due date; the task is to find a sequence of batches that minimizes the weighted number of tardy jobs. This problem was introduced by Hochbaum and Landy in 1994; as a wide generalization of KNAPSACK, it is NP-hard.

In this work we provide a multivariate complexity analysis of the $1|B, r_j| \sum w_j U_j$ problem with respect to several natural parameters. That is, we establish a thorough classification into fixed-parameter tractable and W[1]-hard problems, for parameter combinations of (i) #p = distinct number of processing times, (ii) #w = number of distinct weights, (iii) #d = number of distinct due dates, (iv) #r = number of distinct release dates, and (v) b = batch sizes. Thereby, we significantly extend the work of Hermelin et al. (2018) who analyzed the parameterized complexity of the non-batch variant of this problem without release dates.

As one of our key results, we prove that $1|B, r_j| \sum w_j U_j$ is W[1]-hard parameterized by the number of distinct processing times and distinct due dates. To the best of our knowledge, these are the first parameterized intractability results for scheduling problems with few distinct processing times. Further, we show that $1|B, r_j| \sum w_j U_j$ is fixed-parameter tractable with respect to parameter #p + #d + #r and with respect to parameter #w + #dif there is just a single release date. Both results hold even if the number of jobs per batch is limited by some integer b.

Keywords. Scheduling, single machine scheduling, batch scheduling, weighted number of tardy jobs, fixed-parameter tractability, W[1]-hardness.

1 Introduction

This paper is concerned with the problem of minimizing the total weight of tardy (late) jobs in a single machine batch scheduling environment. Before describing our results, we first briefly overview the classical non-batch variant of this problem, denoted as $1||\sum w_j U_j$ in Graham's classical three-field notation [6]. Following this, we describe the extension of $1||\sum w_j U_j$ to the batch scheduling environment, and discuss how our results fit into the known state of the art.

1.1 Total weight of tardy jobs on a single machine

One of the most fundamental and prominent scheduling criteria on a single machine is that of minimizing the total weight of tardy jobs in a schedule. Let J be a set of jobs, where each job $j \in J$ has a processing time $p_j \in \mathbb{N}$, a weight $w_j \in \mathbb{N}$, and a due date $d_j \in \mathbb{N}$. We are given a

^{*}Ben-Gurion University of the Negev, Beer-Sheva, Israel. hermelin@bgu.ac.il

[†]TU Hamburg, Institute for Algorithms and Complexity, Hamburg, Germany. matthias.mnich@tuhh.de

[‡]TU Hamburg, Institute for Algorithms and Complexity, Hamburg, Germany. simon.omlor@tuhh.de

single machine on which to process all the jobs in J. A schedule for this machine corresponds to assigning a starting time S_j to each job $j \in J$, so that $S_i \notin [S_j, S_j + p_j)$ for any job $i \neq j$. The term $S_j + p_j$, also denoted C_j , is called the *completion time* of job j. A job $j \in J$ is tardy if its completion time exceeds its deadline, i.e., if $C_j > d_j$; otherwise, it is early. The goal is to find a schedule which minimizes the total weight of all tardy jobs; or $\sum_{j \in J} w_j U_j$ where U_j is a binary indicator variable which takes value 1 if and only if job j is tardy. This problem is denoted as the $1 || \sum w_j U_j$ problem.

Karp [9] proved that this problem is (weakly) NP-hard even when all jobs have a common due date (i.e., the $1|d_j = d|\sum w_j U_j$ problem), and in fact this variant is equivalent to the 0/1KNAPSACK problem. The variant where in addition to a single due date, the weight of each job is equal to its processing time (the $1|d_j = d, p_j = w_j|\sum w_j U_j$ problem) is known to be equivalent to the PARTITION problem.

Lawler and Moore [10] provided a pseudo-polynomial time algorithm for $1||\sum w_j U_j$, whereas Sahni [15] showed that the problem admits an FPTAS. The variant where all jobs have unit weight (and a single release date), known as the $1||\sum U_j$ problem, is solvable in $O(n \log n)$ time due to an algorithm by Moore [13]. There is also a classical variant where each job $j \in J$ also has a release time $r_j \in \mathbb{N}$, and $S_j \ge r_j$ is required of any schedule. This variant is known to be NP-hard even if jobs have unit weight and there are only two distinct due dates and only two distinct release times.

Most relevant to this paper is a recent result by Hermelin et al. [7] who studied the $1||\sum w_j U_j$ problem from the perspective of parameterized complexity [4]. There, the following three parameters are considered for the problem:

- #d: number of distinct due dates,
- #p: number of distinct processing times,
- #w: number of distinct weights.

Their main results are given in the theorem below:

Proposition 1 ([7]). Problem $1 || \sum w_j U_j$ can be solved in

- time $f(\#d + \#p) \cdot n^{O(1)}$, time $f(\#d + \#w) \cdot n^{O(1)}$, and in time $f(\#p + \#w) \cdot n^{O(1)}$.
- time $n^{O(\#p)}$, and in time $n^{O(\#w)}$.

A special case of this result was already obtained by Etscheid et al. [5] who presented an $f(\#p) \cdot n^{O(1)}$ -time algorithm for the single due date $1|d_j = d|\sum w_j U_j$ problem.

1.2 Batch scheduling

Batch scheduling has recently received a considerable amount of attention in the scheduling community. The motivation for this line of research stems from the fact that in manufacturing systems items flow between facilities in boxes, pallets, or carts. A set of items assigned to the same container is considered as a *batch*. It is often the case that items in the same batch leave the facility together, and thus have equal completion time. We refer to Potts and Kovalyov [14] and Webster and Baker [16] for further reading on the topic.

Hochbaum and Landy [8] studied the generalization of the $1||\sum w_j U_j$ problem to the batch setting. In this problem, denoted $1|B|\sum w_j U_j$, a schedule consists of a partition of the job set J into batches, and a starting time S_B for each batch B such that $S_{B'} \notin [S_B, C_B =$ $S_B + \Delta + \sum_{i \in B} p_j)$ for any batch $B' \neq B$, where Δ is a given setup time associated with starting any batch. The completion time of any job $j \in B$ is $C_j = C_B$, meaning that all the jobs together in a batch are completed at the same time. The goal is again to minimize the total weight of tardy jobs $\sum w_j U_j$. Note that the order of the jobs within each batch is irrelevant, and that when $\Delta = 0$ this problem becomes the classical $1 || \sum w_j U_j$ problem.

Hochbaum and Landy observed that this problem is weakly NP-hard (being a direct generalization of $1||\sum w_j U_j$), and provided pseudopolynomial-time algorithms for the problem that are linear in the total sum of job processing-times (plus $n \cdot \Delta$) or the maximum due-date. Brucker and Kovalyov provided an analogous algorithm which is linear the total sum of job weights [2]. Nevertheless, in this paper we are interested in the case where job weights, processing times, or due dates can be arbitrarily large, but the number of different values of each of these parameters (namely, #w, #p, or #d) is relatively small. In this context, the following result of Hochbaum and Landy is very relevant.

Proposition 2 ([8]). Problems $1|B, p_i = p|\sum w_i U_i$ and $1|B|\sum U_i$ are polynomial-time solvable.

One can also consider restrictions on batches that are relevant in practice. For instance, one can require a bound on the size |B| or volume $||B|| = \sum_{j \in B} p_j$ of any batch B. Cheng and Kovalyov [3] argued about the importance of the batch-size $|B| \leq b$ bound in real-life applications. Note that for b = n we have the unbounded $1|B| \sum w_j U_j$ problem, whereas for b = 1 one obtains the classical non-batch model $1||\sum w_j U_j$. The following is a very relevant result of Cheng and Kovalyov who showed that $1||B| \leq b|\sum U_j$ is in XP when parameterized by either #p or #d:

Proposition 3 ([3]). Problem $1||B| \leq b| \sum U_j$ can be solved in time $n^{O(\#p)}$, and in time $n^{O(\#d)}$.

1.3 Our contributions

We provide a thorough multivariate complexity analysis of $1|B| \sum w_j U_j$ and related variants: Problem $1|B, r_j| \sum w_j U_j$ where jobs also have release dates, problem $1||B| \leq b| \sum w_j U_j$ where there is a bound on the batch size, and problem $1||B|| \leq b| \sum w_j U_j$ where there is a bound on the batch volume.

