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Comparative Study of the Path Integration Method and the Stochastic Averaging Method for Nonlinear Roll Motion in Random Beam Seas

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Abstract

In this work, the path integration method and the energy-based stochastic averaging method are introduced in order to study the stochastic responses of ship roll motion in random beam seas. Based on the Markov property of the dynamical system, the path integration (PI) method provides approximate solution to the governing Fokker-Planck (FP) equation. On the other hand, the stochastic averaging method leads to a dimension reduction of the original system, but the essential behavior is retained. Then numerical solution or even analytical solution of the low-dimensional FP equation can be obtained. Since the principles of the above two stochastic methods are different, a comparative study is valuable and noticeable. The accuracy and performance of each method are evaluated with the assistance of Monte Carlo simulation (MCS).

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Keywords: Path integration method, Stochastic averaging method, Roll motion, Stochastic response.

1. Introduction

Nonlinear roll motion is one of the main reasons leading to ship stability failures or even capsizing of the vessels [1]. The problem of roll motion in random seas has been a demanding challenges in the past decades [2]. For the cases

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of beam seas, the roll motion is assumed to be decoupled from other motion modes and governed by a single-degree-of-freedom (SDOF) model in which the nonlinear effects associated with damping and restoring terms, and also the random external excitation are incorporated [3]. Due to the randomness of the wave excitation and roll response, stochastic methods should be introduced in order to study the roll response as well as to evaluate the associated reliability of roll motion.

Within the scope of stochastic dynamics, the methodology based on the Markov diffusion is attractive because the probabilistic properties of the roll motion are governed by the Fokker-Planck (FP) equation. In this work, a second order linear filter is applied in order to approximate the stationary wave excitation moment as a filtered white noise and then the original SDOF model is extended into a four-dimensional (4D) Markov system [4].

For the 4D Markov system, the path integration (PI) method is an efficient approximation for solving the FP equation. The principle of the PI lies in the fact that the evolution of the response PDF is computed in short steps via a step-by-step solution technique. More specifically, the response probability density function (PDF) at a given time instant can be obtained by applying the Chapman-Kolmogorov equation when the response PDF at an earlier closer time and the transition PDF are already known [5].

Unlike the PI method which provides approximate solution of the FP equation, the stochastic averaging method leads to a dimension reduction of the original system, but the essential behavior of the system is retained [6, 7]. Then, numerical solution or even analytical solution of the low-dimensional FP equation can be obtained. This convenient approximate technique has been widely used in the field of random vibration for predicting the stationary or non-stationary response of the stochastic systems.

In this work, both the PI method and the stochastic averaging method will be applied in order to calculate the stochastic response of the roll motion in random beam seas. The accuracy and performance of each method are evaluated by Monte Carlo simulation (MCS).

2. Mathematical Model

When the ship is excited by beam wave loads, the roll motion is described by the following SDOF equation:

$$\ddot{\theta}(t) + b_{44}\dot{\theta}(t) + b_{44q}\dot{\theta}(t)\,|\,\dot{\theta}(t)\,| + c_1\theta(t) - c_3\theta^3(t) = m(t) \tag{1}$$

where $\theta(t)$ and $\dot{\theta}(t)$ are the roll angle and the roll velocity, respectively. b_{44} and b_{44q} are the relative linear and quadratic damping coefficients, c_1 and c_3 are the relative linear and nonlinear roll restoring coefficients. $m(t)$ is the relative roll excitation moment, it is assumed to be a stationary process and described by the spectrum $S_{mm}(\omega)$. It should be noted that the roll motion has a softening characteristic since the nonlinear restoring term is negative. For the softening cases, ship capsizing would occur when the roll angle exceeds the angle of vanishing stability beyond which the restoring moment becomes negative [8].

The linear filter technique is widely used in the engineering community due to its simplicity and practicality [4]. The driving process $m(t)$ is a stationary Gaussian process which can be approximated quite satisfactorily by a suitable linear filter. In this work, a second-order linear is applied in order to approximate the desired spectrum, $S_{mm}(\omega)$.

$$\begin{cases} dx_3 = (x_4 - \beta x_3)dt + \gamma dW \\ dx_4 = -\alpha x_3 dt \end{cases} \tag{2}$$

in which x_3 and x_4 are the state variables in the filter equation, $x_3=m(t)$. $dW(t)=W(t+dt)-W(t)$ represents an infinitesimal increment of a standard Wiener process with $E\{dW(t)\} = 0$ and $E\{dW(t)dW(s)\}= 0$ for $t \neq s$ and $E\{dW(t)^2\} = dt$. The spectrum generated by the second-order linear filter (2) is denoted as $S_{Filter}(\omega)$ and given as:

$$S_{Filter}(\omega) = \frac{1}{2\pi}\frac{\gamma^2\omega^2}{(\alpha - \omega^2)^2 + (\beta\omega)^2} \tag{3}$$

where α, β, γ are the parameters of the linear filter and they are determined by minimizing the least square error between the spectral density of the filtered spectrum $S_{Filter}(\omega)$ and the spectral density of the target spectrum, $S_{mm}(\omega)$.

