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Three-Dimensional Internal Gravity Waves in a Stratified Free-Surface Flow

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The three-dimensional gravity waves due to a submerged disturbance in a steady free-surface flow of a stratified fluid are investigated based on a linearized theory. The effect of density stratification $\rho_o(z)$ along the vertical z direction in a gravity field (with constant g) is characterized by two flow parameters: $\sigma = -L\rho_o'/\rho_o$ and $\lambda = gL/U^2$, L and U being respectively a characteristic length and flow velocity. In the special case of constant σ treated here, the far field contains two wave modes, one being an irrotational surface wave and the other a system of internal gravity waves, both existing only in the downstream. The pattern of the internal waves is examined in detail. The uniqueness of the solution is determined by considering a corresponding initial-value problem.

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I. Introduction

The problem of two-dimensional gravity waves due to a concentrated disturbance submerged in a steady free-surface flow of a stratified fluid has been treated recently by Mei and Wu (1964) based on a linearized theory. In this problem the effects of densith stratification and gravity are characterized by two flow parameters, one being the vertical rate of change in the basic density distribution and the other, a characteristic Froude number. Regarding these parameters, two different cases were investigated: (1) the basic density stratification has an exponential growth with depth and the Froude number is arbitrary; (2) the basic density profile is arbitrary and the internal Froude number is small.

The present work is a generalization to the three-dimensiomal case. There is an essential difference between the two- and three-dimensional flows of stratified fluids. Yih (1959) showed that the gravity effect in steady weak motions is to inhibit vertical flows and horizontal density gradients. Physically, exchange of two gluid bulks of equal bolume at different altitudes (and hence of different densities) always results in an increase of potential energy. Hence, if the inertial effect is a weak motion is not strong enough to make the fluid go up or down to pass by an obstacle, the fluid in the front, as well as in the rear, will be constrained by this potential barries to move along with the obstacle, producing the so-called blocked flow. In an experiment with a peddle moving slowly through a slightly stratified fluid, Yih has found the effect of the peddle motion far upstream and downstream. In the two-dimensional results of Mei and Wu, the far field is shown to contain two types of waves, one being an irrotational surface wave and the other a system of internal gravity waves. Some internal waves are indeed found on the upstream side, with their amplitude decaying with distance.

The picture is however quite different in the three-dimensional case since the fluid can now go around the obstacle in a horizontal plane if its kinetic energy is insufficient to overcome the potential difference. In the special case of exponential density distribution treated in this paper, it is found that the far field also consists of a surface wave and a system of internal waves. The wave phase and the vertical exponential decay rate of the irrotational surface wave depend solely on the Froude number; only its amplitude is affected by the density stratification. Consequently, this component of the wave field can also be constructed by superposition of the corresponding two-dimensional surface waves, if so desired. The internal waves, on the other hand, are due to the combined effects of stratification and gravity. From the above physical reasoning, their basic features in three-dimensional flows are expected to be different from the two-dimensional case. It is shown here that both the surface wave and internal wave exist only on the downstream side. Unlike the Kelvin wave pattern of the surface wave, the internal wave has an unbounded lateral extent and its amplitude diminishes like R^{-1} over a spherical surface of large radius R. At large lateral distances from the disturbance, where the inertial effect is small, the constant phase surfaces are nearly horizontal, so that the resulting density gradient is the smallest in the horizontal plane. Only near the disturbance, where the inertial effect is relatively strong, do the equi-phase surfaces become inclined to resemble the radial propagation of these waves.

The method of Fourier transform is adopted here to obtain the solution in the form of an integral representation. Like in all steady, inviscid surface wave problems, indeterminacy arises as singularities appear on the path of integration. Following the method of De Prima and Wu (1956), The uniqueness of the solution is determined by verifying the large time limit of a corresponding initial-value problem.

2. General Formulation

We consider the steady flow of a stratified fluid due to a three-dimensional point source submerged beneath the free surface of an otherwise uniform flow with velocity U in the positive x direction. A uniform gravity field acts in the negative z direction, keeping the original undisturbed flow in $z \leq 0$. The fluid is assumed to be inviscid, incompressible, and is characterized by a basic profile of density stratification:

$$\rho_0 = \rho_0(z), \qquad z \leq 0. \tag{1}$$

In the formulation stage, $\rho_0(z)$ may be regarded as arbitrary except that $\rho_0(z)$ and $(-d\rho_0/dz)$ are everywhere positive. The flow is incompressible in the sense that the density $\rho(x, y, z)$ is invariant along the flow motion, or, with $\vec{q}$ denoting the flow velocity,

$$\vec{q}\cdot(\mathrm{grad}\,\rho) = 0. \tag{2}$$

Thus, with a concentrated flow source of strength Q submerged at (0, 0, -h), as represented by Dirac's functions, the continuity equation becomes

$$\nabla\cdot\vec{q} = Q\,\delta(x)\delta(y)\delta(z+h) \tag{3}$$

in which (2) has been implied. The momentum equation reads

$$\vec{q}\cdot\nabla\vec{q} = -\frac{1}{\rho}\nabla p - g\,\vec{e}_z \tag{4}$$

where p is the pressure, g the gravitational constant, and $\vec{e}_z$ a unit vector in the positive z direction. Under condition (2), (4) can be integrated once to give

$$\frac{1}{2}q^2 + \frac{p}{\rho} + gz = C(\psi) \tag{5}$$

where $C(\psi)$ represents a function which is constant on streamlines only.

Let the free surface of the perturbed flow be denoted by

$$F(x,y,z) \equiv z - \zeta(x,y) = 0 \tag{6}$$

Then the boundary conditions on the free surface are

$$\frac{dF}{dt} = \vec{q} \cdot \nabla F = 0 \qquad \text{on} \quad F(x,y,z) = 0, \tag{7}$$

$$p(x,y,z) = p_s(x,y) \qquad \text{on} \quad F(x,y,z) = 0, \tag{8}$$

where $p_s(x, y)$ denotes a given pressure distribution on the free surface.

We introduce the perturbation quantities defined by

$$\vec{q}/U = (1+u, v, w), \qquad \rho = \rho_0(z) + \rho_1, \qquad p = p_0(z) + p_1 \tag{9}$$

where

$$p_0(z) = p_0(0) - g\int_0^z \rho_0(z)\,dz. \tag{10}$$

For the linearlized formulation, we assume the disturbance to be small compared with the basic flow so that the nonlinear terms may be negelected. Then (2), (3), (4) reduce to the following system of linear equations

$$D\rho_1 + w\frac{d\rho_0}{dz} = 0, \tag{11}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{Q}{U}\,\delta(x)\delta(y)\,\delta(z+h), \tag{12}$$

$$\rho_0 U^2 D u = -\frac{\partial p_1}{\partial x}, \tag{13a}$$

$$\rho_0 U^2 D v = -\frac{\partial p_1}{\partial y}, \tag{13b}$$

$$\rho_0 U^2 D w = -\frac{\partial p_1}{\partial z} - g\rho_1, \tag{13c}$$

in which $D \equiv \partial/\partial x$. By linearization, condition (7) reduces to

$$w = D\zeta \qquad \text{on} \quad z = 0. \tag{14}$$

As for condition (8), one may apply first the differential operator $(1+u)(\partial/\partial x) + v(\partial/\partial y)$, and using (9), to obtain

$$D\left(p_1 + \frac{dp_0}{dz}\zeta - p_s\right) = 0 \qquad \text{on} \quad z = 0. \tag{15}$$