The standard batch model: In the first part of the paper we study the $1|B| \sum w_j U_j$ problem without release dates or batch restrictions. We show that almost all results of Proposition 1 regarding the $1||\sum w_j U_j$ problem extend to the batch setting.

Theorem 4. Problem $1|B| \sum w_j U_j$ can be solved in

- time $n^{O(\#p)}$, and in time $n^{O(\#w)}$.
- time $f(\#d + \#p) \cdot n^{O(1)}$, and in time $f(\#d + \#w) \cdot n^{O(1)}$.

The second part of this theorem is proved by an elegant reduction to the non-batch case, while the first part is based on dynamic programming. Note that the second item of the theorem is a generalization of the result by Hochbaum and Landy stated in Proposition 2.

Release dates: Next, we show that adding release dates makes the problem much harder. Specifically, we prove that $1|B, r_j| \sum w_j U_j$ is highly unlikely to be fixed-parameter tractable for parameter #d + #p or #p + #r.

Theorem 5. Problem $1|B, r_j| \sum w_j U_j$ is W[1]-hard when parameterized by #d + #p, and is W[1]-hard when parameterized by #p + #r. Furthermore, the problem is solvable in time $n^{f(\#p+\#r,+\#w)}$, and in time $n^{f(\#p+\#d+\#w)}$.

To the best of our knowledge, this is the first W[1]-hardness result for any scheduling problem parameterized by the number of distinct processing times #p. In particular, whether or not $P||C_{\text{max}}$ (makespan minimization on an unbounded number of parallel machines) is W[1]-hard for this parameter is a famous open problem (see [12]), and this question is also open for $1||\sum w_j U_j$ [7].

Batch restrictions: In the final part of the paper we show that the $f(\#d + \#w) \cdot n^{O(1)}$ algorithm in the second part of Theorem 4 can be generalized to the setting where each batch contains at most b jobs; this setting was proposed by Cheng and Kovalyov [3]. Further, the algorithm with run time $f(\#d + \#p) \cdot n^{O(1)}$ can be generalized to the setting where the batch size is limited and the jobs may have different release dates.

Theorem 6. The following problems are fixed-parameter tractable:

- $1||B| \leq b|\sum w_j U_j$ for parameter #d + #w.
- $1||B| \leq b, r_j|\sum w_j U_j$ for parameter #d + #p + #r.

In particular, this improves the result of Cheng and Kovalyov stated in Proposition 3, as our algorithm runs in time $f(\#d) \cdot n^{O(1)}$ for the unweighted version $1||B| \leq b|\sum U_j$.

Finally, let us make a few remarks on the problem $1|||B|| \leq V|\sum w_j U_j$, where the batch volume is bounded. First, for this problem we show NP-hardness even for the case of unit weights and a single due date; this rules out the existence of XP-algorithms parameterized by #p + #w. Second, we show that for parameter #p, this problem is at least as hard as $P||C_{\text{max}}$ parameterized by #p. Recall that the fixed-parameter tractability of $P||C_{\text{max}}$ parameterized by #p is a long-standing open problem.

Problem variant	Parameters	Result	Reference
$1 B \sum w_j U_j$	#d	para-NP-hard	Karp [9]
	#p	XP	Theorem 4
	#w	XP	Theorem 4
	#d + #p	FPT	Theorem 4
	#d + #w	FPT	Theorem 4
	#p + #w	?	
$1 B, B \leq b \sum w_j U_j$	#d + #w	FPT	Theorem 6
$1 B,r_i \sum w_i U_i$	#p + #r	W[1]-hard	Theorem 5
	#d + #p	W[1]-hard	Theorem 5
	#p + #w	?	
	#d + #r + #w	para-NP-hard	Karp [9]
	#d + #p + #w	XP	Theorem 5
	#p + #r + #w	XP	Theorem 5
$1 B, B \leq b, r_j \sum U_j$	#d + #p + #r	FPT	Theorem 6

A summary of our results is given in Table 1.

Table 1: Summary of results.

2 The standard batch model

In this section we present algorithms for the basic $1|B| \sum w_j U_j$ problem, providing a complete proof for Theorem 4. The proof is split into two parts, which are proven in three separate lemmas below. Note that in the setting where all jobs are released at the same time and the batch sizes are not restricted we can schedule the early jobs in order of the due dates. This is a very helpful observation by Hochbaum and Landy [8], which will be used multiple times in this section. We illustrate it by an example in Fig. 1.



Figure 1: An example for batch scheduling with 5 jobs. In the first solution job 1 and 5 are tardy. In the second solution only job 1 is tardy. Tardy jobs can be moved to the end of the schedule without increasing the weight of the tardy jobs.

Lemma 7 ([8]). Any instance of $1|B| \sum w_j U_j$ admits an optimal solution in which all early jobs are in earliest due date (EDD) order. That is, for any two jobs i and j scheduled in two different batches $i \in B_1$ and $j \in B_2$ with $S_{B_1} < S_{B_2}$, we have $d_i < d_j$.

We use following notation to order the due dates: $d^{(1)} < d^{(2)} \cdots < d^{(\#d)}$. Further, we set $d^{(0)}$ to be the smallest release date.

2.1 Fixed-Parameter Algorithms

We begin by presenting fixed-parameter algorithms for $1|B| \sum w_j U_j$ for parameter #d + #p, and for #d + #w.

Lemma 8. Problem $1|B| \sum w_j U_j$ is solvable in time $f(\#d+\#p) \cdot n^{O(1)}$, and in time $f(\#d+\#w) \cdot n^{O(1)}$.

Proof. Let J denote the job set of our $1|B| \sum w_j U_j$ instance. We first observe that there is an optimal schedule in which at most one batch completes within each interval $(d^{(i-1)}, d^{(i)}]$, for each $i \in \{1, \ldots, \#d\}$; if there are two or more batches ending in $(d^{(i-1)}, d^{(i)}]$, then these batches can be combined into a single batch without creating new tardy jobs. The second observation is that if there is no batch ending in $(d^{(i-1)}, d^{(i)}]$ then all jobs with due date $d^{(i)}$ that are completed early must be in batches ending at $d^{(i-1)}$ or earlier. We next use these observations to reduce our $1|B| \sum w_j U_j$ instance J into $2^{O(\#d)}$ instances of the non-batch $1|| \sum w_j U_j$ problem, each with the same number of processing times, weights, and due dates as in J. Combined with the fixed-parameter algorithms for $1|| \sum w_j U_j$ given by Hermelin et al. [7], this will provide a proof for the theorem.

For each $i \in \{1, \ldots, \#d\}$, we guess whether there is a batch ending in $(d^{(i-1)}, d^{(i)}]$ in an optimal solution. Let $I \subseteq \{1, \ldots, \#d\}$ be the set of indexes i such that there is a batch ending in $(d^{(i-1)}, d^{(i)}]$ with respect to our guess. For an index $\ell \in \{1, \ldots, \#d\}$, let $I_{\leq \ell} = \{i \in I \mid i \leq \ell\}$

denote the set of indices in I smaller or equally to ℓ and let $i(\ell) = \max\{i \in I \mid i \leq \ell\}$ be the largest index in I that is less or equal than ℓ . We construct an instance J_I of $1||\sum w_j U_j$ corresponding to I by replacing the due date $d_j = d^{(\ell)}$ of each job $j \in J$ with an alternative due date $d'_j = d^{(i(\ell))} - |I_{\leq \ell}| \cdot \Delta$; all other job parameters remain the same in J_I .