By combining the equation (1) with equation (2), the extended dynamic system is formed. Therefore, the roll motion in random beam seas can be described by the following 4D state space equation:

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = (-b_{44}x_2 - b_{44q}x_2 \mid x_2 \mid - c_1 x_1 + c_3 x_1^3 + x_3)dt \\ dx_3 = (x_4 - \beta x_3)dt + \gamma dW \\ dx_4 = -\alpha x_3 dt \end{cases} \tag{4}$$

where $x_1 = \theta(t)$ and $x_2 = \dot{\theta}(t)$.

3. 4D Path Integration Method

The dynamic system represented by equation (4) is a Markov diffusion process and it can be expressed as the following stochastic differential equation:

$$d\mathbf{x} = \mathbf{a}(\mathbf{x}, t)dt + \mathbf{b}(t)d\mathbf{W}(t) \tag{5}$$

where $\mathbf{x}(t) = (x_1(t), \ldots, x_4(t))^T$ is a 4D state space vector process, the vector $\mathbf{a}(\mathbf{x}, t)$ is the drift term and $b(t)d\mathbf{W}(t)$ represents the diffusive term. The vector $d\mathbf{W}(t) = \mathbf{W}(t+dt) - \mathbf{W}(t)$ denotes independent increments of a standard Wiener process.

For the Markov process $\mathbf{x}(t)$, its transition probability density function (PDF), $p(\mathbf{x},t|\mathbf{x}',t')$, is governed by the FP equation, which is expressed as [9] :

$$\frac{\partial}{\partial t} p(\mathbf{x}, t \mid \mathbf{x}', t') = -\sum_{i=1}^{4} \frac{\partial}{\partial x_i} a_i(\mathbf{x}, t) p(\mathbf{x}, t \mid \mathbf{x}', t') + \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial^2}{\partial x_i \partial x_j} (b(t) \cdot b^T(t))_{ij} \, p(\mathbf{x}, t \mid \mathbf{x}', t') \tag{6}$$

where $\mathbf{x}'$ denotes the state space vector at time t' and $t' < t$.

For the numerical solution of the time continuous stochastic differential equation (5), discretization of the equation with respect to time t is required. In this regard, Naess and Moe [10] proposed a fourth-order Runge-Kutta-Maruyama approximation:

$$\mathbf{x}(t) = \mathbf{x}(t') + \mathbf{r}(\mathbf{x}(t'), t')\Delta t + b(t')\Delta\mathbf{W}(t') \tag{7}$$

where $\Delta t = t - t'$ is the time increment and the vector $\mathbf{r}(\mathbf{x}(t'), t')$ which denotes the explicit fourth-order Runge-Kutta increment. Since $\mathbf{W}(t)$ is a Wiener process, for a short time increment Δt, the independent increment $\Delta\mathbf{W}(t') = \mathbf{W}(t) - \mathbf{W}(t')$ is a Gaussian variable for every t'. The time sequence $\{\mathbf{x}(i \cdot \Delta t)\}_{i=0}^{\infty}$ is a Markov chain and it can approximate the time-continuous Markov solution of the SDE (5) with satisfactory accuracy when the time increment Δt is sufficiently small.

Furthermore, the conditional PDF of the process, $p(\mathbf{x},t|\mathbf{x}',t')$, follows a (degenerate) Gaussian distribution, which is written as:

$$p(\mathbf{x}, t \mid \mathbf{x}', t') = \delta(x_1 - x_1' - r_1(\mathbf{x}', t', \Delta t)) \cdot \delta(x_2 - x_2' - r_2(\mathbf{x}', t', \Delta t)) \cdot \tilde{p}(x_3, t \mid x_3', t') \cdot \delta(x_4 - x_4' - r_4(\mathbf{x}', t', \Delta t)) \tag{8}$$

where $\tilde{p}(x_3, t \mid x_3', t')$ is given by the relation:

$$\tilde{p}(x_3,t\,|\,x_3',t')=\frac{1}{\sqrt{2\pi\gamma^2\Delta t}}\cdot\exp\left\{-\frac{(x_3-x_3'-r_3(\mathbf{x}',t',\Delta t))^2}{2\gamma^2\Delta t}\right\} \tag{9}$$

in which $r_i(\mathbf{x}', t', \Delta t) = r_i(\mathbf{x}(t'), t', \Delta t)$, i=1,2,3,4 are the Runge-Kutta increments for the state space variables.

For the PI method, the evolution of the response statistics, such as the PDF of the random process $\mathbf{x}$ at time t can be obtained by the following basic equation:

$$p(\mathbf{x},t)=\int_{R^4}p(\mathbf{x},t\,|\,\mathbf{x}',t')p(\mathbf{x}',t')d\mathbf{x}' \tag{10}$$

where $d\mathbf{x}'=dx_1'\cdot dx_2'\cdot dx_3'\cdot dx_4'$.