The free surface displacement ζ can be eliminated from (14) and (15), giving

$$D p_1 - \rho_0 g w = D p_s \qquad \text{on } z = 0. \tag{16}$$

Furthermore, the perturbations are required to vanish at large depth,

$$u, v, w, \rho_1, p_1 \to 0 \qquad \text{as } z \to -\infty. \tag{17}$$

In case of a channel of finite depth, say H, condition (17) is replaced by $w(x,y,-H) = 0$. Finally, for the present steady state problem, it is necessary to impose the radition condition

$$(u, v, w)\sqrt{R} \to 0 \qquad \text{as } R^2 = x^2 + y^2 + z^2 \to \infty, \text{ for } x < 0, \tag{18}$$

which implies that no wave energy is allowed to be radiated to the upstream. However, it should be pointed out here that the radiation condition in the above conventional form for irrotational water waves cannot have any effect on those waves whose amplitudes diminish at rates faster than $R^{-1/2}$ as $R \to \infty$.

As will be seen later, there arise two wave modes in our solution. One of them is a system of irrotational surface waves which decays exponentially with depth and falls off like $R^{-1/2}$ along the surface. These waves can be determined uniquely by condition (18). The other mode represents a system of internal waves produced by the effect of density stratification. These internal gravity waves have their amplitudes falling off like $R^{-1/2}$ over a spherical surface of large radius R, and hence are also radiating energy. Furthermore, these waves are rotational, and hence dissipative. It is not obvious á priori that these internal waves cannot exist on the upstream side. (In fact, in the two-dimensional case, some internal waves with decaying amplitude may exist in the upstream, because the obstacle has a blockage effect, to form a forward wake, as shown previously by Mei and Wu (1964).) We shall remove this indeterminacy by referring to the large time behavior of a corresponding initial-value problem. An alternative method is by including in (13) a small viscous force. This latter method, however, will not be executed here.

We shall derive a simple equation for w as follows. First, u, v can be eliminated by differentiating (13a) (13b) and using (12), giving

$$\nabla^2_{(x,y)} p_1 = \rho_0 U^2 D\left(\frac{\partial w}{\partial z} - \frac{Q}{U}\delta(x)\delta(y)\delta(z+h)\right) \tag{19}$$

where

$$\nabla^2_{(x,y)} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad D = \frac{\partial}{\partial x}. \tag{20}$$

After ρ_1, p_1 are eliminated from (11), (13c) and (19), we finally obtain

$$D^2\nabla^2 w + \left(\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right)\left[D^2\frac{\partial w}{\partial z} - \frac{g}{U^2}\nabla^2_{(x,y)} w\right] = \frac{Q}{U} D^2 \delta(x)\delta(y)\frac{1}{\rho_0}\frac{\partial}{\partial z}\left[\rho_0 \delta(z+h)\right] \tag{21}$$

for $z < 0$, where ∇^2 is the Laplacian with respect to x, y, and z. By eliminating p_1 from (16) and (19), the boundary condition for w on the free surface becomes

$$D^2\frac{\partial w}{\partial z} - \frac{g}{U^2}\nabla^2_{(x,y)} w = -D\nabla^2 p_s(x,y) \qquad (z=0). \tag{22}$$

When w is solved from (21) under conditions (22), (17) and (18), p_1 can be deduced from (19), ρ_1 is given by (11), and u,v can be determined from a combination of (12), (13a) (13b). In fact, it follows from a cross differentiation of (13a), (13b) that $u_y - v_x = 0$, indicating that the flow is irrotational in the planes z = constant. Hence we may define a function $\varphi(x,y,z)$ such that

$$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}, \tag{23}$$

and from (12),

$$\frac{\partial w}{\partial z} + \nabla^2_{(x,y)}\varphi = 0 \qquad \text{except at } (0,0,-h) \tag{24}$$

Furthermore, u and p_1 are also related by the integral of (13a)

$$p_1 = -\rho_0 U^2 u. \tag{25}$$

It is convenient to introduce a characteristic length L based on which we define the following dimensionless quantities

$$x_* = x/L, \quad y_* = y/L, \quad z_* = z/L, \quad h_* = h/L, \quad Q_* = Q/UL^2, \tag{26a}$$

and

$$\lambda = \frac{gL}{U^2}, \qquad \sigma(z) = -\frac{L}{\rho_0}\frac{d\rho_0}{dz}. \tag{26b}$$

There are varions choices for L (e.g., L may be U^2/g, $(-\rho_0 dz/d\rho_0)$, or simply h), and L need not be specified at this stage. We shall however omit the * designating the non-dimensional x,y,z for brevity. Then the dimensionless forms of (21) and (22) are

$$\left\{D^2\nabla^2-\sigma(z)\left[D^2\frac{\partial}{\partial z}-\lambda\nabla^2_{(x,y)}\right]\right\}w=Q_*D^2\delta(x)\delta(y)\left[\delta'(z+h)-\sigma(-h)\delta(z+h)\right], \tag{27}$$

$$\left[D^2\frac{\partial}{\partial z}-\lambda\nabla^2_{(x,y)}\right]w=D\nabla^2_{(x,y)}(p_s/\rho_0U^2) \qquad (z=0). \tag{28}$$

We next introduce the Fourier transform with respect x, y by

$$\tilde{w}(\mu,\nu,z)=\iint_{-\infty}^{\infty}e^{-i\mu x-i\nu y}\,w(x,y,z)\,dx\,dy. \tag{29}$$

Now under condition (17), Eqs. (19), (20) are transformed to

$$\left\{\frac{\partial^2}{\partial z^2}-\sigma(z)\frac{\partial}{\partial z}-(\mu^2+\nu^2)\left(1-\frac{\lambda\sigma(z)}{\mu^2}\right)\right\}\tilde{w}=Q_*\left[\delta'(z+h)-\sigma(-h)\delta(z+h)\right] \quad (z<0) \tag{30}$$

$$\frac{\partial\tilde{w}}{\partial z}-\lambda\frac{\mu^2+\nu^2}{\mu^2}\tilde{w}=i\,\frac{\mu^2+\nu^2}{\mu}\,\frac{\tilde{p}_s}{\rho_0U^2} \qquad (z=0). \tag{31}$$

Furthermore, condition (17) becomes w = 0 at $z = -\infty$ (or $z = -H$ for the case of finite depth channel). Obviously, from Eqs. (30), (31) it follows that $\tilde{w}$ is an even function of ν, i.e. $\tilde{w}(\mu,-\nu,z) = w(\mu,\nu,z)$, hence w will be an even function of y, as must be expected on physical grounds. Hence, upon the inverse transform,

$$w(x,y,z)=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}e^{i\mu x}\,\tilde{w}(\mu,\nu,z)\cos\nu y\,d\mu\,d\nu. \tag{32}$$

The Fourier transform $\tilde{\varphi}$ is related to $\tilde{w}$ by the Fourier transform of (24),

$$\tilde{\varphi}=\frac{1}{\mu^2+\nu^2}\frac{\partial\tilde{w}}{\partial z} \tag{33}$$

from which u, v can be derived from (23), and hence p_1 follows from (25), and ρ_1 from (11).