Consider some set of indices $I \subseteq \{1, \ldots, \#d\}$, and let J_I be the corresponding $1 || \sum w_j U_j$ instance. We can convert a schedule of J_I as follows: We note that

$$\sum_{j \text{ is early and } d'_j \leqslant d} p_j \leqslant d$$

for all due dates d. We construct |I| + 1 batches. The first |I| batches are denoted by B_i for $i \in I$ and are processed in increasing order, i.e. if i < i' then B_i is processed before $B_{i'}$. Let j be an early job (i.e. j' is early) with $d_j = d^{(\ell)}$ for some ℓ . Then we assign j to batch $B_{i(\ell)}$. We conclude that j will be early as the completion time of $B_{i(\ell)}$ is equal to

$$C_{B_{i(\ell)}} = |I_{\leqslant \ell}| \Delta + \sum_{j' \text{ is early and } d_{j'} \leqslant d_j} p_{j'} \leqslant |I_{\leqslant \ell}| \Delta + d'_j = d^{i(\ell)} \leqslant d_j \ .$$

Conversely, consider any schedule for J that schedules at most one batch ending in each interval of consecutive due dates, and let $I \subseteq \{1, \ldots, \#d\}$ be the corresponding set of indices. Then any early job $j \in J$ with $d_j = d^{(l)}$ has $C_j \leq d^{i(\ell)}$, and so its completion time in the non-batch setting under the same ordering of early jobs is at most $C_j - |I(\leq \ell)|\Delta \leq d^{i(\ell)} - |I(\leq \ell)|\Delta = d'_j$.

It follows that an optimal schedule for our original $1|B| \sum w_j U_j$ instance corresponds to the schedule with the minimum weight of tardy jobs among all optimal schedules for instances J_I , $I \subseteq \{1, \ldots, \#d\}$. The lemma then follows since there are $2^{\#d}$ instances J_I , and each instance can be solved in $f(\#d + \#p) \cdot n^{O(1)}$ or $f(\#d + \#w) \cdot n^{O(1)}$ time using the algorithm by Hermelin et al. [7].

2.2 XP algorithms

Assume that our input job set $\{1, \ldots, n\}$ is ordered such that $d_1 \leq \cdots \leq d_n$ (i.e. ordered according to EDD). Due to Lemma 7, there is an optimal schedule where any job $j \in J$ is either late, or it is scheduled after the early jobs in $\{1, \ldots, j-1\}$. Thus, an optimal schedule for jobs $\{1, \ldots, j\}$ can be found by appending j to some schedule of jobs $\{1, \ldots, j-1\}$. As observed by Hochbaum and Landy [8, add cite], when appending j to such a schedule, there are three possibilities:

- a. Job j is included in the last batch of early jobs.
- b. Job j is included a new batch by itself, scheduled right after the previous last batch.
- c. Job j is tardy.

Below we devise two dynamic programming algorithms that utilize this fact.

Lemma 9. Problem $1|B| \sum w_i U_i$ is solvable in time $n^{O(\#p)}$.

Proof. Let $J = \{1, \ldots, n\}$ denote our job set ordered according to EDD, and let $p^{(1)} < \cdots < p^{(\#p)}$ denote the different processing times of all jobs in J. For increasing values of $j \in \{1, \ldots, n\}$, we compute a table W_j which has $n^{O(\#p)}$ entries and corresponds to jobs in $\{1, \ldots, j\}$.

The table W_j will be indexed by a #p-dimensional vector $I \in \{1, \ldots, n\}^{\#p}$, and integer $b \in \{1, \ldots, n\}$, and a due date $d \in \{0, d_1, \ldots, d_n\}$. The invariant that our algorithm will maintain is that $W_j[I, b, d]$ will equal the minimum total weight of tardy jobs in a schedule for jobs $\{1, \ldots, j\}$ with the following properties:

- 1. The early jobs are scheduled in EDD fashion as in Lemma 7.
- 2. There are exactly b batches containing exactly I[i] early jobs, $i \in \{1, \ldots, \#p\}$, with processing time $p^{(i)}$, scheduled consecutively starting from time 0.
- 3. The earliest due date among all jobs in the last batch is at least d.

Note that there exists vector I and integers b and d such that the optimal schedule for J satisfies all properties of required from a schedule corresponding to entry $W_n[I, b, d]$ and all jobs in the first b batches are early.

In the beginning, we set $W_j[I, b, d] = \sum_{i=1}^j w_i$ if $I = \emptyset$, and $W_j[I, b, d] = \infty$ otherwise. Fix $j \in \{1, \ldots, n\}$, and consider an entry $W_j[I, b, d]$ of W_j . Let $p^{(\ell)} = p_j$ be the processing time of j for $\ell \in \{1, \ldots, \#p\}$. Let I_ℓ be the vector which coincides with I on every coordinate, except for the ℓ^{th} coordinate for which it is equal to $I[\ell] - 1$. If the ℓ^{th} coordinate of I is 0, then we set $W_j[I, b, d] = W_{j-1}[I, b, d] + w_j$.

Now we consider the expression $\sum_{i=1}^{\#p} I[i] \cdot p^{(i)} + b\Delta$. If $\sum_{i=1}^{\#p} I[i] \cdot p^{(i)} + b\Delta > d$, then job j will be late if it is among the jobs scheduled in the first b batches. Since all of the first j jobs with processing time p_j have a due date less or equal to d_j , there cannot be a schedule that schedules exactly I[i] early jobs with processing time $p^{(i)}$ if we consider only the first j jobs. Thus, we set $W_j[I, b, d] = \infty$.

Else, if $\sum_{i=1}^{\#p} I[i] \cdot p^{(i)} + b\Delta \leq d$, we can schedule job *j* early. There are two possibilities to do so.

The first possibility is to schedule job j in an already existing batch. Then the total weight of tardy jobs is $W_{j-1}[I_{\ell}, b, d]$.

The second possibility is to open a new batch for job j. Then we look at the entries $W_{j-1}[I_{\ell}, b-1, d']$ for $d' \leq d$.

There is also the possibility to schedule j tardy. In this case, the weight is given by $W_{j-1}[I, b, d] + w_j$. Then the recursion for $W_j[I, b, d]$ is given by

$$W_{j}[I, b, d] = \min\left\{W_{j-1}[I_{\ell}, b, d], \min_{d' \leq d} \{W_{j-1}[I_{\ell}, b-1, d']\}, W_{j-1}[I, b, d] + w_{j}\right\}$$

Correctness of our dynamic programming algorithm is immediate following the discussion above. The optimal schedule corresponds to the minimum entry $W_n[I, b, d]$ over all $I \in \{1, \ldots, n\}^{\#p}$, $b \in \{1, \ldots, n\}$, and $d \in \{0, d_1, \ldots, d_n\}$. Note that since table W_j has $n^{O(\#p)}$ entries, and each entry requires O(1) time, computing the entire table can be done in $n^{O(\#p)}$. Thus, the algorithm for computing all tables W_j has the same running time, and the lemma follows.

Lemma 10. Problem $1|B| \sum w_j U_j$ is solvable in time $n^{O(\#w)}$.

Proof. Let $J = \{1, \ldots, n\}$ denote our job set ordered according to EDD, and let $w^{(1)} < \cdots < w^{(\#w)}$ denote the different weights of all jobs in J. The algorithm is very similar to the algorithm in the proof of Lemma 9, except here we compute tables P_j that store minimum total processing time of early jobs, as opposed to minimum total weight of tardy jobs. Namely, for $I \in \{1, \ldots, n\}^{\#p}$, $b \in \{1, \ldots, n\}$, and $d \in \{0, d_1, \ldots, d_n\}$, entry $P_j[I, b, d]$ will equal the minimum total processing time of the early jobs in a schedule for jobs $\{1, \ldots, j\}$ that satisfies the all properties required in the proof of Lemma 9, except that the second condition is rephrased to require exactly I[i] early jobs, $i \in \{1, \ldots, \#w\}$, with weight $w^{(i)}$.

Fix $j \in \{1, ..., n\}$, and let $\ell \in \{1, ..., n\}$ denote the index such that $w_j = w^{(\ell)}$. The base cases for computing $P_j[I, b, d]$ are very similar to those described in the proof of Lemma 9:

If $P_{j-1}[I_{\ell}, b, d] + p_j > d$ and $\min_{d' \leq d} \{P_{j-1}[I_{\ell}, b-1, d'] + p_j + \Delta\} > d$ or if $d > d_j$ then we cannot schedule exactly I[i] jobs with weight $w^{(i)}$ early *including* job j if we consider only the first j jobs. Thus, we set $P_j[I, b, d] = P_{j-1}[I, b, d]$.

