Large components in inhomogeneous random graphs

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Summary

This thesis focuses on the analysis of inhomogeneous random graphs, which are motivated by real-world complex networks. On the one hand, we study so-called rank-1 models given by the Chung-Lu model, the generalised random graph and the Norros-Reittu model. These can be seen as generalisations of the Erdős-Rényi graph. On the other hand, we investigate the weighted random connection model, which is in turn related to the Gilbert graph.

In all these models, one equips the vertices with random weights, governing their degrees. To be more precise, one connects the vertices in such a way that the probability of connecting two vertices by an edge is increasing in their weights. While the rank-1 models have no further random ingredients, the vertices of the random connection model are embedded into Euclidean space by an underlying Poisson process. Here, we obtain additional structure by increasing the probability of connecting two vertices as their Euclidean distance shrinks. Suitable assumptions lead to a scale-free degree distribution for both models, as observed in real-world complex networks.

For the rank-1 models, we study the sizes of large components in the subcritical regime, i.e. in the absence of a giant component. To this end, we investigate the underlying point process of all rescaled component sizes and show weak convergence to a Poisson process. This determines the asymptotic distribution of the (properly rescaled) size of the largest component. More generally, our framework allows us to count only specific vertices per component, such as leaves. In this case, we obtain a result on the largest number of leaves in a single component.

We wish to ask the same question for the random connection model: what is the size of its largest component in the subcritical regime? Since the random connection model is an infinite graph, we restrict ourselves to considering the sizes of components lying in an observation window of finite volume. As the size of this window tends to infinity, we establish weak convergence of the point process of rescaled component sizes to a Poisson process, as in the rank-1 case. Then, we deduce the scaling of the size of the largest component in terms of the volume of the observation window and its asymptotic distribution.

The previous results require the weights to have a regularly varying tail. Our last results concern cycles in rank-1 models and only need the existence of some moments of the weights, not a precise form of their tail. Using Stein's method, we provide quantitative Poisson approximation results for the number of cycles whose length lies in some bounded subset of the natural numbers. In the subcritical case, we extend our results to unbounded subsets of the natural numbers. From this, we deduce the asymptotic distribution of the lengths of the shortest and of the longest cycle, including a bound on the convergence rate with respect to the Kolmogorov distance.

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Going back a couple of years, I remember that mathematics was not my favourite subject in elementary school. However, this changed in 5th grade due to my inspiring mathematics teacher Karsten Alpers. Thanks for sparking my interest in the subject!

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Contents

Chapter 1

Introduction

The study of inhomogeneous random graphs is motivated by complex networks. This term refers to huge complicated graphs, i.e. large collections of vertices connected by edges in an involved pattern. These graphs typically describe the interactions of people or objects in the real-world. They arise in numerous contexts and are usually divided into four groups, see e.g. the survey article [75] by Newman:

- social networks,
- information networks,
- technological networks and
- biological networks.

Examples include Facebook as a virtual social network, where the user profiles correspond to vertices and edges are placed between (Facebook) friends. One is also interested in more classical interaction networks between people, which is of importance for understanding how opinions, rumors or also diseases spread among a population. Information networks are given by the World Wide Web or citation networks of academic papers. The internet (i.e. the connections between computers), power grids and telecommunication networks provide examples for technological networks, while biological networks include, for instance, the brain or protein interactions. More examples and references to papers studying the respective networks can be found in the works [75] by Newman and [78] by Newman, Barabási, and Watts. We refer to the monographs [9] by Barabási and [77] by Newman for introductions to the theory of complex networks from a network science perspective. The books [29] by Chung and Lu and [52] by van der Hofstad provide an introduction from a mathematician's point of view.

Surprisingly, complex networks share some important structural properties, even though they arise in very different contexts. Among these aspects are

- power-law degree distributions,
- small-world phenomena and
- clustering effects.

The degree distribution describes which proportion of vertices in the network has a certain degree, where the degree of some vertex denotes the number of edges attached to it. The importance of knowing whether there are vertices with a very high degree in applications is, for instance, highlighted by the spread of a pandemic being driven by superspreaders, rather few individuals that are responsible for a comparatively large number of infections, as demonstrated in [39] by Galvani and May. We abbreviate the proportion of vertices having degree k by $p(k)$. Roughly speaking, a power-law degree distribution means that

$$p(k) \propto k^{-\gamma} \quad \text{for some} \quad \gamma > 1, \tag{1.1}$$

where $\propto$ denotes a proportional relation between two quantities. Because $p(k)$ decays like a power of k, this is also referred to as a power-law or as scale-free. This phenomenon is easier to observe by plotting $k \mapsto p(k)$ on a log-log-scale. Then, instead of observing a polynomial decay as in (1.1), the graph turns out to be close to a line having the exponent $-\gamma$ as slope. One can make the relation in (1.1) mathematically precise by the usage of regularly varying functions, which we introduce in Section 2.3.

On top of the monographs mentioned above, more information on scale-free networks based on real-world data can be found in [92] by Voitalov, van der Hoorn, van der Hofstad and Krioukov. For historical overviews on the appearances of power-laws, not only in the degree distribution of networks, we refer to [29, Section 1.4] by Chung and Lu and to [76] by Newman.

The small-world property refers to the average distance between vertices being small with respect to the overall network size, where one considers the graph distance, i.e. the smallest number of edges that form a link between two vertices. An example for these short distances is given by the famous six degrees of separation by Milgram in [74], where vertices are given by people, which are connected if they know each other. A more modern example is given by the Facebook graph, where, in 2011, the distance between any two people was on average only 4.7 as investigated in [8] by Backstrom, Boldi, Rosa, Ugander and Vigna as well as in [91] by Ugander, Karrer, Backstrom and Marlow. For an overview on small worlds we additionally refer to the monograph [93] by Watts.

Finally, the term clustering corresponds to the appearance of many triangles, i.e. triplets of vertices for which all three edges between them exist. Thinking of a social network of friendships, when one considers person A and two of its friends B and C, it is quite likely that B and C are also friends, at least much more likely than any two random persons D and E being friends. This can be formalised by the so-called clustering coefficient, measuring the proportion of wedges that also form a triangle, where a wedge is given by three vertices connected by two edges. Albert and Barabási provide the clustering coefficient as well as average distances and other statistics for several real-world examples in [2, Table 1].

We now turn our focus towards random graph models for complex networks. A traditional model is the Erdős-Rényi graph $G(n, p)$ introduced in 1959 by Gilbert in [43]. Note that the model is named after Erdős and Rényi, who actually provided a different, albeit closely related model in [34]. The $G(n, p)$ model takes n vertices and connects any pair of vertices independently via an edge with probability p. This leads to the degree of any vertex following a binomial distribution, which in turn converges to a Poisson distribution as the size of the graph tends to infinity, when the connection probability shrinks appropriately at the same time. The Poisson distribution does not possess a power-law decay as in (1.1), instead it decays much faster. We obtain an absence of vertices having very high degrees, which are an important feature in applications as mentioned above. This leads to the demand of finding models which capture this, and ideally other, properties of complex networks as argued in [10] by Barabási and Albert. They provide the Barabási-Albert model which uses a preferential attachment algorithm, meaning that vertices arrive one by one and are more likely to connect to high-degree vertices, intuitively speaking, to well-established objects in whatever sense. Over the past decades, a lot of random graph models capturing the power-law degree distribution of complex networks have emerged. In this thesis, we shall focus on two such classes of random graphs, which we formally introduce in Chapter 3, where we also discuss existing results from the literature more deeply. For now, we give a brief overview in order to frame the goals and findings of this thesis.

The first class is a generalisation of the Erdős-Rényi graph with the following intuition. The aforementioned problem with $G(n, p)$ is that it does not produce sufficiently many high-degree vertices. This is due to the fact that the graph is homogeneous in the sense that each edge has the same probability of appearing. We obtain a prototype example of an inhomogeneous graph by endowing the vertices with random weights that can be interpreted as fame or fitness of the respective nodes. Given these weights, we no longer connect all vertices with the same probability, but in a way such that it becomes more likely to connect vertices with large weights, yielding inhomogeneity in the model. We focus on three closely related models of this type, the generalised random graph established by Britton, Deijfen and Martin-Löf in [23], the Chung-Lu model presented by Chung and Lu in [27] and the Norros-Reittu model introduced by Norros and Reittu in [81]. These models are asymptotically equivalent under some assumptions on the weight distribution as shown by Janson in [58]. We will formally introduce and properly discuss these models in Section 3.1. A more general version of an inhomogeneous random graph was introduced in [21] by Bolobás, Janson and Riordan. The authors present numerous results in their general setup while Section 16.4 therein discusses the consequences for so-called rank-1 models, which include the three models above, or variants thereof. In particular, the degree distribution satisfies a power-law if the weight distribution obeys a power-law.

Let us consider the next desired property of our random graph models, the distances shall be small. In rank-1 models on n vertices one obtains average distances scaling like $\log(n)$ or $\log(\log(n))$, depending on the tails of the weight distribution. Results in this direction were already proven in [28] by Chung and Lu and in [81] by Norros and Reittu. Also, Bollobás, Janson and Riordan obtained results in their general setup in [21]. For an overview on distances in rank-1 models, we refer to the survey [54] by van der Hofstand and Hooghiemstra.

In addition to the distances, another property that was already studied in the aforementioned papers is that of a phase transition concerning the emergence of a giant component, also simply called the giant. The giant is a component that contains a positive fraction of all vertices as the underlying network size goes to infinity. We will go into more detail later in Subsection 3.1.4. For now, let us just state that there is a parameter ν, given by the ratio of the second and first moment of the weight distribution, such that there is a giant if and only if $\nu > 1$. This is called the supercritical phase and one can show that the giant component is unique in the sense that all other components are of strictly smaller order. This justifies speaking of *the* giant. The case $\nu = 1$ is referred to as critical, whereas $\nu < 1$ is denoted as subcritical and corresponds to a regime in which there are rather few edges. The corresponding statements for the three models above can also be found in the textbook [53, Theorem 3.20] by van der Hofstad. While the largest component is of order n in the supercritical phase, it is of smaller order in the critical and subcritical phase. The respective scalings were investigated in [57] by Janson in the subcritical phase, whereas Bhamidi, van der Hofstad and van Leeuwarden studied the critical phase in [12, 13].

This phase transition motivates our first question of interest. We consider the rank-1 models in the subcritical regime with weights obeying a power-law. The aforementioned result in [58] by Janson provides information on the size of the largest component in the subcritical regime. From another perspective, one obtains the largest component size by counting all vertices in each single component and considering the largest of these counting statistics. What happens if one does not count all vertices, but only specific ones, for example leaves, i.e. vertices of degree one? Put in other words, what is the largest number of leaves one can find in a single component? Since leaves can be thought of as points that mark the end of a component, what about more complicated objects of a similar kind? To have a suitable picture in mind, one can think of a component and replace some of its leaves with trees. Then, these trees are such objects marking the end of a component. We investigate the number of more general vertex counts collected in a point process and its convergence in Chapter 4. From this we deduce results on the largest vertex count in a single component. In the special case where we count all vertices in a component, we retrieve the result from Janson in [58].

Our analysis in Chapter 4 relies heavily on the fact that there are very few cycles in each component. However, we only need to show that there are sufficiently few, without a detailed analysis. This leads to the question whether one can derive precise asymptotics for the number of cycles, which is the subject of Chapter 5. Cycle counts and more general subgraph counts were already studied extensively for different classes of graphs such as random regular graphs or the Erdős-Rényi graph, see e.g. the monograph [20, Section 2.4 and Chapter 4] by Bolobás and the references therein. In the rank-1 setting, we will provide a broader overview on the literature in Chapter 5. For now, let us take a look at two recent results that are in a very similar setting to ours. Liu and Dong showed in [70] that the number of triangles converges in distribution to a Poisson random variable, by calculating factorial moments. Their result was generalised in [18] by Bobkov, Danshina and Ulyanov, using the Chen-Stein method that was established in [26] by Chen, building upon Stein's method for normal approximation from [90] by Stein. They treat cycles of any length k under some moment assumptions on the weight distribution. Additionally, they achieve a rate of convergence in the total variation distance of order $O(n^{-1/2})$. In Chapter 5, we improve their results further, also by applying the Chen-Stein method. Our improvements include much weaker moment assumptions and significantly faster convergence rates. In particular, our moment conditions no longer grow more restrictive as the cycle length increases. We also deduce asymptotic properties of the lengths of the longest and of the shortest cycle in the subcritical regime.

Above it was mentioned that the Erdős-Rényi graph is no suitable model for complex networks as it does not capture the scale-free property, which was the motivation for studying rank-1 models. The results from Chapter 5 imply that the number of triangles converges to a Poisson distribution, in particular it does not grow with increasing network size, contradicting the concept of clustering. More generally, one can show that the rank-1 models behave locally as if they were trees, under some minor assumptions. This is made precise by so-called local convergence, see the monograph [53] by van der Hofstad for an introduction to the topic and the particular result in the rank-1 setting in [53, Theorem 3.18]. Therefore, one is interested in finding random graph models that capture clustering while maintaining the power-law degree distribution.

Seeking inspiration in real-world examples, a complex network such as the brain or a road system has an underlying geometry that naturally favours the emergence of highly connected regions. This leads to the idea of constructing a graph based on a spatial vertex set. A first example was already given in 1961 by Gilbert, see [44], and is known as random geometric graph. One takes the points of a Poisson process on $\mathbb{R}^d$ as vertices and connects them via an edge whenever they are sufficiently close to each other, meaning that their distance is smaller than some fixed threshold $r > 0$. This model exhibits natural clustering effects as all triplets of points that are in some disc of diameter r will form a triangle. However, the degree of a vertex in this model is given by the number of points in a circle around it, which follows a Poisson distribution and does not provide a power-law degree distribution. We encountered a similar problem with the Erdős-Rényi graph and it may be overcome by using the same solution as in the rank-1 case: we endow the vertices with weights. Then, we connect them in such a way that the probability of connecting two vertices is increasing in their weights but decreasing in their distance. The model obtained in this fashion is referred to as weighted (or scale-free) random connection model and was introduced by Deprez and Wüthrich in [33]. On the one hand, suitable weights lead to the scale-free behaviour, whereas the spatial component of the model ensures clustering. More explicitly, Deprez and Wüthrich showed that the degrees obey a power-law and that average distances are small. Dalmau and Salvi in turn investigated the clustering coefficient in [30]. We will formally present this model in Section 3.2.

There are many related models, e.g. on deterministic vertex sets such as $\mathbb{Z}^d$ or models without weights, which we will discuss in more detail in Subsection 3.2.2. For now, we only consider a concept similar to the giant component for rank-1 models or the Erdős-Rényi graph. Recall that this term refers to a component containing a positive fraction of all vertices. This definition does not make sense for the random connection model, as there are infinitely many vertices. Instead, one is interested in the question whether there is an infinite component, also called infinite cluster, or not. This question was originally studied for a simpler model on $\mathbb{Z}^d$, where each vertex is independently connected to any of its $2d$ closest points with probability p. This research area is known as (bond-)percolation and is motivated by the physical application of a fluid passing through a medium. The term bond-percolation refers to the removal of edges, i.e. *bonds*, in the underlying graph. Another variant is that of site-percolation, where one removes vertices, i.e. *sites*, from the graph. The first work on percolation is due to Broadbent and Hammersley in [24], whereas textbooks on the topic include the ones written by Grimmett [46] and by Bollobás and Riordan [22].

In order to set the stage for our last chapter, it suffices to note that random graphs usually undergo a phase transition concerning the existence of an infinite cluster. In the classical model on $\mathbb{Z}^d$, where one connects vertices in distance one with probability p, this phase transition is with respect to p. The infimum over all p such that an infinite cluster exists with positive probability is called the critical value and is denoted by p_c. Analogous to the phase transition for the giant component, one calls the model supercritical when $p > p_c$, critical if $p = p_c$ and subcritical for $p < p_c$. At this point it seems noteworthy that it remains an open question whether there are infinite clusters at criticality, i.e. for $p = p_c$, for $d \geq 2$, see also the book [50, Open Problem 1.1] by Heydenreich and van der Hofstad.

We provide a similar phase transition for the scale-free random connection model due to Deprez and Wüthrich in [33] later on. For phrasing our question of interest in Chapter 6, it suffices to note that we restrict ourselves once more to the subcritical regime of the random connection model, so in particular there will be no infinite component. One can ask the same question we already considered for the Norros-Reittu model: what is the size of the largest component? Phrased like this, the question is ill-posed. This follows from the fact that, with positive probability, one can find k points in the unit cube $[0, 1]^d$ which form a component of size k, for all $k \in \mathbb{N}$. Due to translation invariance of the model, one will find components of every size somewhere in $\mathbb{R}^d$, so that there is no largest component. Thus, we restrict ourselves to a sequence of observation windows $S_n = [0, n^{1/d}]^d$ so that the n-th observation window has volume n for $n \in \mathbb{N}$. Analogously to our investigation in Chapter 4, we study the point process of component sizes for the components lying in S_n. Properly rescaled, this point process converges to a Poisson process, which in turn provides information on the size of the largest component one can find in a cube of volume n.

This thesis is mainly based on the following three papers, two of which are joint work with Matthias Schulte and one of them being in preparation:

- [68] *M. Lienau 2024*:
  Poisson approximation for cycles in the generalised random graph. *Preprint.*
- [69] *M. Lienau and M. Schulte 2025*:
  Large components in the subcritical Norros-Reittu model. *Extremes.*
- *M. Lienau and M. Schulte 2025+*:
  Large components in the scale-free random connection model. *In preparation.*

The structure of this work is as follows. We provide general background material needed throughout this thesis in Chapter 2. After establishing basic notation, we present results on point processes with an emphasis on their convergence and Poisson processes. Additionally, we present results on regular variation and on the Chen-Stein method for Poisson approximation. Chapter 3 is devoted to a formal introduction to and a discussion of the random graph models we use throughout this thesis, starting with rank-1 models in Section 3.1 followed by the random connection model in Section 3.2.

With the proper framework in place, we discuss our results on the (large) component sizes, and different component-wise vertex counts, for subcritical rank-1 models in Chapter 4. To this end, we shall start with an analysis for the Norros-Reittu model, which has the intriguing property that the edges form a Poisson process, enlarging the tool kit to study the problem. We close the chapter by a transfer to the Chung-Lu model and the generalised random graph. This chapter is based on *M. Lienau and M. Schulte 2023*.

Chapter 5 discusses cycles in the three rank-1 models as in *M. Lienau 2024*. Our main results include convergence of cycle counts to a Poisson distribution in a qualitative and quantitative setting, employing the Chen-Stein method. From this, we manage to deduce results on the lengths of the shortest and of the longest cycle in the subcritical regime.

In Chapter 6 we investigate the (large) component sizes in an increasing sequence of observation windows for the subcritical random connection model. This part of the thesis corresponds to *M. Lienau and M. Schulte 2024*.

Chapter 2

Preliminaries

In this chapter we lay a foundation for things to come, mainly gathering results from the literature that we require throughout this thesis. We start with establishing basic notation, followed by results on point processes. Afterwards, we present some facts from regular variation. Finally, we provide results from the Chen-Stein method for Poisson approximation and close the chapter with miscellaneous results.

2.1 Notation

We start with notation that is related to sets. Throughout this thesis, let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, $\mathbb{N} = \{1, 2, 3, \dots\}$ the positive integers and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ the non-negative integers. By $\mathbb{N}_{\geq 3}$ we abbreviate the set containing all integers that are larger than or equal to three, whereas we write $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N}$. For any set A we denote its cardinality by $|A|$ and write $A^k_{\neq}$ for the set of all k-tuples with distinct entries from A. By $A \subseteq B$ we refer to A being a subset of B, including the possibility $A = B$. We denote the power set of A by $\mathcal{P}(A)$. Given a topological space $(\mathcal{X}, \mathcal{T})$, we denote the boundary of a set $A \subseteq \mathcal{X}$ by ∂A.

Concerning typical functions, we denote the exponential function by $\exp(\cdot)$ or $e^{\cdot}$. By $\log(\cdot)$ we refer to the natural logarithm and by $\Gamma(\cdot)$ to the gamma function, whereas

$$(n)_k = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$$

denotes the falling factorial for $k, n \in \mathbb{N}$. Finally, we write $\lfloor \cdot \rfloor$ for the floor function that rounds its argument down. The concatenation of two functions $f, g \colon \mathbb{R} \to \mathbb{R}$ is denoted by $f \circ g$, where $f \circ g(t) = f(g(t))$ for $t \in \mathbb{R}$. We use the same notation for the concatenation of functions having domains different from $\mathbb{R}$.

Regarding measure theory, we write λ_d for the Lebesgue measure on $\mathbb{R}^d$ for $d \in \mathbb{N}$. We equip $\mathbb{R}^d$ with the Borel σ-algebra as usual and write $|\cdot|$ for the Euclidean norm. In an integral with respect to the Lebesgue measure, we often use the standard notation dx instead of $\lambda_d(dx)$. For any set $\mathcal{X}$ and $A \subseteq \mathcal{X}$ we denote its indicator function by

$$\mathbf{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{else.} \end{cases}$$

Occasionally, we also write $\mathbf{1}\{\cdot\}$ for an indicator to enhance the readability of longer formulae. If $\mathcal{X}$ is a measurable space, we denote the Dirac measure of $x \in \mathcal{X}$ by δ_x, i.e. one has for all measurable $A \subseteq \mathcal{X}$ that

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{else.} \end{cases} \tag{2.1}$$

Next, we talk about probabilistic notation. Throughout this thesis we fix some underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and denote the corresponding expectation and variance by $\mathbb{E}[\cdot]$ and $\mathbf{Var}(\cdot)$, respectively. Convergence in distribution, in probability and in an almost sure sense are denoted by $\xrightarrow{d}$, $\xrightarrow{\mathbb{P}}$ and $\xrightarrow{a.s.}$, respectively.

This section concludes with some asymptotic notation. We use the typical big O notation given by $O(\cdot), o(\cdot)$ and $\Theta(\cdot)$. Additionally, we employ their probabilistic versions $O_{\mathbb{P}}, o_{\mathbb{P}}$ and $\Theta_{\mathbb{P}}$. For a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ and a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ these are defined as follows. We write $X_n = o_{\mathbb{P}}(a_n)$ if the ratio X_n/a_n converges to zero in probability as $n \to \infty$. Moreover, $X_n = O_{\mathbb{P}}(a_n)$ means that for all $\delta > 0$ we can find $C > 0$ such that for all sufficiently large $n \in \mathbb{N}$, $\mathbb{P}(|X_n| \leq Ca_n) \geq 1 - \delta$. Finally, $X_n = \Theta_{\mathbb{P}}(a_n)$ sharpens the previous bound on X_n. For all $\delta > 0$ there are $c, C > 0$ such that for all sufficiently large $n \in \mathbb{N}$ we have $\mathbb{P}(ca_n \leq X_n \leq Ca_n) \geq 1 - \delta$.

2.2 Point processes

In this section we lay a foundation regarding point processes and related notions. For us, they are of importance for mainly two reasons. On the one hand, the random connection model uses a point process as its vertex set. On the other hand, many of our statements will be phrased in terms of weak convergence of point processes. We start with basic definitions of point processes and some intuition.

2.2.1 Foundations

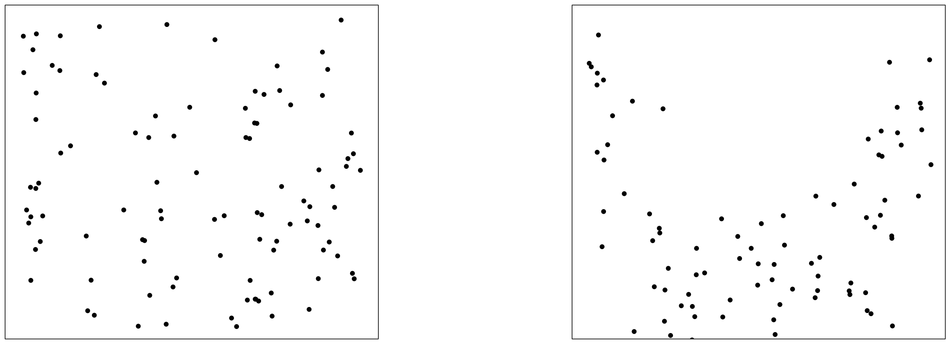
A point process is the formal tool to describe a pattern according to which random points are spread across some space. In Figure 2.1 one can see two simple examples of random points.

On the left-hand side, Figure 2.1(a) shows 100 independent uniformly distributed points in a square. On the right-hand side, Figure 2.1(b) shows 100 independent points whose x-coordinates are uniformly distributed, whereas the respective y-coordinates are given by a quadratic expression in x, plus small independent random errors. To describe the situation in Figure 2.1(a) formally, one could say that we plotted a sample of independent and identically distributed random vectors $X_1, \dots, X_{100}$ with $X_1 \sim \text{Uniform}([0,1])^{\otimes 2}$. For technical reasons, it is preferable to think of point processes as so-called random measures. In the language of random measures, the situation in Figure 2.1(a) is described by

$$\eta = \sum_{i=1}^{100} \delta_{X_i}, \tag{2.2}$$

with $X_1, \dots, X_{100}$ as before. Then, one has for measurable $A \subseteq [0,1]^2$ that

$$\eta(A) = |\{i \in [100] \colon X_i \in A\}|$$



(a) Uniform distribution (b) Parabola with Gaussian noise

Figure 2.1: Realisation of 100 independent random points

denotes the number of points $X_1, \ldots, X_{100}$ that lie in A, using the definition of the Dirac measure in (2.1). We will now formalise the idea of using random measures to describe patterns of random points.

The basic definitions do not require many assumptions. In fact, Last and Penrose introduce point processes in arbitrary measurable spaces in [65]. We will always deal with Polish spaces, i.e. complete and separable metric spaces, which are required for the results in Subsection 2.2.3, taken from [61] by Kallenberg. Therefore, we shall already assume to have such spaces. Our setting resembles [85, Chapter 3] by Resnick, which is a bit different from the one in [61] by Kallenberg.

We fix a complete and separable metric space (E, d) for the remainder of this section and denote its Borel σ-field by $\mathcal{E} = \mathcal{B}(E)$. We call a measure μ on $(E, \mathcal{E})$ a Radon measure if $\mu(K) < \infty$ for all compact $K \in \mathcal{E}$. Let $M_p(E)$ denote the space of all σ-finite Radon measures on $(E, \mathcal{E})$ with values in $\mathbb{N}_0 \cup \{\infty\}$. We write $\mathcal{M}_p(E)$ for the smallest σ-algebra on $M_p(E)$ such that for all $A \in \mathcal{E}$ the evaluation map

$$f_A \colon M_p(E) \to \mathbb{N}_0 \cup \{\infty\}, \ \eta \mapsto \eta(A)$$

is measurable, where we endow $\mathbb{N}_0 \cup \{\infty\}$ with its power set as σ-field.

Definition 2.1 (Point process)**.** A measurable map $\eta \colon (\Omega, \mathcal{A}) \to (M_p(E), \mathcal{M}_p(E))$ is called a point process on E.

Let η be a point process on E and $A \in \mathcal{E}$. Strictly speaking, the argument of η should be an element $\omega \in \Omega$ in order to obtain a measure, so that $\omega \mapsto (\eta(\omega))(A)$ denotes the random variable which indicates how many points of η lie in A. As usual, we abbreviate this random variable by $\eta(A)$. Next, we will formalise the concept of point processes as in (2.2), see e.g. [65, Definition 2.4].

Definition 2.2 (Proper point process)**.** A point process η on E is called proper if there exist random elements $X_1, X_2, \ldots$ in E and an $\mathbb{N}_0 \cup \{\infty\}$-valued random variable κ such that almost surely

$$\eta = \sum_{n=1}^{\kappa} \delta_{X_n}.$$

For a proper point process as above, η is thus given by the random points $X_1, \ldots, X_\kappa$, which we interpret as $X_1, X_2, \ldots$ for $\kappa = \infty$. Similarly, with a slight abuse of notation, we shall also write $X_i \in \eta$ for $i = 1, \ldots, \kappa$, where we encounter a small technical subtlety. It is possible that η has multiplicities, i.e. $X_i = X_j$ for some $i \neq j$ with positive probability. Therefore, we do not treat $\eta = \{X_1, \ldots, X_\kappa\}$ as a standard set, but as a multi-set, meaning that we keep track of multiplicities. In particular, we write $\eta^k_{\neq}$ for k-tuples consisting of distinct points of η, where distinct is to be understood in such a way that $(X_i, X_j) \in \eta^2_{\neq}$ for all $i \neq j$, even if $X_i = X_j$.

2.2.2 Poisson processes

This section gathers a few properties of Poisson processes, a special kind of point processes. A comprehensive treatment of this topic is, for instance, given by Last and Penrose in [65].

Definition 2.3 (Poisson process)**.** Let λ be a σ-finite measure on E. A point process η on E is called a Poisson process with intensity measure λ, if the following hold:

a) For all $A \in \mathcal{E}$ one has that $\eta(A)$ follows a Poisson distribution with parameter $\lambda(A)$.

b) For all $n \in \mathbb{N}$ and pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{E}$ the random variables $\eta(A_1), \ldots, \eta(A_n)$ are independent.

A priori it is not clear whether a Poisson process exists for all choices of E and λ. However, its existence is guaranteed by [65, Theorem 3.6]. Moreover, [65, Corollary 6.5] shows that all Poisson processes with σ-finite intensity measure on metric spaces are proper. Therefore, in our setting, we may tacitly assume that Poisson processes are proper.

We add some minor comments. The first property of a Poisson process motivates its name, while the spatial independence is convenient in calculations. Consider the special case $E = \mathbb{R}^d$ and a Poisson process η with intensity measure $\lambda = c\lambda_d$ for $c > 0$, i.e. a multiple of the Lebesgue measure. Then, we call η homogeneous and simply say that η has intensity c. Note that a homogeneous Poisson process η on $\mathbb{R}^d$ is stationary in the sense that the distribution of η does not change if we translate all of its points by some fixed $x \in \mathbb{R}^d$. This is also referred to as η being translation invariant.

If we return to the very first example in this section, there were 100 independent and uniformly distributed points in a square. This is an example of a so-called binomial process. It is called *binomial* because the number of points in any given subset follows a binomial distribution. While not clear at first sight, this situation is actually related to Poisson processes. When conditioning a Poisson process with finite intensity measure on the number of points in the whole space, one obtains a binomial process as shown in [65, Proposition 3.8]. We continue with deeper results for Poisson processes, starting with the (multivariate) Mecke equation, see [65, Theorem 4.4].

Theorem 2.4 (Mecke equation)**.** *Let η be a Poisson process on E with σ-finite intensity measure λ. For all $k \in \mathbb{N}$ and measurable $f\colon E^k \times M_p(E) \to [0,\infty]$,*

$$\mathbb{E}\Big[\sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f(x_1,\dots,x_k,\eta)\Big] = \int_{E^k} \mathbb{E}[f(x_1,\dots,x_k,\eta+\delta_{x_1}+\dots+\delta_{x_k})]\lambda^k(d(x_1,\dots,x_k)).$$

The next theorem is the so-called Poincaré inequality, see [65, Theorem 18.7]. To this end, we introduce some notation first. For a point process η on E, we denote its image measure on $M_p(E)$ by $\mathbb{P}_\eta$. Given a function $f\colon M_p(E) \to \mathbb{R}$ and $x \in E$ we define

$$D_x f\colon M_p(E) \to \mathbb{R},\ \mu \mapsto f(\mu+\delta_x) - f(\mu),$$

which is known as difference operator. Heuristically, it indicates the change of f when the single point x is added.

Theorem 2.5 (Poincaré inequality)**.** *Let η be a Poisson process on E with σ-finite intensity measure λ. Then, for all $f\colon M_p(E) \to \mathbb{R}$ that are square integrable with respect to $\mathbb{P}_\eta$,*

$$\mathbf{Var}(f(\eta)) \le \int_E \mathbb{E}\big[(D_x f(\eta))^2\big]\lambda(dx).$$

Our last result in this subsection concerns the so-called *probability generating functional*, see [65, Exercise 3.6].

Lemma 2.6. *Let η be a Poisson process on E with σ-finite intensity measure λ. For all measurable functions $u\colon E \to [0,1]$ it holds*

$$\mathbb{E}\Big[\prod_{x\in\eta} u(x)\Big] = \exp\Big(-\int_E (1-u(x))\lambda(dx)\Big).$$

2.2.3 Weak convergence

In order to define weak convergence for point processes, it is desirable to have a metric (or at least a topology) on $M_p(E)$. It turns out that the σ-field $\mathcal{M}_p(E)$ is induced by the so-called vague topology, which is metrisable, turning $M_p(E)$ into a complete, separable metric space, see [61, Theorem 4.2]. The metric structure on $M_p(E)$ allows us to define weak convergence as usual.

Definition 2.7 (Weak convergence of point processes)**.** Let $\eta, \eta_1, \eta_2, \dots$ be point processes on E. We say that $(\eta_n)_{n\in\mathbb{N}}$ converges weakly (or in distribution) to η if

$$\mathbb{E}[f(\eta_n)] \longrightarrow \mathbb{E}[f(\eta)] \quad \text{as} \quad n \to \infty$$

for all $f\colon M_p(E) \to \mathbb{R}$ that are bounded and continuous. We write $\eta_n \xrightarrow{d} \eta$ as $n \to \infty$.

While the previous definition looks very familiar, given its similarity to weak convergence of random variables or random vectors, it demands us to consider functions $f\colon M_p(E) \to \mathbb{R}$. Given the fact that our random elements $\eta_1, \eta_2, \dots$ are (random) measures themselves, it might seem more natural to formulate convergence in terms of the random variables $\eta_n(A)$ for suitable subsets $A \in \mathcal{E}$ or of $\eta_n f := \int_E f d\eta_n$ for suitable test functions $f\colon E \to \mathbb{R}$. The next theorem gives us conditions of precisely this sort, but first we need to introduce notions concerning suitable families of sets.

Recall that for any set S a semi-ring on S is some set $I \subseteq \mathcal{P}(S)$, i.e. a collection of subsets of S, such that

a) $\emptyset \in \mathcal{I}$,

b) $\mathcal{I}$ is closed under finite intersections and

c) for any $A, B \in \mathcal{I}$ there are disjoint $C_1, \ldots, C_n \in \mathcal{I}$ such that $A \setminus B = \cup_{i=1}^n C_i$.

We call a subset $\mathcal{A} \subseteq \mathcal{E}$ dissecting (see [61, p. 24]) if

a) $\mathcal{A}$ only contains bounded sets,

b) every open set $O \in \mathcal{E}$ is a countable union of sets in $\mathcal{A}$ and

c) every bounded set $B \in \mathcal{E}$ is covered by a finite union of sets in $\mathcal{A}$.

We can now give equivalent conditions concerning weak convergence of point processes according to [61, Theorem 4.11]. The fourth equivalence we provide follows directly from the third in combination with the Cramèr-Wold device.

Theorem 2.8. *Let $\eta, \eta_1, \eta_2, \ldots$ be point processes on E and $\mathcal{I} \subseteq \mathcal{E}$ a dissecting semi-ring such that $\mathbb{P}(\eta(\partial I) = 0) = 1$ for all $I \in \mathcal{I}$. Then the following are equivalent:*

a) $\eta_n \xrightarrow{d} \eta$ *as* $n \to \infty$,

b) $\eta_n f \xrightarrow{d} \eta f$ *as* $n \to \infty$ *for all measurable, continuous functions* $f\colon E \to [0, \infty)$ *with bounded support,*

c) $\eta_n f \xrightarrow{d} \eta f$ *as* $n \to \infty$ *for all simple functions* $f\colon E \to [0, \infty)$ *that can be written as* $f(x) = \sum_{i=1}^k \alpha_i \mathbf{1}_{I_i}(x)$ *with* $k \in \mathbb{N}, \alpha_1, \ldots, \alpha_k \geq 0$ *and* $I_1, \ldots, I_k \in \mathcal{I}$ *as well as*

d) $(\eta_n(I_1), \ldots, \eta_n(I_k)) \xrightarrow{d} (\eta(I_1), \ldots, \eta(I_k))$ *as* $n \to \infty$ *for all* $k \in \mathbb{N}$ *and* $I_1, \ldots, I_k \in \mathcal{I}$.

Later on we will consider two particular choices for (E, d). On the one hand, we consider $E = \mathbb{N}_{\geq 3}$ and d being the restriction of the Euclidean distance to $\mathbb{N}_{\geq 3}$. In this setting, E corresponds to the possible cycle lengths and is of interest in Chapter 5. On the other hand, we study $E = (0, \infty]$, which we endow with $d(x, y) = |1/x - 1/y|$ for $x, y \in (0, \infty]$, where we use the convention $1/\infty = 0$. The resulting topology is the one generated by $\{(a, b), (a, \infty]\colon a, b > 0\}$. For intuition, if we think of the metric space $[0, \infty)$ with the standard metric, we see that $f\colon [0, \infty) \to (0, \infty]$ with $f(x) = 1/x$ is a bijective isometry, where we use the convention $1/0 = \infty$. In particular, it follows that both choices for (E, d) are complete and separable as requested in the beginning of this section.

In light of Theorem 2.8 we provide useful choices for dissecting semi-rings. For the sake of completeness, we provide the straightforward proof.

Lemma 2.9. a) *A dissecting semi-ring on* $\mathbb{N}_{\geq 3}$ *is given by*

$$\mathcal{I} = \{\{x\}\colon x \in \mathbb{N}_{\geq 3}\} \cup \{\emptyset\}.$$

b) *A dissecting semi-ring on* $(0,\infty]$ *is given by*

$$\mathcal{I} = \{(a,b]\colon 0 < a \le b \le \infty\}.$$

Proof. a) As every intersection of distinct $A, B \in \mathcal{I}$ is empty, we see that $A \setminus B = A$. Since $\emptyset \in \mathcal{I}$, we obtain that $\mathcal{I}$ is a semi-ring. Obviously, $\mathcal{I}$ only contains bounded sets and since $\mathbb{N}_{\ge 3}$ is countable, every open set can be written as a countable union of singletons. Since we chose the Euclidean distance on $\mathbb{N}_{\ge 3}$, a set $B \subseteq \mathbb{N}_{\ge 3}$ is bounded if and only if it is finite. Thus, we can write it as a finite union of sets in $\mathcal{I}$. This shows that $\mathcal{I}$ is also dissecting.

b) By choosing $a = b$, we obtain $\emptyset \in \mathcal{I}$. For all $A, B \in \mathcal{I}$ it holds that $A \cap B$ and $A \setminus B$ are again half-open intervals or unions thereof, so that $\mathcal{I}$ is a semi-ring. For the dissecting property, it follows from our chosen metric that $\mathcal{I}$ only contains bounded sets. Since we can write any open interval (a,b) as countable union of half-open intervals in $\mathcal{I}$ and the space is separable, we conclude that one can also write every open set $O \in \mathcal{E}$ as a countable union of sets in $\mathcal{I}$. Finally, every bounded set $B \subseteq (0,\infty]$ is contained in $(a,\infty]$ for $a = \inf(B)/2$. The assertion follows. □

In the space $M_p(\mathbb{N}_{\ge 3})$, weak convergence boils down to convergence of finite-dimensional distributions as seen in the following lemma.

Lemma 2.10. *For point processes* $\eta, \eta_1, \eta_2, \ldots$ *on* $\mathbb{N}_{\ge 3}$ *the following are equivalent:*

a) $\eta_n \xrightarrow{d} \eta$ *as* $n \to \infty$ *and*

b) $(\eta_n(k))_{k \in \mathbb{N}_{\ge 3}} \xrightarrow{fdd} (\eta(k))_{k \in \mathbb{N}_{\ge 3}}$ *as* $n \to \infty$*, where* $\xrightarrow{fdd}$ *denotes convergence of the finite-dimensional distributions.*

Proof. This follows from the equivalence of Theorem 2.8 a) and d) combined with Lemma 2.9 a). Here we use that $\partial\{x\} = \emptyset$ for all $x \in \mathbb{N}_{\ge 3}$, as $\{x\}$ is open in the discrete space $\mathbb{N}_{\ge 3}$. □

The following two lemmas are implicitly proven in [14], but not explicitly stated. As the first one is a key ingredient for two of our main theorems and the second one will be used to obtain corollaries thereof, we include their short proofs here.

Lemma 2.11. *Let* $\eta, \eta_1, \eta_2, \ldots, \theta_1, \theta_2, \ldots$ *be point processes on* $(0,\infty]$ *and suppose that*

$$\eta_n \xrightarrow{d} \eta \quad as \quad n \to \infty \tag{2.3}$$

as well as

$$\eta_n((a,\infty]) - \theta_n((a,\infty]) \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty \tag{2.4}$$

and

$$\mathbb{P}(\eta(\{a,b\}) = 0) = 1 \tag{2.5}$$

for all $0 < a \le b < \infty$*. Then,*

$$\theta_n \xrightarrow{d} \eta \quad as \quad n \to \infty.$$

Proof. By Lemma 2.9 b) we know that $\mathcal{I} = \{(a,b]\colon 0 < a \le b \le \infty\}$ forms a dissecting semi-ring on $(0,\infty]$. Given $(a,b] \in \mathcal{I}$, we obtain $\partial(a,a] = \emptyset$, $\partial(a,\infty] = \{a\}$ as well as $\partial(a,b] = \{a,b\}$ for the cases $a = b$, $a < b = \infty$ and $a < b < \infty$, respectively. We conclude that $\mathbb{P}(\eta(\partial(a,b]) = 0) = 1$ for all $(a,b] \in \mathcal{I}$ by (2.5). Using Theorem 2.8, it suffices to show for all $k \in \mathbb{N}$ and $I_1, \ldots, I_k \in \mathcal{I}$,

$$(\theta_n(I_1), \ldots, \theta_n(I_k)) \xrightarrow{d} (\eta(I_1), \ldots, \eta(I_k)) \quad \text{as} \quad n \to \infty.$$

Let thus $I_1, \ldots, I_k \in \mathcal{I}$. Theorem 2.8 and (2.3) provide

$$(\eta_n(I_1), \ldots, \eta_n(I_k)) \xrightarrow{d} (\eta(I_1), \ldots, \eta(I_k)) \quad \text{as} \quad n \to \infty.$$

Thus, using Slutsky's lemma, it suffices to show that

$$(\theta_n(I_1), \ldots, \theta_n(I_k)) - (\eta_n(I_1), \ldots, \eta_n(I_k)) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty.$$

This in turn follows from component-wise convergence, so that it remains to show for $(a,b] \in \mathcal{I}$ that

$$\theta_n((a,b]) - \eta_n((a,b]) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty.$$

In the case $b = \infty$, this is the assumption (2.4). For $b < \infty$, the triangle inequality yields

$$\big|\theta_n((a,b]) - \eta_n((a,b])\big| \le \big|\theta_n((a,\infty]) - \eta_n((a,\infty])\big| + \big|\theta_n((b,\infty]) - \eta_n((b,\infty])\big| \xrightarrow{\mathbb{P}} 0$$

as $n \to \infty$, by the assumption in (2.4). This concludes the proof. □

Recall that a random variable follows a Fréchet distribution with (shape) parameter $\beta > 0$ if

$$\mathbb{P}(X \le t) = \begin{cases} \exp\big(-t^{-\beta}\big), & \text{for } t > 0, \\ 0, & \text{for } t \le 0. \end{cases}$$

In the following lemma, we write $\sup(\xi)$ for a point process ξ on $(0,\infty]$ in order to refer to its largest point in the supremum sense.

Lemma 2.12. *Let $\beta > 0$ and $\xi_1, \xi_2, \ldots$ be a sequence of point processes on $(0,\infty]$ such that*

$$\xi_n \xrightarrow{d} \eta_\beta \quad \text{as} \quad n \to \infty,$$

where η_β denotes a Poisson process on $(0,\infty]$ with intensity measure $\nu((a,b]) = a^{-\beta} - b^{-\beta}$ for all $0 < a < b \le \infty$. Then,

$$\sup(\xi_n) \xrightarrow{d} Z_\beta \quad \text{as} \quad n \to \infty,$$

where Z_β denotes a random variable following a Fréchet distribution with parameter β.

Proof. We show convergence of the underlying distribution functions. Since

$$\sup(\xi_n) \le t \quad \Longleftrightarrow \quad \xi_n((t,\infty]) = 0$$

for all $t > 0$, it follows that

$$\mathbb{P}\big(\sup(\xi_n) \le t\big) = \mathbb{P}\big(\xi_n((t,\infty]) = 0\big).$$

The second part of Lemma 2.9 states that $\mathcal{I} = \{(a,b] \colon 0 < a \le b \le \infty\}$ forms a dissecting semi-ring on $(0,\infty]$ in the topology at hand. Moreover, since ν is diffuse, we obtain that $\mathbb{P}(\eta_\beta(\{a,b\}) = 0) = 1$ for all $0 < a \le b < \infty$. Thus, we can apply Theorem 2.8 to deduce from the assumed convergence $\xi_n \overset{d}{\to} \eta_\beta$ as $n \to \infty$ that

$$\xi_n((t,\infty]) \xrightarrow{d} \eta_\beta((t,\infty]) \quad \text{as} \quad n \to \infty.$$

We conclude

$$\mathbb{P}(\sup(\xi_n) \le t) = \mathbb{P}(\xi_n((t,\infty]) = 0) \to \mathbb{P}(\eta_\beta((t,\infty]) = 0) = \exp(-\nu_\beta(t,\infty]) = \exp\big(-t^{-\beta}\big)$$

as $n \to \infty$. The assertion follows. □

2.3 Regular variation

In this section we recall facts from regular variation. These will be frequently used as regular variation provides a way to formalise the power-law degree distribution of complex networks. Most results are taken from the textbooks [16] by Bingham, Goldie and Teugels as well as [84] by Resnick, to which we refer the reader for additional information.

2.3.1 Regularly varying functions

The following definitions are due to [84, Definition 2.1] and the discussion thereafter.

Definition 2.13 (Slowly varying functions)**.** A function $\ell \colon (0,\infty) \to (0,\infty)$ is called slowly varying (at infinity) if

$$\lim_{t\to\infty} \frac{\ell(ct)}{\ell(t)} = 1$$

for all $c > 0$.

A first example for a slowly varying function is given by any $f \colon (0,\infty) \to (0,\infty)$ for which the limit $\lim_{t\to\infty} f(t)$ exists and is positive. Another example would be

$$f(t) = \begin{cases} 1, & \text{for } t \in (0,1], \\ \log(t), & \text{for } t > 1, \end{cases}$$

where we use the case distinction to ensure that f maps to $(0,\infty)$. Another common way to get rid of this technical issue is to demand the existence of some $C > 0$ such that $\ell(t) > 0$ for all $t > C$ instead of requiring $\ell(t) > 0$ for all $t \in (0,\infty)$ in Definition 2.13.

Definition 2.14 (Regularly varying functions)**.** A function $f \colon (0,\infty) \to (0,\infty)$ is called regularly varying (at infinity) with index $\rho \in \mathbb{R}$ if there exists a slowly varying function ℓ such that

$$f(t) = t^\rho \ell(t)$$

for all $t > 0$. We write RV_ρ for the set of all regularly varying functions with index ρ.

From [84, Proposition 2.6] we obtain part b) and c) of the following proposition. Part a) can be found in [16, Proposition 1.5.7].

Proposition 2.15. a) *Let $\rho_1, \rho_2 \in \mathbb{R}$ and $f_1 \in \mathrm{RV}_{\rho_1}$ as well as $f_2 \in \mathrm{RV}_{\rho_2}$. It holds that*

$$f_1 \cdot f_2 \in \mathrm{RV}_{\rho_1+\rho_2} \quad \textit{and} \quad f_1 + f_2 \in \mathrm{RV}_{\max(\rho_1,\rho_2)}.$$

b) *Let ℓ be slowly varying. For all $\varepsilon > 0$ it holds*

$$\lim_{t\to\infty} t^{\varepsilon}\ell(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} t^{-\varepsilon}\ell(t) = 0.$$

c) *Let $\rho_1 \in \mathbb{R}, \rho_2 > 0$ and $f_1 \in \mathrm{RV}_{\rho_1}$ as well as $f_2 \in \mathrm{RV}_{\rho_2}$. Then*

$$f_1 \circ f_2 \in \mathrm{RV}_{\rho_1\rho_2}.$$

The following theorem investigates the integral over a regularly varying function given by $f(t) = t^{\rho}\ell(t)$. If we think of the slowly varying function ℓ as a technical nuisance, which is almost constant for large input, we obtain for $\rho < -1$ and large values $x > 0$ heuristically

$$\int_x^{\infty} f(t)dt = \int_x^{\infty} t^{\rho}\ell(t)dt \approx \ell(x)\int_x^{\infty} t^{\rho}dt = \frac{\ell(x)x^{\rho+1}}{-\rho-1} = \frac{xf(x)}{-\rho-1}.$$

Part b) of [16, Theorem 2.19] formalises this approximation and states the following:

Theorem 2.16 (Karamata's theorem). *Let $\rho < -1$ and $f \in \mathrm{RV}_{\rho}$ be such that f is locally integrable. Then it holds for $x > 0$ that*

$$\bar{F}(x) = \int_x^{\infty} f(t)dt < \infty.$$

Additionally, one has $\bar{F} \in \mathrm{RV}_{\rho+1}$ and

$$\lim_{x\to\infty} \frac{xf(x)}{\bar{F}(x)} = -\rho - 1.$$

2.3.2 Random variables with regularly varying tails

Throughout this subsection, let W be a random variable such that $\mathbb{P}(W > t) = t^{-\beta}\ell(t)$ for a slowly varying function ℓ, fixed $\beta > 0$ and all $t > 0$. In contrast to a pure power law, as in a Pareto distribution, the slowly varying factor allows for additional flexibility. Our first statement concerns the finiteness of moments depending on β and follows immediately from Karamata's theorem and the identity $\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}(X > t)dt$ for a non-negative random variable X.

Proposition 2.17. *For all $\alpha < \beta$ it holds that $\mathbb{E}[W^{\alpha}] < \infty$.*

We continue with asymptotics of an expression that will occur in several proofs later on.

Lemma 2.18. *Let $\beta > 1$. The function f given by*

$$f\colon (0,\infty) \to (0,\infty), \ u \mapsto \mathbb{E}[\mathbf{1}\{W > u\}W]$$

is regularly varying with index $1 - \beta$.

Proof. For $u > 0$ we have

$$\mathbb{E}[\mathbf{1}\{W > u\}W] = \int_0^\infty \mathbb{P}(\mathbf{1}\{W > u\}W > t)dt = u\mathbb{P}(W > u) + \int_u^\infty \mathbb{P}(W > t)dt.$$

Since $u \mapsto \mathbb{P}(W > u)$ belongs to $\mathrm{RV}_{-\beta}$, the function $u \mapsto u\mathbb{P}(W > u)$ is from $\mathrm{RV}_{1-\beta}$, where we used Proposition 2.15 a). By Theorem 2.16, Karamata's theorem, we derive that $u \mapsto \int_u^\infty \mathbb{P}(W > t)dt$ is also from $\mathrm{RV}_{1-\beta}$. The claim follows from part a) of Proposition 2.15. □

Now, we consider the generalised inverse of W's distribution function, i.e.

$$F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}$$

for $p \in (0, 1)$. Define $q(t) = F^{-1}(1 - 1/t)$ for $t > 1$ as well as $q(t) = 1$ for $t \in (0, 1]$, where the latter is only for the technical reason to obtain a function $q : (0, \infty) \to (0, \infty)$. We gather properties of q in the following proposition, which combines [84, Theorem 3.6 and Remark 3.3 a)].

Proposition 2.19. *For q as above the following hold.*

a) *We have $q \in \mathrm{RV}_{1/\beta}$, i.e. $q(t) = t^{1/\beta}\ell(t)$ for a slowly varying function ℓ and $t > 0$.*

b) *For all $a > 0$ one has $\lim_{n\to\infty} n\mathbb{P}(W > aq(n)) = a^{-\beta}$.*

The following lemma is proven in [14, Lemma 3.6].

Lemma 2.20. *Let η_β denote a Poisson process on $(0, \infty]$ with intensity measure $\nu((a, b]) = a^{-\beta} - b^{-\beta}$ for $0 < a < b \leq \infty$. Then, the following hold.*

a) *Consider independent copies $W_1, W_2, \ldots$ of W. As $n \to \infty$ we have in $M_p((0, \infty])$,*

$$\sum_{v=1}^{n} \delta_{W_v q(n)^{-1}} \xrightarrow{d} \eta_\beta.$$

b) *Let η denote a Poisson process on $\mathbb{R}^d$ with unit intensity, which is independent of W. Given η, we equip each point $x \in \eta$ with an independent copy W_x of W. Writing $S_n = [0, n^{1/d}]^d$ for $n \in \mathbb{N}$, we have*

$$\sum_{x\in\eta\cap S_n} \delta_{W_x q(n)^{-1}} \xrightarrow{d} \eta_\beta$$

in $M_p((0, \infty])$ as $n \to \infty$.

The next statement is part of a well-known result from extreme value theory called Fisher-Tippet-Gnedenko theorem, see e.g. [38, Theorem 1.2.1 and Corollary 1.2.4]. It also follows from combining Lemma 2.20 above with Lemma 2.12.

Proposition 2.21. *Let $W_1, W_2, \ldots$ denote independent copies of W and write $W_{(n)} = \max_{i=1,\ldots,n} W_i$ for $n \in \mathbb{N}$. It holds that*

$$\frac{W_{(n)}}{q(n)} \xrightarrow{d} Z_\beta \quad \text{as} \quad n \to \infty,$$

where Z_β is a random variable following a Fréchet distribution with parameter β.

2.4 Results on Poisson approximation

The content of this section will be used in Chapter 5 and arises in the following scenario: we consider a sequence of non-negative integer-valued random variables $(X_n)_{n\in\mathbb{N}}$, which arise from counting certain objects, in our case cycles. Then, we are interested in the following questions concerning $(X_n)_{n\in\mathbb{N}}$ as $n\to\infty$.

a) How can we show convergence in distribution to a Poisson random variable?

b) How can we quantify the speed of convergence?

c) Suppose now that $X_n, n\in\mathbb{N}$, are random vectors. Is it possible to deduce multi-dimensional convergence results from one-dimensional statements?

The last point above is of course answered by the classical Cramér-Wold device. We use the Poisson Cramér-Wold device instead, see Corollary 2.25. This allows for a more natural setting because it uses a so-called *thinning* of the occurring random variables. This has the advantage that, in all the one-dimensional statements, the limiting distributions remain Poisson distributions.

In order to answer the first two questions above, we provide the framework of the Poisson approximation from [6] by Arratia, Goldstein and Gordon, but in a conditioned setting. Their result relies on Stein's method, originally developed for normal approximation by Stein in [90] and then extended by Chen to the Poisson case in [26]. See also the surveys [36] by Erhardsson and [86] by Ross for more details on Stein's method for Poisson approximation.

Concerning the second question above, we quantify the convergence in terms of the total variation distance. For two $\mathbb{N}_0$-valued random variables X and Y, their total variation distance is given by

$$d_{TV}(X,Y)=\sup_{A\subseteq\mathbb{N}_0}|\mathbb{P}(X\in A)-\mathbb{P}(Y\in A)|.$$

Given $\mathbb{N}_0$-valued random variables $X, X_1, X_2, \dots$, note that $d_{TV}(X_n, X)\to 0$ as $n\to\infty$ implies that $X_n\xrightarrow{d}X$ as $n\to\infty$.

Now let I denote an arbitrary finite index set and $\mathcal{A}$ a σ-field on the underlying sample space Ω. We use $\mathbb{E}_{\mathcal{A}}$ as a short-hand notation for the conditional expectation with respect to $\mathcal{A}$. For all $\alpha\in I$ let X_α be a Bernoulli random variable. We are interested in the total number of successes $S=\sum_{\alpha\in I}X_\alpha$. Additionally, let $B_\alpha\subseteq I$ be such that $\alpha\in B_\alpha$ for all $\alpha\in I$ and define

$$b_1=\sum_{\alpha\in I}\sum_{\beta\in B_\alpha}p_\alpha p_\beta \qquad \text{for} \quad p_\alpha=\mathbb{E}_{\mathcal{A}}[X_\alpha], \tag{2.6}$$

$$b_2=\sum_{\alpha\in I}\sum_{\beta\in B_\alpha\setminus\{\alpha\}}p_{\alpha\beta} \qquad \text{for} \quad p_{\alpha\beta}=\mathbb{E}_{\mathcal{A}}[X_\alpha X_\beta] \quad \text{and} \tag{2.7}$$

$$b_3=\sum_{\alpha\in I}\mathbb{E}_{\mathcal{A}}\left[|\mathbb{E}[X_\alpha-p_\alpha|\sigma(\mathcal{A},X_\beta\colon\beta\in B_\alpha^c)]|\right].$$

We denote a mixed Poisson distribution by $X\sim\mathrm{Poi}(Y)$, meaning that the distribution of the random variable X, conditionally on the random variable Y, is a Poisson distribution with parameter Y.

Lemma 2.22. *If $T_{\mathcal{A}} \sim \mathrm{Poi}(\mathbb{E}_{\mathcal{A}}[S])$, then*

$$d_{TV}(S, T_{\mathcal{A}}) \leq \mathbb{E}\big[\min\big(1, b_1 + b_2 + b_3\big)\big].$$

Note that the random variables $(X_\alpha)_{\alpha \in B_\alpha}$ are not assumed to be independent. In practice, one often chooses B_α as some kind of dependence region of X_α for $\alpha \in I$, meaning that $(X_\beta)_{\beta \in B_\alpha}$ corresponds to those random variables that are closely related to X_α.

Remark 2.23. In our proofs we will always choose B_α such that X_α is, conditionally on $\mathcal{A}$, independent of $(X_\beta)_{\beta \in B_\alpha^c}$ for all $\alpha \in I$. This leads to $b_3 = 0$ as the conditional expectation vanishes. This is related to the notion of dependency graphs, where one thinks of a graph with vertex set I, where one places an edge between distinct $\alpha, \beta \in I$ when X_α and X_β are not independent. Then, B_α is given by the vertices in distance at most one of α, see e.g. [83, Section 2.19].

The proof of the previous lemma is very similar to the proof of [6, Theorem 1]. In the special case that $\mathcal{A}$ is the trivial σ-field, it follows from the aforementioned theorem.

Proof of Lemma 2.22. The total variation distance satisfies, see e.g. [42],

$$d_{TV}(S, T_{\mathcal{A}}) = \frac{1}{2} \sup_{||h||_\infty = 1} |\mathbb{E}[h(S) - h(T_{\mathcal{A}})]|,$$

where the supremum runs over all functions $h\colon \mathbb{N}_0 \to \mathbb{R}$ with supremum norm equal to one. Thus,

$$\begin{aligned} 2d_{TV}(S, T_{\mathcal{A}}) &= \sup_{||h||_\infty = 1} \big|\mathbb{E}\big[\mathbb{E}_{\mathcal{A}}[h(S) - h(T_{\mathcal{A}})]\big]\big| \leq \mathbb{E}\Big[\sup_{||h||_\infty = 1} \big|\mathbb{E}_{\mathcal{A}}[h(S) - h(T_{\mathcal{A}})]\big|\Big] \\ &= \mathbb{E}\Big[\min\Big(2, \sup_{||h||_\infty = 1} \mathbb{E}_{\mathcal{A}}[h(S) - h(T_{\mathcal{A}})]\Big)\Big], \end{aligned}$$

where the last equality uses $||h||_\infty = 1$ and omits the absolute value as one can choose $-h$ when the expression is negative. Now one can mimic the proof of [6, Theorem 1] to bound the supremum and obtain the desired statement. For the sake of completeness, we include the details.