and $\exp(m_2 z)$, where

$$\left.\begin{matrix} m_1 \\ m_2 \end{matrix}\right\} = \frac{1}{2} \pm \Delta, \qquad \Delta = \sqrt{k^2 - \lambda \csc^2\theta + \frac{1}{4}}. \tag{40}$$

Regarding θ to be real and k complex, we define the branch of Δ by requiring $\Delta \to k$ as $k \to +\infty$, along the positive real k-axis. Then, by noting that $\tilde{G}$ is continuous, but $d\tilde{G}/dz$ has a unit jump acress the point $z = -h$, and that $\tilde{G} \to 0$ as $z \to -\infty$, it can be readily shown that the solution (38), (39) is

$$\tilde{G} = e^{\frac{1}{2}(z+h)} \left\{ \frac{1}{2\Delta} \left[e^{(z-h)\Delta} - e^{-|z+h|\Delta} \right] - \frac{1}{m_1 - \kappa} e^{(z-h)\Delta} \right\} \qquad (z \leq 0), \tag{41}$$

with

$$\kappa = \lambda \csc^2\theta. \tag{42}$$

This $\tilde{G}$ may be regarded as an analytic function of complex k and real θ, with the following singular behavior. First, in the complex k-plane, Δ has two branch points at

$$k = \pm b, \qquad b = \sqrt{\kappa - \frac{1}{4}} = \sqrt{\lambda \csc^2\theta - \frac{1}{4}}. \tag{43}$$

Hence for $\lambda > 1/4$, these branch points are located on the real k-axis. For $0 < \lambda < 1/4$, they are on the real k-axis for $0 \leq \theta \leq \sin^{-1}\sqrt{4\lambda}$, and on the imaginary k-axis for $\sin^{-1}\sqrt{4\lambda} \leq \theta \leq \pi$. In accordance with the branch chosen for Δ, we introduce a straight branch cut between $k = b$ and $k = -b$. Then $\tilde{G}$ is single valued in the entire cut k-plane. In this cut plane, $\tilde{G}$ has a pole at $m_1 - \kappa = 0$, or

$$\left(k^2 - \kappa + \frac{1}{4}\right)^{1/2} = \kappa - \frac{1}{2} \qquad (\kappa = \lambda \csc^2\theta).$$

The right hand of this equation is real and positive for all θ, when $\lambda > \frac{1}{2}$, hence the root may exist only when the left side is also positive and real. It therefore follows that the only(simple) pole is at

$$k = \kappa = \lambda \csc^2\theta. \tag{44}$$

When $\lambda < \frac{1}{2}$, $k = \kappa$ is still the pole for $0 \leq \theta \leq \sin^{-1}\sqrt{2\lambda}$, but for $\sin^{-1}\sqrt{2\lambda} \leq \theta \leq \pi$ the pole becomes $k = -\kappa$.

In most of the stratified fluid occuring in nature, the degree of stratification is always very small, i.e., $L = -\rho_0/(d\rho_0/dy)$ is large in physical dimension. Consequently, the above dimensionless

$\lambda = (gL/U^2)$ is also large. Confining ourselves to the case of practical interest, we shall assume $\lambda > \frac{1}{2}$ in what follows. Then the branch cut lies along the real k-axis between (-b,b) and the simple pole of $\tilde{G}$ is at k = κ. Finally, we apply the inverse transform (see (32)) to (41), change the variables into the polar coordinates (k, θ) and introduce the polar coordinates (r,β) defined by

$$x = r\cos\beta, \quad y = r\sin\beta, \tag{45}$$

Then we obtain the follow expression

$$G = \frac{1}{2\pi^2}\,\mathrm{Re}\int_0^{\pi} d\theta \int_{0\,\Gamma}^{\infty} e^{ikr\sin(\theta-\beta)}\,\tilde{G}(\mu,\nu,z)\,k\,dk \quad \begin{pmatrix} r>0, \\ 0<\beta<\pi \end{pmatrix} \tag{46}$$

in which Re stands for the real part of the integral, and the contour Γ is taken along the positive real k-axis except that it lies below the branch cut from k = 0 to k = b and has a small indentation in the lower half k-plane about the pole at k = κ (see Fig. 1). The indentation is required by the radiation condition (18). The choice of Γ below the branch cut is however less straightforward to see. Physically, this choice of Γ yields the result in which all the waves, including those whose amplitude decays faster than $R^{-\frac{1}{2}}$, lie in the downstream. While this result can be supported by the energy consideration based on the dynamic significance of the group velocity, these indeterminate parts can nevertheless be removed definitely, and the contour Γ justified, by berifying the large time limit of the corresponding initial-value problem. The details of such calculation is given in Appendix I.

With the contour Γ so established, it may be further deformed for $-(\pi-\beta) < \theta < \beta$ to a new path along the entire negative imaginary k-axis (from k = 0 to $-i\infty$) without altering the value of the integral; since in so doing, no singularity of the integrand is crossed and the integral along a large circle $|k| = K$ in the fourth quadrant of the k-plane (see Fig.1) vanishes in the limit as $K \to \infty$. In the range $\beta < \theta < (\pi+\beta)$, we may introduce a closed contour by adding to Γ a large circular arc $|k| = K$ from k = K to k = iK, returning to k = +i0 along the positive imaginary axis, and then circumventing the branch cut around the point k = b back to k = -i0 (see Fig. 1). Within this contour, there is only one simple pole at k = κ. By applying the theorem of residue, we obtain, after some straightforward manipulation, the following result

$$G = G_1 + G_2 + G_3 = G_1 + G_{21} + G_{22} + G_{31} + G_{32}, \tag{47a}$$

$$G_1 = \frac{1}{\pi} e^{h} \int_{\beta}^{\pi} (\kappa - \tfrac{1}{2}) e^{\kappa(z-h)} \sin[\kappa r \sin(\theta-\beta)]\, d\theta, \tag{47b}$$

$$G_{21} = \frac{1}{2\pi^2} e^{\frac{1}{2}(z+h)} \int_{\beta}^{\pi} b(\theta)\, d\theta \int_0^1 \sin[brt \sin(\theta-\beta)] \left\{ \cos(b|z+h|\sqrt{1-t^2}) - \cos(b(h-z)\sqrt{1-t^2}) \right\} \frac{t\, dt}{\sqrt{1-t^2}}, \tag{47c}$$

$$G_{22} = \frac{1}{\pi^2} e^{\frac{1}{2}(z+h)}\, \mathrm{Re} \int_{\beta}^{\pi} b(\theta)\, d\theta \int_0^1 \sin[brt \sin(\theta-\beta)] \frac{e^{ib(h-z)\sqrt{1-t^2}}}{\sqrt{1-t^2} - i(\kappa-\frac{1}{2})/b}\, t\, dt, \tag{47d}$$

$$G_{31} = \frac{e^{\frac{1}{2}(z+h)}}{4\pi^2} \int_0^{\pi} b(\theta)\, d\theta \int_0^{\infty} e^{-brt|\sin(\theta-\beta)|} \left\{ \sin(b(h-z)\sqrt{1+t^2}) - \sin(b|z+h|\sqrt{1+t^2}) \right\} \frac{t\, dt}{\sqrt{1+t^2}}, \tag{47e}$$

$$G_{32} = -\frac{e^{\frac{1}{2}(z+h)}}{2\pi^2}\, \mathrm{Re} \int_0^{\pi} b(\theta)\, d\theta \int_0^{\infty} e^{-brt|\sin(\theta-\beta)|} \frac{e^{ib(h-z)\sqrt{1+t^2}}}{(\kappa-\frac{1}{2})\frac{1}{b} + i\sqrt{1+t^2}}\, t\, dt. \tag{47f}$$