Otherwise, the main recursive formula is given by

$$P_{j}[I, b, d] = \min\left\{P_{j-1}[I_{\ell}, b, d] + p_{j}, \min_{d' \leq d} \{P_{j-1}[I_{\ell}, b-1, d'] + p_{j} + \Delta\}, P_{j-1}[I, b, d]\right\} \quad . \quad \Box$$

3 Release dates

In this section we show that the problem of minimizing the weighted number of tardy jobs on a single batch machine when release dates are present is W[1]-hard for parameters #p + #r and #p + #d. That is, we prove Theorem 5. Thereafter, we give XP-algorithms for $1|B, r_j| \sum w_j U_j$ parameterized by #p + #w + #r, and parameterized by #p + #w + #d.

We begin with parameter #p + #r; the hardness for parameter #p + #d will follow almost immediately afterwards. To prove that $1|B, r_j| \sum w_j U_j$ is W[1]-hard with respect to #p + #r, we present a reduction from the k-SUM problem. In this problem, we are given a set $\{x_1, \ldots, x_n\}$ of n positive integers, and a target integer t. The task is to decide if there exist k (not necessarily distinct) integers $x_{\pi(1)}, \ldots, x_{\pi(k)} \in \{x_1, \ldots, x_n\}$ that sum up to t. Abboud, Lewi, and Williams [1] showed that k-SUM is W[1]-hard parameterized by k, even if all integers are in the range $\{1, 2, \ldots, n^{ck}\}$ for some constant c.

3.1 The construction

Let $(x_1, \ldots, x_n; t)$ be an instance of k-SUM, with $x_i \in \{1, 2, \ldots, n^{ck}\}$ for each *i*. Observe that due to their small range, each input integer x_i can be written in the form $x_i = \sum_{j=0}^{ck} \alpha_{i,j} \cdot n^j$ for integers $\alpha_{i,0}, \ldots, \alpha_{c,k} \in \{0, \ldots, n-1\}$, i.e., the base *n* representation of x_i . We will heavily exploit this property in our construction.

Write $X = \sum_i x_i$. Furthermore, we will assume throughout that k - 1 times the largest integer in $\{x_1, \ldots, x_n\}$ is less than t. If this is not the case, one can slightly modify the input by adding kn^{ck} to each integer, and setting the target to $t + k^2n^{ck}$. We construct an instance of $1|B, r_j| \sum w_j U_j$ with O(k) distinct processing times and release times, such that there exists a feasible schedule with $\sum_j w_j U_j \leq kX - t + (n-1)k$ to if and only if there exist k integers $x_{\pi(1)}, \ldots, x_{\pi(k)} \in \{x_1, \ldots, x_n\}$ that sum up to t:

- We create (k-1)t identical jobs, referred to as *leftover jobs*, each with the following parameters:
 - Processing time 1 and weight k(X+n).
 - Release time 0 and due date 3kt.
- For each $\ell \in \{1, \ldots, k\}$, and each input integer $x_i = \sum_{j=0}^{ck} \alpha_{i,j} \cdot n^j$, we create a set $J_{i,\ell}$ of normal jobs that corresponds to x_i . This set consists of $\alpha_{i,j}$ jobs, for each $j \in \{0, \ldots, ck\}$, with the following parameters:
 - Processing time n^j and weight $n^j + n^j/x_i$.
 - Release time $r_{\ell} = (\ell 1)3t$ and due date $(\ell 1)3t + t + x_i$.
- The batch setup time is set to $\Delta = t$.
- The bound on the total weight of tardy jobs is set to kX t + (n-1)k.

Observe that the total processing time of all jobs in the set $J_{i,\ell}$ is precisely x_i , and their total weight is $x_i + 1$. This will be crucial later on. Also note that whereas the weights above are fractional, one can make them integral by multiplying with $\prod x_i$.

3.2 Correctness

Lemma 11. Suppose there exist $x_{\pi(1)}, \ldots, x_{\pi(k)} \in \{x_1, \ldots, x_n\}$ such that $\sum_i x_{\pi(i)} = t$. Then there exists a schedule with $\sum_i w_j U_j \leq kX - t + (n-1)k$.

Proof. We create a schedule with 2k + 1 batches B_1, \ldots, B_{2k+1} . For $\ell \in \{1, \ldots, k\}$, we schedule all jobs in the set $J_{\pi(\ell)}^{(\ell)}$ in batch $B_{2\ell-1}$, and $t - x_{\pi(\ell)}$ leftover jobs in batch $B_{2\ell}$. We schedule the starting time of batch $B_{2\ell-1}$ at time $3t(\ell-1)$, and batch $B_{2\ell}$ at time $3t(\ell-1) + t + x_{\pi(\ell)}$. The remaining jobs are all scheduled in batch B_{2k+1} which starts at time 3kt. Note that in this way all jobs are scheduled after their release times, and only jobs in the last batch B_{2k+1} are tardy. An easy calculation shows that the total weight of jobs in this last batch is

$$\sum_{i \in B_{2k+1}} w_j = kX + kn - \sum_{i=1}^k (x_{\pi(i)} + 1) = kX - K + (n-1)k \quad \Box$$

We illustrate Lemma 11 by an example in Fig. 2.



Figure 2: An illustration of what the schedule given in Lemma 11 looks like.

The converse of Lemma 11 requires more technical detail. We therefore introduce some further notation that will be used throughout the remainder of the section. Assume our constructed instance of $1|B, r_j| \sum w_j U_j$ admits a solution schedule, i.e., a schedule where the total weight of tardy jobs is at most kX + (n-1)k - t. Let $B_1, \ldots, B_b, B_{b+1}$ denote the batches of this schedule, with respective starting times $S_1 < \cdots < S_{b+1}$ and completion times $C_1 < \cdots < C_{b+1}$. Below we modify this schedule, without increasing the total weight of tardy jobs, in order to make our arguments easier.

Lemma 12. Suppose that the constructed instance of $1|B, r_j| \sum w_j U_j$ has a solution schedule. Then it has a solution schedule with batches $B_1, \ldots, B_b, B_{b+1}$, scheduled in that order, where:

- All tardy jobs are in B_{b+1} , and include no leftover jobs.
- All early jobs are in B_1, \ldots, B_b , and include normal jobs with total weight at least t + k.

Proof. Consider any solution schedule with batches B_1, \ldots, B_b , scheduled in that order, that has at most kX - t + (n - 1)k total weight of tardy jobs. We first observe that no leftover job is tardy, as a single leftover job has weight k(X + n) > kX - t + (n - 1)k. Moreover, as the total weight of all normal jobs of the instance is k(X + n), the total weight of the early normal jobs must be at least t + k. Finally, we can move all tardy jobs to a new batch B_{b+1} that starts right after B_b completes, deleting all empty batches resulting from this, without increasing the total weight of tardy jobs.

Due to Lemma 12, some normal jobs must be early. For $\ell \in \{1, \ldots, k\}$, we use E_{ℓ} denote the early jobs of type ℓ in the schedule. Then $\bigcup_{\ell} E_{\ell} \neq \emptyset$. We use $p(E_{\ell})$ and $w(E_{\ell})$ to respectively denote the total processing time and weight of jobs in E_{ℓ} , i.e., $p(E_{\ell}) = \sum_{j \in E_{\ell}} p_j$ and $w(E_{\ell}) = \sum_{j \in E_{\ell}} w_j$.

Lemma 13. For each $\ell \in \{1, \ldots, k\}$ with $E_{\ell} \neq \emptyset$, there is a unique batch $B(\ell) \in \{B_1, \ldots, B_b\}$ with $E_{\ell} \subseteq B$.

Proof. Choose some non-empty E_{ℓ} . Then each job $j \in E_{\ell}$ is released at time r_{ℓ} and has a due date of $r_{\ell} + t + x < r_{\ell} + 2t$ for some $x \in \{x_1, \ldots, x_n\}$ (the inequality follows as all x_i are smaller than t). As batch setup requires t time, and all jobs in E_{ℓ} are early, there must be some batch that contains all jobs of E_{ℓ} . Furthermore, this batch cannot contain jobs of some $E_{\ell'}, \ell' \neq \ell$, since those jobs either have deadlines prior to r_i (in case $\ell' < \ell$), or release times that are later than the due dates of jobs in E_{ℓ} (in case $\ell' > \ell$).