We fix an h with $||h||_\infty = 1$ and aim to show

$$\mathbb{E}_{\mathcal{A}}[h(S) - \mathbb{E}_{\mathcal{A}}[h(T_{\mathcal{A}})]] \leq 2(b_1 + b_2 + b_3). \tag{2.8}$$

Since, conditionally on $\mathcal{A}$, $T_{\mathcal{A}}$ follows a Poisson distribution with parameter $\lambda_{\mathcal{A}} = \mathbb{E}_{\mathcal{A}}[S]$, the Stein equation is given by

$$h(j) - \mathbb{E}_{\mathcal{A}}[h(T_{\mathcal{A}})] = jf(j) - \lambda_{\mathcal{A}} f(j+1) \tag{2.9}$$

and one is looking for a function $f\colon \mathbb{N}_0 \to \mathbb{R}$ that solves the equation above for all $j \in \mathbb{N}_0$. Unlike in the usual setting, this is an equation where both sides are random, i.e. depend on $\omega \in \Omega$ of the underlying probability space. A solution is given by $f_{\mathcal{A}}$ with $f_{\mathcal{A}}(0) = 0$ and

$$f_{\mathcal{A}}(j+1) = -\lambda_{\mathcal{A}}^{-j-1} j! \sum_{\ell=0}^{j} \frac{\lambda_{\mathcal{A}}^{\ell}}{\ell!} \left(h(\ell) - \mathbb{E}_{\mathcal{A}}[h(T_{\mathcal{A}})]\right)$$

for $j \in \mathbb{N}_0$. This can be seen by plugging it into the Stein equation (2.9). On the other hand we can rewrite the left-hand side of (2.8) by the Stein equation as

$$\begin{aligned}\mathbb{E}_{\mathcal{A}}[h(S) - \mathbb{E}_{\mathcal{A}}[h(T_{\mathcal{A}})]] &= \mathbb{E}_{\mathcal{A}}[Sf_{\mathcal{A}}(S) - \lambda_{\mathcal{A}} f_{\mathcal{A}}(S+1)] \\ &= \mathbb{E}_{\mathcal{A}}\left[\sum_{\alpha\in I} X_\alpha f_{\mathcal{A}}(S) - p_\alpha f_{\mathcal{A}}(S+1)\right], \end{aligned} \tag{2.10}$$

where we used $S = \sum_{\alpha\in I} X_\alpha$ and $\lambda_{\mathcal{A}} = \sum_{\alpha\in I} p_\alpha$ in the last step. We write $S_\alpha = S - X_\alpha$ as well as $V_\alpha = \sum_{\beta\in I\setminus B_\alpha} X_\beta$. Since the $X_\alpha, \alpha \in I$, are Bernoulli random variables, we have

$$X_\alpha f_{\mathcal{A}}(S) = X_\alpha f_{\mathcal{A}}(S_\alpha + 1) \tag{2.11}$$

and

$$f_{\mathcal{A}}(S_\alpha + 1) - f_{\mathcal{A}}(S+1) = X_\alpha[f_{\mathcal{A}}(S_\alpha+1) - f_{\mathcal{A}}(S_\alpha+2)] = -X_\alpha \Delta f_{\mathcal{A}}(S_\alpha+1), \tag{2.12}$$

where we write $\Delta f_{\mathcal{A}}$ for the function $\Delta f_{\mathcal{A}} \colon \mathbb{N}_0 \to \mathbb{R}$, $\Delta f_{\mathcal{A}}(k) = f_{\mathcal{A}}(k+1) - f_{\mathcal{A}}(k)$. The two equations above allow us to rearrange the sum in the conditional expectation in (2.10),

$$\begin{aligned}&\sum_{\alpha\in I} X_\alpha f_{\mathcal{A}}(S) - p_\alpha f_{\mathcal{A}}(S+1) \overset{(2.11)}{=} \sum_{\alpha\in I} X_\alpha f_{\mathcal{A}}(S_\alpha+1) - p_\alpha f_{\mathcal{A}}(S+1) \\ &= \sum_{\alpha\in I} X_\alpha f_{\mathcal{A}}(S_\alpha+1) - p_\alpha f_{\mathcal{A}}(S_\alpha+1) + p_\alpha[f_{\mathcal{A}}(S_\alpha+1) - f_{\mathcal{A}}(S+1)] \\ &\overset{(2.12)}{=} \sum_{\alpha\in I} (X_\alpha - p_\alpha) f_{\mathcal{A}}(S_\alpha+1) - p_\alpha X_\alpha \Delta f_{\mathcal{A}}(S_\alpha+1) \\ &= -\sum_{\alpha\in I} p_\alpha X_\alpha \Delta f_{\mathcal{A}}(S_\alpha+1) + \sum_{\alpha\in I}(X_\alpha - p_\alpha)[f_{\mathcal{A}}(S_\alpha+1) - f_{\mathcal{A}}(V_\alpha+1)] \\ &\quad + \sum_{\alpha\in I}(X_\alpha - p_\alpha) f_{\mathcal{A}}(V_\alpha+1) =: a_1 + a_2 + a_3.\end{aligned}$$

The $\mathcal{A}$-measurability of $||\Delta f_{\mathcal{A}}||_\infty$ yields

$$\mathbb{E}_{\mathcal{A}}[a_1] = \mathbb{E}_{\mathcal{A}}\left[-\sum_{\alpha\in I} p_\alpha X_\alpha \Delta f_{\mathcal{A}}(S_\alpha+1)\right] \le ||\Delta f_{\mathcal{A}}||_\infty \sum_{\alpha\in I} p_\alpha^2.$$

Since $f_{\mathcal{A}}(V_\alpha)$ is measurable with respect to $\sigma(\mathcal{A}, (X_\beta)_{\beta\in I\setminus B_\alpha})$, we obtain

$$\begin{aligned}\mathbb{E}_{\mathcal{A}}[a_3] &= \mathbb{E}_{\mathcal{A}}\left[\sum_{\alpha\in I}(X_\alpha - p_\alpha) f_{\mathcal{A}}(V_\alpha+1)\right] \\ &= \sum_{\alpha\in I} \mathbb{E}_{\mathcal{A}}\left[\mathbb{E}[(X_\alpha - p_\alpha) f_{\mathcal{A}}(V_\alpha+1) | \sigma(\mathcal{A}, (X_\beta)_{\beta\in I\setminus B_\alpha})]\right] \\ &= \sum_{\alpha\in I} \mathbb{E}_{\mathcal{A}}\left[f_{\mathcal{A}}(V_\alpha+1)\mathbb{E}[(X_\alpha - p_\alpha) | \sigma(\mathcal{A}, (X_\beta)_{\beta\in I\setminus B_\alpha})]\right] \\ &\le ||f_{\mathcal{A}}||_\infty \sum_{\alpha\in I} \mathbb{E}_{\mathcal{A}}\left[|\mathbb{E}[(X_\alpha - p_\alpha) | \sigma(\mathcal{A}, (X_\beta)_{\beta\in I\setminus B_\alpha})]|\right] = ||f_{\mathcal{A}}||_\infty b_3,\end{aligned}$$

where we used that $||f_{\mathcal{A}}||_\infty$ is $\mathcal{A}$-measurable. For a_2 we note that S_α is the sum over all X_β except for X_α, whereas V_α sums over all X_β except for those lying in B_α. We rewrite $f_{\mathcal{A}}(S_\alpha) - f_{\mathcal{A}}(V_\alpha)$ using a telescopic sum and then bound the increments. To be more precise, let $B_\alpha = \{\alpha, \alpha_1, \dots, \alpha_k\}$ for $k \in \mathbb{N}_0$ be a fixed ordering of the elements in B_α and define $U_{\alpha_\ell} = V_\alpha + 1 + \sum_{i=1}^{\ell-1} X_{\alpha_i}$ for $\ell = 1, \dots, k$. Then it holds that

$$\begin{aligned} a_2 &= \sum_{\alpha\in I}(X_\alpha - p_\alpha)[f_{\mathcal{A}}(S_\alpha + 1) - f_{\mathcal{A}}(V_\alpha + 1)] \\ &= \sum_{\alpha\in I}\sum_{\beta\in B_\alpha\setminus\{\alpha\}}(X_\alpha - p_\alpha)[f_{\mathcal{A}}(U_\beta + X_\beta) - f_{\mathcal{A}}(U_\beta)] \\ &= \sum_{\alpha\in I}\sum_{\beta\in B_\alpha\setminus\{\alpha\}}(X_\alpha - p_\alpha)X_\beta \Delta f_{\mathcal{A}}(U_\beta) \le ||\Delta f_{\mathcal{A}}||_\infty \sum_{\alpha\in I}\sum_{\beta\in B_\alpha\setminus\{\alpha\}} X_\alpha X_\beta + p_\alpha X_\beta \end{aligned}$$

such that the $\mathcal{A}$-measurability of $||\Delta f_{\mathcal{A}}||_\infty$ yields

$$\begin{aligned} \mathbb{E}_{\mathcal{A}}[a_2] &\le ||\Delta f_{\mathcal{A}}||_\infty \sum_{\alpha\in I}\sum_{\beta\in B_\alpha\setminus\{\alpha\}} \mathbb{E}_{\mathcal{A}}[X_\alpha X_\beta] + p_\alpha \mathbb{E}_{\mathcal{A}}[X_\beta] \\ &= ||\Delta f_{\mathcal{A}}||_\infty \sum_{\alpha\in I}\sum_{\beta\in B_\alpha\setminus\{\alpha\}} p_{\alpha\beta} + p_\alpha p_\beta. \end{aligned}$$

By [6, Lemma 1] we can bound both $||f_{\mathcal{A}}||_\infty$ and $||\Delta f_{\mathcal{A}}||_\infty$ pointwise by 2, where the two is needed for the positive and negative parts of f since the lemma only applies for non-negative functions. Summing up those bounds, we get

$$\mathbb{E}_{\mathcal{A}}[h(S) - \mathbb{E}_{\mathcal{A}}[h(V)]] \le \mathbb{E}_{\mathcal{A}}[a_1 + a_2 + a_3] \le 2(b_1 + b_2 + b_3),$$

which shows (2.8) and thereby concludes the proof. □

Theorem 2.24. *For $Z \sim \mathrm{Poi}(\mu)$ with $\mu > 0$,*

$$d_{TV}(S, Z) \le \mathbb{E}[\min(1, b_1 + b_2 + b_3)] + \mathbb{E}[\min(1, |\mathbb{E}_{\mathcal{A}}[S] - \mu|)].$$

Proof. Let $T_{\mathcal{A}} \sim \mathrm{Poi}(\mathbb{E}_{\mathcal{A}}[S])$. Then the triangle inequality and Lemma 2.22 yield

$$d_{TV}(S, Z) \le d_{TV}(S, T_{\mathcal{A}}) + d_{TV}(T_{\mathcal{A}}, Z) \le \mathbb{E}\big[\min(1, b_1 + b_2 + b_3)\big] + d_{TV}(T_{\mathcal{A}}, Z).$$

We compute

$$\begin{aligned} d_{TV}(T_{\mathcal{A}}, Z) &= \sup_{A\subseteq\mathbb{N}_0} |\mathbb{P}(T_{\mathcal{A}} \in A) - \mathbb{P}(Z \in A)| = \sup_{A\subseteq\mathbb{N}_0} |\mathbb{E}[\mathbb{P}_{\mathcal{A}}(T_{\mathcal{A}} \in A) - \mathbb{P}(Z \in A)]| \\ &\le \mathbb{E}\Big[\sup_{A\subseteq\mathbb{N}_0} |\mathbb{P}_{\mathcal{A}}(T_{\mathcal{A}} \in A) - \mathbb{P}(Z \in A)|\Big] = \mathbb{E}\Big[\min\Big(1, \sup_{A\subseteq\mathbb{N}_0} |\mathbb{P}_{\mathcal{A}}(T_{\mathcal{A}} \in A) - \mathbb{P}(Z \in A)|\Big)\Big]. \end{aligned}$$

Since, conditionally on $\mathcal{A}$, $T_{\mathcal{A}}$ follows a Poisson distribution with parameter $\lambda_{\mathcal{A}} = \mathbb{E}_{\mathcal{A}}[S]$ and Z follows a Poisson distribution with parameter μ, we obtain for all $A \subseteq \mathbb{N}_0$,

$$|\mathbb{P}_{\mathcal{A}}(T_{\mathcal{A}} \in A) - \mathbb{P}(Z \in A)| \le |\lambda_{\mathcal{A}} - \mu|,$$

see e.g. [87, Example 1 in Section 3 of Chapter 21]. The assertion follows. □

We wish to add a brief comment on the benefit of having the conditional version instead of the one where $\mathcal{A}$ is trivial. In the lemma above it suffices to show $b_1 + b_2 + b_3 \to 0$ and $|\mathbb{E}_{\mathcal{A}}[S] - \mu| \to 0$ in probability as $n \to \infty$ to obtain a qualitative, albeit not quantitative result. In Section 5.3 we thus manage to derive qualitative results under weaker moment assumptions than we require for our quantitative results in Section 5.4.

Next, we provide a corollary to the so-called Poisson Cramér-Wold device, reducing multivariate Poisson convergence to the univariate setting. A two-dimensional version of the statement can be found in Corollary 2.3 in [5]. The two-dimensional proof generalises to higher dimensions.

For $m \in \mathbb{N}$ and $\mathbb{N}_0$-valued random variables $Y_1, \dots, Y_m$ as well as $q_1, \dots, q_m \in [0,1]$ let

$$Y_i^{(q_i)} \sim \mathrm{Bin}(Y_i, q_i) \quad \text{for } i = 1, \dots, m,$$

where $Y_1^{(q_1)}, \dots, Y_m^{(q_m)}$ are independent, conditionally on $Y_1, \dots, Y_m$. One refers to this as thinning.

Corollary 2.25. *Let $(\boldsymbol{Y}_n)_{n\in\mathbb{N}}$ with $\boldsymbol{Y}_n = (Y_{1,n}, \dots, Y_{m,n})$ for $n \in \mathbb{N}$ be a sequence of $\mathbb{N}_0^m$-valued random vectors. Suppose that there are $\mu_1, \dots, \mu_m > 0$ such that for all $q = (q_1, \dots, q_m) \in [0,1]^m$,*

$$\sum_{i=1}^m Y_{i,n}^{(q_i)} \xrightarrow{d} Y_{q,\mu} \sim \mathrm{Poi}\Big(\sum_{i=1}^m q_i \mu_i\Big) \quad \text{as} \quad n \to \infty.$$

Then

$$\boldsymbol{Y}_n \xrightarrow{d} \boldsymbol{Y} \sim \otimes_{i=1}^m \mathrm{Poi}(\mu_i) \quad \text{as} \quad n \to \infty.$$

2.5 Miscellaneous

This last preliminary section is devoted to presenting further statements that are applied in this thesis. Throughout this section, consider a sequence $W, W_1, W_2, \dots$ of independent and identically distributed random variables. We start with a consequence of the Marcinkiewicz-Zygmund strong law of large numbers, see for example [62, Theorem 5.23].

Lemma 2.26. *Let $\gamma \in \mathbb{R}$ and $p \in (0,1)$. If $\mathbb{E}[W^{\gamma p}] < \infty$, then*

$$n^{-1/p} \sum_{i=1}^n W_i^\gamma \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty.$$

The next lemma concerns bounds for tails and maxima of non-negative random variables with finite second moment. For $n \in \mathbb{N}$ we abbreviate

$$W_{(n)} = \max_{i=1,\dots,n} W_i.$$

Lemma 2.27. *Let W be non-negative and such that $\mathbb{E}[W^2] < \infty$. Then*

a) $\mathbb{P}(W > t) = o\big(t^{-2}\big)$ *as $t \to \infty$,*

b) $\frac{W_{(n)}^2}{n} \xrightarrow{\mathbb{P}} 0$ *as* $n \to \infty$ *and*

c) $\frac{W_{(n)}^2}{L_n} \xrightarrow{\mathbb{P}} 0$ *as* $n \to \infty$, *where* $L_n = \sum_{i=1}^n W_i$.

Proof. a) We have

$$t^2 \mathbb{P}(W > t) \leq \mathbb{E}\big[W^2 \mathbf{1}\{W > t\}\big] \to 0 \quad \text{as} \quad t \to \infty.$$

b) For all $\varepsilon > 0$ we compute

$$\mathbb{P}\big(W_{(n)}^2 > n\varepsilon\big) = \mathbb{P}\big(\exists i \in [n] \colon W_i^2 > n\varepsilon\big) \leq \sum_{i=1}^n \mathbb{P}\big(W_i^2 > n\varepsilon\big) = n\mathbb{P}\big(W > \sqrt{n\varepsilon}\big) \to 0$$

as $n \to \infty$ by part a).

c) We have

$$\frac{W_{(n)}^2}{L_n} = \frac{W_{(n)}^2}{n} \frac{n}{L_n}$$

so that the claim follows directly from b), the weak law of large numbers and Slutsky's lemma. □

The next lemma is a special case of [18, Lemma 1]. It is some variant of the Chernoff bound for random variables with finite second moments.

Lemma 2.28. *Let* $\mathbb{E}[W^2] < \infty$. *For all* $0 < \lambda < 1$ *there exists a constant* $c > 0$ *depending only on* λ *and the first and second moment of* W *such that*

$$\mathbb{P}\bigg(\sum_{i=1}^n W_i \leq \lambda \mathbb{E}[W] n\bigg) \leq \exp\left(-cn\right) \quad \textit{for all} \quad n \in \mathbb{N}.$$

The following inequality is taken from [71, Lemma 8] and can be proven via induction.

Lemma 2.29. *For* $k \in \mathbb{N}$ *and* $b, x_1, \ldots, x_k \in (0, \infty)$ *one has*

$$\prod_{i=1}^k \frac{1}{b + x_i} \geq \frac{1}{b^k} - \frac{\sum_{i=1}^k x_i}{b^{k+1}}.$$

Chapter 3

Random graphs

The goal of this chapter is to give an overview of the random graphs which we study in this thesis. These can be seen as modifications of two famous random graphs, the Erdős-Rényi graph and the random geometric graph. In Section 3.1, we consider the Chung-Lu model, the generalised random graph and the Norros-Reittu model, which one also refers to as rank-1 models. We study properties of these models in Chapters 4 and 5. Afterwards, we discuss the scale-free (weighted) random connection model in Section 3.2, which is the protagonist of Chapter 6.

3.1 Selected rank-1 models

As discussed in the introduction, a motivation for random graphs lies in the modelling of complex networks. Therefore, they should have a scale-free degree distribution in the following sense. Let D denote the typical degree, i.e. the degree of a vertex chosen uniformly at random. Then, it should hold

$$\mathbb{P}(D > t) = \ell(t)t^{-\beta} \tag{3.1}$$

for a slowly varying function ℓ, fixed $\beta > 0$ and all $t > 0$. Of course, this statement only makes sense when the network size tends to infinity, otherwise there will always be a cut-off. Consider the famous Erdős-Rényi graph, also known as $G(n, p)$ for $n \in \mathbb{N}$ and $p \in [0, 1]$. In this model, one takes n vertices labelled from 1 to n and connects any pair of vertices independently with probability p. For the Erdős-Rényi graph it follows by symmetry that D has the same distribution as $\deg(1)$, the degree of the vertex 1. Its limit is Poisson as $n \to \infty$ if $p = \lambda/n$ for $\lambda > 0$, see e.g. [52, Theorem 2.10]. The Poisson distribution does not obey a power-law as in (3.1), instead it has much lighter tails. The goal of the following subsection is to define generalisations of $G(n, p)$ that possess a power-law in their degree distribution.

3.1.1 Definitions

In this section we present several graphs on the vertex set $V = [n]$ for given $n \in \mathbb{N}$. Let $\mathcal{W} = (W_i)_{i\in\mathbb{N}}$ denote a sequence of independent and identically distributed positive random variables. Here, W_i serves as a weight attached to vertex i and

$$L_n = \sum_{i=1}^{n} W_i \tag{3.2}$$

denotes the sum of all weights in the graph. We start by defining the so-called Chung-Lu model, which was introduced in [27] by Chung and Lu.

Definition 3.1 (Chung-Lu model)**.** For $n \in \mathbb{N}$ the Chung-Lu model $\mathrm{CL}(n)$ has the vertex set $[n]$. Conditionally on $\mathcal{W}$, we connect any two distinct vertices $x, y \in [n]$ independently with probability

$$\mathbb{P}_{\mathcal{W},CL}(x \leftrightarrow y) = \mathbb{P}(x \leftrightarrow y|\mathcal{W}) = \min\left(1, \frac{W_x W_y}{L_n}\right). \tag{3.3}$$

The index $\mathcal{W}$ in $\mathbb{P}_{\mathcal{W},CL}$ shall indicate the fact that we condition on the weights $\mathcal{W}$. To simplify notation we will often omit the index referring to the random graph model at hand, as it is going to be fixed in the respective theorems. Similarly, we use the term $\mathbb{E}_{\mathcal{W}}[\cdot] = \mathbb{E}[\cdot|\mathcal{W}]$ for the conditional expectation with respect to the weights $\mathcal{W}$. The intuition behind the connection probability in (3.3) arises from the following approximations. For $\mathbb{E}[W^2] < \infty$, which will be a standard assumption for us, it holds that

$$W_{(n)} = \max_{i\in[n]} W_i = o_{\mathbb{P}}\big(\sqrt{n}\big),$$

see Lemma 2.27. Since L_n is of order n by the law of large numbers, we have

$$\mathbb{P}_{\mathcal{W},CL}(x \leftrightarrow y) = \min\left(1, \frac{W_x W_y}{L_n}\right) \approx \frac{W_x W_y}{L_n} =: p_{xy}, \tag{3.4}$$

as p_{xy} is with high probability eventually smaller than one since $p_{xy} = o_{\mathbb{P}}(1)$, uniformly in $x, y \in [n]$. This leads to the following behaviour of the expected degree of some vertex $x \in [n]$ when conditioning on the weights $\mathcal{W}$,

$$\mathbb{E}_{\mathcal{W}}[\deg(x)] = \sum_{\substack{y=1\\ y\neq x}}^{n} \mathbb{P}_{\mathcal{W},CL}(x \leftrightarrow y) \approx \sum_{\substack{y=1\\ y\neq x}}^{n} \frac{W_x W_y}{L_n} = W_x \frac{L_n - W_x}{L_n} \approx W_x.$$

If, conditionally on $\mathcal{W}$, the degree is sufficiently centred around its mean, the degrees are closely related to the weights. This allows us to obtain inhomogeneity in the degrees by taking a suitable weight distribution. For a more formal treatment, we refer the reader to [21, Theorem 3.13 and Corollary 13.1] and the respective discussion in the rank-1 setting in [21, Subsection 16.4], where it is shown that the degree of vertex x asymptotically follows a mixed Poisson distribution $\mathrm{Poisson}(W_x)$, given that $\mathbb{E}[W] < \infty$. In the aforementioned Corollary 13.1 it is also established that the resulting graph has power-law degrees, if the weight distribution is of the form $\mathbb{P}(W > t) \sim at^{-\beta}$ for $a > 0$ and $\beta > 1$. To allow for additional flexibility, one often uses weights with regularly varying tails instead, as we do in this thesis.

Obviously, one cannot simply take $\mathbb{P}_{\mathcal{W}}(x \leftrightarrow y) = p_{xy}$, as it might be larger than one. While the Chung-Lu model decides to take the minimum of p_{xy} and one, there are other ways to tackle this problem, which have been employed by other authors and give rise to two closely related random graph models. The first one is referred to as generalised random graph whereas the second one is called the Norros-Reittu model. They were introduced in [23] by Britton, Deijfen and Martin-Löf and in [81] by Norros and Reittu, respectively.

Definition 3.2 (Generalised random graph)**.** For $n \in \mathbb{N}$ the generalised random graph $\mathrm{GRG}(n)$ has the vertex set $[n]$. Conditionally on $\mathcal{W}$, we connect any two distinct vertices $x, y \in [n]$ independently with probability

$$\mathbb{P}_{\mathcal{W},GRG}(x \leftrightarrow y) = \frac{W_x W_y}{W_x W_y + L_n}. \tag{3.5}$$

Definition 3.3 (Norros-Reittu model)**.** For $n \in \mathbb{N}$ the Norros-Reittu model $\mathrm{NR}(n)$ has the vertex set $[n]$. Conditionally on $\mathcal{W}$, we connect any two distinct vertices $x, y \in [n]$ independently with probability

$$\mathbb{P}_{\mathcal{W},NR}(x \leftrightarrow y) = 1 - \exp\Big(- \frac{W_x W_y}{L_n} \Big). \tag{3.6}$$

Remark 3.4. The term rank-1 models is motivated by the fact that the matrix $(p_{xy})_{x,y \in [n]}$, which describes the connection probabilities, has rank one. In a more general setting involving kernel functions, see [21], the key property is the fact that the kernel has a product form and the rank-1 property is that of an operator instead.

There exist slight variations of the models introduced above. In fact, the generalised random graph was originally introduced in another version in [23], while our definition follows the one in [37]. Up to rescaling of the weights, [23] uses the connection probability

$$\mathcal{P}_{\mathcal{W},GRG'}(x \leftrightarrow y) = \frac{W_x W_y}{W_x W_y + n\mathbb{E}[W]} \approx \frac{W_x W_y}{n\mathbb{E}[W]} =: p'_{xy}$$

and we denote the corresponding model by $\mathrm{GRG}'(n)$. Similarly, we obtain the models $\mathrm{CL}'(n)$ and $\mathrm{NR}'(n)$ by replacing the quantity L_n in the respective models by $n\mathbb{E}[W]$. We write $\mathcal{G} = \{\mathrm{CL}, \mathrm{GRG}, \mathrm{NR}\}$. Fix $G \in \mathcal{G}$. Then, $G(n)$ refers to one of the three models introduced in Definition 3.1, Definition 3.2 and Definition 3.3. On the other hand, $G'(n)$ refers to the models using $n\mathbb{E}[W]$ instead of L_n in the connection function. Intuitively, we can expect $G(n)$ and $G'(n)$ to behave reasonably similar for large n because

$$\frac{L_n}{n\mathbb{E}[W]} \xrightarrow{a.s.} 1 \quad \text{as} \quad n \to \infty$$

by the strong law of large numbers. The next subsection formalises this intuition.

3.1.2 Relations between the different models

In the following we discuss deeper relations between the models. The results and proofs are taken from [69, Section 2.3] by Lienau and Schulte. We start with a simple comparison of the connection probabilities.

Lemma 3.5. *For all $n \in \mathbb{N}$ and $x, y \in [n]$ one has*

$$p_{xy}(1 - \min(1, p_{xy})) \le \mathbb{P}_{\mathcal{W},GRG}(x \leftrightarrow y) \le \mathbb{P}_{\mathcal{W},NR}(x \leftrightarrow y) \le \mathbb{P}_{\mathcal{W},CL}(x \leftrightarrow y) \le p_{xy}$$

as well as

$$p'_{xy}(1 - \min(1, p'_{xy})) \le \mathbb{P}_{\mathcal{W},GRG'}(x \leftrightarrow y) \le \mathbb{P}_{\mathcal{W},NR'}(x \leftrightarrow y) \le \mathbb{P}_{\mathcal{W},CL'}(x \leftrightarrow y) \le p'_{xy}.$$

Proof. We start with the first series of inequalities. Expressing all three connection probabilities in terms of p_{xy} yields

$$\begin{aligned} \mathbb{P}_{\mathcal{W},GRG}(x \leftrightarrow y) &= \frac{p_{xy}}{p_{xy}+1}, \\ \mathbb{P}_{\mathcal{W},NR}(x \leftrightarrow y) &= 1 - \exp(-p_{xy}) \quad \text{and} \\ \mathbb{P}_{\mathcal{W},CL}(x \leftrightarrow y) &= \min(1, p_{xy}). \end{aligned}$$

Now we deal with the left-most inequality. The third binomial formula provides

$$(1 - p_{xy})(1 + p_{xy}) = 1 - p_{xy}^2 \le 1$$

so that

$$1 - p_{xy} \le \frac{1}{1 + p_{xy}}.$$

Together with $p_{xy} \ge 0$ and $\mathbb{P}_{\mathcal{W},GRG}(x \leftrightarrow y) \ge 0$, this implies the first inequality. The right-most inequality is obvious, leaving us with a comparison of the connection probabilities. From $1 + t \le \exp(t)$ for all $t \in \mathbb{R}$ one obtains $\exp(-t) \le 1/(1+t)$ and thus

$$\frac{t}{1+t} = 1 - \frac{1}{1+t} \le 1 - \exp(-t).$$

With $t = p_{xy}$ we conclude

$$\frac{p_{xy}}{p_{xy}+1} \le 1 - \exp(-p_{xy}) \le \min(1, p_{xy}),$$

where the last inequality uses $\exp(t) > 0$ and $1 - \exp(-t) \le t$ for all $t \in \mathbb{R}$. This shows the first series of inequalities. The second one follows by noting that all statements above remain true when replacing each L_n by $n\mathbb{E}[W]$. □

The relation between the different random graph models in the previous lemma is also known as stochastic domination, see e.g. [52, Subsection 6.8.2]. Essentially one obtains the existence of a coupling of the three models such that every edge being present in the generalised random graph is also present in the other two graphs, whereas every edge of the Norros-Reittu model also exists in the Chung-Lu model.

We conclude this subsection with an application of results from [58] by Janson to compare the six different random graph models introduced in the previous subsection. Two sequences $(G_n)_{n\in\mathbb{N}}$ and $(\tilde{G}_n)_{n\in\mathbb{N}}$ of random graphs are called asymptotically equivalent if one can couple them in a way such that $\mathbb{P}(G_n \neq \tilde{G}_n) \to 0$ as $n \to \infty$, see [58, Definition 1.1 and Theorem 4.2]. To simplify notation, we also say that G_n and $\tilde{G}_n$ are asymptotically equivalent. This definition extends naturally to finitely many graph sequences being asymptotically equivalent.

We gather some results and consequences from [58] in the following lemma.

Lemma 3.6. *If* $\mathbb{E}[W^2] < \infty$*, the following hold.*

a) $\mathrm{NR}(n)$, $\mathrm{CL}(n)$ *and* $\mathrm{GRG}(n)$ *are asymptotically equivalent.*

b) $\mathrm{NR}'(n)$, $\mathrm{CL}'(n)$ *and* $\mathrm{GRG}'(n)$ *are asymptotically equivalent.*

Proof. a) Follows directly from [58, Example 3.6] and Lemma 2.27 a).

b) [58, Example 3.1] shows that $\mathrm{NR}'(n), \mathrm{CL}'(n)$ and $\mathrm{GRG}'(n)$ are asymptotically equivalent when

$$\sum_{1\le i<j\le n} \left(\frac{W_iW_j}{n\mathbb{E}[W]}\right)^3 \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n\to\infty,$$

see [58, Equation (3.6)]. We have

$$\sum_{1\le i<j\le n} \left(\frac{W_iW_j}{n\mathbb{E}[W]}\right)^3 \le \frac{W_{(n)}^2}{n\mathbb{E}[W]^3}\left(\frac{\sum_{i=1}^n W_i^2}{n}\right)^2 \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n\to\infty$$

since the first factor converges in probability to zero as $n\to\infty$ by Lemma 2.27 b) whereas the second factor remains bounded by the strong law of large numbers and the finite second moment of W. □

The previous lemma does not provide asymptotic equivalence of all six models. Instead, we use another concept to relate $\mathrm{NR}(n)$ and $\mathrm{NR}'(n)$, following [58, Definition 1.1], which is formulated in a more abstract setting. For $n\in\mathbb{N}$ we consider some measurable space $(\mathcal{X}_n,\mathcal{A}_n)$ with two probability measures $\mathbf{P}_n$ and $\mathbf{Q}_n$ defined on it. One calls $(\mathbf{P}_n)_{n\in\mathbb{N}}$ contiguous with respect to $(\mathbf{Q}_n)_{n\in\mathbb{N}}$ if

$$\mathbf{Q}_n(A_n)\to 0 \quad \text{as} \quad n\to\infty \qquad \Longrightarrow \qquad \mathbf{P}_n(A_n)\to 0 \quad \text{as} \quad n\to\infty$$

for all sequences of sets $(A_n)_{n\in\mathbb{N}}$ with $A_n\in\mathcal{A}_n$. For technical reasons, we need to keep track of the underlying weights on top of the generated graphs. Therefore, we think of the random graphs as probability measures on $\mathcal{X}_n=(0,\infty)^n\times\mathcal{G}_n$, where $\mathcal{G}_n$ is the set of all graphs with vertex set $[n]$. Then, $\mathcal{X}_n$ is equipped with a suitable σ-field $\mathcal{A}_n$ and contains elements of the form $(\mathbf{w}_n, G_n)$ with a graph G_n having vertex set $[n]$ and weights given by $\mathbf{w}_n=(w_1,\dots,w_n)$. We couple our models in such a way that the vertices have the same weights $\mathbf{W}_n=(W_1,\dots,W_n)$ for all models, but the generation of the graph G_n, given $\mathbf{W}_n$, uses the connection probability from the respective model.

Lemma 3.7. *If* $\mathbb{E}[W^2]<\infty$*, one has that* $(\mathbf{W}_n,\mathrm{NR}'(n))$ *is contiguous with respect to* $(\mathbf{W}_n,\mathrm{NR}(n))$.

Our framework containing graphs *and* the weights is not explicitly covered in [58], but the techniques employed therein generalise to this setting as shown in the following. Before giving the details, we would like to briefly discuss implications. On the one hand, we obtain that asymptotic equivalence of two random graph sequences preserves convergence in probability and in distribution of random variables depending on the respective graphs. For contiguity on the other hand, it only follows that convergence of some functional of the graph (and its underlying weights) in probability to a constant is preserved when passing to a random graph sequence that is contiguous to the former. Even when two random graph sequences are mutually contiguous, convergence in distribution does not necessarily carry over, see also [58, Remark 1.4]. This is why we have to use a more sophisticated argument to transfer results regarding convergence in distribution from NR to $\mathrm{NR}'(n)$ in Chapter 4. To this end, we make use of the underlying weights, which is why we incorporate them in $\mathcal{X}_n$.

In order to prepare the proof of Lemma 3.7, we formulate a lemma. Similarly to the notation in [58] we write for $p, q \in [0,1]$,

$$\rho(p,q) = (\sqrt{p} - \sqrt{q})^2 + \big(\sqrt{1-p} - \sqrt{1-q}\big).$$

By [58, Equation (2.5)] there exists some constant $C_0 > 0$ such that for all $p < 0.9$ and $q \in [0,1]$,

$$\rho(p,q) \le C_0 \frac{(p-q)^2}{p}. \tag{3.7}$$

Given distinct $i, j \in [n]$, we denote the conditional connection probabilities in $\mathrm{NR}(n)$ and $\mathrm{NR}'(n)$, respectively, by

$$q_{ij} = 1 - \exp\Big(- \frac{W_i W_j}{L_n}\Big) \quad \text{and} \quad q'_{ij} = 1 - \exp\Big(- \frac{W_i W_j}{n\mathbb{E}[W]}\Big).$$

Lemma 3.8. *Let assumption (W) hold. Then, for all $\gamma > 0$ there exist $C_1, C_2 > 0$ and $N \in \mathbb{N}$ such that for all $n \ge N$,*

$$\mathbb{P}\Big(\sum_{1 \le i < j \le n} \rho(q_{ij}, q'_{ij}) > C_1 \text{ or } \max_{i,j \in [n]} \frac{q'_{ij}}{q_{ij}} > C_2 \text{ or } \max_{i,j\in[n]} \frac{1 - q'_{ij}}{1 - q_{ij}} > C_2 \Big) \le \gamma.$$

Proof. For $C_1, C_2 > 0$ we have

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{1 \le i < j \le n} \rho(q_{ij}, q'_{ij}) > C_1 \text{ or } \max_{i,j \in [n]} \frac{q'_{ij}}{q_{ij}} > C_2 \text{ or } \max_{i,j\in[n]} \frac{1 - q'_{ij}}{1 - q_{ij}} > C_2 \Big) \\
&\le \mathbb{P}\Big(\sum_{1 \le i < j \le n} \rho(q_{ij}, q'_{ij}) > C_1, \frac{W_{(n)}^2}{L_n} < 0.9\Big) + \mathbb{P}\Big(\frac{W_{(n)}^2}{L_n} \ge 0.9\Big) \\
&\quad + \mathbb{P}\Big(\max_{i,j \in [n]} \frac{q'_{ij}}{q_{ij}} > C_2 \Big) + \mathbb{P}\Big(\max_{i,j\in[n]} \frac{1 - q'_{ij}}{1 - q_{ij}} > C_2 \Big) =: R_1 + R_2 + R_3 + R_4.
\end{aligned}$$

By Lemma 2.27 c) we see that $R_2 \to 0$ as $n \to \infty$. In particular, there exists $N \in \mathbb{N}$ such that $R_2 \le \gamma/4$ for all $n \ge N$. In the following, all statements that hold for large enough n could increase N, which we will not mention explicitly. For R_3 we employ Lemma 3.5 to obtain a lower bound on q_{ij} and an upper bound for q'_{ij},

$$\max_{i,j \in [n]} \frac{q'_{ij}}{q_{ij}} \le \max_{i,j\in[n]} \frac{\frac{W_i W_j}{n\mathbb{E}[W]}}{\frac{W_i W_j}{L_n}\big(1 - \min(1, W_{(n)}^2/L_n)\big)} = \frac{L_n}{n\mathbb{E}[W]} \frac{1}{1 - \min(1, W_{(n)}^2/L_n)} \xrightarrow{\mathbb{P}} 1$$

as $n \to \infty$ due to the law of large numbers and Lemma 2.27 c). Similarly, we have

$$\max_{i,j\in[n]} \frac{1 - q'_{ij}}{1 - q_{ij}} \le \max_{i,j \in [n]} \frac{1}{\exp(-W_i W_j / L_n)} = \frac{1}{\exp(-W_{(n)}^2/L_n)} \xrightarrow{\mathbb{P}} 1$$

as $n \to \infty$. Both convergences together yield the existence of some $C_2 > 0$ such that for n large enough both $R_3 \leq \gamma/4$ and $R_4 \leq \gamma/4$. It remains to address R_1. From (3.7) we derive

$$\begin{aligned} R_1 &= \mathbb{P}\Big(\sum_{1\leq i<j\leq n} \rho(q_{ij}, q'_{ij}) > C_1, \frac{W_{(n)}^2}{L_n} < 0.9 \Big) \\ &\leq \mathbb{P}\Big(\sum_{1\leq i<j\leq n} \frac{(q_{ij} - q'_{ij})^2}{q_{ij}} > \frac{C_1}{C_0} \Big). \end{aligned} \tag{3.8}$$

The mean value theorem yields

$$\exp(x) - \exp(y) = \exp(z)(x-y)$$

for some z lying between x and y for $x, y \in \mathbb{R}$. With $\exp(0) = 1$ we obtain

$$\begin{aligned} \sum_{1\leq i<j\leq n} \frac{(q_{ij} - q'_{ij})^2}{q_{ij}} &\leq \sum_{i,j=1}^{n} \frac{\left(\exp\left(-\frac{W_iW_j}{L_n} \right) - \exp\left(-\frac{W_iW_j}{n\mathbb{E}[W]} \right) \right)^2}{1 - \exp\left(-\frac{W_iW_j}{L_n} \right)} \\ &= \sum_{i,j=1}^{n} \frac{\left(\left(\frac{W_iW_j}{L_n} - \frac{W_iW_j}{n\mathbb{E}[W]} \right) \exp(z_{ij}) \right)^2}{\left(\frac{W_iW_j}{L_n} \right) \exp(z'_{ij})} \end{aligned}$$

for some random z_{ij} between $-W_iW_j/L_n$ and $-W_iW_j/n\mathbb{E}[W]$ as well as some random z'_{ij} between $-W_iW_j/L_n$ and 0. We have $\exp(z_{ij}) < 1$ since $z_{ij} < 0$. Together with the relation $z'_{ij} \geq -W_{(n)}^2/L_n$ we bound the expression above further by

$$\begin{aligned} &\exp\left(\frac{W_{(n)}^2}{L_n} \right) \sum_{i,j=1}^{n} \frac{\left(\frac{W_iW_j}{L_n} - \frac{W_iW_j}{n\mathbb{E}[W]} \right)^2}{\frac{W_iW_j}{L_n}} = \exp\left(\frac{W_{(n)}^2}{L_n} \right) \sum_{i,j=1}^{n} W_iW_jL_n \Big(\frac{n\mathbb{E}[W] - L_n}{n\mathbb{E}[W]L_n} \Big)^2 \\ &= \exp\left(\frac{W_{(n)}^2}{L_n} \right) \frac{L_n}{n\mathbb{E}[W]^2} \Big(\frac{L_n - n\mathbb{E}[W]}{\sqrt{n}} \Big)^2. \end{aligned}$$

This expression converges in distribution as $n \to \infty$ by Slutsky's lemma: the first and second factor converge in probability due to Lemma 2.27 c) and the law of large numbers whereas the last factor converges in distribution due to the central limit theorem and the continuous mapping theorem. Together with (3.8) we can choose $C_1 > 0$ large enough such that $R_1 \leq \gamma/4$ for all n large enough. This concludes the proof. □

We are now ready to prove Lemma 3.7.

Proof of Lemma 3.7. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of measurable $A_n \subseteq \mathcal{X}_n$ such that

$$\mathbb{P}\big((\mathbf{W}_n, \mathrm{NR}(n)) \in A_n \big) \longrightarrow 0 \quad \text{as} \quad n \to \infty, \tag{3.9}$$

where $\mathbf{W}_n = (W_1, \dots, W_n)$ denotes the underlying random weights. We have to show that

$$\mathbb{P}\big((\mathbf{W}_n, \mathrm{NR}'(n)) \in A_n\big) \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.10}$$

For $u > 0$ the Markov inequality yields

$$\mathbb{P}\big(\mathbb{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{NR}(n)) \in A_n) \geq u\big) \leq \frac{1}{u}\mathbb{P}\big((\mathbf{W}_n, \mathrm{NR}(n)) \in A_n\big) \longrightarrow 0$$

as $n \to \infty$ by (3.9), where we use the notation $\mathcal{W}$ for the conditional expectation with respect to the weights as before. Therefore,

$$\mathbb{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{NR}(n)) \in A_n) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty. \tag{3.11}$$

We will show that the same statement holds for $\mathrm{NR}'(n)$, i.e.

$$\mathbb{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{NR}'(n)) \in A_n) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty. \tag{3.12}$$

As the conditional probabilities are at most one, this yields (3.10) and thus the claim. In order to show (3.12), let $\gamma > 0$. By Lemma 3.8 there exist $C_1, C_2 > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mathbb{P}(B_n) \leq \gamma$ with

$$B_n = \left\{ \sum_{1 \leq i < j \leq n} \rho(q_{ij}, q'_{ij}) > C_1 \text{ or } \max_{i,j \in [n]} \frac{q'_{ij}}{q_{ij}} > C_2 \text{ or } \max_{i,j \in [n]} \frac{1 - q'_{ij}}{1 - q_{ij}} > C_2 \right\}.$$

Now let $\varepsilon > 0$. By applying Lemma 5.2 in [58], there exists some $\delta > 0$ such that we have on B_n^c,

$$\mathbb{P}_{\mathcal{W}}(\mathrm{NR}(n) \in \tilde{A}) < \delta \quad \Longrightarrow \quad \mathbb{P}_{\mathcal{W}}(\mathrm{NR}'(n) \in \tilde{A}) < \varepsilon$$

for all $\tilde{A} \subseteq \mathcal{G}_n$, which in turn yields

$$\mathbb{P}_{\mathcal{W}}\big((\mathbf{W}_n, \mathrm{NR}(n)) \in A_n\big) < \delta \implies \mathbb{P}_{\mathcal{W}}\big((\mathbf{W}_n, \mathrm{NR}'(n)) \in A_n\big) < \varepsilon. \tag{3.13}$$

For $n \geq N$ we conclude with the contraposition of (3.13) that

$$\begin{aligned} \mathbb{P}\big(\mathbb{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{NR}'(n)) \in A_n) \geq \varepsilon\big) &\leq \mathbb{P}(B_n) + \mathbb{P}\big(B_n^c, \mathbb{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{NR}'(n)) \in A_n) \geq \varepsilon\big) \\ &\leq \gamma + \mathbb{P}\big(B_n^c, \mathbb{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{NR}(n)) \in A_n) \geq \delta\big) \\ &\leq \gamma + \mathbb{P}\big(\mathbb{P}_{\mathcal{W}}((\mathbf{W}_n, \mathrm{NR}(n)) \in A_n) \geq \delta\big). \end{aligned}$$

By (3.11), the second summand above converges to zero as $n \to \infty$. As $\gamma > 0$ can be chosen arbitrarily small, we obtain (3.12), which yields the claim. □

3.1.3 Related models

We would like to briefly comment on some related models. First of all, there exist versions where the weights are not random, but deterministic. In this setting, one usually assumes that the empirical distribution of the weights converges to that of an underlying random variable W, which is heavy-tailed. Depending on the results one aims to prove, this convergence needs to be in terms of the (empirical) distribution functions or (empirical) moments, see e.g. [52, Condition 6.4] in the book by van der Hofstad. The setting of random weights can be incorporated in the one with deterministic weights by adding another layer of probability to the aforementioned convergences as mentioned in [52, Remark 6.5]. In their original versions, both the Chung-Lu model and the generalised random graph were introduced with deterministic weights, corresponding to the desired degrees. Additionally, the generalised random graph was introduced with the connection probability

$$\mathbb{P}_{\mathcal{W}}(x \leftrightarrow y) = \frac{W_x W_y}{W_x W_y + n},$$

which is only almost equal to our primed setup where we replace L_n by $n\mathbb{E}[W]$. However, using a tilde to denote the respective weights rescaled by $\mathbb{E}[W]^{-1/2}$ yields

$$\frac{\tilde{W}_x \tilde{W}_y}{\tilde{W}_x \tilde{W}_y + n} = \frac{W_x W_y / \mathbb{E}[W]}{W_x W_y / \mathbb{E}[W] + n} = \frac{W_x W_y}{W_x W_y + n\mathbb{E}[W]},$$

so that this change simply amounts to rescaling the weights.

Finally, the Norros-Reittu model was originally introduced as a multi-graph, meaning that it allows for multiple edges between two vertices as well as self-loops, i.e. an edge from a vertex to itself. The graph was actually defined as an evolving process in [81], where the vertices arrive one by one. With each arriving vertex, new edges are placed and some of the old edges are removed. However, the authors showed that this process is such that for n vertices with corresponding weights $W_1, \dots, W_n$, the number of edges between any two vertices $i, j \in [n]$ follows a Poisson distribution with parameter $W_i W_j / L_n$. Here, $i = j$ is explicitly allowed. If one wants to obtain a graph, one can merge all multiple edges between any two vertices into a single one and remove all self-loops. This leads to the probabilities we used in Definition 3.3. Note that sometimes a graph without multiple edges and self-loops is also called a *simple* graph.

Since our models are motivated by the goal to obtain a certain degree distribution, we would like to briefly present a related well-known model that achieves a similar result. The configuration model was proposed in [19] by Bollobás in order to study the number of labelled regular graphs. It is constructed as follows. For $n \in \mathbb{N}$ and a given degree sequence $d_1 \geq \ldots \geq d_n$ such that their sum is even, one takes n vertices. For $i = 1, \dots, n$, we equip the i-th vertex with d_i many so-called half-edges, or stumps. One forms the edges of the graph by iteratively choosing two half-edges uniformly at random and combining them to a proper edge. Since the number of half-edges was even by assumption, eventually all half-edges are paired. The resulting object might contain self-loops as well as multiple edges, but possesses exactly the desired degree sequence. In the case where the degree sequence of the configuration model consists of random variables, it is also known as Newman-Strogatz-Watts model introduced by the respective authors in [80]. When one is interested in generating a graph, not a multi-graph, one can merge multiple edges and remove loops to obtain the *erased* configuration model. However, this affects the degree sequence. Another possibility is to resample until the output is a graph, which is known as *repeated* configuration model. This can be seen as conditioning on obtaining a graph. Conditionally on this event, the resulting graph is chosen uniformly at random among all graphs with the respective degree sequence, see also [52, Proposition 7.15] by van der Hofstad. In [23], Britton, Deijfen and Martin-Löf note that the generalised random graph also has this property of being uniformly chosen at random, see also [52, Theorem 6.15].

For this reason, one can deduce properties for the generalised random graph by statements on the configuration model, see also [52, Corollary 7.17] and the discussion thereafter.

The paper [5] by Angel, van der Hofstad and Holmgren studies limit laws for the number of cycles and self-loops that appear in the configuration model. Under some assumptions it turns out that, asymptotically, their counts are independent and follow Poisson distributions. This can in turn be used to deduce that the probability of the resulting graph being simple is positive. By the probabilistic method originally pioneered by Erdős, see also [4], it follows that at least one graph with the respective given degree sequence exists. The author of the present thesis has applied a similar scheme to derive results concerning the existence of (regular) hypergraphs with given degree sequences in [7] together with Ascolese, Schulte and Taraz. In contrast to a graph, the (hyper-)edges of a k-regular hypergraph contain not two, but k vertices. In this case, self-loops correspond to some vertex appearing more than once in a single (hyper-)edge.

3.1.4 Properties

In this section we discuss some properties of the rank-1 models at hand. We shall focus on the ones that are of interest for this thesis. These include:

a) power-law degree distribution,

b) phase transition and

c) locally tree-like behaviour.

We already discussed the power-law degrees of the model and thus focus on the remaining two properties. We start with the phase transition, where we should more explicitly say phase transition of the size of the largest component. Here, one speaks of a phase transition, when the behaviour changes drastically when a parameter reaches a certain threshold. The name originates from the phase transition between liquids, solids and gases, where one witnesses vast differences between the different phases. We denote the size of the k-th largest component by $\mathcal{C}_{(k)}$, where we break ties arbitrarily. A classical result for the Erdős-Rényi graph is the following phase transition concerning the size of its largest component. Suppose that the connection probability is given by $p = \lambda/n$. What can we say about the size of the largest component, depending on λ? As $n \to \infty$, it turns out that

a) $\mathcal{C}_{(1)} = \Theta_{\mathbb{P}}(\log(n))$ if $\lambda < 1$,

b) $\mathcal{C}_{(1)} = \Theta_{\mathbb{P}}(n^{2/3})$ if $\lambda = 1$ and

c) $\mathcal{C}_{(1)} = \Theta_{\mathbb{P}}(n)$ if $\lambda > 1$.

The different regimes are referred to as subcritical in a), critical in b) and supercritical in c). Moreover, in the supercritical phase it is known that the second-largest component is of strictly smaller order, to be more precise $C_{(2)} = O_{\mathbb{P}}(\log(n))$. In particular, the largest component in the supercritical regime contains a positive fraction of all vertices and is also referred to as the giant component, due to its uniqueness. In the critical and subcritical phase instead, for each fixed $k \in \mathbb{N}$ the k largest components are actually of the same order. An overview on these results can be found in [52, Chapters 4 and 5] by van der Hofstad, where the results are obtained by comparing the so-called exploration process to a branching process. The mean of its offspring distribution turns out to be exactly λ, which explains the change at $\lambda = 1$.

Historically speaking, this phase transition was already established by Erdős and Rényi in 1960 in [35] for the related model $G(n, M)$, where one chooses a graph uniformly at random among all graphs having n vertices and M edges. Here, it takes place when the number of edges is of order $M = n/2$, which corresponds exactly to the average degree being one. They did not only study the size of the largest component, but also the graph structure, giving in fact five different phases. Another notion related to this field is that of a critical window. Here, λ is not constant, but of the form $\lambda = 1 + tn^{-1/3}$ where one still observes a scaling of order $n^{2/3}$ for $\mathcal{C}_{(1)}$, see e.g. [52, Theorem 5.1]. However, we would like to focus on a similar phase transition for the rank-1 models in more detail now.

The existence of a giant component for the Chung-Lu model and the Norros-Reittu model based on assumptions on the weight distribution was studied immediately when the models were introduced in [27] by Chung and Lu and in [81] by Norros and Reittu, respectively. A complete picture on the emergence of the giant is given in [21, Theorem 3.1] by Bollobás, Janson and Riordan in their general setting. We write $\nu = \mathbb{E}[W^2]/\mathbb{E}[W]$. It turns out that in all rank-1 models mentioned above there is a giant component if and only if $\nu > 1$, without further assumptions on the tail of the weights. Moreover, in the case that a giant exists, it is unique. As in the case of $G(n, p)$, one calls the graph supercritical for $\nu > 1$, critical for $\nu = 1$ and subcritical for $\nu < 1$. The respective statement can also be found in [53, Theorem 3.20] by van der Hofstad, where it is phrased for our particular setting. The result is once more obtained via a coupling to a branching process, to which we will return momentarily.

Naturally, the results above lead to the question of finding the size of the largest component when the model is critical or subcritical. This question was answered by Janson for the subcritical regime in [58], while the critical regime was investigated in [12] and [13] by Bhamidi, van der Hofstad and van Leeuwaarden. Suppose that the weights obey a power-law of the form $\mathbb{P}(W > t) \sim at^{-\beta}$ for $a, \beta > 0$. The results on the component sizes can be summarised as follows:

a) $\mathcal{C}_{(1)} = \Theta_{\mathbb{P}}(n^{1/\beta})$ for $\nu < 1, \beta > 2$,

b) $\mathcal{C}_{(1)} = \Theta_{\mathbb{P}}(n^{(\beta-1)/\beta})$ for $\nu = 1, \beta \in (2, 3)$ and

c) $\mathcal{C}_{(1)} = \Theta_{\mathbb{P}}(n^{2/3})$ for $\nu = 1, \beta > 3$.

It is noteworthy that the component sizes in the subcritical regime are much larger than for $G(n, p)$, whereas the component sizes at criticality are, for $\beta > 3$, of the same order. Intuitively, the reason for larger components in the subcritical phase for the rank-1 model with comparison to $G(n, p)$ lies in the very motivation of its construction, the power-law degree distribution. There will be vertices having a comparatively large weight, resulting in a large degree that is already of the same order as the component size. More details on the maximal degree in some random graph models can be found in [14] by Bhattacharjee and Schulte. Also note that regime b) interpolates nicely between the regimes a) and c). Indeed, if one were to plug in $\beta = 2$ in a) and b), we obtain the exponent $1/2$, whereas plugging in $\beta = 3$ in b) yields $2/3$ as in part c).

Now, we return to the aforementioned convergence to a branching process. The respective statement is given in the book [53, Theorem 3.18] by van der Hofstad, as a special case of [53, Theorem 3.14]. The result is phrased in the form of local convergence, a concept that was introduced by Benjamini and Schramm in [11] and, independently, by Aldous and Steele in [3]. We will not treat this subject formally in this thesis, instead we refer to [53, Chapter 2]. As the term local convergence suggests, asymptotically, the graphs look locally like the limiting object. In [53, Theorem 3.14] it is shown that the three rank-1 models of interest for this thesis converge locally in probability to a unimodular Galton-Watson tree with offspring distribution $(p_k)_{k\in\mathbb{N}_0}$ given by

$$p_k = \mathbb{E}\left[e^{-W}\frac{W^k}{k!}\right]. \tag{3.14}$$

Before giving a definition of the unimodular Galton-Watson tree, see [53, Definition 1.26], we need to define the size-biased version X^* of a non-negative random variable X, see [53, Equation (1.4.16)].

Definition 3.9 (Size-biased distribution)**.** For a non-negative random variable X such that $0 < \mathbb{E}[X] < \infty$, a random variable X^* is distributed according to the size-biased distribution of X if

$$\mathbb{P}(X^* \leq t) = \frac{\mathbb{E}[X\mathbf{1}\{X \leq t\}]}{\mathbb{E}[X]} \quad \text{for} \quad t \in \mathbb{R}.$$

Definition 3.10 (Unimodular Galton-Watson tree)**.** For a given probability distribution $(p_k)_{k\in\mathbb{N}_0}$ of a random variable D on $\mathbb{N}_0$, the unimodular Galton-Watson tree is the branching process where the root has offspring distribution $(p_k)_{k\in\mathbb{N}_0}$, whereas all other vertices' offspring distribution is given by

$$p_k^* = \mathbb{P}(D^* = k+1).$$

Using Definition 3.9, we can calculate p_k^* in the previous definition via

$$p_k^* = \mathbb{P}(D^* = k+1) = \frac{\mathbb{E}[D\mathbf{1}\{D = k+1\}]}{\mathbb{E}[D]} = \frac{k+1}{\mathbb{E}[D]}\mathbb{P}(D = k+1). \tag{3.15}$$

In our setting, D is given by the typical degree in the rank-1 graphs, which follows a mixed Poisson distribution with parameter W as in (3.14). By definition, if one chooses a vertex uniformly at random, its degree is distributed according to D. However, if one picks one of its neighbours, this is no longer true as we obtain some information about its weight. The size-biased version turns out to be the tool that makes up for this. Plugging (3.14) into (3.15) and using $\mathbb{E}[D] = \mathbb{E}[W]$ yields

$$p_k^* = \frac{k+1}{\mathbb{E}[W]}\mathbb{E}\left[e^{-W}\frac{W^{k+1}}{(k+1)!}\right] = \frac{\mathbb{E}[e^{-W}W^{k+1}]}{\mathbb{E}[W]k!}, \tag{3.16}$$

which gives the probability that a vertex, other than the root, has k offsprings or, alternatively, degree $D^* = k+1$. We compute the average number of offsprings according to p_k^* using an index shift and monotone convergence, resulting in

$$\sum_{k=0}^{\infty} kp_k^* = \sum_{k=0}^{\infty} k\frac{\mathbb{E}\left[e^{-W}W^{k+1}\right]}{\mathbb{E}[W]k!} = \frac{1}{\mathbb{E}[W]}\mathbb{E}\left[W^2e^{-W}\sum_{k=0}^{\infty}\frac{W^k}{k!}\right] = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} = \nu. \tag{3.17}$$

This intuition also explains the phase transition at the point $\nu = 1$, as the behaviour of a Galton-Watson process changes at the point where its offspring distribution has expected value equal to one: transitioning from almost sure extinction to potential survival.

3.2 The scale-free random connection model

This section is devoted to an introduction to the scale-free random connection model, which we study in Chapter 6.

3.2.1 Definition

The locally tree-like behaviour of rank-1 models is undesirable for complex network models, as it contradicts clustering phenomena. In order to obtain clustering effects, we consider a random graph with underlying geometry. Here, we keep the weights of the rank-1 models to preserve a scale-free degree distribution. We define the model similarly to Deprez and Wüthrich in [33]. For our analysis later on, we use a construction of the model by so-called edge-marked Poisson processes, which we provide in Subsection 3.2.4. For now, we define the model in a more intuitive way.