By making use of the integral representation of the Gessel function of the first kind (cf. Warson (1948), p. 359)

$$J_0(\sqrt{a^2+b^2}) = \frac{2}{\pi} \int_0^{\pi/2} \cos(a\cos\theta)\cos(b\sin\theta)\, d\theta$$

G_{21} can be also expressed as

$$G_{21} = \frac{e^{\frac{1}{2}(z+h)}}{4\pi} \int_{\beta}^{\pi} b(\theta)\sin(\theta-\beta) \left\{ \frac{J_1(brS_+(\theta))}{S_+(\theta)} - \frac{J_1(brS_-(\theta))}{S_-(\theta)} \right\} d\theta \tag{48a}$$

where

$$S_{\pm}(\theta) = \left[\sin^2(\theta-\beta) + \frac{(h \pm y)^2}{r^2} \right]^{1/2}. \tag{48b}$$

In the above integral representation of G, G_1 is the contribution from the residue, G_2 comes from the integration around the branch cut, and G_3 from the integration along the imaginary k-axis. From the following study of the far field, it will be seen that G_1 represents the irrotational surface wave, G_2 is the internal gravity wabe due to the stratification effect, and G_3 exhibits a local effect which is signifinant only near the disturbance. As the integral representatipms pf w, φ, p_1, ρ_1 cam be readily deduced from G, they will not be explicitly given here.

4. Asymptotic Behavior of the Wave Field; Internal Waves

We now examine the asymptotic behavior of G in the far field for λr and $\sqrt{\lambda}\, r$ both large. The integral representation of G_1 is familiar in the theory of water waves, its asymtotic wave pattern was calculated first by Kelvin (1887). By using the method of stationary phase it can be shown that for $\lambda r \gg 1$,

$$G_1 \sim \frac{e^h}{\sqrt{2\pi\lambda r}} \sum_{n=1}^{2} \frac{2\lambda \csc^2\theta_n - 1}{\sqrt{|\psi''(\theta_n)|}} e^{\lambda(z-h)\csc^2\theta_n} \sin\left[\lambda r \psi(\theta_n) + (-)^n \frac{\pi}{4}\right] + O\left(\frac{1}{\lambda r}\right) \tag{49}$$

where $$\psi(\theta,\beta) = \csc^2\theta \sin(\theta-\beta), \quad \psi''(\theta,\beta) = (1-2\cot^2\theta)\psi(\theta), \tag{50a}$$

$$\left.\begin{matrix}\theta_1\\ \theta_2\end{matrix}\right\} = \tan^{-1}\left[\frac{1}{2}\cot\beta \mp \frac{1}{2}\sqrt{\cot^2\beta - 8}\,\right]. \tag{50b}$$

The above asymptotic representation is valid for $0 < \beta < \sin^{-1}(1/3)$. In the region near the plane $\beta = \beta_0 = \sin^{-1}(1/3)$, the stationary point of higher order must be considered since $\Psi''(\theta_n, \beta_0) = 0$. It is well known that G_1 falls off like $r^{-1/3}$ in this region. For $\beta_0 < \beta < \pi$, there is no stationary point and G_1 diminishes like r^{-1}. A uniformly valid asymptotic representation for the type of integral as G_1 has been given by Ursell (1964) so we shall not pursue this matter any further. The essential point of the present result is that the phase of the above irrotational free surface waves and their exponential decay in z are both unaf unaffected by the density stratification. Only the amplitude of these tranverse and divergent waves are now changed due to the effect of stratification.

G_{21} and G_{22} can also be evaluated by the stationary phase method. The inner integrals in the expression for G_{21}, G_{22} can be written (by putting $t = \sin\varphi$) in the form

$$F_1(A,B) = \int_0^{\pi/2} e^{i\psi(\varphi)} f(\varphi)\, d\varphi, \qquad \psi(\varphi) = A\sin\varphi + B\cos\varphi \tag{51}$$

where (A^2+B^2) is a larger parameter. The stationary point φ_0 is given by

$$\psi'(\varphi_0) = A\cos\varphi_0 - B\sin\varphi_0 = 0, \quad \text{or} \quad \varphi_0 = \tan^{-1}\frac{A}{B}.$$

Hence there is one stationary point within the range of integration if A, BB are of the same sign, othersi

A, B are of the same sign, otherwise the integral has no stationary point. We further note that

$$\psi''(\varphi_0) = -\psi(\varphi_0) = -(A \sin\varphi_0 + B\cos\varphi_0) = -\sqrt{A^2+B^2}$$

which is always negative. Therefore, by the stationary phase method

$$F_1(A,B) \sim (2\pi)^{1/2}(A^2+B^2)^{-1/4} f(\varphi_0)\, e^{i(A^2+B^2)^{1/2} - i\pi/4} + O(A^2+B^2)^{-1/2} \quad (AB>0), \tag{52}$$

and $F_1(A,B)$ is of order $(A^2+B^2)^{-1/2}$ for $AB<0$ (or when A, B are of opposite sign), as can be readily shown by partial integration. Application of this result to (47c), (47d) yields

$$G_{21} \approx \frac{e^{\frac{1}{2}(z+h)}}{\sqrt{8\pi^3 r}} \,\mathrm{Re}\, e^{-i\frac{3\pi}{4}} \int_\beta^\pi \sqrt{b(\theta)} \left[\frac{e^{ibrS_+(\theta)}}{S_+(\theta)^{3/2}} - \frac{e^{ibrS_-(\theta)}}{S_-(\theta)^{3/2}}\right] \sin(\theta-\beta)\,d\theta \left[1+O\left(\frac{1}{\sqrt{r}}\right)\right] \tag{53}$$

$$G_{22} \sim \frac{e^{\frac{1}{2}(z+h)}}{\sqrt{2\pi^3 r}} \left(\frac{h-z}{r}\right) \mathrm{Re} \int_\beta^\pi \sqrt{b(\theta)}\, \frac{e^{ibrS_-(\theta)-i\pi/4}}{\frac{1}{b}(\kappa-\frac{1}{2})S_-(\theta) + i\frac{h-z}{r}}\, \frac{\sin(\theta-\beta)}{S_-(\theta)^{3/2}}\,d\theta \left[1+O\left(\frac{1}{\sqrt{r}}\right)\right] \tag{54}$$

where $b(\theta)$ is given by (43) and $S_+(\theta)$ by (48b), Incidently, (53) can also be obtained by substituting the asymptotic representation of J_1 in (48a).