Lemma 13 implies that we can assume there is a specific batch associated with each nonempty E_{ℓ} . Let d_{ℓ} be the earliest deadline in E_{ℓ} . Then $d_{\ell} = (\ell - 1)3t + t + x_{\pi(\ell)}$ for some integer $x_{\pi(\ell)} \in \{x_1, \ldots, x_n\}$. Thus, there is also a specific due date and input integer associated with E_{ℓ} .

Lemma 14. For each $\ell \in \{1, \ldots, k\}$ with $E_{\ell} \neq \emptyset$ we have:

•
$$p(E_\ell) \leq x_{\pi(\ell)}$$
.

• $w(E_{\ell}) \leq p(E_{\ell}) + 1$, and this holds with equality if and only if $E_{\ell} = J_{\pi(\ell),\ell}$.

Proof. According to Lemma 13, there is a unique batch $B(\ell)$ which includes all jobs of E_{ℓ} . As the release time of all jobs in E_{ℓ} is $r_{\ell} = (\ell - 1) \cdot 3t$, and the setup time of $B(\ell)$ is t, it must be that $p(E_{\ell}) \leq ||B(\ell)|| \leq x_{\pi(\ell)}$; otherwise, jobs in E_{ℓ} with due date d_{ℓ} would be late. Now, for each job $j \in E_{\ell}$, let $x(j) \in \{x_1, \ldots, x_n\}$ denote the integer for which j is associated with (i.e., $j \in J_{\ell,x(j)}$). Then $x(j) \geq x_{\pi(\ell)}$ by definition of $x_{\pi(\ell)}$. Since $p(E_{\ell}) \leq x_{\pi(\ell)}$, we have

$$w(E_{\ell}) = \sum_{j \in E_{\ell}} p_j + p_j / x(j) \leq \sum_{j \in E_{\ell}} p_j + p_j / x_{\pi(\ell)} = p(E_{\ell}) + p(E_{\ell}) / x_{\pi(\ell)} \leq p(E_{\ell}) + 1 .$$

Note that the first inequality is strict if and only if there is a job $j \in E_{\ell} \setminus J_{\pi(\ell),\ell}$ as $x(j) \ge x_{\pi(\ell)}$ and the second inequality is strict if and only if $p(E_{\ell}) < x_{\pi(\ell)}$. Hence equality holds if and only if $E_{\ell} = J_{\pi(\ell),\ell}$. The statement of the lemma thus follows.

Lemma 15. $E_{\ell} \neq \emptyset$ for each $\ell \in \{1, \ldots, k\}$.

Proof. By Lemma 12, we have $t + k \leq \sum_{\ell} w(E_{\ell})$. By Lemma 13, we have $p(E_{\ell}) \leq x_{\pi(\ell)}$ and $w(E_{\ell}) \leq p(E_{\ell}) + 1$. Thus,

$$t \leqslant \sum_{\ell=1}^k w(E_\ell) - k \leqslant \sum_{\ell=1}^k p(E_\ell) \leqslant \sum_{\ell=1}^k x_{\pi(\ell)},$$

where $x_{\pi(\ell)} = 0$ if $E_{\ell} = \emptyset$ in the summation above. Since any k - 1 integers in $\{x_1, \ldots, x_n\}$ sum up to a number which is smaller than t, it must be that $x_{\pi(\ell)} > 0$ for all $\ell \in \{1, \ldots, k\}$, and the statement of the lemma follows.

Lemma 16. If there is a solution schedule, then there is one with batches B_1, \ldots, B_{2k+1} scheduled in that order, where for each $\ell \in \{1, \ldots, k\}$:

- $B_{2\ell-1}$ is scheduled at time $3t(\ell-1)$, and $B_{2\ell}$ is scheduled immediately after the completion of $B_{2\ell-1}$.
- $B_{2\ell-1}$ contains only normal jobs of type of ℓ , and $B_{2\ell-1}$ contains only leftover jobs.
- All tardy jobs are in B_{2k+1} , and are normal.

Proof. Let B_1, \ldots, B_{b+1} be the batches of our schedule as in Lemma 12. We modify the batches of this schedule so as to fit the requirements of the lemma without increasing the total weight of tardy jobs in the schedule.

We first note that for each ℓ batch $B(\ell)$ is completely processed in the interval $[3t(\ell - 1), 3t(\ell - 1) + 2t]$. Thus, if there is no batch between $B(\ell)$ and $B(\ell + 1)$, we might as well add one as the time between the completion time of $B(\ell)$ and the starting time of $B(\ell + 1)$ is at least t. (We set $B(k + 1) = B_{b+1}$.) Since there is a batch $B(\ell)$ for each $\ell \leq k + 1$, by Lemma 15, there are 2k batches consisting only of early jobs.

Suppose that the completion time of a batch B_i is in the interval $(3t(\ell - 1), 3t(\ell - 1) + t]$ for some ℓ . Then B_i cannot contain type ℓ jobs, as it started before $3t(\ell - 1)$. Hence, B_i only contains leftover jobs. We can move some leftover jobs from B_i to $B_{i+1} = B(\ell)$ and simultaneously reduce the starting time of $B(\ell)$ by the number of moved jobs, until the starting time of B_i equals $3t(\ell - 1)$.

If no batch is completed in $(3t(\ell-1), 3t(\ell-1) + t]$, then we can start $B(\ell)$ at time $3t(\ell-1)$. This can only decrease the completion times of the jobs. If there are leftover jobs in batch $B(\ell) = B_{2\ell-1}$, then we can move them to batch $B_{2\ell}$. They will not be late as the completion time of $B_{2\ell}$ is at most 3kt.

Lemma 17. Suppose that the constructed instance of $1|B, r_j| \sum w_j U_j$ admits a schedule with $\sum_j w_j U_j \leq kX - t + (n-1)k$. Then there exist $x_{\pi(1)}, \ldots, x_{\pi(k)} \in \{x_1, \ldots, x_n\}$ so that $\sum_i x_{\pi(i)} = t$.

Proof. Let B_1, \ldots, B_{2k+1} be the batches of a schedule as promised by Lemma 16 for our $1|B, r_j| \sum w_j U_j$ instance with $\sum_j w_j U_j \leq kX - t + (n-1)k$. Then batch B_{2k} completes at time $C_{2k} \leq 3kt$, since 3kt is the latest due date of the input jobs. Since there are 2k batches with early jobs, and the setup time for each of these batches is t, we have $\sum_{\ell=1}^{2k} ||B_\ell|| \leq kt$. Thus, as the total processing times of all leftover jobs is (k-1)t, we have

$$\sum_{\ell=1}^{k} p(E_{\ell}) = \sum_{\ell=1}^{k} ||B_{2\ell-1}|| - \sum_{\ell=1}^{k} ||B_{2\ell}|| = \sum_{\ell=1}^{2k} ||B_{\ell}|| - (k-1)t \le t .$$

Recall that, by Lemma 12 and Lemma 13, we also have

$$t \leq \sum_{\ell=1}^{k} w(E_{\ell}) - k \leq \sum_{\ell=1}^{k} p(E_{\ell}) .$$

It follows that $\sum p(E_{\ell}) = t$, and $\sum w(E_{\ell}) = \sum p(E_{\ell}) + k$. The latter equality can only happen if $w(E_{\ell}) = p(E_{\ell}) + 1$ for each $\ell \in \{1, \ldots, k\}$, which in turn implies by Lemma 15 that $p(E_{\ell}) = x_{\pi(\ell)}$ for each $\ell \in \{1, \ldots, k\}$. Thus, $\sum x_{\pi(\ell)} = t$, and the statement of the lemma follows.

3.3 Parameter #p + #d

Lemma 11 and Lemma 17 combined prove that our construction indeed shows W[1]-hardness for parameter #p + #r. We next show that this construction can be transformed to show hardness for parameter #p + #d.