Definition 3.11 (Weighted and scale-free random connection model)**.** The vertices of the *weighted* random connection model ξ are given by a Poisson process η on $\mathbb{R}^d$ with unit intensity for $d \in \mathbb{N}$. Let W denote a positive random variable. We equip each point $x \in \eta$ with a copy W_x of W, called the weight of x. All weights are independent of each other and of η. Given the weights, we connect any two distinct vertices $x, y \in \eta$ independently via an edge with probability

$$\mathbb{P}(x \leftrightarrow y | x, y \in \eta, W_x, W_y) = 1 - \exp\bigg(-\lambda \frac{W_x W_y}{|x-y|^\alpha}\bigg), \tag{3.18}$$

where $\alpha, \lambda > 0$ are fixed model parameters. If the weight distribution is of the form

$$\mathbb{P}(W > t) = t^{-\beta} \ell(t)$$

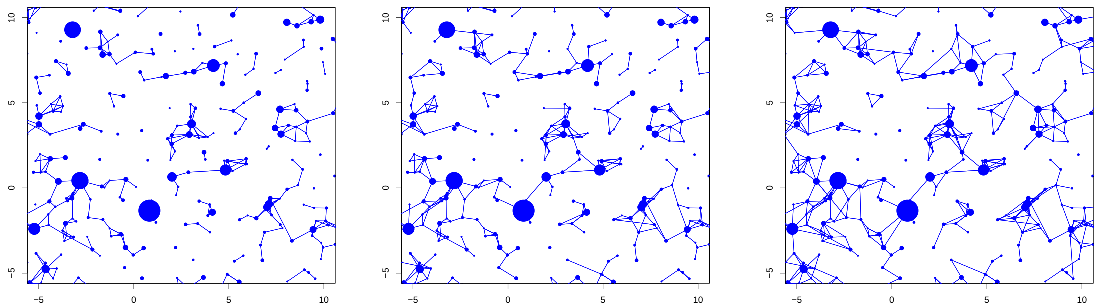
for fixed $\beta > 0$, a slowly varying function ℓ and all $t > 0$, we also call ξ *scale-free* random connection model.

The connection probability above means that vertices are more likely to connect when their weights are large and when they are close to each other. We have included realisations of the scale-free random connection model for varying parameter choices in Figure 3.1, Figure 3.2 and Figure 3.3 to get some intuition. Here, we fixed a simulation of the underlying point process η with its weights as well as a simulation of independent random variables U_{xy} which are uniformly distributed on $[0,1]$ and determine whether $x \leftrightarrow y$ for $x, y \in \eta$, according to (3.18). This allows us to see the immediate effect of the model parameters. Note that the radii of the discs around the vertices correspond to their weights. In Figure 3.1 we can observe varying choices of λ. As to be expected, an increasing value of λ yields more edges overall.

We show different values for α in Figure 3.2, which do not affect the edge probabilities as uniformly as different choices for λ do. Instead, increasing α renders long edges less likely, whereas edges between vertices being very close to each other become more likely.

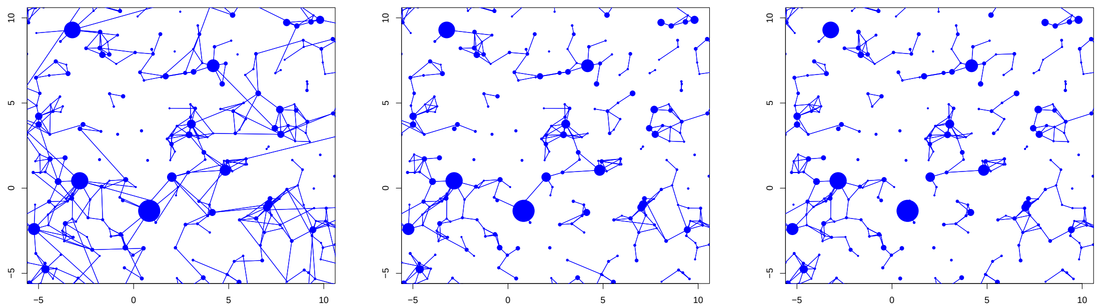
Finally, Figure 3.3 shows varying choices of β. Unlike in the previous two figures, we can no longer fix the weights of the vertices, as β affects their distribution. By fixing a seed in the generation of the model, we observe a similar pattern concerning the locations of large weights nonetheless.

We use the term *scale-free* random connection model as the respective assumptions on the weights ensure a *scale-free* degree distribution of the model. Here, we need to slightly adapt the scale-free notion from the introduction because we deal with an infinite graph. For such a graph, the proportion of vertices having a certain degree is not well defined. Neither is it possible to choose a vertex uniformly at random. Instead, one adds the vertex $\mathbf{0} \in \mathbb{R}^d$ to the graph, equips it with a weight, constructs the edges according to (3.18) and studies its degree. If the degree distribution of $\mathbf{0}$ obeys a power-law, then we call the respective graph scale-free. Intuitively, the added vertex fulfils the role of a typical vertex in the graph, matching the previous idea of choosing a vertex uniformly at random to determine the typical degree. This concept is called Palm theory and we refer to [65, Chapter 9] by Last and Penrose for more details. In the next subsection we will also discuss graph models on $\mathbb{Z}^d$, where we can directly study the degree of $\mathbf{0} \in \mathbb{Z}^d$, without the need of adding it.



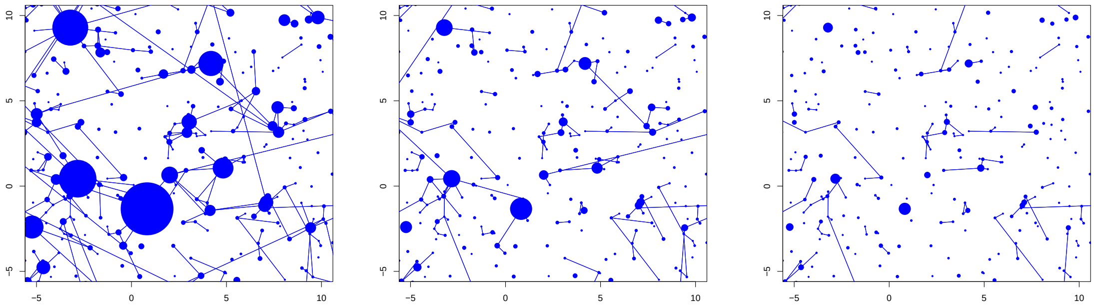
(a) $\lambda = 0.05$ (b) $\lambda = 0.1$ (c) $\lambda = 0.2$

Figure 3.1: Realisations of the scale-free random connection model for $\alpha = 6$, $\beta = 1.5$ and varying choices of λ in dimension $d = 2$, restricted to a finite observation window.



(a) $\alpha = 4$ (b) $\alpha = 6$ (c) $\alpha = 8$

Figure 3.2: Realisations of the scale-free random connection model for $\beta = 1.5$, $\lambda = 0.1$ and varying choices of α in dimension $d = 2$, restricted to a finite observation window.



(a) $\beta = 1.1$ (b) $\beta = 1.5$ (c) $\beta = 2$

Figure 3.3: Realisations of the scale-free random connection model for $\alpha = 6$, $\lambda = 0.01$ and varying choices of β in dimension $d = 2$, restricted to a finite observation window.

3.2.2 Related models

We would like to give a brief overview on related models, some of which can be seen as predecessors of the scale-free (or weighted) random connection model. To this end, we distinguish two classes of models, depending on their vertex set: $\mathbb{Z}^d$ as deterministic vertex set and point processes as random vertex sets.

Deterministic vertex sets

The first model we want to mention was introduced by Broadbent and Hammersley in [24]. The model is motivated by physics, by the way a fluid spreads through a medium to be more precise. We focus on the nearest-neighbour bond percolation although the authors discussed more general models. We consider $\mathbb{Z}^d$ as vertices and connect any point with its $2d$ nearest neighbours, i.e. all vertices that have distance one in the Euclidean sense. The obtained lattice corresponds to the regular structure of a crystal. For $p \in (0, 1)$, we remove all edges independently with probability $1 - p$. Equivalently, one can start without any edges and construct them with probability p each. The name *bond* percolation is explained by the fact that we remove some of the edges (also called bonds) of the underlying graph. A related concept is that of site percolation, where one removes vertices (also called sites) instead. Moreover, one does not need to restrict to the nearest-neighbour graph on $\mathbb{Z}^d$, but can also start with different graphs, even random graphs.

We say that the graph percolates if the origin is, with positive probability, contained in an infinite component, also called infinite cluster. From the application point of view, the connected components correspond to paths that allow for a liquid to pass through. Naturally, as p increases, percolation becomes more likely. The question of interest is at which point percolation emerges. The so-called critical probability p_c, or p_H in honour of its funder Hammersley in [49], is defined such that the graph percolates for $p > p_c$ and does not if $p < p_c$. Follow-up questions are whether the graph percolates at $p = p_c$ and if the infinite cluster is unique. We refer to the monographs [46] by Grimmet and [22] by Bollobás and Riordan for more information on percolation. For now, let us just remark on the fact that percolation in dimension $d = 1$ is trivial, as $p_c = 1$. For $d \geq 2$ however, the situation is non-trivial and there are still open problems. One of them is surprisingly easy to state: it is unknown whether there is percolation at criticality for $d \geq 2$, see in the book [50, Open Problem 1.1] by Heydenreich and van der Hofstad, where further open problems are mentioned.

In the nearest-neighbour model above, it was only possible to connect points of $\mathbb{Z}^d$ that have distance one. By long-range (bond) percolation on $\mathbb{Z}^d$ one refers to models where the probability of connecting any two vertices decays like a negative power of their distance. To be more precise, two points $x, y \in \mathbb{Z}^d$ are connected with a probability that is approximately $\delta|x-y|^{-s}$ for some $\delta > 0$. For given s, one is interested in percolation for varying values of δ. Contrary to the nearest-neighbour bond percolation above, dimension one shows a non-trivial behaviour and was studied by Aizenman and Newman in [1], Newman and Schulman in [79] and by Schulman in [89]. The uniqueness of the infinite cluster in general dimension $d \in \mathbb{N}$ was investigated by Gandolfi, Keane and Newman in [40], see also the references therein. However, in dimensions $d \geq 2$ the question of whether there is some $\delta > 0$ such that the graph percolates is answered by simply restricting to nearest-neighbour edges which already give percolation. Instead, one focuses on other quantities measuring the impact of long edges as argued in [17] by Biskup. The author studies the graph distances in long-range percolation models on $\mathbb{Z}^d$, see also the references therein for previous results in different parameter regimes. It turns out that long-range percolation models can indeed produce small worlds, i.e. comparatively short graph distances as witnessed in complex networks. However, they do not possess the desired power-law degree distribution. This problem can be solved by a similar strategy as in the previous section, attaching weights to all points in $\mathbb{Z}^d$.

There are different ways of using weights to obtain a connection function based on the vertex weights. The first model we would like to mention was introduced by Yukich in [94]. Here, the weights determine the radius of a so-called ball of influence and one connects two vertices when they lie in each other's ball of influence. After the first layer of randomness of the weights, the connections are carried out in a deterministic way. In [31] instead, Deijfen, van der Hofstad and Hooghiemstra proposed the so-called scale-free percolation model on $\mathbb{Z}^d$. Conditionally on the weights, one connects the vertices in such a way that the probability of connecting two vertices is decreasing in their distance, but increasing in their weights. More precisely, two vertices $x, y \in \mathbb{Z}^d$ with weights W_x, W_y are connected with the probability given in (3.18). The authors study the degree distribution, which turns out to obey a power-law if the weights do, percolation and graph distances. In this model, percolation is studied for given parameters d and α and varying λ. The authors also formulate related questions that might be of interest. Among these is that of percolation at criticality, which was addressed by Deprez and Wüthrich in [32] for weights following a Pareto distribution instead of a more general power-law. Another proposal in [31] is to study the continuous analogue of this version, which was then carried out by Deprez and Wüthrich in [33], where they studied the scale-free random connection model, also referred to as scale-free percolation in continuum space. In particular, the results on graph distance, percolation and scale-free degrees turn out to be similar. We provide the corresponding statements for the two latter properties in Subsection 3.2.3, as they are closely related to our question of interest in Chapter 6.

Random vertex sets

We turn the focus towards similar graph models on random vertex sets. A first example of such a spatial random graph was introduced by Gilbert in [44] and is called the Gilbert graph or random geometric graph. Here, one takes a homogeneous Poisson process η on $\mathbb{R}^d$ with unit intensity. We connect any two of its points if their distance is smaller than a given threshold $r > 0$. Concerning percolation we have the problem that the origin $\mathbf{0} \in \mathbb{R}^d$ is not necessarily part of η if we want to define it as before. Once more we make use of Palm theory and add the point $\mathbf{0}$ to η. Then we can define percolation as before, this time with respect to increasing radii r and a critical radius r_c. One can also consider other underlying point processes, such as binomial point processes or Gibbs processes. The motivation for this model in dimension $d = 2$ was given by communication networks, where one can think of the vertices as stations that can receive and send signals with range r. Then, two stations are connected if they are able to communicate.

Another way of thinking about the random geometric graphs is by placing balls of radius $r/2$ at all vertices. Then, we connect two points when their respective balls intersect. Formulated this way, we may add an additional layer of randomness by choosing random radii. This leads to the so-called (Poisson) Boolean model introduced rigorously in [47] by Hall. Consider a homogeneous Poisson process with intensity $\lambda > 0$ and equip its points x with independent weights W_x according to some weight distribution. We connect two vertices x and y when $|x - y| \leq W_x + W_y$. In this model, one studies percolation with respect to increasing λ. As for the nearest-neighbour bond percolation on $\mathbb{Z}^d$, one observes a trivial behaviour in dimension one, whereas higher dimensions show a non-trivial phase transition as shown by Hall in the aforementioned paper. This phenomenon is subject to some moment conditions on the radii, which intuitively prohibit that some large balls cover the whole space. Uniqueness of the infinite cluster was studied by Meester and Roy in [72].

One may ask the question whether one can also consider a varying intensity, as in the Boolean model, for the Gilbert graph. However, by rescaling, this amounts to changing the connection threshold r. Moreover, one can think of placing more arbitrary sets around the points and not restrict them to balls. Finally, instead of studying the underlying graph, one can also stick to the union of balls and study its geometric properties, e.g. covering probabilities of some given set $A \subseteq \mathbb{R}^d$ or (intrinsic) volumes. For more information in this direction we refer the reader to the monograph [88] by Schneider and Weil and the references therein. More details concerning the Boolean model can be found in the textbooks [73] by Meester and Roy as well as [48] by Hall. The book [83] offers a comprehensive treatment of random geometric graphs.

So far, the connections were constructed in a deterministic way, once the points and the balls around them were given. One way to change this is to consider continuum versions of long-range percolation of $\mathbb{Z}^d$, where the connection probability of two points was depending on their distance. Such a model was introduced in [82] by Penrose and is known as random connection model. Here, the underlying point process η is not equipped with weights, but connections are established with probability given by $g(|x - y|)$ for two points x, y and a connection function g. As in the version on $\mathbb{Z}^d$ and the random geometric graph, the random connection model undergoes a phase transition, under some assumptions. For detailed information on the random connection model we refer once more to the monograph [73] by Meester and Roy. In the case where the connection function satisfies $g(x) = \mathbf{1}\{|x| \leq r\}$ for $x \in \mathbb{R}^d$, we obtain the random geometric graph as a special case. However, the connection function g usually does not jump from 1 to 0 at some point but interpolates, which is why the random connection model is also referred to as *soft* random geometric graph.

Another related model is the so-called soft Boolean model introduced by Gracar, Grauer and Mörters in [45]. Of particular interest for us are the results from Jahnel, Lüchtrath and Ortgiese in [56] for this model, as they obtain bounds on component sizes, which are related to our research in Chapter 6. We postpone the discussion of their model and its relation to the scale-free random connection model to the end of the next section.

3.2.3 Properties

In this subsection we provide some basic properties concerning the degree distribution and the phase transition, before getting to more technical results in the next subsection. Recall that we add the origin $\mathbf{0}$ to the graph. We denote the probability distribution of the respective Palm measure by $\mathbb{P}_{\mathbf{0}}$. The results we state are due to [33] by Deprez and Wüthrich. Throughout this section, let the weights follow a Pareto distribution with scale parameter $\beta > 0$, i.e. they obey a pure power-law of the form

$$\mathbb{P}(W > t) = t^{-\beta}, \quad \text{for } t \geq 1. \tag{3.19}$$

The following theorem is [33, Theorem 3.1].

Theorem 3.12. *Consider the scale-free random connection model with added origin and weights as in* (3.19)*.*

a) *If $\alpha \leq d$ or $\alpha\beta \leq d$, then $\mathbb{P}_{\mathbf{0}}(\deg(\mathbf{0}) = \infty) = 1$.*

b) *If $\alpha > d$ and $\alpha\beta > d$, then the tail of $\deg(\mathbf{0})$ is regularly varying with index $-\alpha\beta/d$.*

The second part of the previous theorem ensures that the model is scale-free as desired, when the weights obey a power-law. To allow for more flexibility, we will allow more general weights with regularly varying tail in Chapter 6. The phase transition was studied in [33, Theorem 3.2] and is as in the theorem below, where λ_c denotes the critical parameter at which percolation starts appearing.

Theorem 3.13. *Assume that $\alpha > d$ and $\alpha\beta > d$ for the scale-free random connection model with weights as in* (3.19)*.*

1. *For $d \geq 2$ we have*

    a) *$\lambda_c = 0$ for $\alpha\beta < 2d$ and*

    b) *$\lambda_c \in (0, \infty)$ for $\alpha\beta > 2d$.*

2. *In dimension $d = 1$ it holds that*

    a) *$\lambda_c = 0$ for $\alpha\beta < 2$,*

    b) *$\lambda_c \in (0, \infty)$ for $\alpha\beta > 2$ and $\alpha \in (1, 2]$ as well as*

    c) *$\lambda_c = \infty$ for $\alpha > 2$ and $\alpha\beta > 2$.*

Recall that there is no percolation for $\lambda < \lambda_c$, but for all $\lambda > \lambda_c$ there is. Regarding percolation at criticality, the cases $\lambda_c = 0$ and $\lambda_c = \infty$ are trivial. Thus, only $\lambda_c \in (0, \infty)$ is interesting for this question. It was partially solved in [33, Theorem 3.3 and the discussion before], where it is stated that there is no infinite component at $\lambda = \lambda_c$ when $d = 1$ or when both $\alpha \in (d, 2d)$ and $\alpha\beta > 2d$ for $d \geq 2$. The case $\alpha \geq 2d$ and $\alpha\beta > 2d$ is still open in dimension $d \geq 2$ to the best of our knowledge.

The transition between the different states, especially the cases $\lambda_c = 0$ and $\lambda_c = \infty$, fit to the intuition concerning our model parameters. Increasing values of α punish long edges, so that larger values of α should make it more difficult to percolate, as is the case. Similarly, increasing values of β result in smaller weights, which also yield fewer connections. A further comment concerns the introductory assumption that $\alpha > d$ and $\alpha\beta > d$. By Theorem 3.12, we see that the degrees are almost surely infinite if we drop either of these conditions. Therefore, one has $\lambda_c = 0$ in that case.

In Chapter 6 we will study the size of the largest component we can find in some growing sequence of observation windows. For $\lambda_c = 0$, this does not make any sense as not all components are of finite size. Therefore, we will use assumptions corresponding to the ones above that ensure $\lambda_c > 0$. Then, we will treat choices of λ that satisfy $\lambda < \lambda_c$. Unfortunately, we are not able to consider the whole subcritical regime, i.e. there are some choices of $\lambda < \lambda_c$ we are not able to treat.

In preparation for the component sizes in Chapter 6, we would like to discuss a related result on component sizes in soft Boolean models. The models are reasonably similar, although they cannot be directly translated into one another. Nonetheless, we would like to compare them and use an unjustified approximation to guess parameter choices for the scale-free random connection model that are necessary for finite second moments of the component size. We are interested in these, as we use second moment methods later on. In particular, assumptions that are needed for finite second moments of component sizes cannot be relaxed further without changing our general strategy. In some sense, this allows us to discuss how restrictive our assumptions are.

Therefore, we will present one particular result from [56] by Jahnel, Lüchtrath and Ortgiese here in more detail. In their version of the soft Boolean model, one considers a homogeneous Poisson process η with unit intensity on $\mathbb{R}^d$ and equips all vertices with independent weights following a Pareto distribution of the form

$$\mathbb{P}(W > t) = \min\left(1, t^{-1/\gamma}\right)$$

for $\gamma \in (0, 1)$ and $t > 0$. Then, they endow each possible edge between two vertices x, y with weights $W(x, y)$ satisfying

$$\mathbb{P}(W(x, y) > t) = \min\left(1, t^{-\delta}\right)$$

for $\delta > 1$ and $t > 0$. The collection of η, $(W_x)_{x\in\eta}$ and $(W(x, y))_{x,y\in\eta}$ is jointly independent. Finally, we connect any two distinct vertices $x, y \in \eta$ if

$$|x - y|^d \leq \kappa W(x, y) \min(W_x, W_y)$$

for some $\kappa > 0$. An immediate difference is that one takes the minimum of the vertex weights instead of their product. However, in [56, Theorem 2.3] the authors also obtain results for the usage of the product kernel, i.e. when connecting two vertices x, y if

$$|x - y|^d \leq \kappa W(x, y) W_x W_y. \tag{3.20}$$

We wish to compare this, on a heuristic level, to the connection probability in (3.18) that we use. According to (3.20), given the weights of two vertices x, y their connection probability is given by

$$\mathbb{P}\Big(W(x,y) \geq \frac{|x-y|^d}{\kappa W_x W_y}\Big| x, y \in \eta, W_x, W_y\Big) = \min\Big(1, \Big(\frac{\kappa W_x W_y}{|x-y|^d}\Big)^\delta\Big) \\ \approx 1 - \exp\Big(-\kappa^\delta \frac{W_x^\delta W_y^\delta}{|x-y|^{\delta d}}\Big),$$

where the last approximation is similar to the one for the comparison of the Chung-Lu model and the Norros-Reittu model. In particular, it is reasonably sharp for vertices that are far apart from each other. If we match the term on the right-hand side to our connection probability, we can translate their parameter choices to ours and vice versa. Naturally, this is no rigorous comparison, but it might give us a guess what to expect from our model. For convenience, recall our connection probability

$$\mathbb{P}(x \leftrightarrow y | x, y \in \eta, W_x, W_y) = 1 - \exp\Big(-\lambda \frac{W_x W_y}{|x-y|^\alpha}\Big),$$

so that $\lambda = \kappa^\delta, \alpha = \delta d$ and the power-law exponent of our weights satisfies $\beta = 1/(\gamma\delta)$. Jahnel, Lüchtrath and Ortgiese deduce the following in [56, Theorem 2.3].

Theorem 3.14. *Consider the soft Boolean model with $d \geq 1, \delta > 1$ and $0 < \gamma < 1/2$. Then, there exists some $\tilde{\kappa}_c > 0$ such that for all $0 < \kappa < \tilde{\kappa}_c$ there are constants $c, C \in (0, \infty)$ such that for all $m > 1$,*

$$cm^{1-1/\gamma} \leq \mathbb{P}_{\mathbf{0}}(|\mathcal{C}(\mathbf{0})| > m) \leq Cm^{1-1/\gamma}.$$

With

$$\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|^2\big] = \int_0^\infty \mathbb{P}_{\mathbf{0}}\big(|\mathcal{C}(\mathbf{0})|^2 > t\big)dt = \int_0^\infty \mathbb{P}_{\mathbf{0}}\big(|\mathcal{C}(\mathbf{0})| > \sqrt{t}\big)dt$$

we obtain a finite second moment of $|\mathcal{C}(\mathbf{0})|$ if $1/2 - 1/(2\gamma) < -1$. This is equivalent to $\gamma < 1/3$. From the relation $\beta = 1/(\gamma\delta)$ and $\alpha = \delta d$ above, we obtain

$$\gamma < \frac{1}{3} \iff \frac{d}{\alpha\beta} < \frac{1}{3} \iff \alpha\beta > 3d,$$

which in turn implies $\mathbb{E}[W^{3d/\alpha}] < \infty$. The latter will be one of our assumptions in Chapter 6. As argued above, we do not believe that this can be relaxed further without drastically changing the proof strategy.

3.2.4 Formal construction and technical properties

In order to construct the weighted random connection model formally, we take an intermediate step. Consider a Poisson process $\hat{\eta}$ on $\mathbb{R}^d \times [0, \infty)$ with intensity measure $\lambda_d \otimes \mathbf{Q}$, where $\mathbf{Q}$ denotes the weight distribution. We think of a point $\hat{x} = (x, W_x) \in \hat{\eta}$ as a point at location x with weight W_x. We use the notation $\hat{\cdot}$ throughout this thesis to indicate a pair of location and weight, whereas the respective quantity without hat refers to the projection on the spatial component. This applies to both $\hat{x}, x$ and $\hat{\eta}, \eta$.

Including the weights in the Poisson process has the advantage that we can apply the strong machinery from Poisson processes. Recall that the index $\mathbf{0}$ indicates the Palm distribution, where the point $\mathbf{0}$ was added. Naturally, we also add a weight $W_{\mathbf{0}}$, so that we should rather write $\mathbb{P}_{\hat{\mathbf{0}}}$ instead. In order to keep the notation simple, we omit the hats in the indices. If we are interested, for example, in the expected degree of $\mathbf{0}$, conditionally on its weight $W_{\mathbf{0}}$, we were to compute

$$\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}}\big] = \mathbb{E}_{\mathbf{0}}\bigg[\sum_{\hat{x}\in\hat{\eta}} \mathbf{1}\{x \leftrightarrow \mathbf{0}\}\bigg|W_{\mathbf{0}}\bigg].$$

Unfortunately, we cannot directly use the standard Mecke equation, see Theorem 2.4. This is due to the fact that the edges in the weighted random connection model incorporate randomness that exceeds the randomness of the underlying Poisson process $\hat{\eta}$. A solution is given by going to an even larger point process, which also includes the information on the edges. The resulting object is called an edge-marked Poisson process. This approach was already used for the random connection model without weights by Last and Ziesche in [66], see also [51] by Heydenreich, van der Hofstad, Last and Matzke. As mentioned above, the additional dimension for the weights can be included in a straightforward way. Last, Nestmann and Schulte provide a setup of the random connection model in a general Borel space in [64], where they prove in particular an analogue of the Poincaré inequality that we will use later on. For now, let us focus on the construction.

The Poisson process $\hat{\eta}$ from above is a proper point process, so that we may write it as $\hat{\eta} = \{\hat{x}_n : n \in \mathbb{N}\}$ and denote the lexicographic order on its points by $\prec$. Recall that the space $M_p(E)$ denotes the space of all σ-finite Radon measures on some complete and separable metric space $(E, \mathcal{E})$. We may choose $E = \mathbb{R}^d \times [0, \infty)$ with the usual topology as subspace of $\mathbb{R}^{d+1}$, corresponding to the space on which $\hat{\eta}$ lives. We consider $E^{[2]} = \{e \in M_p(E) : e(E) = 2\}$, the space containing all possible edges in the weighted random connection model, and consider a double sequence $(U_{m,n})_{m,n\in\mathbb{N}}$ of independent random variables uniformly distributed on $[0, 1]$ that are also independent of $\hat{\eta}$. Then,

$$\xi = \{((\hat{x}_m, \hat{x}_n), U_{m,n}) : \hat{x}_m \prec \hat{x}_n, m, n \in \mathbb{N}\}$$

is a point process on $E^{[2]} \times [0, 1]$, which we can interpret as the weighted random connection model ξ from Definition 3.11 as follows. The points $(x_m)_{m\in\mathbb{N}}$ provide the vertices, whereas we connect any pair of distinct vertices x_m, x_n with $m, n \in \mathbb{N}$ if

$$U_{m,n} \leq 1 - \exp\bigg(- \lambda \frac{W_x W_y}{|x - y|^\alpha}\bigg),$$

which gives us precisely our connection rule from (3.18). The name edge-marked Poisson process originates from the points $((\hat{x}_m, \hat{x}_n), U_{m,n}) \in \xi$, which correspond to a possible edge with a mark. With the framework in place, we can present the Mecke equation and the Poincaré inequality for edge-marked Poisson processes, see Theorem 2.4 as well as Theorem 2.5 for their standard versions for Poisson processes. We use notation similar to the one in [51]. For points $\hat{x}_0, \hat{x}_{-1} \in \mathbb{R}^d \times [0, \infty)$ we write $\hat{\eta}^{x_0}$ and $\hat{\eta}^{x_0,x_{-1}}$ to denote the underlying vertex sets with added points $\hat{x}_0$ and $\hat{x}_0, \hat{x}_{-1}$, respectively, where we once more omit the hats in the exponent for a better readability. In terms of the weighted random connection model ξ, we extend the double sequence $(U_{m,n})$ to negative indices and obtain, for example

$$\xi^{x_0,x_{-1}} = \{((\hat{x}_m, \hat{x}_n), U_{m,n})\colon \hat{x}_m \prec \hat{x}_n, m, n \in \mathbb{Z}, m, n \geq -1\}.$$

Similarly, we may add any finite number of vertices. As for the Palm measure, we add indices such as $\mathbb{P}_{\mathbf{0},x_0}$ or $\mathbb{E}_{x_0,x_{-1}}$ to indicate which points were added. The Mecke equation now states the following, see also [25, Equation (3.1)].

Lemma 3.15. *For $n \in \mathbb{N}$ and a measurable function $f\colon (E^{[2]} \times [0,1]) \times E^n \to [0,\infty]$ it holds*

$$\mathbb{E}\Big[\sum_{(\hat{x}_1,\dots,\hat{x}_n)\in\hat{\eta}^n_{\neq}} f(\xi, \hat{x}_1, \dots, \hat{x}_n)\Big] = \int_{(\mathbb{R}^d)^n} \mathbb{E}_{x_1,\dots,x_n}[f(\xi^{x_1,\dots,x_n}, \hat{x}_1, \dots, \hat{x}_n)]d(x_1, \dots, x_n).$$

In the equation above, the expected value is also to be taken with respect to the random weights $W_{x_1}, \dots, W_{x_n}$. Sometimes, we want to use the Mecke formula in a situation where we already added a point to ξ, i.e. where we consider $f(\xi^y, \hat{x}_1, \dots, \hat{x}_n)$ on the left-hand side above. This can be achieved by considering the additional point $\hat{y}$ together with another sequence of random variables $(U_n)_{n\in\mathbb{N}}$, which keeps track of the connections between $\hat{y}$ and the points of $\hat{\eta}$. A rigorous construction can be found in the proof of [25, Lemma 6.2]. For given $\hat{y} \in \mathbb{R}^d \times [0,\infty)$ this yields, see also [25, Equation 3.4],

$$\mathbb{E}_y\Big[\sum_{(\hat{x}_1,\dots,\hat{x}_n)\in\hat{\eta}^k_{\neq}} f(\xi^y, \hat{x}_1, \dots, \hat{x}_n)\Big] = \int_{(\mathbb{R}^d)^n} \mathbb{E}_{y,x_1,\dots,x_n}[f(\xi^{y,x_1,\dots,x_n}, \hat{x}_1, \dots, \hat{x}_n)]d(x_1, \dots, x_n). \tag{3.21}$$

This can be generalised to the case where already finitely many distinct points $\hat{y}_1, \dots, \hat{y}_k$ were added, using the same strategy as above.

The Poincaré inequality for edge-marked Poisson processes is given in [64, Theorem 5.2]. Let $f\colon M_p(E^{[2]} \times [0,1]) \to \mathbb{R}$ be measurable and such that $\mathbb{E}[f(\xi)^2] < \infty$, then

$$\mathbf{Var}(f(\xi)) \leq \int_{\mathbb{R}^d} \mathbb{E}\big[(\Delta_x f(\xi))^2\big]dx. \tag{3.22}$$

In this setting, the difference operator Δ_x is to be understood as $\Delta_x f(\xi) = f(\xi^x) - f(\xi)$, where an additional vertex x with mark W_x is added to the underlying Poisson process $\hat{\eta}$. Note that the expectation on the right-hand side above is also with respect to W_x. This means that we add a point $\hat{x}$ with deterministic location but random mark as in $\mathbb{E}_x$. We refrain from using the notation $\mathbb{E}_x$ on the right-hand side above as the difference operator also considers the weighted random connection model ξ where no point was added.

We will also use this formula for the point process ξ^y, where a point $\hat{y} \in \mathbb{R}^d \times [0,\infty)$ was added to $\hat{\eta}$ and we condition on its weight. We can think of ξ^y as the point process ξ constructed from $\hat{\eta}$ with another dimension, namely where each point $\hat{x} \in \hat{\eta}$ carries an additional mark $U_x \sim \text{Uniform}([0,1])$ to determine whether x and y are connected. Since we condition on the weight of y, there is no randomness outside of this point process involved. Therefore, an application of the Poincaré inequality in (3.22) yields the following lemma.

Lemma 3.16. *For all measurable* $f\colon M_p(E^{[2]}\times[0,1])\to\mathbb{R}$ *with* $\mathbb{E}[f(\xi^y)^2]<\infty$ *we have*

$$\mathbf{Var}_y(f(\xi^y)|W_y)\le\int_{\mathbb{R}^d}\mathbb{E}_y\big[(\Delta_x f(\xi^y))^2\big|W_y\big]\,dx. \tag{3.23}$$

Chapter 4

Large components in the Norros-Reittu model

In this chapter we discuss results on the component sizes and other vertex counts for subcritical rank-1 models. We start by providing an abstract point process convergence result in Section 4.1 for the Norros-Reittu model, which immediately yields a corollary concerning the maximum of the point process. After proving the aforementioned result in Section 4.2, we investigate more explicit settings such as the size of the largest component or the maximal number of leaves in a single component in Section 4.3. At this point, we also discuss related results and give some intuition by means of local weak convergence. The proofs of the claims in Section 4.3 are postponed to Section 4.4. We finish the chapter by transferring our results to the Chung-Lu model and the generalised random graph in Section 4.5.

The results and proofs of this chapter are taken from [69] by Lienau and Schulte.

4.1 Main result

We start by providing a result for the Norros-Reittu model. To this end, we require the following assumption on the weight distribution throughout this chapter:

(W) The distribution of the positive random variable W has a regularly varying tail with index $-\beta$ for $\beta > 2$ and satisfies $\mathbb{E}[W^2] < \mathbb{E}[W]$.

Recall that the regularly varying tail ensures the desired power-law of the degrees, whereas $\mathbb{E}[W^2] < \mathbb{E}[W]$ corresponds to the subcritical regime of the graph. Note that $\beta > 2$ in assumption (W) implies the finiteness of the second moment of W as stated in Proposition 2.17. We denote the component of a vertex $x \in [n]$ by $\mathcal{C}_n(x)$, i.e. the set of all vertices in the component of x.

For all $n \in \mathbb{N}$ and realisations G_n of the Norros-Reittu model, we consider a function $(v, G_n) \mapsto \mathcal{X}_n(v) \subseteq [n]$ for $v \in [n]$ such that $\mathcal{X}_n(v) \subseteq \mathcal{C}_n(v)$. For $n \in \mathbb{N}$ and $v \in [n]$, $\mathcal{X}_n(v)$ denotes the vertices we would like to count in $\mathcal{C}_n(v)$. We write $S_n(v)$ for the cardinality of $\mathcal{X}_n(v)$ and recall the quantity $q(n)$ defined before Proposition 2.19. We require the following two assumptions.

(A1) There exists some $\zeta > 0$ such that

$$\frac{1}{q(n)} \sup_{v=1,\dots,n} |\mathbb{E}_{\mathcal{W}}[S_n(v)] - W_v\zeta| \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty.$$

(A2) Let $v \in [n]$ and $x \in \mathcal{C}_n(v)$. Checking whether $x \in \mathcal{X}_n(v)$ only depends on all paths that start in v and contain x.

Concerning (A2) we want to note that in this thesis a path consists of distinct vertices connected by edges, as usual.

Finally, we choose from every component in G_n one vertex with the largest weight. If the largest weight is not unique, we choose the one with the smallest label. The collection of these vertices with maximal weight is denoted by $V_n^{\max} \subseteq [n]$. We study the point process

$$\Xi_n = \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}\} \delta_{S_n(v)q(n)^{-1}\zeta^{-1}}$$

for $n \in \mathbb{N}$, where the constant $\zeta > 0$ is specified in assumption (A1). The indicator $v \in V_n^{\max}$ ensures that we consider each component exactly once in Ξ_n. For fixed $v \in [n]$ and $x \in \mathcal{C}_n(v)$, note that $S_n(v)$ and $S_n(x)$ do not need to coincide, even though v and x lie in the same component. In Section 4.3 we provide several possible choices for $\mathcal{X}_n(v)$ and thus also for $S_n(v)$, some of which depend on the choice of the component's representative.

Our main theorem of this chapter concerns weak convergence of Ξ_n in $M_p((0, \infty])$, see Subsection 2.2.3 for more information on this topic.

Theorem 4.1. *For $n \in \mathbb{N}$, consider the Norros-Reittu model* NR(n) *with weights satisfying assumption* (W) *and $(\mathcal{X}_n(v))_{v\in[n]}$ such that assumptions* (A1) *and* (A2) *hold. Then*

$$\Xi_n \xrightarrow{d} \eta_\beta \quad \text{as} \quad n \to \infty, \tag{4.1}$$

where η_β is a Poisson process with intensity measure ν_β satisfying $\nu_\beta((a, b]) = a^{-\beta} - b^{-\beta}$ for all $0 < a < b \leq \infty$.

Note that the points of Ξ_n at zero are not taken into account in (4.1) as we work on the space of point processes on $(0, \infty]$. For some intuition on the proof strategy, we refer to the start of the next section. There, we sketch our approach before carrying out the details. The point process convergence above allows us to employ Lemma 2.12 in order to deduce the following asymptotic behaviour concerning the maximum of Ξ_n.

Corollary 4.2. *Under the same assumptions as in Theorem 4.1,*

$$\frac{1}{q(n)\zeta} \max_{v \in V_n^{\max}} S_n(v) \xrightarrow{d} Z_\beta \quad \text{as} \quad n \to \infty, \tag{4.2}$$

where Z_β is a random variable following a Fréchet distribution with parameter β.

4.2 Proof of the main result

The key idea to prove Theorem 4.1 is to show that Ξ_n is close to $\Theta_n = \sum_{v=1}^n \delta_{W_v q(n)^{-1}}$, the collection of rescaled weights, which converges to η_β as $n \to \infty$, see part a) of Lemma 2.20. Formally, we apply Lemma 2.11 to obtain the desired result. Note that the point process Θ_n has the significant advantage that its points are independent and identically distributed. In order to show that Ξ_n and Θ_n behave similarly, we control the difference

$$S_n(v) - W_v\zeta = S_n(v) - \mathbb{E}_{\mathcal{W}}[S_n(v)] + \mathbb{E}_{\mathcal{W}}[S_n(v)] - W_v\zeta =: R_1 + R_2.$$

While assumption (A1) takes care of R_2, we will apply the Chebyshev inequality to address R_1. Therefore, we need to control the variance of $S_n(v)$. To this end, we use the Poissonian nature of the Norros-Reittu model which allows us to employ the Poincaré inequality for Poisson functionals, see Theorem 2.5. Here, assumption (A2) will be used to control the effect of the difference operator $D_{\{i,j\}}$ on $S_n(v)$, i.e. the change of $S_n(v)$ when we add an edge between the vertices $i, j \in [n]$ if it did not already exist and 0, otherwise.

Technical details aside, a similar strategy was already employed in [14] by Bhattacharjee and Schulte to study large degrees of various random graphs. However, our analysis is more involved as the considered statistics of the components are less local than degrees. If one thinks of the variance bound mentioned above, the difference operator $D_{\{i,j\}} \deg(v)$ for some vertex v is trivially bounded by one. For, say, its component size instead, an additional edge may provide a link to a bigger cluster, making it more difficult to bound $D_{\{i,j\}}\mathcal{C}_n(v)$. Moreover, verifying assumption (A1) is also much simpler for the degree than for the component size.

We start with discussing convergence of often occurring series. We typically approximate the number of vertices in the set $\mathcal{X}_n(v_0)$ for $v_0 \in [n]$ by counting paths originating in v_0 and leading to vertices in $\mathcal{X}_n(v_0)$, resulting in a sum of the following kind,

$$T_n(v_0) = \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\}\mathbf{1}\{v_k \in \mathcal{X}_n(v_0)\}, \tag{4.3}$$

where $x \leftrightarrow y$ means that the vertices x and y are connected by an edge. We first sum over the length of the path starting in v_0 and then over its vertices, which we demand to be distinct. Recall that $\mathbb{E}_{\mathcal{W}}$ denotes the conditional expectation with respect to the weights. Since we only consider the Norros-Reittu model for now, we use $\mathbb{P}_{\mathcal{W}}$ as a short-hand notation for $\mathbb{P}_{\mathcal{W},NR}$ when referring to connection probabilities. Writing $L_n^{[2]} = \sum_{v=1}^n W_v^2$ and using the upper bound one for the last indicator in (4.3), conditional independence of the remaining indicators leads to

$$\begin{aligned}\mathbb{E}_{\mathcal{W}}[T_n(v_0)] &\le \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n]^k_{\neq}} \prod_{i=1}^{k} \mathbb{P}_{\mathcal{W}}(v_i \leftrightarrow v_{i-1}) \le \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n]^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n}\\ &\le W_{v_0} \sum_{k=1}^{n} \sum_{v_1,\dots,v_k=1}^{n} \prod_{i=1}^{k-1} \frac{W_{v_i}^2}{L_n} \times \frac{W_{v_k}}{L_n} = W_{v_0} \sum_{k=0}^{n-1} \left(\frac{L_n^{[2]}}{L_n}\right)^k =: W_{v_0} S_{n,\mathcal{W}},\end{aligned}$$

where we used Lemma 3.5 to bound the connection probabilities. From the strong law of large numbers it follows that

$$\frac{L_n^{[2]}}{L_n} = \frac{n^{-1}L_n^{[2]}}{n^{-1}L_n} \xrightarrow{a.s.} \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} \quad \text{as} \quad n \to \infty. \tag{4.4}$$

The geometric sum formula together with $\mathbb{E}[W^2] < \mathbb{E}[W]$ by assumption (W) yields

$$S_{n,\mathcal{W}} = \frac{1 - \left(L_n^{[2]}/L_n\right)^n}{1 - L_n^{[2]}/L_n} \xrightarrow{a.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[W] - \mathbb{E}[W^2]} \quad \text{as} \quad n \to \infty. \tag{4.5}$$

This argument also provides almost sure convergence for related expressions of the form

$$\sum_{k=0}^{n} p(k) \left(\frac{L_n^{[2]}}{L_n}\right)^k, \tag{4.6}$$

where p is an arbitrary polynomial of finite degree. Note that the limit in (4.4) equals the mean of the offspring distribution of the local limit, see also (3.17). From the calculation above one can also conclude that the graph is subcritical when $\mathbb{E}[W^2] < \mathbb{E}[W]$.

For $n \in \mathbb{N}, x, y \in [n]$ with $x \neq y,\ A \subseteq [n]$ and $k \geq 2$ we write

$$\begin{aligned}\mathcal{P}_k^{(n)}(x, y, A) &= \{(v_1, \ldots, v_k) \in [n]_{\neq}^k \colon v_1 = x, v_k = y, v_2, \ldots, v_{k-1} \notin A, v_1 \leftrightarrow \ldots \leftrightarrow v_k\},\\ \mathcal{P}_k^{(n)}(x, A) &= \{(v_1, \ldots, v_k) \in [n]_{\neq}^k \colon v_1 = x, v_2, \ldots, v_k \notin A, v_1 \leftrightarrow \ldots \leftrightarrow v_k\}.\end{aligned}$$

The first set consists of all k-tuples of vertices that form a path with endpoints x and y whose inner vertices do not lie in A. Similarly, the second set consists of all k-tuples which form a path whose starting vertex is x and whose remaining vertices do not belong to A. We often require bounds for the expected number of such paths, conditionally on the weights. To this end, we derive the following lemma.

Lemma 4.3. *Let $x, y \in [n]$ with $x \neq y,\ A \subseteq [n]$ and $k \geq 2$. Then*

$$\mathbb{E}_{\mathcal{W}}[|\mathcal{P}_k^{(n)}(x, y, A)|] \leq \frac{W_x W_y}{L_n} \left(\frac{L_n^{[2]}}{L_n}\right)^{k-2} \quad \textit{and} \quad \mathbb{E}_{\mathcal{W}}[|\mathcal{P}_k^{(n)}(x, A)|] \leq W_x \left(\frac{L_n^{[2]}}{L_n}\right)^{k-2}.$$

Proof. It suffices to prove the first assertion as the second one follows immediately from the former by summing over $y \in [n] \setminus \{x\}$ and using $\sum_{y \in [n] \setminus \{x\}} W_y \leq L_n$. As all edges in the paths are distinct and therefore conditionally independent we obtain

$$\begin{aligned}\mathbb{E}_{\mathcal{W}}[|\mathcal{P}_k^{(n)}(x, y, A)|] &\leq \mathbb{E}_{\mathcal{W}}\Bigg[\sum_{(v_1, \ldots, v_k) \in [n]_{\neq}^k} \mathbf{1}\{v_1 = x, v_k = y\} \prod_{i=2}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\}\Bigg]\\ &= \sum_{(v_1, \ldots, v_k) \in [n]_{\neq}^k} \mathbf{1}\{v_1 = x, v_k = y\} \prod_{i=2}^{k} \mathbb{P}_{\mathcal{W}}(v_i \leftrightarrow v_{i-1})\\ &\leq \sum_{(v_1, \ldots, v_k) \in [n]_{\neq}^k} \mathbf{1}\{v_1 = x, v_k = y\} \prod_{i=2}^{k} \frac{W_{v_i} W_{v_{i-1}}}{L_n}\end{aligned}$$

$$\leq \frac{W_x W_y}{L_n}\bigg(\mathbf{1}\{k=2\} + \sum_{v_2,\dots,v_{k-1}=1}^{n} \prod_{i=2}^{k-1} \frac{W_{v_i}^2}{L_n}\bigg) = \frac{W_x W_y}{L_n}\bigg(\frac{L_n^{[2]}}{L_n}\bigg)^{k-2},$$

which is the desired inequality. □

The next lemma essentially says that vertices with large weights are typically not connected or, equivalently, every component has at most one vertex with large weight. This may seem counterintuitive as vertices with larger weights are more likely to connect. However, in the subcritical regime, there are few paths between any two given vertices $x, y \in [n]$, see Lemma 4.3. Moreover, there are very few vertices with *large* weights. Thus, they will in fact not be connected.

Lemma 4.4. *For $a > 0$ and $n \in \mathbb{N}$ define the event*

$$\mathcal{A}_n = \{\exists x, y \in [n]\colon x \neq y, x \in \mathcal{C}_n(y), W_x \geq W_y > aq(n)\}^c.$$

Under assumption (W) *we have $\mathbb{P}(\mathcal{A}_n) \to 1$ as $n \to \infty$.*

Proof. We have

$$\mathbb{P}(\mathcal{A}_n^c) = \mathbb{E}\big[\mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{A}_n^c}]\big] = \mathbb{E}\big[\min(1, \mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{A}_n^c}])\big],$$

so that it suffices to show that $\mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{A}_n^c}]$ converges in probability to zero as $n \to \infty$. Using Lemma 4.3 and its notation, we have

$$\begin{aligned}
\mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{A}_n^c}] &\leq \sum_{x,y=1}^{n} \mathbf{1}\{W_x \geq W_y > aq(n)\} \sum_{k=2}^{n} \mathbb{E}_{\mathcal{W}}\big[\big|\mathcal{P}_k^{(n)}(x, y, \emptyset)\big|\big] \\
&\leq \sum_{x,y=1}^{n} \mathbf{1}\{W_x \geq W_y > aq(n)\} \sum_{k=2}^{n} \frac{W_x W_y}{L_n}\bigg(\frac{L_n^{[2]}}{L_n}\bigg)^{k-2} \\
&\leq \frac{n}{L_n}\bigg(n^{-1/2} \sum_{x=1}^{n} \mathbf{1}\{W_x > aq(n)\} W_x\bigg)^2 S_{n,\mathcal{W}}. \qquad (4.7)
\end{aligned}$$

By (4.5), $S_{n,\mathcal{W}}$ converges almost surely to a constant as $n \to \infty$, as does n/L_n by the strong law of large numbers. We focus on the expectation of the sum, which gets squared above. It holds

$$n^{-1/2}\mathbb{E}\bigg[\sum_{x=1}^{n} \mathbf{1}\{W_x > aq(n)\} W_x\bigg] = n^{1/2}\mathbb{E}[\mathbf{1}\{W > aq(n)\} W]. \qquad (4.8)$$

By Lemma 2.18 we know that $t \mapsto \mathbb{E}[\mathbf{1}\{W > t\} W]$ is regularly varying with index $1 - \beta$. Since $n \mapsto aq(n)$ is regularly varying with index $1/\beta$, see Proposition 2.19, we obtain from Proposition 2.15 a) and c) that (4.8), as a function in t, is regularly varying with index $1/\beta - 1/2$. Since $\beta > 2$, this term is negative. Therefore, the squared sum in (4.7) converges in probability to zero as $n \to \infty$ by Proposition 2.15 b). Altogether, we can apply Slutsky's lemma to derive that the whole expression in (4.7) converges in probability to zero as $n \to \infty$. The assertion follows. □

We introduce a slightly different variant of assumption (A2) here.

(A2') Let $v \in [n]$ and $x \in \mathcal{C}_n(v)$. Then, $v \notin \mathcal{X}_n(v)$ and checking whether $x \in \mathcal{X}_n(v)$ only depends on all paths that start in v and contain x.

The difference is that we do not allow v itself to lie in $\mathcal{X}_n(v)$. This ensures that all $x \in \mathcal{X}_n(v)$ have positive distance to v, which simplifies the phrasing and application of the following lemmas. One could also work with (A2), but the notation becomes messier. Ultimately we will argue in the proof of Theorem 4.1 that we can assume (A2') instead of (A2) without loss of generality.

For the following lemma we introduce $L_n^{[3]} = \sum_{v=1}^n W_v^3$. Note that assumption (W) ensures that L_n and $L_n^{[2]}$ grow linearly in n, as the first and second moment of W are finite. If the third moment of W is infinite, the growth of $L_n^{[3]}$ becomes superlinear instead.

Lemma 4.5. *Assume* (W), (A2') *and define*

$$\tilde{S}_{n,\mathcal{W}} = \sum_{k=0}^{n} (k+3)^3 \left(\frac{L_n^{[2]}}{L_n} \right)^k, X_n = 4\tilde{S}_{n,\mathcal{W}}^2 \left(1 + \frac{W_{(n)}^2}{L_n} \tilde{S}_{n,\mathcal{W}} + \tilde{S}_{n,\mathcal{W}}^2 \right) \text{ and } Y_n = 4\frac{n}{L_n} \tilde{S}_{n,\mathcal{W}}^5$$

for $n \in \mathbb{N}$. Then X_n and Y_n converge in probability to positive constants as $n \to \infty$ and for all $n \in \mathbb{N}$ and $v \in [n]$,

$$\mathbf{Var}_{\mathcal{W}}\left(S_n(v)\right) \leq W_v X_n + W_v \frac{L_n^{[3]}}{n} Y_n.$$

Proof. The convergence of $\tilde{S}_{n,\mathcal{W}}$ was established in (4.6). The convergence of X_n and Y_n follows from the convergence of $\tilde{S}_{n,\mathcal{W}}$, of $W_{(n)}^2/L_n$ in Lemma 2.27 c) and of n/L_n by the strong law of large numbers. Now we address the actual variance bound.

Conditionally on the weights $\mathcal{W}$, the quantity $S_n(v)$ depends on the independent Bernoulli random variables $(\mathbf{1}\{i \leftrightarrow j\})_{1 \leq i < j \leq n}$, which determine the graph. We can write

$$\mathbb{P}_{\mathcal{W}}(i \leftrightarrow j) = 1 - \exp\left(-\frac{W_i W_j}{L_n} \right) = \mathbb{P}_{\mathcal{W}}(E_n\{i,j\} \neq 0),$$

where

$$E_n\{i,j\} \sim \text{Poisson}\left(\frac{W_i W_j}{L_n} \right)$$

denotes a mixed Poisson distribution. We can think of the collection $(E_n\{i,j\})_{1 \leq i < j \leq n}$ as a Poisson process on the discrete space $\{\{i,j\} \colon 1 \leq i < j \leq n\}$ with intensity measure $\lambda(\{i,j\}) = W_i W_j / L_n$ for $1 \leq i < j \leq n$. This means that $S_n(v)$ is a Poisson functional, which is in particular square-integrable as it is bounded by n. We use the Poincaré inequality for Poisson functionals, see Theorem 2.5, to derive

$$\mathbf{Var}_{\mathcal{W}}\left(S_n(v)\right) \leq \sum_{1 \leq i < j \leq n} \mathbb{E}_{\mathcal{W}}\left[(D_{\{i,j\}} S_n(v))^2 \right] \frac{W_i W_j}{L_n}, \tag{4.9}$$

where $D_{\{i,j\}}$ denotes the difference operator

$$D_{\{i,j\}} S_n(v) = S_{n,i,j}(v) - S_n(v)$$

and $S_{n,i,j}(v)$ is the number of elements in $\mathcal{X}_n(v)$ after increasing $E_n\{i,j\}$ by 1. By assumption (A2'), $v \notin \mathcal{X}_n(v)$ and for a vertex $x \in \mathcal{C}_n(v)$ with $x \neq v$ the property $x \in \mathcal{X}_n(v)$ only depends on all paths starting in v and containing the vertex x. For $i < j$ we obtain

$$|D_{\{i,j\}}S_n(v)| \leq |\{x \in [n] \setminus \{v\} \colon \text{There is a path starting in } v \text{ that contains } x \text{ and the edge } \{i,j\}\}|,$$

where the existence of the path is to be checked after increasing $E_n\{i,j\}$ by 1. We obtain four different scenarios before increasing $E_n\{i,j\}$ (independent of whether i and j were already connected or not). Figure 4.1 contains pictures of the different cases, where solid lines refer to edges and dashed lines to paths. We write $v \dots x$ for a path that starts in v and ends in x and introduce the following four sets, which decompose $[n] \setminus \{v\}$ from the bound above into the respective cases.

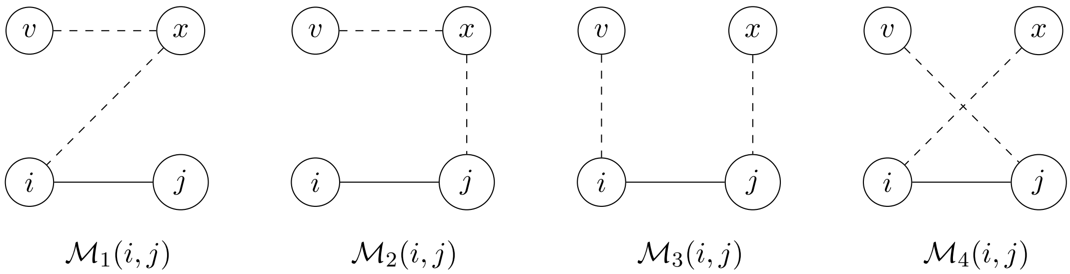


Figure 4.1: Visualisation of $x \in \mathcal{M}_\ell(i,j)$ for $\ell \in \{1,2,3,4\}$.

$$\mathcal{M}_1(i,j) = \{x \in [n] \setminus \{v\} \colon \text{There is a path } v \dots i \text{ that includes } x \text{ but not } j.\}$$
$$\mathcal{M}_2(i,j) = \{x \in [n] \setminus \{v\} \colon \text{There is a path } v \dots j \text{ that includes } x \text{ but not } i.\}$$
$$\mathcal{M}_3(i,j) = \{x \in [n] \setminus \{v\} \colon \text{There are two disjoint paths } v \dots i \text{ and } j \dots x.\}$$
$$\mathcal{M}_4(i,j) = \{x \in [n] \setminus \{v\} \colon \text{There are two disjoint paths } v \dots j \text{ and } i \dots x.\}$$

By disjoint paths we mean that they do not share a single vertex. Also, we allow a path to consist of only a single vertex, e.g. in $\mathcal{M}_3(i,j)$ the special case $v = i$ immediately yields the existence of a path from v to i. We obtain

$$|D_{\{i,j\}}S_n(v)| \leq \sum_{k=1}^{4} |\mathcal{M}_k(i,j)|$$

and with Jensen's inequality

$$(D_{\{i,j\}}S_n(v))^2 \leq 4\sum_{k=1}^{4} |\mathcal{M}_k(i,j)|^2 = 4\sum_{k=1}^{4} |\mathcal{M}_k(i,j)^2|.$$

Observe that $\mathcal{M}_1(i,j) = \mathcal{M}_2(j,i)$ as well as $\mathcal{M}_3(i,j) = \mathcal{M}_4(j,i)$. Therefore, (4.9) simplifies to

$$\mathbf{Var}_{\mathcal{W}}(S_n(v)) \leq 4 \sum_{(i,j)\in[n]^2_{\neq}} \mathbb{E}_{\mathcal{W}}\big[|\mathcal{M}_1(i,j)^2|\big] \frac{W_iW_j}{L_n} + 4 \sum_{(i,j)\in[n]^2_{\neq}} \mathbb{E}_{\mathcal{W}}\big[|\mathcal{M}_3(i,j)^2|\big] \frac{W_iW_j}{L_n}. \tag{4.10}$$

We start with bounding the first sum and consider $\mathbb{E}_{\mathcal{W}}[|\mathcal{M}_1(i,j)^2|]$. Let $(x,y) \in \mathcal{M}_1(i,j)^2$. This means that there is a first path P from v through x to i and a second path from v through y to i. Note that P must contain at least two vertices because x is not allowed to equal v. We distinguish two cases, see also Figure 4.2 for a visualisation:

a) y lies on P,

b) y does not lie on P.

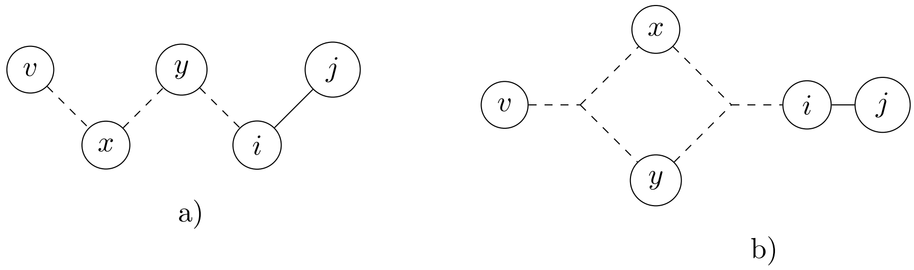


Figure 4.2: Scenarios for $(x,y) \in \mathcal{M}_1(i,j)^2$.