To simplify the subsequent calculation of G_{21}, G_{22}, we shall limit ourselves to the case of practical interest with $\lambda \gg 1$ so that

$$b = \sqrt{\lambda \csc^2\theta - \frac{1}{4}} \sim \sqrt{\lambda}\,\csc\theta, \qquad \kappa - \frac{1}{2} = \lambda\csc^2\theta - \frac{1}{2} \sim \lambda\csc^2\theta. \tag{55}$$

Then it suffices to consider the integral of the form

$$F_2(r,\beta,\eta) = \int_\beta^\pi e^{i\sqrt{\lambda}\, r\, \psi(\theta;\beta,\eta)} f(\theta)\,d\theta \qquad (\sqrt{\lambda}\, r \gg 1) \tag{56}$$

where

$$\psi(\theta;\beta,\eta) = \csc\theta\left[\sin^2(\theta-\beta) + \eta^2\right]^{1/2}. \tag{57}$$

The point of stationary phase is given by

$$\psi'(\theta_0;\beta,\eta) = \csc^2\theta_0\left[\sin\beta\sin(\theta_0-\beta) - \eta^2\cos\theta_0\right]\left[\sin^2(\theta-\beta)+\eta^2\right]^{-1/2} = 0 \tag{58}$$

or

$$\tan\theta_0 = \frac{\eta^2 + \sin^2\beta}{\sin\beta\cos\beta}. \tag{58}$$

The function $\theta_0 = \theta_0(\beta, \eta)$ is symmetric about the point ($\beta = \pi/2$, $\theta_0 = \pi/2$) for all η. Also, $\theta_0 = \pi/2$ at $\beta = 0, \pi$ if $\eta \neq 0$. Further, $\partial\theta_0/\partial\beta = (1+\eta^2)^{-1}$ at $\beta = \pi/2$. Hence, as shown in Fig. 2, $\theta_0 \gtrless \beta$ according as $\beta \lessgtr \pi/2$. Consequently, there is only one stationary point given by (58) for $0 \leqslant \beta \leqslant \pi/2$, and no stationary point exists for $\pi/2 < \beta \leqslant \pi$. This further implies that the waves contained in F_2 exist only in the downstream (or $x > 0$). Using (58), we deduce that

$$\sin(\theta_0 - \beta) = \eta^2(\cos\beta)\left[\eta^4 + (2\eta^2+1)\sin^2\beta\right]^{-1/2},$$

$$S(\theta_0) = \sqrt{\sin^2(\theta_0-\beta)+\eta^2} = \eta(1+\eta^2)^{1/2}(\eta^2+\sin^2\beta)^{1/2}\left[\eta^4+(2\eta^2+1)\sin^2\beta\right]^{-1/2}, \tag{59}$$

$$\psi(\theta_0;\beta,\eta) = (\csc\theta_0)\,S(\theta_0) = \eta(1+\eta^2)^{1/2}(\eta^2+\sin^2\beta)^{-1/2},$$

$$\psi''(\theta_0;\beta,\eta) = \left[\eta^4+(2\eta^2+1)\sin^2\beta\right]^2\eta^{-1}(1+\eta^2)^{-1/2}(\eta^2+\sin^2\beta)^{-5/2}.$$

Finally, by the principle of stationary phase, we obtain for $0 < \beta < \pi/2$,

$$F_2(r,\beta,\eta) \sim \left[\frac{2\pi}{\sqrt{\lambda}\, r\, \psi''(\theta_0)}\right]^{1/2} f(\theta_0)\, e^{i\sqrt{\lambda}\, r\, \psi(\theta_0) + i\pi/4}\left[1 + O\left(\frac{1}{\sqrt{\lambda} r}\right)^{1/2}\right], \tag{60}$$

and $F_2 \sim O(r^{-1})$ for $\pi/2 < \beta < \pi$. This result is valid uniformly in β and η since $\psi''(\theta_0)$ never vanishes. By using this formula, we obtain from (53), (54), after some simplification, that for $x > 0$,

$$G_{21} \sim \frac{\lambda^{1/4} e^{\frac{1}{2}(z+h)}}{2\pi r}\left\{B_+ \sin(\sqrt{\lambda}\, r\, \psi_+(\theta_0)) - B_- \sin(\sqrt{\lambda}\, r\, \psi_-(\theta_0))\right\}\left[1 + O\left(\frac{1}{\lambda r^2}\right)^{1/4}\right], \tag{61}$$

$$G_{22} \sim \frac{\lambda^{1/4} e^{\frac{1}{2}(z+h)}}{\pi r}\left\{\frac{\eta_- B_-}{\sqrt{\lambda\psi_-^2(\theta_0)+\eta_-^2}}\cos\left(\sqrt{\lambda}\, r\, \psi_-(\theta_0) - \tan^{-1}\frac{\eta_-}{\sqrt{\lambda}\,\psi_-(\theta_0)}\right)\right\}\left[1 + O\left(\frac{1}{\lambda r^2}\right)^{1/4}\right] \tag{62}$$

where

$$\psi_\pm(\theta_0) = \psi(\theta_0;\beta,\eta_\pm), \qquad B_\pm = B(\beta,\eta_\pm), \qquad \eta_\pm = \frac{|h \pm z|}{r}, \tag{63}$$

$$B(\beta,\eta) = \frac{\eta\cos\beta}{(1+\eta^2)^{1/2}(\eta^2+\sin^2\beta)^{1/2}\left[\eta^4+(2\eta^2+1)\sin^2\beta\right]^{1/4}}. \tag{64}$$

For $x < 0$, G_{21} and G_{22} both decay like $r^{-3/2}$ for large r.

As both η and $\sin\beta$ are ratio of coordinates, (61) and (62) indicate clearly that G_{21} and G_{22} have amplitude of order r^{-1} in the wave field. The energy of wave G_2 is therefore transmitted downstream at a rate proportional to R^{-2} cross a large sphetical surface of radius R. Consequently, this wave mode G_2, like G_1, is also radiating inergy at the expense of an additional resistance on the disturbance. Unlike G_1, however, the wave mode G_2 is rotational and generates vorticity; it therefore makes contribution to the internal dissipation.

In order to exhibit the effect of different strength of stratification, it is useful to testore σ at this stage. This is done simply by adopting a new characteristic length L' (for example h) based on which we write σ', λ', x' etc. as the new dimensionless quantities. Then $\lambda = \lambda'/\sigma'$, $\kappa = \kappa'\sigma'$, $r = r'\sigma'$ etc. Expressing (61), (62) explicitly in terms of x', y', z', λ' and σ', and then dfopping the primes, we obtain, for $x > 0$,

$$G_{21} \sim \frac{\lambda^{1/4}\sigma^{-5/4}}{2\pi} e^{\frac{1}{2}\sigma(z+h)} \left\{ A_+ \sin(\sqrt{\lambda\sigma}\,\Psi_+) - A_- \sin(\sqrt{\lambda\sigma}\,\Psi_-) \right\} \left[1+O\left(\frac{1}{\lambda r^2}\right)^{1/4}\right] \qquad (65)$$

$$G_{22} \sim \frac{1}{\pi} \lambda^{1/4}\sigma^{-5/4} e^{\frac{1}{2}\sigma(z+h)} \left\{ \frac{A_-}{\sqrt{\lambda(\Psi_-/z_-)^2+\sigma}} \cos\left[\sqrt{\lambda\sigma}\,\Psi_- - \tan^{-1}\left(\sqrt{\frac{\sigma}{\lambda}}\frac{z_-}{\Psi_-}\right)\right] \right\} \left[1+O\left(\frac{1}{\lambda r^2}\right)^{1/4}\right] \qquad (66)$$

where

$$\Psi_\pm = \Psi(x,y,z_\pm), \quad A_\pm = A(x,y,z_\pm), \quad z_\pm = |z \pm h|, \qquad (67)$$

$$\Psi(x,y,z) = |z| \left[\frac{x^2+y^2+z^2}{y^2+z^2} \right]^{1/2}, \qquad (68)$$

$$A(x,y,z) = \frac{x\,|z|}{(y^2+z^2)^{1/2}(x^2+y^2+z^2)^{1/2}\left[(y^2+z^2)^2+x^2y^2\right]^{1/4}}. \qquad (69)$$

In the above equations (65)-(68), all length, spatial coordinates, λ and σ are referred to an arbitrary base length L'. In usual circumstances σ is very small when L' is of order 100 meters of less. Hence the vertical decay indicated by the factor $\exp(\sigma z/2)$ in (65), (66) is much slower compared with the decay of the surface wave G_1 (see (49)). The wave profile of G_2 thus may have a wider vertical spread.