Lemma 18. For non-negative integers k, k', any instance of $1|B, r_j| \sum w_j U_j$ with k distinct release dates and k' distinct due dates can be transformed into an instance of $1|B, r_j| \sum w_j U_j$ with k' distinct release dates and k distinct due dates, which has the same objective value.

Proof. Let J be a set of n jobs forming an instance of $1|B, r_j| \sum w_j U_j$. We create a set J' of n jobs, as follows. For each job $j \in J$ we create one job $j' \in J'$ with $p_{j'} = p_j$, $w_{j'} = w_j$, $r_{j'} = -d_j$ and $d_{j'} = -r_j$. Observe that the problem of finding a maximum-weight set of early jobs is the same for both J and J':

Let σ be a schedule for J, and let $J_e(\sigma)$ be the set of jobs in J that are early in σ . For $j \in J_e(\sigma)$ let S_j denote its starting time of j and C_j its completion time. Then we obtain a schedule σ' for J' by setting the start time of j' to be $S_{j'} = -C_j$ for all jobs $j \in J_e(\sigma)$ and scheduling the remaining jobs late. No two jobs will be processed at the same time, as the intervals $(S_j, C_j), (S_{j'}, C_{j'})$ are pairwise disjoint for all $j, j' \in J_e(\sigma)$. Thus the intervals $(-C_j, S_j), (-C_{j'}, S_{j'})$ are also pairwise disjoint for all $j, j' \in J'_e(\sigma)$. Further, for each $j \in J_e(\sigma)$ we have $S_j \ge r_j$ and $d_j \ge C_j$ and thus also $r_{j'} = -d_j \le -C_j = S_{j'}$ and $d_{j'} = -r_j \ge -S_j = C_{j'}$.

Similarly, given the set $J'_e(\sigma')$ of early jobs for a schedule σ' for J' we obtain a schedule for J such that all jobs j for which $j' \in J'_e(\sigma')$ are scheduled early, by setting $S_j = -C_{j'}$.

This shows that the problem $1|B, r_j| \sum w_j U_j$ with parameter #d + #p is as hard as $1|B, r_j| \sum w_j U_j$ with parameter #r + #p.

Corollary 19. Problem $1|B, r_j| \sum w_j U_j$ is W[1]-hard for parameter #d + #p.

3.4 XP algorithms

Last in this section we give an XP-algorithm for the problem $1|B, r_j| \sum w_j U_j$ parameterized by #p + #r + #w. We use the following notation: Similarly to the due dates, we order the release dates as follows: $r^{(1)} < r^{(2)} \cdots < r^{(\#d)}$.

Lemma 20. Problem $1|B, r_i| \sum w_i U_i$ is solvable in time $n^{f(\#p, \#r, \#w)}$.

Proof. Let *I* be the set of job types with respect to processing time, weight and release date. Let $U = \{v \in \{1, ..., n\}^I\}$ and $V = \{v = (v_1, ..., v_{\#r}) \in U^{\#r} \mid \sum_{\ell=1}^{\#r} (v_\ell)_i \leq n_i\}$ denote the space of possible solution vectors. For each element $v \in V$ we decide whether it is possible to get a schedule that starts $(v_\ell)_i$ early jobs of type $i \in I$ in the interval $[r^{(\ell)}, r^{(\ell+1)})$.

First, notice that if such a schedule exists then we might assume that the jobs of types *i* are scheduled in order of their due date and that only the $\sum_{\ell=1}^{\#r} (v_{\ell})_i$ jobs of type *i* with the latest due dates are scheduled early. Thus we know which jobs are started in each interval $[r^{(\ell)}, r^{(\ell+1)})$.

Second, notice that if we schedule the jobs that are started in $[r^{(\ell)}, r^{(\ell+1)})$ in (EDD)-order starting new batches only if it is necessary then we also get a schedule for these jobs that ends as early as possible. Thus all we need to do in order to decide whether such a schedule exists is to the following: First schedule all jobs that start in $[r^{(1)}, r^{(2)})$ in (EDD). Then let t_1 be the date where the last of these jobs is finished. Then we schedule all jobs that start in $[r^{(2)}, r^{(3)})$ in (EDD) but the starting time of the first batch is $\min\{r_2, t_1\}$. Then let t_2 be the date where the last of these jobs is finished. We then continue in the obvious way. If all jobs scheduled are early and no job is started before its release time then there is such a schedule; otherwise, no such schedule exists. From all schedules we obtain, we take the one that maximizes $\sum_{\ell,i} (v_\ell)_i w_i$. The total run time is $n^{O(\#r^2 \# p \# w)}$. Using Lemma 18 we also get the following result:

Corollary 21. Problem $1|B, r_i| \sum w_i U_i$ is solvable in $n^{f(\#p, \#d, \#w)}$ time.

4 Batch restrictions

In this section we consider the variants of $1|B| \sum w_j U_j$ where the batches are either restricted in terms of their size $(|B| \leq b)$ or their volume $(||B|| \leq b)$.

In this section we will use the notion of job types: Each job $j \in J$ has a *type*, which is given by the vector $\tau(j) = (p_j, w_j, d_j, r_j)$. In some settings parts of the tuple can be omitted, which allows us to shortcut the job type. For example, a job of type (p_j, w_j, d_j, r_j) is also of type (p_j, d_j, r_j) . We denote the set of all job types by \mathcal{T} . For each type $\tau \in \mathcal{T}$ let $d_{\tau}, p_{\tau}, r_{\tau}$ and w_{τ} denote the due date, processing time, release date and weight of jobs with type τ . Note that if all jobs are released at time zero, then a schedule can be given by a function $\sigma : \{1, \ldots, \#d\} \to \mathbb{N}^{\mathcal{T}};$ let $\sigma(\ell)_{\tau}$ indicate the number of jobs of type τ that are completed in the time interval $(d^{(\ell-1)}, d^{(\ell)}]$.

4.1 Bounded batch sizes

First, we show that $1||B| \leq b|\sum w_j U_j$ is fixed-parameter tractable for parameter #d + #w, proving the first part of Theorem 6.

Lemma 22. Problem $1||B| \leq b|\sum w_j U_j$ can be solved in time $f(\#d + \#w) \cdot n^{O(1)}$.

Proof. Given an instance \mathcal{I} of $1|B, |B| \leq b| \sum w_j U_j$, we set up the following mixed-integer linear program (MILP) to find an optimal schedule. The variables of the MILP are defined as follows. Let $I = \{(w,d) \mid (w,p,d) \in \mathcal{T} \text{ for some } d\}$ be the set of job types with respect to weight and due date. For each type $i \in I$ and each $\ell \in \{1, \ldots, \#d\}$ we have one integer variable $x_{i,\ell}$, indicating the number of jobs of type i finishing job in the time interval $(d^{(\ell-1)}, d^{(\ell)}]$. (Note that this means that their batches finish in the interval.) For each job type $\tau = (d_{\tau}, p_{\tau}, w_{\tau}) \in \mathcal{T}$, we have one fractional variable $y_{(\tau,\ell)} \in [0, n_{\tau}]$ to indicate the number jobs of type τ which are processed in time before their due date $d^{(\ell)}$. (Recall that n_{τ} is the number of jobs of type τ .) For each index $\ell \in \{1, \ldots, \#d\}$ we have one integer variable z_{ℓ} to indicate the number of batches that are completed before or at time $d^{(\ell)}$. Finally, we set $z_0 = 0$.

The MILP is given by

(1)
$$\min \sum_{\tau \in \mathcal{T}} (n_{\tau} - y_{(\tau, \#d)}) w$$

(2)
$$z_{\ell} \ge z_{\ell-1} + \frac{1}{b} \sum_{i \in I} x_{i,\ell},$$

(3)
$$\sum_{\ell_0 \leqslant \ell} x_{i,\ell_0} = \sum_{\tau \in \mathcal{T}, p_\tau = p_i \land d_\tau = d_i} y_{(\tau,\ell)}, \qquad i \in I, \ell = 1, \dots, \#d,$$

(4)
$$z_{\ell}\Delta + \sum_{\tau \in \mathcal{T}} p_{\tau} y_{(\tau,\ell)} \leq d^{(\ell)} \qquad \qquad \ell = 1, \dots, \# d$$

The MILP has $\#d(|I|+1) = O(\#d^2 \cdot \#w)$ integer variables and $|\mathcal{T}|\#d = O(|\mathcal{T}|^2)$ fractional variables. It can be solved by Lenstra's algorithm [11] for integer programming in fixed dimension in time $f(\#d, \#w) \cdot n^{O(|\mathcal{T}|)}$.