Since the expression for conditional PDF is obtained, the PDF of $\mathbf{x}(t)$ can be obtained by the following iterative algorithm if an initial PDF (i.e. at time t_0) is given:

$$p(\mathbf{x},t)=\int_{R^4}\cdots\int_{R^4}\prod_{s=1}^{n}p(\mathbf{x}^{(s)},t_s\,|\,\mathbf{x}^{(s-1)},t_{s-1})\cdot p(\mathbf{x}^{(0)},t_0)d\mathbf{x}^{(0)}\ldots d\mathbf{x}^{(n-1)} \tag{11}$$

where $\mathbf{x}=\mathbf{x}^{(n)}=\mathbf{x}(t_n)$, $t=t_n=t_0+n\cdot\Delta t$, $\mathbf{x}^{(s)}=\mathbf{x}(t_s)$ and $t_s=t_0+s\cdot\Delta t$.

Equation (11) describes the mathematical principle of the PI approach. The numerical iterative algorithm and the associated computational steps have been systematically described in Refs. [1, 5].

4. Stochastic Averaging Method

The stochastic averaging method is focused on the energy envelope process of the roll dynamics and it leads to a lower dimensional description of the energy of roll dynamics. By applying the stochastic averaging method, a one-dimensional SDE for the roll energy process can be obtained and relevant PDFs for the low-dimensional cases can be easily calculated. The principle of the energy based stochastic averaging method is presented below.

Generally, the damping coefficients of the roll motion is small when compared with the restoring coefficients. A perturbation parameter $\varepsilon \ll 1$ is introduced in the SDOF model (1) in order to provide a scaled system for the stochastic averaging procedure. The scaled system with $t_\varepsilon = \varepsilon t$ is given as:

$$\begin{cases}\dfrac{d}{dt_\varepsilon}x=y\\[2mm]\dfrac{d}{dt_\varepsilon}y=-\alpha_1x+\alpha_3x^3-\varepsilon(\beta_1y+\beta_2y|y|)+\sqrt{\varepsilon}\upsilon m_s(t_\varepsilon)\end{cases} \tag{12}$$

where $x=x_1$, $y=\varepsilon^{-1}x_2$, $\alpha_1=\varepsilon^{-2}c_1$, $\alpha_3=\varepsilon^{-2}c_3$, $\beta_1=\varepsilon^{-2}b_{44}$, $\beta_3=\varepsilon^{-1}b_{44q}$, $\upsilon=\varepsilon^{-5/2}$, $m_s(t_\varepsilon)=m(\varepsilon t)$.

Since we are interested in the total energy of the roll dynamics, the Hamilton function $H(x, y)$ is introduced:

$$H(x,y)=\frac{y^2}{2}+\alpha_1\frac{x^2}{2}-\alpha_3\frac{x^4}{4} \tag{13}$$

Then, equation (12) is re-written as:

$$\begin{cases}\dfrac{d}{dt_\varepsilon}x=\dfrac{\partial H(x,y)}{\partial y}\\[2mm]\dfrac{d}{dt_\varepsilon}y=-\dfrac{\partial H(x,y)}{\partial x}-\varepsilon\dfrac{\partial H(x,y)}{\partial y}(\beta_1+\beta_2|y|)+\sqrt{\varepsilon}\upsilon m_s(t_\varepsilon)\end{cases} \tag{14}$$

The contour lines of the Hamilton function, $H(x, y)$ are shown in Fig. 1. The fixed points of system (14) without dissipation and random perturbation (i.e. ε=0) are: P_1= $(\sqrt{\alpha_1/\alpha_3},0)$, P_2= $(-\sqrt{\alpha_1/\alpha_3},0)$ and S= (0, 0). Moreover, the change of the total energy is the time derivate of the Hamilton function:

$$\frac{d}{dt_\varepsilon}H=\varepsilon y^2(-\beta_1-\beta_2\left|y\right|)+\sqrt{\varepsilon}y\upsilon m_s(t_\varepsilon) \tag{15}$$

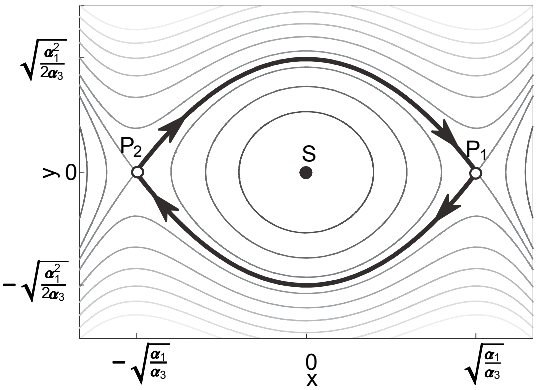

Fig. 1. Contour lines of $H(x, y)$.