A characteristic wavelength of G_2 is

$$\ell_2 = \frac{2\pi}{\sqrt{\lambda\sigma}} \tag{70}$$

which is the wavelength in the x = 0 and in y = 0 planes. This wavelength is large compared with that of G_1 when $\sigma/\lambda \ll 1$, which is generally the case. This wave mode G_2 will be called internal gravity wave.

To investigate further the features of this wave, we consider the typical term

$$W = A(x,y,z) \sin\left[\frac{2\pi}{\ell_2} \Psi(x,y,z)\right] \qquad (x>0). \tag{71}$$

Since W is symmetric with respect to zz= 0 and y = 0, we shall consider only $z>0$, $y>0$. In terms of the spherical polar coordinates (R, θ, φ) defined by (see Fig. 3)

$$x = R\cos\theta, \qquad y = R\sin\theta\cos\varphi, \qquad z = R\sin\theta\sin\varphi \tag{72}$$

the phase function and amplitude of wave W become

$$\Psi(x,y,z) = R\sin\varphi = \frac{z}{\sin\theta}, \tag{73}$$

$$A = \frac{1}{R}\,\frac{\cos\theta\sin\varphi}{(\sin\theta)^{1/2}(1-\sin^2\varphi\cos^2\theta)^{1/4}} \tag{74}$$

Hence on the equi-phase surface $\Psi/\ell = \frac{1}{2}(n+\frac{1}{2})$, n = 0,1,2.... (they being the wave crests), we have

$$\frac{R}{\ell}\sin\varphi = \frac{z}{\ell}\,\frac{1}{\sin\theta} = \frac{1}{2}\left(n+\frac{1}{2}\right) \qquad (n=0,1,2,\cdots). \tag{75}$$

At x = 0, (or $\theta = \pi/2$), the wave crests start from horizontal lines $z_n/\ell = \pm(2n+1)/4$, but with vanishing amplitude (see (74)). These constant phase surfaces intercept the planes φ = const along concentric circles, $p_n/\ell = (2n+1)/(4\sin\varphi)$, with wavelength inversely proportional to sin φ, and with amplitude increasing with x, as shown in Fig. 3a. The second form of (75) is perhaps more useful for constructing the equiphase surface. It shows that the intersections between the equiphase surfaces and the cone θ = constant are level curves, and hence are hyperbolas, along each of which the amplitude A decreases monotonically with increasing x. These salient features are illustrated in Fig. 4. Different equi-phase surfaces (Ψ_n = const.) all descend downwards in the streamwise direction to form a long-trenched valley, reaching its wedge-shaped bottom along the x-axis from $x_n = (2n+1)\ell/4$ to $x = \infty$. The steepest path of descent (down the phase surface) is the circular arc in the symmetry plane $y = 0$. The amplitude of the wave increases at the fastest rate along this steepest path of descent until it reaches the valley bottom along which the amplitude A becomes unbounded. For x so large that a great number of wave-crest surfaces have intercepted the x-axis, the oscillation is very fast along a small circle around the x-axis. Consequently it may be conjectured that in consturcting physical problems by superposition of singularities, the fast oscillations behind each source element tend to cancel out one another under the mutual influence with the neighboring source elements.

The integral representing G_3 given by (47)e, f) can be shown to fall off like R^{-2} as $R \to \infty$, it therefore represents a local effect and is not important in the outer wave field. The calculation of G_3 hence will not be further pursued here.

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Appendix I

Large Time Limit of a Corresponding Initial-Value Flow Problem

We evaluate below the large time limit of a corresponding initial value flow problem in which the concentrated source behaves like a step function H (t) of the time t. The corresponding equations governing w are still (21) and (22) given in the text except that a step function H(t) must be inserted in the forcing terms and that the operator D now assumes the new form

$$D = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$$

with an appropriate time unit. After application of a double Fourier transform with respect to x,y and Laplace transform with respect to t, the function $G_1(x,y,z,t)$ as defined by (36) now has its transform $\tilde{\bar{G}}_1(\mu,\nu,z,s)$ satisfying

$$\left[\frac{d^2}{dz^2} - \frac{d}{dz} - (k^2 - \kappa_1)\right]\tilde{\bar{G}}_1 = \frac{1}{s}\,\delta(z+h) \qquad (z<0),$$

$$\left(\frac{d}{dz} - \kappa_1\right)\tilde{\bar{G}}_1 = 0 \qquad (z=0)$$

with

$$\kappa_1 = \lambda\left(\sin\theta - i\frac{s}{k}\right)^{-2},$$

where s is the Laplace transform variable and the bar over G denotes the Laplace transform. Consequently the solution of $s\tilde{\bar{G}}_1 = \tilde{\bar{G}}_o(\mu,\nu,z,s)$ still has the same expression as (41) except that κ_1 replaces κ and

$$\Delta = \left[k^2 - \lambda\left(\sin\theta - i\frac{s}{k}\right)^{-2} + \frac{1}{4}\right]^{1/2}.$$

By the Laplace inversion,

$$G_1(\vec{x},t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\bar{G}_o(\vec{x},s)\,\frac{ds}{s} \qquad (c>0).$$

Next, following De Prima and Wu (1956), we evaluate the large time limit of $G_1(\vec{x},t)$ by applying the Tauberian theorem which states that

$$\lim_{t\to+\infty} G_1(\vec{x},t) = \lim_{s\to 0+} \bar{G}_o(\vec{x},s)$$

if and only if

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t t\,\frac{\partial G_1}{\partial t}\,dt = 0.$$

Hence, to obtain the large time limit, it is necessary to evaluate the Laplace transform only for small positive s. In this limit, it is immediately seen that the above Δ vanishes at

$$k = \pm b + is\frac{\lambda}{b^2}\csc^3\theta + O(s^2), \qquad \left(b = \sqrt{\lambda\csc^2\theta - \tfrac{1}{4}}\right).$$

Hence for $\lambda > 1/4$ and $0 < \theta < \pi$, the two branch points are in the upper half k-plane, symmetric about the imaginary axis. With a straight branch cut made in between these two points, it is found that $(m_1 - \kappa_1) = \frac{1}{2} - \kappa_1 + \Delta$ vanishes for $\lambda > \frac{1}{2}$ at

$$k = \lambda\csc^2\theta + 2is\csc\theta + O(s^2) = \kappa + 2is\csc\theta + O(s^2).$$

Hence for $0 < \theta < \pi$, the simple pole is in the first quadrant of the k-plane. Therefore, in the limit as $s \to 0+$, the integration path for the k-integral in (46) of the text may be deformed to Γ as described therein. In the same limit, it is obvious that $\tilde{\tilde{G}}_0(\mu, \nu, z, s) \to \tilde{G}(\mu, \nu, z)$ given by (40). It then follows that the present solution is the steady state limit of $G_1(x,y,z,t)$ provided the above necessary and sufficient condition of the Tauberian Theorem is also satisfied. The fulfillment of this condition can be shown in a manner similar to the verification by De Prima and Wu (1956). In short, it follows from the result that the time dependent part of G_1 behaves like $t^{\alpha}\exp(i\sigma t)$ for $0 < \alpha < 1$ and some real σ. The details of such evaluation, however, will be omitted here.