 $\ell = 1, \dots, \#d,$

It remains to show that optimal solutions of value W to the MILP correspond to optimal schedules with weighted number of tardy jobs equal to W. A crucial observation is that, given

an optimal solution to the MILP, we can assume that all variables $y_{(\tau,\ell)}$ take integer values. This is due to the fact that, given a job type $\tau \in \mathcal{T}$ and an index $\ell \in \{1, \ldots, \#d\}$, we can assume that if $y_{\tau\ell} < n_j$ then $y_{\tau'\ell} = 0$ for all τ' with $p_{\tau'} > p_{\tau}, w_{\tau'} = w_{\tau}$ and $d_{\tau'} = d_{\tau}$. For if that was not the case, then we can increase $y_{\tau\ell}$ and decrease $y_{\tau'\ell}$ by the same amount, without changing the objective value or violating constraint (3) or constraint (4). The intuition here is that we can process the jobs of type $i \in I$ in increasing order of their processing time.

Note that (2) assures that we use $\left[\frac{1}{b}\sum_{i\in I} x_{i\ell}\right]$ batches ending in $(d^{(\ell-1)}, d^{(\ell)}]$ which is the minimum number of batches needed to complete all the jobs ending in that interval. Constraint (3) is for determining the exact types of the jobs that are processed rather than just the type with respect to weight and due date. As mentioned we can assume that the *y*-variables are integral in an optimum solution. Constraint (4) makes sure that all the early jobs are indeed completed before their due date.

To obtain a schedule from a solution to the MILP, we first process $x_{i,1}$ jobs of type *i* for each *i* in order of their processing times with ties broken arbitrarily, and always starting a new batch when necessary and closing the last batch at the end. Then we can continue with $x_{i,2}$ jobs of type *i* for each *i* the same way, and so on. Conversely, a schedule translates into a solution (also fulfilling (2)) using the interpretations for the variables.

If #p + #d (rather than #w + #d) is bounded by our parameter then we get an even stronger result. More precisely we can solve instances where jobs additionally can have different release dates as long as the number of different release dates is also bound by our parameter.

Theorem 23. Problem $1|B, |B| \leq b, r_j | \sum w_j U_j$ is fixed-parameter tractable for parameter #d + #p + #r.

Proof. We set $T = \{r_j \mid j \in J\} \cup \{d_j \mid j \in J\}$ to be the set of critical time points. Further we order $T = \{t_1, \ldots, t_k\}$ in increasing order, i.e., $t_1 < t_2 < \cdots < t_k$. We again design a MILP, but this time with slightly more variables. Instead of variables z_{ℓ} , this time we will use integral variables $z_{\ell,\ell'}$ for any $\ell < \ell'$ to indicate the number of batches that start at or after t_ℓ but before $t_{\ell+1}$ and finish before or at $t_{\ell'}$ but after $t_{\ell'-1}$. Now we set $I = \{(p, r, d) \mid (p, w, r, d) \in \mathcal{T}$ for some $d\}$ be the set of job types with respect to weight and due date. Instead of $x_{i,\ell,\ell}$, we have integral variables $x_{i,\ell,\ell'}$ to indicate the number of early jobs of a given type that are processed in batches starting at or after t_ℓ but before $t_{\ell+1}$ and completed before or at $t_{\ell'}$ but after $t_{\ell'-1}$. We note that we remove variables $x_{i,\ell,\ell'}$ if $d_i < t_{\ell'}$ or $r_i > t_{\ell}$. We use variables y_{τ} to indicate the number of early jobs of type $\tau \in \mathcal{T}$.

The MILP has the following constraints:

$$\min \sum_{\tau \in \mathcal{T}} (n_{\tau} - y_{\tau}) w_{\tau}$$

$$\sum_{i \in I} x_{i,r,d} \leq b z_{\ell,\ell'}$$
for any $1 \leq \ell < \ell' \leq k$

$$\sum_{\ell,\ell'} x_{i,\ell,\ell'} = \sum_{\tau \in \mathcal{T}, w_{\tau} = w_i \land r_i = r_{\tau} \land d_{\tau} = d_i} y_{\tau}$$
for each $i \in I$

$$t + \sum_{t \leq t_{\ell} < t_{\ell'} \leq t'} \left(z_{\ell,\ell'} \Delta + \sum_{i \in I} p_i x_{i,\ell,\ell'} \right) \leq t'$$
for each $t, t' \in T$ with $t < t'$

$$y_{\tau} \leq n_{\tau}$$
for each $\tau \in \mathcal{T}$

We further need two more kinds of constraints to guarantee that if there is a long batch, i.e., a batch that starts before t_{ℓ} and ends at or before $t_{\ell'}$ but after $t_{\ell'-1} \ge t_{\ell}$, then there cannot be any

other batch starting and ending in $[t_j, t_{j'}]$ for any pair $(j, j') \in \{\ell, \ldots, \ell'\}^2 \setminus \{(t_\ell, t_{\ell+1}), (t_{\ell'-1}, t_{\ell'})\}$.

 $z_{\ell_1,\ell_2} + z_{\ell_3,\ell_4} \leq 1$ (5)

$$\begin{aligned} z_{\ell_1,\ell_2} + z_{\ell_3,\ell_4} &\leq 1 & \text{if } \ell_1 \leq \ell_3 < \ell_3 + 2 \leq \ell_4 \leq \ell_2 \\ \frac{z_{\ell,\ell+1}}{n} + z_{\ell_1,\ell_2} \leq 1 & \text{if } \ell_1 < \ell \text{ and } \ell_2 > \ell + 1 \end{aligned}$$

(6)

Using the interpretations of the variables given a schedule, one can easily construct a feasible solution of the MILP with same value.

We claim that in any optimal solution of the MILP, all variables of the form y_{τ} are integral, and $y_{\tau} \leq n_{\tau}$ implies $y_{\tau'} = 0$ for all other types τ' with the same processing time, release date and due date but higher weight. For proof, suppose, for sake of contradiction, that there is some non-integral y_{τ} . Let τ be of (sub)type $i \in I$. Since $\sum_{\tau \in \mathcal{T}, w_{\tau} = w_i \wedge r_i = r_{\tau} \wedge d_{\tau} = d_i} y_{\tau} = \sum_{\ell, \ell'} x_{i,\ell,\ell'}$ is integral there must be another non integral variable $y_{\tau'}$ such that τ' is also of type *i*. Assume, without loss of generality, that $w_{\tau} > w_{\tau'}$. Now since n_{τ} is integral, we have $y_{\tau} < n_{\tau}$. Thus we can increase y_{τ} and decrease $y_{\tau'}$ by the same amount until either $y_{\tau} = n_{\tau}$ or $y_{\tau'} = 0$. The solution we get is still feasible, but its value is smaller, contradicting the optimality of the initial solution. The same argumentation can be used to show that $y_{\tau} \leq n_{\tau}$ implies $y_{\tau'} = 0$ for all other types τ' with the same processing time, release date and due date but higher weight. This proves the claim.

Now to create a schedule we create $z_{\ell,\ell'}$ batches $\mathcal{B}_{\ell,\ell'}$ for each variable $z_{\ell,\ell'}$ and fill them with appropriate jobs, i.e., such that there $x_{i,\ell,\ell'}$ jobs of type *i* assigned to them. We schedule the batches in the following way: If batch B is in $\mathcal{B}_{\ell,\ell'}$ and batch B' is in $\mathcal{B}_{\ell_1,\ell_2}$ then we schedule B before B' if $\ell < \ell_1$, or $\ell = \ell_1$ and $\ell' < \ell_2$, breaking ties arbitrarily. Given this ordering, we schedule batch $B \in \mathcal{B}_{\ell,\ell'}$ at the completion time of the previous batch if it finishes later than t_{ℓ} , or at time t_{ℓ} otherwise.