From the Hamiltonian, we obtain:

$$Q(x,H)=y^2=2H-\alpha_1x^2+\alpha_3\frac{x^4}{2} \tag{16}$$

Then, combing the first equation of equation (12) with equation (15), the following system is formed:

$$\begin{cases}\dfrac{d}{dt_\varepsilon}x=\sqrt{Q(x,H)}\\ \dfrac{d}{dt_\varepsilon}H=\varepsilon Q(x,H)(-\beta_1-\beta_2\sqrt{Q(x,H)})+\sqrt{\varepsilon}\sqrt{Q(x,H)}\upsilon m_s(t_\varepsilon)\end{cases} \tag{17}$$

An important property of the reformulated system (17) is that the energy level H changes slowly compared to the oscillations of the variable x due to the small value ε. This enables the application of the stochastic averaging method to this system since the fast oscillatory dynamics of the roll motion can be averaged over the roll period. For the multiple scale model (17), the period $T(H)$ at the energy level H ($0 \le H \le H_c$) of one oscillation of the fast variable x in the absence of noise and damping (i.e. $\varepsilon = 0$), is given by:

$$T(H)=\int_0^{T(H)}dt=2\int_{-b(H)}^{b(H)}\frac{dx}{\sqrt{Q(x,H)}}=\frac{4}{q}K(k) \tag{18}$$

where H_c is the energy level corresponding to the roll angle of vanishing stability, which is

$$H_c=\alpha_1^2/4\alpha_3\,, q=a\sqrt{\frac{\alpha_3}{2}}, a=\sqrt{\frac{4H}{b^2\alpha_3}} \tag{19}$$

and the function $K(k)$ is the complete elliptic integral of the first kind with elliptic modulus k given by k=b/a .The limits of integration $\pm$ $b(H)$ are the points where $y=\sqrt{Q(x,H)}=0$ and the periodic orbit interacts the x-axis, i.e. $b(H)$

is the maximum value of x for each energy level H and it is given by:

$$b=\sqrt{-\frac{-\alpha_1+\sqrt{\alpha_1^2-4\alpha_3 H}}{\alpha_3}} \tag{20}$$

The process H converges weakly to a diffusion Markov process as the perturbation parameter $\varepsilon\to 0$, and then the corresponding one-dimensional *Itô* equation for the Markov process is expressed as:

$$dH=m(H)dt_\varepsilon+\sigma(H)dW(t_\varepsilon) \tag{21}$$

with the drift coefficient and the diffusion coefficient:

$$m(H)=\frac{4}{Tq}\int_{-\infty}^{0}\upsilon^2 R_{m_s m_s}(\tau)\int_0^{K(k)}\frac{cn_{t_\varepsilon+\tau}dn_{t_\varepsilon+\tau}}{cn_{t_\varepsilon}dn_{t_\varepsilon}}dud\tau+\frac{1}{T}\int_0^T Q(x(t_\varepsilon),H)(-\beta_1-\beta_2\sqrt{Q(x(t_\varepsilon),H)})dt_\varepsilon \tag{22}$$

$$\sigma^2(H)=\frac{4b^2 q}{T}\int_{-\infty}^{\infty}\upsilon^2 R_{m_s m_s}(\tau)\int_0^{K(k)} cn_{t_\varepsilon}dn_{t_\varepsilon}cn_{t_\varepsilon+\tau}dn_{t_\varepsilon+\tau}dud\tau \tag{23}$$

Here, $R_{m_s m_s}$ is the autocorrelation function of the stochastic process $m_s(t_\varepsilon)$. $cn(\cdot, k)$ and $dn(\cdot, k)$ are the Jacobian elliptic functions, the following abbreviations are used:

$$cn:=cn(qt_\varepsilon,k);\, dn:=dn(qt_\varepsilon,k)\ \text{and}\ u:=qt_\varepsilon \tag{24}$$

in which, if the substitute τ or $\tau+t_\varepsilon$ is used, we refer to the argument $q\tau$ or $q(\tau+t_\varepsilon)$, respectively.

As mentioned in Section 2, the roll response can be approximately assumed to be a stationary diffusion process when the mean time to capsize is very long. Therefore, a stationary solution of the one-dimensional FP equation which governs the equation (21) can be obtained and expressed as:

$$p_{st}(H)=\frac{C}{\sigma^2(H)}\exp\left(2\int_0^H\frac{m(h)}{\sigma^2(h)}dh\right) \tag{25}$$

in which, C is a non-dimensional parameter which can be obtained by normality condition.