We write $M_{1,\mathrm{a}}(i,j)$ for the number of all $(x,y) \in \mathcal{M}_1(i,j)^2$ of type a), similarly $M_{1,\mathrm{b}}(i,j)$ for b), so that $|\mathcal{M}_1(i,j)^2| \leq M_{1,\mathrm{a}}(i,j) + M_{1,\mathrm{b}}(i,j)$. We have

$$M_{1,\mathrm{a}}(i,j) \leq \sum_{k=2}^{n} (k-1)^2 \cdot |\mathcal{P}_k^{(n)}(v,i,\{i,j\})|$$

because x and y need to lie on a path from v to i which does not include the vertices i,j as inner vertices. Since $x \neq v \neq y$, whereas $x = y$ is possible, the factor $(k-1)^2$ accounts for the possibilities to place x and y on the respective path of length k. From Lemma 4.3 we conclude that

$$\mathbb{E}_{\mathcal{W}}[M_{1,\mathrm{a}}(i,j)] \leq \sum_{k=2}^{n} (k-1)^2 \frac{W_v W_i}{L_n} \left(\frac{L_n^{[2]}}{L_n}\right)^{k-2} \leq W_v \frac{W_i}{L_n} \tilde{S}_{n,\mathcal{W}}.$$

In case b) we use a similar argument to obtain

$$\begin{aligned} M_{1,\mathrm{b}}(i,j) \leq \sum_{k=2}^{n} \sum_{(p_1,\ldots,p_k) \in \mathcal{P}_k^{(n)}(v,i,\{i,j\})} & (k-1) \\ \times \sum_{1 \leq c < d \leq k} \sum_{\ell=3}^{n} & (\ell-2) \cdot |\mathcal{P}_\ell^{(n)}(p_c, p_d, \{i,j,p_1,\ldots,p_k\})| \end{aligned}$$

because x needs to lie one some path P from v to i whereas y needs to lie on another path Q which splits off P and merges with P again. Since y does not lie on P due to case b), Q contains at least three vertices and y can be neither its starting point nor its endpoint, denoted by p_c and p_d above. Note that the path Q can be chosen in such a way that it intersects P only in p_c and p_d. We therefore obtain conditional independence of all paths above and use Lemma 4.3 to obtain

$$\begin{aligned}\mathbb{E}_{\mathcal{W}}[M_{1,\mathrm{b}}(i,j)] &\le \sum_{k=2}^{n}(k-1)\frac{W_vW_i}{L_n}\Big(\frac{L_n^{[2]}}{L_n}\Big)^{k-2}\sum_{1\le c<d\le k}\sum_{\ell=3}^{n}(\ell-2)\frac{W_{(n)}^2}{L_n}\Big(\frac{L_n^{[2]}}{L_n}\Big)^{\ell-2}\\ &\le W_v\frac{W_i}{L_n}\frac{W_{(n)}^2}{L_n}\sum_{k=2}^{n}k^3\Big(\frac{L_n^{[2]}}{L_n}\Big)^{k-2}\sum_{\ell=1}^{n}\ell\Big(\frac{L_n^{[2]}}{L_n}\Big)^{\ell}\le W_v\frac{W_i}{L_n}\frac{W_{(n)}^2}{L_n}\tilde{S}_{n,\mathcal{W}}^2.\end{aligned}$$

Combining the bounds from part a) and b) yields

$$\mathbb{E}_{\mathcal{W}}[|\mathcal{M}_1(i,j)^2|]\le W_v\frac{W_i}{L_n}\tilde{S}_{n,\mathcal{W}}\Big(1+\frac{W_{(n)}^2}{L_n}\tilde{S}_{n,\mathcal{W}}\Big).$$

Therefore, the first sum in (4.10) is bounded by

$$\sum_{(i,j)\in[n]^2_{\neq}}^{n}W_v\frac{W_i}{L_n}\tilde{S}_{n,\mathcal{W}}\Big(1+\frac{W_{(n)}^2}{L_n}\tilde{S}_{n,\mathcal{W}}\Big)\frac{W_iW_j}{L_n}\le W_v\tilde{S}_{n,\mathcal{W}}^2\Big(1+\frac{W_{(n)}^2}{L_n}\tilde{S}_{n,\mathcal{W}}\Big), \qquad (4.11)$$

where we used the bound $L_n^{[2]}/L_n\le\tilde{S}_{n,\mathcal{W}}$ for the sum over i, while the sum over j cancels out with L_n^{-1}.

We proceed with a similar strategy for $\mathcal{M}_3(i,j)^2$. There are some more cases due to the fact that v itself may be equal to i whereas this was impossible in the case of $\mathcal{M}_1(i,j)^2$. Let $(x,y)\in\mathcal{M}_3(i,j)^2$. This means that we can find a path P from v to i via $p_2,\dots,p_{u-1}$ where $u=1$ corresponds to the case $v=i$. We fix such a path P. Additionally, there is a path from j to x and another path from j to y. We distinguish the following cases, see also Figure 4.3. By definition of $\mathcal{M}_3(i,j)$ all paths Q and R in the cases below may be chosen disjoint from the path P from v to i:

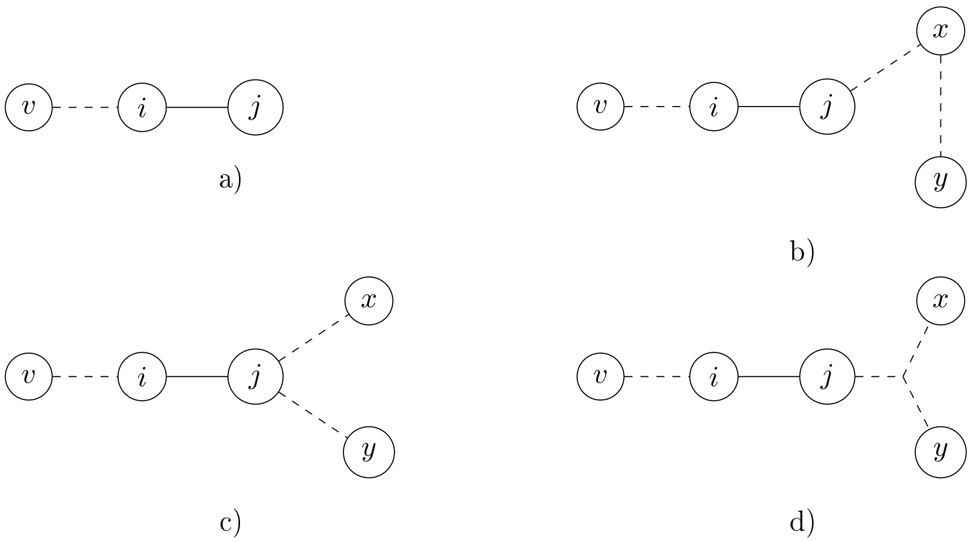


Figure 4.3: Different scenarios for $(x,y)\in\mathcal{M}_3(i,j)^2$.

a) $x = y = j$,

b) $|\{x, y, j\}| \geq 2$ and there is a path Q that starts in j and contains x and y,

c) $|\{x, y, j\}| = 3$ and there is a path Q from j to x and a path R from j to y such that j is their only common vertex,

d) $|\{x, y, j\}| = 3$, there is a path Q from j to x and there is a second path R from one of Q's inner vertices to y that is disjoint from Q apart from its first vertex.

For fixed i, j, P we write $M_{3,\mathrm{z}}(i, j, P)$ for the number of all $(x, y) \in \mathcal{M}_3(i, j)^2$ considered in case z) for $\mathrm{z} \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$. In case a) we simply have

$$\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{a}}(i, j, P) = \mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} \tag{4.12}$$

and by counting all possibilities to place x and y on Q we obtain for case b),

$$\begin{aligned}
&\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{b}}(i, j, P) \\
&\leq \mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} \sum_{k=2}^{n} k^2 |\mathcal{P}_k^{(n)}(j, \{v, i, j, p_2, \ldots, p_{u-1}\})|,
\end{aligned}$$

so that Lemma 4.3 yields

$$\begin{aligned}
&\mathbb{E}_{\mathcal{W}}[\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{b}}(i, j, P)] \\
&\leq \mathbb{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j \sum_{k=2}^{n} k^2 \left(\frac{L_n^{[2]}}{L_n}\right)^{k-2} \\
&\leq \mathbb{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j \tilde{S}_{n,\mathcal{W}}.
\end{aligned} \tag{4.13}$$

For c) we obtain similarly

$$\begin{aligned}
&\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{c}}(i, j, P) \\
&\leq \mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} \sum_{k=2}^{n} k \sum_{(q_1, \ldots, q_k) \in \mathcal{P}_k^{(n)}(j, \{v, p_2, \ldots, p_{u-1}, i\})} \\
&\quad \times \sum_{\ell=2}^{n} \ell \cdot |\mathcal{P}_\ell^{(n)}(j, \{v, p_2, \ldots, p_{u-1}, i, q_1, \ldots, q_k\})|.
\end{aligned}$$

We derive due to conditional independence and Lemma 4.3

$$\begin{aligned}
&\mathbb{E}_{\mathcal{W}}[\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{c}}(i, j, P)] \\
&\leq \mathbb{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j^2 \left(\sum_{k=2}^{n} k \left(\frac{L_n^{[2]}}{L_n}\right)^{k-2}\right)^2 \\
&\leq \mathbb{P}_{\mathcal{W}}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j^2 \tilde{S}_{n,\mathcal{W}}^2.
\end{aligned} \tag{4.14}$$

Case d) is similar to c), but we need to choose a vertex q_t on the path from j to x where the path to y originates. We obtain

$$\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{d}}(i, j, P)$$

$$\leq \mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} \sum_{k=3}^{n} k \sum_{(q_1,\ldots,q_k)\in\mathcal{P}_k^{(n)}(j,\{v,p_2,\ldots,p_{u-1},i\})}$$
$$\times \sum_{\ell=2}^{n} \sum_{1<t<k} \ell \cdot |\mathcal{P}_\ell^{(n)}(q_t, \{q_1,\ldots,q_k,v,p_2,\ldots,p_{u-1},i\})|.$$

When calculating the conditional expectation, we again use conditional independence. This time, the weight W_{q_t} of the vertex in which the second path originates will appear with a third power. It already has a second power since it is an inner vertex of the first path and as starting point of the second path, we obtain another factor W_{q_t}. In total, we get

$$\begin{aligned}
&\mathbb{E}_\mathcal{W}[\mathbf{1}\{v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i\} M_{3,\mathrm{d}}(i,j,P)] \\
&\leq \mathbb{P}_\mathcal{W}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j \frac{L_n^{[3]}}{L_n} \sum_{k=3}^{n} k^2 \left(\frac{L_n^{[2]}}{L_n}\right)^{k-3} \sum_{\ell=2}^{n} \ell \left(\frac{L_n^{[2]}}{L_n}\right)^{\ell-2} \\
&\leq \mathbb{P}_\mathcal{W}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i) W_j \frac{L_n^{[3]}}{L_n} \tilde{S}_{n,\mathcal{W}}^2. && (4.15)
\end{aligned}$$

Summing over all different choices for P to connect v and i via $p_2,\ldots,p_{u-1}$ as well as using the four bounds from (4.12), (4.13), (4.14) and (4.15) yields

$$\begin{aligned}
&\mathbb{E}_\mathcal{W}[|\mathcal{M}_3(i,j)^2|] \\
&\leq \left(\mathbf{1}\{v=i\} + \mathbb{P}_\mathcal{W}(v \leftrightarrow i) + \sum_{u=3}^{n} \sum_{(p_2,\ldots,p_{u-1})\in([n]\setminus\{v,i\})_{\neq}^{u-2}} \mathbb{P}_\mathcal{W}(v \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_{u-1} \leftrightarrow i)\right) \\
&\quad \times \left(1 + W_j \tilde{S}_{n,\mathcal{W}} + W_j^2 \tilde{S}_{n,\mathcal{W}}^2 + W_j \frac{L_n^{[3]}}{L_n} \tilde{S}_{n,\mathcal{W}}^2\right) \\
&\leq \left(1\{v=i\} + \sum_{u=2}^{n} \mathbb{E}_\mathcal{W}[|\mathcal{P}_u^{(n)}(v,i,\emptyset)|]\right)\left(1 + W_j \tilde{S}_{n,\mathcal{W}} + W_j^2 \tilde{S}_{n,\mathcal{W}}^2 + W_j \frac{L_n^{[3]}}{L_n} \tilde{S}_{n,\mathcal{W}}^2\right) \\
&\leq \left(1\{v=i\} + \frac{W_v W_i}{L_n} \tilde{S}_{n,\mathcal{W}}\right)\left(1 + W_j \tilde{S}_{n,\mathcal{W}} + W_j^2 \tilde{S}_{n,\mathcal{W}}^2 + W_j \frac{L_n^{[3]}}{L_n} \tilde{S}_{n,\mathcal{W}}^2\right),
\end{aligned}$$

where the last inequality uses Lemma 4.3. This bounds the second sum in (4.10) by

$$\begin{aligned}
&\sum_{(i,j)\in[n]_{\neq}^2} \left(1\{v=i\} + \frac{W_v W_i}{L_n} \tilde{S}_{n,\mathcal{W}}\right)\left(1 + W_j \tilde{S}_{n,\mathcal{W}} + W_j^2 \tilde{S}_{n,\mathcal{W}}^2 + W_j \frac{L_n^{[3]}}{L_n} \tilde{S}_{n,\mathcal{W}}^2\right)\frac{W_i W_j}{L_n} \\
&\leq \sum_{i=1}^{n} \left(W_i 1\{v=i\} + \frac{W_v W_i^2}{L_n} \tilde{S}_{n,\mathcal{W}}\right) \sum_{j=1}^{n} \frac{W_j}{L_n}\left(1 + W_j \tilde{S}_{n,\mathcal{W}} + W_j^2 \tilde{S}_{n,\mathcal{W}}^2 + W_j \frac{L_n^{[3]}}{L_n} \tilde{S}_{n,\mathcal{W}}^2\right) \\
&= W_v \left(1 + \frac{L_n^{[2]}}{L_n} \tilde{S}_{n,\mathcal{W}}\right)\left(1 + \frac{L_n^{[2]}}{L_n} \tilde{S}_{n,\mathcal{W}} + \frac{L_n^{[3]}}{L_n} \tilde{S}_{n,\mathcal{W}}^2 + \frac{L_n^{[2]} L_n^{[3]}}{L_n^2} \tilde{S}_{n,\mathcal{W}}^2\right) \\
&\leq W_v \tilde{S}_{n,\mathcal{W}}^2 \left(\tilde{S}_{n,\mathcal{W}}^2 + \frac{L_n^{[3]}}{L_n} \tilde{S}_{n,\mathcal{W}}^3\right), && (4.16)
\end{aligned}$$

where we used $1+L_n^{[2]}\tilde{S}_{n,\mathcal{W}}/L_n \leq \tilde{S}_{n,\mathcal{W}}^2$ in the last step. Combining (4.10) with (4.11) and (4.16), we obtain

$$\begin{aligned}\mathbf{Var}_{\mathcal{W}}(S_n(v)) &\leq 4W_v\left(\tilde{S}_{n,\mathcal{W}}^2\left(1+\frac{W_{(n)}^2}{L_n}\tilde{S}_{n,\mathcal{W}}\right)+\tilde{S}_{n,\mathcal{W}}^2\left(\tilde{S}_{n,\mathcal{W}}^2+\frac{L_n^{[3]}}{L_n}\tilde{S}_{n,\mathcal{W}}^3\right)\right)\\ &= 4W_v\tilde{S}_{n,\mathcal{W}}^2\left(1+\frac{W_{(n)}^2}{L_n}\tilde{S}_{n,\mathcal{W}}+\tilde{S}_{n,\mathcal{W}}^2\right)+4W_v\frac{L_n^{[3]}}{L_n}\tilde{S}_{n,\mathcal{W}}^5 = W_vX_n+W_v\frac{L_n^{[3]}}{n}Y_n,\end{aligned}$$

which finishes the proof. □

The following lemma essentially shows that the conditional m-th moment of the number of vertices in a component is bounded by a polynomial of degree m in the largest weight of the component, up to a term which converges almost surely.

Lemma 4.6. *Assume* (W), (A2'), *let $m \in \mathbb{N}$ and define for $n \in \mathbb{N}$,*

$$R_{n,m} = m^m \sum_{t=1}^{m} t! \left(\sum_{k=0}^{n}\left(\frac{L_n^{[2]}}{L_n}\right)^k (k+3)^t\right)^t.$$

Then $R_{n,m}$ converges almost surely to a constant as $n \to \infty$ and for all $v \in [n]$ it holds that

$$\mathbb{E}_{\mathcal{W}}\left[\mathbf{1}\{v \in V_n^{\max}\}S_n(v)^m\right] \leq R_{n,m}\sum_{t=1}^{m} W_v^t.$$

Proof. For $m \in \mathbb{N}$, the almost sure convergence of $R_{n,m}$ as $n \to \infty$ follows from (4.6). To simplify notation we assume without loss of generality that $v = n$, which yields $[n]\setminus\{v\} = [n-1]$. For the moment bound, recall $v \notin \mathcal{X}_n(v)$ by (A2') so that

$$\begin{aligned}\mathbb{E}_{\mathcal{W}}\left[\mathbf{1}\{v \in V_n^{\max}\}S_n(v)^m\right] &\leq \mathbb{E}_{\mathcal{W}}\left[\mathbf{1}\{v \in V_n^{\max}\}\left(\sum_{x\in[n-1]}\mathbf{1}\{x \in \mathcal{C}_n(v)\}\right)^m\right]\\ &= \sum_{(x_1,\dots,x_m)\in[n-1]^m}\mathbb{E}_{\mathcal{W}}\left[\mathbf{1}\{v \in V_n^{\max}\}\prod_{i=1}^{m}\mathbf{1}\{x_i \in \mathcal{C}_n(v)\}\right]\\ &\leq \sum_{t=1}^{m} m^m \sum_{(x_1,\dots,x_t)\in[n-1]^t_{\neq}}\mathbb{E}_{\mathcal{W}}\left[\mathbf{1}\{v \in V_n^{\max}\}\prod_{i=1}^{t}\mathbf{1}\{x_i \in \mathcal{C}_n(v)\}\right],\end{aligned}$$

where the last step accounts for equal entries of a vector $(x_1,\dots,x_m) \in [n-1]^m$ by the additional factor m^m and instead only sums over the $t \in [m]$ distinct entries of x. Hence it suffices to show for all $t \in [m]$ that

$$\sum_{(x_1,\dots,x_t)\in[n-1]^t_{\neq}}\mathbb{E}_{\mathcal{W}}\left[\mathbf{1}\{v \in V_n^{\max}\}\prod_{i=1}^{t}\mathbf{1}\{x_i \in \mathcal{C}_n(v)\}\right] \leq W_v^t t!\left(\sum_{k=0}^{n}\left(\frac{L_n^{[2]}}{L_n}\right)^k (k+3)^t\right)^t.$$

We order the vertices $x_1,\dots,x_t$ in such a way that the graph distance of x_i and v is non-decreasing in i which gives us the $t!$ on the right-hand side of the inequality above.

In order to bound the product on the left-hand side above, we apply an iterative argument. If $x_1, \dots, x_t \in \mathcal{C}_n(v)$, there exists a shortest path P_1 starting in v and ending in x_1, consisting of $k_1 \geq 2$ vertices because $x_1 \neq v$. There may be several shortest paths and if so, we always choose the smallest one with respect to the lexicographic order, e.g. the path $n \leftrightarrow 3 \leftrightarrow 5$ is preferred over $n \leftrightarrow 4 \leftrightarrow 5$ (for $n \geq 6$). Now consider a shortest path from v to x_2 and remove the part leading from v to its last intersection with the already existing path P_1. The remaining part, connecting P_1 and x_2, is called P_2 and contains $k_2 \geq 2$ vertices. Applying this construction iteratively, we connect for $i = 2, \dots, t$ a vertex x_i via a path P_i of $k_i \geq 2$ vertices to some vertex a_i of the previously added paths $P_1, \dots, P_{i-1}$. Note that we obtain a tree structure without cycles by choosing the shortest path by lexicographic order (if necessary), see also Figure 4.4 for a visualisation.

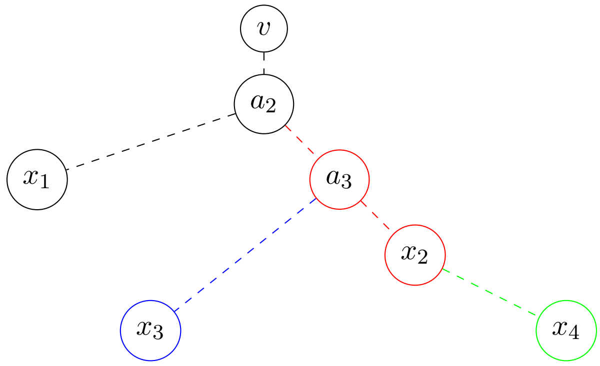


Figure 4.4: For $t = 4$ we have the path P_1 connecting x_1 and v in black. The red path P_2 connects x_2 to some vertex a_2 on P_1. Moreover, P_3 in blue connects x_3 to a vertex a_3 on P_2. Finally, the green path P_4 connects x_4 to the vertex $a_4 = x_2$ on P_2.

For $i \in [t]$ and any of these paths P_i we write a_i for its starting vertex. Note that $a_1 = v = n$. For $i \geq 2$, there are at most $k_1 + \ldots + k_{i-1} \leq \prod_{j=1}^{i-1}(k_j + 1)$ possibilities to choose the point a_i on any of the paths $P_1, \dots, P_{i-1}$ where P_i may be attached to. We write $\cup_{j=1}^{i-1} P_j$ for the set of all vertices of $P_1, \dots, P_{i-1}$. Once one has chosen any such a starting vertex a_i, the conditional expectation of the number of such paths P_i is bounded by

$$\mathbb{E}_{\mathcal{W}}\left[\sum_{k_i=2}^{n} |\mathcal{P}_{k_i}^{(n)}(a_i, x_i, \cup_{j=1}^{i-1} P_j)| \cdot \mathbf{1}\{W_{a_i} \leq W_v\}\right] \leq W_v \frac{W_{x_i}}{L_n} \sum_{k_i=2}^{n} \left(\frac{L_n^{[2]}}{L_n}\right)^{k_i - 2},$$

where we used Lemma 4.3 and the fact that the weight of W_v is maximal in its component, so in particular not smaller than the weight W_{a_i}. By construction, the paths $P_1, \dots, P_t$ do not share an edge and are therefore conditionally independent.

Combining this bound with the number of possible choices for a_i we obtain

$$\sum_{(x_1,\dots,x_t) \in [n-1]^t_{\neq}} \mathbb{E}_{\mathcal{W}}\left[\mathbf{1}\{v \in V_n^{\max}\} \prod_{i=1}^{t} \mathbf{1}\{x_i \in \mathcal{C}_n(v)\}\right]$$
$$\leq W_v^t t! \sum_{(x_1,\dots,x_t) \in [n-1]^t_{\neq}} \prod_{i=1}^{t} \frac{W_{x_i}}{L_n} \sum_{k_i=2}^{n} \left(\frac{L_n^{[2]}}{L_n}\right)^{k_i - 2} (k_i + 1)^{t-i}$$

$$\leq W_v^t t! \bigg(\sum_{k=0}^n \bigg(\frac{L_n^{[2]}}{L_n} \bigg)^k (k+3)^t \bigg)^t ,$$

which concludes the proof. □

Now we have collected all auxiliary lemmas and can proceed with the proof of the main theorem.

Proof of Theorem 4.1. For $n \in \mathbb{N}$ we consider the point processes

$$\Theta_n = \sum_{v=1}^n \delta_{W_v q(n)^{-1}} \quad \text{and} \quad \Xi_n = \sum_{v=1}^n \mathbf{1}\{v \in V_n^{\max}\} \delta_{S_n(v) q(n)^{-1} \zeta^{-1}}.$$

By part a) of Lemma 2.20 we have

$$\Theta_n \xrightarrow{d} \eta_\beta \quad \text{as} \quad n \to \infty$$

in $M_p((0,\infty])$. In order to show the respective statement for Ξ_n instead of Θ_n, we apply Lemma 2.11. Since η_β has a diffuse intensity measure, we obtain

$$\mathbb{P}(\eta_\beta(\{a,b\}) = 0) = 1$$

for all $0 < a \leq b < \infty$ so that it suffices to show for all $a > 0$ that

$$\Xi_n((a,\infty]) - \Theta_n((a,\infty]) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty \tag{4.17}$$

in order to prove Theorem 4.1. Since $q(n) \to \infty$ as $n \to \infty$, it does not matter whether we consider $S_n(v)$ or $S_n(v) - 1$ in Ξ_n. Therefore, we may assume without loss of generality that (A2') is satisfied, i.e. that $v \notin \mathcal{X}_n(v)$ for all $v \in [n]$. It holds that

$$\begin{aligned} &|\Xi_n((a,\infty]) - \Theta_n((a,\infty])| \\ &\leq \sum_{v=1}^n \mathbf{1}\{W_v > aq(n)\} \mathbf{1}\{v \notin V_n^{\max}\} + \sum_{v=1}^n \mathbf{1}\{W_v > aq(n)\} \mathbf{1}\{v \in V_n^{\max}, S_n(v) \leq aq(n)\zeta\} \\ &\quad + \sum_{v=1}^n \mathbf{1}\{W_v \leq aq(n)\} \mathbf{1}\{v \in V_n^{\max}, S_n(v) > aq(n)\zeta\} =: I_1 + I_2 + I_3. \end{aligned}$$

We show that I_1, I_2 and I_3 converge in probability to zero as $n \to \infty$. For I_1 it follows from Lemma 4.4 that

$$\mathbb{P}(I_1 \neq 0) = \mathbb{P}(\exists x, y \in [n] \colon x \neq y, W_x \geq W_y > aq(n), x \in \mathcal{C}_n(y)) = \mathbb{P}(\mathcal{A}_n^{\mathrm{c}}) \to 0$$

as $n \to \infty$. We continue with decomposing I_2 and I_3. To this end, let $\varepsilon \in (0,a)$. It holds that

$$\begin{aligned} I_2 &= \sum_{v=1}^n \mathbf{1}\{v \in V_n^{\max}, S_n(v) \leq aq(n)\zeta < W_v \zeta\} \\ &\leq \sum_{v=1}^n \mathbf{1}\{aq(n) < W_v \leq (a+\varepsilon)q(n)\} + \sum_{v=1}^n \mathbf{1}\{W_v > (a+\varepsilon)q(n), S_n(v) \leq aq(n)\zeta\} \end{aligned}$$

$=: I_{2,1} + I_{2,2}.$

The introduction of ε leads to some positive minimal distance between $W_v\zeta$ and $S_n(v)$ in $I_{2,2}$, which allows us to use the Chebyshev inequality later on. For I_3 we fix some small positive γ satisfying $0 < \gamma < \beta^{-1}$ and define $\tilde{q}(n) = n^{-\gamma}q(n)$ for $n \in \mathbb{N}$. Note that $\tilde{q}(n) \leq q(n)$ as well as $\tilde{q}(n)q(n)^{-1} \to 0$ and, by Proposition 2.19 a), $\tilde{q}(n) \to \infty$ as $n \to \infty$. For $n \in \mathbb{N}$ we have

$$\begin{aligned}
I_3 &= \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}, W_v\zeta \leq aq(n)\zeta < S_n(v)\} \\
&\leq \sum_{v=1}^{n} \mathbf{1}\{(a-\varepsilon)q(n) < W_v \leq aq(n)\} \\
&\quad + \sum_{v=1}^{n} \mathbf{1}\{a\tilde{q}(n) < W_v \leq (a-\varepsilon)q(n), S_n(v) > aq(n)\zeta\} \\
&\quad + \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}, W_v \leq a\tilde{q}(n), S_n(v) > aq(n)\zeta\} =: I_{3,1} + I_{3,2} + I_{3,3}.
\end{aligned}$$

By Proposition 2.19 b) we get

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\Bigg[\sum_{v=1}^{n} \mathbf{1}\{(a-\varepsilon)q(n) < W_v \leq (a+\varepsilon)q(n)\}\Bigg] \\
&= \lim_{n\to\infty} n\mathbb{P}((a-\varepsilon)q(n) < W \leq (a+\varepsilon)q(n)) = (a-\varepsilon)^{-\beta} - (a+\varepsilon)^{-\beta},
\end{aligned}$$

so that $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{E}\left[I_{2,1}\right] = 0$ and $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{E}\left[I_{3,1}\right] = 0$.

From $\tilde{q}(n) \leq q(n)$ we conclude that

$$I_4 := \sum_{v=1}^{n} \mathbf{1}\left\{a\tilde{q}(n) < W_v, |S_n(v) - W_v\zeta| > \varepsilon q(n)\zeta\right\}$$

is an upper bound for $I_{2,2}$ and $I_{3,2}$. It remains to show that $I_{3,3} \xrightarrow{\mathbb{P}} 0$ and $I_4 \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. Note that both these random variables take values in $\mathbb{N}_0$. For any sequence of $\mathbb{N}_0$-valued random variables $(Z_n)_{n\in\mathbb{N}}$ and any sequence of events $(Q_n)_{n\in\mathbb{N}}$ it holds that

$$\begin{aligned}
\mathbb{P}(Z_n \neq 0) &= \mathbb{P}(\mathbf{1}_{Q_n^c} Z_n \neq 0) + \mathbb{P}(\mathbf{1}_{Q_n} Z_n \neq 0) \leq \mathbb{P}(Q_n^c) + \mathbb{E}\left[\mathbf{1}\{\mathbf{1}_{Q_n} Z_n \neq 0\}\right] \\
&= \mathbb{P}(Q_n^c) + \mathbb{E}\left[\min\left(1, \mathbf{1}_{Q_n} Z_n\right)\right] = \mathbb{P}(Q_n^c) + \mathbb{E}\left[\mathbb{E}_{\mathcal{W}}\left[\min\left(1, \mathbf{1}_{Q_n} Z_n\right)\right]\right] \\
&\leq \mathbb{P}(Q_n^c) + \mathbb{E}\left[\min\left(1, \mathbb{E}_{\mathcal{W}}\left[\mathbf{1}_{Q_n} Z_n\right]\right)\right], \qquad (4.18)
\end{aligned}$$

where we used Jensen's inequality in the last step. Thus, in order to obtain $Z_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, it suffices to show that both $\mathbb{E}_{\mathcal{W}}[\mathbf{1}_{Q_n} Z_n] \xrightarrow{\mathbb{P}} 0$ and $\mathbb{P}(Q_n) \to 1$ as $n \to \infty$.

We start with the summand $I_{3,3}$. By the argument above it suffices to show that $\mathbb{E}_{\mathcal{W}}[I_{3,3}] \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, where we choose $Q_n = \Omega$ for all $n \in \mathbb{N}$. For $m \in \mathbb{N}$ we compute

$$\mathbb{E}_{\mathcal{W}}[I_{3,3}] = \mathbb{E}_{\mathcal{W}}\Bigg[\sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}, W_v \leq a\tilde{q}(n), S_n(v) > aq(n)\zeta\}\Bigg]$$

$$
\begin{aligned}
&= \sum_{v=1}^{n} \mathbf{1}\{W_v \le a\tilde{q}(n)\} \mathbb{P}_{\mathcal{W}}\Big(\mathbf{1}\{v \in V_n^{\max}\} \cdot S_n(v) > aq(n)\zeta\Big) \\
&\le \sum_{v=1}^{n} \mathbf{1}\{W_v \le a\tilde{q}(n)\} \frac{\mathbb{E}_{\mathcal{W}}\left[\mathbf{1}\{v \in V_n^{\max}\} \cdot S_n(v)^m\right]}{(aq(n)\zeta)^m},
\end{aligned}
$$

where the last inequality follows from the Markov inequality. Let n be large enough such that $a\tilde{q}(n) > 1$. We can bound the conditional expectation by applying Lemma 4.6, which leads to

$$
\begin{aligned}
\mathbb{E}_{\mathcal{W}}[I_{3,3}] &\le \left(\frac{1}{aq(n)\zeta}\right)^m \sum_{v=1}^{n} \mathbf{1}\{W_v \le a\tilde{q}(n)\} R_{n,m} \sum_{t=1}^{m} W_v^t \\
&\le R_{n,m} \left(\frac{1}{aq(n)\zeta}\right)^m \sum_{v=1}^{n} m(a\tilde{q}(n))^m = m\zeta^{-m} R_{n,m} \cdot n^{1-m\gamma},
\end{aligned}
$$

where the last step uses $\tilde{q}(n) = n^{-\gamma} q(n)$. Since $R_{n,m}$ converges for all $m \in \mathbb{N}$ almost surely to a constant as $n \to \infty$ by Lemma 4.6, choosing $m > \gamma^{-1}$ ensures that $\mathbb{E}_{\mathcal{W}}[I_{3,3}]$ converges almost surely to zero.

In order to deal with I_4 we define the event

$$
\mathcal{G}_{n,\varepsilon} = \left\{ \sup_{v=1,\dots,n} |\mathbb{E}_{\mathcal{W}}[S_n(v)] - W_v\zeta| \le \frac{\varepsilon q(n)\zeta}{2} \right\},
$$

which satisfies $\mathbb{P}(\mathcal{G}_{n,\varepsilon}) \to 1$ as $n \to \infty$ due to (A1). By the discussion after (4.18) we are left to show that $\mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{G}_{n,\varepsilon}} I_4] \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. We use the $\mathcal{W}$-measurability of $\mathcal{G}_{n,\varepsilon}$ to compute

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{G}_{n,\varepsilon}} I_4] = \mathbf{1}_{\mathcal{G}_{n,\varepsilon}} \sum_{v=1}^{n} \mathbb{P}_{\mathcal{W}}\left(W_v > a\tilde{q}(n), |S_n(v) - W_v\zeta| > \varepsilon q(n)\zeta\right) \\
&\le \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \mathbb{P}_{\mathcal{W}}\left(|\mathbb{E}_{\mathcal{W}}[S_n(v)] - W_v\zeta| \le \frac{\varepsilon q(n)\zeta}{2}, |S_n(v) - W_v\zeta| > \varepsilon q(n)\zeta\right) \\
&\le \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \mathbb{P}_{\mathcal{W}}\left(|S_n(v) - \mathbb{E}_{\mathcal{W}}[S_n(v)]| > \frac{\varepsilon q(n)\zeta}{2}\right).
\end{aligned}
$$

We use the Chebyshev inequality and Lemma 4.5 to bound this further by

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{G}_{n,\varepsilon}} I_4] \le \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \frac{\mathbf{Var}_{\mathcal{W}}\left(S_n(v)\right)}{\left(\frac{\varepsilon q(n)\zeta}{2}\right)^2} \\
&\le \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \frac{W_v X_n + W_v n^{-1} L_n^{[3]} Y_n}{\left(\frac{\varepsilon q(n)\zeta}{2}\right)^2} \\
&= \frac{4}{(\varepsilon\zeta)^2} X_n \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \frac{W_v}{q(n)^2} + \frac{4}{(\varepsilon\zeta)^2} Y_n \sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\} \frac{W_v L_n^{[3]}}{nq(n)^2} \\
&=: \frac{4}{(\varepsilon\zeta)^2}(X_n I_X + Y_n I_Y).
\end{aligned}
$$

From Lemma 4.5 we know that X_n and Y_n converge in probability to positive constants as $n \to \infty$. Therefore, it suffices to show that $I_X + I_Y \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$.

For $\beta \in (2,3]$ we choose some $\tau \in (0,\beta)$ and define

$$p = \begin{cases} 1, & \text{for } \beta > 3, \\ \frac{\beta-\tau}{3}, & \text{for } \beta \in (2,3]. \end{cases}$$

Then one has $\mathbb{E}[W^{3p}] < \infty$ by Proposition 2.17. Thus, it follows from Lemma 2.26, the Marcinkiewicz-Zygmund strong law of large numbers, that

$$\frac{L_n^{[3]}}{n^{1/p}} \xrightarrow{a.s.} \begin{cases} \mathbb{E}[W^3], & \text{for } \beta > 3, \\ 0, & \text{for } \beta \in (2,3], \end{cases} \quad \text{as} \quad n \to \infty.$$

We obtain that

$$I_X + I_Y = \frac{1}{q(n)^2}\left(1 + \frac{L_n^{[3]}}{n}\right)\sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\}W_v \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty$$

if

$$\frac{1 + n^{1/p-1}}{q(n)^2}\sum_{v=1}^{n} \mathbf{1}\{W_v > a\tilde{q}(n)\}W_v \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty.$$

In the following we prove this by showing that its expectation

$$h(n) = \frac{n + n^{1/p}}{q(n)^2}\mathbb{E}[\mathbf{1}\{W > a\tilde{q}(n)\}W]$$

vanishes for $n \to \infty$. More precisely, we show that the expression above is regularly varying, as a function in n, and that its index is negative. In Lemma 2.18 it is shown that the map $t \mapsto \mathbb{E}[\mathbf{1}\{W > t\}W]$ belongs to $\mathrm{RV}_{1-\beta}$. Together with $\tilde{q} \in \mathrm{RV}_{1/\beta-\gamma}$ and Proposition 2.15 c) we obtain that $n \mapsto \mathbb{E}[\mathbf{1}\{W > a\tilde{q}(n)\}W]$ belongs to $\mathrm{RV}_{(1-\beta)(1/\beta-\gamma)}$. For the fraction in the definition of h we use $1/p \geq 1$, so that $n^{1/p}$ is the dominant term in the numerator. Since $q \in \mathrm{RV}_{1/\beta}$, the whole fraction is regularly varying with index $1/p - 2/\beta$. Therefore, $h \in \mathrm{RV}_{(1-\beta)(1/\beta-\gamma)+1/p-2/\beta}$.

If $\beta > 3$,

$$(1-\beta)(1/\beta - \gamma) + 1/p - 2/\beta = -\frac{1+\beta}{\beta} + 1 + \gamma(\beta - 1) = -\beta^{-1} + \gamma(\beta-1),$$

while for $\beta \in (2,3]$,

$$(1-\beta)(1/\beta - \gamma) + 1/p - 2/\beta = -\frac{1+\beta}{\beta} + \frac{3}{\beta-\tau} + \gamma(\beta-1).$$

Now we can choose γ and τ sufficiently small so that the expressions become negative for both cases. Then, we have $h(n) \to 0$ as $n \to \infty$, which implies

$$\mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{G}_{n,\varepsilon}} I_4] \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad I_4 \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty.$$

This concludes the proof. □

4.3 Discussion and applications of the main theorem

In this section we provide several choices for $\mathcal{X}_n(v)$, i.e. vertices in $\mathcal{C}_n(v)$ that one would like to count, to which Theorem 4.1 can be applied. The proofs regarding the applicability are postponed to the next section. The first example arises when simply counting all vertices.

Proposition 4.7. *Under assumption* (W), $\mathcal{X}_n(v) = \mathcal{C}_n(v)$ *satisfies* (4.1) *and* (4.2) *with the choice* $\zeta = \frac{\mathbb{E}[W]}{\mathbb{E}[W]-\mathbb{E}[W^2]}$.

This case results in $S_n(v) = |\mathcal{C}_n(v)|$ being the component size of $v \in [n]$. Therefore, Corollary 4.2 provides asymptotic results on the size of the largest component. In [57], Janson shows in a similar setting that the k-th largest component size satisfies $|\mathcal{C}_{(k)}| = \zeta d_{(k)} + o_{\mathbb{P}}(q(n))$ where $d_{(k)}$ denotes the k-th largest degree of the graph. The idea here is to do a breadth-first exploration of the component, starting in the vertex having the largest degree, which is similar in spirit to our approach. His proof is based on lower and upper bounds for the component sizes by a coupling to the configuration model and stochastic domination of and by, respectively, a suitable sum of independent indicators. It might be possible to extend this method to our other examples, but there are some technical details that do not easily generalise. To be more precise, Janson reveals the vertices one by one and keeps track of that count. When one does not count all vertices, it is unclear how this affects the process.

We would also like to comment on the value of ζ. By assumption (A1), ζW_v can be approximated by $\mathbb{E}_{\mathcal{W}}[S_n(v)]$. Recall that the local limit of the Norros-Reittu model is the unimodular Galton-Watson tree whose offspring distribution is Poisson(W), see (3.14) and Definition 3.10. This means that, conditionally on the weight W_v, the vertex v has W_v neighbours and attached to each of them is a Galton-Watson tree having on average $\mathbb{E}[W^2]/\mathbb{E}[W]$ many offsprings, as calculated in (3.17). By classical branching theory results, the total progeny in each of these trees is given by ζ, see e.g. [52, Theorem 3.5]. This gives reason to expect that $\mathbb{E}_{\mathcal{W}}[|\mathcal{C}_n(v)|] \approx W_v\zeta$. Unfortunately, the local convergence result does not immediately imply that assumption (A1) holds. We would like to mention two issues. First of all, it is unclear how to treat the maximum in the assumption (A1) by means of local convergence. Additionally, the heuristics above include conditioning on the weight of W_v, but for (A1) one needs to condition on all the weights. This does not seem to fit the local convergence framework. One obtains similar problems when trying to prove Theorem 4.1 directly via local convergence, as one needs to treat several large components simultaneously.

The next example concerns vertices in a certain distance to v, where we write d for the graph distance. Recall that we consider those vertices v whose weight is maximal in their component. These are also referred to as hubs and play a dominant role in the sense that their degree is typically much larger than that of other vertices in their component. Therefore, we can think of them as some sort of centre of their component and study the structure around them.

Proposition 4.8. *Let* $m \in \mathbb{N}$. *Under assumption* (W), $\mathcal{X}_n(v) = \{x \in \mathcal{C}_n(v) \colon \mathrm{d}(x, v) = m\}$ *satisfies* (4.1) *and* (4.2) *with* $\zeta = \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^{m-1}$.

Consider the special case $m = 1$. Then, we study the point process containing the degrees of the vertices having maximal weight in their component. In this case, our result yields the same limiting process as obtained in [14] by Bhattacharjee and Schulte, where the point process $\mathcal{D}_n$ of *all* degrees, also rescaled by $\zeta^{-1}q(n)^{-1}$, was investigated. In particular, our point process is a subset of theirs, if we think of a point process as a collection of random points. The fact that they have the same limit in $M_p((0,\infty])$ might be surprising at first, but highlights an important aspect of the space: it has a blind spot at 0, since this point is excluded. The points which are considered in $\mathcal{D}_n$ but not in Ξ_n correspond to the degrees of vertices whose weights are not maximal in their component. For vertices whose weight is large, i.e. at least of size roughly $q(n)$, it will be maximal in their component, see Lemma 4.4. Thus, these vertices are considered in both Ξ_n and $\mathcal{D}_n$. On the other hand, vertices having a weight of order $o_{\mathbb{P}}(q(n))$ should also have degrees of order $o_{\mathbb{P}}(q(n))$. Therefore, the corresponding points of $\mathcal{D}_n$ tend towards zero and vanish in the limit. This explains heuristically why Ξ_n and $\mathcal{D}_n$ have the same limit.

Also note that summing up the values of ζ over all distances $m \in \mathbb{N}$ in the proposition above allows us to retrieve the quantity ζ for the whole component from Proposition 4.7, as to be expected. Furthermore, the value ζ can again be intuitively explained by local convergence. A vertex in distance m of v lies in generation $m-1$ of the Galton-Watson trees started in the neighbours of v. The expected size of generation $m-1$ in turn equals ζ, see e.g. [52, Theorem 3.3].

The following example studies vertices of a fixed degree.

Proposition 4.9. *Let $m \in \mathbb{N}$. Under assumption* (W)*, $\mathcal{X}_n(v) = \{x \in \mathcal{C}_n(v) \colon \deg(x) = m\}$ satisfies* (4.1) *and* (4.2) *with $\zeta = \frac{1}{(m-1)!} \frac{\mathbb{E}[W^m e^{-W}]}{\mathbb{E}[W] - \mathbb{E}[W^2]}$.*

As in the previous example, summing ζ over all choices of $m \in \mathbb{N}$ recovers the result from Proposition 4.7. Once more we obtain some intuition on ζ due to the local limit. The offspring distribution of vertices close to v was given by $(p_k^*)_{k\in\mathbb{N}_0}$ in (3.16). Plugging in $k = m-1$ yields $m-1$ offsprings, resulting in degree m when combined with the link to its ancestor. If we multiply the total number of offsprings, $\mathbb{E}[W]/(\mathbb{E}[W] - \mathbb{E}[W^2])$ from Proposition 4.7, with the probability that an offspring has degree m, i.e. with p_{m-1}^*, we obtain our value for ζ.

Our last example requires some preparation. By the local limit, we know that the component looks like a tree. We shall be interested in certain kinds of (rooted) trees that can be found as some kind of decoration at the boundary of a component, which we call terminal trees. They are given by subgraphs which are isomorphic to a given rooted tree and are *terminal* in the sense that they only spread away from the vertex v (in the graph distance sense) and have no further edges attached to them. Let us consider Figure 4.5, where we can see the component of v and where we are interested in terminal wedges whose root is given by the vertex of degree two. In the picture, there are two vertices that give birth to such a terminal wedge, v_3 and v_9. The vertex v itself does not qualify although v, v_1, v_2 form a wedge, because there are further edges. Similarly, the vertex v_4 is a root of the wedge v_4, v_8, v_9, which is not terminal since v_9 has more neighbours than just v_4. The subgraph induced by v, v_2, v_5 with v_2 as root forms also a wedge, but is not terminal since the wedge spreads from its root v_2 towards v.

Proposition 4.10. *Let $m \in \mathbb{N}$ and consider a rooted tree T with vertex set $V(T) = [m]$ and root* 1*. For $x \in [n]$ we say that $x \in \mathcal{X}_n(v)$ if and only if there is exactly one path from v to x and the subgraph generated by x and its descendants with x as root is isomorphic to*

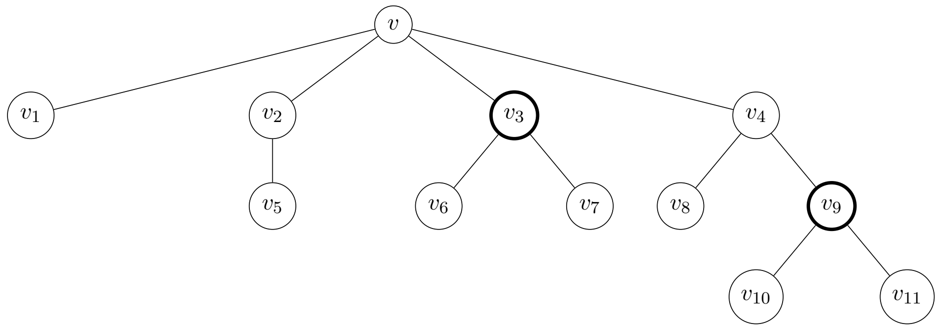


Figure 4.5: Counting terminal wedges in $\mathcal{C}_n(v)$, here with v_3 and v_9 as roots.

T. Under assumption (W)*, this choice of $\mathcal{X}_n(v)$ satisfies* (4.1) *and* (4.2) *with*

$$\zeta = \frac{1}{c(T)} \frac{\mathbb{E}[W^{\deg_T(1)+1} e^{-W}]}{\mathbb{E}[W] - \mathbb{E}[W^2]} \prod_{i=2}^{m} \frac{\mathbb{E}[W^{\deg_T(i)} e^{-W}]}{\mathbb{E}[W]},$$

where $c(T)$ is the order of the automorphism group of T that preserves the root and $\deg_T$ *denotes the degree of a vertex in the tree T.*

Also in this case, the value of ζ can be calculated from the local limit by determining the respective offspring numbers which are required.

Moreover, it would be possible to study vertices of a certain degree as in Proposition 4.9 or the terminal trees from Proposition 4.10 in a fixed distance $k \in \mathbb{N}$ to v, as in Proposition 4.8. Then, we study either vertices in distance k having degree m or terminal trees whose root has distance k to v. This of course affects the choice of ζ. Following the intuition on the values of ζ above, one needs to multiply the values of ζ in the respective setting with

$$\left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^{k-1} \frac{\mathbb{E}[W] - \mathbb{E}[W^2]}{\mathbb{E}[W]},$$

which corresponds to the ratio of the values for ζ in Proposition 4.7 and Proposition 4.8, as we do not consider all vertices, but only those in distance k.

Finally, there is a related question which one can study by a similar approach, the one of the overall weight in the components, i.e.

$$\mathrm{vol}_\alpha(\mathcal{C}_n(v)) = \sum_{x \in \mathcal{C}_n(v)} W_x^\alpha$$

for some $\alpha > 0$. Instead of the indicator $\mathbf{1}\{v_k \in \mathcal{X}_n(v)\}$ in e.g. (4.3), we use $W_{v_k}^\alpha$. On an intuitive level, we get the following result for $v_0 \in [n]$, using approximations as in the previous section

$$\begin{aligned} \mathbb{E}_{\mathcal{W}}[\mathrm{vol}_\alpha(\mathcal{C}_n(v_0))] &\approx W_{v_0}^\alpha + \sum_{k=1}^{\infty} \sum_{(v_1,\dots,v_k) \in ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_{i-1}} W_{v_i}}{L_n} W_{v_k}^\alpha \\ &\approx W_{v_0}^\alpha + W_{v_0} \sum_{k=1}^{\infty} \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^{k-1} \frac{\mathbb{E}[W^{\alpha+1}]}{\mathbb{E}[W]}, \end{aligned} \tag{4.19}$$

where the last fraction is due to the fact that the power of W_{v_k} is given by $\alpha + 1$. One could actually prove a similar result to ours, the only difference being the value of ζ in assumption (A1). We refrained from carrying out the details here, as they do not match our framework very well. To be more precise, we repeatedly make use of the fact that every vertex can contribute at most one to the count $S_n(v)$, which is no longer true in the setting above. Nonetheless, the overall proof strategy works out. It may be possible that the Poincaré inequality yields small issues, but one can simply calculate the variance by hand, using $\mathbf{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ and path counting techniques. In this particular setting it is quite convenient, because there are no interactions between two different vertices $x, y \in \mathcal{C}_n(v)$ contributing to $\mathrm{vol}_\alpha(\mathcal{C}_n(v))$. In our setup, the Poincaré inequality was easier to deal with since, e.g. in the case of leaves, for $x, y \in \mathcal{C}_n(v)$ we obtain information on the probability of y being a leaf when we know that x is a leaf (as this prohibits the edge $\{x, y\}$).

We would like to close this section with some remarks on the value of ζ for $\mathrm{vol}_\alpha(\mathcal{C}_n(v))$. Let us start with the case $\alpha < 1$. Then, the contribution of W_v itself stays negligible due to $W^\alpha_{(n)}/q(n)$ converging to zero in probability. Thus, ζ is given by the second summand in (4.19), divided by W_{v_0}. For $\alpha = 1$, ζ is the same as in Proposition 4.7, which is quite surprising. This is due to the following simple calculation, where we use $\alpha = 1$,

$$W^\alpha_{v_0} + W_{v_0} \sum_{k=1}^{\infty} \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} \right)^{k-1} \frac{\mathbb{E}[W^{\alpha+1}]}{\mathbb{E}[W]} = W_{v_0} \left(1 + \sum_{k=1}^{\infty} \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} \right)^{k} \right) = W_{v_0} \frac{\mathbb{E}[W]}{\mathbb{E}[W] - \mathbb{E}[W^2]}.$$

For $\alpha > 1$, we require stronger moment conditions. As evident by (4.19), we require $\mathbb{E}[W^{\alpha+1}] < \infty$, which can be ensured by $\beta > \alpha + 1$. In this case, the contribution of the second summand in (4.19) becomes negligible and we need to rescale by the quantile $q(n)$ corresponding to the distribution function of W^α instead of that of W. Thus, we simply obtain $\zeta = 1$. Intuitively, the hub has by far the largest weight in its component and by using a power larger than one, it overcomes all other weights in its component. In our setup we witnessed a similar phenomenon, since the degree of the hub already gave the order of the component size.

4.4 Verifying assumptions for the applications

In this section we prove the propositions from the previous section. By Theorem 4.1 and Corollary 4.2, it suffices to show that they satisfy the assumptions (A1) and (A2). While assumption (A2) is usually more or less trivial, assumption (A1) is the lion's share of work. To this end, we establish a general framework that we use for all four examples. The idea is to explore the component $\mathcal{C}_n(v)$, starting from the vertex v. For a vertex $v_0 \in [n]$ we recall the random variable

$$T_n(v_0) = \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\}\mathbf{1}\{v_k \in \mathcal{X}_n(v_0)\}$$

from (4.3), which checks for all possible paths $v_0 \dots v_k$ whether they exist or not and if the endpoint belongs to $\mathcal{X}_n(v_0)$. If the component of v_0 is a tree, i.e. it has no cycles, then there is a unique path between any two of its vertices so that $T_n(v_0)$ equals $S_n(v_0)$

if $v_0 \notin \mathcal{X}_n(v_0)$. However, if there is a cycle, it is possible that $T_n(v_0)$ counts some vertices more than once. We define the event

$$\mathcal{B}_n(v_0) = \Big\{\exists k \geq 3, (s_1, \ldots, s_k) \in \mathcal{C}_n(v_0)^k_{\neq} \colon s_1 \leftrightarrow \ldots \leftrightarrow s_k \leftrightarrow s_1\Big\}^c,$$

which means that there are no cycles in v_0's component, and the random variable

$$\overline{T}_n(v_0) = \sum_{k=1}^{n} \sum_{(v_1,\ldots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\},$$

which is an upper bound for $T_n(v_0)$. The following lemma essentially shows that cycles are unlikely.

Lemma 4.11. *Assume* (W). *For $n \in \mathbb{N}$ and $v_0 \in [n]$ we have*

$$\mathbb{E}_{\mathcal{W}}\left[\mathbf{1}_{\mathcal{B}_n(v_0)^c}\overline{T}_n(v_0)\right] \leq W_{v_0}\frac{W^2_{(n)}}{L_n}U_n, \text{ where } U_n = 2\bigg(\sum_{k=0}^{\infty}(k+2)^2\Big(\frac{L_n^{[2]}}{L_n}\Big)^k\bigg)^2.$$

Proof. Assume without loss of generality that $v_0 = n$. Then

$$\mathbf{1}_{\mathcal{B}_n(v_0)^c}\overline{T}_n(v_0) = \mathbf{1}_{\mathcal{B}_n(v_0)^c}\sum_{k=1}^{n} \sum_{(v_1,\ldots,v_k)\in[n-1]^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\}.$$

For $\mathbf{1}_{\mathcal{B}_n(v_0)^c}\overline{T}_n(v_0)$ to be non-zero, there must be a cycle somewhere in the component of v_0 by the definition of $\mathcal{B}_n(v_0)$. Consider the path $v_0 \ldots v_k$ currently counted in $\overline{T}_n(v_0)$. For the existence of a cycle in the component of v_0 we obtain two possible cases, see also Figure 4.6:

1. Two vertices of the path, say v_p and v_q for $p < q$, can be connected by a different path $s_1 \ldots s_\ell$ with $s_1 = v_p$ and $s_\ell = v_q$, resulting in a cycle. We may assume that $s_1 \ldots s_\ell$ has no intersections other than s_1, s_ℓ with $v_0, \ldots, v_k$ by shortening it if needed.

2. There is a path $s_1 \ldots s_\ell$ with $\ell \geq 3$ starting in $s_1 = v_p$ for some $0 \leq p \leq k$ such that $s_\ell \leftrightarrow s_q$ for some $q < \ell - 1$. We may assume that $s_1 \ldots s_\ell$ has no intersections other than s_1 with $v_0 \ldots v_k$ by shortening $s_1 \ldots s_\ell$ or ending up in the first case.

Figure 4.6: The path $v_1 \ldots v_k$ is displayed in black dashed lines whereas the path $s_1 \ldots s_\ell$ is shown by red dashed lines. In 2), we also include the edge $s_\ell \leftrightarrow s_q$ (in red) to highlight the existence of a cycle.

The two cases above yield the following upper bound for $\mathbf{1}_{\mathcal{B}_n(v_0)^c}\overline{T}_n(v_0)$, where we use the notation from Lemma 4.3,

$$\sum_{k=1}^{n}\sum_{(v_1,\dots,v_k)\in[n-1]^k_{\neq}}\prod_{i=1}^{k}\mathbf{1}\{v_i\leftrightarrow v_{i-1}\}\sum_{0\le p<q\le k}\sum_{\ell=2}^{n}|\mathcal{P}_\ell^{(n)}(v_p,v_q,\{v_0,\dots,v_k\})|$$
$$+\sum_{k=1}^{n}\sum_{(v_1,\dots,v_k)\in[n-1]^k_{\neq}}\prod_{i=1}^{k}\mathbf{1}\{v_i\leftrightarrow v_{i-1}\}\sum_{p=0}^{k}\sum_{\ell=3}^{n}\sum_{x\in[n]\setminus\{v_0,\dots,v_k\}}$$
$$\times\sum_{(s_1,\dots,s_\ell)\in\mathcal{P}_\ell^{(n)}(v_p,x,\{v_0,\dots,v_k\})}\sum_{q=1}^{\ell-2}\mathbf{1}\{x\leftrightarrow s_q\}=:I_1+I_2.$$

Without any shared edges, we can use conditional independence and apply Lemma 3.5 and Lemma 4.3 to obtain

$$\begin{aligned}\mathbb{E}_{\mathcal{W}}[I_1]&\le\sum_{k=1}^{n}\sum_{(v_1,\dots,v_k)\in[n-1]^k_{\neq}}\prod_{i=1}^{k}\frac{W_{v_i}W_{v_{i-1}}}{L_n}\sum_{p,q=0}^{k}\sum_{\ell=2}^{n}\left(\frac{L_n^{[2]}}{L_n}\right)^{\ell-2}\frac{W_{(n)}^2}{L_n}\\&\le W_{v_0}\frac{W_{(n)}^2}{L_n}\sum_{k=1}^{n}\left(\frac{L_n^{[2]}}{L_n}\right)^{k-1}(k+1)^2\sum_{\ell=2}^{n}\left(\frac{L_n^{[2]}}{L_n}\right)^{\ell-2}\le W_{v_0}\frac{W_{(n)}^2}{L_n}\frac{U_n}{2}.\end{aligned}$$

For the second summand we use

$$\mathbb{E}_{\mathcal{W}}\left[\sum_{q=1}^{\ell-2}\mathbf{1}\{x\leftrightarrow s_q\}\right]\le\ell\frac{W_xW_{(n)}}{L_n}$$

so that we can bound $\mathbb{E}_{\mathcal{W}}[I_2]$ by

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{(v_1,\dots,v_k)\in[n-1]^k_{\neq}}\prod_{i=1}^{k}\frac{W_{v_i}W_{v_{i-1}}}{L_n}\sum_{p=0}^{k}\sum_{\ell=3}^{n}\sum_{x\in[n]\setminus\{v_0,\dots,v_k\}}\frac{W_{v_p}W_x}{L_n}\left(\frac{L_n^{[2]}}{L_n}\right)^{\ell-2}\ell\frac{W_xW_{(n)}}{L_n}\\&\le W_{v_0}\frac{W_{(n)}^2}{L_n}\sum_{k=1}^{n}\left(\frac{L_n^{[2]}}{L_n}\right)^{k-1}(k+1)\sum_{\ell=3}^{n}\left(\frac{L_n^{[2]}}{L_n}\right)^{\ell-2}\ell\frac{\sum_{x=1}^{n}W_x^2}{L_n}\le W_{v_0}\frac{W_{(n)}^2}{L_n}\frac{U_n}{2}.\end{aligned}$$

Summing up both bounds yields the claim. □

The following technical lemma provides sufficient conditions for verifying assumption (A1). The intuition is that one wants to determine $S_n(v_0)$ by counting paths of length k, checking locally whether the endpoint belongs to $\mathcal{X}_n(v_0)$ and summing over all possible path lengths.