We need to show that indeed all $\sum_{\ell,\ell'} x_{i,\ell,\ell'}$ jobs of type *i* scheduled in these kind of batches are early for each type i. Suppose, for sake of contradiction, that there is late job j in batch $B \in \mathcal{B}_{\ell,\ell'}$ for some ℓ and ℓ' . Let t_{ℓ_0} be the latest time point less or equal to t_{ℓ} such that there is idle time before t_{ℓ_0} , or—if no such time exists—we set t_{ℓ_0} to be the smallest release time.

We claim that

$$t_{\ell_0} + \sum_{\ell_0 \leqslant \ell_1, \ell_2 \leqslant \ell'} \left(z_{\ell_1, \ell_2} \Delta + \sum_{i \in I} p_i x_{i, \ell_1, \ell_2} \right) > t_{\ell'} \ .$$

To see that notice, that only jobs in batches in $\mathcal{B}_{\ell_1,\ell_2}$ with $\ell_1 \ge \ell_0$ and $\ell_2 \le \ell'$ are scheduled before the completion time of j. This holds true as any batch $B' \in \mathcal{B}_{\ell_1,\ell_2}$ with $\ell_1 < \ell_0$ is completed before t_{ℓ_0} by definition of ℓ_0 , and any batch $B' \in \mathcal{B}_{\ell_1,\ell_2}$ with $\ell_1 \ge \ell$ and $\ell_2 > \ell'$ is scheduled later than j. Further, we have $z_{\ell_1,\ell_2} = 0$ if $\ell_1 < \ell$ and $\ell_2 > \ell'$ by constraint (5) and (6) using that $z_{\ell,\ell'} \ge 1$. However, our claim contradicts the feasibility of our solution thus j cannot be late.

4.2Bounded batch volume

In the last part of this section we show why the problem $1|B| \sum U_i$ becomes hard when we add a bound to the maximum batch volume. First, we consider parameter #d + #w, and afterwards parameter #p.

In PARTITION we are given a set $T = \{x_1, \ldots, x_n\}$ of natural numbers such that $\sum_{x \in T} x = 2K$; the task is to decide if there exists a set $T' \subseteq T$ such that $\sum_{x \in T'} x = \sum_{x \in T \setminus T'} x = K$.

We now devise a reduction from PARTITION to show hardness of batch scheduling even in the unweighted case and a single due date.

Theorem 24. Problem $1|B, ||B|| \leq V |\sum U_j$ is NP-hard for #d = 1.

Proof. Let $(T = \{x_1, \ldots, x_n\}; K)$ be an instance of PARTITION; we construct an instance of $1|B, ||B|| \leq V |\sum U_j$ as follows:

- We set $\Delta = 1$ and V = K + 1.
- For each number x_i there is one job j_i with $p_{j_i} = x_i$ and $d_{j_i} = 2K + 2$.

Observe that there is a schedule with zero tardy jobs if and only if there is a subset $T' \subseteq T$ such that $\sum_{x \in T} x = \sum_{x \in T' \setminus T} x = K$. This is due to the fact that the only way to get such a schedule is to use exactly two batches of volume V and a batch can only be of volume V if the processing times of the jobs assigned to it add up to K.

For parameter #p we prove the following result:

Theorem 25. Any instance \mathcal{I} of $P||C_{\max}$ with #p different processing times can be transformed to an instance of $1|B, ||B|| \leq V|\sum U_j$ with #p different processing times and a single due date, such that all jobs of \mathcal{I} complete by time T if and only if all jobs of \mathcal{I}' are early.

Proof. Consider an instance \mathcal{I} of $P||C_{\max}$ with job set J, number m of machines, and target makespan T. We create an instance \mathcal{I}' of $1|B, ||B|| \leq V|\sum U_j$ consisting of a batch setup time Δ , a batch volume V, and a job set J'. We set $\Delta = Tm$ and V = T. The set J' contains one job j' for each job $j \in J$, where the processing time of j' is the same as the processing time of j and the due date of j' is equal to $d_j = mV$.

In the forward direction, any schedule for \mathcal{I} with makespan at most T can be translated to a feasible schedule for \mathcal{I}' that schedules all jobs early by creating one batch B for each machine i. All jobs scheduled on i will be assigned to B. Then the batch volume of each batch is at most T, and all m batches are completed early.

In the backward direction, any schedule for \mathcal{I}' has at most m batches with early jobs, as

$$(m+1)\Delta = (m+1)Tm > m(Tm+T) = m(\Delta + T) = mV = d_i$$
.

Thus, for each batch B with early jobs we can schedule all jobs assigned to B on one machine, whose completion time is at most T. In summary, for m batches with early jobs, we obtain a schedule for \mathcal{I} with makespan at most T.

5 Discussion and Open Problems

We provided an extensive multivariate analysis of the single-machine batch scheduling problem to minimize the weighted number of tardy jobs. In particular, we significantly refined and extended the work of Hochbaum and Landy [8], as well as Hermelin et al [7].

Several open questions remain, even for the setting without batches. It appears especially challenging to resolve the question of whether $1||w_jU_j|$ is fixed-parameter tractable for #p, or #w, or turns out to be W[1]-hard for either of those parameterizations. This question was already stated by Hermelin et al. [7], and is not resolved here. Naturally, we do not know the answer to this question for the more general $1|B| \sum U_j w_j$ problem; however, we also do not know the status of parameter #p + #w for which $1||\sum w_jU_j$ is known to be fixed-parameter tractable [7]. Another interesting question is to see if $1||B| \leq b| \sum U_j$ is fixed-parameter tractable for parameter #p or b, or even solvable in polynomial time.

References

- A. Abboud, K. Lewi, and R. Williams. Losing weight by gaining edges. In Proc. ESA 2014, pages 1–12, 2014.
- [2] P. Brucker and M. Y. Kovalyov. Single machine batch scheduling to minimize the weighted number of late jobs. *Math. Meth. Oper. Res.*, 43(1):1–8, 1996.
- [3] T. Cheng and M. Kovalyov. Single machine batch scheduling with sequential job processing. *IIE Trans.*, 33(5):413–420, 2001.
- [4] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized algorithms*, volume 4. Springer, 2015.
- [5] M. Etscheid, S. Kratsch, M. Mnich, and H. Röglin. Polynomial kernels for weighted problems. J. Comput. Syst. Sci., 84:1–10, 2017.
- [6] R. Graham, E. Lawler, J. Lenstra, and A. Kan. Optimization and approximation in deterministic sequencing and scheduling: a survey. Ann. Discrete Math., 3:287–326, 1979.
- [7] D. Hermelin, S. Karhi, M. Pinedo, and D. Shabtay. New algorithms for minimizing the weighted number of tardy jobs on a single machine. *Ann. Oper. Res.*, 2018.
- [8] D. S. Hochbaum and D. Landy. Scheduling with batching: minimizing the weighted number of tardy jobs. Oper. Res. Lett., 16(2):79–86, 1994.
- [9] R. M. Karp. Reducibility among Combinatorial Problems, pages 85–103. 1972.
- [10] E. Lawler and J. Moore. A functional equation and its application to resource allocation and sequencing problems. *Mgmt. Sci.*, 16(1):77–84, 1969.
- [11] H. W. Lenstra. Integer programming with a fixed number of variables. Math. Oper. Res., 8(4):538–548, 1983.
- [12] M. Mnich and R. van Bevern. Parameterized complexity of machine scheduling: 15 open problems. Computers Oper. Res., 100:254 – 261, 2018.
- [13] J. M. Moore. An n job, one machine sequencing algorithm for minimizing the number of late jobs. Mgmt. Sci., 15(1):102–109, 1968.
- [14] C. N. Potts and M. Y. Kovalyov. Scheduling with batching: A review. European J. Oper. Res., 120(2):228–249, 2000.
- [15] S. Sahni. Algorithms for scheduling independent tasks. J. ACM, 23(1):116–127, 1976.
- [16] S. Webster and K. R. Baker. Scheduling groups of jobs on a single machine. Oper. Res., 43(4):692–703, 1995.