In order to get the joint PDF of x, y when the stationary distribution (25) is obtained by the stochastic averaging method, the following transformation is introduced:

$$\begin{cases} X=X=g_1(X,Y)\\ H=\dfrac{Y^2}{2}+\alpha_1\dfrac{X^2}{2}-\alpha_3\dfrac{X^4}{4}=g_2(X,Y)\end{cases} \tag{26}$$

where, X and Y are random variables corresponds to x, y defined by Eq. (13). Following Stratonovich [11], the PDF of the displacement x given an energy level value H is inversely proportional to the velocity y and thus

$$f_{X|H}(x\,|\,H)=\frac{1}{T(H)}\frac{1}{\sqrt{Q(x,H)}} \tag{27}$$

Using $f_{xH}(x, H)=f_{x|H}(x|H)p_{st}(H)$, the joint PDF of x and y is obtained as:

$$f_{XY}(x,y)=f_{XH}(x,H)\begin{vmatrix}\dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y}\\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y}\end{vmatrix}=p_{st}(H)/T(H) \tag{28}$$

5. Numerical Results

In this work, a fisher research vessel is selected to study the rolling behavior in random beam seas. The main parameters of the vessel, the restoring arm and the information of wave excitation moment spectrum, $S_{mm}(\omega)$ are given in Ref. [12]. In order to have a detailed study of the performance for the PI method and the stochastic averaging method, Gaussian white noise and filtered white noise are selected as the driving process for the SDOF model (1).

5.1. System excited by Gaussian white noise

Under the excitation of Gaussian white noise, the 4D state space equation (4) can be simplified as:

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = (-b_{44}x_2 - b_{44q}x_2 \,|\, x_2 \,| - c_1 x_1 + c_3 x_1^3)dt + \sigma_0 dW \end{cases} \tag{29}$$

where σ_0 is the noise level. In addition, for the two-dimensional (2D) system (29), the principles of the 2D PI method and the stochastic averaging method have been described in Chai et al. [13] and Dostal et al. [7] respectively.

In this case, the noise level σ_0=0.067 and the stochastic responses can be obtained by applying the PI method and the stochastic averaging method. The joint PDFs of the roll angle process and the roll velocity process provided by the two different methods are presented in Figs. 2 and 3, respectively. Fig. 4 shows the contour lines of the joint PDFs and Fig. 5 presents the marginal PDFs of the roll angle process calculated by the two methods . Moreover, the marginal PDFs of the roll angle process plotted on a logarithmic scale are presented in Fig. 6.

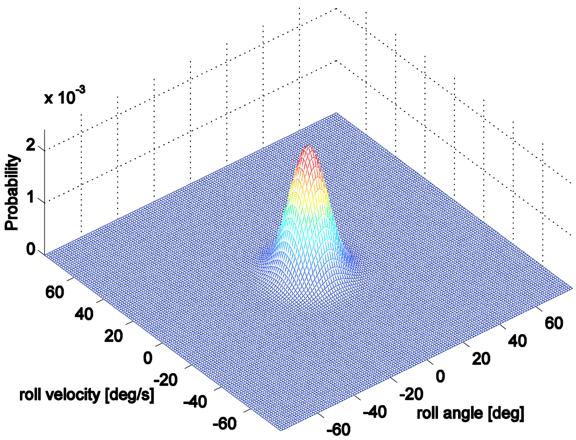

Fig. 2. Joint PDF of the roll response of the 2D system, calculated by the PI method.

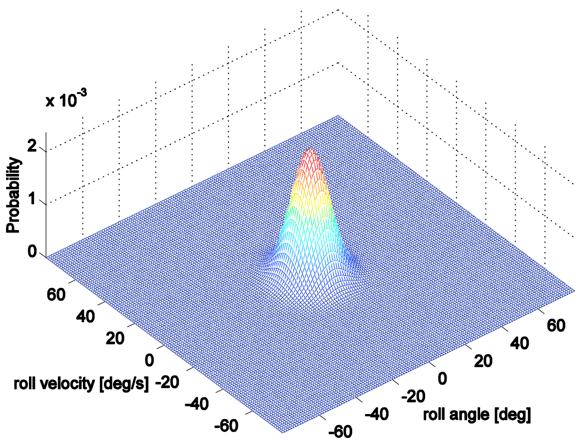

Fig. 3. Joint PDF of the roll response of the 2D system, obtained by the stochastic averaging method

In addition to the joint PDF and marginal PDFs, the mean upcrossing rate is an important parameter for the response statistics and it is also a key parameter for estimation of the extreme response statistics and the associated reliability of the structures [14]. Basically, the mean upcrossing rate is a time-dependent parameter for the roll motion with softening restoring characteristics. Nevertheless, in this work and for current ship model, the mean time to capsize is long enough and the dynamic system can be regarded as a highly reliable system. The corresponding roll response reaches stationary in an approximate sense [4, 15]. Based on the Rice formula (30), the mean upcrossing rate is given as:

$$\nu^+(\zeta) = \int_0^\infty \dot{\theta} f_{\theta\dot{\theta}}(\zeta, \dot{\theta}) d\dot{\theta} \tag{30}$$

where $\nu^{+}(\zeta)$ denotes the expected (or average) number of upcrossing for the ζ-level per unit time and $f_{\theta\dot{\theta}}(\theta,\dot{\theta})$ is the joint PDF of the roll response which can be provided by the PI method and the stochastic averaging method. The mean upcrossing rates calculated by the two methods are presented in Fig. 7 on a logarithmic scale.