Lemma 4.12. *Let $\zeta>0$. Assume* (W) *and that there exists for all $n\in\mathbb{N},k\in[n]$ and $(v_0,\dots,v_k)\in[n]^{k+1}_{\neq}$ an event $\mathcal{I}_n(v_0,\dots,v_k)$ such that*

$$\mathcal{I}_n(v_0,\dots,v_k)\text{ is conditionally on }\mathcal{W}\text{ independent of }\{v_0\leftrightarrow v_1\},\dots,\{v_{k-1}\leftrightarrow v_k\},\quad(4.20)$$
$$\mathbf{1}_{\mathcal{B}_n(v_0)}\mathbf{1}\{v_0\leftrightarrow\dots\leftrightarrow v_k,v_k\in\mathcal{X}_n(v_0)\}=\mathbf{1}_{\mathcal{B}_n(v_0)}\mathbf{1}\{v_0\leftrightarrow\dots\leftrightarrow v_k\}\mathbf{1}_{\mathcal{I}_n(v_0,\dots,v_k)}\quad(4.21)$$

and

$$\frac{1}{q(n)} \sup_{v_0 \in [n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_i} W_{v_{i-1}}}{L_n} \mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - W_{v_0}\zeta \right| \xrightarrow{\mathbb{P}} 0, \quad (4.22)$$

as $n \to \infty$*. Then, assumption* (A1) *is satisfied for that choice of* ζ*.*

Proof. We need to show

$$\frac{1}{q(n)} \sup_{v=1,\dots,n} |\mathbb{E}_{\mathcal{W}}[S_n(v)] - W_v\zeta| \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty.$$

For $v \in [n]$, adding v to $\mathcal{X}_n(v)$ or removing v from $\mathcal{X}_n(v)$ changes the value of $S_n(v)$ by one. Due to the triangle inequality and $q(n) \to \infty$ as $n \to \infty$, this does not change whether the statement above is true or false. Therefore, we may assume without loss of generality that $v \notin \mathcal{X}_n(v)$ for all $v \in [n]$. For $v_0 \in [n]$ we write

$$\widetilde{T}_n(v_0) = \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\} \mathbf{1}_{\mathcal{I}_n(v_0,\dots,v_k)}$$

and obtain with $v \notin \mathcal{X}_n(v)$ and (4.21) for all $v \in [n]$ that

$$\mathbf{1}_{\mathcal{B}_n(v)} S_n(v) = \mathbf{1}_{\mathcal{B}_n(v)} T_n(v) = \mathbf{1}_{\mathcal{B}_n(v)} \widetilde{T}_n(v).$$

We conclude

$$\begin{aligned}
\mathbb{E}_{\mathcal{W}}[S_n(v)] &= \mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)} S_n(v)] + \mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} S_n(v)] \\
&= \mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)} \widetilde{T}_n(v)] + \mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} S_n(v)] \\
&= \mathbb{E}_{\mathcal{W}}[\widetilde{T}_n(v)] - \mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} \widetilde{T}_n(v)] + \mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} S_n(v)]
\end{aligned}$$

so that $S_n(v), \widetilde{T}_n(v) \le \overline{T}_n(v)$ and Lemma 4.11 yield

$$|\mathbb{E}_{\mathcal{W}}[S_n(v)] - \mathbb{E}_{\mathcal{W}}[\widetilde{T}_n(v)]| \le 2\mathbb{E}_{\mathcal{W}}[\mathbf{1}_{\mathcal{B}_n(v)^c} \overline{T}_n(v)] \le 2W_v \frac{W_{(n)}^2}{L_n} U_n \le 2\frac{W_{(n)}^3}{L_n} U_n.$$

This in turn provides us with

$$\frac{1}{q(n)} \sup_{v \in [n]} |\mathbb{E}_{\mathcal{W}}[S_n(v)] - W_v\zeta| \le \frac{1}{q(n)} \sup_{v \in [n]} |\mathbb{E}_{\mathcal{W}}[\widetilde{T}_n(v)] - W_v\zeta| + \frac{2W_{(n)}^3}{q(n)L_n} U_n.$$

Here, the last summand converges in probability to zero as $n \to \infty$, see (4.6) for U_n and Proposition 2.21 as well as Lemma 2.27 c) for the other factors. Next, we bound the supremum on the right-hand side above. By (4.20) we have

$$\mathbb{E}_{\mathcal{W}}[\widetilde{T}_n(v_0)] = \mathbb{E}_{\mathcal{W}}\Big[\sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\} \mathbf{1}_{\mathcal{I}_n(v_0,\dots,v_k)}\Big]$$

$$= \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \mathbb{P}_{\mathcal{W}}(v_i \leftrightarrow v_{i-1}) \mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k))$$

so that the triangle inequality yields

$$\begin{aligned}
&\frac{1}{q(n)} \sup_{v_0\in[n]} |\mathbb{E}_{\mathcal{W}}[\widetilde{T}_n(v_0)] - W_{v_0}\zeta| \\
&\leq \frac{1}{q(n)} \sup_{v_0\in[n]} \Big| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - W_{v_0}\zeta \Big| \\
&\quad + \frac{1}{q(n)} \sup_{v_0\in[n]} \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \Big| \prod_{i=1}^{k} \mathbb{P}_{\mathcal{W}}(v_i \leftrightarrow v_{i-1}) - \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \Big|.
\end{aligned}$$

By assumption (4.22), the first term on the right-hand side converges in probability to zero as $n \to \infty$. For the second summand we obtain from Lemma 3.5 for all $v_0, \dots, v_k \in [n]$,

$$\begin{aligned}
&\Big| \prod_{i=1}^{k} \mathbb{P}_{\mathcal{W}}(v_i \leftrightarrow v_{i-1}) - \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \Big| \leq \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \left(1 - \left(1 - \min\left(1, \frac{W_{(n)}^2}{L_n}\right)\right)^k\right) \\
&= W_{v_0} \prod_{i=1}^{k-1} \frac{W_{v_i}^2}{L_n} \frac{W_{v_k}}{L_n} \left(1 - \left(1 - \min\left(1, \frac{W_{(n)}^2}{L_n}\right)\right)^k\right) \leq W_{v_0} \prod_{i=1}^{k-1} \frac{W_{v_i}^2}{L_n} \frac{W_{v_k}}{L_n} k \frac{W_{(n)}^2}{L_n},
\end{aligned}$$

where the last inequality uses that $1 - y^k \leq k(1-y)$ for all $y \in [0,1]$ by applying the mean value theorem to the function $f\colon [0,1] \to \mathbb{R}$ with $f(x) = x^k$. We obtain

$$\begin{aligned}
&\frac{1}{q(n)} \sup_{v_0\in[n]} \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in([n]\setminus\{v_0\})^k_{\neq}} \Big| \prod_{i=1}^{k} \mathbb{P}_{\mathcal{W}}(v_i \leftrightarrow v_{i-1}) - \prod_{i=1}^{k} \frac{W_{v_i}W_{v_{i-1}}}{L_n} \Big| \\
&\leq \frac{1}{q(n)} \sup_{v_0\in[n]} \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k)\in[n]^k} W_{v_0} \prod_{i=1}^{k-1} \frac{W_{v_i}^2}{L_n} \frac{W_{v_k}}{L_n} k \frac{W_{(n)}^2}{L_n} = \frac{W_{(n)}^3}{q(n)L_n} \sum_{k=1}^{n} k \left(\frac{L_n^{[2]}}{L_n}\right)^{k-1},
\end{aligned}$$

which converges in probability to zero as $n \to \infty$ due to Proposition 2.21, Lemma 2.27 c) and (4.6). □

The following lemma yields a more tractable condition than (4.22) and provides a way to compute the value of ζ.

Lemma 4.13. *Assume* (W). *Suppose that for all $n \in \mathbb{N}, k \in [n]$ and $(v_0, \dots, v_k) \in [n]^{k+1}_{\neq}$ there exists an event $\mathcal{I}_n(v_0, \dots, v_k)$ such that* (4.20) *as well as* (4.21) *hold. Moreover, assume that there exist a bounded, measurable function $g\colon [0,\infty)\times\mathbb{N} \to [0,\infty)$, a polynomial p of finite degree and random variables $(R_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}, k \in [n]$ and $(v_0, \dots, v_k) \in [n]^{k+1}_{\neq}$,*

$$|\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - g(W_{v_k}, k)| \leq p(k)R_n \tag{4.23}$$

and

$$R_n \xrightarrow{\mathbb{P}} 0 \quad as \quad n \to \infty. \tag{4.24}$$

Then, (A1) *is satisfied with*

$$\zeta = \sum_{k=1}^{\infty} \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^{k-1} \frac{\mathbb{E}[Wg(W,k)]}{\mathbb{E}[W]}.$$

Proof. By Lemma 4.12 it suffices to show (4.22) for this choice of ζ. Using the triangle inequality, we decompose the sum in (4.22) into three parts, the second one making up for missing summands. We have

$$\begin{aligned}
&\frac{1}{q(n)} \sup_{v_0 \in [n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_i} W_{v_{i-1}}}{L_n} \mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - W_{v_0}\zeta \right| \\
&\le \frac{1}{q(n)} \sup_{v_0 \in [n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_i} W_{v_{i-1}}}{L_n} \Big(\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - g(W_{v_k},k)\Big) \right| \\
&\quad + \frac{1}{q(n)} \sup_{v_0 \in [n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in [n]^k \setminus ([n]\setminus\{v_0\})^k_{\neq}} \prod_{i=1}^{k} \frac{W_{v_i} W_{v_{i-1}}}{L_n} g(W_{v_k},k) \right| \\
&\quad + \frac{1}{q(n)} \sup_{v_0 \in [n]} \left| \sum_{k=1}^{n} \sum_{(v_1,\dots,v_k) \in [n]^k} \prod_{i=1}^{k} \frac{W_{v_i} W_{v_{i-1}}}{L_n} g(W_{v_k},k) - W_{v_0}\zeta \right| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Due to (4.23) we have

$$I_1 \le \frac{W_{(n)}}{q(n)} \sum_{k=1}^{n} \left(\frac{L_n^{[2]}}{L_n}\right)^{k-1} p(k) R_n,$$

which converges to zero in probability as $n \to \infty$, by Proposition 2.21, (4.6) and (4.24). For I_2, we consider the elements $(v_1,\dots,v_k) \in [n]^k \setminus ([n] \setminus \{v_0\})^k_{\neq}$ for a fixed $v_0 \in [n]$, which means that $(v_1,\dots,v_k)$ has (at least) two equal entries or contains v_0. There are $\binom{k}{2}$ choices for $i,j \in [k]$ with $v_i = v_j$ for $i < j$. In this case, we may bound $W_{v_i}^2/L_n$ by $W_{(n)}^2/L_n$ and omit the summation over v_i. On the other hand, it may be that there exists $i \in [k]$ such that $v_i = v_0$. If $i < k$, we have $(k-1)$ choices and can once more omit the summation over v_i and bound the respective factor by $W_{(n)}^2/L_n$. For $i = k$, we obtain a different scenario due to the factor $g(W_{v_k},k)$. We derive

$$\begin{aligned}
I_2 &\le \frac{W_{(n)}^3}{L_n q(n)} \sum_{k=2}^{n} \left(\binom{k}{2} + k\right) \left(\frac{L_n^{[2]}}{L_n}\right)^{k-2} \frac{\sum_{i=1}^n W_i g(W_i,k)}{L_n} \\
&\quad + \sum_{k=1}^{n} \frac{\sup_{v_0 \in [n]} W_{v_0}^2 g(W_{v_0},k)}{L_n q(n)} \left(\frac{L_n^{[2]}}{L_n}\right)^{k-1}.
\end{aligned}$$

In the first summand, the first factor vanishes as $n \to \infty$ due to Proposition 2.21 and Lemma 2.27 c), while the two sums are bounded because of (4.6) as well as the boundedness of g and the law of large numbers. For the second summand, the first fraction converges in probability to zero as $n \to \infty$ by the boundedness of g and Lemma 2.27 c), whereas the remaining sum converges almost surely as $n \to \infty$ by (4.5).

For I_3 we obtain

$$I_3 = \frac{W_{(n)}}{q(n)} \left| \sum_{k=1}^{n} \left(\frac{L_n^{[2]}}{L_n} \right)^{k-1} \frac{\sum_{i=1}^{n} W_i g(W_i, k)}{L_n} - \zeta \right|.$$

Proposition 2.21 shows that the first fraction converges in distribution. From the strong law of large numbers and the boundedness of g we deduce

$$\sum_{k=1}^{n} \left(\frac{L_n^{[2]}}{L_n} \right)^{k-1} \frac{\sum_{i=1}^{n} W_i g(W_i, k)}{L_n} \xrightarrow{a.s.} \sum_{k=1}^{\infty} \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} \right)^{k-1} \frac{\mathbb{E}[W g(W,k)]}{\mathbb{E}[W]} = \zeta$$

as $n \to \infty$. Thus, the absolute value in I_3 converges almost surely to zero, which finishes the proof. □

Now, we proceed with verifying the propositions from the previous section, where we rely on Lemma 4.13 for showing that assumption (A1) is satisfied.

Proof of Proposition 4.7. For $n \in \mathbb{N}, k \in [n]$ and $(v_0, \dots, v_k) \in [n]_{\neq}^{k+1}$ we can simply choose $\mathcal{I}_n(v_0, \dots, v_k) = \Omega$, $g = 1$ as well as $p = R_n = 0$. In this case it is easy to see that all assumptions of Lemma 4.13 are met, showing (A1). For (A2) it suffices to note that the existence of any path starting in v and containing x already shows that $x \in \mathcal{X}_n(v)$. □

Proof of Proposition 4.8. For $n \in \mathbb{N}, k \in [n]$ as well as $(v_0, \dots, v_k) \in [n]_{\neq}^{k+1}$ we choose $\mathcal{I}_n(v_0, \dots, v_k) = \{k = m\}$, $g = \mathbf{1}\{k = m\}$ and $p = R_n = 0$. Again, we conclude (A1) from Lemma 4.13. If we have access to all paths starting in v and ending in a given vertex $x \in \mathcal{C}_n(v)$, we can in particular decide which one is the shortest. Thus, (A2) holds. □

Proof of Proposition 4.9. We start with the case $m = 1$, i.e. we consider leaves. We define for $n \in \mathbb{N}, k \in [n]$ and $(v_0, \dots, v_k) \in [n]_{\neq}^{k+1}$,

$$\mathcal{I}_n(v_0, \dots, v_k) = \{v_k \text{ has no neighbour in } [n] \setminus \{v_{k-1}, v_k\}\}.$$

Then, $\mathcal{I}_n(v_0, \dots, v_k)$ satisfies (4.20) and (4.21) since the path $v_0 \dots v_k$ ensures that v_k already has one neighbour. Additionally, let $g(x, k) = e^{-x}$ and $p = 1$. For (4.23) we calculate

$$\begin{aligned} &\left| \mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \dots, v_k)) - e^{-W_{v_k}} \right| = \left| \mathbb{E}_{\mathcal{W}} \Big[\prod_{x \in [n] \setminus \{v_{k-1}, v_k\}} \mathbf{1}\{v_k \nleftrightarrow x\} \Big] - e^{-W_{v_k}} \right| \\ &= \left| \exp\left(- W_{v_k} \frac{\sum_{x \in [n] \setminus \{v_{k-1}, v_k\}} W_x}{L_n} \right) - e^{-W_{v_k}} \right| \\ &\le e^{-W_{v_k}} \left(e^{2W_{(n)}/L_n} - 1 \right) \le e^{2W_{(n)}/L_n} - 1 =: R_n. \end{aligned}$$

From Lemma 2.27 c) we deduce that $W_{(n)}/L_n$ converges to zero in probability as $n \to \infty$, as $W_{(n)}^2 \geq W_{(n)}$ for $W_{(n)} \geq 1$ whereas it is trivial for $W_{(n)} \leq 1$. We conclude that (4.24) holds. We may thus apply Lemma 4.13 to obtain (A1).

For vertices of degree $m \geq 2$ we proceed similarly. We define

$$\mathcal{I}_n(v_0, \dots, v_k) = \{v_k \text{ has exactly } m-1 \text{ neighbours in } [n] \setminus \{v_{k-1}, v_k\}\}$$

and see that (4.20) and (4.21) hold. We rewrite $\mathbf{1}_{\mathcal{I}_n(v_0,\dots,v_k)}$ as

$$\frac{1}{(m-1)!} \sum_{(a_1,\dots,a_{m-1}) \in ([n]\setminus\{v_{k-1},v_k\})^{m-1}_{\neq}} \prod_{i=1}^{m-1} \mathbf{1}\{v_k \leftrightarrow a_i\} \prod_{x \in [n]\setminus\{a_1,\dots,a_{m-1},v_{k-1},v_k\}} \mathbf{1}\{v_k \not\leftrightarrow x\}$$

and define $g(x,k) = x^{m-1}e^{-x}/(m-1)!$, which is a bounded function as required. For (4.23) we provide upper and lower bounds for $\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k))$. We compute with the equality above and Lemma 3.5,

$$\begin{aligned}
&\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) \\
&\leq \frac{1}{(m-1)!} \sum_{a_1,\dots,a_{m-1}=1}^{n} \prod_{i=1}^{m-1} \frac{W_{a_i}W_{v_k}}{L_n} \exp\Big(- W_{v_k} \sum_{x \in [n]\setminus\{a_1,\dots,a_{m-1},v_{k-1},v_k\}} \frac{W_x}{L_n}\Big) \\
&\leq \frac{W_{v_k}^{m-1}e^{-W_{v_k}}}{(m-1)!} \sum_{a_1,\dots,a_{m-1}=1}^{n} \prod_{i=1}^{m-1} \frac{W_{a_i}}{L_n} \exp\Big(\frac{(m+1)W_{(n)}^2}{L_n}\Big) \\
&= g(W_{v_k},k) \exp\Big(\frac{(m+1)W_{(n)}^2}{L_n}\Big) =: g(W_{v_k},k)\overline{I}_n,
\end{aligned}$$

where the upper bound $\overline{I}_n$ satisfies $\overline{I}_n \xrightarrow{\mathbb{P}} 1$ as $n \to \infty$ by Lemma 2.27 c). Similarly, by Lemma 3.5,

$$\begin{aligned}
\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) &\geq \frac{W_{v_k}^{m-1}e^{-W_{v_k}}}{(m-1)!} \sum_{(a_1,\dots,a_{m-1}) \in ([n]\setminus\{v_{k-1},v_k\})^{m-1}_{\neq}} \prod_{i=1}^{m-1} \frac{W_{a_i}}{L_n} \Big(1 - \frac{W_{(n)}^2}{L_n}\Big)^{m-1} \\
= g(W_{v_k},k)\Big(1 - \frac{W_{(n)}^2}{L_n}\Big)^{m-1} &\Big(1 - \sum_{(a_1,\dots,a_{m-1}) \in [n]^{m-1}\setminus([n]\setminus\{v_{k-1},v_k\})^{m-1}_{\neq}} \prod_{i=1}^{m-1} \frac{W_{a_i}}{L_n}\Big).
\end{aligned}$$

The last sum above involves all tuples a of length $m-1$ which either contain the same entry twice or contain at least one entry equal to v_{k-1} or v_k. This leads to

$$\sum_{(a_1,\dots,a_{m-1}) \in [n]^{m-1}\setminus([n]\setminus\{v_{k-1},v_k\})^{m-1}_{\neq}} \prod_{i=1}^{m-1} \frac{W_{a_i}}{L_n} \leq \frac{W_{(n)}}{L_n}\left(\binom{m}{2} + 2m\right) \leq 3m^2 \frac{W_{(n)}}{L_n}$$

so that

$$\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) \geq g(W_{v_k},k)\Big(1 - \frac{W_{(n)}^2}{L_n}\Big)^{m-1}\Big(1 - 3m^2\frac{W_{(n)}}{L_n}\Big) =: g(W_{v_k},k)\underline{I}_n,$$

where the lower bound $\underline{I}_n$ satisfies $\underline{I}_n \xrightarrow{\mathbb{P}} 1$ as $n \to \infty$ since both $W_{(n)}^2/L_n$ and $W_{(n)}/L_n$ converge to zero in probability as $n \to \infty$, which follows from Lemma 2.27. For $k \in [n]$ and $p = \sup_{x \in [0,\infty)} g(x,k)$ we obtain

$$\begin{aligned}\left|\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - g(W_{v_k},k)\right| &\le g(W_{v_k},k)(|1-\overline{I}_n| + |1-\underline{I}_n|)\\ &\le p(|1-\overline{I}_n| + |1-\underline{I}_n|) =: pR_n,\end{aligned}$$

which shows (4.23). From $\underline{I}_n, \overline{I}_n \xrightarrow{\mathbb{P}} 1$ as $n \to \infty$ it follows $R_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ as demanded in (4.24). Using Lemma 4.13 yields (A1). Assumption (A2) holds true since, for any $x \in \mathcal{C}_n(v)$, knowledge of all paths that start in v and contain x allows us to determine $\deg(x)$. Here it is important to note that the paths do not need to *end* in x, they may as well pass through. □

Proof of Proposition 4.10. We consider a tree T with m vertices $1,\dots,m$ and root 1. We write $V(T) = [m]$ for its vertices and $E(T)$ for its edges. For $n \in \mathbb{N}, k \in [n]$ and $(v_0,\dots,v_k) \in [n]_{\neq}^{k+1}$ let

$$\begin{aligned}\mathcal{I}_n(v_0,\dots,v_k) = \{\exists (a_1,\dots,a_m) \in ([n]\setminus\{v_0,\dots,v_{k-1}\})_{\neq}^m \text{ with } a_1 = v_k \text{ such that the}\\ \text{graph induced by } a_1,\dots,a_m \text{ with } a_1 \text{ as root is isomorphic to } T\\ \text{and, after deleting the edge } \{v_{k-1},v_k\}, \text{ the vertices } a_1,\dots,a_m\\ \text{form a component in } G_n\},\end{aligned}$$

$p(k) = k+1$ and with $c(T)$ as in Proposition 4.10,

$$g(x,k) = \frac{x^{\deg_T(1)}e^{-x}}{c(T)} \prod_{i=2}^{m} \frac{\mathbb{E}[W^{\deg_T(i)}e^{-W}]}{\mathbb{E}[W]}.$$

We note that (4.20) and (4.21) are satisfied by the choice of $\mathcal{I}_n(v_0,\dots,v_k)$. We write $\varphi\colon V(T) \to [n]$ for the function $i \mapsto a_i$ which maps a vertex of T to its corresponding vertex in G_n, where G_n denotes the realisation of $\mathrm{NR}(n)$. Consequently, $\varphi(E(T)) = \{\{\varphi(i),\varphi(j)\}\colon i,j \in V(T)\}$ denotes the edges of T embedded into G_n via φ. For the graph induced by $\varphi(T)$ to be isomorphic to T, we require all edges in $\varphi(E(T))$ to exist in G_n while no further connections between any vertices in $\varphi(V(T))$ are allowed. Furthermore, all edges between $\varphi(V(T))$ and $[n]\setminus\varphi(V(T))$ are forbidden since the tree is supposed to be terminal, up to the exception of $a_1 = v_k$ being connected to v_{k-1}. We gather all these forbidden edges in the set

$$F_a = \{\{a_x,y\}\colon x \in V(T), y \in [n]\setminus\{a_x\}\} \setminus (\varphi(E(T)) \cup \{v_k,v_{k-1}\}).$$

Considering all possible choices for $a_1,\dots,a_m$ and permutations thereof yields

$$\mathbf{1}_{\mathcal{I}_n(v_0,\dots,v_k)} = c(T)^{-1} \sum_{\substack{(a_1,\dots,a_m)\in([n]\setminus\{v_0,\dots,v_{k-1}\})_{\neq}^m\\ a_1=v_k}} \prod_{\{i,j\}\in E(T)} \mathbf{1}\{a_i \leftrightarrow a_j\} \prod_{\{i,j\}\in F_a} \mathbf{1}\{i \not\leftrightarrow j\}. \quad (4.25)$$

All occurring factors are independent when conditioning on $\mathcal{W}$. For (4.23) we provide upper and lower bounds of $\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k))$. We start with the last product concerning the forbidden edges. Let $a_1 = v_k$ and $(a_2, \dots, a_m) \in ([n] \setminus \{v_0, \dots, v_k\})_{\neq}^{m-1}$. We get

$$\mathbb{E}_{\mathcal{W}}\Big[\prod_{\{i,j\}\in F_a} \mathbf{1}\{i \nleftrightarrow j\}\Big] = \prod_{\{i,j\}\in F_a} \mathbb{P}_{\mathcal{W}}(i \nleftrightarrow j) = \prod_{\{i,j\}\in F_a} \exp\Big(-\frac{W_iW_j}{L_n}\Big).$$

Because all factors are bounded by one, we may add similar factors for a lower bound. Since $F_a \subseteq \{\{a_x, y\}\colon x \in V(T), y \in [n]\}$, this yields

$$\mathbb{E}_{\mathcal{W}}\Big[\prod_{\{i,j\}\in F_a} \mathbf{1}\{i \nleftrightarrow j\}\Big] \geq \prod_{x=1}^{m}\prod_{y=1}^{n} \exp\Big(-\frac{W_{a_x}W_y}{L_n}\Big) = \prod_{x=1}^{m} \exp(-W_{a_x}). \tag{4.26}$$

For an upper bound we observe that $F_a \supseteq \{\{a_x, y\}\colon x \in V(T), y \in [n] \setminus (\varphi(V(T)) \cup \{v_{k-1}, a_x\})\}$. In comparison to $\{\{a_x, y\}\colon x \in V(T), y \in [n]\}$, there are at most $m(m+2)$ fewer elements. Therefore,

$$\begin{aligned}\mathbb{E}_{\mathcal{W}}\Big[\prod_{\{i,j\}\in F_a} \mathbf{1}\{i \nleftrightarrow j\}\Big] &\leq \prod_{x=1}^{m}\prod_{y=1}^{n} \exp\Big(-\frac{W_{a_x}W_y}{L_n}\Big) \exp\Big(m(m+2)\frac{W_{(n)}^2}{L_n}\Big) \\ &= \prod_{x=1}^{m} \exp(-W_{a_x}) \exp\Big(m(m+2)\frac{W_{(n)}^2}{L_n}\Big) =: \prod_{x=1}^{m} \exp(-W_{a_x}) \overline{I}_n,\end{aligned} \tag{4.27}$$

where $\overline{I}_n$ converges in probability to 1 as $n \to \infty$ due to Lemma 2.27 c). Next, we consider the first product in (4.25). From Lemma 3.5 and the fact that a tree with m vertices has $m-1$ edges we get the upper bound

$$\mathbb{E}_{\mathcal{W}}\Big[\prod_{\{i,j\}\in E(T)} \mathbf{1}\{a_i \leftrightarrow a_j\}\Big] \leq \prod_{\{i,j\}\in E(T)} \frac{W_{a_i}W_{a_j}}{L_n} = W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \tag{4.28}$$

and similarly the lower bound

$$\begin{aligned}\mathbb{E}_{\mathcal{W}}\Big[\prod_{\{i,j\}\in E(T)} \mathbf{1}\{a_i \leftrightarrow a_j\}\Big] &\geq W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \times \Big(1 - \min\Big(1, \frac{W_{(n)}^2}{L_n}\Big)\Big)^{m-1} \\ &=: W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \underline{I}_n,\end{aligned} \tag{4.29}$$

where $\underline{I}_n$ converges in probability to 1 as $n \to \infty$ by Lemma 2.27 c). Using the upper bounds (4.27) and (4.28) for the conditional expectation of (4.25) yields

$$\begin{aligned}\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0, \dots, v_k)) &\leq c(T)^{-1} \sum_{\substack{(a_1,\dots,a_m)\in([n]\setminus\{v_0,\dots,v_{k-1}\})_{\neq}^m \\ a_1 = v_k}} W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \prod_{x=1}^{m} \exp(-W_{a_x}) \overline{I}_n \\ &\leq \frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} \prod_{i=2}^{m} \frac{\sum_{a_i=1}^{n} W_{a_i}^{\deg_T(i)} e^{-W_{a_i}}}{L_n} \overline{I}_n =: \frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} X_n \overline{I}_n,\end{aligned}$$

where the weak law of large numbers gives us

$$X_n \xrightarrow{\mathbb{P}} \prod_{i=2}^{m} \frac{\mathbb{E}[W^{\deg_T(i)} e^{-W}]}{\mathbb{E}[W]} =: X \quad \text{as} \quad n \to \infty.$$

The lower bounds (4.26) and (4.29) provide

$$\begin{aligned}
&\mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) \geq c(T)^{-1} \sum_{\substack{(a_1,\dots,a_m)\in([n]\setminus\{v_0,\dots,v_{k-1}\})^m_{\neq} \\ a_1=v_k}} W_{a_1}^{\deg_T(1)} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)}}{L_n} \prod_{x=1}^{m} \exp(-W_{a_x}) \underline{I}_n \\
&= \frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} \underline{I}_n \Bigg(X_n - \sum_{(a_2,\dots,a_m)\in[n]^{m-1}\setminus([n]\setminus\{v_0,\dots,v_k\})^{m-1}_{\neq}} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)} e^{-W_{a_i}}}{L_n} \Bigg).
\end{aligned}$$

In the last term we sum over all tuples $(a_2,\dots,a_m)$ which contain two equal entries or one entry from the set $\{v_0,\dots,v_k\}$. The number of such tuples is bounded by

$$\binom{m-1}{2} n^{m-2} + (k+1)(m-1)n^{m-2},$$

where the first summand accounts for two equal entries whereas the second one considers the case where one entry lies in $\{v_0,\dots,v_k\}$. As $C := \max_{j=1,\dots,m} \sup_{x\in[0,\infty)} x^j e^{-x} < \infty$ and $\underline{I}_n \leq 1, \deg_T(i) \leq m$, we obtain

$$\begin{aligned}
&\frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} \underline{I}_n \sum_{(a_2,\dots,a_m)\in[n]^{m-1}\setminus([n]\setminus\{v_0,\dots,v_k\})^{m-1}_{\neq}} \prod_{i=2}^{m} \frac{W_{a_i}^{\deg_T(i)} e^{-W_{a_i}}}{L_n} \\
&\leq \frac{C^m}{c(T)} \frac{n^{m-2}}{L_n^{m-1}} \left(\binom{m-1}{2} + (k+1)(m-1) \right) \\
&\leq (k+1) \frac{C^m}{c(T)} \frac{n^{m-2}}{L_n^{m-1}} \left(\binom{m-1}{2} + (m-1) \right) =: (k+1) J_n = p(k) J_n,
\end{aligned}$$

where J_n converges in probability to 0 as $n \to \infty$. The lower and upper bound provide

$$\begin{aligned}
&\Big| \mathbb{P}_{\mathcal{W}}(\mathcal{I}_n(v_0,\dots,v_k)) - g(W_{v_k}, k) \Big| \\
&\leq p(k) J_n + \frac{W_{v_k}^{\deg_T(1)} e^{-W_{v_k}}}{c(T)} \Big(|X - X_n \overline{I}_n| + |X - X_n \underline{I}_n| \Big) \\
&\leq p(k) \bigg(J_n + \frac{C}{c(T)} \Big(|X - X_n \overline{I}_n| + |X - X_n \underline{I}_n| \Big) \bigg) =: p(k) R_n,
\end{aligned}$$

which is condition (4.23). Since $J_n, X_n, \overline{I}_n$ and $\underline{I}_n$ do not depend on k and converge in probability to $0, X, 1$ and 1 as $n \to \infty$, respectively, we conclude that $R_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, showing (4.24). Thus, Lemma 4.13 is applicable, which implies that (A1) is satisfied. As before, we see that assumption (A2) is also met. □

4.5 Transfer of the results to related models

The following theorem concerns the applicability of our results when considering $\mathrm{NR}'(n)$ or any graph $G(n)$ or $G'(n)$ for $G \in \mathcal{G} = \{\mathrm{CL}, \mathrm{GRG}\}$ instead of $\mathrm{NR}(n)$. For all these models we could mimic the proof for $\mathrm{NR}(n)$ as most calculations only use the bounds from Lemma 3.5. There are two points where adaptions are in order. On the one hand, we make explicit use of the connection probabilities in the proofs of Proposition 4.7, Proposition 4.8, Proposition 4.9 and Proposition 4.10. The calculations should be manageable with the other connection probabilities as well. On the other hand, the variance bound in Lemma 4.5 relies on the Poincaré inequality, which needs an underlying Poisson process. While such a structure exists for $\mathrm{NR}'(n)$, the other four models do not posses such a structure. However, one could also calculate the variance by hand via path counting techniques. We shall instead use the notions of asymptotic equivalence and contiguity provided in Subsection 3.1.2.

Theorem 4.14. *Suppose that assumption* (W) *holds true. Then, the following hold.*

a) The convergences in (4.1) *and in* (4.2) *either hold for all three models $G(n)$ with $G \in \mathcal{G}$ or for none of them. The same applies to the three models $G'(n)$ with $G \in \mathcal{G}$.*

b) Suppose that the assumptions (A1) *and* (A2) *are satisfied for* $\mathrm{NR}(n)$. *Then, the convergences in* (4.1) *and* (4.2) *remain true when considering $G(n)$ and $G'(n)$ for $G \in \mathcal{G}$ instead of* $\mathrm{NR}(n)$.

c) For all $G \in \mathcal{G}$, Proposition 4.7, Proposition 4.8, Proposition 4.9 and Proposition 4.10 remain true when considering $G(n)$ and $G'(n)$ instead of $\mathrm{NR}(n)$.

Proof. The first claim is clear as the three models are asymptotically equivalent in either case by Lemma 3.6.

From Theorem 4.1 and Corollary 4.2 we obtain the statement for $\mathrm{NR}(n)$. By part a) above, this extends to $\mathrm{CL}(n)$ and $\mathrm{GRG}(n)$. Moreover, again by part a), it suffices to transfer the result from $\mathrm{NR}(n)$ to $\mathrm{NR}'(n)$. In the proof of Theorem 4.1, we establish (4.1) for $\mathrm{NR}(n)$ by verifying (4.17). Formally speaking, this statement concerns the convergence of a functional of the random graph sequence *and* its weights in probability to zero as $n \to \infty$. Thus, it is a statement regarding $(\mathbf{W}_n, \mathrm{NR}(n))$, where $\mathbf{W}_n = (W_1, \dots, W_n)$ denotes the underlying weights. By Lemma 3.7, we obtain that $(\mathbf{W}_n, \mathrm{NR}'(n))$ is contiguous with respect to $(\mathbf{W}_n, \mathrm{NR}(n))$. Therefore, the convergence in (4.17) also holds for $(\mathbf{W}_n, \mathrm{NR}'(n))$. This shows (4.1) for $\mathrm{NR}'(n)$, which in turn implies (4.2) due to Lemma 2.12. This shows the second part of the theorem above.

For the last claim it suffices to note that we showed the four propositions by verifying assumptions (A1) and (A2) for $\mathrm{NR}(n)$. Thus, part c) follows from part b). This concludes the proof. □

Chapter 5

Cycles in the generalised random graph

This chapter is devoted to the study of cycles in the Chung-Lu model, the generalised random graph and the Norros-Reittu model. To be more precise, we are interested in the following questions:

a) What is the asymptotic behaviour of the number of cycles having a specific length?

b) What are the asymptotic distributions of the shortest and of the longest cycle in the graph?

c) Can we answer the previous questions in a quantitative way?

After formalising the questions above, we discuss existing results in Section 5.1. The framework of the Chen-Stein method for Poisson approximation in our particular situation is presented in Section 5.2. Based on this approach, we answer all three questions above. We prove a qualitative result concerning convergence of cycle counts to Poisson random variables in Section 5.3 under very mild assumptions. Afterwards, we quantify similar results under stronger assumptions in Section 5.4. Finally, we derive convergence in distribution of the length of the shortest and longest cycle in the subcritical regime in Section 5.5.

The results and proofs of this chapter are taken from [68] by Lienau. In this thesis, we add details for the proofs concerning the models where one normalises the connection probability with $n\mathbb{E}[W]$ instead of L_n. In the aforementioned paper, they were omitted.

5.1 Introduction

We start with introducing some notation in order to formalize the three questions above. Fix some underlying sequence $(G(n))_{n\in\mathbb{N}}$ of random graphs. For $A \subseteq \mathbb{N}_{\geq 3}$ let $\mathcal{C}_n(A)$ denote the number of cycles in $G(n)$ whose length lies in A. We are interested in properties of the underlying point process $\mathcal{C}_n$ on $\mathbb{N}_{\geq 3}$. For a point process ξ on $\mathbb{N}_{\geq 3}$, $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in \mathbb{N}_{\geq 3}$ we write $\xi(x_1, \ldots, x_k)$ instead of $\xi(\{x_1, \ldots, x_k\})$ to simplify notation.

In the case of the Erdős-Rényi graph, a Poisson limit for $\mathcal{C}_n(k)$ was already established in [35], see Theorem 3a therein. More information on cycles and more general subgraphs in Erdős-Rényi graphs and random regular graphs can be found in [20, Chapter 4 and

Section 2.4] and the references therein. Our aim is to analyse $\mathcal{C}_n$ when the underlying graph sequence is $G(n)$ or $G'(n)$ for $G \in \mathcal{G}$, i.e. for any of the rank-1 models introduced in Subsection 3.1.1.

Throughout this chapter, let η denote a Poisson process on $\mathbb{N}_{\geq 3}$ with intensity measure given by

$$\lambda(\{k\}) = \lambda_k = \frac{1}{2k}\left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k \tag{5.1}$$

for $k \in \mathbb{N}_{\geq 3}$ and W following the weight distribution of the underlying rank-1 model.

Recently, Janssen, van Leeuwaarden and Shneer derived results for $\mathbb{E}[\mathcal{C}_n(k)]$, the expected number of cycles of a fixed length k, in the model $\mathrm{CL}'(n)$ when the weight distribution has a regularly varying tail with parameter $\beta \in (1, 2)$ in [60]. More precisely, the authors present an integral representation for the expected number of cycles (as well as cliques) and provide an asymptotically equivalent expression that simplifies the appearing integrals. This builds on results of Bianconi and Marsili in [15], who required a cutoff for the weight distribution. Additionally, Gao, van der Hofstad, Southwell and Stegehuis investigate the asymptotic expected number of triangles, in our notation $\mathbb{E}[\mathcal{C}_n(3)]$, and show that most of the triangles are found among vertices of degree $\sqrt{n}$, see [41]. A similar approach is employed by van der Hofstad, van Leeuwaarden and Stegehuis in [55] to determine the asymptotic expected number of more general subgraphs, including cycles of any length. The aforementioned results are all in the case where $\mathbb{E}[W^2] = \infty$, so in particular in the supercritical regime.

There are also results for weights with lighter tails, which align with the regime we investigate. In [70], Liu and Dong show for bounded weights that

$$\mathcal{C}_n(3) \xrightarrow{d} \eta(3) \quad \text{as} \quad n \to \infty,$$

i.e. that the number of triangles converges to a Poisson random variable with parameter λ_3, by calculating factorial moments. This result was generalised by Bobkov, Danshina and Ulyanov to cycles of arbitrary length in [18], where it was shown that for all $k \in \mathbb{N}_{\geq 3}$ there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$d_{TV}(\mathcal{C}_n(k), \eta(k)) \leq \frac{C}{\sqrt{n}}$$

if $\mathbb{E}[W^{2k+1}] < \infty$ or a slightly weaker tail condition holds.

In this chapter we strengthen this result in different ways. In Section 5.3 we are only interested in a qualitative result and manage to reduce the moment condition to $\mathbb{E}[W^2] < \infty$, which no longer depends on k. In order to keep λ_k finite, this cannot be relaxed any further. In fact, Theorem 5.1 shows weak convergence as discussed in Section 2.2 of $\mathcal{C}_n$ to η as $n \to \infty$. As mentioned in Lemma 2.10, this implies asymptotic independence of the numbers of cycles having different lengths.

Our quantitative results are presented in Section 5.4. Here, we reduce the moment condition $\mathbb{E}[W^{2k+1}] < \infty$ from [18] to $\mathbb{E}[W^4] < \infty$ by using a similar strategy but avoiding certain summands. This assumption no longer becomes more restrictive with growing cycle length k and is a weaker assumption even for the shortest cycle of length $k = 3$. Additionally, we increase the rate of convergence to $O(n^{-1})$. Finally, we do not restrict to $\mathcal{C}_n(k)$, cycles of a fixed length $k \in \{3, 4, \dots\}$, but allow for $\mathcal{C}_n(A)$ for some bounded set $A \subseteq \{3, 4, \dots\}$, meaning that we count the number of cycles in some finite set. The corresponding result is given in Theorem 5.2 a).

We have additional results in the subcritical regime, i.e. when $\mathbb{E}[W^2] < \mathbb{E}[W]$, where we still require $\mathbb{E}[W^4] < \infty$. In this case, the intensity measure λ of the Poisson process η is finite, so that $\eta(A)$ follows a Poisson distribution with finite parameter for all $A \subseteq \mathbb{N}_{\geq 3}$. This gives hope to extend the result from Theorem 5.2 a) to arbitrary sets $A \subseteq \mathbb{N}_{\geq 3}$, which no longer need to be finite. We provide such a result in Theorem 5.2 b) at the cost of a logarithmic factor in the convergence rate. As a by-product of the proof, but interesting in its own right, we show in Theorem 5.7 that with high probability there are no cycles whose length grows at least logarithmically in n. Having information about cycles of all lengths allows us to derive the limiting distribution of and a rate of convergence in the Kolmogorov distance for the lengths of the shortest and the longest cycle in Section 5.5.

5.2 Framework

For some set $A \subseteq \mathbb{N}_{\geq 3}$ we will bound $d_{TV}(\mathcal{C}_n(A), \eta(A))$ by applying Theorem 2.24. Using the notation introduced in the beginning of Section 2.4, we start by providing the choices for I and X_α in our setting. Recall that we write $D^k_{\neq}$ for the k-tuples with pairwise distinct entries from some set D. For $k \in \mathbb{N}_{\geq 3}$ and $n \in \mathbb{N}$ let

$$I_k = \{\alpha \in [n]^k_{\neq} \colon \alpha_1 = \min_{i=1,\dots,k} \alpha_i, \alpha_2 < \alpha_k\},$$

which represents the possible cycles of length k. The constraints on α in I_k correspond to fixing a starting vertex α_1 and an orientation of the cycle α when enumerating the vertices to ensure that each cycle corresponds to exactly one $\alpha \in I_k$. We suppress the dependence of I_k on n in the notation for simplicity and do so for other quantities without further mention. For $k \in \mathbb{N}_{\geq 3}$ and $\alpha \in I_k$ we write

$$X_\alpha = \mathbf{1}\{\text{The cycle } \alpha \text{ exists}\} = \mathbf{1}\{\alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \dots \leftrightarrow \alpha_k \leftrightarrow \alpha_1\} \tag{5.2}$$

so that $\mathcal{C}_n(A)$, the number of cycles whose length lies in A, satisfies

$$\mathcal{C}_n(A) = \sum_{k \in A} \sum_{\alpha \in I_k} X_\alpha = \sum_{\alpha \in I} X_\alpha \quad \text{for} \quad I = \bigcup_{k \in A} I_k.$$

It remains to choose $B_\alpha \subseteq I_k$ for $\alpha \in I$ and a σ-field $\mathcal{A}$ in order to apply Theorem 2.24. For given $\mathcal{A}$, we will choose B_α in such a way that X_α is, conditionally on $\mathcal{A}$, independent of $(X_\beta)_{\beta \in B_\alpha^c}$. This implies $b_3 = 0$ as mentioned in Remark 2.23. Since we will use different choices for the σ-field $\mathcal{A}$ in Section 5.3 and Section 5.4, this will in turn affect the choice of B_α. We state both choices now in order to discuss why we obtain a fourth moment assumption for our quantitative result and how we get rid of it regarding our qualitative result.

In Section 5.4 we choose $\mathcal{A}$ as the trivial σ-field. In this case, let

$$B_\alpha = \{\beta \in I \colon \alpha \text{ and } \beta \text{ share at least one vertex}\}, \tag{5.3}$$

ensuring that all $\beta \in B_\alpha^c$ do not share a single vertex with α so that X_α and $(X_\beta)_{\beta \in B_\alpha^c}$ are independent. Taking a quick look at the upper bound in Theorem 2.24, we need to

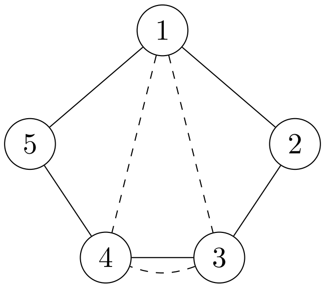


Figure 5.1: The cycles $\alpha = (1, 2, 3, 4, 5)$ displayed by solid edges and $\beta = (1, 3, 4)$ having dashed edges. The shared edge $\{1, 2\}$ is shown twice although it is just a single edge.

bound b_2 and thus the (unconditional) probability that α and β exist simultaneously for $\alpha \in I$ and $\beta \in B_\alpha$. Let us consider the toy example $\alpha = (1, 2, 3, 4, 5)$ and $\beta = (1, 3, 4)$ in Figure 5.1, where α and β share exactly three vertices and one edge in some graph model $G'(n)$ with $G \in \mathcal{G}$ and $n \geq 5$. We use the upper bound $\mathbb{P}_{\mathcal{W}}(x \leftrightarrow y) \leq W_x W_y/(n\mathbb{E}[W])$ for $x, y \in [n]$ from Lemma 3.5, which yields

$$\begin{aligned} p_{\alpha\beta} &= \mathbb{E}[X_\alpha X_\beta] = \mathbb{E}\big[\mathbf{1}\{1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 1 \quad \text{and} \quad 3 \leftrightarrow 1 \leftrightarrow 4\}\big] \\ &\leq \mathbb{E}\left[\prod_{i=1}^{5} \frac{W_i^2}{n\mathbb{E}[W]} \times \frac{W_1 W_3}{n\mathbb{E}[W]} \frac{W_1 W_4}{n\mathbb{E}[W]}\right] = \frac{\mathbb{E}[W^2]^2 \mathbb{E}[W^3]^2 \mathbb{E}[W^4]}{(n\mathbb{E}[W])^7} \end{aligned}$$

by conditioning on the weights. This is why we require a finite fourth moment when following this approach. Since $p_{\alpha\beta} \leq 1$, we immediately see that this constraint arises from the proof and it remains an open question whether one can weaken this assumption and still derive a rate of convergence.

In Section 5.3 we choose $\mathcal{A} = \mathcal{W}$ for a qualitative result. Conditionally on the weights, any two random variables X_α and X_β are independent as long as the corresponding cycles $\alpha, \beta \in I$ share no edges – even if they share vertices. Therefore, we choose

$$B_\alpha = \{\beta \in I \colon \alpha \text{ and } \beta \text{ share at least one edge}\}. \tag{5.4}$$

As our bound from Theorem 2.24 is of the form $\mathbb{E}[\min(1, Y_n)]$ for some sequence of random variables $(Y_n)_{n\in\mathbb{N}}$, we may simply show that Y_n converges to zero in probability as $n \to \infty$. However, this does not yield quantitative results. If we stick to the explicit example from Figure 5.1, we obtain in this setting

$$p_{\alpha\beta} = \mathbb{E}_{\mathcal{W}}[X_\alpha X_\beta] \leq \prod_{i=1}^{5} \frac{W_i^2}{n\mathbb{E}[W]} \times \frac{W_1 W_3}{n\mathbb{E}[W]} \frac{W_1 W_4}{n\mathbb{E}[W]} \leq \frac{W_{(n)}^4}{(n\mathbb{E}[W])^2} \prod_{i=1}^{5} \frac{W_i^2}{n\mathbb{E}[W]},$$

where we bounded some weights by $W_{(n)}$, the maximum of all weights. The first fraction converges in probability to zero as $n \to \infty$ by Lemma 2.27 b), whereas all remaining factors are stochastically bounded by the law of large numbers when $\mathbb{E}[W^2] < \infty$. Using this strategy is how we manage to get rid of the fourth moment assumption to obtain a (solely) qualitative result under weaker moment assumptions.

5.3 A qualitative result for cycle counts

Our first theorem deals with weak convergence of the point process $\mathcal{C}_n$ in $M_p(\mathbb{N}_{\geq 3})$. This is equivalent to convergence of the finite-dimensional distributions of $(\mathcal{C}_n(k))_{k\in\mathbb{N}_{\geq 3}}$ as shown in Lemma 2.10.

Theorem 5.1. *Let $\mathbb{E}[W^2] < \infty$ and consider $G(n)$ or $G'(n)$ for $G \in \mathcal{G}$. Then,*

$$\mathcal{C}_n \xrightarrow{d} \eta \quad \text{as } n \to \infty.$$

Proof. By Lemma 2.10 it suffices to show convergence of the finite-dimensional distributions. Let thus $A \subseteq \mathbb{N}_{\geq 3}$ be finite. Using the Poisson Cramér-Wold device, see Corollary 2.25, it suffices to show for $q_k \in [0,1]$ for $k \in A$ that

$$\sum_{k\in A} \mathcal{C}_n^{(q_k)}(k) = \sum_{k\in A}\sum_{\alpha\in I_k} X_\alpha^{(q_k)} \xrightarrow{d} Z \sim \mathrm{Poi}\Big(\sum_{k\in A} q_k\lambda_k\Big) \quad \text{as} \quad n\to\infty,$$

where $X^{(q_k)}$ denotes a thinned random variable as introduced before Corollary 2.25. We use the notation from Section 2.4, $\mathcal{A} = \mathcal{W}$ and the choice for B_α in (5.4). Note that $X_\alpha^{(q_k)}$ is a Bernoulli random variable, despite the thinning, so that the results from Section 2.4 are applicable. By Theorem 2.24 it suffices to show that b_1, b_2, b_3 and

$$\left|\mathbb{E}_{\mathcal{W}}\Big[\sum_{k\in A}\mathcal{C}_n^{(q_k)}(k)\Big] - \sum_{k\in A} q_k\lambda_k\right| \tag{5.5}$$

converge in probability to zero as $n\to\infty$. From now on, consider only $G(n)$ for $G\in\mathcal{G}$. The case $G'(n)$ goes analogously, since $n\mathbb{E}[W]/L_n$ converges almost surely to one as $n\to\infty$. By Remark 2.23 and the discussion in the previous section we have $b_3 = 0$. With the bound

$$\mathbb{P}_{\mathcal{W}}(x \leftrightarrow y) \leq \frac{W_x W_y}{L_n}$$

for distinct $x, y\in[n]$ from Lemma 3.5 we conclude that

$$\mathbb{E}_{\mathcal{W}}[X_\alpha] = \mathbb{E}_{\mathcal{W}}\Big[\prod_{i=1}^k \mathbf{1}\{\alpha_i \leftrightarrow \alpha_{i-1}\}\Big] \leq \prod_{i=1}^k \frac{W_{\alpha_i}^2}{L_n} \tag{5.6}$$

for $\alpha\in I_k$ and $k\in A$, where we wrote $\alpha_0 = \alpha_k$. We obtain

$$\begin{aligned} b_1 &= \sum_{k,\ell\in A}\sum_{\alpha\in I_k}\sum_{\beta\in B_\alpha\cap I_\ell} \mathbb{E}_{\mathcal{W}}[X_\alpha^{(q_k)}]\mathbb{E}_{\mathcal{W}}[X_\beta^{(q_\ell)}] \leq \sum_{k,\ell\in A}\sum_{\alpha\in I_k}\sum_{\beta\in B_\alpha\cap I_\ell}\mathbb{E}_{\mathcal{W}}[X_\alpha]\mathbb{E}_{\mathcal{W}}[X_\beta] \\ &\leq \sum_{k,\ell\in A}\sum_{\alpha\in I_k}\sum_{\beta\in I_\ell}\mathbf{1}\{\alpha \text{ and } \beta \text{ share an edge}\}\prod_{i=1}^k\frac{W_{\alpha_i}^2}{L_n}\prod_{j=1}^\ell\frac{W_{\beta_i}^2}{L_n} \end{aligned}$$

by the definition of B_α. If $\alpha\in I_k$ and $\beta\in I_\ell$ share an edge, the weights of the two incident vertices appear with a fourth power in the product above. Since each $\gamma\in I$ corresponds to exactly one cycle in the graph, we may assume without loss of generality that $(\alpha_1, \alpha_2) = (\beta_1, \beta_2)$ by changing the enumerations of α and β. As in the last chapter, we write $L_n^{[\gamma]} = \sum_{i=1}^n W_i^\gamma$ for $\gamma \in \mathbb{R}$. We obtain

$$\begin{aligned} b_1 &\leq \sum_{k,\ell\in A} \sum_{\alpha_1,\alpha_2\in[n]} \frac{W_{\alpha_1}^4 W_{\alpha_2}^4}{L_n^4} \sum_{\alpha_3,\dots,\alpha_k\in[n]} \sum_{\beta_3,\dots,\beta_\ell\in[n]} \prod_{i=3}^k \frac{W_{\alpha_i}^2}{L_n} \prod_{j=3}^\ell \frac{W_{\beta_j}^2}{L_n} \\ &= \left(\frac{L_n^{[4]}}{L_n^2}\right)^2 \sum_{k,\ell\in A} \left(\frac{L_n^{[2]}}{L_n}\right)^{k+\ell-4} = \left(\frac{n^{-2}L_n^{[4]}}{(n^{-1}L_n)^2}\right)^2 \sum_{k,\ell\in A} \left(\frac{n^{-1}L_n^{[2]}}{n^{-1}L_n}\right)^{k+\ell-4}. \end{aligned}$$

The sum is almost surely bounded due to the strong law of large numbers and $\mathbb{E}[W^2] < \infty$. Additionally, we apply Lemma 2.26 to obtain that the numerator of the first fraction tends almost surely to zero as $n \to \infty$. The strong law of large numbers yields that the denominator converges almost surely to $\mathbb{E}[W]^2$ as $n \to \infty$. Therefore, the whole fraction converges almost surely to zero as $n \to \infty$. In the following, we will simply refer to Lemma 2.26 for expressions similar to $L_n^{[4]}/L_n^2$. For

$$b_2 = \sum_{k,\ell\in A} \sum_{\alpha\in I_k} \sum_{\beta\in B_\alpha\cap I_\ell\setminus\{\alpha\}} \mathbb{E}_{\mathcal{W}}[X_\alpha^{(q_k)} X_\beta^{(q_\ell)}] \leq \sum_{k,\ell\in A} \sum_{\alpha\in I_k} \sum_{\beta\in B_\alpha\cap I_\ell\setminus\{\alpha\}} \mathbb{E}_{\mathcal{W}}[X_\alpha X_\beta] \tag{5.7}$$

we need to account for dependencies between X_α and X_β.

For $\alpha \in I_k$ and $\beta \in B_\alpha \cap I_\ell \setminus \{\alpha\}$ we look at the graph union $\alpha \cup \beta$, where we take the unions of the respective vertex and edge sets. This graph is given by α and some additional connections between vertices of α induced by β. We call these connections arcs. They are paths whose two endpoints belong to α and whose edges and inner points belong to β, but not to α. In Figure 5.2 are two examples. On the left-hand side, there are six arcs, namely all edges of β. On the right-hand side, the three arcs are given by $(2, 8, 7, 1)$, $(6, 11, 10, 5)$ and $(3, 9, 2)$.

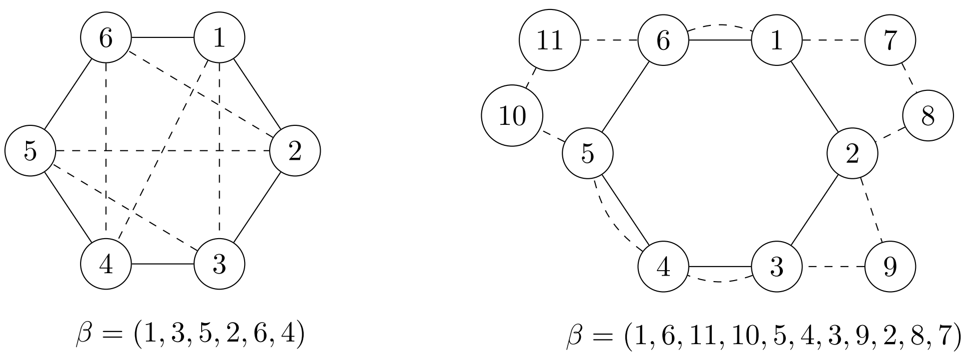


Figure 5.2: Arcs for varying β and fixed $\alpha = (1, \dots, 6)$. The cycle α is displayed by solid edges, whereas β consists of dashed edges. When both cycles share an edge, we added a curved edge to increase visibility.

We denote the number of such arcs by $m = m(\alpha, \beta)$, which is bounded by k and ℓ as both endpoints of an arc belong to α and β. Note that the left-hand side of Figure 5.2 provides an extreme example of $k = \ell$ many arcs. The numbers of edges of the arcs are denoted by $i_1, \dots, i_m$, where we enumerate the arcs in the order that one passes through them according to the labelling of β. It follows that $1 \le i_1, \dots, i_m \le \ell - 1$. Note that we can almost reconstruct β from α and the arcs: at each point where α and β intersect and just a single arc is attached, one needs to decide in which direction β connects to the next arc, using edges of α. Since β can go either clockwise or counter clockwise with respect to the orientation of α, there are (at most) two possible choices per arc endpoint. However, as each such choice connects two endpoints of arcs, there are in total at most two choices per arc instead of four. We will sum over all possibilities for the arcs and account for possible connections thereof instead of summing over the respective choices for β. We use that the existence of the edges of α and of the arcs are, conditionally on the weights, independent. The conditional expectation of the number of paths having i edges with two fixed endpoints α_s and α_t is bounded by

$$\frac{W_{\alpha_s} W_{\alpha_t}}{L_n} \left(\frac{L_n^{[2]}}{L_n} \right)^{i-1} \le \frac{W_{(n)}^2}{L_n} \left(\frac{L_n^{[2]}}{L_n} \right)^{i-1},$$

due to summing over all choices for the $i-1$ vertices in between, see also Lemma 4.3. By summing over all possible lengths, the contribution of all arcs is thus bounded by

$$Y_{k,\ell} \coloneqq \sum_{m=1}^{\min(k,\ell)} 2^m (k)_{2m} \left(\frac{W_{(n)}^2}{L_n} \right)^m \sum_{i_1,\dots,i_m=1}^{\ell-1} \prod_{j=1}^{m} \left(\frac{L_n^{[2]}}{L_n} \right)^{i_j - 1},$$

where the factor $(k)_{2m} = k \cdot \ldots \cdot (k - 2m + 1)$ accounts for the possibilities to choose the endpoints of the arcs among α and 2^m for the possibilities to connect the arcs and obtain β. By Lemma 2.27 c), the factor $W_{(n)}^2 / L_n$ converges in probability to 0 as $n \to \infty$ and thus also $Y_{k,\ell}$ for all $k, \ell \in A$ as all sums are finite. From (5.7) we derive

$$b_2 \le \sum_{k,\ell \in A} \sum_{\alpha \in I_k} \mathbb{E}_{\mathcal{W}}[X_\alpha] Y_{k,\ell} \le \sum_{k,\ell \in A} \sum_{\alpha \in [n]^k_{\neq}} \prod_{j=1}^{k} \frac{W_{\alpha_j}^2}{L_n} Y_{k,\ell} \le \sum_{k,\ell \in A} \left(\frac{L_n^{[2]}}{L_n} \right)^k Y_{k,\ell} \xrightarrow{\mathbb{P}} 0$$

as $n \to \infty$ since A is a finite set.