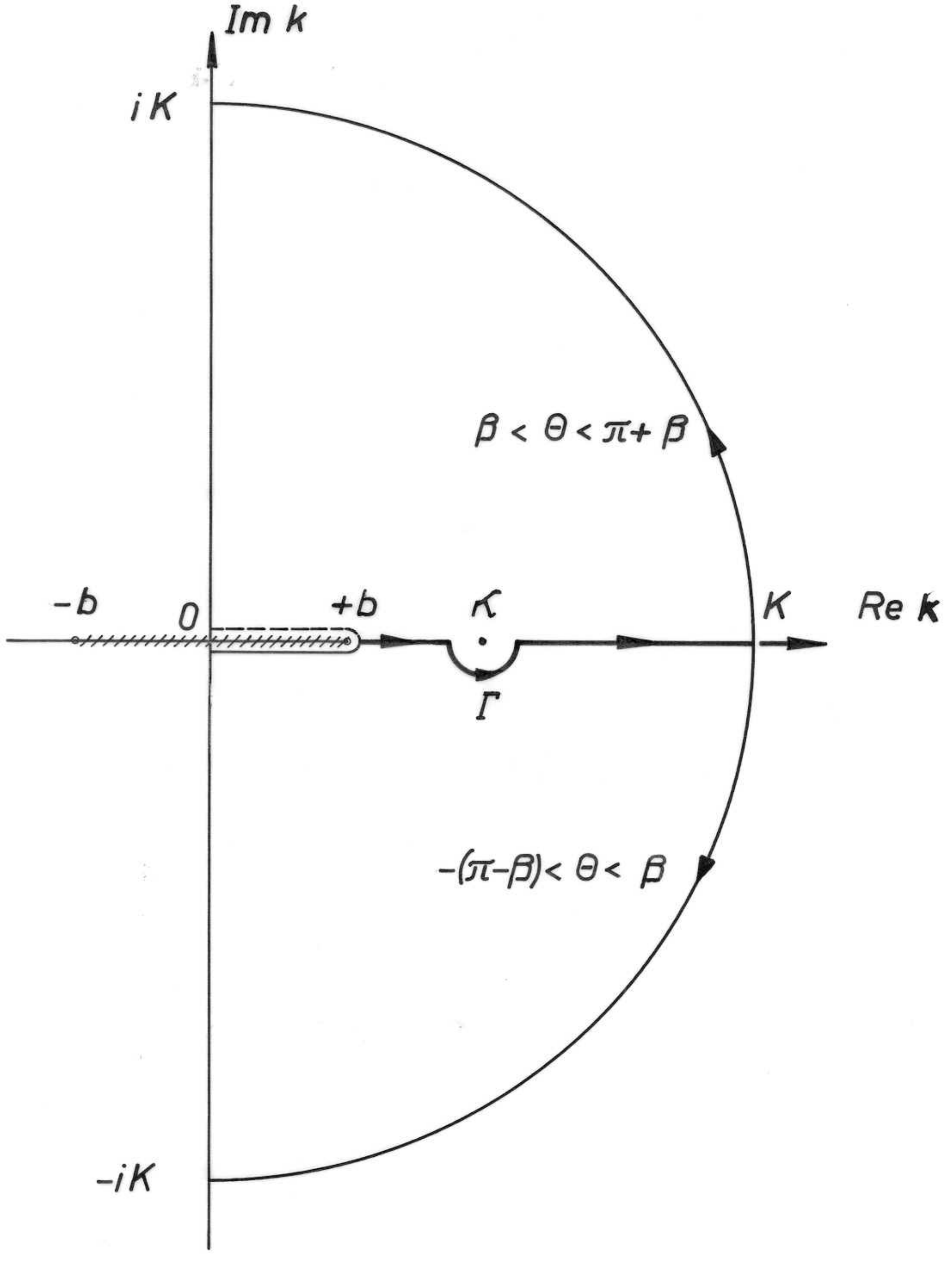

Figure 1

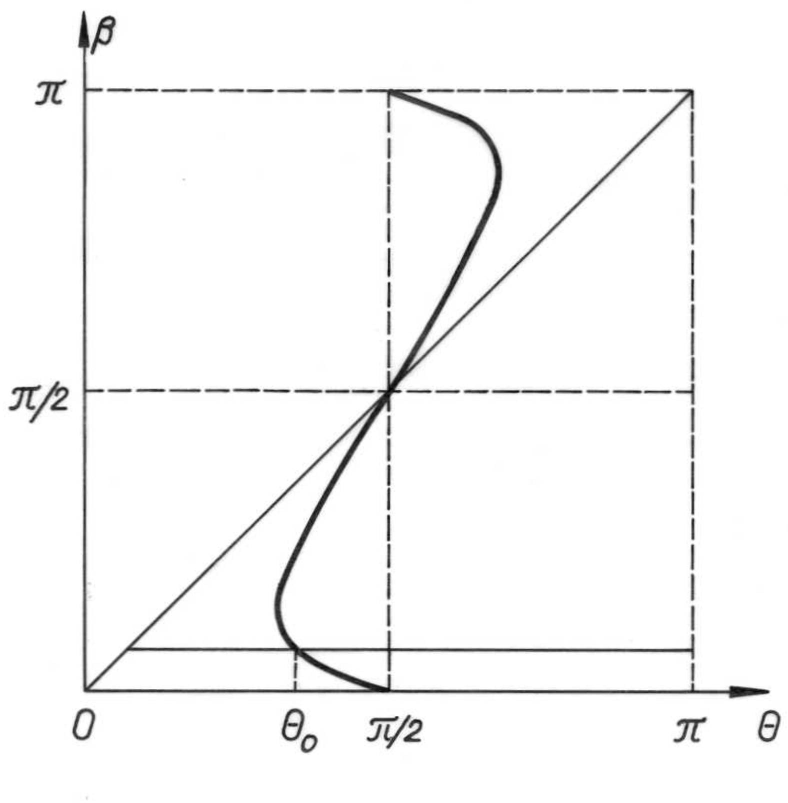

Figure 2