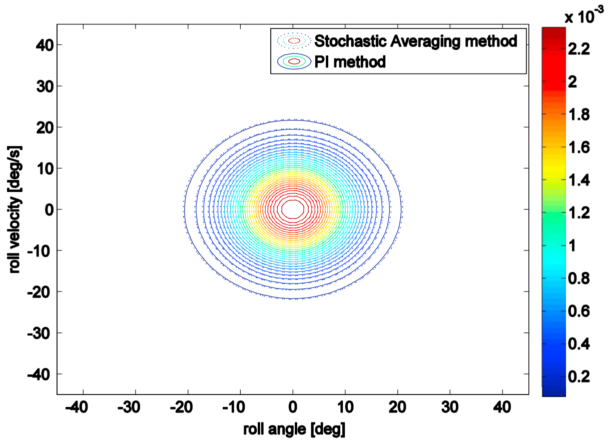

Fig. 4. Contour lines of the joint PDFs for the roll response of the 2D system, calculated by the PI method and the stochastic averaging method.

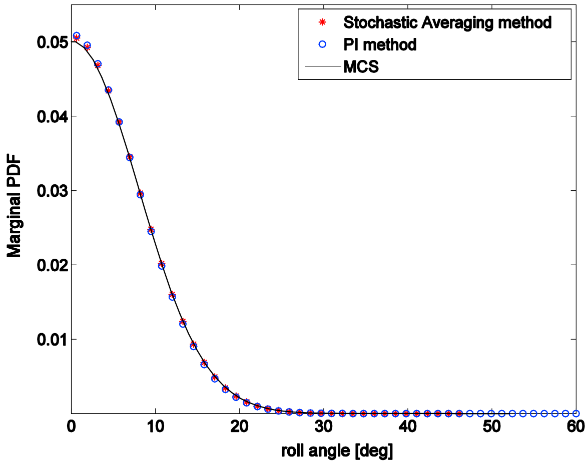

Fig. 5. Marginal PDFs of the roll angle process for the 2D system, calculated by the PI method and the stochastic averaging method.

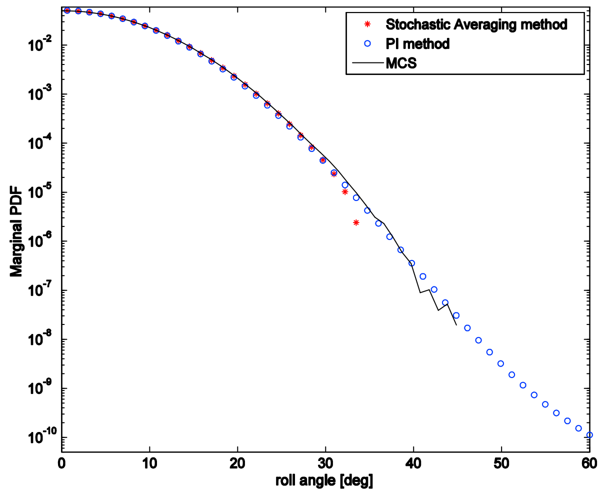

Fig. 6. Marginal PDFs of the roll angle process for the 2D system, given in a logarithmic scale

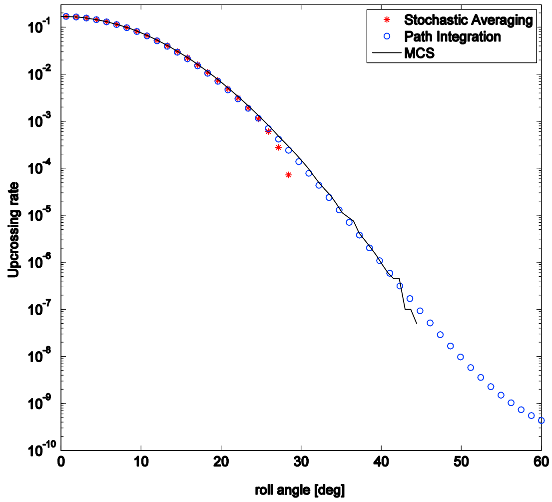

Fig. 7. Mean upcrossing rates of the roll motion for 2D system, calculated by the PI method and the stochastic averaging method

In order to evaluate the performance of these two methods, MCS is introduced and relevant response statistics are plotted in Figs. 5-7. It is seen in Figures 2-7 that both of the PI method and the stochastic averaging method can provide good results for the response statistics of the roll motion excited by Gaussian white noise. Especially, from the comparisons given in Figs. 6 and 7, it is seen that both of the two methods are able to provide satisfactory estimation of the response statistics, even in the tail region with very low probability levels.

5.2. System excited by filtered white noise

In this part, the performance of the PI method and the stochastic averaging method for the 4D system (4) is studied. The sea state with the significant wave height H_s = 4.0 m and the peak period T_p =11.0 s is selected in order to obtain the wave excitation spectrum $S_{mm}(\omega)$ [1]. After the work of spectrum fitting, the parameters in the filter (2) and the 4D system are determined. Therefore, the PI method and the stochastic averaging method can be applied in order to calculate the response statistics.