For the remaining term in (5.5) we use the triangle inequality to deduce

$$\left| \mathbb{E}_{\mathcal{W}} \left[\sum_{k \in A} \mathcal{C}_n^{(q_k)}(k) \right] - \sum_{k \in A} q_k \lambda_k \right| \le \sum_{k \in A} \left| \mathbb{E}_{\mathcal{W}} \left[\mathcal{C}_n^{(q_k)}(k) \right] - q_k \lambda_k \right|.$$

We show for fixed $k \in A$ that the summand on the right-hand side converges almost surely to zero as $n \to \infty$. To this end we consider lower and upper bounds for

$$\mathbb{E}_{\mathcal{W}} \left[\mathcal{C}_n^{(q_k)}(k) \right] = \mathbb{E}_{\mathcal{W}} \left[\sum_{\alpha \in I_k} X_\alpha^{(q_k)} \right] = \frac{q_k}{2k} \sum_{\alpha \in [n]^k_{\neq}} \mathbb{E}_{\mathcal{W}} [X_\alpha], \tag{5.8}$$

where the factor $2k$ makes up for starting vertex and orientation when labelling a fixed cycle. For an upper bound, we use (5.6) and add some missing summands. This yields

$$\mathbb{E}_{\mathcal{W}} \left[\mathcal{C}_n^{(q_k)}(k) \right] \le \frac{q_k}{2k} \sum_{\alpha \in [n]^k} \prod_{i=1}^{k} \frac{W_{\alpha_i}^2}{L_n} = \frac{q_k}{2k} \left(\frac{L_n^{[2]}}{L_n} \right)^k \xrightarrow{a.s.} \frac{q_k}{2k} \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} \right)^k = q_k \lambda_k$$

as $n \to \infty$ due to the strong law of large numbers. For a lower bound, we use the connection probability of the generalised random graph, see Lemma 3.5, and obtain

$$\mathbb{E}_{\mathcal{W}}\big[\mathcal{C}_n^{(q_k)}(k)\big] \geq \frac{q_k}{2k} \sum_{\alpha \in [n]^k_{\neq}} \prod_{i=1}^{k} \frac{W_{\alpha_i}^2}{L_n + W_{\alpha_i} W_{\alpha_{i-1}}},$$

where we write $\alpha_0 = \alpha_k$ as usual. Accounting for the missing summands gives us

$$\begin{aligned}&\frac{2k}{q_k}\mathbb{E}_{\mathcal{W}}\big[\mathcal{C}_n^{(q_k)}(k)\big]\\ &\geq \sum_{\alpha \in [n]^k} \prod_{i=1}^{k} \frac{W_{\alpha_i}^2}{L_n + W_{\alpha_i} W_{\alpha_{i-1}}} - \sum_{\alpha \in [n]^k} \mathbf{1}\{\exists i \neq j \colon \alpha_i = \alpha_j\} \prod_{i=1}^{k} \frac{W_{\alpha_i}^2}{L_n + W_{\alpha_i} W_{\alpha_{i-1}}},\end{aligned} \tag{5.9}$$

where the absolute value of the second summand is bounded by

$$\binom{k}{2} \frac{L_n^{[4]}}{L_n^2} \left(\frac{L_n^{[2]}}{L_n}\right)^{k-2}$$

due to choosing two of the k vertices that shall be equal. By Lemma 2.26, this converges almost surely to zero as $n \to \infty$. For a lower bound of the first summand in (5.9) we use Lemma 2.29 with $b = L_n$ and $x_i = W_{\alpha_i} W_{\alpha_{i-1}}$. This yields the lower bound

$$\sum_{\alpha \in [n]^k} \prod_{i=1}^{k} \frac{W_{\alpha_i}^2}{L_n} - \sum_{\alpha \in [n]^k} \frac{\prod_{i=1}^{k} W_{\alpha_i}^2 \sum_{j=1}^{k} W_{\alpha_j} W_{\alpha_{j-1}}}{L_n^{k+1}} = \left(\frac{L_n^{[2]}}{L_n}\right)^k - k\left(\frac{L_n^{[3]}}{L_n^{3/2}}\right)^2 \left(\frac{L_n^{[2]}}{L_n}\right)^{k-2}.$$

As above, the last summand goes almost surely to zero as $n \to \infty$ whereas the first summand converges almost surely to $2k\lambda_k$ as $n \to \infty$. Altogether we obtain

$$\mathbb{E}_{\mathcal{W}}\big[\mathcal{C}_n^{(q_k)}(k)\big] \xrightarrow{a.s.} q_k \lambda_k$$

as $n \to \infty$, which concludes the proof. □

5.4 Quantitative results for cycle counts

The main goal of this section is to quantify the convergence in Theorem 5.1 by evaluating the point process $\mathcal{C}_n$ on some set $A \subseteq \mathbb{N}_{\geq 3}$, which amounts to counting the number of cycles whose length lies in A. When A is a finite set, the qualitative convergence

$$\mathcal{C}_n(A) \xrightarrow{d} \eta(A) \quad \text{as} \quad n \to \infty$$

follows immediately from Theorem 5.1 by using Theorem 2.8 or by applying the continuous mapping theorem to the finite-dimensional convergence. However, the theorem below not only quantifies this convergence, but extends it to *unbounded* sets $A \subseteq \mathbb{N}_{\geq 3}$ when the graph is subcritical. This is possible since the subcriticality yields finiteness of the intensity measure of η. This allows us to derive asymptotic results concerning the lengths of the shortest and of the longest cycles in Section 5.5.

Theorem 5.2. *Let $\mathbb{E}[W^4] < \infty$ and consider $G(n)$ or $G'(n)$ for any $G \in \mathcal{G}$.*

a) *For all $N \in \mathbb{N}$ there exists a constant $C > 0$ such that for all $A \subseteq \{3, 4, \dots, N\}$ and $n \geq 3$,*

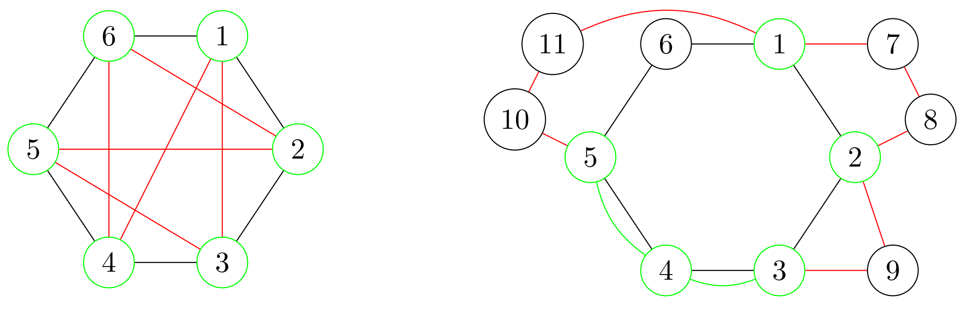
$$d_{TV}(\mathcal{C}_n(A), \eta(A)) \leq \frac{C}{n}.$$

b) *Let $\mathbb{E}[W^2] < \mathbb{E}[W]$. There exists a constant $C > 0$ such that for all $n \geq 3$ and $A \subseteq \mathbb{N}_{\geq 3}$,*

$$d_{TV}(\mathcal{C}_n(A), \eta(A)) \leq \frac{C \log(n)^3}{n}.$$

The proofs of part a) and part b) of the theorem above follow the same overall structure. Before sketching the proof, we wish to state its key ingredient. To this end, we require some notation.

In this section we choose $\mathcal{A}$ to be trivial, see Section 5.2, so that $p_\alpha = \mathbb{E}[X_\alpha]$ only depends on k for $\alpha \in I_k$, not on the exact choice of α. Thus, we simply write p_k instead of p_α. The term $p_{\alpha\beta} = \mathbb{E}[X_\alpha X_\beta]$, the probability for the existence of the two cycles $\alpha \in I_k$ and $\beta \in I_\ell$, in turn depends on the relation of α and β. If they share edges, we obtain dependencies between the existence of α and the existence of β, even after conditioning on the weights. Therefore, we need to investigate the structure of $\alpha \cup \beta$. For a cycle $\alpha \in I_k$ we refer to its indices here and in the following always modulo k, e.g. $\alpha_{k+2} = \alpha_2$. We introduce segments, which correspond to the parts of α and β consisting of consecutively shared vertices. This concept is probably best explained by a picture, see Figure 5.3. On the left-hand side, there are six segments, all consisting of a single vertex. We say that they have length one. On the right-hand side, there are three segments, namely $(1), (2)$ and $(3, 4, 5)$, which are of length one, one and three, respectively.



$\beta = (1, 3, 5, 2, 6, 4)$ $\beta = (1, 7, 8, 2, 9, 3, 4, 5, 10, 11)$

Figure 5.3: Segments highlighted in green for varying β and fixed $\alpha = (1, \dots, 6)$. The cycle α is displayed by black edges, whereas β consists of green and red edges. When both cycles share an edge, we added a second edge to increase visibility.

We feel obliged to also provide a formal definition. For $k, \ell \geq 3$ and two cycles $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_\ell)$ let $i \in [k], j \in [\ell]$ be such that $\alpha_i = \beta_j$ but $\alpha_{i-1} \neq \beta_{j-1}, \beta_{j+1}$. Now take $m \in [k]$ maximal with the property $\alpha_{i+1} = \beta_{j+1}, \dots, \alpha_{i+m} = \beta_{j+m}$

or $\alpha_{i+1} = \beta_{j-1}, \dots, \alpha_{i+m} = \beta_{j-m}$ so that $\alpha_i, \dots, \alpha_{i+m}$ are consecutively shared vertices of α and β. We call $(\alpha_i, \dots, \alpha_{i+m})$ a segment of length $m+1$. Note that segments are maximal in the sense that they cannot be extended in either direction. For two cycles $\alpha \in I_k, \beta \in I_\ell$ that intersect each other in s segments of lengths $i_1, \dots, i_s$ we write $i = (i_1, \dots, i_k)$ and $p_{k,\ell,s,i} = p_{\alpha\beta}$. This is well-defined. In the following, we write $|i| = \sum_{j=1}^{s} i_j$ for the total number of shared vertices to simplify notation.

For the phrasing of the following lemma, recall that $(n)_k$ denotes the falling factorial.

Lemma 5.3. *Let $n \in \mathbb{N}, A \subseteq \{3, \dots, n\}$ and consider $G(n)$ or $G'(n)$ for $G \in \mathcal{G}$. Then,*

$$d_{TV}(\mathcal{C}_n(A), \eta(A)) \leq \frac{1}{2n} \sum_{k,\ell \in A} p_k p_\ell n^{k+\ell} + \sum_{k,\ell \in A} \sum_{s=1}^{k} \sum_{i \in [k]^s} p_{k,\ell,s,i} (2k\ell)^{s-1} n^{k+\ell-|i|}$$
$$+ \sum_{k \in A} \left| \frac{(n)_k}{2k} p_k - \lambda_k \right|.$$

Before proving this lemma, we briefly sketch its proof and how we use it to prove Theorem 5.2. The proof of Lemma 5.3 is done by using Theorem 2.24 and some combinatorial arguments regarding the number of summands at hand. Once we have shown Lemma 5.3, we will carry out the following steps to show Theorem 5.2 a), where A is bounded:

(i) Recall that p_k is the probability to obtain a cycle of length k. As the (conditional) probability for any fixed edge scales like n^{-1}, we can expect this quantity to behave like n^{-k}. This shows that the first summand in Lemma 5.3 is of order n^{-1}. We prove this rigorously in Lemma 5.4.

(ii) The quantity $p_{k,\ell,s,i}$ concerns the existence of two cycles α and β such that $\alpha \cup \beta$ contains $k + \ell - |i| + s$ edges, so that it should scale like $n^{-k-\ell+|i|-s}$. With $s \geq 1$, this provides the desired convergence rate $O(n^{-1})$. The corresponding statement is proven in Lemma 5.5.

(iii) For the last term, we investigate p_k in more detail. For $\alpha = (1, \dots, k)$ we have

$$\mathbb{P}_{\mathcal{W}}(X_\alpha) \approx \prod_{i=1}^{k} \frac{W_i^2}{n\mathbb{E}[W]}.$$

For large n and small k we obtain

$$\frac{(n)_k}{2k} p_k \approx \frac{n^k}{2k} \mathbb{E}\big[\mathbb{P}_{\mathcal{W}}(X_\alpha)\big] \approx \frac{n^k}{2k} \left(\frac{\mathbb{E}[W^2]}{n\mathbb{E}[W]} \right)^k = \lambda_k.$$

It remains to quantify all approximations above, which is carried out in Lemma 5.6.

When trying to apply the same strategy for part b), one encounters the problem that the intuitive usage of n being way larger than k breaks down. Therefore, we impose a cutoff of the set A at roughly the size $\log(n)$, which still grows sufficiently slower than n. For all cycles of smaller lengths, we proceed as above. At this point, we get an additional logarithmic term, as the number of summands in Lemma 5.3 also grows logarithmically. All cycles whose length is larger than some logarithmic term are shown to be sufficiently unlikely in Theorem 5.7, which we deem an interesting result on its own. We continue with carrying out the details.

Proof of Lemma 5.3. We have

$$\mathbb{E}[\mathcal{C}_n(A)] = \sum_{k\in A}\sum_{\alpha\in I_k}\mathbb{E}[X_\alpha] = \sum_{k\in A}\frac{(n)_k}{2k}p_k,$$

where we used $|I_k| = (n)_k/(2k)$. Thus, Theorem 2.24 yields

$$d_{TV}(\mathcal{C}_n(A),\eta(A)) \le b_1 + b_2 + b_3 + \sum_{k\in A}\left|\frac{(n)_k}{2k}p_k - \lambda_k\right|.$$

We first note that $b_3 = 0$ by the definition of B_α in (5.3) and the brief comment thereafter. It remains to bound b_1 and b_2. We compute

$$b_1 = \sum_{k,\ell\in A}\sum_{\alpha\in I_k}\sum_{\beta\in B_\alpha\cap I_\ell}\mathbb{E}[X_\alpha]\mathbb{E}[X_\beta] = \sum_{k,\ell\in A}p_kp_\ell\sum_{\alpha\in I_k}|B_\alpha\cap I_\ell|.$$

To estimate the cardinalities of the sets, recall that B_α contains all cycles that intersect $\alpha\in I_k$ in at least one vertex. We have k choices for a vertex of α which fixes one vertex of $\beta\in B_\alpha\cap I_\ell$. For the remaining $\ell-1$ vertices, there are fewer than $n^{\ell-1}$ many choices, resulting in $|B_\alpha\cap I_\ell|\le kn^{\ell-1}$. Together with $|I_k|\le n^k/(2k)$ we derive

$$b_1 \le \sum_{k,\ell\in A}p_kp_\ell\frac{n^k}{2k}kn^{\ell-1} = \frac{1}{2n}\sum_{k,\ell\in A}p_kp_\ell n^{k+\ell},$$

which is the first summand on the right-hand side of the claimed inequality. Additionally,

$$b_2 = \sum_{k,\ell\in A}\sum_{\alpha\in I_k}\sum_{\beta\in I_\ell\cap B_\alpha\setminus\{\alpha\}}\mathbb{E}[X_\alpha X_\beta] \le 2\sum_{k,\ell\in A\colon k\le\ell}\sum_{\alpha\in I_k}\sum_{\beta\in I_\ell\cap B_\alpha\setminus\{\alpha\}}\mathbb{E}[X_\alpha X_\beta].$$

The value of $\mathbb{E}[X_\alpha X_\beta]$ depends on a more thorough analysis of the intersections of α and β, where we use the notion of a segment introduced before Lemma 5.3. For $k,\ell\in A$ with $k\le\ell$, $\alpha\in I_k$ and $\beta\in B_\alpha\cap I_\ell\setminus\{\alpha\}$ there can be at most $s=1,\dots,k$ of these segments. The case $s=0$ is excluded since the two cycles share at least one vertex. Recall that we denote the number of vertices of the segments by $1\le i_1,\dots,i_s\le k$ and write $i=(i_1,\dots,i_s)$. We define

$$M(s,i,\alpha) = \left\{\beta\in\bigcup_{\ell\in A}I_\ell\colon \alpha \text{ and } \beta \text{ share } s \text{ segments of lengths } i_1,\dots,i_s \text{ in that order}\right\},$$

where we order the segments with respect to the enumeration of α. We derive

$$\begin{aligned} b_2 &\le 2\sum_{k,\ell\in A\colon k\le\ell}\sum_{\alpha\in I_k}\sum_{\beta\in I_\ell\cap B_\alpha\setminus\{\alpha\}}\sum_{s=1}^{k}\sum_{i\in[k]^s}\mathbf{1}\{\beta\in M(s,i,\alpha)\}\mathbb{E}[X_\alpha X_\beta] \\ &\le 2\sum_{k,\ell\in A\colon k\le\ell}\sum_{\alpha\in I_k}\sum_{s=1}^{k}\sum_{i\in[k]^s}|B_\alpha\cap I_\ell\cap M(s,i,\alpha)|p_{k,\ell,s,i}. \end{aligned}$$

For $k,\ell\in A$ with $k\le\ell$ as well as $\alpha\in I_k, s\in[k]$ and $i\in[k]^s$ we bound the cardinality of $B_\alpha\cap I_\ell\cap M(s,i,\alpha)$. Since $\alpha=(\alpha_1,\dots,\alpha_k)\in I_k$ has a fixed order, we may talk about the first vertex of a segment, i.e. the endpoint α_j of the segment with minimal j among the two endpoints. There are at most k^s possibilities to choose the first vertices of the s segments among the k vertices of α, call them $v_1, \dots, v_s$. We may assume without loss of generality that v_1 is the first entry of $\beta = (\beta_1, \dots, \beta_\ell)$. This leaves us with placing the $s-1$ remaining vertices $v_2, \dots, v_s$ among the $\ell - 1$ remaining slots in β, giving us fewer than ℓ^{s-1} choices. We need to decide whether v_j is the first or the last vertex of the j-th segment with respect to the orientation of β. This gives us another factor 2^{s-1} because we may assume that the orientation of β coincides with the orientation of the first segment. This determines the $|i|$ entries of β that also belong to α. Finally, we fill the remaining $\ell - |i|$ entries of β, amounting to fewer than $n^{\ell-|i|}$ choices. In total we obtain

$$\begin{aligned} b_2 &\le 2 \sum_{k,\ell\in A\colon k\le\ell} \sum_{\alpha\in I_k} \sum_{s=1}^{k} \sum_{i\in[k]^s} k^s \ell^{s-1} 2^{s-1} n^{\ell-|i|} p_{k,\ell,s,i} \\ &\le 2 \sum_{k,\ell\in A} \frac{n^k}{2k} \sum_{s=1}^{k} \sum_{i\in[k]^s} k^s \ell^{s-1} 2^{s-1} n^{\ell-|i|} p_{k,\ell,s,i} = \sum_{k,\ell\in A} \sum_{s=1}^{k} \sum_{i\in[k]^s} (2k\ell)^{s-1} n^{k+\ell-|i|} p_{k,\ell,s,i}, \end{aligned}$$

which concludes the proof. □

We continue with carrying out the 3-point plan outlined after stating Lemma 5.3. The following three lemmas each address one of these points and contain parts a) and b), which will be used for the respective parts of Theorem 5.2. This means that the considered cycle lengths are bounded in part a), whereas they are allowed to grow logarithmically in n for part b). There will be minor differences between the models $G(n)$ and $G'(n)$ for $G \in \mathcal{G}$. This comes from the fact that L_n is not independent of the numerator in the connection probability, which results in some extra effort for $G(n)$. We start with uniform bounds on p_k and remind the reader that $\lfloor\cdot\rfloor$ denotes the floor function, rounding its argument down.

Lemma 5.4. *Suppose that $\mathbb{E}[W^2] < \infty$ and consider $G(n)$ or $G'(n)$ for any $G \in \mathcal{G}$.*

a) *Let $N \in \mathbb{N}$. There exists a constant $C > 0$ such that for all $n \in \mathbb{N}, 3 \le k \le N$,*

$$p_k \le C n^{-k}.$$

b) *Let $a > 0$ and $\mathbb{E}[W^2] < \mathbb{E}[W]$. There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $3 \le k \le \lfloor a \log(n) \rfloor$,*

$$p_k \le C n^{-k}.$$

Proof. We start with the case $G'(n)$ for $G \in \mathcal{G}$. Using Lemma 3.5 yields

$$p_k \le \mathbb{E}\left[\prod_{i=1}^{k} \frac{W_i^2}{n\mathbb{E}[W]}\right] = \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k n^{-k}. \tag{5.10}$$

In part a) the result is immediate, as we may take the maximum over finitely many choices of k for the constant C. Using the assumption $\mathbb{E}[W^2] < \mathbb{E}[W]$ in part b) yields the upper bound for $C = 1$.

For $G(n)$ we proceed similarly and provide a general bound, which we use for both claims. Let $3 \leq k \leq n, \lambda \in (0,1), \alpha \in I_k$ and

$$A_n = \{L_n > \lambda \mathbb{E}[W]n\}.$$

Since $X_\alpha = \mathbf{1}\{\alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \ldots \leftrightarrow \alpha_k \leftrightarrow \alpha_1\}$ is bounded by one, we have

$$p_k = p_\alpha = \mathbb{E}[X_\alpha] \leq \mathbb{P}(A_n^c) + \mathbb{E}[\mathbf{1}_{A_n} X_\alpha].$$

By Lemma 2.28 we know that $\mathbb{P}(A_n^c)$ decays exponentially in n and thus faster than

$$\exp(-a\log(n)^2) = n^{-a\log(n)},$$

which is the fastest rate possibly claimed in the lemma. Therefore, it suffices to show that

$$\mathbb{E}[\mathbf{1}_{A_n} X_\alpha] \leq Dn^{-k}$$

for some constant D in the respective setting. We write $\alpha_0 = \alpha_k$. Lemma 3.5 and the definition of A_n (note that A_n is $\mathcal{W}$-measurable) yield

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{A_n} X_\alpha] &= \mathbb{E}\left[\mathbf{1}_{A_n}\mathbb{E}_{\mathcal{W}}\left[\prod_{i=1}^{k}\mathbf{1}\{\alpha_i \leftrightarrow \alpha_{i-1}\}\right]\right] = \mathbb{E}\left[\mathbf{1}_{A_n}\prod_{i=1}^{k}\mathbb{P}_{\mathcal{W}}(\alpha_i \leftrightarrow \alpha_{i-1})\right] \qquad (5.11)\\ &\leq \mathbb{E}\left[\mathbf{1}_{A_n}\prod_{i=1}^{k}\frac{W_i^2}{L_n}\right] \leq \mathbb{E}\left[\prod_{i=1}^{k}\frac{W_i^2}{\lambda\mathbb{E}[W]n}\right] = \left(\frac{\mathbb{E}[W^2]}{\lambda\mathbb{E}[W]}\right)^k n^{-k}.\end{aligned}$$

Similarly to $G'(n)$, the first claim follows immediately. For the second claim we choose $\lambda = \mathbb{E}[W^2]/\mathbb{E}[W]$, which is smaller than 1 by assumption, so that we may choose $D = 1$. This finishes the proof. □

The next lemma is used to bound $p_{k,\ell,s,i}$ in the second summand of Lemma 5.3.

Lemma 5.5. *Suppose that $\mathbb{E}[W^4] < \infty$. For all $G(n)$ and $G'(n)$ with $G \in \mathcal{G}$ the following hold.*

a) *Let $N \in \mathbb{N}$. There exists a constant $C > 0$ such that for all $n \in \mathbb{N}, 3 \leq k, \ell \leq N$ and $\alpha \in I_k, \beta \in I_\ell$ which share $0 \leq j \leq \min(k,\ell)$ edges we have*

$$p_{\alpha\beta} \leq Cn^{-k-\ell+j}.$$

b) *Let $a > 0$ and $\mathbb{E}[W^2] < \mathbb{E}[W]$. There exist constants $\kappa, C > 0$ such that for all $n \in \mathbb{N}$, $3 \leq k, \ell \leq \lfloor a\log(n)\rfloor$ and distinct $\alpha \in I_k, \beta \in I_\ell$ which intersect each other in $1 \leq s \leq k$ segments of lengths $1 \leq i_1, \ldots, i_s \leq \min(k,\ell)$,*

$$p_{k,\ell,s,i} \leq C\kappa^s n^{-k-\ell+|i|-s},$$

where $|i| = \sum_{j=1}^{s} i_j$ denotes the number of all vertices that α and β share.

Proof. We start with the case $G'(n)$ for $G \in \mathcal{G}$. Write $\alpha \cup \beta$ for the graph union of the cycles induced by α and β. We denote its vertices by $V(\alpha \cup \beta)$, its edges by $E(\alpha \cup \beta)$ and the degree of $v \in V(\alpha \cup \beta)$ in $\alpha \cup \beta$ by $\deg_{\alpha\cup\beta}(v)$. Then,

$$\begin{aligned}\mathbb{E}[X_\alpha X_\beta] &= \mathbb{E}\Big[\prod_{\{x,y\}\in E(\alpha\cup\beta)} \mathbf{1}\{x \leftrightarrow y\}\Big] = \mathbb{E}\Big[\prod_{\{x,y\}\in E(\alpha\cup\beta)} \mathbb{P}_{\mathcal{W}}(x \leftrightarrow y)\Big] \\ &\le \mathbb{E}\Big[\prod_{\{x,y\}\in E(\alpha\cup\beta)} \frac{W_x W_y}{n\mathbb{E}[W]}\Big] \le (\mathbb{E}[W]n)^{-|E(\alpha\cup\beta)|} \prod_{x\in V(\alpha\cup\beta)} \mathbb{E}\Big[W_x^{\deg_{\alpha\cup\beta}(x)}\Big]. \quad (5.12)\end{aligned}$$

We continue with showing the claim in part a) of the lemma. Suppose that there is a fixed $N \in \mathbb{N}$ with $k, \ell \le N$ and that α and β share j edges. Then $|E(\alpha \cup \beta)| = k + \ell - j \le 2N$ and the degree of any vertex $v \in V(\alpha \cup \beta)$ must lie in $\{2, 3, 4\}$. Since $|V(\alpha \cup \beta)| \le 2N$,

$$\mathbb{E}[X_\alpha X_\beta] \le \max\left(1, \mathbb{E}[W]^{-2N}\right) n^{-k-\ell+j} \max(1, \mathbb{E}[W^2], \mathbb{E}[W^3], \mathbb{E}[W^4])^{2N},$$

which shows the first claim for $G'(n)$.

For the claim in part b), let $a > 0$, $k, \ell \le \lfloor a \log(n) \rfloor$ and $\alpha \in I_k$ and $\beta \in I_\ell$ share s segments which contain $i_1, \dots, i_s \ge 1$ vertices. Therefore, $|V(\alpha \cup \beta)| = k + \ell - |i|$. Now we analyse how many vertices of the possible degrees 2, 3 and 4 are in $\alpha \cup \beta$. Whenever a segment has length one, this vertex needs to have degree 4. We write $m_i(1)$ for the number of segments of length 1 so that we have $m_i(1)$ vertices of degree 4 in $\alpha \cup \beta$. When a segment has at least length two, there will be two vertices of degree 3, namely start- and endpoint of the segment. Here we used that $\alpha \ne \beta$. We obtain $2(s - m_i(1))$ vertices of degree 3. The remaining $k + \ell - |i| - 2s + m_i(1)$ vertices have degree 2. Finally, the number of edges is given by $|E(\alpha \cup \beta)| = k + \ell - |i| + s$. By assumption we have $n\mathbb{E}[W^2] < n\mathbb{E}[W]$. We use this to bound the very first factor on the right-hand side in (5.12) (note that its exponent is negative), which yields

$$\begin{aligned}&\mathbb{E}[X_\alpha X_\beta] \le (n\mathbb{E}[W^2])^{-k-\ell+|i|-s}\mathbb{E}[W^4]^{m_i(1)}\mathbb{E}[W^3]^{2(s-m_i(1))}\mathbb{E}[W^2]^{k+\ell-|i|-2s+m_i(1)} \\ &= \left(\frac{\mathbb{E}[W^4]\mathbb{E}[W^2]}{\mathbb{E}[W^3]^2}\right)^{m_i(1)} \left(\frac{\mathbb{E}[W^3]^2}{\mathbb{E}[W^2]^3}\right)^{s} n^{-k-\ell+|i|-s} =: \gamma^{m_i(1)}\delta^s n^{-k-\ell+|i|-s} \le \kappa^s n^{-k-\ell+|i|-s},\end{aligned}$$

where $\kappa = \max(1, \gamma)\delta$ and the last inequality uses $m_i(1) \le s$. This concludes the proof for $G'(n)$.

For $G(n)$ and $\lambda \in (0, 1)$ we define

$$A_n = \{L_n > \lambda \mathbb{E}[W]n\}.$$

We obtain for all $3 \le k, \ell \le n$ and $\alpha \in I_k, \beta \in I_\ell$ that

$$p_{\alpha\beta} = \mathbb{E}[X_\alpha X_\beta] \le \mathbb{P}(A_n^c) + \mathbb{E}[\mathbf{1}_{A_n} X_\alpha X_\beta].$$

The rates of convergence claimed in the lemma are all slower than

$$n^{-3a\log(n)} = \exp(-3a\log(n)^2).$$

Since the term $\mathbb{P}(A_n^c)$ decays faster, as shown in Lemma 2.28, it suffices to show

$$\mathbb{E}[\mathbf{1}_{A_n} X_\alpha X_\beta] \le Dn^{-k-\ell+j} \quad \text{and} \quad \mathbb{E}[\mathbf{1}_{A_n} X_\alpha X_\beta] \le D\kappa^s n^{-k-\ell+|i|-s}$$

for some constant $D > 0$ in the respective setting. Using the same notation as above, we derive with the $\mathcal{W}$-measurability of A_n,

$$\mathbb{E}[\mathbf{1}_{A_n} X_\alpha X_\beta] = \mathbb{E}\Bigg[\mathbf{1}_{A_n} \prod_{\{x,y\}\in E(\alpha\cup\beta)} \mathbf{1}\{x \leftrightarrow y\}\Bigg] = \mathbb{E}\Bigg[\mathbf{1}_{A_n} \prod_{\{x,y\}\in E(\alpha\cup\beta)} \mathbb{P}_{\mathcal{W}}(x \leftrightarrow y)\Bigg]$$
$$\le \mathbb{E}\Bigg[\mathbf{1}_{A_n} \prod_{\{x,y\}\in E(\alpha\cup\beta)} \frac{W_x W_y}{L_n}\Bigg] \le (\lambda \mathbb{E}[W] n)^{-|E(\alpha\cup\beta)|} \prod_{x\in V(\alpha\cup\beta)} \mathbb{E}\Big[W_x^{\deg_{\alpha\cup\beta}(x)}\Big].$$

For part a), we obtain as above for $G'(n)$,

$$\mathbb{E}[\mathbf{1}_{A_n} X_\alpha X_\beta] \le \max\Big(1, (\lambda\mathbb{E}[W])^{-2N}\Big) n^{-k-\ell+j} \max(1, \mathbb{E}[W^2], \mathbb{E}[W^3], \mathbb{E}[W^4])^{2N}.$$

For the second claim, one can choose $\lambda = \mathbb{E}[W^2]/\mathbb{E}[W] < 1$ since $\mathbb{E}[W^2] < \mathbb{E}[W]$. As $\lambda\mathbb{E}[W] = \mathbb{E}[W^2]$, we can use the same calculations as for $G'(n)$ to obtain

$$\mathbb{E}[\mathbf{1}_{A_n} X_\alpha X_\beta] \le \kappa^s n^{-k-\ell+|i|-s}.$$

This concludes the proof. □

The following lemma will be employed to bound the third summand in the upper bound of Lemma 5.3.

Lemma 5.6. *Suppose that $\mathbb{E}[W^3] < \infty$ and consider $G(n)$ or $G'(n)$ for $G \in \mathcal{G}$.*

a) *Let $N \in \mathbb{N}$. There exists a constant $C > 0$ such that for all $3 \le k \le N$ and $n \in \mathbb{N}$,*

$$\left|\frac{(n)_k}{2k} p_k - \lambda_k\right| \le \frac{kC}{n}.$$

b) *Let $a > 0$ and $\mathbb{E}[W^2] < \mathbb{E}[W]$. There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $3 \le k \le \lfloor a\log(n)\rfloor$,*

$$\left|\frac{(n)_k}{2k} p_k - \lambda_k\right| \le \frac{kC}{n}.$$

Proof. We start with the much simpler case of $G'(n)$ for $G \in \mathcal{G}$. We provide upper and lower bounds for p_k and show that they are both sufficiently close to λ_k, providing in the desired result. We use the upper bound

$$p_k \le \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k n^{-k} = \frac{2k\lambda_k}{n^k}$$

from (5.10). Replacing p_k in the claims by this upper bound, we derive

$$\left|\frac{(n)_k}{n^k}\lambda_k - \lambda_k\right| \le \frac{\lambda_k}{n^k}\left(n^k - (n-k)^k\right) = \frac{\lambda_k}{n^k}\left(n^k - \sum_{i=0}^k \binom{k}{i}(-k)^i n^{k-i}\right)$$
$$\le \frac{\lambda_k}{n^k}\sum_{i=1}^k \binom{k}{i} k^i n^{k-i} \le \lambda_k \sum_{i=1}^k \left(\frac{k^2}{n}\right)^i = \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k \frac{k}{2n}\sum_{j=0}^{k-1}\left(\frac{k^2}{n}\right)^j,$$

where the last step uses the definition of λ_k and pulls one factor k^2/n out of the sum. This bound is as desired for the following reasons. First of all, the latter sum goes to one as $n \to \infty$ for all k that are of order $o(\sqrt{n})$ by a comparison to the geometric series. On the other hand, the first fraction is dealt with in part a) by taking the maximum over all $k = 3, \dots, N$, whereas it is bounded by one in part b).

For a lower bound, we write $W_0 = W_k$ and apply Lemma 3.5 and Lemma 2.29 to obtain

$$\begin{aligned} p_k &\geq \mathbb{E}\Big[\prod_{i=1}^k W_i^2 \prod_{j=1}^k \frac{1}{W_j W_{j-1} + n\mathbb{E}[W]}\Big] \\ &\geq \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k n^{-k} - \frac{1}{(n\mathbb{E}[W])^{k+1}} \mathbb{E}\Big[\sum_{j=1}^k W_j W_{j-1} \prod_{i=1}^k W_i^2\Big]. \end{aligned}$$

The first summand coincides with our upper bound. The second summand is negligible, because

$$\frac{(n)_k}{2k} \frac{1}{(n\mathbb{E}[W])^{k+1}} k \mathbb{E}[W^3]^2 \mathbb{E}[W^2]^{k-2} \leq \frac{1}{2n} \frac{\mathbb{E}[W^3]^2}{\mathbb{E}[W]^3} \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^{k-2}.$$

In part a), this is of the same order as the claimed rate. This is also true for part b), because the last fraction is bounded by one. This yields the claim for $G'(n)$.

Our analysis becomes more involved for the graphs $G(n)$. We provide an approach that works for part a) and b) with the correct parameter choices. Let $M_n = N$ for the first claim and $M_n = \lfloor a \log(n) \rfloor$ for the second claim as well as $3 \leq k \leq M_n$. We may assume without loss of generality that $M_n < n$ since $M_n = o(n)$ as $n \to \infty$.

Once again, we investigate lower and upper bounds on p_k to obtain the desired statements. For $\lambda \in (0, 1)$ and $1 \leq j < n$ we define $L_{n,j} = \sum_{i=j+1}^n W_i$ and

$$A_n = \{L_{n,M_n} > (n - M_n)\lambda\mathbb{E}[W]\},$$

where we choose $\lambda = 1/2$ for the first claim of the lemma and $\lambda = \mathbb{E}[W^2]/\mathbb{E}[W]$ for the second claim. Note that A_n is independent of $W_1, \dots, W_k$, which will be crucial later on. For an upper bound, we use that p_k is the expectation of an indicator. Thus, we can bound its contribution on the set A_n^c by $\mathbb{P}(A_n^c)$. Using the $\mathcal{W}$-measurability of $\mathcal{A}_n$ and the bounds for edge probabilities from Lemma 3.5, we obtain

$$p_k \leq \mathbb{E}\left[\mathbf{1}_{A_n} \prod_{i=1}^k \frac{W_i^2}{L_n}\right] + \mathbb{P}(A_n^c).$$

For a lower bound, we additionally use Lemma 2.29 with $b = L_n$ and $x_i = W_i W_{i-1}$ for $i = 1, \dots, k$ so that

$$\begin{aligned} p_k &\geq \mathbb{E}\left[\prod_{i=1}^k \frac{W_i^2}{L_n + W_i W_{i-1}}\right] \geq \mathbb{E}\left[\mathbf{1}_{A_n} \prod_{i=1}^k \frac{W_i^2}{L_n + W_i W_{i-1}}\right] \\ &\geq \mathbb{E}\left[\mathbf{1}_{A_n} \prod_{i=1}^k \frac{W_i^2}{L_n}\right] - \mathbb{E}\left[\mathbf{1}_{A_n} \frac{1}{L_n^{k+1}} \sum_{j=1}^k W_j W_{j-1} \prod_{i=1}^k W_i^2\right] =: \mathbb{E}\left[\mathbf{1}_{A_n} \prod_{i=1}^k \frac{W_i^2}{L_n}\right] - R_1(k), \end{aligned}$$

where the first summand already appeared in the upper bound. We continue by showing

$$\left|\frac{(n)_k}{2k}\mathbb{E}\left[\mathbf{1}_{A_n}\prod_{i=1}^k \frac{W_i^2}{L_n}\right] - \lambda_k\right| \leq \sum_{i=2}^5 R_i(k) \tag{5.13}$$

with

$$\begin{aligned}
R_2(k) &= \frac{(n)_k}{2k}\mathbb{E}\left[\mathbf{1}_{A_n}\left(\frac{1}{L_{n,k}^k} - \frac{1}{L_n^k}\right)\prod_{i=1}^k W_i^2\right],\\
R_3(k) &= (n)_k\lambda_k\mathbb{E}\left[\mathbf{1}_{A_n}\left(\frac{1}{L_{n,k}^k} - \frac{1}{L_n^k}\right)\prod_{i=1}^k W_i\right],\\
R_4(k) &= (n)_k\lambda_k\mathbb{E}\left[\mathbf{1}_{A_n^c}\prod_{i=1}^k \frac{W_i}{L_n}\right] \text{ and}\\
R_5(k) &= \lambda_k\mathbb{E}\left[\sum_{v\in[n]^k_=}\prod_{i=1}^k \frac{W_{v_i}}{L_n}\right],
\end{aligned}$$

where $[n]^k_= = [n]^k \setminus [n]^k_{\neq}$ denotes the k-tuples in $[n]^k$ with at least two equal entries. After establishing (5.13), combining the lower and upper bound yields

$$\left|\frac{(n)_k}{2k}p_k - \lambda_k\right| \leq \frac{n^k}{2k}\big(\mathbb{P}(A_n^c) + R_1(k)\big) + \sum_{i=2}^5 R_i(k) \tag{5.14}$$

and it remains to derive bounds for $\mathbb{P}(A_n^c), R_1(k), \ldots, R_5(k)$. We show (5.13) by rewriting

$$\frac{(n)_k}{2k}\mathbb{E}\left[\mathbf{1}_{A_n}\prod_{i=1}^k \frac{W_i^2}{L_n}\right] = \frac{(n)_k}{2k}\mathbb{E}\left[\mathbf{1}_{A_n}\prod_{i=1}^k \frac{W_i^2}{L_{n,k}}\right] - R_2(k).$$

Because A_n and $L_{n,k}$ are independent of $W_1, \ldots, W_k$, we derive

$$\begin{aligned}
&\frac{(n)_k}{2k}\mathbb{E}\left[\mathbf{1}_{A_n}\prod_{i=1}^k \frac{W_i^2}{L_{n,k}}\right] = \frac{(n)_k}{2k}\prod_{j=1}^k \frac{\mathbb{E}[W_j^2]}{\mathbb{E}[W_j]} \times \mathbb{E}\left[\mathbf{1}_{A_n}\prod_{i=1}^k \frac{W_i}{L_{n,k}}\right] = (n)_k\lambda_k\mathbb{E}\left[\mathbf{1}_{A_n}\prod_{i=1}^k \frac{W_i}{L_{n,k}}\right]\\
&= (n)_k\lambda_k\mathbb{E}\left[\mathbf{1}_{A_n}\prod_{i=1}^k \frac{W_i}{L_n}\right] + R_3(k) = (n)_k\lambda_k\mathbb{E}\left[\prod_{i=1}^k \frac{W_i}{L_n}\right] - R_4(k) + R_3(k).
\end{aligned}$$

Observe that $|[n]^k_{\neq}| = (n)_k$ and use $\sum_{v\in[n]^k}\prod_{i=1}^k W_{v_i} = L_n^k$ to rewrite

$$(n)_k\lambda_k\mathbb{E}\left[\prod_{i=1}^k \frac{W_i}{L_n}\right] = \lambda_k\mathbb{E}\left[\sum_{v\in[n]^k_{\neq}}\prod_{i=1}^k \frac{W_{v_i}}{L_n}\right] = \lambda_k - \lambda_k\mathbb{E}\left[\sum_{v\in[n]^k_=}\prod_{i=1}^k \frac{W_{v_i}}{L_n}\right] = \lambda_k - R_5(k).$$

Applying the triangle inequality shows (5.13).

We proceed with establishing that all summands in (5.14) are bounded by Ck/n for some constant $C > 0$ as claimed. From Lemma 2.28 we obtain some $D > 0$ with

$$n^{2k+1}\mathbb{P}(A_n^c) \leq n^{2M_n+1}\exp(-D(n - M_n)) = \exp((2M_n + 1)\log(n) - Dn + DM_n) \to 0$$

as $n \to \infty$. In particular there exists a constant c_A not depending on k and n such that

$$\mathbb{P}(A_n^c) \leq c_A n^{-2k-1},$$

which addresses the first term in (5.14), since

$$\frac{n^k}{2k}\mathbb{P}(A_n^c) \leq \frac{c_A}{n^{k+1}}$$

decays even faster than the desired order.

Concerning $R_4(k)$ we use the definition of λ_k in (5.1) to obtain similarly

$$R_4(k) = (n)_k \lambda_k \mathbb{E}\left[\mathbf{1}_{A_n^c} \prod_{i=1}^{k} \frac{W_i}{L_n}\right] \leq n^k \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k \mathbb{P}(A_n^c) \leq \left(\frac{\mathbb{E}[W^2]}{n\mathbb{E}[W]}\right)^k c_A n^{-1}. \tag{5.15}$$

Since the first fraction converges to zero as $n \to \infty$, the expression is indeed bounded by C'/n for some constant $C' > 0$. Concerning $R_5(k)$, we have

$$R_5(k) = \lambda_k \mathbb{E}\left[\mathbf{1}_{A_n} \sum_{v \in [n]^{\underline{k}}_{=}} \prod_{i=1}^{k} \frac{W_{v_i}}{L_n}\right] + \lambda_k \mathbb{E}\left[\mathbf{1}_{A_n^c} \sum_{v \in [n]^{\underline{k}}_{=}} \prod_{i=1}^{k} \frac{W_{v_i}}{L_n}\right]. \tag{5.16}$$

For the second summand on the right-hand side above, note that the sum is bounded by one so that

$$\lambda_k \mathbb{E}\left[\mathbf{1}_{A_n^c} \sum_{v \in [n]^{\underline{k}}_{=}} \prod_{i=1}^{k} \frac{W_{v_i}}{L_n}\right] \leq \frac{1}{2k}\left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k \mathbb{P}(A_n^c),$$

which can be bounded as in (5.15). For the first summand on the right-hand side in (5.16), we use that there are fewer than k^2 choices for the indices of two equal entries of $v \in [n]^{\underline{k}}_{=}$. This gives us

$$\sum_{v \in [n]^{\underline{k}}_{=}} \prod_{i=1}^{k} W_{v_i} \leq k^2 \sum_{v \in [n]} W_v^2 L_n^{k-2}$$

and thus, using A_n to bound $L_n \geq (n - M_n)\lambda\mathbb{E}[W]$,

$$\begin{aligned}\lambda_k \mathbb{E}\left[\mathbf{1}_{A_n} \sum_{v \in [n]^{\underline{k}}_{=}} \prod_{i=1}^{k} \frac{W_{v_i}}{L_n}\right] &\leq \lambda_k k^2 \mathbb{E}\left[\mathbf{1}_{A_n} \sum_{v=1}^{n} \frac{W_v^2}{L_n^2}\right] \leq \frac{k}{2}\left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k \frac{n\mathbb{E}[W^2]}{((n-M_n)\lambda\mathbb{E}[W])^2} \\ &\leq \frac{k}{2n}\left(\frac{n}{n-M_n}\right)^2 \frac{\mathbb{E}[W^2]^{k+1}}{\lambda^2\mathbb{E}[W]^{k+2}}.\end{aligned}$$

The second factor on the right-hand side converges to 1 as $n \to \infty$ since $M_n = o(n)$. The last term is bounded by its maximum over $k = 3, \ldots, N$ in part a). In part b) we have $\mathbb{E}[W^2] < \mathbb{E}[W]$ so that the fraction is bounded by $\lambda^{-2}/\mathbb{E}[W]$, showing the desired bound for $R_5(k)$.

Before we continue with the remaining terms $R_1(k), R_2(k)$ and $R_3(k)$, we observe that the inequality $1+x \le e^x$ for $x \in \mathbb{R}$ and $M_n^2 = o(n)$ imply

$$1 \le \left(\frac{n}{n-M_n}\right)^k \le \left(\frac{n}{n-M_n}\right)^{M_n} = \left(1+\frac{M_n}{n-M_n}\right)^{M_n} \le \exp\left(\frac{M_n^2}{n-M_n}\right) \to 1 \quad (5.17)$$

as $n \to \infty$. For $R_1(k)$ we again use A_n to bound L_n^{-k-1} and compute

$$\begin{aligned} \frac{n^k}{2k} R_1(k) &= \frac{n^k}{2k}\mathbb{E}\left[\mathbf{1}_{A_n}\frac{1}{L_n^{k+1}}\sum_{j=1}^k W_j W_{j-1}\prod_{i=1}^k W_i^2\right] \le \frac{n^k}{2k}\frac{k\mathbb{E}[W^3]^2\mathbb{E}[W^2]^{k-2}}{((n-M_n)\lambda\mathbb{E}[W])^{k+1}} \\ &= \frac{1}{2n}\left(\frac{n}{n-M_n}\right)^{k+1}\frac{\mathbb{E}[W^3]^2\mathbb{E}[W^2]^{k-2}}{(\lambda\mathbb{E}[W])^{k+1}}. \end{aligned} \quad (5.18)$$

By (5.17), the second fraction in (5.18) is bounded by some constant. It remains to show that the third fraction is bounded by a constant, where we do a case distinction. For $3 \le k \le N$ in part a), the bound is trivial by taking the maximum. In part b) instead, plugging in $\lambda = \mathbb{E}[W^2]/\mathbb{E}[W]$ simplifies the expression to $\mathbb{E}[W^3]^2/\mathbb{E}[W^2]^3$. For $R_2(k)$ we apply the mean value theorem to $f(x) = x^{-k}$. This gives us

$$\frac{1}{L_{n,k}^k} - \frac{1}{L_n^k} \le k\sum_{j=1}^k W_j L_{n,k}^{-k-1}$$

so that, using A_n to bound $L_{n,k}^{-k-1}$,

$$\begin{aligned} R_2(k) &= \frac{(n)_k}{2k}\mathbb{E}\left[\mathbf{1}_{A_n}\left(\frac{1}{L_{n,k}^k}-\frac{1}{L_n^k}\right)\prod_{i=1}^k W_i^2\right] \le \frac{n^k}{2k}\mathbb{E}\left[\mathbf{1}_{A_n} k\sum_{j=1}^k W_j L_{n,k}^{-k-1}\prod_{i=1}^k W_i^2\right] \\ &\le \frac{n^k}{2k}k^2\mathbb{E}[W^3]\mathbb{E}[W^2]^{k-1}\left(\frac{1}{(n-M_n)\lambda\mathbb{E}[W]}\right)^{k+1} \\ &= \frac{k}{2n}\left(\frac{n}{n-M_n}\right)^{k+1}\frac{\mathbb{E}[W^3]\mathbb{E}[W^2]^{k-1}}{(\lambda\mathbb{E}[W])^{k+1}}. \end{aligned}$$

The same argument as for (5.18) applies. Similarly, we obtain for $R_3(k)$ that

$$\begin{aligned} R_3(k) &= (n)_k\lambda_k\mathbb{E}\left[\mathbf{1}_{A_n}\left(\frac{1}{L_{n,k}^k}-\frac{1}{L_n^k}\right)\prod_{i=1}^k W_i\right] \\ &\le \frac{n^k}{2k}\left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k k^2\mathbb{E}[W^2]\mathbb{E}[W]^{k-1}\left(\frac{1}{(n-M_n)\lambda\mathbb{E}[W]}\right)^{k+1} \\ &= \frac{k}{2n}\left(\frac{n}{n-M_n}\right)^{k+1}\left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^{k+1}\frac{1}{\mathbb{E}[W]\lambda^{k+1}}. \end{aligned}$$

As above, a case distinction for part a) and b) yields the desired bound. This concludes the proof. □

The following theorem shows in the subcritical regime that there are asymptotically no cycles whose lengths grow at least logarithmically in n, as $n \to \infty$.

Theorem 5.7. *Let $\mathbb{E}[W^2] < \mathbb{E}[W], a > 0$ and consider $G(n)$ or $G'(n)$ for $G \in \mathcal{G}$. There exists a constant $C > 0$ such that for all $n \geq 3$,*

$$\mathbb{P}(\mathcal{C}_n(\lfloor a\log(n)\rfloor + 1, \dots, n) > 0) \leq Cn^{a\log(p)} \quad \textit{for} \quad p = \frac{2\mathbb{E}[W^2]}{\mathbb{E}[W^2] + \mathbb{E}[W]}.$$

Note that the assumption $\mathbb{E}[W^2] < \mathbb{E}[W]$ implies $p < 1$, so that $\log(p) < 0$. Therefore, the upper bound in the previous theorem tends to zero as $n \to \infty$.

Proof of Theorem 5.7. We start once more with $G'(n)$ for $G \in \mathcal{G}$. Using the Markov inequality yields

$$\begin{aligned}\mathbb{P}(\mathcal{C}_n(\lfloor a\log(n)\rfloor + 1, \dots, n) > 0) &= \mathbb{P}\Bigg(\sum_{k=\lfloor a\log(n)\rfloor+1}^{n} \sum_{\alpha \in I_k} X_\alpha \geq 1\Bigg) \\ &\leq \sum_{k=\lfloor a\log(n)\rfloor+1}^{n} \sum_{\alpha \in I_k} \mathbb{E}[X_\alpha] \leq \sum_{k=\lfloor a\log(n)\rfloor+1}^{n} \left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}\right)^k,\end{aligned}$$

where we applied (5.10) and $|I_k| \leq n^k$ in the last inequality. As $\mathbb{E}[W^2] < \mathbb{E}[W]$, the fraction on the right-hand side above is bounded by $p < 1$ as given in the Theorem. Therefore, the expression above is bounded by

$$\sum_{k=\lfloor a\log(n)\rfloor+1}^{n} p^k \leq p^{\lfloor a\log(n)\rfloor+1} \sum_{k=0}^{\infty} p^k = p^{\lfloor a\log(n)\rfloor+1} \frac{1}{1-p}.$$

Since $\lfloor a\log(n)\rfloor + 1 \geq a\log(n)$ and $p < 1$, we have

$$p^{\lfloor a\log(n)\rfloor+1} \leq p^{a\log(n)} = n^{\log(p)a},$$

which proves the claim for $G'(n)$.

For $G(n)$, we choose

$$\lambda = \frac{\mathbb{E}[W^2] + \mathbb{E}[W]}{2\mathbb{E}[W]} < 1.$$

Define $A_n = \{L_n > \lambda\mathbb{E}[W]n\}$ so that Lemma 2.28 yields a positive constant c such that

$$\begin{aligned}\mathbb{P}(\mathcal{C}_n(\lfloor a\log(n)\rfloor + 1, \dots, n) > 0) &\leq \mathbb{P}(A_n^c) + \mathbb{P}(\mathbf{1}_{A_n}\mathcal{C}_n(\lfloor a\log(n)\rfloor + 1, \dots, n) > 0) \\ &\leq e^{-cn} + \sum_{k=\lfloor a\log(n)\rfloor+1}^{n} \sum_{\alpha \in I_k} \mathbb{E}[\mathbf{1}_{A_n} X_\alpha]. \qquad (5.19)\end{aligned}$$

Since the first summand decays faster than the desired order, it suffices to show the claimed bound for the second summand. With (5.11) we compute for $k \geq 3$ and $\alpha \in I_k$,

$$\mathbb{E}[\mathbf{1}_{A_n} X_\alpha] \leq n^{-k}\left(\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} \frac{2\mathbb{E}[W]}{\mathbb{E}[W^2] + \mathbb{E}[W]}\right)^k = n^{-k}\left(\frac{2\mathbb{E}[W^2]}{\mathbb{E}[W^2] + \mathbb{E}[W]}\right)^k = n^{-k}p^k.$$

Therefore, we can proceed as for $G'(n)$ to derive the assertion. □

We have now gathered all tools to prove Theorem 5.2.

Proof of Theorem 5.2. We start with proving part a). By Lemma 5.3 we have for all $A \subseteq \{3, \dots, N\}$,

$$d_{TV}(\mathcal{C}_n(A), \eta(A)) \leq \frac{1}{2n} \sum_{k,\ell=3}^{N} p_k p_\ell n^{k+\ell} + \sum_{k,\ell=3}^{N} \sum_{s=1}^{k} \sum_{i \in [k]^s} p_{k,\ell,s,i} (2k\ell)^{s-1} n^{k+\ell-|i|} + \sum_{k=3}^{N} \left| \frac{(n)_k}{2k} p_k - \lambda_k \right|.$$

We show that all summands can be bounded by C/n for some $C > 0$. For the third summand, part a) of Lemma 5.6 yields the existence of some $C_1 > 0$ such that for all $n \in \mathbb{N}$,

$$\sum_{k=3}^{N} \left| \frac{(n)_k}{2k} p_k - \lambda_k \right| \leq \sum_{k=3}^{N} \frac{kC_1}{n} \leq \frac{C_1 N^2}{n}.$$

Next we address the first summand. From part a) of Lemma 5.4 we know that there exists $C_2 > 0$ with $p_k \leq C_2 n^{-k}$ for all $3 \leq k \leq N$ and $n \in \mathbb{N}$. Therefore,

$$\frac{1}{2n} \sum_{k,\ell=3}^{N} p_k p_\ell n^{k+\ell} \leq \frac{N^2 C_2^2}{2n}.$$

For the remaining summand, note that when α and β intersect in s segments of lengths $i_1, \dots, i_s$, they share $|i| - s$ edges. Thus, part a) of Lemma 5.5 yields the existence of some $C_3 > 0$ with

$$\sum_{k,\ell=3}^{N} \sum_{s=1}^{k} \sum_{i \in [k]^s} p_{k,\ell,s,i} (2k\ell)^{s-1} n^{k+\ell-|i|} \leq \sum_{k,\ell=3}^{N} \sum_{s=1}^{k} \sum_{i \in [k]^s} C_3 n^{-k-\ell+|i|-s} (2k\ell)^{s-1} n^{k+\ell-|i|}$$
$$= \frac{C_3}{n} \sum_{k,\ell=3}^{N} \sum_{s=0}^{k-1} k^{s+1} (2kl)^s n^{-s},$$

where the sums are bounded in n, which concludes the proof for a).

For the proof of part b), let

$$p = \frac{2\mathbb{E}[W^2]}{\mathbb{E}[W^2] + \mathbb{E}[W]} \quad \text{and} \quad \rho = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}$$

so that $p, \rho < 1$ due to $\mathbb{E}[W^2] < \mathbb{E}[W]$. Choose $a > 0$ large enough such that

$$a \log(p) < -1 \quad \text{and} \quad a \log(\rho) < -1.$$

Writing $M_a = \{3, 4, \dots, \lfloor a \log(n) \rfloor\}$ and $\overline{M}_a = \{\lfloor a \log(n) \rfloor + 1, \dots\}$ we have

$$d_{TV}(\mathcal{C}_n(A), \eta(A))$$
$$\leq d_{TV}(\mathcal{C}_n(A \cap M_a), \eta(A \cap M_a)) + \mathbb{P}(\mathcal{C}_n(A) \neq \mathcal{C}_n(A \cap M_a)) + \mathbb{P}(\eta(A) \neq \eta(A \cap M_a))$$

$$\leq d_{TV}(\mathcal{C}_n(A \cap M_a), \eta(A \cap M_a)) + \mathbb{P}(\mathcal{C}_n(\overline{M}_a) > 0) + \mathbb{P}(\eta(\overline{M}_a) > 0). \tag{5.20}$$

For the second summand above we use Theorem 5.7 to obtain a constant $C_1 > 0$ with

$$\mathbb{P}(\mathcal{C}_n(\overline{M}_a) > 0) \leq C_1 n^{a \log(p)} \leq C_1 n^{-1}$$

by the choice of a. For the third summand the definition of η yields

$$\mathbb{P}(\eta(\overline{M}_a) > 0) = 1 - \mathbb{P}(\eta(\overline{M}_a) = 0) = 1 - \exp\Big(- \sum_{k=\lfloor a\log(n)\rfloor+1}^{\infty} \lambda_k \Big).$$

We use $1 - \exp(-x) \leq x$ for $x \in \mathbb{R}$ and $\lambda_k = \rho^k/(2k) \leq \rho^k$ to derive

$$1 - \exp\Big(- \sum_{k=\lfloor a\log(n)\rfloor+1}^{\infty} \lambda_k \Big) \leq \sum_{k=\lfloor a\log(n)\rfloor+1}^{\infty} \rho^k \leq \rho^{\lfloor a \log(n)\rfloor+1} \sum_{k=0}^{\infty} \rho^k.$$

Since $\rho < 1$, this is further bounded by

$$\rho^{a\log(n)} \frac{1}{1-\rho} = n^{a\log(\rho)} \frac{1}{1-\rho} \leq \frac{n^{-1}}{1-\rho},$$

again by the choice of a. It remains to consider the first summand of (5.20). By Lemma 5.3 we have for n large enough such that $M_a \subseteq \{3, \ldots, n\}$,

$$\begin{aligned} &d_{TV}(\mathcal{C}_n(A \cap M_a), \eta(A \cap M_a)) \\ &\leq \frac{1}{2n} \sum_{k,\ell \in M_a} p_k p_\ell n^{k+\ell} + \sum_{k,\ell \in M_a} \sum_{s=1}^{k} \sum_{i \in [k]^s} p_{k,\ell,s,i} (2k\ell)^{s-1} n^{k+\ell-|i|} + \sum_{k \in M_a} \Big| \frac{(n)_k}{2k} p_k - \lambda_k \Big|. \end{aligned}$$

For the third summand above we apply part b) of Lemma 5.6 which yields the existence of some constant $D_1 > 0$ such that

$$\sum_{k \in M_a} \Big| \frac{(n)_k}{2k} p_k - \lambda_k \Big| \leq \sum_{k=3}^{\lfloor a\log(n)\rfloor} \frac{D_1 k}{n} \leq \frac{D_1 a^2 \log(n)^2}{n}.$$

For the first summand we use part b) of Lemma 5.4 to obtain the existence of a constant $D_2 > 0$ with

$$\frac{1}{2n} \sum_{k,\ell \in M_a} p_k p_\ell n^{k+\ell} \leq \frac{1}{2n} \sum_{k,\ell=3}^{\lfloor a\log(n)\rfloor} D_2^2 n^{-k-\ell} n^{k+\ell} \leq \frac{D_2^2 a^2 \log(n)^2}{2n}.$$

Finally, for the second summand, we obtain via part b) of Lemma 5.5 the existence of constants $\kappa, D_3 > 0$ such that

$$\sum_{k,\ell \in M_a} \sum_{s=1}^{k} \sum_{i \in [k]^s} p_{k,\ell,s,i} (2k\ell)^{s-1} n^{k+\ell-|i|} \leq \sum_{k,\ell=3}^{\lfloor a\log(n)\rfloor} \sum_{s=1}^{k} \sum_{i \in [k]^s} D_3 \kappa^s n^{-k-\ell+|i|-s} (2k\ell)^{s-1} n^{k+\ell-|i|}$$

$$= \sum_{k,\ell=3}^{\lfloor a\log(n)\rfloor} \frac{D_3}{2k\ell} \sum_{s=1}^{k} \left(\frac{2k^2\ell\kappa}{n}\right)^s = \sum_{k,\ell=3}^{\lfloor a\log(n)\rfloor} \frac{k\kappa D_3}{n} \sum_{s=0}^{k-1} \left(\frac{2k^2\ell\kappa}{n}\right)^s$$
$$\leq \frac{\kappa D_3}{n} a^3 \log(n)^3 \sum_{s=0}^{n} \left(\frac{2\lfloor a\log(n)\rfloor^3\kappa}{n}\right)^s,$$

where the series converges to 1 due to $\lfloor a\log(n)\rfloor^3/n \to 0$ as $n \to \infty$ and a comparison to the geometric series. This shows the claim. □

5.5 Longest and shortest cycle

Theorem 5.2 b) allows us to derive the asymptotic distributions of the lengths of the shortest cycle $\mathcal{C}_{\min}^{(n)}$ and of the longest cycle $\mathcal{C}_{\max}^{(n)}$ in the subcritical regime, when considering a graph $G(n)$ or $G'(n)$ for $G \in \mathcal{G}$. When there are no cycles at all, we choose the convention $\mathcal{C}_{\max}^{(n)} = \mathcal{C}_{\min}^{(n)} = 0$. We define the $\{0,3,4,\dots\}$−valued random variables $\mathcal{S}$ and $\mathcal{L}$ for the limits of the shortest and longest cycle, respectively, via

$$\mathbb{P}(\mathcal{S}=0) = \mathbb{P}(\mathcal{L}=0) = \exp\Big(-\sum_{k=3}^{\infty}\lambda_k\Big)$$

and

$$\mathbb{P}(3 \leq \mathcal{S} \leq t) = 1 - \exp\Big(-\sum_{k=3}^{t}\lambda_k\Big)$$

as well as

$$\mathbb{P}(\mathcal{L} \leq t) = \exp\Big(-\sum_{k=t+1}^{\infty}\lambda_k\Big)$$

for all $t \in \mathbb{N}_{\geq 3}$. We measure the rate of convergence in the Kolmogorov distance given by

$$d_{Kol}(X,Y) = \sup_{t\in\mathbb{R}} |\mathbb{P}(X \leq t) - \mathbb{P}(Y \leq t)|$$

for two random variables X and Y.