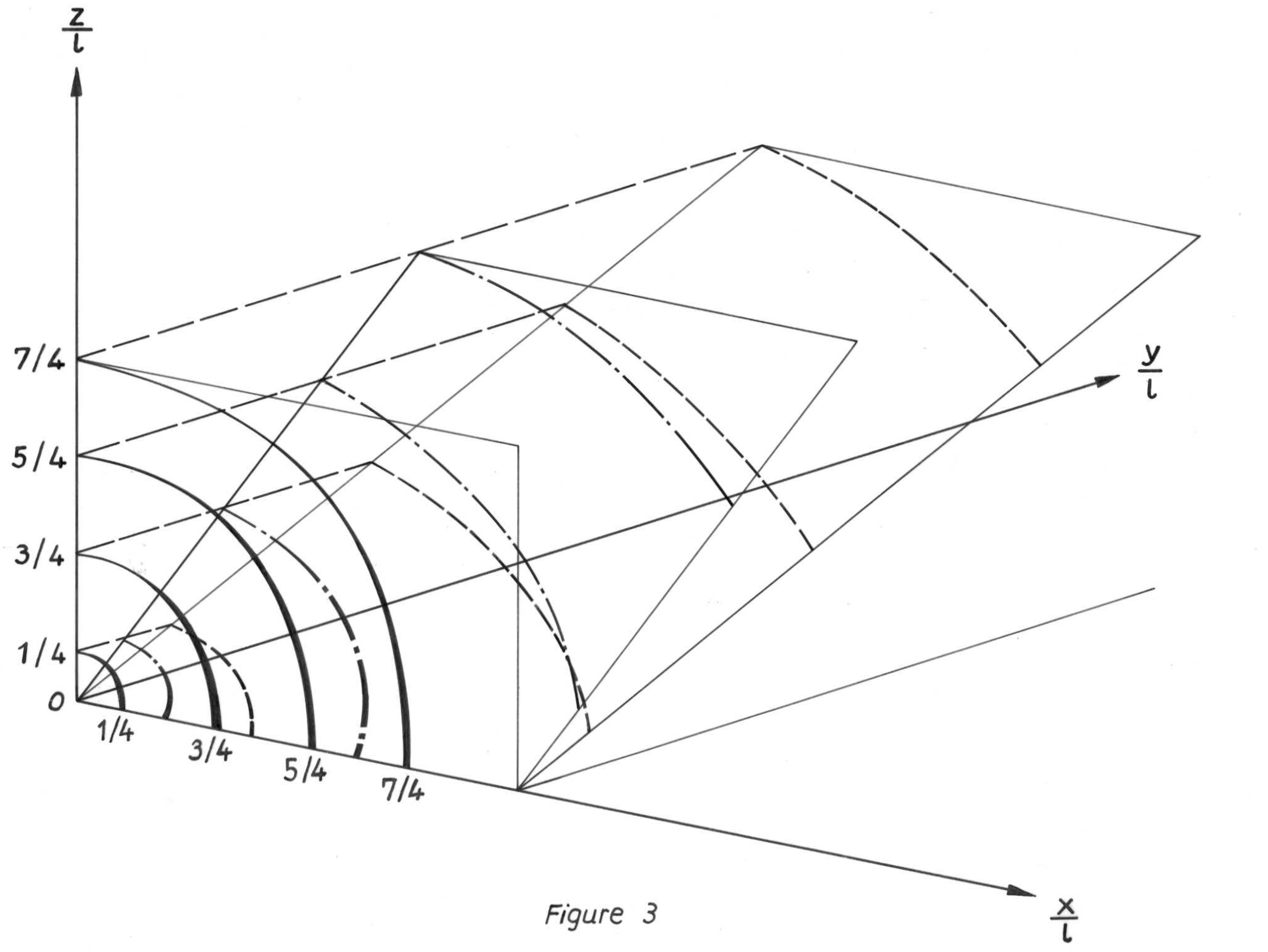

Figure 3

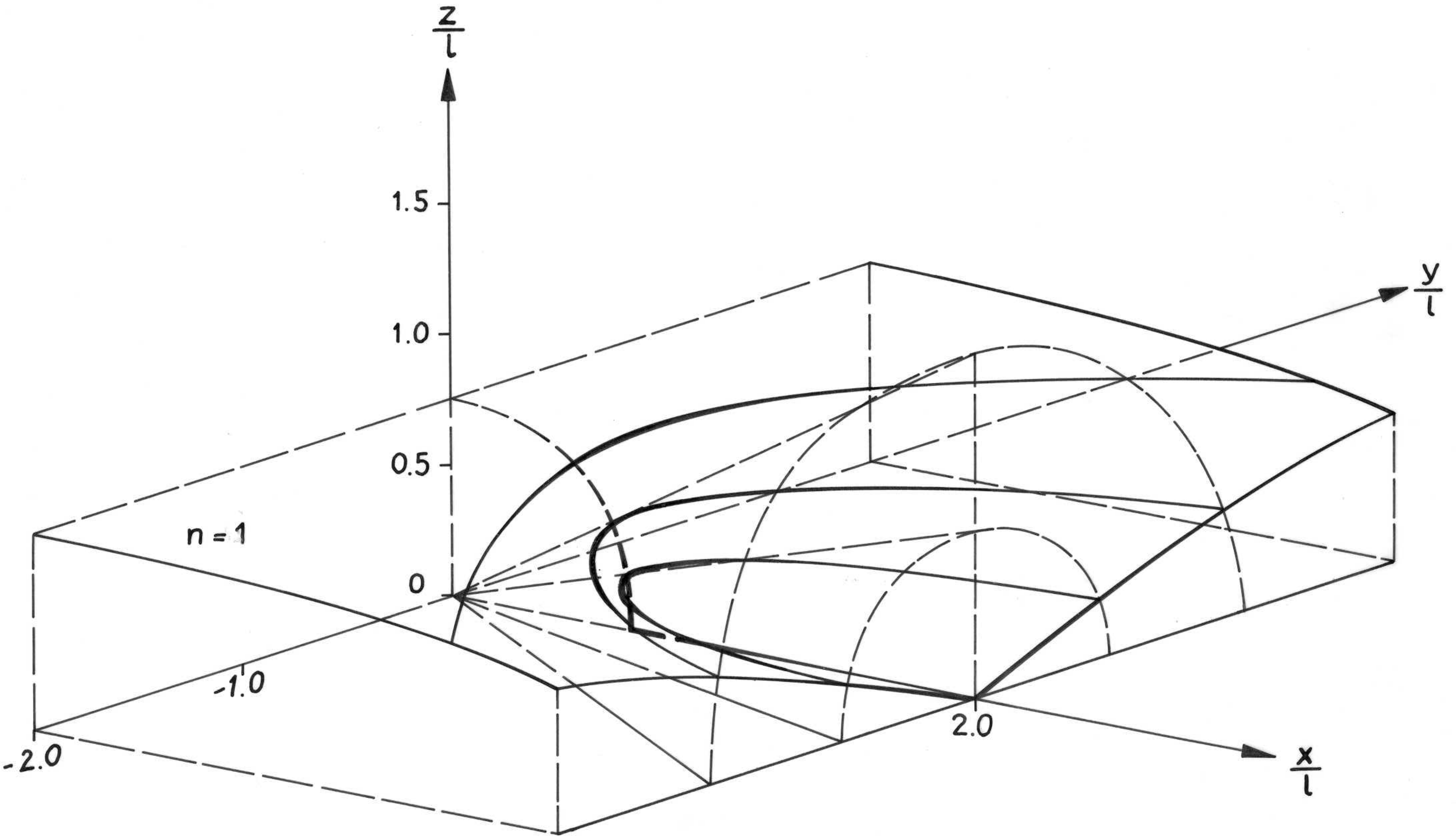

Figure 4