The joint PDFs of the roll response calculated by the 4D PI method and the stochastic averaging method are shown in Figs. 8 and 9, respectively. The contour lines of the joint PDFs are shown in Fig. 10 and the marginal PDF of the roll angle process provided by these two methods are presented in Fig. 11.

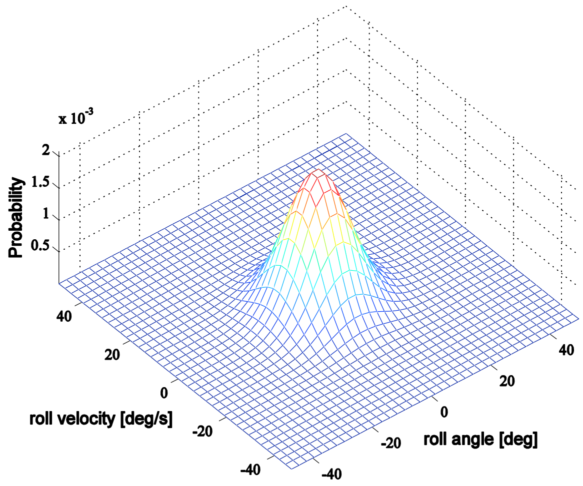

Fig. 8. Joint PDF of the roll response for the 4D system, calculated by the PI method.

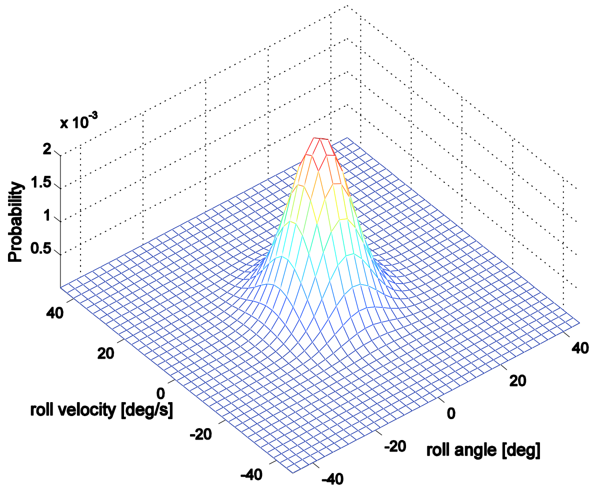

Fig. 9. Joint PDF of the roll response for the 4D system, provided by the stochastic averaging method

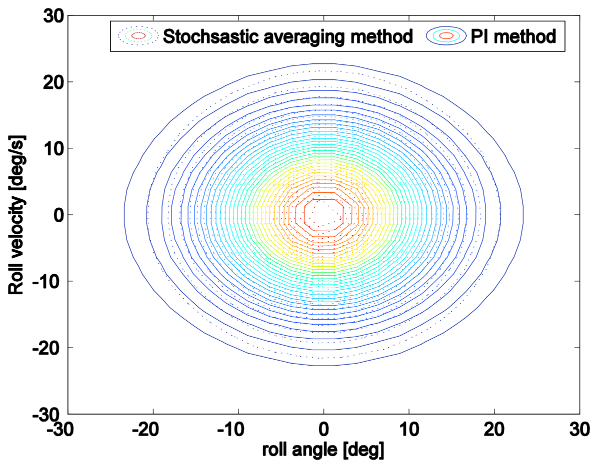

Fig. 10. Contour lines of the joint PDFs for the roll response of the 4D system, calculated by the PI method and the stochastic averaging method.

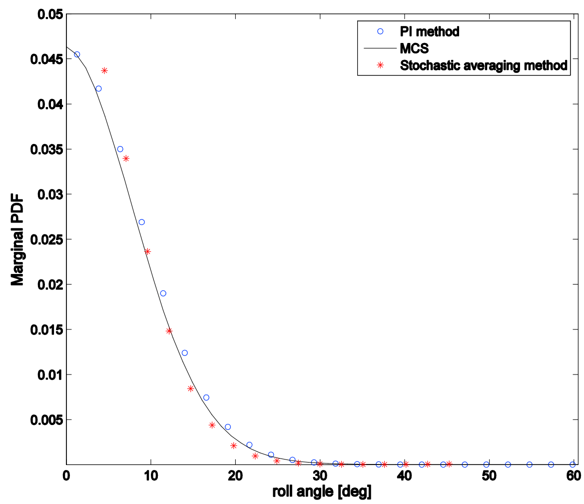

Fig. 11. Marginal PDFs of the roll angle process for the 4D system, calculated by the PI method and the stochastic averaging method.

In addition to Figs 8-11, Figs. 12 and 13 present the marginal PDFs of the roll angle process and mean upcrossing rates on a logarithmic scale, respectively. MCS results are introduced in order to evaluate the performance of the PI method and the stochastic averaging method, especially in the tail region with low probability levels. It is seen in Figs.