Theorem 5.8. *Let $\mathbb{E}[W^4] < \infty$ and $\mathbb{E}[W^2] < \mathbb{E}[W]$. We consider the graph $G'(n)$ or $G(n)$ for $G \in \mathcal{G}$.*

a) *There exists a constant $C > 0$ such that for all $n \geq 3$,*

$$d_{Kol}(\mathcal{C}_{\min}^{(n)}, \mathcal{S}) \leq \frac{C\log(n)^3}{n}.$$

b) *There exists a constant $C > 0$ such that for all $n \geq 3$,*

$$d_{Kol}(\mathcal{C}_{\max}^{(n)}, \mathcal{L}) \leq \frac{C\log(n)^3}{n}.$$

Proof. We relate the lengths of the shortest and of the longest cycle, respectively, to properties of the point process $\mathcal{C}_n$. Additionally, we express the probabilities of the limiting random variables $\mathcal{S}$ and $\mathcal{L}$ in terms of the point process η. The claim then follows from part b) of Theorem 5.2. For the claim on the shortest cycle we compute

$$\begin{aligned} & d_{Kol}(\mathcal{C}_{\min}^{(n)}, \mathcal{S}) = \sup_{t\in\mathbb{R}} |\mathbb{P}(\mathcal{C}_{\min}^{(n)} \leq t) - \mathbb{P}(\mathcal{S} \leq t)| \\ & \leq |\mathbb{P}(\mathcal{C}_{\min}^{(n)} = 0) - \mathbb{P}(\mathcal{S} = 0)| + \sup_{t\in\mathbb{N}_{\geq 3}} |\mathbb{P}(3 \leq \mathcal{C}_{\min}^{(n)} \leq t) - \mathbb{P}(3 \leq \mathcal{S} \leq t)|. \end{aligned}$$

We defined the shortest cycle to have length zero if and only if there is no cycle. Additionally, the shortest cycle having a length between 3 and n is equivalent to the existence of at least one cycle of that length. With the definitions of η and $\mathcal{S}$ the term above equals

$$\begin{aligned} & |\mathbb{P}(\mathcal{C}_n(3,4,\dots) = 0) - \mathbb{P}(\eta(3,4,\dots) = 0)| + \sup_{t\in\mathbb{N}_{\geq 3}} |\mathbb{P}(\mathcal{C}_n(3,\dots,t) > 0) - \mathbb{P}(\eta(3,\dots,t) > 0)| \\ & \leq 2 \sup_{A\subseteq\mathbb{N}_{\geq 3}} d_{TV}\big(\mathcal{C}_n(A), \eta(A)\big) \leq \frac{2C\log(n)^3}{n} \end{aligned}$$

for some constant $C > 0$ by part b) of Theorem 5.2. For the longest cycle we proceed similarly and have

$$\begin{aligned} & d_{Kol}(\mathcal{C}_{\max}^{(n)}, \mathcal{L}) = \sup_{t\in\mathbb{R}} |\mathbb{P}(\mathcal{C}_{\max}^{(n)} \leq t) - \mathbb{P}(\mathcal{L} \leq t)| \\ & \leq |\mathbb{P}(\mathcal{C}_{\max}^{(n)} = 0) - \mathbb{P}(\mathcal{L} = 0)| + \sup_{t\in\mathbb{N}_{\geq 3}} |\mathbb{P}(\mathcal{C}_{\max}^{(n)} > t) - \mathbb{P}(\mathcal{L} > t)| \\ & = |\mathbb{P}(\mathcal{C}_n(3,4,\dots) = 0) - \mathbb{P}(\eta(3,4,\dots) = 0)| \\ & \quad + \sup_{t\in\mathbb{N}_{\geq 3}} |\mathbb{P}(\mathcal{C}_n(t+1,\dots) > 0) - \mathbb{P}(\eta(t+1,\dots) > 0)| \\ & \leq 2 \sup_{A\subseteq\mathbb{N}_{\geq 3}} d_{TV}\big(\mathcal{C}_n(A), \eta(A)\big) \leq \frac{2C\log(n)^3}{n}, \end{aligned}$$

which concludes the proof. □

Chapter 6

Large components in the scale-free random connection model

This chapter investigates a question very similar to that in Chapter 4. We are interested in the scale-free random connection model in the subcritical regime. More precisely, we consider the random graph in a finite observation window and study its component sizes. Our main result concerns convergence of the point process of rescaled component sizes as the size of the observation window goes to infinity. After presenting this result and a corollary on the size of the largest component in Section 6.1, we provide a discussion concerning related results and possible directions for further research in Section 6.2. We conclude this chapter with the proof of the point process convergence in Section 6.3.

The results and proofs of this chapter are joint work with Matthias Schulte. The respective paper is in preparation.

6.1 Main result

The question of interest in this chapter is closely related to that in Chapter 4. Instead of counting specific vertices in every component of the Norros-Reittu model and studying convergence of an underlying point process, we do the same for the scale-free random connection model. As it is a natural question whether the results can be extended to other weight distributions than those with regularly varying tail, we state all our results for the *weighted* random connection model. Phrased this way, one can see at which points we require the extra assumption on the tails of the weights that yield the scale-free random connection model according to Definition 3.11. To further simplify notation, we shall only write random connection model in the following and omit the term *weighted*.

Let us return to the question at hand. Contrary to the Norros-Reittu model, we restrict ourselves to the component sizes, i.e. we only treat the case where we count all vertices in each component as in Proposition 4.7. For the Norros-Reittu model, this amounts to studying the point process

$$\Xi_n = \sum_{v=1}^{n} \mathbf{1}\{v \in V_n^{\max}\}\delta_{|\mathcal{C}_n(v)|q(n)^{-1}\zeta^{-1}}$$

for $n \in \mathbb{N}$ and $\zeta = \mathbb{E}[W]/(\mathbb{E}[W] - \mathbb{E}[W^2])$, where $V_n^{\max}$ denotes the set of vertices having maximal weight in their component and $q(n)$ was a quantile of the weight distribution. We will now consider a similar point process for the subcritical random connection model.

The first observation is that we have no parameter controlling the number of vertices in the random connection model. If we were to consider all components at once, there would be infinitely many components of all finite sizes as the underlying graph is infinite and translation invariant. Therefore, we restrict ourselves to the observation window $S_n = [0, n^{1/d}]^d$ of volume n for $n \in \mathbb{N}$. Recall that we write $\hat{\eta}$ for the underlying Poisson process containing the vertices $\hat{x} = (x, W_x)$ with their spatial positions $x \in \mathbb{R}^d$ and their weights W_x. Moreover, we write η for the projection of $\hat{\eta}$ on the first d coordinates, i.e. the spatial coordinates of the vertices. Then, $\eta \cap S_n$ contains on average n vertices, as η has unit intensity.

Under the assumptions we shall use, the random connection model does not percolate and all components are almost surely finite. We define $V_{\max}$ analogously to $V_n^{\max}$ for the Norros-Reittu model. In order to keep the notation simple, $V_{\max} \subseteq \eta$ only contains the spatial coordinates of the vertices having the largest weight in their component. If there are multiple vertices having the largest weight, we choose the vertex whose spatial coordinates are smallest with respect to the lexicographic order. However, this choice does not affect our results as long as we choose exactly one vertex per component.

Then, we study the point process

$$\Xi_n = \sum_{x \in \eta \cap S_n} \mathbf{1}\{x \in V_{\max}\} \delta_{q(n)^{-1}\zeta^{-1}|\mathcal{C}(x)|}$$

for $n \in \mathbb{N}$, where $\mathcal{C}(x)$ denotes the component of some vertex $x \in \eta$ in the random connection model. As in Chapter 4, ζ is some positive constant whereas $q(n)$ is related to a quantile of the weight distribution. As the overall strategy is similar, we stick to the notation, although ζ and $q(n)$ differ slightly from their values in Chapter 4. Sticking to the notation from before should facilitate the comparison of both proofs.

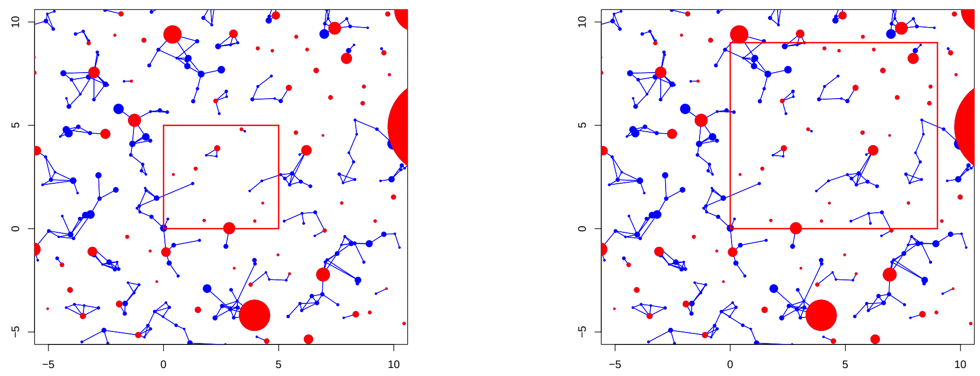
A visualisation of the situation can be found in Figure 6.1, where the red squares correspond to the observation windows S_n. The vertices of maximal weight are highlighted in red, so that Ξ_n studies the rescaled component sizes of all red vertices inside the red square.

Note that considering the components of the vertices in $V_n^{\max}$ simply led to studying all components in the graph exactly once in Chapter 4. In our current setup, however, it is a convention concerning how to treat components that lie just partially in the observation window S_n. We discuss related setups in the next section.

We continue by stating the assumptions that we require. Our first assumption is required to turn the (weighted) random connection model into the scale-free random connection model according to Definition 3.11. As mentioned above, we include this assumption explicitly in our statements to highlight which results remain true for other weight distributions. It is very similar to assumption (W) in Chapter 4, but the subcriticality of our model cannot be expressed simply by the weight distribution this time.

(W) W has a regularly varying tail with index $-\beta$ for $\beta > 0$.

Note that our scaling factor $q(n)$ differs from the one in Chapter 4. Essentially, this is due to the fact that $\mathbb{E}_{\mathcal{W}}[|\mathcal{C}_n(x)|] \approx W_x\zeta$ for $x \in [n]$ in the Norros-Reittu model whereas, for the random connection model instead, Lemma 6.9 below shows a statement of the form



(a) $n = 25$ (b) $n = 81$

Figure 6.1: Components in the random connection model in the red observation window $S_n = [0, \sqrt{n}]^2$. Vertices with maximal weight in their component are highlighted in red.

$\mathbb{E}\big[|\mathcal{C}(v)|\big|W_v\big] \approx W_v^{d/\alpha}\zeta$ for $v \in \eta$. Given $t \geq 1$, we define $q(t)$ as the $1 - 1/t$ quantile of $W^{d/\alpha}$ so that

$$q(t) = \inf\{s > 0 \colon \mathbb{P}(W^{d/\alpha} \geq s) \geq 1 - 1/t\}.$$

Note that $W^{d/\alpha}$ has a regularly varying tail with index $-\alpha\beta/d$, so that the asymptotics from Proposition 2.19 are applicable to $q(t)$. We also use the assumptions

$$\begin{aligned}
&\text{(A1)} \qquad \alpha > d,\\
&\text{(A2)} \qquad \mathbb{E}[W^{3d/\alpha}] < \infty \quad \text{and}\\
&\text{(A3)} \qquad \rho := \lambda^{d/\alpha}\kappa_d\Gamma(1 - d/\alpha)\mathbb{E}[W^{2d/\alpha}] < 1,
\end{aligned}$$

where λ is as in Definition 3.11, κ_d denotes the volume of the d-dimensional unit ball and Γ refers to the gamma function. We will discuss the assumptions in the next section, including comments on the extent to which one may relax them.

With the framework in place, we can state the main result of this chapter, which resembles Theorem 4.1.

Theorem 6.1. *Suppose that* (W), (A1)–(A3) *hold and write* $\gamma = \alpha\beta/d$. *Let* η_γ *denote a Poisson process on* $(0, \infty]$ *with intensity measure given by* $\mu((a, b]) = a^{-\gamma} - b^{-\gamma}$ *for* $0 < a < b \leq \infty$. *Then, there exists some positive constant* $\zeta > 0$ *such that in* $M_p((0, \infty])$,

$$\Xi_n = \sum_{x \in \eta \cap S_n \cap V_{\max}} \delta_{q(n)^{-1}\zeta^{-1}|\mathcal{C}(x)|} \xrightarrow{d} \eta_\gamma \quad \textit{as} \quad n \to \infty.$$

The previous theorem allows us to derive asymptotic properties concerning the size of the largest component in S_n by means of Lemma 2.12.

Corollary 6.2. *If* (W), (A1)–(A3) *are true, there is a positive constant* $\zeta > 0$ *such that*

$$\max_{x \in \eta \cap S_n \cap V_{\max}} \frac{|\mathcal{C}(x)|}{q(n)\zeta} \xrightarrow{d} Z \quad \textit{as} \quad n \to \infty, \tag{6.1}$$

where Z denotes a random variable distributed according to a Fréchet distribution with parameter $\gamma = \alpha\beta/d$.

Since $q(n)$ is regularly varying with tail index $1/\gamma = d/(\alpha\beta)$, the scaling fits to our intuition: larger values of α penalise long edges, resulting in smaller components. When β increases, the weights become smaller, also resulting in smaller components. Finally, when the dimension increases, there are more directions in which a component can spread, which leads to larger components.

6.2 Discussion

We would like to comment on the assumptions (A1)-(A3) under assumption (W). First of all, note that a sufficient condition for (A2) in terms of the model parameters α, β and d is given by $3d < \alpha\beta$. For α, β and d satisfying assumptions (W), (A1) and (A2), assumption (A3) amounts to choosing λ small enough. We continue with a brief discussion on the necessity of (A1)–(A3).

Assumption (A1) cannot be relaxed, as $\alpha > d$ is required to ensure that the degrees are not almost surely infinite, see Theorem 3.12. This is also independent of the underlying weight distribution, as the function $1 - \exp(-|x|^{-\alpha})$ is not integrable over $\mathbb{R}^d$ when $\alpha \leq d$. Infinite degrees are problematic, as they in turn yield infinite component sizes, making the behaviour of Ξ_n trivial.

Since we are particularly interested in the size of the largest component in our observation window, we do not want the graph to percolate. This is ensured by $\alpha\beta > 2d$ and a small enough choice of λ, see Theorem 3.13 for the case when the weights follow a Pareto distribution. However, $\alpha\beta > 2d$ is equivalent to $\mathbb{E}[W^{2d/\alpha}] < \infty$ for a Pareto distribution. In principal, this gives hope concerning a possible relaxation of assumption (A2), which demands $\mathbb{E}[W^{3d/\alpha}] < \infty$. However, our proof relies on a variance bound in Lemma 6.7, which in turn requires a finite second moment of the component size. As discussed at the end of Subsection 3.2.2, this requires $\alpha\beta > 3d$ or, equivalently for a Pareto distribution, our assumption (A2). On the one hand one might argue that this restriction arises artificially from the proof. On the other hand, the intuition behind the statement is that the component sizes can be approximated reasonably well by weights, just as in Chapter 4. In this chapter, the respective statement can be found in Lemma 6.9. This idea starts to fall apart when the component sizes have infinite variance, i.e. when they are no longer reasonably centred around their conditional mean.

The last assumption arises artificially. We require (A3) to bound certain quantities, which should be bounded in an appropriate way throughout the whole subcritical regime. Our assumption does not cover the whole subcritical regime and it would be interesting to find a way to extend our result to all λ in the subcritical regime.

In our results, we restricted the random connection model to an observation window S_n of a specific form, namely a hypercube. By translation invariance, this set could also be centred around the origin, i.e. $S'_n = [-n^{1/d}/2, n^{1/d}/2]^d$ would yield the same result. One can also think of rotating the cube or even consider other sets, as long as they have volume n. This is due to the fact that our proof only relies on the large weights to be found in S_n. More precisely, we will always use the Mecke equation and translation invariance to place the points in S_n under consideration in the origin.

In order to discuss some related results from the literature, we introduce some notation. If we focus on the extremal quantities, Corollary 6.2 investigates the scaling of

$$\max_{x\in\eta\cap S_n\cap V_{\max}} |\mathcal{C}(x)| \tag{6.2}$$

for $n \in \mathbb{N}$, $S_n = [0, n^{1/d}]^d$ and $V_{\max}$ denoting the vertices that are of maximal weight in their component. As mentioned in the previous section, vertices *outside* of S_n may contribute to the considered component sizes above as long as the respective vertex of largest weight belongs to S_n. One could also restrict the whole random connection model ξ to solely build upon vertices in $\eta \cap S_n$, which we denote by ξ_n. We write $\mathcal{C}_n(x)$ for the component size of some vertex $x \in \eta \cap S_n$ in ξ_n. Instead of investigating the maximum in (6.2), one could study

$$\max_{x\in\eta\cap S_n} |\mathcal{C}_n(x)|. \tag{6.3}$$

Compared to our setup, a complication arises. As vertices close to the boundaries have less points near them and thus tend to have a smaller degree, boundary effects come into play. These in turn ruin the translation invariance, which is a key ingredient for our proof. Usually, one can expect these effects to be negligible, as the volume of a cube or a ball grows faster than its surface, at least in Euclidean space. We are unaware of results concerning the setting in (6.3) for the weighted random connection model, but there are two similar results we would like to mention. On the one hand, Deprez and Wüthrich studied the quantity above in the supercritical setting, see [33, Section 3.3]. It turns out that the number of vertices in the largest component in a box scales like a fraction of its volume. Küpper and Penrose, on the other hand, studied a version of the random connection model in [63] where the connection of any two vertices $x, y \in \mathbb{R}^d$ is governed by a symmetric connection function whose support is restricted to the unit cube. They study the term in (6.3) in the subcritical regime and obtain scaling of order $\log(n)$. Speaking of a connection function, the question arises whether we can extend Theorem 6.1 to more general connection functions than the explicit choice that we use. Such an extension seems possible and is planned to be carried out in the aforementioned paper by Lienau and Schulte which is still in preparation.

While the largest component of the random connection model in the supercritical regime is infinite, it is known that there is just one infinite cluster, see e.g. [73, Theorem 6.3]. Therefore, the *second*-largest component in some window S_n is of finite size and one might study its size. Results of this type were already studied for the Erdős-Rényi graph and the random geometric graph, see e.g. [59, Theorem 5.4] by Janson, Luczak and Rucinski for $G(n, p)$ and [83, Theorem 10.18] by Penrose as well as the recent paper [67] by Lichev, Lodewijks, Mitsche and Schapira for the random geometric graph. Note that [67] yields more precise results than [83], but only addresses dimension two. Also, [67, Remark 1.2] states that their results can be extended to more general connection functions than the strict indicator from the random geometric graph. Interestingly, the findings differ for the Erdős-Rényi graph and the random geometric graph: the second-largest component of $G(n, p)$ in the supercritical phase is of order $\log(n)$, the same size as the largest component in the subcritical regime. This is also referred to as duality principle. In the random geometric graph instead, the second-largest component in S_n is of order $\log(n)^{d/(d-1)}$, whereas the largest component in the subcritical phase is of order $\log(n)$, see e.g. [83, Theorem 10.3].

It would be interesting to derive results on the size of the second-largest component for the supercritical weighted random connection model, regarding both (6.2) and (6.3), but this seems to be a very difficult task. In particular, the methods used in this chapter all break down as they are restricted to the subcritical phase. To the best of our knowledge, a similar result is also unknown in the rank-1 models studied in Chapter 4, it is only known that the second-largest component is of strictly smaller order than the giant, as already established in [21] by Bollobás, Janson and Riordan. Since there are differences between the unweighted models $G(n,p)$ and the random geometric graph, it is of interest to compare the behaviour of the weighted models with and without geometry, i.e. the rank-1 models and the random connection model studied in this thesis. Do the differences prevail when moving to weighted versions or do the weights mitigate them?

Speaking of rank-1 models, recall that we also considered other counting statistics than the component sizes in Chapter 4, for example the count of all leaves in a single component. One might wonder whether it is possible to extend results of this kind to the random connection model. If one was able to show that the respective counting statistics satisfy a version of Lemma 6.9 below, the upper bounds for the component size derived in this chapter are sufficient to derive the respective results. However, proving a version of Lemma 6.9 seems a difficult task, as it is already the most involved lemma for the comparatively simple case of component sizes. Unfortunately, the usage of path counting techniques as for the Norros-Reittu model is not an option, as they rely on the absence of cycles. While counting paths in the random connection model leads to feasible upper bounds under our assumptions, the existence of cycles leads to overcounting. Therefore, it is not suitable for deriving precise asymptotics.

6.3 Proof of the main result

In this section we prove the point process convergence in Theorem 6.1. The proof idea is the same as in Chapter 4, we compare the point process of rescaled component sizes to that of rescaled weights. Therefore, we require similar ingredients as in Chapter 4. At the respective lemmas we refer to the corresponding earlier statements for the Norros-Reittu model.

One particular integral appears occasionally in the following so that we want to treat it here for future reference, see also the proof of [33, Proposition 4.3]. Recall that κ_d denotes the volume of the d-dimensional unit ball.

Lemma 6.3. *For $\alpha > d$ we have for all $a > 0$ and $y \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \left(1 - \exp\left(-\frac{a}{|x-y|^\alpha}\right)\right) dx = a^{d/\alpha}\kappa_d \Gamma(1 - d/\alpha).$$

Proof. A substitution and switching to (hyper-)spherical coordinates yields

$$\begin{aligned}\int_{\mathbb{R}^d} \left(1 - \exp\left(-\frac{a}{|x-y|^\alpha}\right)\right) dx &= a^{d/\alpha} \int_{\mathbb{R}^d} \big(1 - \exp\big(-|t|^{-\alpha}\big)\big) dt \\ &= a^{d/\alpha} d\kappa_d \int_0^\infty r^{d-1}\big(1 - \exp\big(-r^{-\alpha}\big)\big) dr.\end{aligned}$$

Using integration by parts, we compute

$$\int_0^\infty r^{d-1}\big(1-\exp\big(-r^{-\alpha}\big)\big)dr = \frac{r^d}{d}\big(1-\exp\big(-r^{-\alpha}\big)\big)\Big|_0^\infty + \int_0^\infty \frac{r^{d-\alpha-1}}{d}\alpha\exp\big(-r^{-\alpha}\big)dr$$
$$= 0 + \frac{1}{d}\int_0^\infty \alpha r^{d-\alpha-1}\exp\big(-r^{-\alpha}\big)dr = \frac{1}{d}\Gamma(1-d/\alpha),$$

where the last step substitutes $s = r^{-\alpha}$ and uses the definition of the gamma function. The assertion follows. □

We briefly recall some notation from the formal construction of the random connection model in Subsection 3.2.4. We denote the random connection model by ξ. When we add $k \in \mathbb{N}$ deterministic points $x_1, \dots, x_k \in \mathbb{R}^d$ with random weights $W_{x_1}, \dots, W_{x_k}$, which are independent of everything else, to the underlying Poisson process $\hat{\eta}$, we denote the resulting random connection model by $\xi^{x_1,\dots,x_k}$. To indicate the fact that vertices were added, we also write $\mathbb{E}_{x_1,\dots,x_k}$, meaning that all quantities inside the expectation depending on the random connection model shall be evaluated in $\xi^{x_1\dots,x_k}$ unless stated otherwise. In particular, the expected value $\mathbb{E}_{x_1,\dots,x_k}$ also integrates over the random weights $W_{x_1}, \dots, W_{x_k}$. We use the same convention when writing $\mathbb{P}_{x_1,\dots,x_k}$ or $\mathbf{Var}_{x_1,\dots,x_k}$.

In many proofs, we use path counting for upper bounds. Occasionally, we are going to combine several paths starting in a single vertex in such a way that we obtain a tree structure. The following lemma allows us to bound appearing expressions.

Lemma 6.4. *Let T denote a tree with vertices $V(T) = \{0, \dots, k\}$ for $k \in \mathbb{N}$ and edge set $E(T)$. For vertices $v_1, \dots, v_k \in \mathbb{R}^d$, we consider the embedding $\varphi\colon V(T) \to \{\mathbf{0}, v_1, \dots, v_k\}$ with $\varphi(0) = \mathbf{0}$ and $\varphi(i) = v_i$ for $i \in [k]$. Then we have for all measurable, non-negative functions $f\colon (\mathbb{R}^d)^{k+1} \to \mathbb{R}$,*

$$\mathbb{E}_{\mathbf{0}}\Big[\sum_{\hat{v}\in\hat{\eta}^k_{\neq}} f(W_{\varphi(0)}, \dots, W_{\varphi(k)}) \prod_{\{a,b\}\in E(T)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}\Big| W_{\mathbf{0}}\Big]$$
$$= (\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha))^k W_{\mathbf{0}}^{\deg_T(0)d/\alpha}\mathbb{E}\Big[f(W_{\mathbf{0}}, W_1, \dots, W_k)\prod_{i=1}^k W_i^{\deg_T(i)d/\alpha}\Big|W_{\mathbf{0}}\Big],$$

where $\deg_T(i)$ denotes the degree of vertex i in T and $W_1, \dots, W_k$ are independent copies of $W_{\mathbf{0}}$.

Proof. We apply Lemma 3.15, the Mecke equation for edge-marked Poisson processes, to obtain

$$\mathbb{E}_{\mathbf{0}}\Big[\sum_{\hat{v}\in\hat{\eta}^k_{\neq}} f(W_{\varphi(0)}, \dots, W_{\varphi(k)}) \prod_{\{a,b\}\in E(T)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}\Big| W_{\mathbf{0}}\Big]$$
$$= \int_{(\mathbb{R}^d)^k} \mathbb{E}_{\mathbf{0},v_1,\dots,v_k}\Big[f(W_{\varphi(0)}, \dots, W_{\varphi(k)}) \prod_{\{a,b\}\in E(T)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}\Big| W_{\mathbf{0}}\Big] dv_1 \dots dv_k.$$

The conditional expectation does not change when we replace the spatially dependent weights by independent copies $W_1, \dots, W_k$ of $W_0 = W_{\mathbf{0}}$. By conditioning on these weights, the indicators $\mathbf{1}\{x \leftrightarrow y\}$ become the respective connection probabilities, yielding

$$\int_{(\mathbb{R}^d)^k} \mathbb{E}\Big[f(W_0,\dots,W_k) \prod_{\{a,b\}\in E(T)} \Big(1-\exp\Big(-\lambda\frac{W_aW_b}{|\varphi(a)-\varphi(b)|^\alpha}\Big)\Big)\Big|W_0\Big]dv_1\dots dv_k.$$

We interchange integration and expectation, resulting in

$$\mathbb{E}\Big[f(W_0,\dots,W_k)\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} \prod_{\{a,b\}\in E(T)} \Big(1-\exp\Big(-\lambda\frac{W_aW_b}{|\varphi(a)-\varphi(b)|^\alpha}\Big)\Big)dv_1\dots dv_k\Big|W_0\Big].$$

Note that the variables v_i we integrate over appear in the exponential function in the form of $\varphi(i)$. Since T is a tree with at least two vertices, we can find (at least) one leaf that is distinct from zero. Since this leaf appears in just a single factor of the product, we can use Lemma 6.3 to evaluate the respective integral. After removing a leaf from a tree, there are two scenarios. Either there is just one vertex left or there are at least two remaining leaves. We may thus iterate this procedure and evaluate all integrals via Lemma 6.3. In each step, an edge $\{a,b\}$ gets removed from the tree and an integral gets replaced by the factor

$$W_a^{d/\alpha}W_b^{d/\alpha}\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha).$$

Therefore, every weight gets an exponent depending on its degree. Since the total number of edges in T is given by k, the result is given by

$$(\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha))^k W_{\mathbf{0}}^{\deg_T(0)d/\alpha}\mathbb{E}\Big[f(W_{\mathbf{0}},W_1,\dots,W_k)\prod_{i=1}^k W_i^{\deg_T(i)d/\alpha}\Big|W_{\mathbf{0}}\Big].$$

This shows the desired equality and concludes the proof. □

We continue with gathering lemmas to facilitate the proof of Theorem 6.1. The first statements concern moment bounds of the component sizes. For a gentle first contact, we focus on the simplest case for the time being, the first moment. We are using path counting techniques similar to the ones in Lemma 4.3, where we need to put in more effort due to a more complicated model.

Lemma 6.5. *Under assumptions* (A1) *and* (A3) *there exists a constant* $C>0$ *such that*

$$\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}}\big] \le 1 + W_{\mathbf{0}}^{d/\alpha}C.$$

Proof. Since $\mathbf{0}\notin\eta$ almost surely, we have in $\xi^{\mathbf{0}}$,

$$|\mathcal{C}(\mathbf{0})| = 1 + \sum_{\hat{x}\in\hat{\eta}} \mathbf{1}\{x\in\mathcal{C}(\mathbf{0})\}.$$

In order to derive the claim, we study the expectation of $|\mathcal{C}(\mathbf{0})|-1$. Using the Mecke equation provided in Lemma 3.15, we obtain

$$\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|-1\big|W_{\mathbf{0}}\big] = \mathbb{E}_{\mathbf{0}}\Big[\sum_{\hat{x}\in\hat{\eta}}\mathbf{1}\{x\in\mathcal{C}(\mathbf{0})\}\Big|W_{\mathbf{0}}\Big].$$

For $\hat{x} \in \hat{\eta}$, the condition $x \in \mathcal{C}(\mathbf{0})$ in $\xi^{\mathbf{0}}$ implies that there is a path leading from $\mathbf{0}$ to x. This means that there are $k \in \mathbb{N}$ and distinct $\hat{v}_0, \ldots, \hat{v}_k \in \hat{\eta} \cup \hat{\mathbf{0}}$ such that $v_0 \leftrightarrow v_1 \leftrightarrow \ldots \leftrightarrow v_k$ in $\xi^{\mathbf{0}}$, where we write $v_k = x$ and $v_0 = \mathbf{0}$ to simplify notation. We refer to their weights accordingly. We obtain

$$\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})| - 1 \big| W_{\mathbf{0}}\big] \leq \sum_{k=1}^{\infty} \mathbb{E}_{\mathbf{0}}\Big[\sum_{(\hat{v}_1,\ldots,\hat{v}_k) \in \hat{\eta}^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\}\Big| W_{\mathbf{0}}\Big].$$

For fixed k, the expectation on the right-hand side was addressed in Lemma 6.4. Since the endpoints $\mathbf{0}$ and v_k have degree one in their path while the remaining $k-1$ vertices have degree two, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbf{0}}\Big[\sum_{(\hat{v}_1,\ldots,\hat{v}_k) \in \hat{\eta}^k_{\neq}} \prod_{i=1}^{k} \mathbf{1}\{v_i \leftrightarrow v_{i-1}\}\Big| W_{\mathbf{0}}\Big] \\
&= W_{\mathbf{0}}^{d/\alpha} \kappa_d \lambda^{d/\alpha} \Gamma(1-d/\alpha) \mathbb{E}\big[W^{d/\alpha}\big] \big(\kappa_d \lambda^{d/\alpha} \Gamma(1-d/\alpha) \mathbb{E}[W^{2d/\alpha}]\big)^{k-1} \\
&= W_{\mathbf{0}}^{d/\alpha} \kappa_d \lambda^{d/\alpha} \Gamma(1-d/\alpha) \mathbb{E}\big[W^{d/\alpha}\big] \rho^{k-1},
\end{aligned}$$

using the definition of ρ in assumption (A3). Therefore,

$$\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})| - 1 \big| W_{\mathbf{0}}\big] \leq W_{\mathbf{0}}^{d/\alpha} \kappa_d \lambda^{d/\alpha} \Gamma(1-d/\alpha) \mathbb{E}\big[W^{d/\alpha}\big] \sum_{k=1}^{\infty} \rho^{k-1} =: W_{\mathbf{0}}^{d/\alpha} C,$$

where $\rho < 1$ by (A3) ensures the convergence of the series. This concludes the proof. □

Note that the previous lemma shows that the graph does not percolate under the assumptions (A1) and (A3). Now we provide similar bounds for higher moments. The idea is exactly the same, but the details are more involved.

Part b) of the following moment bound will be used after applying the Markov inequality and is the random connection model analogue of Lemma 4.6. We require the finite second moment from part a) to justify an application of the Poincaré inequality later on.

Lemma 6.6. *Suppose that* (A1) *and* (A3) *hold.*

a) *If* (A2) *is also satisfied, there exists a polynomial p_2 of degree* 2 *with non-negative coefficients such that*

$$\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|^2 \big| W_{\mathbf{0}}\big] \leq p_2(W_{\mathbf{0}}^{d/\alpha}).$$

b) *For all $k \in \mathbb{N}$ there exists a polynomial q_k of degree k with non-negative coefficients such that*

$$\mathbb{E}_{\mathbf{0}}\big[\mathbf{1}\{\mathbf{0} \in V_{\max}\} |\mathcal{C}(\mathbf{0})|^k \big| W_{\mathbf{0}}\big] \leq q_k(W_{\mathbf{0}}^{d/\alpha}).$$

Proof. We start by proving part a). As in the previous lemma, we write

$$|\mathcal{C}(\mathbf{0})| = 1 + \sum_{\hat{x} \in \hat{\eta}} \mathbf{1}\{x \in \mathcal{C}(\mathbf{0})\},$$

where the first summand accounts for $\mathbf{0}$ itself and we use that $\mathbf{0} \notin \eta$ almost surely. With Jensen's inequality we derive

$$|\mathcal{C}(\mathbf{0})|^2 \leq 2 + 2 \sum_{(\hat{x}_1,\hat{x}_2)\in\hat{\eta}^2} \mathbf{1}\{x_1, x_2 \in \mathcal{C}(\mathbf{0})\}.$$

In order to apply the Mecke equation, we need to get rid of equal entries of $(\hat{x}_1, \hat{x}_2)$. We have

$$\sum_{(\hat{x}_1,\hat{x}_2)\in\hat{\eta}^2} \mathbf{1}\{x_1, x_2 \in \mathcal{C}(\mathbf{0})\} = \sum_{\hat{x}_1\in\hat{\eta}} \mathbf{1}\{x_1 \in \mathcal{C}(\mathbf{0})\} + \sum_{(\hat{x}_1,\hat{x}_2)\in\hat{\eta}^2_{\neq}} \mathbf{1}\{x_1, x_2 \in \mathcal{C}(\mathbf{0})\}. \tag{6.4}$$

In Lemma 6.5 we have shown that

$$\mathbb{E}_{\mathbf{0}}\Big[\sum_{\hat{x}\in\hat{\eta}} \mathbf{1}\{x \in \mathcal{C}(\mathbf{0})\}\Big| W_{\mathbf{0}}\Big] \leq W_{\mathbf{0}}^{d/\alpha} C$$

for some $C > 0$. It suffices to show that there exists a positive constant $C' > 0$ such that

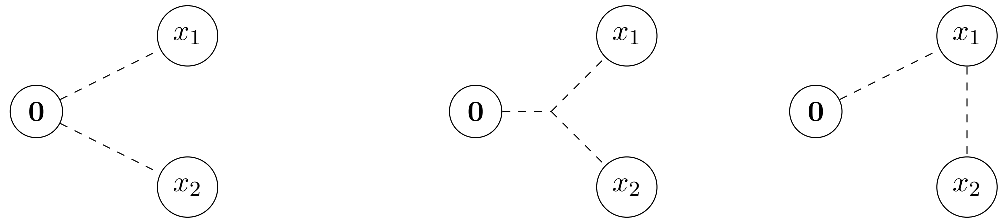
$$\mathbb{E}_{\mathbf{0}}\Big[\sum_{(\hat{x}_1,\hat{x}_2)\in\hat{\eta}^2_{\neq}} \mathbf{1}\{x_1, x_2 \in \mathcal{C}(\mathbf{0})\}\Big| W_{\mathbf{0}}\Big] \leq \big(W_{\mathbf{0}}^{d/\alpha} + W_{\mathbf{0}}^{2d/\alpha}\big) C' \tag{6.5}$$

to obtain part a) of the claim. For a fixed realisation of ξ and distinct $x_1, x_2 \in \eta$, we order them non-decreasing in their graph distance to $\mathbf{0}$. Then, we build a tree as follows. In the construction below, the term *shortest path* is with respect to the number of edges and, if there are several such paths, we always take the one whose vertices have the smallest lexicographic order when written in a single vector.

1. Connect $\mathbf{0}$ to x_1 by a shortest path $\mathcal{P}_1$.
2. Consider a shortest path P from $\mathbf{0}$ to x_2. Its part starting in its last intersection with $\mathcal{P}_1$ is denoted by $\mathcal{P}_2$. Here, *last* refers to P starting in $\mathbf{0}$ and ending in x_2. We consider the graph union of $\mathcal{P}_1$ and $\mathcal{P}_2$.

The resulting object is a tree, denoted by T. For our analysis, it can have either of the following two forms, see also Figure 6.2 where the dashed lines correspond to paths. Here, case a) on the left-hand side corresponds to both paths only intersecting in $\mathbf{0}$. Then, all vertices in T have degree two except for x_1 and x_2 having degree one. This case is responsible for a contribution of the form $W_{\mathbf{0}}^{2d/\alpha}$ to (6.5). In case b) on the other hand, the degree of $\mathbf{0}$ equals one so that this scenario contributes a term involving $W_{\mathbf{0}}^{d/\alpha}$ to (6.5). Note that it is possible to obtain a vertex of degree three in case b), which is why we require assumption (A2). We introduce some notation, which will later be generalised to the claim in part b). We denote the vertices of $\mathcal{P}_1$ by $\mathbf{0} \leftrightarrow v_1^{(1)} \leftrightarrow \ldots \leftrightarrow v_{k_1}^{(1)} = x_1$ so that $k_1 \in \mathbb{N}$ denotes the number of vertices other than $\mathbf{0}$ in $\mathcal{P}_1$. The variable s shall indicate where the second path starts. Therefore, we have $s = 0$ in case a). Fore case b), $s \in [k_1]$ is determined by the vertex $v_s^{(1)}$ of $\mathcal{P}_1$ in which the path $\mathcal{P}_2$ starts. Similarly to $\mathcal{P}_1$, we denote the vertices of $\mathcal{P}_2$ by $v_s^{(1)} \leftrightarrow v_1^{(2)} \leftrightarrow \ldots \leftrightarrow v_{k_2}^{(2)} = x_2$. For $k \in \mathbb{N}^2, s \in \{0, \ldots, k_1\}$ and given vertices

$$u = (v_1^{(1)}, \ldots, v_{k_1}^{(1)}, v_1^{(2)}, \ldots, v_{k_2}^{(2)}) \in (\mathbb{R}^d)^{|k|_1},$$



a) Last intersection of P_1 and P_2 in $\mathbf{0}$ b) Last intersection of P_1 and P_2 not in $\mathbf{0}$

Figure 6.2: Relations between P_1 and P_2

where we write $|\cdot|_1$ for the 1-norm, we denote the resulting tree connecting the vertices in u and $\mathbf{0}$ as above by $T(k,s,u)$. We define the isomorphism

$$\begin{aligned}\varphi\colon \{0,1,\dots,k_1+k_2\} &\to \{\mathbf{0},u_1,\dots,u_{k_1+k_2}\}\\ 0 &\mapsto \mathbf{0},\\ i &\mapsto u_i \quad \text{for} \quad i>0,\end{aligned}$$

which induces a tree $T(k,s)$ on $\{0,\dots,|k|_1\}$. We denote the respective edge set by $E(k,s)$. Summing over the possible choices for $k=(k_1,k_2)$, s and u yields

$$\begin{aligned}&\mathbb{E}_{\mathbf{0}}\Big[\sum_{(\hat{x}_1,\hat{x}_2)\in\hat{\eta}^2_{\neq}} \mathbf{1}\{x_1,x_2\in\mathcal{C}(\mathbf{0})\}\Big|W_{\mathbf{0}}\Big] \qquad (6.6)\\ &\leq 2\sum_{k\in\mathbb{N}^2}\sum_{s=0}^{k_1}\mathbb{E}_{\mathbf{0}}\Big[\sum_{\hat{u}\in\hat{\eta}^{|k|_1}_{\neq}}\prod_{\{a,b\}\in E(k,s)}\mathbf{1}\{\varphi(a)\leftrightarrow\varphi(b)\}\Big|W_{\mathbf{0}}\Big],\end{aligned}$$

where the factor two makes up for ordering the vertices x_1 and x_2 with respect to their graph distance from $\mathbf{0}$. For fixed $k\in\mathbb{N}^2$ and $0\leq s\leq k_1$, we can calculate the expectation by means of Lemma 6.4. This yields

$$\begin{aligned}&\mathbb{E}_{\mathbf{0}}\Big[\sum_{\hat{u}\in\hat{\eta}^{|k|_1}_{\neq}}\prod_{\{a,b\}\in E(k,s)}\mathbf{1}\{\varphi(a)\leftrightarrow\varphi(b)\}\Big|W_{\mathbf{0}}\Big]\\ &=(\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha))^{|k|_1}W_{\mathbf{0}}^{\deg_T(0)d/\alpha}\prod_{i=1}^{|k|_1}\mathbb{E}[W_i^{\deg_T(i)d/\alpha}], \qquad (6.7)\end{aligned}$$

where $\deg_T(i)$ denotes the degree of vertex i in $T(k,s)$ and $W_1,\dots,W_k$ are independent copies of $W_{\mathbf{0}}$. The possible degrees obviously lie in $\{1,2,3\}$, see also Figure 6.2. In case a), where $s=0$, $\mathbf{0}$ has degree two whereas there are two leaves and the $|k|_1-2$ other vertices have degree two. In case b) on the other hand, $\mathbf{0}$ has degree one and we have to distinguish the two displayed cases. For $s=k_1$, the second path gets attached to x_1, we have $|k|_1-1$ vertices of degree two and x_2 is another leaf. If $0<s<k_1$, the second path gets attached to an inner vertex of the first path $\mathcal{P}_1$. Then, we obtain one vertex with degree three and three leaves (including $\mathbf{0}$) while all $|k|_1-3$ remaining vertices have degree two. We bound (6.7) by

$$(\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha))^{|k|_1}\Big[W_{\mathbf{0}}^{2d/\alpha}\mathbb{E}[W^{2d/\alpha}]^{|k|_1-2}\mathbb{E}[W^{d/\alpha}]^2$$

$$
\begin{aligned}
&+ W_{\mathbf{0}}^{d/\alpha}\mathbb{E}[W^{2d/\alpha}]^{|k|_1-1}\mathbb{E}[W^{d/\alpha}] + W_{\mathbf{0}}^{d/\alpha}\mathbb{E}[W^{3d/\alpha}]\mathbb{E}[W^{2d/\alpha}]^{|k|_1-3}\mathbb{E}[W^{d/\alpha}]^2\Big]\\
&\le (W_{\mathbf{0}}^{d/\alpha} + W_{\mathbf{0}}^{2d/\alpha})(\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha)\mathbb{E}[W^{2d/\alpha}])^{|k|_1}\\
&\quad\times\bigg(\frac{\mathbb{E}[W^{d/\alpha}]^2}{\mathbb{E}[W^{2d/\alpha}]^2} + \frac{\mathbb{E}[W^{d/\alpha}]}{\mathbb{E}[W^{2d/\alpha}]} + \frac{\mathbb{E}[W^{3d/\alpha}]\mathbb{E}[W^{d/\alpha}]^2}{\mathbb{E}[W^{2d/\alpha}]^3}\bigg)\\
&=: (W_{\mathbf{0}}^{d/\alpha} + W_{\mathbf{0}}^{2d/\alpha})\rho^{|k|_1}M,
\end{aligned}
$$

where $\rho < 1$ is as in assumption (A3). Since this is an upper bound for a single summand of the right hand-side in (6.6), we obtain an upper bound for the whole expression by summing over k and s. This yields

$$
(W_{\mathbf{0}}^{d/\alpha} + W_{\mathbf{0}}^{2d/\alpha})2M\sum_{k_1,k_2=1}^{\infty}\sum_{s=0}^{k_1}\rho^{k_1+k_2} \le (W_{\mathbf{0}}^{d/\alpha} + W_{\mathbf{0}}^{2d/\alpha})2M\bigg(\sum_{\ell=1}^{\infty}(\ell+1)\rho^{\ell}\bigg)^2.
$$

By assumption (A3) the series is finite, providing the desired bound in (6.5). This shows part a).

For the claim in b) we proceed similarly, but two things change. On the one hand, the notation becomes more taxing, since we consider more than just two paths. On the other hand, we have the additional information that $\mathbf{0}$ has maximal weight in its component. Intuitively, the latter allows us to replace higher powers of the weights, originating in higher degrees in certain trees, by higher powers of $W_{\mathbf{0}}$. We continue with the details.

As before, we obtain via Jensen's inequality

$$
|\mathcal{C}(\mathbf{0})|^k \le 2^{k-1} + 2^{k-1}\sum_{(\hat{x}_1,\dots,\hat{x}_k)\in\hat{\eta}^k}\mathbf{1}\{x_1,\dots,x_k\in\mathcal{C}(\mathbf{0})\},
$$

but instead of the equality in (6.4) we consider

$$
\begin{aligned}
&\mathbf{1}\{\mathbf{0}\in V_{\max}\}\sum_{(\hat{x}_1,\dots,\hat{x}_k)\in\hat{\eta}^k}\mathbf{1}\{x_1,\dots,x_k\in\mathcal{C}(\mathbf{0})\}\\
&\le\sum_{t=1}^{k}t^k\sum_{(\hat{x}_1,\dots,\hat{x}_t)\in\hat{\eta}^t_{\neq}}\mathbf{1}\{\mathbf{0}\in V_{\max}\}\mathbf{1}\{x_1,\dots,x_t\in\mathcal{C}(\mathbf{0})\},
\end{aligned}
$$

where t^k is a conservative bound for the number of possibilities to distribute the t distinct points in $\hat{\eta}^t_{\neq}$ with possible repetitions among a vector with k entries. Thus, it suffices to show that

$$
\mathbb{E}_{\mathbf{0}}\bigg[\sum_{(\hat{x}_1,\dots,\hat{x}_t)\in\hat{\eta}^t_{\neq}}\mathbf{1}\{\mathbf{0}\in V_{\max}\}\mathbf{1}\{x_1,\dots,x_t\in\mathcal{C}(\mathbf{0})\}\bigg|W_{\mathbf{0}}\bigg] \le W_{\mathbf{0}}^{td/\alpha}C_t \tag{6.8}
$$

for all $t\in[k]$ and constants $C_t > 0$. We fix $t\in[k]$, order $x_1,\dots,x_t$ non-decreasing in their graph distance to $\mathbf{0}$ and consider a tree constructed by t paths leading from $\mathbf{0}$ to $x_1,\dots,x_t$. Similarly to the previous case, we choose minimal paths with respect to the lexicographic order.

1. Start with the graph G_0 consisting only of the vertex $\mathbf{0}$.

2. For $i = 1, \dots, t$, take a shortest path from $\mathbf{0}$ to x_i. Add its vertices and edges after its last intersection with G_{i-1} to G_{i-1} to obtain G_i.

Since we ordered the vertices non-decreasing in their graph distance to $\mathbf{0}$, it is ensured that we add at least one new vertex and one new edge for each $i \geq 1$. The resulting graph is a tree due to taking shortest paths and only adding the fragments after the last intersection with previously selected paths. The possible outcomes could be, for example, simply a path with endpoints $\mathbf{0}$ and x_t, a tree with $\mathbf{0}, x_1, \dots, x_t$ as its leaves or as in Figure 6.3.

Encoding this tree structure is more complicated than in part a) of this lemma. We denote the vertices added in step 2 for $i \in \{1, \dots, t\}$ by $\mathcal{P}_i = \{v_1^{(i)}, \dots, v_{k_i}^{(i)}\}$ so that $k_i \in \mathbb{N}$ denotes the number of vertices added. Similarly, we write $\mathcal{P}_0 = \{\mathbf{0}\} = \{v_1^{(0)}\}$ and $k_0 = 1$. For $i > 0$ we choose the convention that $v_1^{(i)}, \dots, v_{k_i}^{(i)}$ are labelled such that $v_1^{(i)} \leftrightarrow \dots \leftrightarrow v_{k_i}^{(i)} = x_i$ is the added path. Next, we need to know where to attach this branch, i.e. where $v_1^{(i)}$ connects to G_{i-1}. For $i \in \{1, \dots, t\}$, let $p_i = j$ when $\mathcal{P}_i$ is attached to some vertex in $\mathcal{P}_j$, see also Figure 6.3. For $p_i \geq 0$ we choose $s_i \in \{1, \dots, k_{p_i}\}$ such that $\mathcal{P}_i$ is attached to $v_{s_i}^{(p_i)}$, the s_i-th vertex of $\mathcal{P}_{p_i}$.

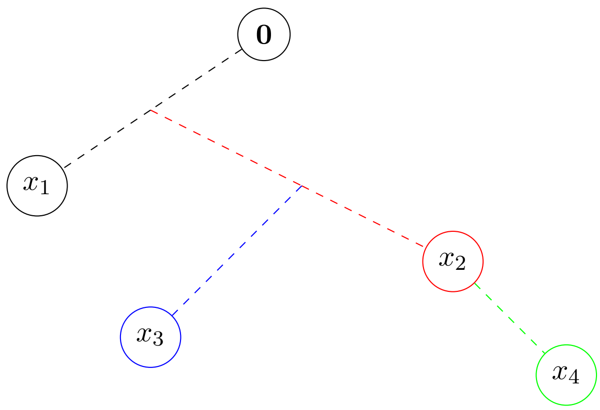


Figure 6.3: For $t = 4$, G_0 just has the vertex $\mathbf{0}$ whereas G_1 consists of the black part of the graph. The red part is added to obtain G_2 and adding the blue part yields G_3. Finally, G_4 contains all displayed vertices and edges. We have $p_1 = 0$, $p_2 = 1$, $p_3 = 2$ and $p_4 = 2$.

We write $k = (k_1, \dots, k_t)$ and define

$$u = (u_1, \dots, u_{|k|_1}) = (v_1^{(1)}, \dots, v_{k_1}^{(1)}, v_1^{(2)}, \dots, v_{k_2}^{(2)}, \dots, v_1^{(t)}, \dots, v_{k_t}^{(t)}).$$

We also abbreviate $p = (p_1, \dots, p_t)$ and $s = (s_1, \dots, s_t)$. The possible choices for p and s lie in

$$A = \bigtimes_{i=1}^{t} \{0, \dots, i-1\} \quad \text{and} \quad B(k,p) = \bigtimes_{i=1}^{t} \{1, \dots, k_{p_i}\}, \tag{6.9}$$

respectively, where $\times$ denotes the Cartesian product. Given $k \in \mathbb{N}^t, p \in A, s \in B(k,p)$ and $u \in (\mathbb{R}^d)^{|k|_1}$, there is a unique tree $T(k,p,s,u)$ that connects the vertices $\{\mathbf{0}, u_1, \dots, u_{|k|_1}\}$ as described above. The isomorphism

$$\varphi \colon \{0, 1, \dots, |k|_1\} \to \{\mathbf{0}, u_1, \dots, u_{|k|_1}\}$$

$$0 \mapsto \mathbf{0},$$
$$i \mapsto u_i \quad \text{for} \quad i > 0,$$

induces a tree $T(k,p,s)$ on $\{0,1,\dots,|k|_1\}$ and we denote its edge set by $E(k,p,s)$.

We use the same argument as in part a) and bound (6.8) by summing over the vertices belonging to certain trees. This time, $\mathbf{0} \in V_{\max}$ allows us to only sum over vertices whose weight is bounded by $W_{\mathbf{0}}$ from above. We obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbf{0}}\Bigg[\mathbf{1}\{\mathbf{0} \in V_{\max}\} \sum_{(\hat{x}_1,\dots,\hat{x}_t)\in\hat{\eta}^t_{\neq}} \mathbf{1}\{x_1,\dots,x_t \in \mathcal{C}(\mathbf{0})\}\Bigg| W_{\mathbf{0}}\Bigg] \\
&\le t! \sum_{k\in\mathbb{N}^t} \sum_{p\in A} \sum_{s\in B(k,p)} \mathbb{E}_{\mathbf{0}}\Bigg[\sum_{\hat{u}\in\hat{\eta}^{|k|_1}_{\neq}} \mathbf{1}\{W_{u_1},\dots,W_{u_{|k|_1}} \le W_{\mathbf{0}}\} \\
&\qquad \prod_{\{a,b\}\in E(k,p,s)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}\Bigg| W_{\mathbf{0}}\Bigg],
\end{aligned} \tag{6.10}$$

where $t!$ allows us to sort the vertices non-decreasingly in their distance to $\mathbf{0}$.

We apply Lemma 6.4 for fixed $k \in \mathbb{N}^t, p \in A$ and $s \in B(k,p)$ to obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbf{0}}\Bigg[\sum_{\hat{u}\in\hat{\eta}^{|k|_1}_{\neq}} \mathbf{1}\{W_{u_1},\dots,W_{u_{|k|_1}} \le W_{\mathbf{0}}\} \prod_{\{a,b\}\in E(k,p,s)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}\Bigg| W_{\mathbf{0}}\Bigg] \\
&= (\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha))^{|k|_1} W_{\mathbf{0}}^{\deg_T(0)d/\alpha}\mathbb{E}\Bigg[\mathbf{1}\{W_1,\dots,W_k \le W_{\mathbf{0}}\} \prod_{i=1}^{|k|_1} W_i^{\deg_T(i)d/\alpha}\Bigg| W_{\mathbf{0}}\Bigg]
\end{aligned}$$

for independent copies $W_1,\dots,W_k$ of $W_{\mathbf{0}}$. Once more we need to analyse the degrees in $T(k,p,s)$ or, equivalently, in $T(k,p,s,u)$. After adding the first path, $\mathbf{0}$ and x_1 have degree one whereas all $k_1 - 1$ vertices in between are of degree two. Whenever we add a new path $\mathcal{P}_i$ and attach it to $v^{(p_i)}_{s_i}$, we increase its degree by one, add $k_i - 1$ vertices of degree two and one leaf x_i. Due to the new indicator we can increase the power of $W_{\mathbf{0}}$ instead of increasing the power of the weight corresponding to $v^{(p_i)}_{s_i}$ in the product above. This leads to the exponents d/α for the weights corresponding to $x_1,\dots,x_t$ while $W_{\mathbf{0}}$ has exponent td/α and all remaining exponents are given by $2d/\alpha$. We obtain the upper bound

$$W_{\mathbf{0}}^{td/\alpha}(\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha))^{|k|_1}\mathbb{E}[W^{d/\alpha}]^t\mathbb{E}[W^{2d/\alpha}]^{|k|_1-t} = W_{\mathbf{0}}^{td/\alpha}\left(\frac{\mathbb{E}[W^{d/\alpha}]}{\mathbb{E}[W^{2d/\alpha}]}\right)^t \rho^{|k|_1}.$$

By (6.9), for each $i \in [t]$ there are $i \le t$ choices for p_i and $k_{p_i} \le \prod_{j=1}^t k_j$ choices for s_i so that (6.10) and the bound above yield

$$\begin{aligned}
&\mathbb{E}_{\mathbf{0}}\Bigg[\mathbf{1}\{\mathbf{0} \in V_{\max}\} \sum_{(\hat{x}_1,\dots,\hat{x}_t)\in\hat{\eta}^t_{\neq}} \mathbf{1}\{x_1,\dots,x_t \in \mathcal{C}(\mathbf{0})\}\Bigg| W_{\mathbf{0}}\Bigg] \\
&\le W_{\mathbf{0}}^{td/\alpha} t!\left(\frac{\mathbb{E}[W^{d/\alpha}]}{\mathbb{E}[W^{2d/\alpha}]}\right)^t \sum_{k\in\mathbb{N}^t} t^t\left(\prod_{j=1}^t k_j\right)^t \rho^{|k|_1} = W_{\mathbf{0}}^{td/\alpha} t! t^t\left(\frac{\mathbb{E}[W^{d/\alpha}]}{\mathbb{E}[W^{2d/\alpha}]}\right)^t \left(\sum_{\ell=1}^\infty \ell^t\rho^\ell\right)^t,
\end{aligned}$$

which shows (6.8) as $\rho < 1$. This concludes the proof. □

The variance bound below is required for an application of the Chebyshev inequality. In Lemma 4.5 we have seen a similar result for the Norros-Reittu model.