8-11 that the PI method and the stochastic averaging method provides very close results with respect to the PDFs of the roll response and some discrepancies can be observed in Figs. 10 and 11. For the marginal PDF of the roll angle process, it is seen in Fig. 11 that stochastic averaging method slightly overestimate the roll response in the region around $\theta < 10$ degrees, while it is observed in Figs. 12 and 13 that this method underestimate the response in the region with $\theta > 15$ degrees when compared with the MCS and the PI method. More importantly, it is seen that the accuracy of the PI method is significantly higher than the stochastic averaging method. For current 4D case, the performance of the stochastic averaging method, especially in the tail region, is far away from satisfactory when compared with the PI method and MCS.

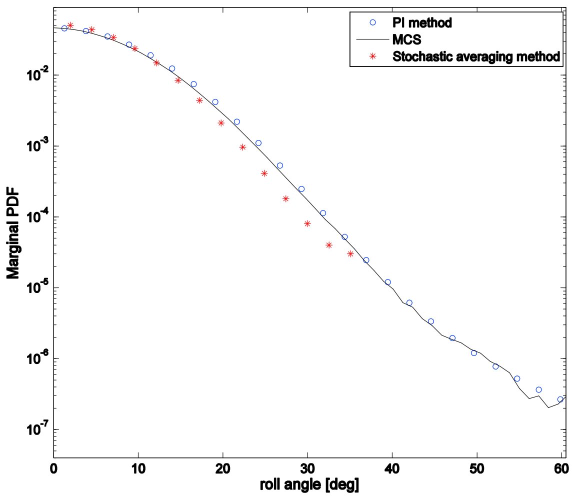

Fig. 12. Marginal PDFs of the roll angle process for the 4D system, given in a logarithmic scale

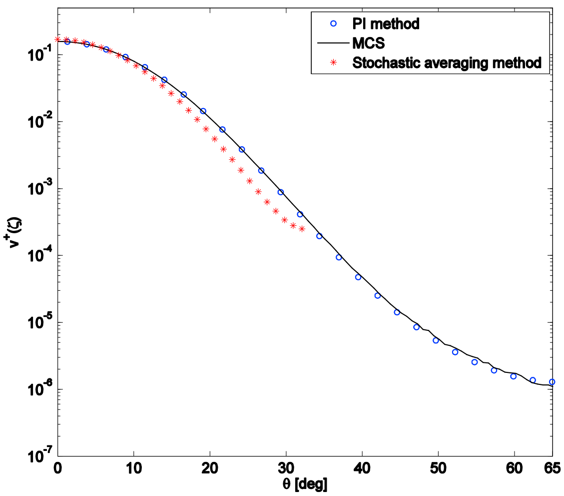

Fig. 13. Mean upcrossing rates of the roll motion for 4D system, calculated by the PI method and the stochastic averaging method

5.3. Discussions

The principles of the PI method and the stochastic averaging method are quite different: the former method is based on the Markov property of the dynamical system and the response statistics are obtained by solving the governing FP equation, while the latter method is based on the fact that the fast oscillatory roll motion can be averaged over the roll period, due to the light roll damping and small external excitation terms.

For the accuracy of the above two methods, it is seen that the PI method can provide satisfactory results of the response statistics, both for the system excited by Gaussian white noise and for the system driven by filtered white noise. However, for current vessel and selected excitations, the stochastic averaging method can only provide satisfactory results for the 2D system. In fact, the performance and accuracy of the stochastic averaging method depend on the ratio between the correlation of the external excitation and the relaxation time of the dynamical system [11] . For the white noise case, such ratio is close to 0, which corresponds to the best performance. For 4D system subjected to filtered white noise excitation, the performance of the stochastic averaging method decreases with increased value of the ratio.

Another important issue is the computation efficiency. The computation cost of the PI method increases dramatically with the dimensions and grid number in each dimension. For the 2D system (29), the computation time for one simulation is less than 1 minute, but for the 4D system (4), 1.5 hours is required for one simulation on a laptop. On the other hand, the stochastic averaging method reduces the 2D and 4D systems into one-dimensional systems and the results can be given by analytical formulas or by numerical calculations. The drift and diffusion coefficients given by equations (22) and (23) are obtained analytically, and then the stationary solution (25) can be obtained within 20 seconds for the 2D system and the 4D system requires about 7 minutes to get the response statistics. Moreover, the drift and diffusion coefficients can also be calculated by direct numerical calculations which takes a longer time than the analytical solutions. Therefore, the stochastic averaging method has more advantage than the PI method with respect to the computation efficiency.

6. Conclusions

In this work, two approaches based on different principles were applied in order to calculate the response statistics of the ship roll motion excited by Gaussian white noise and filtered white noise. MCS is introduced in order to evaluate the performance of the PI method and the stochastic averaging method.

Based on the numerical results given in Section 5, it is seen that the PI can provide satisfactory results for the 2D system as well as for the 4D system. However, the stochastic averaging method is able to provide good accuracy for the 2D system and its performance for the 4D system, especially in the tail region is not satisfactory. One the other hand, the stochastic averaging method has obvious advantage on the computation efficiency.

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