Lemma 6.7. *Let* (A1), (A2) *and* (A3) *be true. Then, there exists a constant* $C > 0$ *such that*

$$\mathbf{Var}_{\mathbf{0}}\big(|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}}\big) \le W_{\mathbf{0}}^{d/\alpha} C.$$

Proof. From Lemma 6.6 a) it follows that $|\mathcal{C}(\mathbf{0})|$ is square-integrable. From the Poincaré inequality for edge-marked Poisson processes, see Lemma 3.16, we obtain

$$\mathbf{Var}_{\mathbf{0}}\big(|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}}\big) \le \int_{\mathbb{R}^d} \mathbb{E}_{\mathbf{0}}\big[(\Delta_x|\mathcal{C}(\mathbf{0})|)^2\big|W_{\mathbf{0}}\big]\,dx, \tag{6.11}$$

where $\Delta_x|\mathcal{C}(\mathbf{0})|$ denotes the change of $|\mathcal{C}(\mathbf{0})|$ when going from $\xi^{\mathbf{0}}$ to $\xi^{\mathbf{0},x}$. To avoid confusion, we write $\mathcal{C}_{\mathbf{0}}(\mathbf{0})$ for the component of $\mathbf{0}$ in $\xi^{\mathbf{0}}$ and $\mathcal{C}_{\mathbf{0},x}$ for the component of $\mathbf{0}$ in $\xi^{\mathbf{0},x}$. When there is a path from $\mathbf{0}$ leading over x to y in $\xi^{\mathbf{0},x}$, we write $\mathbf{0} \overset{x}{\longleftrightarrow} y$. We obtain

$$\Delta_x|\mathcal{C}(\mathbf{0})| \le \mathbf{1}\{x \in \mathcal{C}_{\mathbf{0},x}(\mathbf{0})\} + |\{y \in \eta \colon \mathbf{0} \overset{x}{\longleftrightarrow} y, y \notin \mathcal{C}_{\mathbf{0}}(\mathbf{0})\}|.$$

In particular, the case $x = y$ is excluded as it happens with probability zero that η contains the point x. With Jensen's inequality we derive

$$\begin{aligned}(\Delta_x|\mathcal{C}(\mathbf{0})|)^2 &\le 2 \cdot \mathbf{1}\{x \in \mathcal{C}_{\mathbf{0},x}(\mathbf{0})\} + 2|\{(y,z) \in \eta^2 \colon \mathbf{0} \overset{x}{\longleftrightarrow} y, \mathbf{0} \overset{x}{\longleftrightarrow} z, y, z \notin \mathcal{C}_{\mathbf{0}}(\mathbf{0})\}| \\ &=: 2 \cdot \mathbf{1}\{x \in \mathcal{C}_{\mathbf{0},x}(\mathbf{0})\} + 2|M|.\end{aligned}$$

We consider the contribution of both summands to (6.11) separately. We start with the first summand. With the Mecke equation we conclude

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbb{E}_{\mathbf{0},x}\big[\mathbf{1}\{x \in \mathcal{C}_{\mathbf{0},x}(\mathbf{0})\}\big|W_{\mathbf{0}}\big]\,dx = \mathbb{E}_{\mathbf{0}}\Big[\sum_{\hat{x} \in \hat{\eta}} \mathbf{1}\{x \in \mathcal{C}_{\mathbf{0}}(\mathbf{0})\}\Big|W_{\mathbf{0}}\Big] &= \mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}_{\mathbf{0}}(\mathbf{0})| - 1\big|W_{\mathbf{0}}\big] \\ &\le W_{\mathbf{0}}^{d/\alpha} C_1\end{aligned}$$

for some positive constant C_1 by Lemma 6.5.

We continue with the contribution of $|M|$ to (6.11). We bound the number of $(y,z) \in M$ as follows. Since $\mathbf{0} \overset{x}{\longleftrightarrow} y$, there are $k \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ such that we can find a path

$$\mathbf{0} \leftrightarrow v_1 \leftrightarrow \ldots \leftrightarrow v_k \leftrightarrow x \leftrightarrow v_{k+1} \leftrightarrow \ldots \leftrightarrow v_{k+\ell} = y \tag{6.12}$$

with $v_1, \ldots, v_{k+\ell} \in \eta$. Here, $k = 0$ corresponds to $\mathbf{0} \leftrightarrow x$. By choosing the vector $(k, \ell, v_1, \ldots, v_{k+\ell})$ minimal with respect to the lexicographic order, the choices in (6.12) become unique. Now there are two scenarios, decomposing M into M_{a} and M_{b}.

a) z is among $v_{k+1}, \ldots, v_{k+\ell}$.

b) z is not among $v_{k+1}, \ldots, v_{k+\ell}$.

In case a), there are ℓ possibilities to place z among $v_{k+1}, \ldots, v_{k+\ell}$ and we write $v = (v_1, \ldots, v_{k+\ell})$, similarly for $\hat{v}$. We denote the path in (6.12) by $P(k, \ell, v)$. The isomorphism

$$\varphi \colon \{-1, 0, 1, \ldots, k+\ell\} \to \{x, \mathbf{0}, v_1, \ldots, v_{k+\ell}\}$$

$$\begin{aligned} -1 &\mapsto x, \\ 0 &\mapsto \mathbf{0}, \\ i &\mapsto v_i \quad \text{for} \quad i > 0, \end{aligned}$$

induces a tree on $\{-1, 0, \dots, k+\ell\}$ whose edge set we refer to by $E(k,\ell)$. We obtain

$$|M_{\mathrm{a}}| \le \sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty} \ell \sum_{\hat{v}\in\hat{\eta}_{\neq}^{k+\ell}} \prod_{\{a,b\}\in E(k,\ell)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}$$

so that the Mecke equation yields

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathbb{E}_{\mathbf{0},x}\big[|M_{\mathrm{a}}|\big|W_{\mathbf{0}}\big]dx \\ &\le \sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty} \ell \int_{(\mathbb{R}^d)^{k+\ell+1}} \mathbb{E}_{\mathbf{0},x,v_1,\dots,v_{k+\ell}}\Big[\prod_{\{a,b\}\in E(k,\ell)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}\Big|W_{\mathbf{0}}\Big]dx dv_1\dots dv_{k+\ell} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty} \ell\mathbb{E}_{\mathbf{0}}\Big[\sum_{(\hat{x},\hat{v})\in\hat{\eta}_{\neq}^{k+\ell+1}} \prod_{\{a,b\}\in E(k,\ell)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}\Big|W_{\mathbf{0}}\Big], \end{aligned}$$

where the tuple (x,v) consists of $x \in \mathbb{R}^d$ and $v = (v_1,\dots,v_k) \in (\mathbb{R}^d)^k$. We use Lemma 6.4 to compute the expectation. For fixed $k \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$, the vertices $\mathbf{0}$ and $v_{k+\ell}$ have degree one, whereas all remaining $k+\ell$ vertices are of degree two. Thus, the expression above simplifies to

$$\begin{aligned} &W_{\mathbf{0}}^{d/\alpha}\sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty} \ell\mathbb{E}[W^{d/\alpha}]\mathbb{E}[W^{2d/\alpha}]^{k+\ell}(\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha))^{k+\ell+1} \\ &= W_{\mathbf{0}}^{d/\alpha}\mathbb{E}[W^{d/\alpha}]\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha)\sum_{k=0}^{\infty}\rho^k\sum_{\ell=1}^{\infty}\ell\rho^\ell =: W_{\mathbf{0}}^{d/\alpha}C_2, \end{aligned}$$

with $C_2 < \infty$ as (A3) ensures convergence of the occurring series.

For case b) we proceed similarly, but the notation becomes a bit more involved. We take an additional path as in (6.12) that connects $\mathbf{0}$ over x to z. This path needs to have one last common vertex with the path in (6.12), denoted by u. From $z \notin \mathcal{C}_{\mathbf{0}}(\mathbf{0})$, i.e. z not being connected to $\mathbf{0}$ before adding x, it follows that u is among $x, v_{k+1}\dots, v_{k+\ell}$. For $u = x$ we write $s = 0$, whereas s is given by $u = v_{k+s}$ otherwise, in particular $s \in \{0,\dots,\ell\}$. We obtain a remaining path fragment having $m \in \mathbb{N}$ edges that connects u and z. We denote the vertices by

$$u \leftrightarrow v_{k+\ell+1} \leftrightarrow \dots \leftrightarrow v_{k+\ell+m} = z. \tag{6.13}$$

For fixed $k \in \mathbb{N}_0, \ell \in \mathbb{N}, s \in \{0,\dots,\ell\}, m \in \mathbb{N}$ and $v = (v_1,\dots,v_{k+\ell+m}) \in (\mathbb{R}^d)^{k+\ell+m}$ we denote the tree obtained from (6.12) and (6.13) by $T(k,\ell,m,s,v)$. We consider the isomorphism

$$\varphi\colon \{-1,0,1,\dots,k+\ell+m\} \to \{x,\mathbf{0},v_1,\dots,v_{k+\ell+m}\}$$

$$\begin{aligned}-1 &\mapsto x,\\ 0 &\mapsto \mathbf{0},\\ i &\mapsto v_i \quad \text{for} \quad i > 0,\end{aligned}$$

which induces a tree on $\{-1, 0, \dots, k + \ell + m\}$. We call the corresponding edge set $E(k, \ell, m, s)$. Summing over all possible choices yields the upper bound

$$|M_{\mathrm{b}}| \le \sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\sum_{s=0}^{\ell}\sum_{m=1}^{\infty}\sum_{\hat{v}\in\hat{\eta}_{\neq}^{k+\ell+m}}\prod_{\{a,b\}\in E(k,\ell,m,s)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}.$$

Similarly to part a) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathbb{E}_{\mathbf{0},x}\big[|M_{\mathrm{b}}|\big|W_{\mathbf{0}}\big]\,dx\\ &\le \sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\sum_{s=0}^{\ell}\sum_{m=1}^{\infty}\mathbb{E}_{\mathbf{0}}\Big[\sum_{(\hat{x},\hat{v})\in\hat{\eta}_{\neq}^{k+\ell+m+1}}\prod_{\{a,b\}\in E(k,\ell,m,s)} \mathbf{1}\{\varphi(a) \leftrightarrow \varphi(b)\}\Big|W_{\mathbf{0}}\Big]\end{aligned}$$

and use Lemma 6.4 to compute the expectation for fixed values of k, ℓ, s and m. Therefore, we need to analyse the degrees in the tree $T(k, \ell, m, s, v)$. If $s = \ell$, then $u = v_{k+\ell} = y$ so that there is just one long path and there are two vertices of degree one, $\mathbf{0}$ and z, whereas all other vertices have degree two. If $s < \ell$, then there are three leaves, $\mathbf{0}, y$ and z, while u has degree three and the remaining vertices have degree two. Therefore, the expression above simplifies to

$$\begin{aligned}&W_{\mathbf{0}}^{d/\alpha}\sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}(\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha))^{k+\ell+m+1}\\ &\quad\times\Big(\mathbb{E}[W^{d/\alpha}]\mathbb{E}[W^{2d/\alpha}]^{k+\ell+m} + \ell\mathbb{E}[W^{d/\alpha}]^2\mathbb{E}[W^{2d/\alpha}]^{k+\ell+m-2}\mathbb{E}[W^{3d/\alpha}]\Big)\\ &\le W_{\mathbf{0}}^{d/\alpha}\mathbb{E}[W^{d/\alpha}]\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha)\Big(\sum_{k=0}^{\infty}\rho^k\Big)^3\\ &\quad + W_{\mathbf{0}}^{d/\alpha}\frac{\mathbb{E}[W^{d/\alpha}]^2\mathbb{E}[W^{3d/\alpha}]\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha)}{\mathbb{E}[W^{2d/\alpha}]^2}\Big(\sum_{k=0}^{\infty}\rho^k\Big)^2\sum_{\ell=1}^{\infty}\ell\rho^\ell\\ &=: W_{\mathbf{0}}^{d/\alpha}C_3\end{aligned}$$

with $C_3 < \infty$ because of (A2) and (A3). This shows the claim for $C = C_1 + C_2 + C_3$. □

In the following lemma we write $\hat{S}_n = S_n \times (0, \infty)$, which corresponds to the observation window S_n and all possible values of the weights. Its statement roughly means that vertices with large weight are not connected, see Lemma 4.4 for the version in the Norros-Reittu model.

Lemma 6.8. *Suppose that* (W), (A1) *and* (A3) *hold true and let* $a > 0$. *Then*

$$\mathbb{E}\Big[\sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n}\mathbf{1}\{W_x^{d/\alpha} > aq(n), x \notin V_{\max}\}\Big] \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. By the Mecke equation in Lemma 3.15 and translation invariance, we have

$$\mathbb{E}\Big[\sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n}\mathbf{1}\{W_x^{d/\alpha}>aq(n),x\notin V_{\max}\}\Big]=\int_{S_n}\mathbb{P}_x\big(W_x^{d/\alpha}>aq(n),x\notin V_{\max}\big)dx$$
$$=n\mathbb{P}_{\mathbf{0}}\big(W_{\mathbf{0}}^{d/\alpha}>aq(n),\mathbf{0}\notin V_{\max}\big).$$

The event $\mathbf{0}\notin V_{\max}$ implies the existence of another vertex $\hat{x}\in\hat{\eta}$ which is connected via a path to $\mathbf{0}$ and satisfies $W_x\geq W_{\mathbf{0}}$. We conclude

$$\begin{aligned}&\mathbb{P}_{\mathbf{0}}\big(W_{\mathbf{0}}^{d/\alpha}>aq(n),\mathbf{0}\notin V_{\max}\big)\\&\leq\sum_{k=1}^{\infty}\mathbb{E}_{\mathbf{0}}\Big[\sum_{(\hat{x}_1,\dots,\hat{x}_k)\in\hat{\eta}^k_{\neq}}\mathbf{1}\{W_{x_k}^{d/\alpha}\geq W_{\mathbf{0}}^{d/\alpha}>aq(n),\mathbf{0}\leftrightarrow x_1\leftrightarrow\ldots\leftrightarrow x_k\}\Big]\\&=\sum_{k=1}^{\infty}\mathbb{E}\Big[\mathbb{E}_{\mathbf{0}}\Big[\sum_{(\hat{x}_1,\dots,\hat{x}_k)\in\hat{\eta}^k_{\neq}}\mathbf{1}\{W_{x_k}^{d/\alpha}\geq W_{\mathbf{0}}^{d/\alpha}>aq(n),\mathbf{0}\leftrightarrow x_1\leftrightarrow\ldots\leftrightarrow x_k\}\Big|W_{\mathbf{0}}\Big]\Big].\end{aligned}\tag{6.14}$$

Now, Lemma 6.4 yields for $k\in\mathbb{N}$ and independent copies $W,W_1,\dots,W_k$ of $W_{\mathbf{0}}$,

$$\begin{aligned}&\mathbb{E}_{\mathbf{0}}\Big[\sum_{(\hat{x}_1,\dots,\hat{x}_k)\in\hat{\eta}^k_{\neq}}\mathbf{1}\{W_{x_k}^{d/\alpha}\geq W_{\mathbf{0}}^{d/\alpha}>aq(n),\mathbf{0}\leftrightarrow x_1\leftrightarrow\ldots\leftrightarrow x_k\}\Big|W_{\mathbf{0}}\Big]\\&=\big(\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha)\big)^k\mathbb{E}\Big[\mathbf{1}\{W_k^{d/\alpha}\geq W_{\mathbf{0}}^{d/\alpha}>aq(n)\}W_{\mathbf{0}}^{d/\alpha}W_k^{d/\alpha}\prod_{i=1}^{k-1}W_i^{2d/\alpha}\Big|W_{\mathbf{0}}\Big]\\&\leq\big(\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha)\big)^k\mathbf{1}\{W_{\mathbf{0}}^{d/\alpha}>aq(n)\}\mathbb{E}\big[\mathbf{1}\{W_k^{d/\alpha}>aq(n)\}W_k^{2d/\alpha}\big]\mathbb{E}\big[W^{2d/\alpha}\big]^{k-1}\\&=\rho^{k-1}\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha)\mathbf{1}\{W_{\mathbf{0}}^{d/\alpha}>aq(n)\}\mathbb{E}\big[\mathbf{1}\{W^{d/\alpha}>aq(n)\}W^{2d/\alpha}\big].\end{aligned}$$

Thus, (6.14) simplifies to

$$\begin{aligned}&\mathbb{P}_{\mathbf{0}}\big(W_{\mathbf{0}}^{d/\alpha}>aq(n),\mathbf{0}\notin V_{\max}\big)\\&\leq\sum_{k=1}^{\infty}\rho^{k-1}\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha)\mathbb{P}(W_{\mathbf{0}}^{d/\alpha}>aq(n))\mathbb{E}\big[\mathbf{1}\{W^{d/\alpha}>aq(n)\}W^{2d/\alpha}\big]\\&=\frac{\lambda^{d/\alpha}\kappa_d\Gamma(1-d/\alpha)}{1-\rho}\mathbb{P}(W_{\mathbf{0}}^{d/\alpha}>aq(n))\mathbb{E}\big[\mathbf{1}\{W^{d/\alpha}>aq(n)\}W^{2d/\alpha}\big]\end{aligned}$$

since $\rho<1$ by assumption (A3). We have

$$\mathbb{E}[W^{2d/\alpha}\mathbf{1}\{W^{d/\alpha}>aq(n)\}]\to 0\quad\text{as}\quad n\to\infty,$$

since $q(n)\to\infty$ as $n\to\infty$ and $\mathbb{E}[W^{2d/\alpha}]<\infty$ due to (A3). Finally,

$$\lim_{n\to\infty}n\mathbb{P}(W^{d/\alpha}>aq(n))=a^{-\alpha\beta/d}$$

by Proposition 2.19. The assertion follows. □

The following lemma is the analogue of assumption (A1) in the Norros-Reittu model, essentially stating that we can approximate the expected component size of $\mathbf{0}$ in $\xi^{\mathbf{0}}$, conditionally on $W_{\mathbf{0}}$, by a constant multiple of $W_{\mathbf{0}}^{d/\alpha}$.

Lemma 6.9. *If* (A1)-(A3) *hold true, there exists a constant* $\zeta > 0$ *such that*

$$\lim_{w\to\infty} \frac{\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}} = w\big] - w^{d/\alpha}\zeta}{w^{d/\alpha}} = 0.$$

Proof. From Lemma 6.5 it follows that there is almost surely no infinite component. Thus, we can calculate the component size of $\mathbf{0}$ in $\xi^{\mathbf{0}}$ by summing over all components of size k in ξ which are connected via an edge to $\mathbf{0}$ in $\xi^{\mathbf{0}}$, denoted by $\mathcal{C}_k(\mathbf{0})$, i.e.

$$\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}}\big] = 1 + \sum_{k=1}^{\infty} k\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_{\mathbf{0}}\big],$$

where we used monotone convergence. Assuming that we may move the limit inside the series in the second equality below, it follows

$$\begin{aligned}\lim_{w\to\infty} \frac{\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}} = w\big]}{w^{d/\alpha}} &= \lim_{w\to\infty} \sum_{k=1}^{\infty} k \frac{\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_{\mathbf{0}} = w\big]}{w^{d/\alpha}} \\ &= \sum_{k=1}^{\infty} k \lim_{w\to\infty} \frac{\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_{\mathbf{0}} = w\big]}{w^{d/\alpha}}. \end{aligned} \tag{6.15}$$

We will show for all $k \in \mathbb{N}$ that

$$\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_{\mathbf{0}}\big] = W_{\mathbf{0}}^{d/\alpha} f_k(W_{\mathbf{0}}) \tag{6.16}$$

for some function $f_k \colon \mathbb{R} \to \mathbb{R}$ with $f_k(w) \to \zeta_k$ for some $\zeta_k > 0$ as $w \to \infty$. Then we conclude from (6.15) that

$$\lim_{w\to\infty} \frac{\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}} = w\big]}{w^{d/\alpha}} = \sum_{k=1}^{\infty} k \lim_{w\to\infty} f_k(w) = \sum_{k=1}^{\infty} k\zeta_k =: \zeta,$$

which yields the claim. Note that $\zeta < \infty$ by Lemma 6.5. We establish applicability of dominated convergence in (6.15) and the statement in (6.16).

We start with the claim in (6.16) and write

$$p_k(\hat{x}_1, \dots, \hat{x}_k) = \mathbb{P}(x_1, \dots, x_k \text{ form a component in } \xi^{x_1,\dots,x_k} | W_{x_1}, \dots, W_{x_k})$$

for $k \in \mathbb{N}$ and points $x_1, \dots, x_k \in \mathbb{R}^d$ with weights $W_{x_1}, \dots, W_{x_k}$. In the calculation below, we eventually drop the indices from the expectation, e.g. we write $\mathbb{E}$ instead of $\mathbb{E}_{\mathbf{0},x_1,\dots,x_k}$, since we explicitly specify the respective random connection model in the occurring expressions. For $[k] = \{1, \dots, k\}$ the Mecke equation yields

$$\begin{aligned}&\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_{\mathbf{0}}\big] \\ &= \mathbb{E}_{\mathbf{0}}\Bigg[\frac{1}{k!} \sum_{(\hat{x}_1,\dots,\hat{x}_k)\in\hat{\eta}^k_{\neq}} \mathbf{1}\{x_1, \dots, x_k \text{ form a component in } \xi\}\mathbf{1}\{\exists j \in [k] \colon x_j \leftrightarrow \mathbf{0} \text{ in } \xi^{\mathbf{0}}\}\Bigg|W_{\mathbf{0}}\Bigg]\end{aligned}$$

$$= \int_{(\mathbb{R}^d)^k} \frac{1}{k!}\mathbb{E}\Big[p_k(\hat{x}_1,\dots,\hat{x}_k)\mathbb{P}(\exists j\in[k]\colon x_j\leftrightarrow \mathbf{0} \text{ in } \xi^{\mathbf{0},x_1,\dots,x_k}|W_{\mathbf{0}},W_{x_1},\dots,W_{x_k})\Big|W_{\mathbf{0}}\Big]$$

$$dx_1\dots dx_k \tag{6.17}$$

$$= \int_{(\mathbb{R}^d)^k} \frac{1}{k!}\mathbb{E}\Big[p_k(\hat{x}_1,\dots,\hat{x}_k)\Big(1-\exp\Big(-\lambda W_{\mathbf{0}}\sum_{j=1}^k \frac{W_{x_j}}{|x_j|^\alpha}\Big)\Big)\Big|W_{\mathbf{0}}\Big]dx_1\dots dx_k,$$

where we calculated the probability in (6.17) by considering the complementary event, that none of $x_1,\dots,x_k$ is connected to $\mathbf{0}$. Note that p_k is translation invariant in the sense that

$$p_k(\hat{x}_1,\dots,\hat{x}_k) = p_k((\mathbf{0},W_{x_1}),(x_2-x_1,W_{x_2}),\dots,(x_k-x_1,W_{x_k})).$$

In the following, we will substitute $z_i = x_i - x_1$ for $i=2,\dots,k$. This has the notational issue that the weight associated with z_i is actually $W_{z_i+x_1}$. In order to get rid of this nuisance, we simply write $W_1,\dots,W_k$ for k independent copies of $W_{\mathbf{0}}$ and write $\hat{z}_i' = (z_i,W_i)$ for $i=1,\dots,k$ and $z_1 = \mathbf{0}\in\mathbb{R}^d$. We derive

$$\begin{aligned}
&\mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_{\mathbf{0}}\big]\\
&= \int_{(\mathbb{R}^d)^k}\frac{1}{k!}\mathbb{E}\Big[p_k(\hat{z}_1',\dots,\hat{z}_k')\Big(1-\exp\Big(-\lambda W_{\mathbf{0}}\sum_{j=1}^k\frac{W_j}{|x_1+z_j|^\alpha}\Big)\Big)\Big|W_{\mathbf{0}}\Big]dx_1dz_2\dots dz_k\\
&= W_{\mathbf{0}}^{d/\alpha}\frac{\lambda^{d/\alpha}}{k!}\int_{(\mathbb{R}^d)^k}\mathbb{E}\Big[p_k(\hat{z}_1',\dots,\hat{z}_k')\Big(1-\exp\Big(-\sum_{j=1}^k\frac{W_j}{|y+(\lambda W_{\mathbf{0}})^{-1/\alpha}z_j|^\alpha}\Big)\Big)\Big|W_{\mathbf{0}}\Big]\\
&\qquad dydz_2\dots dz_k\\
&=: W_{\mathbf{0}}^{d/\alpha}f_k(W_{\mathbf{0}})
\end{aligned}$$

by substituting $y=(\lambda W_{\mathbf{0}})^{-1/\alpha}x_1$. This shows (6.16) but we still need to verify that $f_k(w)\to\zeta_k$ for some $\zeta_k>0$ as $w\to\infty$. This is clear if we are allowed to interchange the limit with integration and expectation. We justify this by applying dominated convergence twice. First, we show that

$$\begin{aligned}
\lim_{w\to\infty} f_k(w) = \frac{\lambda^{d/\alpha}}{k!}\int_{(\mathbb{R}^d)^{k-1}}\lim_{w\to\infty}\Big(\int_{\mathbb{R}^d}\mathbb{E}\Big[p_k(\hat{z}_1',\dots,\hat{z}_k')\\
\times\Big(1-\exp\Big(-\sum_{j=1}^k\frac{W_j}{|y+(\lambda w)^{-1/\alpha}z_j|^\alpha}\Big)\Big)\Big]dy\Big)dz_2\dots dz_k
\end{aligned}\tag{6.18}$$

and in a second step we argue that one can also apply dominated convergence to the inner integral over y and taking the expectation. We start off with various bounds to find a majorant to use for (6.18). Given $a_1,\dots,a_k>0$ it holds

$$1-\exp\Big(-\sum_{j=1}^k a_j\Big)\le 1-\exp\big(-k\max_{j=1,\dots,k}a_j\big)\le\sum_{j=1}^k 1-\exp(-ka_j) \tag{6.19}$$

so that

$$1-\exp\Big(-\sum_{j=1}^k\frac{W_j}{|y+(\lambda w)^{-1/\alpha}z_j|^\alpha}\Big)\le\sum_{j=1}^k 1-\exp\Big(-\frac{kW_j}{|y+(\lambda w)^{-1/\alpha}z_j|^\alpha}\Big). \tag{6.20}$$

For $z_1 = \mathbf{0} \in \mathbb{R}^d$ and $z_2, \dots, z_k \in \mathbb{R}^d$ we abbreviate $Z_k = \{z_1, \dots, z_k\}$ and $\hat{Z}_k = \{\hat{z}'_1, \dots, \hat{z}'_k\}$ with $\hat{z}'_i = (z_i, W_i)$ as before. We write $\xi^{\hat{Z}_k}$ for the random connection model $\xi^{z_1,\dots,z_k}$ where z_i has weight W_i instead of W_{z_i} for $i = 1, \dots, k$. For a graph $H = (V, E)$ and $V' \subseteq V$ we denote the subgraph of H induced by V' by $H[V']$. We can now address $p_k(\hat{z}'_1, \dots, \hat{z}'_k)$, the conditional probability that $\hat{z}_1, \dots, \hat{z}_k$ form a component in $\xi^{\hat{Z}_k}$. We have

$$\begin{aligned} &p_k(\hat{z}'_1, \dots, \hat{z}'_k) \qquad (6.21)\\ &= \mathbb{P}(\xi^{\hat{Z}_k}[Z_k] \text{ is connected} | W_1, \dots, W_k) \mathbb{P}(Z_k \text{ has no edges to } \eta \text{ in } \xi^{\hat{Z}_k} | W_1, \dots, W_k), \end{aligned}$$

since $Z_k \cap \eta = \emptyset$ almost surely. We rewrite the second factor and use the Hölder inequality to obtain

$$\begin{aligned} &\mathbb{P}(Z_k \text{ has no edges to } \eta \text{ in } \xi^{\hat{Z}_k} | W_1, \dots, W_k) = \mathbb{E}_{z_1,\dots,z_k}\Big[\prod_{i=1}^k \prod_{\hat{x} \in \hat{\eta}} \mathbf{1}\{x \nleftrightarrow z_i\} \Big| W_1, \dots, W_k\Big] \\ &\leq \prod_{i=1}^k \mathbb{E}_{z_i}\Big[\prod_{\hat{x} \in \hat{\eta}} \mathbf{1}\{x \nleftrightarrow z_i\} \Big| W_i\Big]^{1/k} = \prod_{i=1}^k \mathbb{E}\Big[\prod_{\hat{x} \in \hat{\eta}} \exp\Big(-\lambda \frac{W_x W_i}{|x - z_i|^\alpha}\Big) \Big| W_i\Big]^{1/k}. \end{aligned}$$

For fixed $i \in [k]$ and $w_i > 0$, Lemma 2.6 yields

$$\mathbb{E}\Big[\prod_{\hat{x} \in \hat{\eta}} \exp\Big(-\lambda \frac{W_x W_i}{|x - z_i|^\alpha}\Big) \Big| W_i = w_i\Big] = \exp\Big(-\mathbb{E}\Big[\int_{\mathbb{R}^d} \Big(1 - \exp\Big(-\lambda \frac{W w_i}{|x - z_i|^\alpha}\Big)\Big) dx\Big]\Big),$$

where the expectation on the right-hand side accounts for the weight $W = W_x$ of the added point $\hat{x} = (x, W_x)$. With Lemma 6.3, this simplifies to

$$\exp\big(-\lambda^{d/\alpha} \mathbb{E}[W^{d/\alpha}] w_i^{d/\alpha} \kappa_d \Gamma(1 - d/\alpha)\big)$$

so that

$$\begin{aligned} &\mathbb{P}(Z_k \text{ has no edges to } \eta \text{ in } \xi^{\hat{Z}_k} | W_1, \dots, W_k) \\ &\leq \prod_{i=1}^k \exp\big(-k^{-1} \lambda^{d/\alpha} \mathbb{E}[W^{d/\alpha}] \kappa_d \Gamma(1 - d/\alpha)] W_i^{d/\alpha}\big) =: \prod_{i=1}^k \exp\big(-\tilde{c}_k W_i^{d/\alpha}\big), \qquad (6.22) \end{aligned}$$

where $\tilde{c}_k$ is a positive constant. The first factor on the right-hand side of (6.21) is the probability that the graph induced by k given points is connected. We upper bound this by summing over all possible spanning trees and the probability that their respective edges are present. Let $\mathcal{T}_k$ denote the set of all trees on the vertex set $\{1, \dots, k\}$. For $T \in \mathcal{T}_k$ let $E(T)$ denote its edge set. Then we have

$$\mathbb{P}(\xi^{\hat{Z}_k}[Z_k] \text{ is connected} | W_1, \dots, W_k) \leq \sum_{T \in \mathcal{T}_k} \prod_{\{a,b\} \in E(T)} \Big(1 - \exp\Big(-\lambda \frac{W_a W_b}{|z_a - z_b|^\alpha}\Big)\Big). \qquad (6.23)$$

Combining (6.21), (6.22) and (6.23) yields

$$p_k(\hat{z}'_1, \dots, \hat{z}'_k) \leq \sum_{T \in \mathcal{T}_k} \prod_{\{a,b\} \in E(T)} \Big(1 - \exp\Big(-\lambda \frac{W_a W_b}{|z_a - z_b|^\alpha}\Big)\Big) \times \prod_{i=1}^k \exp\big(-\tilde{c}_k W_i^{d/\alpha}\big).$$

Together with (6.20) we obtain for all $y, z_2, \dots, z_k \in \mathbb{R}^d$,

$$
\begin{aligned}
&p_k(\hat{z}_1', \dots, \hat{z}_k')\Big(1 - \exp\Big(- \sum_{j=1}^k \frac{W_j}{|y + z_j(\lambda w)^{-1/\alpha}|^\alpha}\Big)\Big) \\
&\le \sum_{j=1}^k \Big(1 - \exp\Big(- \frac{kW_j}{|y + z_j(\lambda w)^{-1/\alpha}|^\alpha}\Big)\Big) \sum_{T\in\mathcal{T}_k} \prod_{\{a,b\}\in E(T)} \Big(1 - \exp\Big(- \lambda\frac{W_aW_b}{|z_a - z_b|^\alpha}\Big)\Big) \\
&\quad \times \prod_{i=1}^k \exp\big(- \tilde{c}_k W_i^{d/\alpha}\big).
\end{aligned}
$$

In light of (6.18), we integrate with respect to y and take expectation, resulting in

$$
\begin{aligned}
&\sum_{j=1}^k \sum_{T\in\mathcal{T}_k} \mathbb{E}\Big[\prod_{\{a,b\}\in E(T)} \Big(1 - \exp\Big(- \lambda\frac{W_aW_b}{|z_a - z_b|^\alpha}\Big)\Big) \prod_{i=1}^k \exp\big(- \tilde{c}_k W_i^{d/\alpha}\big) \\
&\qquad \times \int_{\mathbb{R}^d} \Big(1 - \exp\Big(- \frac{kW_j}{|y + z_j(\lambda w)^{-1/\alpha}|^\alpha}\Big)\Big) dy\Big] \\
&= \sum_{j=1}^k \sum_{T\in\mathcal{T}_k} \mathbb{E}\Big[\prod_{\{a,b\}\in E(T)} \Big(1 - \exp\Big(- \lambda\frac{W_aW_b}{|z_a - z_b|^\alpha}\Big)\Big) \prod_{i=1}^k \exp\big(- \tilde{c}_k W_i^{d/\alpha}\big) \\
&\qquad \times (kW_j)^{d/\alpha}\kappa_d\Gamma(1 - d/\alpha)\Big],
\end{aligned}
$$

where we used Lemma 6.3 for the integral over y. As this term no longer depends on w, it may serve as our majorant in (6.18). It remains to show its integrability over $z_2, \dots, z_k \in \mathbb{R}^d$ (recall that $z_1 = \mathbf{0}$ is fixed). Integrating over $z_2, \dots, z_k$ and interchanging expectation and integration yields

$$
\begin{aligned}
&\sum_{j=1}^k \sum_{T\in\mathcal{T}_k} \mathbb{E}\Big[(kW_j)^{d/\alpha}\kappa_d\Gamma(1 - d/\alpha) \prod_{i=1}^k \exp\big(- \tilde{c}_k W_i^{d/\alpha}\big) \\
&\qquad \times \int_{(\mathbb{R}^d)^{k-1}} \prod_{\{a,b\}\in E(T)} \Big(1 - \exp\Big(- \lambda\frac{W_aW_b}{|z_a - z_b|^\alpha}\Big)\Big) dz_2 \dots dz_k\Big].
\end{aligned}
$$

For each $T \in \mathcal{T}_k$ we choose a leaf $j > 1$ of T and integrate over z_j first. Since only a single factor in the product over $(a, b) \in T$ depends on j, we can apply Lemma 6.3. After removing j from T, one can iterate this procedure, leading to

$$
\begin{aligned}
&\sum_{j=1}^k \sum_{T\in\mathcal{T}_k} \mathbb{E}\Big[(kW_j)^{d/\alpha}\kappa_d\Gamma(1 - d/\alpha) \prod_{i=1}^k \exp\big(- \tilde{c}_k W_i^{d/\alpha}\big) \prod_{\{a,b\}\in E(T)} \kappa_d\Gamma(1 - d/\alpha)W_a^{d/\alpha}W_b^{d/\alpha}\lambda^{d/\alpha}\Big] \\
&= k^{d/\alpha}\big(\kappa_d\Gamma(1 - d/\alpha)\big)^k \lambda^{(k-1)d/\alpha} \sum_{j=1}^k \sum_{T\in\mathcal{T}_k} \mathbb{E}\Big[\prod_{i=1}^k W_i^{(\deg_T(i) + \mathbf{1}\{i=j\})d/\alpha} \exp\big(- \tilde{c}_k W_i^{d/\alpha}\big)\Big],
\end{aligned}
$$

where $\deg_T(i)$ denotes the degree of i in T and we used that $|E(T)| = k - 1$. Since the term inside the expectation is bounded, the whole expression is finite. Thus, dominated convergence implies (6.18). Next, we show

$$\lim_{w\to\infty}\int_{\mathbb{R}^d}\mathbb{E}\Big[p_k(\hat{z}_1',\dots,\hat{z}_k')\Big(1-\exp\Big(-\sum_{j=1}^k\frac{W_j}{|y+z_j(\lambda w)^{-1/\alpha}|^\alpha}\Big)\Big)\Big]dy$$
$$=\int_{\mathbb{R}^d}\mathbb{E}\Big[p_k(\hat{z}_1',\dots,\hat{z}_k')\Big(1-\exp\Big(-\sum_{j=1}^k\frac{W_j}{|y|^\alpha}\Big)\Big)\Big]dy$$

for fixed $z_2,\dots,z_k\in\mathbb{R}^d$ and $z_1=\mathbf{0}$. In order to apply dominated convergence, we use the trivial bound

$$p_k(\hat{z}_1',\dots,\hat{z}_k')\Big(1-\exp\Big(-\sum_{j=1}^k\frac{W_j}{|y+z_j(\lambda w)^{-1/\alpha}|^\alpha}\Big)\Big)$$
$$\le 1-\exp\Big(-\sum_{j=1}^k\frac{W_j}{|y+z_j(\lambda w)^{-1/\alpha}|^\alpha}\Big).$$

As a function in y, this expression is bounded by one. For $|y|>2$ and w large enough such that

$$\max_{i=1,\dots,k}\left|\frac{z_i}{(\lambda w)^{1/\alpha}}\right|\le 1$$

we can use (6.19) and $1-\exp(-x)\le x$ for $x\in\mathbb{R}$ to bound it by

$$1-\exp\Big(-\frac{\sum_{j=1}^k W_j}{||y|-1|^\alpha}\Big)\le\sum_{j=1}^k 1-\exp\Big(-\frac{kW_j}{||y|-1|^\alpha}\Big)\le\sum_{j=1}^k\frac{kW_j}{||y|-1|^\alpha}.$$

These bounds yield an integrable (with respect to y and taking expectation) majorant. Therefore, dominated convergence allows us to interchange the limit and integration a second time which shows that f_k is as claimed after (6.16).

It remains to justify interchanging the limit and the series in (6.15), i.e.

$$\lim_{w\to\infty}\sum_{k=1}^\infty k\frac{\mathbb{E}_\mathbf{0}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_\mathbf{0}=w\big]}{w^{d/\alpha}}=\sum_{k=1}^\infty k\lim_{w\to\infty}\frac{\mathbb{E}_\mathbf{0}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_\mathbf{0}=w\big]}{w^{d/\alpha}}.$$

For fixed $M\in\mathbb{N}$ we truncate the series at M, leading to

$$\sum_{k=1}^\infty k\frac{\mathbb{E}_\mathbf{0}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_\mathbf{0}=w\big]}{w^{d/\alpha}}=\sum_{k=1}^{M-1} k\frac{\mathbb{E}_\mathbf{0}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_\mathbf{0}=w\big]}{w^{d/\alpha}}+\sum_{k=M}^\infty k\frac{\mathbb{E}_\mathbf{0}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_\mathbf{0}=w\big]}{w^{d/\alpha}}$$
$$=:S_M^<+S_M^\ge.$$

As $S_M^<$ is a finite sum, we are free to interchange the limit with the sum. From (6.16) and the properties of f_k specified thereafter we obtain

$$\lim_{w\to\infty}S_M^<=\lim_{w\to\infty}\sum_{k=1}^{M-1}k\frac{\mathbb{E}_\mathbf{0}\big[|\mathcal{C}_k(\mathbf{0})|\big|W_\mathbf{0}=w\big]}{w^{d/\alpha}}=\sum_{k=1}^{M-1}k\zeta_k\to\zeta\quad\text{as}\quad M\to\infty.\qquad(6.24)$$

Since

$$\limsup_{w\to\infty} \left|S_M^< + S_M^\geq - \zeta\right| \leq \limsup_{w\to\infty} \left|S_M^< - \sum_{k=1}^{M-1} k\zeta_k\right| + \limsup_{w\to\infty} \left|\sum_{k=1}^{M-1} k\zeta_k - \zeta\right| + \limsup_{w\to\infty} S_M^\geq$$

and the first summand converges to zero by (6.24) whereas the second summand is independent of w and converges to zero as $M \to \infty$, it suffices to show that

$$\limsup_{M\to\infty} \limsup_{w\to\infty} S_M^\geq \to 0 \quad \text{as} \quad n \to \infty. \tag{6.25}$$

Since $S_M^\geq$ accounts for all components in ξ containing at least M vertices and getting connected to $\mathbf{0}$ in $\xi^{\mathbf{0}}$, we obtain

$$\begin{aligned}
S_M^\geq &\leq w^{-d/\alpha}\mathbb{E}_{\mathbf{0}}\Big[\sum_{\hat{x}\in\hat{\eta}} \mathbf{1}\{x \leftrightarrow \mathbf{0}\}\mathbf{1}\{|\mathcal{C}(x)| \geq M\}|\mathcal{C}(x)|\Big|W_{\mathbf{0}} = w\Big] \\
&= w^{-d/\alpha} \int_{\mathbb{R}^d} \mathbb{E}_{\mathbf{0},x}\big[\mathbf{1}\{x \leftrightarrow \mathbf{0}\}\mathbf{1}\{|\mathcal{C}(x)| \geq M\}|\mathcal{C}(x)|\big|W_{\mathbf{0}} = w\big]\,dx \\
&= w^{-d/\alpha} \int_{\mathbb{R}^d} \mathbb{E}_{\mathbf{0},x}\Big[\mathbb{E}_{\mathbf{0},x}\big[\mathbf{1}\{x \leftrightarrow \mathbf{0}\}\big|W_x, W_{\mathbf{0}} = w\big] \\
&\qquad\qquad \mathbb{E}_x\big[\mathbf{1}\{|\mathcal{C}(x)| \geq M\}|\mathcal{C}(x)|\big|W_x\big]\Big|W_{\mathbf{0}} = w\Big]\,dx \\
&\leq w^{-d/\alpha} \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(1 - \exp\Big(-\lambda\frac{wW_y}{|x|^\alpha}\Big)\Big)\frac{\mathbb{E}_y\big[|\mathcal{C}(y)|^2\big|W_y\big]}{M}\Big|W_{\mathbf{0}} = w\Big]\,dx,
\end{aligned}$$

where the last inequality manipulated the conditional expectation $\mathbb{E}_x[\cdot|W_x]$ by using the fact that $|\mathcal{C}(x)| \geq M$ and that the whole expression is invariant under a translation of x, which allows us to replace (x, W_x) by (y, W_y) for some fixed y no longer depending on the integrand. We replaced all remaining weights W_x by W_y accordingly. We rearrange integral and expectation so that we may evaluate the integral via Lemma 6.3, resulting in

$$\begin{aligned}
&w^{-d/\alpha}\mathbb{E}\Big[\frac{\mathbb{E}_y\big[|\mathcal{C}(y)|^2\big|W_y\big]}{M}\int_{\mathbb{R}^d}\Big(1 - \exp\Big(-\lambda\frac{wW_y}{|x|^\alpha}\Big)\Big)dx\Big|W_{\mathbf{0}} = w\Big] \\
&= \lambda^{d/\alpha}\kappa_d\Gamma(1 - d/\alpha)\mathbb{E}\Big[\frac{\mathbb{E}_y\big[|\mathcal{C}(y)|^2\big|W_y\big]}{M}W_y^{d/\alpha}\Big] \\
&\leq \lambda^{d/\alpha}\kappa_d\Gamma(1 - d/\alpha)M^{-1}\mathbb{E}\big[p_2(W_y^{d/\alpha})W_y^{d/\alpha}\big],
\end{aligned}$$

where we used Lemma 6.6 in the last inequality. As the product $p_2(W_y^{d/\alpha})W_y$ yields a polynomial of degree three in $W_y^{d/\alpha}$, whose integrability is ensured by assumption (A2), the term above goes to zero as $M \to \infty$. This shows (6.25) and concludes the proof. □

Now that we have gathered all auxiliary lemmas, we proceed with the proof of our main theorem. The overall strategy is similar to that of the proof of Theorem 4.1, up to technical details. Recall that $\hat{S}_n = S_n \times (0, \infty)$.

Proof of Theorem 6.1. We compare the point process

$$\Xi_n = \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{x \in V_{\max}\}\delta_{q(n)^{-1}\zeta^{-1}|\mathcal{C}(x)|}$$

to the point process consisting of the rescaled weights in the same observation window, namely

$$\Theta_n = \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \delta_{q(n)^{-1}W_x^{d/\alpha}}.$$

Part b) of Lemma 2.20 yields

$$\Theta_n \xrightarrow{d} \eta_\gamma \quad \text{as} \quad n\to\infty.$$

From Lemma 2.11 we obtain the claim once we have shown that

$$\Xi_n((a,\infty]) - \Theta_n((a,\infty]) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n\to\infty$$

for all $a>0$. Let thus $a>0$. By analysing how a vertex may contribute to $\Xi_n((a,\infty])$ but not to $\Theta_n((a,\infty])$ and vice versa we obtain

$$\begin{aligned}|\Xi_n((a,\infty]) - \Theta_n((a,\infty])| \le &\sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{W_x^{d/\alpha} > aq(n), x\notin V_{\max}\}\\ &+ \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{W_x^{d/\alpha} > aq(n), x\in V_{\max}, |\mathcal{C}(x)|\le aq(n)\zeta\}\\ &+ \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{W_x^{d/\alpha} \le aq(n), x\in V_{\max}, |\mathcal{C}(x)| > aq(n)\zeta\}\\ =:& I_1+I_2+I_3.\end{aligned}$$

We have $\mathbb{E}[I_1]\to 0$ as $n\to\infty$ by Lemma 6.8. For the two remaining terms let $\varepsilon\in(0,a)$. Then,

$$\begin{aligned}I_2 \le &\sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{aq(n) < W_x^{d/\alpha} \le (a+\varepsilon)q(n)\}\\ &+ \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{W_x^{d/\alpha} > (a+\varepsilon)q(n), |\mathcal{C}(x)|\le aq(n)\zeta\}.\end{aligned}$$

Note that ε creates some minimal distance between $\zeta W_x^{d/\alpha}$ and $|\mathcal{C}(x)|$ in the second sum above, which allows us to employ the Chebyshev inequality later on. We conclude

$$\begin{aligned}I_2 \le &\sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{aq(n) < W_x^{d/\alpha} \le (a+\varepsilon)q(n)\}\\ &+ \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\big\{W_x^{d/\alpha} > (a+\varepsilon)q(n), \big||\mathcal{C}(x)| - W_x^{d/\alpha}\zeta\big| > \varepsilon q(n)\zeta\big\} =: I_{2,1}+I_{2,2}.\end{aligned}$$

For I_3 we proceed similarly but need one further ingredient. Let $0<\gamma<d/(\alpha\beta)$ be a positive constant and $\tilde{q}(n) = n^{-\gamma}q(n)$. Later on, we will choose γ small enough. Intuitively, $\tilde{q}(n)$ grows slightly slower than $q(n)$. We derive

$$I_3 = \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{W_x^{d/\alpha} \le aq(n), x\in V_{\max}, |\mathcal{C}(x)| > aq(n)\zeta\}$$

$$\begin{aligned}
&\le \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{W_x^{d/\alpha} \le a\tilde{q}(n), x \in V_{\max}, |\mathcal{C}(x)| > aq(n)\zeta\} \\
&\quad + \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{a\tilde{q}(n) < W_x^{d/\alpha} \le (a-\varepsilon)q(n), x \in V_{\max}, |\mathcal{C}(x)| > aq(n)\zeta\} \\
&\quad + \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{(a-\varepsilon)q(n) < W_x^{d/\alpha} \le aq(n)\} =: I_{3,1} + I_{3,2} + I_{3,3}.
\end{aligned}$$

We have

$$I_{2,1} + I_{3,3} = \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{(a-\varepsilon)q(n) < W_x^{d/\alpha} \le (a+\varepsilon)q(n)\}$$

and the Mecke equation yields

$$\begin{aligned}
\mathbb{E}[I_{2,1} + I_{3,3}] &= \int_{S_n} \mathbb{E}_x\big[\mathbf{1}\{(a-\varepsilon)q(n) < W_x^{d/\alpha} \le (a+\varepsilon)q(n)\}\big]dx \\
&= n\mathbb{P}((a-\varepsilon)q(n) < W^{d/\alpha} \le (a+\varepsilon)q(n))
\end{aligned}$$

by translation invariance and $\lambda_d(S_n) = n$. With Proposition 2.19 we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[I_{2,1} + I_{3,3}] &= \lim_{n\to\infty} n\mathbb{P}((a-\varepsilon)q(n) < W^{d/\alpha} \le (a+\varepsilon)q(n)) \\
&= (a-\varepsilon)^{-\alpha\beta/d} - (a+\varepsilon)^{-\alpha\beta/d}.
\end{aligned}$$

Therefore, we have

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \mathbb{E}[I_{2,1} + I_{3,3}] = 0.$$

Since $I_{2,2}$ and $I_{3,2}$ are both bounded from above by

$$I_4 = \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\big\{W_x^{d/\alpha} > a\tilde{q}(n), \big||\mathcal{C}(x)| - W_x^{d/\alpha}\zeta\big| > \varepsilon q(n)\zeta\big\}$$

due to $a\tilde{q}(n) < (a+\varepsilon)q(n)$, it remains to show that $I_{3,1} \xrightarrow{\mathbb{P}} 0$ and $I_4 \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ for all fixed $\varepsilon > 0$. Concerning $I_{3,1}$, we use the Mecke equation to compute

$$\begin{aligned}
\mathbb{E}[I_{3,1}] &= \mathbb{E}\Bigg[\sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{W_x^{d/\alpha} \le a\tilde{q}(n), x \in V_{\max}, |\mathcal{C}(x)| > aq(n)\zeta\}\Bigg] \\
&= \int_{S_n} \mathbb{E}_x\big[\mathbf{1}\{W_x^{d/\alpha} \le a\tilde{q}(n), x \in V_{\max}, |\mathcal{C}(x)| > aq(n)\zeta\}\big]dx \\
&= n\mathbb{E}_{\mathbf{0}}\big[\mathbf{1}\{W_{\mathbf{0}}^{d/\alpha} \le a\tilde{q}(n), \mathbf{0} \in V_{\max}, |\mathcal{C}(\mathbf{0})| > aq(n)\zeta\}\big].
\end{aligned}$$

We condition on $W_{\mathbf{0}}$ and use the Markov inequality for $x \mapsto x^k$ with $k \in \mathbb{N}$, which allows us to employ Lemma 6.6, resulting in

$$\mathbb{E}[I_{3,1}] \le n\mathbb{E}\Bigg[\mathbf{1}\{W_{\mathbf{0}}^{d/\alpha} \le a\tilde{q}(n)\}\mathbb{P}_{\mathbf{0}}\bigg(\mathbf{1}\{\mathbf{0} \in V_{\max}\}|\mathcal{C}(\mathbf{0})| > aq(n)\zeta\bigg|W_{\mathbf{0}}\bigg)\Bigg]$$

$$\leq n\mathbb{E}\bigg[\mathbf{1}\{W_{\mathbf{0}}^{d/\alpha} \leq a\tilde{q}(n)\}(aq(n)\zeta)^{-k}\mathbb{E}_{\mathbf{0}}\bigg[\mathbf{1}\{\mathbf{0} \in V_{\max}\}|\mathcal{C}(\mathbf{0})|^k\bigg|W_{\mathbf{0}}\bigg]\bigg]$$

$$\leq n\mathbb{E}\big[\mathbf{1}\{W_{\mathbf{0}}^{d/\alpha} \leq a\tilde{q}(n)\}(aq(n)\zeta)^{-k}q_k(W_{\mathbf{0}}^{d/\alpha})\big] \leq \frac{nq_k(a\tilde{q}(n))}{(aq(n)\zeta)^k}.$$

Since q_k is a polynomial of degree k with positive coefficients and $\tilde{q}(n) = n^{-\gamma}q(n)$, the numerator is of order $n^{1-k\gamma}q(n)^k$. Choosing k large enough such that $k\gamma > 1$ therefore ensures that $\mathbb{E}[I_{3,1}] \to 0$ as $n \to \infty$. For I_4 we employ the triangle inequality to obtain

$$I_4 = \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\big\{W_x^{d/\alpha} > a\tilde{q}(n), \big||\mathcal{C}(x)| - W_x^{d/\alpha}\zeta\big| > \varepsilon q(n)\zeta\big\}$$

$$\leq \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\bigg\{W_x^{d/\alpha} > a\tilde{q}(n), \big||\mathcal{C}(x)| - \mathbb{E}_x\big[|\mathcal{C}(x)|\big|W_x\big]\big| > \frac{1}{2}\varepsilon q(n)\zeta\bigg\}$$

$$+ \sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\bigg\{W_x^{d/\alpha} > a\tilde{q}(n), \big|\mathbb{E}_x\big[|\mathcal{C}(x)|\big|W_x\big] - W_x^{d/\alpha}\zeta\big| > \frac{1}{2}\varepsilon q(n)\zeta\bigg\} =: R_1 + R_2.$$

From Lemma 6.9 we obtain the existence of a null sequence $(b_n)_{n\in\mathbb{N}}$ such that for all $x \in \mathbb{R}^d$ equipped with weight W_x it holds almost surely that

$$\mathbf{1}\{W_x^{d/\alpha} > a\tilde{q}(n)\}\frac{\big|\mathbb{E}_x\big[|\mathcal{C}(x)|\big|W_x\big] - W_x^{d/\alpha}\zeta\big|}{W_x^{d/\alpha}} \leq b_n.$$

Note that the left-hand side does, by translation invariance, actually not depend on the position of x. From this uniform bound we obtain

$$\mathbb{P}(R_2 > 0) = \mathbb{P}\bigg(\max_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{W_x^{d/\alpha} > a\tilde{q}(n)\}\frac{W_x^{d/\alpha}}{q(n)}\frac{\big|\mathbb{E}_x\big[|\mathcal{C}(x)|\big|W_x\big] - W_x^{d/\alpha}\zeta\big|}{W_x^{d/\alpha}} > \frac{\varepsilon\zeta}{2}\bigg)$$

$$\leq \mathbb{P}\bigg(\max_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \frac{W_x^{d/\alpha}}{q(n)} > \frac{\varepsilon\zeta}{2b_n}\bigg).$$

From Lemma 2.20 and Lemma 2.12 it follows that

$$\max_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \frac{W_x^{d/\alpha}}{q(n)} \xrightarrow{d} Z \quad \text{as} \quad n \to \infty,$$

where Z denotes a random variable following a Fréchet distribution with parameter $\alpha\beta/d$. Combined with $b_n \to 0$, this implies $\mathbb{P}(R_2 > 0) \to 0$ as $n \to \infty$. For R_1 we wish to use the Chebyshev inequality. We compute via the Mecke equation in Lemma 3.15,

$$\mathbb{E}[R_1] = \mathbb{E}\bigg[\sum_{\hat{x}\in\hat{\eta}\cap\hat{S}_n} \mathbf{1}\{W_x^{d/\alpha} > a\tilde{q}(n)\}\mathbf{1}\bigg\{\big||\mathcal{C}(x)| - \mathbb{E}_x[|\mathcal{C}(x)|\big|W_x]\big| > \frac{\varepsilon q(n)\zeta}{2}\bigg\}\bigg]$$

$$= \int_{S_n} \mathbb{E}_x\bigg[\mathbf{1}\{W_x^{d/\alpha} > a\tilde{q}(n)\}\mathbf{1}\bigg\{\big||\mathcal{C}(x)| - \mathbb{E}_x[|\mathcal{C}(x)|\big|W_x]\big| > \frac{\varepsilon q(n)\zeta}{2}\bigg\}\bigg]dx$$

$$= \int_{S_n} \mathbb{E}_x\bigg[\mathbf{1}\{W_x^{d/\alpha} > a\tilde{q}(n)\}\mathbb{P}_x\bigg(\big||\mathcal{C}(x)| - \mathbb{E}_x\big[|\mathcal{C}(x)|\big|W_x\big]\big| > \frac{\varepsilon q(n)\zeta}{2}\bigg|W_x\bigg)\bigg]dx$$

$$= n\mathbb{E}_{\mathbf{0}}\Big[\mathbf{1}\{W_{\mathbf{0}}^{d/\alpha} > a\tilde{q}(n)\}\mathbb{P}_{\mathbf{0}}\Big(\big||\mathcal{C}(\mathbf{0})| - \mathbb{E}_{\mathbf{0}}\big[|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}}\big]\big| > \frac{\varepsilon q(n)\zeta}{2}\Big|W_{\mathbf{0}}\Big)\Big]$$
$$\leq n\mathbb{E}_{\mathbf{0}}\Big[\mathbf{1}\{W_{\mathbf{0}}^{d/\alpha} > a\tilde{q}(n)\}\frac{4}{\varepsilon^2 q(n)^2\zeta^2}\mathbf{Var}_{\mathbf{0}}\big(|\mathcal{C}(\mathbf{0})|\big|W_{\mathbf{0}}\big)\Big],$$

where translation invariance allowed us to shift x to the origin. The variance bound established in Lemma 6.7 provides the existence of some positive constant $C > 0$ such that

$$\mathbb{E}[R_1] \leq \frac{4Cn}{\varepsilon^2\zeta^2 q(n)^2}\mathbb{E}\big[\mathbf{1}\{W^{d/\alpha} > a\tilde{q}(n)\}W^{d/\alpha}\big]. \tag{6.26}$$

Since $W^{d/\alpha}$ has a regularly varying tail with index $-\alpha\beta/d$, Lemma 2.18 yields that $t \mapsto \mathbb{E}[\mathbf{1}\{W^{d/\alpha} > t\}W^{d/\alpha}]$ is regularly varying with index $1-\alpha\beta/d$. Moreover, $\tilde{q}(n) = n^{-\gamma}q(n)$ with $q(t) \in \mathrm{RV}_{d/(\alpha\beta)}$ by Proposition 2.19. Putting these facts together using Proposition 2.15 yields that, as a function of n,

$$\mathbb{E}\big[\mathbf{1}\{W^{d/\alpha} > a\tilde{q}(n)\}W^{d/\alpha}\big] \in \mathrm{RV}_{(1-\alpha\beta/d)(-\gamma+d/(\alpha\beta))} = \mathrm{RV}_{d/(\alpha\beta)-1+\gamma(\alpha\beta/d-1)}.$$

Therefore, the upper bound from (6.26) satisfies, once more as a function in n,

$$\frac{4Cn}{\varepsilon^2\zeta^2 q(n)^2}\mathbb{E}\big[\mathbf{1}\{W^{d/\alpha} > a\tilde{q}(n)\}W^{d/\alpha}\big] \in \mathrm{RV}_{-d/(\alpha\beta)+\gamma(\alpha\beta/d-1)}.$$

Choosing γ sufficiently small yields a negative index so that $\mathbb{E}[R_1] \to 0$ as $n \to \infty$. This concludes the proof. □

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